# A Sequent Calculus for K-restricted Common Sense Modal Predicate Logic

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## Abstract

In recent years, Common sense Modal Predicate Calculus (CMPC) has been proposed by J. van Benthem in [4, pp. 120–121] and further developed by J. Seligman in [1, 3, 2]. It allows us to 'take ∃ to mean just "exists" while denying the Constant Domain thesis' [1, p. 8].[1] This is done in terms of *talking about only things in each world in which they exist*. From a proof-theoretical view, the Hilbert-style system for CMPC given by Seligman is a system for modal predicate logic **S5** *which has the following axiom* $\mathrm{K}_{inv}$ *instead of axiom* K:

$$\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi) \quad \textit{provided that all free variables in } \varphi \textit{ are free variables in } \psi.$$

It is quite interesting because it might make a clean sweep of all philosophical discussions on possible world semantics between actualists and possibilists. However, neither van Benthem nor Seligman have developed K-restricted CMPC and expansions of the logic with some well known axioms. Moreover, proof-theoretic studies for such logics have not been done yet.

In this talk, I shall propose a sequent calculus for K-restricted CMPC. The main mathematical contributions of this talk are the completeness result (Theorem 1) and cut elimination theorem (Theorem 2) for the calculus. If time allows I shall also introduce sequent calculi for K-restricted CMPC with T axiom and D-like axioms. In what follows, I will outline the contents of this talk.

The language $\mathcal{L}$ of K-restricted Common sense Modal Predicate Calculus **CK** consists of a countably infinite set Var $= \{x, y, \dots\}$ of variables, a countably infinite set Pred $= \{P, Q, \dots\}$ of predicate symbols each of which has a fixed finite

[1] The Constant Domain thesis is a thesis that '[e]very possible world has exactly the same objects as every other possible world.' [1, p. 5]

arity, and logical symbols, $\bot, \supset, \Box, \forall$. The set $\mathsf{Form}$ of formulas of $\mathcal{L}$ is defined recursively as follows:

$$\mathsf{Form} \ni \varphi ::= Px_1 \dots x_n \mid \bot \mid (\varphi \supset \varphi) \mid \forall x\varphi \mid \Box\varphi$$

where $P$ is a predicate symbol with arity $n$ and $x, x_1, \dots, x_n$ are variables. The other connectives are defined as usual. We also define the sets $\mathsf{FV}(\varphi)$ and $\mathsf{FV}(\Gamma)$ of free variables in a formula $\varphi$ and a set $\Gamma$ of formulas, respectively, as usual.

Semantics for **CK** is given as follows. A frame is a tuple $(W, R, D)$, where $W$ is a nonempty set; $R$ is a binary relation on $W$; $D$ is a $W$-indexed family $\{D_w\}_{w\in W}$ of nonempty sets. Thus $R$ does not need to satisfy the *inclusion requirement*: if $wRv$ then $D_w \subseteq D_v$. A model is a tuple $(F, V)$, where $F$ is a frame and $V$ is a valuation that maps each world $w$ and each predicate $P$ to a subset $V_w(P)$ of $D_w$. An *assignment* $\alpha$ is a partial function from variables to entities and $\alpha(x|d)$ stands for the same assignment as $\alpha$ except for assigning $d$ to $x$. In addition to these notions, we follow [1, p. 15] and say that a formula $\varphi$ is an *$\alpha_w$-formula* if $\alpha(x) \in D_w$ for any variable $x \in \mathsf{FV}(\varphi)$. Then, similarly as in [1, pp. 15–16], the satisfaction relation and validity are defined as follows.

**Definition 1** (Satisfaction relation)**.** Let $M$ be a model, $\alpha$ be an assignment, and $w$ be a world in $W$. The *satisfation relation* $M, \alpha, w \models \varphi$ between $M, \alpha, w$ and an $\alpha_w$-formula $\varphi$ is defined as follows:

$$\begin{array}{lll}
M, \alpha, w \models Px_1 \dots x_n & \text{iff} & (\alpha(x_1), \dots, \alpha(x_n)) \in V_w(P) \\
M, \alpha, w \not\models \bot & & \\
M, \alpha, w \models \psi \supset \gamma & \text{iff} & M, \alpha, w \models \psi \text{ implies } M, \alpha, w \models \gamma \\
M, \alpha, w \models \forall x\psi & \text{iff} & M, \alpha(x|d), w \models \psi \quad \text{for any } d \in D_w \\
M, \alpha, w \models \Box\psi & \text{iff} & M, \alpha, v \models \psi \\
 & & \text{for any } v \text{ such that } wRv \text{ and } \psi \text{ is an } \alpha_v\text{-formula}
\end{array}$$

**Definition 2** (Validity)**.** Let $\Gamma \cup \{ \varphi \}$ be a set of formulas. We say that $\varphi$ *is valid in a frame* if for any model $M$ based on the frame, assignment $\alpha$ and world $w$ such that $\varphi$ is an $\alpha_w$-formula, $M, \alpha, w \models \varphi$. We also say that $\varphi$ *is valid in a class of frames* if $\varphi$ is valid in all frames in the class.

The following propositions that Seligman proves in [1, pp. 16–17] are noteworthy[2].

**Proposition 3** (Converse Barcan formula)**.** A formula $\Box\forall x\varphi \supset \forall x\Box\varphi$ is valid in the class of all frames.

*Proof.* Fix any model $M$, assignment $\alpha$, world $w$ such that $\Box\forall x\varphi \supset \forall x\Box\varphi$ is an $\alpha_w$-formula. Suppose $M, \alpha, w \models \Box\forall x\varphi$ and fix any element $d \in D_w$, any world $v$ such that $wRv$ and $\varphi$ is an $\alpha(x|d)_v$-formula. We show $M, \alpha(x|d), v \models \varphi$. Since

[2] Strictly speaking, he considers the dual formulas of those in Proposition 3,4.

$\mathsf{FV}(\forall x\varphi) \subseteq \mathsf{FV}(\varphi)$ and $\varphi$ is an $\alpha(x|d)_v$-formula, we have that $\forall x\varphi$ is an $\alpha(x|d)_v$-formula and thus that $\forall x\varphi$ is an $\alpha_v$-formula. Hence we get $M, \alpha, v \models \forall x\varphi$ so $M, \alpha(x|d), v \models \varphi$. ■

**Proposition 4.** A formula $\forall x\Box\varphi \supset \Box\forall x\varphi$ is not valid in the class $\mathbb{F}$ of all frames $F = (W, R, D)$ such that $R$ is an equivalence relation.

*Proof.* Consider a model $M = (W, R, D, V)$, where $W = \{0, 1\}$; $R = W \times W$; $D_0 = \{a\}$ and $D_1 = \{b\}$; $V_0(P) = \{a\}$ and $V_1(P) = \varnothing$ for some predicate symbol $P$ with arity 1, and $V_i(Q) = \varnothing$ for the other predicate symbols $Q$ with arity $n$. Then, we can establish $M, \alpha, 0 \models \forall x\Box Px$ but $M, \alpha, 0 \not\models \Box\forall xPx$. Therefore, $\forall x\Box\varphi \supset \Box\forall x\varphi$ is not valid in $\mathbb{F}$. ■

Given finite multisets $\Gamma, \Delta$ of formulas, we call an expression $\Gamma \Rightarrow \Delta$ a sequent. Then a sequent calculus G(**CK**) for **CK** is given in Table 1. The rule $\Box_{inv}$ in it plays roles of axiom $\mathrm{K}_{inv}$ and the necessitation rule in the Hilbert-style system for CMPC given by Seligman. The notion of a derivation in G(**CK**) is defined as usual.

Table 1: A Sequent Calculus G(**CK**) for **CK**

**Initial Sequents**

$$\varphi \Rightarrow \varphi \qquad\qquad \bot \Rightarrow$$

**Structural Rules**

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}\ (\Rightarrow w) \qquad \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (w \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}\ (\Rightarrow c) \qquad \frac{\varphi, \varphi\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (c \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Theta \Rightarrow \Sigma}{\Gamma, \Theta \Rightarrow \Delta, \Sigma}\ (Cut)$$

**Logical Rules**

$$\frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \supset \psi}\ (\Rightarrow \supset) \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Theta \Rightarrow \Sigma}{\varphi \supset \psi, \Gamma, \Theta \Rightarrow \Delta, \Sigma}\ (\supset \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi(y/x)}{\Gamma \Rightarrow \Delta, \forall x\varphi}\ (\Rightarrow \forall^{\dagger}) \qquad \frac{\varphi(t/x), \Gamma \Rightarrow \Delta}{\forall x\varphi, \Gamma \Rightarrow \Delta}\ (\forall \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}\ (\Box^{\ddagger}_{inv})$$

†: $y$ does not occur in $\Gamma, \Delta, \forall x\varphi$. ‡: $\mathsf{FV}(\Gamma) \subseteq \mathsf{FV}(\varphi)$.

We also say that *a sequent* $\Gamma \Rightarrow \Delta$ *is valid if* $(\gamma_1 \wedge \cdots \wedge \gamma_m) \supset (\delta_1 \vee \cdots \vee \delta_n)$ *is valid*, where $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ and $\Delta = \{\delta_1, \ldots, \delta_n\}$. Then, the following theorems hold under the settings above.

**Theorem 1** (Completeness)**.** Let $\Gamma \cup \{\varphi\}$ be a set of formulas. If $\Gamma \Rightarrow \varphi$ is valid in the class of all frames, then $\Gamma \Rightarrow \varphi$ is derivable in G(**CK**).

**Theorem 2** (Cut elimination)**.** Let $\Gamma, \Delta$ be finite multisets of formulas. If $\Gamma \Rightarrow \Delta$ is derivable in G(**CK**), then it is also derivable in G(**CK**) without any application of *Cut*.

## References

[1] Jeremy Seligman. Common Sense Modal Predicate Logic. 1st draft, October 2016.

[2] Jeremy Seligman. Common Sense Modal Predicate Logic. Presentation, 2017. Non-classical Modal and Predicate Logics: The 9th International Workshop on Logic and Cognition, Guangzhou, China (4th Dec. 2017).

[3] Jeremy Seligman. Common Sense Modal Predicate Logic. 2nd draft, November 2017.

[4] Johan van Benthem. *Modal Logic for Open Minds.* CSLI Publication, Stanford, 2010.