A Measurement-Theoretic Modification of Harvey's Aggregation Theorem

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Harsanyi's Aggregation Theorem (1)

- Harsanyi (1955) attempts to develop expected utility theory of von Neumann and Morgenstern (1944) to provide a formalization of (weighted) utilitarianism.
- Weymark (1991) refers to this result as Harsanyi's Aggregation Theorem.

Harsanyi's Aggregation Theorem (2)

- Harsanyi, J.C.: Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility. Journal of Political Economy 63, 309–321 (1955)
- von Neumann, J., Morgenstern, O.: Theory of Games and Economic Behavior. Princeton University Press, Princeton (1944)
- Weymark, J.A.: A Reconsideration of the Harsanyi-Sen Debate on Utilitarianism. In: Elster, J., Roemer, J.E. (eds.): Interpersonal Comparisons of Well-Being, pp. 255–320. Cambridge University Press, Cambridge (1991)

Measurement-Theoretic Concepts

Here we would like to define such measurement-theoretic concepts as

1. scale types,
2. representation and uniqueness theorems, and
3. measurement types

on which the argument of this talk is based:

Scale Types (1)

First, we classify scale types in terms of the class of admissible transformations φ.

- A scale is $\langle \mathfrak{U}, \mathfrak{B}, f \rangle$ or f, where $\mathfrak{U}$ is an observed relational structure that is qualitative, $\mathfrak{B}$ is a numerical relational structure that is quantitative, and f is a homomorphism from $\mathfrak{U}$ into $\mathfrak{B}$.
- A is the domain of $\mathfrak{U}$ and B is the domain of $\mathfrak{B}$.
- When the admissible transformations are all the functions $\varphi : f(A) \to B$, where $f(A)$ is the range of f, of the form $\varphi(x) := \alpha x; \alpha > 0$.
- φ is called a similarity transformation, and a scale with the similarity transformations as its class of admissible transformations is called a ratio scale.
- Length is an example of a ratio scale.

Scale Types (2)

- When the admissible transformations are all the functions $\varphi : f(A) \to B$ of the form $\varphi(x) := \alpha x + \beta; \alpha > 0$, φ is called a positive affine transformation, and a corresponding scale is called an interval scale.
- Temperature on the Fahrenheit scale and temperature on the Celsius scale are examples of interval scales.

Scale Types (3)

- When a scale is unique up to order, the admissible transformations are monotone increasing functions φ satisfying the condition that $x \geq y$ iff $\varphi(x) \geq \varphi(y)$.
- Such scales are called ordinal scales.
- The Mohs scale is an example of a ordinal scale.
- A scale is called a log-interval scale if the admissible transformations are functions φ of the form $\varphi(x) := \alpha x^{\beta}; \alpha, \beta > 0$.
- Psychophysical functions are examples of log-interval scales.

Representation and Uniqueness Theorems

Second, we state about representation and uniqueness theorems. There are two main problems in measurement theory:

1. the representation problem: Given a quantitative (numerical) relational structure $\mathfrak{B}$, find conditions on a qualitative relational structure $\mathfrak{U}$ (necessary and) sufficient for the existence of a homomorphism f from $\mathfrak{U}$ to $\mathfrak{B}$ that preserves all the relations and operations in $\mathfrak{U}$.
2. the uniqueness problem: Find the transformation of the homomorphism f under which all the relations and operations in $\mathfrak{U}$ are preserved.

- A solution to the former can be furnished by a representation theorem that specifies conditions on $\mathfrak{U}$ are (necessary and) sufficient for the existence of f.
- A solution to the latter can be furnished by a uniqueness theorem that specifies the transformation up to which f is unique.

Measurement Types

Third, we classify measurement types: Suppose A is a set, $\succ$ is a binary relation on A, $\bigcirc$ is a binary operation on A, $\succ'$ is a quaternary relation on A, and f is a real-valued function. Then we call

the representation $a \succ b$ iff $f(a) > f(b)$

ordinal measurement. We call

the representation $a \succ b$ iff $f(a) > f(b)$ and $f(a \bigcirc b) = f(a) + f(b)$

extensive measurement. We call

the representation $(a,b) \succ' (c,d)$ iff $f(a) - f(b) > f(c) - f(d)$

algebraic-difference measurement. We call

the representation $(a,b) \succ' (c,d)$ iff $\dfrac{f(a)}{f(b)} > \dfrac{f(c)}{f(d)}$.

algebraic-quotient measurement.

Harsanyi's Aggregation Theorem Again

In terms of these measurement-theoretic concepts, Harsanyi's Aggregation Theorem can be stated in the following way:

Theorem (Harsanyi's Aggregation Theorem)

Suppose that individual and social binary preference relations $\succsim_i$ $(i = 1, \ldots, n)$ and $\succsim$ on the set of lotteries satisfy von Neumann-Morgenstern axioms, and also suppose that $\succsim_i$ and $\succsim$ satisfy the Strong Pareto condition. Furthermore, suppose that $\succsim_i$ and $\succsim$ are represented by individual and social expected utility functions $U_i (i = 1, \ldots, n)$ and U respectively. Then, there are real numbers $\alpha_i (> 0) (i = 1, \ldots, n)$ and β such that

$$U(p) = \sum_{i=1}^{n} \alpha_i U_i(p) + \beta,$$

for any lottery p.

Weighted Utilitarianism on Set of Lotteries

The next corollary directly follows from this theorem:

Corollary (Weighted Utilitarianism on Set of Lotteries)

Lotteries are socially ranked according to a weighted utilitarian rule:

$$U(p) \geq U(q) \text{ iff } \sum_{i=1}^{n} \alpha_i U_i(p) \geq \sum_{i=1}^{n} \alpha_i U_i(q),$$

for any lotteries p.q.

Representation and Uniqueness Lemmas

Harsanyi's Aggregation Theorem follows from the next lemmas:

Lemma (Representation)

Suppose that $\succsim_i$ $(i = 1, \ldots, n)$ and $\succsim$ satisfy Weak Order, Continuity, and Independence. Then, there exist individual and social expected utility functions $U_i (i = 1, \ldots, n)$ and U such that

$$\begin{cases} p \succsim_i q \text{ iff } U_i(p) \geq U_i(q), \\ p \succsim q \text{ iff } U(p) \geq U(q), \end{cases}$$

for any lotteries p, q.

Lemma (Uniqueness)

Suppose that $\succsim_i$ $(i = 1, \ldots, n)$ and $\succsim$ on the set of lotteries satisfy not only the conditions for the representation above but also Nondegeneracy. Then, the individual and social expected utility functions U_i and U are unique up to a positive affine transformation, that is, U_i and U are interval scales.

Sen's Criticism

- There are at least two well-known criticisms on Harsanyi's Aggregation Theorem.
- The first criticism is by Sen (1976):
- Von Neumann-Morgenstern axioms on individual and social binary preference relations in Lemma (Representation) are for ordinal measurement and, therefore, any monotone increasing (even non-affine) transform of an expected utility function is a satisfactory representation of individual and social binary preference relations.
- However, (weighted) utilitarianism requires a theory of cardinal utility, and so Harsanyi is not justified in giving his theorems utilitarian interpretations.

Sen, A.: Welfare inequalities and Rawlsian axiomatics. Theory and Decision 7, 243–262 (1976)

Probability Agreement Theorem

- The second criticism is based on the following probability agreement theorem that is provided by Broome (19910:

Theorem (Probability Agreement Theorem)

Suppose that individual and social binary preference relations $\succsim_i$ $(i = 1, \ldots, n)$ and $\succsim$ on the set of lotteries satisfy von Neumann-Morgenstern axioms. Then, $\succsim_i$ and $\succsim$ cannot satisfy the strong Pareto condition unless every individual agrees about the probability of every elementary event.

Broome, J.: Weighing Goods. Blackwell, Oxford (1991)

- In fact, under many circumstances, the members of a society have different beliefs (probabilities) of elementary events.

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Harvey's Aggregation Theorem (1)

In order to escape these two criticisms, we can resort to Harvey's Aggregation Theorem (1999) that has quaternary preference relations as primitive that can be represented by utility differences, and is concerned only with quaternary preference relations on the set of outcomes but is not concerned with binary preference relations on the set of lotteries in Harsanyi's Aggregation Theorem.

Harvey, C.M.: Aggregation of individuals' preferences intensities into social preference intensity. Social Choice and Welfare 16, 65–79 (1999)

History of Cardinal Utility (1)

- Lange (1934) is the first to connect formally the ranking of utility differences with positive affine transformations of utility functions.
- However, he does not use the expression "cardinal utility".
- Alt (1936) is considered to be the first to prove the representation theorem for quaternary preference relations that can be represented by utility differences, and the uniqueness theorem on the uniqueness of the utility functions up to positive affine transformations.
- However, he also dose not connect utility differences with the expression "cardinal utility".
- Samuelson (1938) is the first to connect utility differences in which utility functions are unique up to positive affine transformations "cardinal utility", though he takes a negative position toward cardinal utility.

History of Cardinal Utility (2)

Lange, O.: The determinateness of the utility function. Review of Economic Studies 1, 218–224 (1934)

Alt, F.: Über die Mäßbarkeit des Nutzens. Zeitschrift für Nationalökonomie 7, 161–169 (1936)

Samuelson, P.: The numerical representation of ordered classifications and the concept of utility. Review of Economic Studies 6, 65–70 (1938)

Harvey's Aggregation Theorem (2)

Harvey (1999, p.69) defines difference-worth conditions as follows:

> *We will use conditions on a quaternary preference relation $\succsim$ as any set of conditions that are satisfied iff there exists a worth function w such that*
>
> $$(a,b) \succsim (c,d) \text{ iff } w(a) - w(b) \geq w(c) - w(d)$$
>
> *for any outcome a, b, c, d, and we will refer to any such conditions as a set of difference-worth conditions.*

Harvey's Aggregation Theorem (3)

Then Harvey's Aggregation Theorem can be stated in the following way:

Theorem (Harvey's Aggregation Theorem)

Suppose that individual and social quaternary preference relations $\succsim_i$ $(i = 1, \dots, n)$ and $\succsim$ on the set of outcomes satisfy a certain set of difference-worth conditions. Then, $\succsim_i$ and $\succsim$ satisfy the strong Pareto condition iff there are real numbers $\alpha_i (> 0)(i = 1, \dots, n)$ and β such that

$$w(a) = \sum_{i=1}^{n} \alpha_i w_i(a) + \beta,$$

for any outcome a.

Weighted Utilitarianism on Set of Outcomes

The next corollary directly follows from this theorem:

Corollary (Weighted Utilitarianism on Set of Outcomes)

Outcomes are socially ranked according to a weighted utilitarian rule.

Representation and Uniqueness Lemmas

Harvey's Aggregation theorem follows from the next lemmas:

Lemma (Representation)

Suppose that $\succsim_i$ $(i = 1, \dots, n)$ and $\succsim$ on the set of outcomes satisfy a certain set of difference-worth conditions. Then, there exist individual and social worth functions $w_i (i = 1, \dots, n)$ and w such that

$$(1) \quad \begin{cases} (a,b) \succsim_i (c,d) \text{ iff } w_i(a) - w_i(b) \geq w_i(c) - w_i(d), \\ (a,b) \succsim (c,d) \text{ iff } w(a) - w(b) \geq w(c) - w(d), \end{cases}$$

for any outcome a, b, c, d.

Lemma (Uniqueness)

Suppose that $\succsim_i$ $(i = 1, \dots, n)$ and $\succsim$ on the set of outcomes satisfy the conditions for the representation above. Then, $w_i (i = 1, \dots, n)$ and w are unique up to a positive affine transformation, that is, w_i and w are interval scales.

Because any set of difference-worth conditions is for algebraic-difference measurement that is a kind of cardinal measurement, this theorem can escape the first criticism.

Hammond's Position

- When Hammond (1982) attempts to salvage utilitarianism in the way that the (strong) Pareto condition can apply only to outcomes.
- Harvey takes the same position as Hammond that enables this theorem to escape the second criticism.

Hammond, P.: Utilitarianism, uncertainty and information. In: Sen, A., Williams, B. (eds.) Utilitarianism and Beyond, pp. 85–102. Cambridge University Press, Cambridge (1982)

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First Criticism

- Now we inspect Harvey's Aggregation Theorem from a measurement-theoretic point of view.
- We offer two criticisms on Harvey's Aggregation Theorem: The first criticism is as follows:
- As Roberts (1979, p.139) says, the only set of necessary and sufficient difference-worth conditions is due to Scott (1964) and requires the assumption that the set of outcomes is finite.

Roberts, F.S.: Measurement Theory. Addison-Wesley, Reading (1979)

Scott, D.: Measurement structures and linear inequalities. Journal of Mathematical Psychology 1, 233–247 (1964)

- So when there is no domain-size limitation, the set of necessary and sufficient difference-worth conditions is still unknown.

Second Criticism (1)

- The second criticism is as follows:
- The most essential task of aggregation theorem from a measurement-theoretic point of view is to prove the existence of individual and social worth functions that represent individual and social quaternary preference relations which satisfy not only difference-worth conditions but also the strong Pareto condition.
- However, Harvey's Aggregation Theorem is not of such a form.

Second Criticism (2)

- For, in Lemma (Representation), individual and social quaternary preference relations satisfy only difference-worth conditions.
- So the existence of individual and social worth functions that represent individual and social quaternary preference relations which satisfy both difference-worth conditions and the strong Pareto condition is not guaranteed.

Second Criticism (3)

- Harvey (1999, p.72) comments on the feature of his own theorem:

  > *I view the result in Harsanyi (1955) and the result presented here as uniqueness results rather than as existence results. ... an expected-utility function or a worth function is unique up to a positive affine function.*

- Then, does what Harvey says keep to the point?
- What should be proved is the uniqueness of individual and social worth functions that represent individual and social quaternary preference relations which satisfy not only difference-worth conditions but also the strong Pareto condition.

Second Criticism (4)

- However, in Lemma (Uniqueness), individual and social quaternary preference relations satisfy also only difference-worth conditions.
- So the uniqueness of individual and social worth functions that represent individual and social quaternary preference relations which satisfy both difference-worth conditions and the strong Pareto condition is not guaranteed either.
- After all, Harvey's Aggregation Theorem can give any answer neither to the representation problem nor to the uniqueness problem.

Contents of This Talk

Our Aggregation Representation Theorem (1)

- The aim of this talk is that we show new aggregation theorems, which escape these two criticisms, inspired by Harvey's Aggregation Theorem.
- Our aggregation representation and uniqueness theorems can be stated in the following way:

Theorem (Aggregation Representation Theorem)

Suppose that individual and social quaternary preference relations $\succsim_i$ $(i = 1, \ldots, n)$ and $\succsim$ on the set of outcomes satisfy Weak Order, Order Reversal, Weak Monotonicity, Soluvability and Archimedean condition in Krantz et al. (1971), and also suppose that $\succsim_i$ and $\succsim$ satisfy the strong Pareto condition. Then, there EXIST individual and social utility functions $u_i(i = 1, \ldots, n)$ and u such that NOT ONLY

$$(1)\ \begin{cases} (a,b) \succsim_i (c,d) \text{ iff } u_i(a) - u_i(b) \geq u_i(c) - u_i(d), \\ (a,b) \succsim (c,d) \text{ iff } u(a) - u(b) \geq u(c) - u(d), \end{cases}$$

for any outcome a, b, c, d BUT ALSO there are real numbers $\alpha_i(> 0)(i = 1, \ldots, n)$ and β such that

$$u(a) = \sum_{i=1}^{n} \alpha_i u_i(a) + \beta,$$

for any outcome a.

A Measurement-Theoretic Modification of Harvey's Aggregation Theorem
Our Aggregation Theorems

Our Aggregation Representation Theorem (2)

Krantz, D.H., et al.: Foundations of Measurement, vol. 1. Academic Press, New York (1971)

One of key techniques for proving this theorem is a version of Moment Theorem in abstract linear spaces in Domotor (1979).

Domotor, Z.: Ordered sum and tensor product of linear utility structures. Theory and Decision 11, 375–399 (1979)

A Measurement-Theoretic Modification of Harvey's Aggregation Theorem
Our Aggregation Theorems

Weighted Utilitarianism on Set of Outcomes

The next corollary directly follows from this theorem.

Corollary (Weighted Utilitarianism on Set of Outcomes)

Outcomes are socially ranked according to a weighted utilitarian rule.

A Measurement-Theoretic Modification of Harvey's Aggregation Theorem
Our Aggregation Theorems

Our Aggregation Uniqueness Theorem

Theorem (Aggregation Uniqueness Theorem)

Suppose that $\succsim_i$ $(i = 1, \ldots, n)$ and $\succsim$ on the set of outcomes satisfy the conditions for the representation above. Then, $u_i(i = 1, \ldots, n)$ and u are unique up to a positive affine transformation, that is, u_i and u are interval scales.

Escapes from Criticisms

- Because our aggregation theorems do not include any set of necessary and sufficient difference-worth (algebraic difference) conditions but include only some sufficient conditions, it escapes the first criticism.
- Because our aggregation representation theorem guarantees the existence of individual and social utility functions that represent individual and social quaternary preference relations which satisfy not only difference-worth (algebraic difference) conditions but also the strong Pareto condition, and our aggregation uniqueness theorem guarantees the uniqueness of such functions, they escape the second criticism.

Possible Criticism (1)

- Finally, we would like to discuss the following possible criticism, which is similar to the first criticism on Harsanyi's Aggregation Theorem by Sen, to our aggregation representation and uniqueness theorems.
- We can prove the following propositions similar to Lemma (Representation) and Lemma (Uniqueness) of Harsanyi's Aggregation Theorem:

Possible Criticism (2)

Proposition (Representation)

Suppose that individual and social quaternary preference relations $\succsim_i$ $(i = 1, \dots, n)$ and $\succsim$ on the set of outcomes satisfy Weak Order, Order Reversal, Weak Monotonicity, Soluvability and Archimedean condition in Krantz et al. (1971). Then, there exist individual and social utility functions $u_i (i = 1, \dots, n)$ and u such that

$$(2)\begin{cases} (a,b) \succsim_i (c,d) \text{ iff } \dfrac{u_i(a)}{u_i(b)} \geq \dfrac{u_i(c)}{u_i(d)}, \\ (a,b) \succsim (c,d) \text{ iff } \dfrac{u(a)}{u(b)} \geq \dfrac{u(c)}{u(d)}, \end{cases}$$

for any outcome a, b, c, d.

Proposition (Uniqueness)

Suppose that $\succsim_i$ $(i = 1, \dots, n)$ and $\succsim$ on the set of outcomes satisfy the conditions for the representation above. Then, $u_i (i = 1, \dots, n)$ and u are unique up to a transformation of functions of the form $\alpha x^{\beta}; \alpha, \beta > 0$, that is, u_i and u are log-interval scales.

Possible Criticism (3)

- These propositions imply that Weak Order, Order Reversal, Weak Monotonicity, Soluvability and Archimedean condition in Krantz et al. (1971) can satisfy not only (1) [algebraic-difference measurement] but also (2) [algebraic-quotient measurement].
- So our aggregation theorems cannot justify weighted utilitarianism.
- How can we escape this criticism?

Escape from Criticism (1)

- Von Neumann-Morgenstern axioms on individual and social binary preference relations in Lemma (Representation) are considered, as we have argued earlier, to be for ordinal measurement according to the first criticism by Sen.
- In this criticism, the fact that any monotone increasing (even non-affine) transform of an expected utility function is a satisfactory representation of individual and social binary preference relations is used to prove that von Neumann-Morgenstern axioms on individual binary preference relations in Lemma (Representation) are not for cardinal measurement but for ordinal measurement.
- In Lemma (Uniqueness), von Neumann-Morgenstern axioms together with U_i and U being expected utility functions imply the cardinality of U_i and U.
- So von Neumann-Morgenstern axioms only does not justify the cardinality of U_i and U.

Escape from Criticism (2)

- On the other hand, our axioms on individual and social quaternary preference relations can be for utility-difference measurement (algebraic-difference measurement) that is regarded historically as the definition of cardinal utility.
- Our aggregation representation theorem can justify the existence of the cardinal utilities u_i and u implying weighted utilitarianism.
- Propositions (Representation) and (Uniqueness) are not about cardinal utility but about algebraic-quotient measurement.
- Propositions (Representation) can justify the existence of the non-cardinal utilities u_i and u implying non-weighted utilitarianism.
- Then these propositions do not relate to the cardinality of u_i and u.
- So we do not have to take these propositions into consideration.

Contents of This Talk

Summary

- Harsanyi's Aggregation Theorem is an attempt to justify (weighted) utilitarianism.
- However, there are at least two well-known criticisms on Harsanyi's Aggregation Theorem.
- In order to escape these two criticisms, we can resort to Harvey's Aggregation Theorem.
- In this talk, we have offered two criticisms on this theorem from a measurement-theoretic point of view.
- Then, we have proposed new aggregation theorems, which escape these two criticisms on Harvey's Aggregation Theorem.
- Moreover, we have shown that these theorems can escape a possible criticism that seems to plausible.
- In this sense, these theorems can justify weighted utilitarianism in a strict way.

Thank You for Your Attention!