Doctoral Thesis

Analytical Study on Cosmological Perturbations with Exact Solutions

(厳密解に基づく宇宙揺らぎの解析的研究)

佐々木 伸 (Tadashi Sasaki)

Theoretical particle physics group
Department of Physics
Graduate School of Science
Hokkaido University

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Tadashi Sasaki
Hokkaido University
Graduate school of science
Department of physics
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Abstract

We consider the linear perturbation theory on the homogeneous and isotropic spacetime filled with various kinds of perfect fluid matters. In particular, we solve exactly the time evolution equation for the tensor fluctuation, which represents the amplitude of the gravitational waves, and for a massless minimally coupled scalar field, by using the associated third order differential equation for several combinations of background matters. It turns out that the solutions are given by the Weierstrass's elliptic and related functions, denoted by $\wp(z), \zeta(z)$, and $\sigma(z)$. We consider physical properties of these solutions. In the first part the problem of propagation of gravitational waves, which is supposed to be generated in the early universe, is considered in the standard cosmological setting and in the setting with a non-standard matter content, which is considered as a effective model of the quintessence. We obtain the explicit expressions for the energy spectrum of the waves and discuss the physical relevance of them. In the second part we consider a pre-inflationary model, where the radiation and the spatial curvature energy density dominates, followed by the de Sitter inflation. The scalar field equation is solved exactly there and consider the problem of the vacuum choice, especially particle creation caused by the difference from the standard Bunch-Davies vacuum is discussed. Furthermore, the vacuum expectation value of the energy density of the quantum field is calculated with the adiabatic regularization procedure. The quantum correction to the cosmological constant as a function of the density parameters is obtained, and it is found that the correction term canges its sign depending on the spatial curvature. Finally we calculated the primordial power spectrum and large scale suppression compared to the scale invariant spectrum is observed for the flat and closed universe, which is consistent with the results of the previous works. On the other hand, it is found that when the curvature is negative and sufficiently large, it can cause a some enhancement in the large scale.

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1 Introduction

Modern cosmology started with the discovery of the expansion of our universe by Edwin Hubble in 1929[2]. Although the data used by Hubble were not so reliable because only nearby galaxies, of which peculiar velocities are coparable to the expansion speed, were included, a number of observations so far have established that our universe has been really expanding. The expansion rate of the universe is characterized by the so called Hubble's constant (or parameter) H_0, which is defined as the coefficient of proportionality between the distance d from the earth and the radial velocity v,

$$v = H_0 d. \tag{1.1}$$

Note that this naive relation holds only for relatively nearby galaxies or stars because as one can see the radial velocity can become larger than the speed of light when directly use this relation for far away galaxies, which violates the theory of relativity. In the far region where the full account of the relativity is needed, the value of redshift z itself is suitable to represent the distance, and the Hubble parameter is defined as the changing ratio of the scale of the universe in time (more precise definition will be given later). At present, the local measurement of H_0 is quite accurate and recent estimate by using the Hubble Space Telescope [4] gives

$$H_0 = 73.24 \pm 1.74 \text{ km s}^{-1} \text{ Mpc}^{-1}. \tag{1.2}$$

Although there is a statistically significant discrepancy between the value quoted above and the value estimated from the recent observation of the cosmic microwave background (CMB)[37] with the standard ΛCDM model of cosmology,, the fact that our universe has been expanding is an established result in modern cosmology.

Once we know the universe is expanding, a reasonable guess leads to the conclusion that the universe was much "smaller" in far past. Note that the fact that the universe is expanding doesn't necessarily imply the finiteness of the spatial volume. This can be understood only when we consider the universe within the theory of general relativity, where the spacetime is identified as a (pseudo) Riemannian manifold and the associated metric tensor becomes a dynamical variable. In this framework, the expansion of the universe can be understood as the stretch of distances between any two objects.

If the past universe was smaller than that at present, from the view point of energy conservation, it should be much denser and hotter in early universe and what is more surprising is that there may exist the beginning of the universe. This has led to the so-called Big Bang cosmology. If this is indeed the case, it may be possible to find some imprints of that age. In the late 1940s George Gamow[3] recognized that if the universe was high temperature in the past, at present it should be filled with radiation of lower temperature because of gravitational redshift. Then, the temperature at present was estimated from consideration of the nucleosynthesis in the early universe to be about 5K, which corresponds to the peak wavelength of several centimeters. The first signal of this cosmic microwave background was observed in the 1960s, although it was not until the 1990s that the full spectrum was confirmed to be that of the black body radiation described by the Planck's formula with the temperature about 2.7K by the COBE satellite.

The observed CMB radiation turned out to be quite isotropic. Furthermore, the distribution of clusters of galaxies in the largest length scale are also known to be almost homogeneous and isotropic. These observational results eneble us to describe our universe in a rather simple model of the homogeneous and isotropic spacetime, which will be reviewed in the next section (Sec.2). At the same time, however, it raises the following two questions: (i) How can the smaller structures, such as galaxies, be formed? and (ii) How and why did such homogeneity and isotropy realized? Although the CMB distribution is almost isotropic, there exist small (at a ratio of about 10^{-5}) fluctuations. Since gravitational force between massive bodies is always attractive, alost homogeneous distribution of matters at the initial time would collapse to form a dense cluster, namely gravitational instability. This mechanism can be responsible for the first question but it is true only for sufficiently small region, because in larger scale the effect of cosmological expansion described by the Hubbles law becomes significant. For the full treatment, we have to employ the general relativity. Since the fluctuation from the homogeneity and isotropy can be assumed to be small in cosmological scale, we can apply the linear perturbation theory on the homogeneous background, which will be explained in Sec.3.

As for the second question, the high accuracy of the isotropy of the CMB contains a serious problem, which can be recognized by applying the formalism explained in the next section. In essence, because no physical interaction can propagate over the speed of light, if the age of the universe is finite then the whole space cannot be causaly connected. Although it is necessary to consider quantitatively, it turns out that the causaly connected region at the time when the CMB photons were emitted now subtends an angle of order 1° (see [5], provided that the universe is filled with ordinary baryonic matters and radiation. Thus, the isotropy of the CMB cannot be explained by physical mechanisms with the seemingly reasonable assumption, and this is called the horizon problem. Of course we cannot logically exclude the possibility that just a very homogeneous and isotropic initial condition occurs accidentally, it is desirable to seek for some dynamics that results in the observed homogeneity and isotropy.

In this regards, the possibility that the universe underwent an accelerated (usually exponential like) expansion before the hot Big Bang, has been widely considered. The accelerated expansion in this context is called the (cosmic) inflation (see [6] for fundamental aspects of inflation). In theories of inflation, the tiny fluctuations observed in the CMB is attributed to quantum fluctuations in the inflationary epoch. Because of exponentially fast expansion of space, initially small (quantum mechanical) fluctuations is enlarged to give rise to classical (macroscopic) fluctuations. Although there is a problem concerning how to "classicalize" the quantuma fluctuations (see for example [7] on this topic), the prediction of typical models of inflation, so called the (almost) scale invariant power spectrum, has been confirmed by the observation of the CMB[8].

The cosmic inflation can, in fact, is desirable for resolving not only the horizon problem but also the so called flatness problem. As we will see in Sec.2, the requirement of homogeneity and isotropy gives three possible types of space, which are distinguished by the constant spatial curvature denoted by K. Up to rescaling of the length scale, $K = \pm 1$ and $K = 0$ are possible. If there is nonzero spatial curvature comparable to other matter energy density in the past, then the general relativity predicts that the universe at present is dominated by the curvature energy density (see Eq.(2.11)). However, the observations of Type Ia supernovae as well as that of the CMB revealed that the universe is actually almost flat now. This implies that at early stage the energy density of the spatial curvature was much smaller than the other components. The cosmic inflation can give a dynamical mechanism that explain this flatness of the space.

In terms of accelerated expansion, there is a problem on the universe not in the past but at present. The observation of Type Ia supernovae noted above showed that the expansion of the universe is now accelerating. The simplest possiblity is that the universe is filled with the cosmological constant Λ, which can also be interpreted as a matter field with the constant energy density and negative pressure. From the supernovae observations, more than half the energy density of the present universe turns out to be such comological constant like contribution, and this is called dark energy. In the Standard model of particle physics, the elementary contents of the universe like electrons are described by quantum field theory. In this theory, typically there appears vacuum energy even if there is no real particle and if we consider the backreaction of the vacuum energy, i.e. its energy momentum tensor, it is known that it behaves as a cosmological constant term in the Einstein equation. In this sence, the cosmological constant is sometimes called the vacuum energy as well. However, if we try to explain the origin of now observed acceleration of expansion by using quantum field theories, it turns out that the vacuum energy of quantum field is typically much (120 orders of magnitude) larger than the observed value [9]. This cosmological constant problem is an important problem of recent theoretical physics.

We note also that there is attempts to explaining the observed late time acceleration by means of other than the cosmological constant. One example is the quintessence models[10]. This kind of models include a scalar field whose potential energy is considered to be responsible for the late time acceleration as the inflaton field does in the inflation. Since the CMB observation is well explained by the standard ΛCDM model, the quintessence field should not contribute other than the near present time. But the potential energy must be smoothly turned on to mimic the cosmological constant at late time, so there could arise some difference from the ΛCDM model in the evolution of linear perturbations. We consider this problem in Sec.7 in a rather simplified model.

Given that the universe seems to have an inflationary era in the past, one natural question is then what happened before the inflation? This is not only for theoretical interest but also observational importance. Although the observational result is well explained within the ΛCDM model with the prediction of the typical inflation theory, it is known that the CMB anisotropy spectrum at low multipole l is unexpectedly suppressed. Since there is a difficulty of the cosmic-variance in such low l region, it is difficult to extract some definite conclusion. Nevertheless, many theoretical proposal has been given so far and we consider one of such a model in Sec.8.

Another important ingredient in modern cosmology is gravitational waves. After the first direct detection of gravitational waves from a binary black hole merger[11], it is now expected that the gravitational wave background, like CMB, from the early universe can be detected in the near future. Mechanisms for generating gravitational waves in the early universe including inflation have been discussed in the literature (see for example [12]). In order to extract some information on the early universe from the spectra of to be observed gravitational waves, it is important to precisely investigate the propagation of the waves after its generation, otherwise we cannot understand whether an observed characteristic of the waves is really originated from the early universe. Also, since a lot of mechanisms of generating gravitational waves has been proposed, precise calculation of their propagation is essentially important to distinguish them. Since the amplitude of the waves are considarably tiny, we can apply the linear perturbation theory in this case as well.

In these regards, to analyze the linear perturbation equation in the various cosmological setting is an intriguing problem. Because of the homogeneity and isotropy, to solve the spatial part is in general not so difficult (although the non zero spatial curvature case is less studied in the literature). However, the time evolution equation is in general hard to solve analytically except for the simplest cases where the universe is assumed to be filled with a single component of the field (see Sec.3). But what is physically relevant is how the amplitude would change when the linear perturbation propagates over two or more different stages of the expansion history, where usually we have to use some approximation or numerical computations. For example, the WKB approximation has been used to study the effect of gravitational waves on the CMB fluctuations in Ref.[42], but this approximation scheme cannot be applied to long wavelength modes.

In this thesis, we consider the linear perturbation theory in the cosmological setting within fully analytical treatment. There are two reasons why we persist in solving the evolution equation exactly. The first one is that any exact solutions are valuable both theoretically and practically. For example, usually the evolution equation contains several parameters that characterize the physical system or the propagating modes, and the exact solutions then give thorough understanding on how the solutions depend on these parameters. The exact solutions themselves which we will derive in the following sections are not written in terms of the elementary functions but physically observable quantities calculated from them such as the energy density spectrum take rather simple form (see 6.62)) so it is useful for practical calculations. The second reason is that the quantization of the linear perturbations needs exact mode functions. In particular, to construct a interacting quantum field theory on a curved background, the free field theory on that spacetime must be exactly solved if we apply the perturbation theory to that interaction. Quantum theoretical aspects of the perturbations on the cosmological setting are important especially in the early stage of the universe such as the inflationary epoch.

After reviewing the fundamental description of cosmological expansion and the linear perturbation theory in Secs.2 and 3, we explain in Sec.4 the method we utilize to obtain some exact solutions for the time evolution equation of tensor mode, which can be regarded as a test scalar field as well, in Sec.5. Then, Sec.6 and Sec.7 are devoted to analyze the propagation equation of gravitational waves in the late universe using the exact solutions so obtained. In Sec.8 we consider a possibility of a pre-inflationary epoch filled with radiation and curvature energy component, and derive the expectation value of the energy-momentum tensor in the initial vacuum state and the primordial power spectrum from the exact solution obtained in Sec.5. In contrast to the previous two sections, we treat a quantum field in this section so we consider the problem of regularization as well.

2 FRW spacetime and cosmology

In this section, we review the fundamental account of cosmology based on general relativity. For more detailed discussions, refer to a standard text book such as [5] As we have mentioned in the introduction, our universe is almost homogeneous and isotropic in the largest length scale. The dynamics of spacetimes such as the expansion of the universe can be described by the theory of general relativity, where the metric tensor $g_{\mu\nu}$ of the spacetime becomes dynamical and obeys the Einstein equation,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{2.1}$$

Here, $G_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu}R/2$ is the Einstein tensor and the Ricci tensor for the connection coefficients ${\Gamma^\mu}_{\nu\lambda}$ is defined as the trace of the Riemann curvature tensor $R_{\mu\nu} = {R^\rho}_{\mu\rho\nu}$, where

$${R^\mu}_{\nu\rho\sigma} = \partial_\rho {\Gamma^\mu}_{\sigma\nu} - \partial_\sigma {\Gamma^\mu}_{\rho\nu} + {\Gamma^\mu}_{\rho\lambda}{\Gamma^\lambda}_{\sigma\nu} - {\Gamma^\mu}_{\sigma\lambda}{\Gamma^\lambda}_{\rho\nu}. \tag{2.2}$$

Our convention for the signature of the metric is $(-,+,+,+)$. The connection coefficients are uniquely determined by the assumption that the spacetime is torsion free ($\Leftrightarrow {\Gamma^\lambda}_{\mu\nu} = {\Gamma^\lambda}_{\nu\mu}$) and also the metric tensor is covariantly constant ($\Leftrightarrow \nabla_\lambda g_{\mu\nu} = 0$). The latter assumption is equivalent to require the inner product of arbitrary two vectors are invariant under the parallel transport. Explicitly, the components of the connection are given by

$${\Gamma^\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\rho}(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\rho\nu} - \partial_\rho g_{\nu\lambda}), \tag{2.3}$$

where $g^{\mu\nu}$ is the inverse matrix of $g_{\mu\nu}$. Λ is the cosmological constant, which can also be treated as vacuum energy of homogeneous distribution of matter. G is the Newton's gravitational constant and note that we employ the natural unit where the speed of light in the vacuum is equal to 1. $T_{\mu\nu}$ is the energy momentum tensor which provides the source of gravity.

Homogeneity and isotropy of the spacetime imply that the metric can be reduced into the following form with a suitable choice of the coordinates (t, x^i):

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}dx^i dx^j. \tag{2.4}$$

Here, because of homogeneity we can introduce the global time coordinate t, which defines the same time on the whole space. $a(t)$ is called the scale factor and naively represents the size of the universe at the time t. $a(t)$ is assumed to be dimensionless quantity and it represents the time evolution of the spatial length scale. We can freely choose its normalization (the most typical convention is such that $a = 1$ at the present time) but we don't explicitly fix it now. Note that unless the spatial curvature is positive (see below), the spatial volume of the spacetime is infinite. The spatial metric γ_{ij} must be homogeneous and isotropic, the latter of which implies that the spatial metric can take the following form by using the polar coordinate:

$$\gamma_{ij}dx^i dx^j = f(r)dr^2 + r^2 d\Omega^2, \tag{2.5}$$

where $d\Omega^2$ represents the metric of the 2-dimensional sphere with the unit radius. This metric respects only the isotropy if $f(r)$ takes arbitrary form, so we now determine the possible forms of $f(r)$ by demanding homogeneity as well. By requiring the components of the Einstein tensor to be constant on the $t =$constant hypersurface and not to be singular at any point, gives a first order differential equation for $f(r)$ and the general solution is given by

$$f(r) = \frac{1}{1 - Kr^2}, \tag{2.6}$$

with a constant K. Since the absolute value of K can be absorbed in the redefinition of the radial coordinate r, there are essentially three distinct spaces; (1) the closed three-sphere ($K = 1$), (2) the Euclidian space ($K = 0$), and (3) the hyperbolic space ($K = -1$).

The obtained full metric

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2\right), \tag{2.7}$$

is called the Friedmann-Robertson-Walker metric. Only the dynamical variable is the scale factor $a(t)$, of which time evolution is determined by the behavior of energy momentum tensor $T_{\mu\nu}$ through the Einstein equation (2.1). To write down the Einstein equation, we need the Ricci tensor, of which components are found to be

$$R_{tt} = -\frac{3\ddot{a}}{a}, \quad R_{ti} = 0, \quad R_{ij} = (a\ddot{a} + 2\dot{a}^2 + 2K)\gamma_{ij}. \tag{2.8}$$

where the over dot(s) represents the differentiation with respect to t. Since only the dynamical variable is $a(t)$, also the matter energy momentum tensor which is consistent with the FRW spacetime has essentially one function of t. Such a matter energy momentum tensor can always be written as the following perfect fluid form:

$$T_{\mu\nu} = pg_{\mu\nu} + (p + \rho)u_\mu u_\nu, \tag{2.9}$$

where ρ and p are energy density and pressure of the fluid and assumed to be functions only of the time t. u_μ is the four velocity of the fluid and in the rest frame (t, x^i) it takes $u_t = -1$ and $u_i = 0$, which leads to

$$T_{tt} = \rho, \quad T_{ij} = pg_{ij}. \tag{2.10}$$

Then, the Einstein equation for this system reduces to

$$H^2 := \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}, \tag{2.11}$$

$$-(2a\ddot{a} + \dot{a}^2 + K) = 8\pi G a^2 p. \tag{2.12}$$

The first equation (2.11) is called the Friedmann equation, which describes the time evolution of the scale factor for a given energy density. In this equation, we defined the Hubble parameter $H(t)$ at time t. We note that the second equation results from the first equation with the help of the continuity equation for the fluid,

$$\dot{\rho} = -\frac{3\dot{a}}{a}(p + \rho), \tag{2.13}$$

which is a consequence of the contracted Bianchi identity $\nabla_\mu G^\mu{}_\nu = 0$. As we have mentioned, this equation shows that the pressure is actually not independent from the energy density in order the fluid to be consistent with the FRW spacetime. However, to close the system, we need another equation because there are three unknown variables, $a(t), \rho(t)$, and $p(t)$. Although the continuity equation (2.13) gives a relationship between ρ and p, it doesn't reflect the properties of the fluid but is the consequence of the dynamical consistency in the expanding spacetime. In this sence, what we need is the so called equation of state, which relate ρ and p. The simplest cases are the following form:

$$p = w\rho, \tag{2.14}$$

with some constant w. In this case, the continuity equation (2.13) can be solved for ρ as

$$\rho \propto a^{-3(1+w)}. \tag{2.15}$$

Given the spatial curvature K and the parameter w, the Friedmann equation (2.11) completely determines the expansion behavior $a(t)$. In the standard cosmological models, there appear three important cases:

Nonrelativistic matter $(w = 0)$

$$p = 0, \quad \rho \propto \frac{1}{a^3}, \tag{2.16}$$

Relativistic matter $(w = 1/3)$

$$p = \frac{1}{3}\rho, \quad \rho \propto \frac{1}{a^4} \tag{2.17}$$

Vacuum energy $(w = -1)$

$$p = -\rho, \quad \rho = \text{const}. \tag{2.18}$$

In the case of flat space ($K = 0$), the Friedmann equation (2.11) for the simple equation of state (2.14), or equivalently (2.15), can be easily integrated,

$$a(t) \propto t^{\frac{2}{3(1+w)}} \ (w \neq -1), \quad a(t) \propto e^{Ht} \ (w = -1), \tag{2.19}$$

where H is a constant proportional to the vacuum energy density.

Conversely, if we provide the energy density as a function of the scale factor, the continuity equation (2.13) can be regarded as a equation of state, in which case w is no longer a constant in general. In the ΛCDM model, the energy density are assumed to be a superposition of the above three kinds of matters,

$$\rho = \frac{\rho_r}{a^4} + \frac{\rho_m}{a^3} + \rho_\Lambda, \tag{2.20}$$

where ρ_r, ρ_m, and ρ_Λ are constants which characterize magnitude of each content. Given the nonzero constants $\rho_r, \rho_m, \rho_\Lambda$, this energy density behaves as that of relativistic matter in the early stage $a \sim 0$, followed by the matter dominant epoch, and finally the cosmological constant (or vacuume energy) remains in the universe after a becomes sufficiently large. Note that if ρ_m is rather small, or ρ_Λ is large, there doesn't exist the matter dominant era. In this case, by using the continuity equation (2.13), the equation of state parameter $w = p/\rho$ is given by

$$w = \frac{\rho_r/3a^4 - \rho_\Lambda}{\rho_r/a^4 + \rho_m/a^3 + \rho_\Lambda}. \tag{2.21}$$

Implicit assumption for this energy density is that the amount of each component doesn't change with time; the time dependence of the energy density comes only from the expansion of the universe. Therefore if some kind of interaction between different kind of matters cannot be ignored, the above formula doesn't work any more. In fact, it has been discussed that in the early radiation dominated era, time dependence of the effective degrees of freedom of radiation component such as photons or neutrinos can affect the spectrum of the primordial gravitational waves[43]. However, we ignore this effect in the following, one reason for which is that the exact solutions for gravitational waves in our study can be applied after the radiation component becomes negligible as we will see later, although in the another application, a pre-inflationary model such an effect can be significant.

For generic scale factor a, we can introduce another time coordinate η called the conformal time,

$$\eta = \int \frac{dt}{a(t)}. \tag{2.22}$$

The reason why this is called conformal time can be easily seen by writing down the metric with this coordinate,

$$ds^2 = a^2(\eta)\left(-d\eta^2 + \frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\right). \tag{2.23}$$

As you can see, when the spatial curvature is vanishing, the metric takes the conformally flat form. Another advantage of this coordinate system can be shown by considering null geodesics in this spacetime. Because of homogeneity and isotropy, by a suitable coordinate transformation, any single null geodesic can be converted to a radial geodesic which pass through the origin of the space. Such a null geodesic satisfies

$$0 = ds^2 \;\Rightarrow d\eta^2 = \frac{dr^2}{1-Kr^2}. \tag{2.24}$$

By integrating this equation, the radial motion of lightlike particles can be described by the function $r(\eta)$. When $K = 0$, null geodesics are just straight lines, which is consistent with the fact that the metric in this coordinates is conformally flat. In order to understand the meaning of the conformal time, let us first consider the case of constant scale factor. In this case, we can normalize the coordinate so as to $a = 1$. By integrating Eq.(2.24) between two points (η_1, r_1) and (η_2, r_2), we obtain

$$\eta_2 - \eta_1 = \int_{r_1}^{r_2} \frac{dr}{\sqrt{1-Kr^2}}, \tag{2.25}$$

that is, the difference in the value of the conformal time is equal to the spatial distance that can be traveled by a massless particle (e.g. photon). Because ordinary matters cannot travel faster than massless particles, we can conclude that the value of the conformal time gives the maximum distance matters can travel.

Let us now remove the assumption of constant scale factor to consider the effect of non trivial expansion of the universe. In the standard cosmological scenario, where the energy density takes the form Eq.(2.20), the universe is in the radiation dominant era in the early stage ($a \sim 0$). In this era, the Friedmann equation (2.11) gives $a(t) \propto t^{1/2}$. Here, the integration constant is fixed by the condition where the Big Bang singularity $a = 0$ occurs at $t = 0$. Then, one can see that the defining integral of the conformal time $\eta(t) = \int dt/a(t)$ is convergent at $t = 0$, so again the integrtion constant can be fixed so as to $\eta = 0$ at $t = 0$. With these conventions, the maximum proper distance $d(t)$ at time t that can be traveled by a massless particle starting at the origin just after the Big Bang is given by

$$d(t) = a(t)\int_0^{r(t)} \frac{dr}{\sqrt{1-Kr^2}} = a(t)\eta(t). \tag{2.26}$$

In other words, the value of conformal time multiplied by the scale factor represents the limit of the observable region. In this sence, $d(t)$ is called the particle horizon scale.

For later use, we write down the Friedmann equation by using the conformal time,

$$\left(\frac{1}{a^2}\frac{da}{d\eta}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}, \tag{2.27}$$

In particular, for the energy density (2.20) this equation reduces to

$$\left(\frac{a'}{a^2}\right)^2 = H_0^2\left(\frac{\Omega_r}{a^4} + \frac{\Omega_m}{a^3} + \frac{\Omega_K}{a^2} + \Omega_\Lambda\right), \tag{2.28}$$

where H_0 is the Hubble constant at $a = 1$, $a' = da/d\eta$, and

$$\Omega_r = \frac{8\pi G}{3} \frac{\rho_r}{H_0^2}, \; \Omega_m = \frac{8\pi G}{3} \frac{\rho_m}{H_0^2}, \; \Omega_K = -\frac{K}{H_0^2}, \; \Omega_\Lambda = \frac{\Lambda}{3H_0^2}, \tag{2.29}$$

are the density parameters. Note that by definition of H_0, these parameters are under the constraint $\Omega_r + \Omega_m + \Omega_K + \Omega_\Lambda = 1$. For later convenience, we define a polynomial of a by $g(a) := \Omega_r + \Omega_m a + \Omega_K a^2 + \Omega_\Lambda a^4$ to write the Friedmann equation (2.28) as $(a')^2 = H_0^2 g(a)$.

3 Evolution equation for the gravitational wave on FRW spacetime

In this section, we consider linear perturbation on the FRW spacetime (2.7). This topic is also treated in the standard textbooks on cosmology such as [5] and we follow the line of argument of it. However, our formulation below is slightly generalized compared to it in two points. The first point is that we don't assume $K = 0$ in the followings. Although our universe at present seems almost flat, it can be nonzero strictly speaking. Furthermore, when we consider the very early stage of the universe, it could give rise to some non trivial and non negligible effects. We consider more on this point later (see Sec.8). The second point is that we maintain the general covariance of spatial metric. This is not only for formal preference but also for convenience in practical calculations especially for spatially curved cases. Even in the case of flat space, we can use the polar coordinates, where the covariant derivatives D_i take non trivial forms. We denote by $h_{\mu\nu}$ the metric perturbation, i.e. $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ is the background metric (2.7). We use $\bar{g}_{\mu\nu}$ and $\bar{g}^{\mu\nu}$ to raise and lower the indices of perturbed quantities, for example $h^{\mu\nu} = \bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}h_{\rho\sigma}$. Although our main concern is weak gravitational waves propagating on the FRW spacetime, which is described by the components of $h_{\mu\nu}$ with helicity 2, in general the perturbation contains other helicity components, namely scalar and vector modes. Therefore we have to decompose $h_{\mu\nu}$ into these different modes. Since the background spacetime has the unique choice of time slices defined through the cosmological time coordinate t, we can first classify the perturbation into h_{tt}, h_{ti}, and h_{ij}. The first one h_{tt} is clearly a scalar component. The second one h_{ti} generally contains both vector and scalar components, which can be separated as

$$h_{ti} = a\left(D_i F + G_i\right), \tag{3.1}$$

where F and G_i are a scalar and a vector field on the 3-space and D_i is the covariant derivative of the background 3-space metric γ_{ij}. Here, the scale factor a is inserted because the spatial coordinates x^i are the comoving ones. In this decomposition, we require the vector part to be divergenceless, namely $D^i G_i = 0$ [1] so that the scalar component F is uniquely determined through the Laplace equation

$$D^i h_{ti} = a D^2 F. \tag{3.2}$$

Similarly, h_{ij} can be decomposed as

$$h_{ij} = a^2\left(A\gamma_{ij} + D_i D_j B + D_i C_j + D_j C_i + \chi_{ij}\right). \tag{3.3}$$

Here, C_i is a divergenceless vector ($D^i C_i = 0$) and χ_{ij} is a transverse and traceless tensor, in other words

$$\chi^i{}_i = D^i \chi_{ij} = 0. \tag{3.4}$$

[1]Indices of quantities such as D_i or G_i on the 3-space are raised and lowered by using the spatial comoving metric γ_{ij} and γ^{ij}, but this does not apply to the spatial components of 4-dimensional quantities such as h_{ti}, for instance $h^{ti} = g^{tt}g^{ij}h_{tj} = -(\gamma^{ij}/a^2)h_{tj}$.

In this decomposition, the scalar components A and B are determined by the trace and the divergence of h_{ij} through the following equations:

$$\gamma^{ij}h_{ij} = a^2(3A + D^2B), \quad D^iD^jh_{ij} = a^2(D^2A + D^iD^jD_iD_jB). \tag{3.5}$$

After that, the vector component C_i can be defined by

$$D^ih_{ij} = a^2(D_jA + D^2D_jB + (D^2 + 2K)C_j). \tag{3.6}$$

In deriving this equation, we use the Ricci identity

$$[D_i, D_j]X^k = \boldsymbol{R}^k{}_{lij}X^l \tag{3.7}$$

for arbitrary vector X^k with the Riemann curvature tensor $\boldsymbol{R}^k{}_{lij}$ for the spatial metric γ_{ij}, and the fact that in the constant curvature 3-space its components are given by

$$\boldsymbol{R}_{ijkl} = K(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}). \tag{3.8}$$

Lastly, χ_{ij} can be obtained by solving Eq.(3.3).

Before proceeding to write down the Einstein equation for the perturbed quantities, we note that this decomposition is useful especially for linearized theory. If we consider higher order contributions, they are in general coupled with each other. For example, at the second order, C^iC_i would appear in the field equation for the scalar part. This decoupling at the linearized level is realized because there are no vector or tensor field other than the metric on this background.

We can decompose the pertrubed matter energy momentum tensor $\delta T_{\mu\nu}$ as in the case of metric perturbation, but in this case we can further classify $\delta T_{\mu\nu}$ into two types. The first one is a perturbation that can be considered as a perturbed perfect fluid, and the second one is the other forms. By taking the differential of Eq.(2.9), we obtain

$$\delta_{\rm pf}T_{\mu\nu} = \delta p\bar{g}_{\mu\nu} + \bar{p}h_{\mu\nu} + (\delta\rho + \delta p)\bar{u}_\mu\bar{u}_\nu + (\bar{\rho} + \bar{p})(\delta u_\mu\bar{u}_\nu + \bar{u}_\mu\delta u_\nu). \tag{3.9}$$

Here $\delta_{\rm pf}$ indicates the perturbation of perfect fluid form. Provided that the velocity of the fluid is timelike, the perturbation for u_μ must satisfies $\delta(g^{\mu\nu}u_\mu u_\nu) = 0$, which with the background velocity $\bar{u}_t = -1, \bar{u}_i = 0$ gives

$$0 = h_{tt} - 2\delta u_t. \tag{3.10}$$

Therefore only the spatial component δu_i is dynamical in the velocity perturbation. Then, we can write down the components of the perturbation with perfect fluid form,

$$\delta_{\rm pf}T_{tt} = -\bar{\rho}h_{tt} + \delta\rho, \quad \delta_{\rm pf}T_{ti} = \bar{p}h_{ti} - (\bar{\rho} + \bar{p})\delta u_i, \quad \delta_{\rm pf}T_{ij} = \bar{p}h_{ij} + a^2\gamma_{ij}\delta p. \tag{3.11}$$

In general we have to include the matter perturbation that cannot be written in this form. However, for example in $\delta_{\rm pf}T_{tt}$ we can reinterpret $\delta\rho$ as an arbitrary scalar perturbation instead of the infinitesimal change of the perfect fluid energy density. Similarly in $\delta_{\rm pf}T_{ti}$, by giving up the original definition as the velocity perturbation, we can regard δu_i as a general vector perturbation, which can be decomposed into the gradient of a scalar field and a divergenceless vector. From these arguments, we can see that what we have to add to express the most general matter perturbation are only the scalar, vector, and tensor modes in δT_{ij}. As a result, the most general form of matter perturbation can be cast into the following form:

$$\delta T_{tt} = -\bar{\rho}h_{tt} + \delta\rho, \tag{3.12}$$

$$\delta T_{ti} = \bar{p}h_{ti} - (\bar{\rho} + \bar{p})(D_iU + V_i), \tag{3.13}$$

$$\delta T_{ij} = \bar{p}h_{ij} + a^2\left(\gamma_{ij}\delta p + D_iD_j\pi^S + D_i\pi^V_j + D_j\pi^V_i + \pi^T_{ij}\right), \tag{3.14}$$

where V_i and π^V_i are divergenceless vectors, and π^T_{ij} is a transverse and traceless tensor. π^S, π^V_i, and π^T_{ij} represent the departure from the perfect fluid form and are called the anisotropic inertia.

The next task for deriving the perturbed Einstein equation is to calculate the perturbed Einstein tensor,

$$\delta G_{\mu\nu} = \delta R_{\mu\nu} - \frac{1}{2}h_{\mu\nu}\bar{R} - \frac{1}{2}\bar{g}_{\mu\nu}\delta R. \tag{3.15}$$

The perturbed connection coefficients can be obtained by using the generic formula

$$\delta\Gamma^{\mu}{}_{\nu\lambda} = \frac{1}{2}\left(\bar{\nabla}_\nu h^{\mu}{}_{\lambda} + \bar{\nabla}_\lambda h^{\mu}{}_{\nu} - \bar{\nabla}^{\mu} h_{\nu\lambda}\right). \tag{3.16}$$

For completeness, we show the result below,

$$\delta\Gamma^{t}{}_{ij} = \frac{1}{2}\left(\partial_t h_{ij} + 2a\dot{a}\gamma_{ij}h_{tt} - D_i h_{tj} - D_j h_{ti}\right), \tag{3.17}$$

$$\delta\Gamma^{t}{}_{ti} = \frac{1}{2}\left(-\partial_i h_{tt} + \frac{2\dot{a}}{a}h_{ti}\right), \tag{3.18}$$

$$\delta\Gamma^{t}{}_{tt} = -\frac{1}{2}\partial_t h_{tt}, \tag{3.19}$$

$$\delta\Gamma^{i}{}_{tt} = \frac{1}{2a^2}\gamma^{ij}\left(2\partial_t h_{jt} - \partial_j h_{tt}\right), \tag{3.20}$$

$$\delta\Gamma^{i}{}_{tj} = \frac{1}{2a^2}\gamma^{ik}\left(\partial_t h_{kj} - \frac{2\dot{a}}{a}h_{kj} + D_j h_{kt} - D_k h_{jt}\right), \tag{3.21}$$

$$\delta\Gamma^{i}{}_{jk} = \frac{1}{2a^2}\gamma^{il}\left(D_j h_{lk} + D_k h_{lj} - D_l h_{jk} - 2a\dot{a}\gamma_{jk}h_{lt}\right). \tag{3.22}$$

The linearized Ricci tensor then can be calculated by

$$\delta R_{\mu\nu} = \bar{\nabla}_\lambda \delta\Gamma^{\lambda}{}_{\nu\mu} - \bar{\nabla}_\nu \delta\Gamma^{\lambda}{}_{\lambda\mu}. \tag{3.23}$$

We again list explicit expressions for each component,

$$\delta R_{tt} = -\frac{1}{2a^2}\left(D^2 + \frac{3\dot{a}}{a}\partial_t\right)h_{tt} + \frac{1}{a^2}D^i\partial_t h_{it} - \frac{1}{2a^2}\left(\partial_t^2 - \frac{2\dot{a}}{a}\partial_t + \frac{2\dot{a}^2}{a^2} - \frac{2\ddot{a}}{a}\right)h^{(3)}, \tag{3.24}$$

$$\begin{aligned}\delta R_{ti} = &-\frac{\dot{a}}{a}D_i h_{tt} - \frac{1}{2a^2}\left(D^2 h_{it} - D^j D_i h_{jt}\right) + \left(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2}\right)h_{it} \\ &+ \frac{1}{2}D^j\partial_t\left(\frac{h_{ji}}{a^2}\right) - \frac{1}{2}\partial_t D_i\left(\frac{h^{(3)}}{a^2}\right)\end{aligned} \tag{3.25}$$

$$\begin{aligned}\delta R_{ij} = &\frac{1}{2}\left(D_i D_j + a\dot{a}\gamma_{ij}\partial_t + 2(a\ddot{a} + 2\dot{a}^2)\gamma_{ij}\right)h_{tt} - \frac{1}{2}\left(\partial_t + \frac{\dot{a}}{a}\right)\left(D_i h_{tj} + D_j h_{ti}\right) \\ &- \frac{\dot{a}}{a}\gamma_{ij}D^k h_{kt} + \frac{1}{2}\left(\partial_t^2 - \frac{\dot{a}}{a}\partial_t + \frac{4\dot{a}^2}{a^2}\right)h_{ij} + \frac{1}{2}a\dot{a}\gamma_{ij}\partial_t\left(\frac{h^{(3)}}{a^2}\right) \\ &+ \frac{1}{2a^2}\left(D^k D_j h_{ki} + D^k D_i h_{kj} - D^2 h_{ij} - D_i D_j h^{(3)}\right).\end{aligned} \tag{3.26}$$

where we introduced $h^{(3)} = \gamma^{ij}h_{ij}$ for notational simplicity. We also need the perturbed Ricci scalar,

$$\begin{aligned}\delta R &= -h^{\mu\nu}\bar{R}_{\mu\nu} + \bar{g}^{\mu\nu}\delta R_{\mu\nu} \\ &= \left(\frac{3\dot{a}}{a}\partial_t + \frac{6\ddot{a}}{a} + \frac{6\dot{a}^2}{a^2} + \frac{1}{a^2}D^2\right)h_{tt} - \frac{2}{a^2}\left(\partial_t + \frac{2\dot{a}}{a}\right)D^i h_{it} \\ &\quad + \frac{1}{a^4}\left(D^i D^j h_{ij} - D^2 h^{(3)}\right) + \frac{1}{a^2}\left(\partial_t^2 - \frac{2\ddot{a}}{a} - \frac{2\dot{a}^2}{a^2} - \frac{2K}{a^2}\right)h^{(3)}.\end{aligned} \tag{3.27}$$

Although we are now ready to explicitly write down the perturbed Einstein equation $\delta G_{\mu\nu} = 8\pi G\delta T_{\mu\nu}$, even if we separate the equations into scalar, vector, and tensor parts they take quite complicated forms. Furthermore, this system of linear differential equations for the perturbed quantities is not sufficient to completely determine the evolutions of them because of the gauge degrees of freedom. In general relativity, the gauge degrees of freedom come from general coordinate transformation $x^\mu \to x^\mu + \epsilon^\mu$, where ϵ^μ is an arbitrary infinitesimal vector. In general, tensor fields including vectors and scalars as well transforms by the negative of the Lie derivative generated by ϵ^μ. For a tensor field, the Lie derivative generated by ϵ^μ is given by

$$\mathcal{L}_\epsilon g_{\mu\nu} = \epsilon^\lambda \partial_\lambda g_{\mu\nu} + g_{\mu\lambda}\partial_\nu \epsilon^\lambda + g_{\lambda\nu}\partial_\mu \epsilon^\lambda. \tag{3.28}$$

Assuming the infinitesimal vector ϵ^μ is the same order as the perturbation $h_{\mu\nu}$, the gauge transformation $\Delta h_{\mu\nu}$[2] are given by

$$\Delta h_{\mu\nu} = -\left(\epsilon^\lambda \partial_\lambda \bar{g}_{\mu\nu} + \bar{g}_{\mu\lambda}\partial_\nu \epsilon^\lambda + \bar{g}_{\lambda\nu}\partial_\mu \epsilon^\lambda\right). \tag{3.29}$$

On the background of FRW metric, the gauge transformation of each component turns out to be

$$\Delta h_{tt} = -2\partial_t \epsilon_t, \ \Delta h_{ti} = -D_i \epsilon_t - a^2 \partial_t \left(\frac{\epsilon_i}{a^2}\right), \ \Delta h_{ij} = 2a\dot{a}\epsilon_t \gamma_{ij}. \tag{3.30}$$

The spatial component is again decomposed as $\epsilon_i = \partial_i \epsilon^S + \epsilon_i^V$ with a divergenceless vector ϵ_i^v. By comparing these transformation laws with the decompositions of h_{ti} and h_{ij}, namely Eqs.(3.1) and (3.3), we can obtain

$$\Delta F = -\frac{\epsilon_t}{a} - a\partial_t \left(\frac{\epsilon^S}{a^2}\right), \quad \Delta G_i = -a\partial_t \left(\frac{\epsilon_i^V}{a^2}\right), \tag{3.31}$$

$$\Delta A = \frac{2\dot{a}}{a}\epsilon_t, \quad \Delta B = -\frac{2\epsilon^S}{a^2}, \tag{3.32}$$

$$\Delta C_i = -\frac{\epsilon_i^V}{a^2}, \quad \Delta \chi_{ij} = 0. \tag{3.33}$$

We immediately see that the transvese and traceless component χ_{ij} is gauge invariant. For the scalar and vector components, we can construct gauge invariant combinations. From these gauge transformation laws, we can immediately see

$$\begin{aligned} \Phi &:= \frac{1}{2}\left(A + 2\dot{a}F - a\dot{a}\dot{B}\right), \\ a^2\Psi &:= -\frac{1}{2}h_{tt} + \partial_t(aF) - \frac{1}{2}\partial_t(a^2\dot{B}), \\ S_i &:= G_i - a\dot{C}_i \end{aligned} \tag{3.34}$$

are gauge invariant combinations. Especially, Φ and Ψ are called Bardeen's potentials in the literature[13].

Weak gravitational waves on this spacetime can be described as a linear perturbations of the metric, in particular tensor modes. By picking up the transverse and traceless part of the linearized Einstein equation $\delta G_{ij} = 8\pi G T_{ij}$, we can read off the following equation:

$$\left(a^2\partial_t^2 + 3a\dot{a}\partial_t + 2K - D^2\right)\chi_{ij} = 16\pi G \pi_{ij}^T. \tag{3.35}$$

Our main objective is to analyze the propagation of gravitational waves in vacuum space, so we assume $\pi_{ij}^T = 0$ hereafter. Since time derivatives and spatial derivatives are completely separated

[2] We denote the gauge transformation not by δ but by Δ in order to distinguish the gauge transformation from physical perturbation.

in the equation, we can reduce it to an ordinary differential equation by expanding the solution χ_{ij} as

$$\chi_{ij}(t, x^i) = \sum_A \sum_{\boldsymbol{k}} e^A_{ij}(\boldsymbol{k}) h_A(\boldsymbol{k}) \chi(t) \Phi_k(x^i), \tag{3.36}$$

where $e^A_{ij}(\boldsymbol{k})$ $(A = 1, 2)$ is a normalized set of polarization tensor, $h_A(\boldsymbol{k})$ characterize initial conditions, and $\Phi_k(x^i)$ is the eigen function satisfying $(D^2 - 2K)\Phi_k = -k^2\Phi_k$. $\chi(t)$ represents the time development of the amplitude, which is thoroughly investigated in the rest of the paper. Furthermore, we use the conformal time η instead of the cosmological time t. Then, the evolution equation for $\chi(\eta)$ can be easily derived by substituting the expansion into the above equation,

$$\chi'' + \frac{2a'}{a}\chi' + k^2\chi = 0. \tag{3.37}$$

There are two extreme cases which doesn't depend on the details of the expansion. These are distinguished by the ratio of the wave number k and $a'/a = aH$, called the comoving Hubble horizon scale. The first case is super-horizon regime $(k \ll aH)$, in which case the third term in the equation can be neglected and the general solution is found to be

$$\chi(\eta) \sim A + B \int \frac{d\eta}{a^2(\eta)}, \ (k \ll aH) \tag{3.38}$$

with two arbitrary constants A, B. The second solution decays with the expansion of the universe while the first one keeps constant value. The other extreme case is sub-horizon regime $(k \gg aH)$, where the evolution equation has the following solution:

$$\chi(\eta) \sim \frac{1}{a(\eta)} e^{\pm ik\eta}. \ (k \gg aH) \tag{3.39}$$

This is just a decaying oscillation, and what can be directly observed by gravitational wave detectors is the waves in this regime. These behaviors can be physically understood from Eq.(3.37) as one particle equation of motion. The second term can be interpreted as a friction term caused by the expansion of the universe, while the third term is the force obeying the Hooke's law. In order to start oscillation in this system, the latter force must be larger than the friction, otherwise the particle at rest cannot start to move.

The comoving Hubble radius $1/(aH) = 1/\dot{a}$ becomes larger and larger in the radiation or matter dominated eras, in other words the universe is decelerating, while the comoving wave number k is constant. Then, the wave modes that are in the super-horizon regime in the early epoch can "cross the horizon", namely $k = aH$, at some time and possibly become sub-horizon modes after sufficient deceleration time. One important implication of this dynamics is that the modes that cross the horizon around the present time keep the information of the early universe because such modes are not affected by various local interactions during they are outside the horizon. In the inflationary models, gravitational waves are generated from the quantum fluctuations and the wave length are stretched out to the super-horizon scale by the exponential expansion as we will briefly see later. Such super-horizon modes re-enter the horizon at some time after the inflation depending of their comoving wave number k.

Now we consider background spacetime with a single matter component, i.e. non relativistic matter, relativistic matter, and vacuum energy, in which cases we can obtain analytic expressions for χ by using elementary functions. When the parameter w appearing in the equation of state is constant, the Friedmann equation (2.27) can be solved as

$$a(\eta) = \left(\frac{3w+1}{2} H_0 \eta\right)^{2/(3w+1)}. \ \ (w \neq -1/3) \tag{3.40}$$

The exceptional case $w = -1/3$ corresponding to the case where the spacetime is filled with no matter and $K \neq 0$ is not of physical interest. The evolution equation (3.37) for this background expansion becomes

$$\chi'' + \frac{4}{3w+1}\frac{1}{\eta}\chi' + k^2\chi = 0. \tag{3.41}$$

Introducing a dimensionless variable $\xi = k\eta$ and factoring out a multiplicative power by $\chi = \tilde{\chi}/\xi^\lambda$ with $\lambda = 3(1-w)/(6w+2)$, this equation can be easily transformed to the Bessel's equation of order λ,

$$\left[\frac{d^2}{d\xi^2} + \frac{1}{\xi}\frac{d}{d\xi} + \left(1 - \frac{\lambda^2}{\xi^2}\right)\right]\tilde{\chi} = 0. \tag{3.42}$$

In particular, in the most important cases of $w = 1/3$ (radiation), $w = 0$ (matter), and $w = -1$ (vacuum energy), the order takes $\lambda = 1/2, 3/2,$ and $-3/2$, of which solution is given by the spherical Bessel functions with integer order. For refference, we write these solutions explicitly below.

Relativistic matter $(w = 1/3)$

$$\chi_\pm(\eta) = \frac{1}{\sqrt{k\eta}}H^{(1,2)}_{1/2}(k\eta) = \sqrt{\frac{2}{\pi}}\frac{i}{k\eta}e^{\pm ik\eta}. \tag{3.43}$$

Non relativistic matter $(w = 0)$

$$\chi_\pm(\eta) = \frac{1}{(k\eta)^{3/2}}H^{(1,2)}_{3/2}(k\eta) = -i\sqrt{\frac{2}{\pi}}\frac{1}{(k\eta)^3}\left(1 \mp ik\eta\right)e^{\pm ik\eta}. \tag{3.44}$$

Vacuum energy $(w = -1)$

$$\chi_\pm(\eta) = (k\eta)^{3/2}H^{(1,2)}_{3/2}(k\eta) = -i\sqrt{\frac{2}{\pi}}\left(1 \mp ik\eta\right)e^{\pm ik\eta}. \tag{3.45}$$

In these solutions, $H^{(1,2)}_\nu(x)$ is the Hankel function of 1st and 2nd kind.

Concerning observations of gravitational waves, one physically important quantity is the energy density of the waves. Although there is no local and covariant definition of energy momentum tensor for gravitational field in general situations, we can define the energy momentum tensor of gravitational waves propagating on a given background spacetime. For the derivation, we concentrate on the vacuum Einstein equation,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0. \tag{3.46}$$

As in the previous calculations, we devide the metric into two parts, $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, and assume this metric solves the Einstein equation. In general, the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R can be expanded in powers of $h_{\mu\nu}$ as

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + O(h^3), \quad R = \bar{R} + R^{(1)} + R^{(2)} + O(h^3). \tag{3.47}$$

Substituting these expressions into the full Einstein equation, we can arange various terms up to the second order as

$$\bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} - \frac{1}{2}(\bar{g}_{\mu\nu} + h_{\mu\nu})(\bar{R} + R^{(1)}) + (\bar{g}_{\mu\nu} + h_{\mu\nu})\Lambda = -\left(R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}\right) + O(h^3). \tag{3.48}$$

This form of the Einstein equation manifestly shows that the second order terms in the right hand side behaves as a source of gravity and induces the first order correction $h_{\mu\nu}$ to the background metric $\bar{g}_{\mu\nu}$ appearing in the left hand side. Thus we define the energy momentum tensor for the gravitational waves on the background $\bar{g}_{\mu\nu}$ by

$$T^{\rm GW}_{\mu\nu} = -\frac{1}{8\pi G}\left(R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}\right). \tag{3.49}$$

We now derive the energy density, namely $T^{\rm GW}_{tt}$, on the FRW background. In general, $T^{\rm GW}_{\mu\nu}$ contains all components of the perturbations $h_{\mu\nu}$ but we restrict ourselves to the contribution from the transverse traceless tensor mode, $h_{ij} = a^2\chi_{ij}$ and ignore the other components. The second order Ricci tensor can be calculated by using the following formula:

$$R^{(2)}_{\mu\nu} = \bar{\nabla}_\rho \delta^{(2)}\Gamma^\rho{}_{\nu\mu} - \bar{\nabla}_\nu \delta^{(2)}\Gamma^\rho{}_{\rho\mu} + \delta\Gamma^\rho{}_{\rho\sigma}\delta\Gamma^\sigma{}_{\nu\mu} - \delta\Gamma^\rho{}_{\nu\sigma}\delta\Gamma^\sigma{}_{\rho\mu}, \tag{3.50}$$

where $\delta^{(2)}\Gamma^\mu{}_{\nu\lambda}$ is the second order variation of $\Gamma^\mu{}_{\nu\lambda}$, of which components are given by

$$\delta^{(2)}\Gamma^\mu{}_{\nu\lambda} = -\frac{1}{2}h^{\mu\rho}\left(\bar{\nabla}_\nu h_{\rho\lambda} + \bar{\nabla}_\lambda h_{\rho\nu} - \bar{\nabla}_\rho h_{\nu\lambda}\right). \tag{3.51}$$

The first order variation $\delta\Gamma^\mu{}_{\nu\lambda}$ is already given in Eqs.(3.17)-(3.22). The second order Ricci scalar is then given by

$$R^{(2)} = \bar{g}^{\mu\nu}R^{(2)}_{\mu\nu} - h^{\mu\nu}R^{(1)}_{\mu\nu} + h^{\mu\rho}h_\rho{}^\nu \bar{R}_{\mu\nu}. \tag{3.52}$$

Note that $R^{(1)}_{\mu\nu}$ is nothing but $\delta R_{\mu\nu}$ given in Eqs.(3.24)-(3.26). In deriving the energy density of gravitational waves, we further assume the wave length λ and the period T are much smaller than the curvature radius. On the FRW background, this means $\lambda^2 \ll 1/K$ and $T \ll a/\dot{a}$. Then we can ignore terms like $(\dot{a}/a)\chi_{ij}$ compared to $\partial_t\chi_{ij}$. Furthermore, the energy of such a rapidly oscillating waves can be defined through averaging over several wave length and periods, which then allows us to ignore total derivative terms. With these assumptions, we can obtain the following energy density:

$$\rho_{\rm GW} := T^{\rm GW}_{tt} = \frac{1}{32\pi G}\langle \partial_t\chi_{ij}\partial_t\chi^{ij}\rangle, \tag{3.53}$$

where $\langle\cdots\rangle$ represents the average mentioned above. The spectrum of gravitational waves is usually described by the fraction of the energy density per logarithmic frequency interval normalized by the critical energy density,

$$\Omega_{\rm GW} := \frac{1}{\rho_c}\frac{d\rho_{GW}}{d\log k}. \tag{3.54}$$

The energy density can be expressed as an integral of the Fourier components of χ_{ij} as

$$\rho_{\rm GW} = \frac{1}{32\pi G}\int \frac{dk}{k}\frac{P(k)}{4\pi^2}\langle|\dot{\chi}_k|^2\rangle, \tag{3.55}$$

where we introduced $P(k)$, called primordial power spectrum, to reflect the initial spectrum of gravitational waves. As we will see soon below, various mechanisms of generating gravitational waves in the early universe has been proposed and each model predict different spectrum $P(k)$. Then, the spectrum at time t is given by

$$\Omega_{\rm GW} = \frac{P(k)}{24\pi^2 H^2}\langle|\dot{\chi}_k(t)|^2\rangle = \frac{P(k)}{24\pi^2 a^2 H^2}\langle|\chi'_k(\eta)|^2\rangle, \tag{3.56}$$

where we used the conformal time in the last expression.

In order to obtain what we can really observe, namely the spectrum at present, we have to evaluate the time evolution of the amplitude χ_k. As we have seen, χ_k are given by rather simple forms in the single component cases (see Eqs.(3.43)-(3.45)). However, in the long history of the expansion of our universe after the hot Big Bang, the parameter w of equation of state has not been kept constant because different kind of matters are suppressed by different powers of the scale factor a as we can see from Eq.(2.20). For this reason, we have to analize the evolution equation (3.37) with the energy density (2.20).

Instead of η, we can also use the scale factor as a time coordinate. Note that the change of coordinate $\eta \to a$ is well defined because, as one can see from the Friedmann equation (2.27), a' is always positive with a suitable initial condition ($a' > 0$ at the initial time) unless the positive curvature term ($K > 0$) and the matter energy density in the right hand side cancel each other, which is not realistic situation. By using Eq.(2.28), the evolution equation for the tensormode can be rewritten as

$$\left[\frac{d^2}{da^2} + \left(\frac{g'(a)}{2g(a)} + \frac{2}{a}\right)\frac{d}{da} + \frac{(k/H_0)^2}{g(a)}\right]\chi = 0. \tag{3.57}$$

This differential equation has at most six singular points at $a = 0, \infty$ and at the roots of $g(a) = 0$. Depending on the value of the density parameters $\Omega_r \sim \Omega_\Lambda$, the nature of these singular points could change. In the most general case, where $g(a) = 0$ has four distinct solutions, all the singular points are regular ones, though all but $a = 0$ lie in the unphysical region, which means that they are not on the positive real axis of a. When the density parameters satisfy a specific relation, which can be obtained from the discriminant, some of the roots of $g(a) = 0$ coincide to form an irregular singular point, and the behavior of the solution around that point would significantly change.

On the other hand, the Friedmann equation in the ΛCDM model (2.28) can be solved in terms of the elliptic functions, which can be seen by directly considering the integration on a. Then, with the general property that the derivatives of any elliptic function is also elliptic functions in mind, we can see that the time evolution equation (3.37) is a differential equation with elliptic function coefficients. One of the most famous and important equation is the Lamé equation,

$$\left[\frac{d^2}{dz^2} - n(n+1)\wp(z) - B\right] f(z) = 0, \tag{3.58}$$

where n is a non negative integer, B is an arbitrary constant, and $\wp(z)$ is the Weierstrass's elliptic function (see appendix A for its definition and fundamental properties). Lamé equation arises when one solves the Laplace equation in the so called ellipsoidal coordinate system[16]. Later, we will encounter this equation in a model of density perturbation in the early stage of cosmology (see Sec.8). R. Burger, G. Labahn, and M. van Hoeij[14] proposed a method to obtain a closed form solution of ordinary differential equations which have elliptic function coefficients, including the Lamé equation. There, the authors considered the 3rd order differential equation associated to the original 2nd order equation (say, Lamé equation). Then what they showed is that it can be (though not always) solved by a rational function of the elliptic function that appear in the coefficients of the equation. Because any rational function of the elliptic function is again an elliptic funciton with the same periods, this result can be stated as that, in some cases, 3rd order differential equations with elliptic function coefficients have solutions which are elliptic functions as well. Note that such a property doesn't hold in general cases where the coefficients in the differential equation are periodic. Except for the 1st order differential equations, as there are two or more independent solutions, only what one can argue is that a solution shifted by the period is some linear superposition of a complete set of basis solutions. More explicitly, for a complete set of solutions $\{\chi_i\}_{i=1\sim n}$ to a n-th order equation with coefficients whose periodicity is ω, there exists a matrix T_{ij} that satisfies

$$\chi_i(x+\omega) = \sum_{j=1}^{n} T_{ij}\chi_j(x). \tag{3.59}$$

The matrix T_{ij} is called the monodromy matrix for the differential equation. Unless the set of eigenvalues of the monodromy matrix contains 1, there is no periodic solution to the equation.

When the coefficients of the equation are elliptic functions, there exist two independent periods, hence two monodromy matrices arises. Thus the condition in order the equation allows an elliptic function solution is further restricted: the two monodromy matrices must be commutative. After obtaining the solution for the 3rd order differential equation, we can construct the general solution to the original, 2nd order equation by performing an integral. We review this method in Sec. 4.

There has been plenty of studies on physical implications of cosmological gravitational waves discussed so far. First of all, we must cosider possible mechanisms for producing gravitational waves. For example, gravitational wave production in the cosmic inflation, the first-order phase transitions, and topological defects have been discussed in the literature. In the cosmic inflation, the vacuum fluctuations of (a) quantum field(s) (usually a single or several scalar fields) are extended by the exponential expansion of the spacetime and this fluctuation is considered as a seed of the various structures in the univers at present. In general, such quantum fluctuations induce not only scalar perturbations but also tensor perturbations, namely gravitational waves. We can easily see this mechanism in the simple de Sitter approximation of inflation. As we have shown in Eq.(3.45), the amplitude of a transverse and traceless tensor field $\chi(\eta)$ in the de Sitter expansion phase is in general given by a superposition of the following form:

$$\chi(\eta) = C_+\chi_+(\eta) + C_-\chi_-(\eta). \tag{3.60}$$

Instead of $\chi(\eta)$, we define $v_k(\eta) = a(\eta)\chi(\eta)$, in order to get the canonical form of the kinetic energy. Using the scale factor for de Sitter expansion $a(\eta) = -1/(H\eta)$ and normalizing the basis solutions $v_\pm(\eta)$ as

$$v_\pm(\eta) = -\sqrt{\frac{\pi H^2}{4k^3}}\, a(\eta)\chi_\pm(\eta), \tag{3.61}$$

we write the general solution as

$$v(\eta) = a_+(k)v_+(\eta) + a_-(k)v_-(\eta). \tag{3.62}$$

In this normalization, the basis solutions $v_\pm(\eta)$ satisfies $v_+v_-' - v_-v_+' = -i$ and behave in the far past $\eta \to -\infty$ as

$$v_\pm(\eta) \to \frac{1}{\sqrt{2k}} e^{\pm ik\eta}, \tag{3.63}$$

which is the plane wave form. Thus, when we promote $v(\eta)$ to be a quantum field, a_+ and a_- are considered to be creation and annihilation operators in the far past. The typical assumption is the quantum field was in the vacuum state defined by a_+, which is called the Bunch-Davies vacuum in the literature. In this vacuum state, the expectation value of the quantum fluctuations of $\chi = v/a$ is equal to v_+v_-/a^2, which can be easily calculated as

$$\langle|\chi(\eta)|^2\rangle = \frac{H^2}{2k^3}\left(1 + \frac{k^2}{H^2a^2(\eta)}\right). \tag{3.64}$$

We can see that the expectation value is "frozen" once the wave length becomes much larger than the Hubble scale ($k^2 \ll H^2a^2$), to a non zero constant. Then, the primordial power spectrum $P(k)$ is given by

$$P(k) := \lim_{\eta\to -0} \frac{k^3}{2\pi^2}\langle|\chi(\eta)|^2\rangle = \left(\frac{H}{2\pi}\right)^2. \tag{3.65}$$

The most important feature is that the spectrum is in fact independent of the wavelength of the fluctuation, namely scale invariant. If one deals with more detailed dynamics of the inflation by introducing the inflaton field(s) and consider slow-roll models, the power spectrum is slightly modified and becomes nearly scale invariant. Other modification may come from different choices of the quantum states. As is well known, there is no unique and preferable quantum vacuum state in non stationary spacetime. There has been much discussion on the choice of vacuum state or even the possibility of excited states[17, 18, 19]. In addition, it may possible that there existed certain pre-inflationary era before de Sitter expansion started, in which case the power spectrum would be modified because the wave function used above may not be a simple plane wave at that time. We will discuss more on this point later in Sec.8.

This significant property of scale invariance holds for a scalar field case as well because the evolution equation for a minimally coupled free massless scalar field is exactly the same as that of $\chi(\eta)$. The scalar fluctuations generated in this way are reflected in the temperature fluctuations observed in CMB and verified (though not completely) through the observation of CMB temperature fluctuations.

4 The third order differential equation associated with the second order differential equation

Let us begin with general form of second order linear ordinary differential equation,

$$\chi''(x) + p(x)\chi'(x) + q(x)\chi(x) = 0, \tag{4.1}$$

where $p(x)$ and $q(x)$ are assumed to be known functions. The problem of solving this equation appears in almost all all the areas of physics, for example the Schrödinger equation for a given potential. In this paper this kind of equations arise in the context of linear perturbation on a given background spacetime. If the background spacetime has enough symmetry, for example time translation symmetry in stationary spacetimes, the partial differential equations(PDE) for linear perturbations on that spacetime can be solved by the method of separation of variables, which reduces the PDE into a set of ODEs. Although Eq.(4.1) for some specific forms of $p(x)$ and $q(x)$ are known to be solved by using well known functions, it is in general hopeless task to obtain the analytic solution in a closed form.

In such a situation, usually the solution is expressed as a (typically infinite) series of some known functions. The simplest one is the power series expansion around some fixed point, which is known as a Frobenius solution. In many cases, the coefficient functions $p(x)$ and $q(x)$ have various kinds of singular points, which represents some physical boundaries when the independent variable x corresponds to one of the spacetime coordinates. Such singular points are called singular points of the ODE (4.1). Although we can solve for the expansion coefficients order by order, such a series solution has a finite value of convergence radius. If the expansion location is a regular singular point, the convergence radius is known to be the distance to the closest singular point. The convergence radius can even be 0 when the point is irregular, or indefinite singular point, which means that the infinite series is actually an asymptotic series.

Since Eq.(4.1) is a second order ODE, the solution space forms a 2-dimensional vector space, in other words any solution can be expressed by a superposition of two independent solutions. Expansion of the solution around a singular point can give a two linearly independent solutions (in the case of irregular singular point, it is possible that the power series expansion gives only one solution).When the ODE (4.1) has two or more singular points, we can obtain several sets of the two linearly independent solutions which are defined by the analytic continuations of each series solutions. Then, one of the most important problems in this situation is to understand how two of such sets are related each other.

One physical example is scattering problem on the real axis with a given potential function. For definiteness, let us consider the Schrödinger equation,

$$\psi''(x) + (V(x) + k^2)\psi(x) = 0, \tag{4.2}$$

where k is a real constant. If the potential decays sufficiently fast at infinity ($x \to \pm\infty$), the solution takes the form of plane wave there, $\psi(x) \sim e^{\pm ikx}$. We denote the ingoing and outgoing waves at $x \to \pm\infty$ by $\psi_{\pm}^{\mathrm{in}}(x)$ and $\psi_{\pm}^{\mathrm{out}}(x)$ respectively. When the incident wave comes from the negative infinity, the solution $\psi(x)$ satisfies the following boundary condition:

$$\psi(x) \to \psi_{-}^{\mathrm{in}}(x) + R(k)\psi_{-}^{\mathrm{out}}(x) \ (x \to -\infty) \tag{4.3}$$

$$\to T(k)\psi_{+}^{\mathrm{out}}(x) \ (x \to +\infty), \tag{4.4}$$

where we normalize the amplitude of the incident wave to be unity and $R(k)$ and $T(k)$ represent the reflection and transmission coefficients respectively. Although the asymptotic form of the solution can be easily deduced without knowing the preciese form of the potential, these scattering coefficients can be obtained only when the full solution $\psi(x)$ is known. As this example shows, it is physically desirable to obtain the exact solution for ODEs.

Let us come back to the general ODE (4.1). Here, we consider, instead of $\chi(x)$ itself, the product of the two solution, $y = \chi_1\chi_2$ following the work[14]. By using the ODE (4.1), we can derive the differential equation for y. Because the ODE (4.1) is 2nd order, all the derivatives of y can be expressed as sums of $\chi_1\chi_2$, $\chi_1'\chi_2 + \chi_1\chi_2'$, and $\chi_1'\chi_2'$. As a result, y''' can be represented by using y, y' and y'', which means that y satisfies a linear 3rd order differential equation. This fact can be understood in another way. Since the ODE (4.1) is 2nd order, there are two independent solutions, say χ_1 and χ_2. Then there exist three independent symmetric products, namely $(\chi_1)^2, (\chi_2)^2$, and $\chi_1\chi_2$. Thus y should satisfies a 3rd order equation.

Now we derive the 3rd order equation for $y = \chi_1\chi_2$. The first derivative is $y' = \chi_1'\chi_2 + \chi_1\chi_2'$, by using which the second derivative can be simplified as

$$y'' = -py' - 2qy + 2\chi_1'\chi_2'.$$

Now we can solve these equations for $\chi_1\chi_2$, $\chi_1'\chi_2 + \chi_1\chi_2'$, and $\chi_1'\chi_2'$, the third derivative

$$y''' = -(py' + 2qy)' - 4p\chi_1'\chi_2' - 2q(\chi_1'\chi_2 + \chi_1\chi_2')$$

can be expressed by usind y, y', and y''. The result is given below:

$$y''' + 3py'' + (p' + 4q + 2p^2)y' + 2(q' + 2pq)y = 0. \tag{4.5}$$

If one can obtain a solution for this third order equation, then the solutions to the original second order equation can be constructed as follows. In doing so, we have to distinguish whether χ_1 and χ_2, which consist of $y = \chi_1\chi_2$, is linearly independent or not. For the solutions of 2nd order ODE (4.1), it is known that we can construct a constant out of the Wronskian,

$$C(\chi_1, \chi_2) := \exp\left(\int p(x)dx\right)(\chi_1\chi_2' - \chi_1'\chi_2). \tag{4.6}$$

The two solutions χ_1 and χ_2 are linearly independent if and only if the constant $C(\chi_1, \chi_2$ takes a nonzero value. Similarly, we can define the constant L for two solutions $y = \chi_1\chi_2$ and $y = \chi_3\chi_4$ of the 3rd order equation (4.5) as

$$L(y_1, y_2) := C(\chi_1, \chi_3)C(\chi_2, \chi_4) + C(\chi_1, \chi_4)C(\chi_2, \chi_3). \tag{4.7}$$

By observing that this definition is symmetric under the interchanges $(\chi_1 \leftrightarrow \chi_2)$ and $(\chi_3 \leftrightarrow \chi_4)$, $L(y_1, y_2)$ can be expressed in terms of y_1, y_2 and their derivatives,

$$L(y_1, y_2) = \exp\left(2\int p(x)dx\right)\left[y_1(y_2'' + py_2' + 2qy_2) + y_2(y_1'' + py_1' + 2qy_1) - y_1'y_2'\right]. \tag{4.8}$$

Because $L(y_1, y_2)$ is symmetric, namely $L(y_1, y_2) = L(y_2, y_1)$, the square of the constant $C(\chi_1, \chi_2)$ can evaluated from the solution $y = \chi_1\chi_2$ by setting $y_2 = y_1 = y$ in the above formula,

$$C^2(\chi_1, \chi_2) = -\exp\left(2\int p(x)dx\right)\left[2yy'' + 2pyy' + 4qy^2 - (y')^2\right]. \tag{4.9}$$

When the constant $C(\chi_1,\chi_2)$ for a given solution y vanishes, the solution is actually the square of a solution for the 2nd order ODE (4.1), i.e. $y=\chi_1^2$. In this case we can immediately obtain the solution $\chi_1=\sqrt{y}$. By solving the first order differential equation (4.6) for χ_2 with an arbitrary nonzero value of $C(\chi_1,\chi_2)$, a linearly independent solution can be also obtained. Note that the value of $C(\chi_1,\chi_2)\neq 0$ in this process is not important because this is proportional to the normalization of χ_2 and χ_1.

On the other hand, if $C(\chi_1,\chi_2)$ for a given solution y doesn't vanish, we cannot simply take the square root to obatin the solution for the original 2nd order equation. In this case, we regard the derivative $y'=\chi_1\chi_2'+\chi_1'\chi_2$ and the definition of $C(\chi_1,\chi_2)$ Eq.(4.6) as a system of 1st order equations for χ_1 and χ_2. These two equations can be solved for the logarithmic derivatives of χ_1 and χ_2 as follows:

$$\begin{aligned}\frac{\chi_1'}{\chi_1}&=\frac{y'}{2y}-\frac{C}{2y}\exp\left(-\int p(x)dx\right),\\ \frac{\chi_1'}{\chi_1}&=\frac{y'}{2y}-\frac{C}{2y}\exp\left(-\int p(x)dx\right),\end{aligned}\tag{4.10}$$

where we omit the argument of $C(\chi_1,\chi_2)$ because this is assumed to be a known constant. Finally integration of these equations give the two independent solutions,

$$\chi_\pm=\sqrt{y}\exp\left[\pm\frac{1}{2}\int\frac{C}{y}\exp\left(-\int p(x)dx\right)dx\right],\tag{4.11}$$

where $\chi_+=\chi_1$ and $\chi_-=\chi_2$.

One might think that it is as hard (or harder) to solve the third order equation (4.5) as (than) the original equation (4.1), but this is not always the case. One of such possibilities is that the ODE (4.1) allows solutions of the form $\chi_\pm=|\chi(x)|e^{\pm i\theta(x)}$, where the squared amplitude $|\chi(x)|^2$ is a polynomial of x. In this case, by expanding y as a power series one will find that the recurrence relation for the expansion coefficients terminates at some finite order, so that the remaining non trivial process is to perform the integration in (4.11). On the other hand, such a solution could not be found by directly expanding χ in a power series because both the amplitude $|\chi(x)|$ and the phase factor $e^{\pm i\theta(x)}$ are in general infinite series of x. Therefore considering the third order equation can give another way to solve second order ODEs which have not been solved yet.

Before we apply this method to more complicated problems, we give a few simple examples.. The first one is the simple harmonic oscillator,

$$\chi''(x)+k^2\chi(x)=0.$$

The generic solutions can be expressed by linear combinations of two fundamental solutions $\chi_1(x)=\cos kx$ and $\chi_2(x)=\sin kx$. Alternatively, we can use another set of solutions as a basis, $\chi_\pm(x)=e^{\pm kx}$. The associated 3rd order equation in this case becomes

$$y'''+4k^2y'=0,$$

whose independent solutions are given by

$$y_0=1,\quad y_\pm=e^{\pm 2ikx}.$$

Note that if we consider the power series solutions around $x=0$, $y_\pm$ is given by infinite series while y_0 has a terminate series (just one term). We take y_0 to construct the solution χ. The constant C^2 for this solution can be calculated by using Eq.(4.9), which gives

$$C^2(y_0)=-4k^2.$$

Because the sign change of C just corresponds to the interchange of two solutions $\chi_\pm$, we can take its sign arbitrarily. By taking $C(y_0) = 2ik$, the general formula (4.11) gives $\chi_\pm = e^{\pm kx}$.

As a next example, we consider the Bessel's differential equation,

$$\chi''(x) + \frac{1}{x}\chi'(x) + \left(1 - \frac{\nu^2}{x^2}\right)\chi(x) = 0.$$

The associated 3rd order equation now becomes

$$\left[\theta(\theta + 2\nu)(\theta - 2\nu) + 4x^2(\theta + 1)\right]y = 0.$$

When the order is a positive half odd integer, i.e. $\nu = N + 1/2$, this equation allows the following polynomial solution:

$$y = \frac{2^{2N+1}\Gamma^2(N+1/2)}{\pi^2}\sum_{n=0}^{N}\frac{(-1)^n(-N)_n}{(1/2-N)_n(-2N)_n n!}x^{-1-2N+4n}.$$

For this solution, Eq.(4.9) gives $C^2 = -16/\pi^2$. By taking $C = 4i/\pi$, the general solution (4.11) turns out to be the Hankel function,

$$\chi_+(x) = H^{(1)}_{N+1/2}(x), \quad \chi_-(x) = H^{(2)}_{N+1/2}(x).$$

In both examples, we observe the constant C is pure imaginary. When $C^2 < 0$ and y is positive, then the solutions $\chi_\pm$ in Eq.(4.11) are complex conjugate each other and y corresponds to indeed the squared amplitude of these solutions, namely $y = |\chi_\pm|^2$. Thus, by analyzing the solution y for which $C^2(y)$ is negative, we can directly see the behavior of the amplitude of χ.

Before finishing this section, we briefly discuss the relationship of this method with the WKB method for solving second order differential equations. The WKB method is typically applied to the Schroedinger-type differential equation, where the coefficient $p(x)$ is identically set to zero, namely

$$\chi''(x) + q(x)\chi(x) = 0,$$

and assume the following ansatz:

$$\chi(x) = \frac{1}{\sqrt{2W(x)}}\exp\left(-i\int W(x)dx\right). \tag{4.12}$$

Then, the differential equation for $W(x)$ becomes non-linear,

$$W^2(x) = q(x) + \frac{3}{4}\left(\frac{W'(x)}{W(x)}\right)^2 - \frac{W''(x)}{2W(x)}. \tag{4.13}$$

Usually, we don't try to solve this equation exactly but instead look for an approximate solution. Since the right hand side of Eq.(4.13) contains the derivatives of $W(x)$, we can consider a derivative expansion. In other words, we regard the number of derivatives as an expansion order. Then, the leading behavior is given by $W(x) \sim \sqrt{q(x)}$, and higher order terms can be determined algebraically by assuming the derivative terms of $W(x)$ in the right hand side as a perturbation. For example, the next leading approximation is given by inserting $W = \sqrt{q}$ into the right hand side of Eq.(4.13),

$$W(x) \sim \sqrt{q(x)}\left[1 + \frac{1}{8q(x)}\left(\frac{3q'^2(x)}{q^2(x)} - \frac{q''(x)}{q(x)}\right)\right]. \tag{4.14}$$

This WKB expansion is valid when the derivatives of $q(x)$ is small compared to the value of $q(x)$ itself. If the independent variable x represents some sort of time coordinate, then this expansion corresponds to the so-called adiabatic expansion. This point of view will be utilized later in Sec.8.

Now it is obvious that the WKB ansatz (4.12) and the solution (4.11) constructed from y are equivalent with the identification $y = 1/W$ under the assumptions $p(x) = 0$ and $C = 2i$. Since the value of C can be freely changed by normalizing the solution y unless $C =$ (as noted, in this case a solution is directly obtained by $\chi = \sqrt{y}$.), this identification is essentially always possible. In fact, the WKB equation (4.13) can be shown to be equivalent to Eq.(4.9) with $C = 2i$ and $p = 0$ by transforming $W = 1/y$. In this sence, the WKB equation (4.13) can be considered as a 1st integral of the 3rd order linear equation for y (4.5). The restriction $p = 0$ is not critical since any 2nd order equation can be converted to this form by a suitable change of the dependent variable χ.

Although the correspondence $W = 1/y$ itself seems one-to-one, because y is obtained as a solution to the 3rd order equation (4.5) which has three independent solutions while W has only two independent solutions corresponding to the choice of the branch of the square root in the leading order $W \sim \sqrt{q}$, there should be some degeneracy in this identification. To clarify this degeneracy, we introduce an inner product in the space of solutions to the 3rd order equation (4.5) by

$$I(y_1, y_2) := \frac{1}{2}\left[C^2(y_1 + y_2) - C^2(y_1) - C^2(y_2)\right]. \tag{4.15}$$

This inner product is symmetric and bilinear, and reduces to the value of C^2 when two solutions is identical, namely $I(y, y) = C^2(y)$. Also, note that this definition is valid for $p(x) \neq 0$. Since y must be a product of two solutions of a second order equation, we can always take a basis solution y_1, y_2, and y_3 such that $C^2(y_1) = C^2(y_2) = 0$ and $C^2(y_3) \neq 0$. Explicitly, this can be realized by $y_1 \propto (\chi_1)^2$, $y_2 \propto (\chi_2)^2$, and $y_3 \propto \chi_1\chi_2$ with any two independent solutions χ_1 and χ_2. Then, by direct computation one can verify that the matrix $I_{ij} = I(y_i, y_j)$ can be cast into the following form with a suitable normalization of y_i:

$$I_{ij} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{4.16}$$

which implies that the signature of the metric in the solution space of y is Lorentzian. In this space we can choose two independent null direction, which corresponds to the choice of the square of two independent solutions $(\chi_1)^2$ and $(\chi_2)^2$. Once two null directions are fixed, there remains unique space like direction orthogonal to both the null direction, which corresponds to the product of these solutions $\chi_1\chi_2$. One can form a time-like and space-like basis from two null directions y_1 and y_2 as $y_\pm = (y_1 \pm y_2)/\sqrt{2}$.

Each choice of the two null direction in 3-dimensional Minkowski space is in one-to-one correspondence to the point of the circle S^1, reflecting the degrees of freedom of rotation in the 2-dimensional plane of the solution space for χ. The metric I_{ij} itself doesn't distinguish any of these solutions, i.e. Lorentz invariant. Note that the boost transformation changes the normalization of χ_1 and χ_2. These Lorentz transformations in the space of y are exactly same as the $SL(2, \mathbb{C})/Z_2$ transformations in the space of χ that preserve the Wronskian $W(\chi_1, \chi_2)$ (or equivalently $C(\chi_1, \chi_2)$). The division by Z_2 originates from the fact that y is invariant under the sign change of χ.

In any one of basis where the metric is represented by the matrix (4.16), what is really identified with the WKB factor $W(x)$ is y_3 since y_1 and y_2 are null. Among the three basis Lorentz transformations (1 rotation and 2 boosts), the boost along the direction of y_- (imaginary rotation about y_3 axis) is the only transformation that preserves y_3, while the remaining transformation, the rotation about y_+ axis and the boost along the direction of y_3 do change y_3. Thus the degrees of freedom in the choice of y_3, hence W, is really two.

5 3rd order differential equation for the gravitational waves on FLRW spacetime

In this section, we apply the method explained in Sec.4 to the time evolution equation of gravitational waves on FLRW spacetime, namely Eq.(3.57). We apply the general formula (4.5) to this case by identifying

$$p(a) = \frac{g'(a)}{2g(a)} + \frac{2}{a}, \quad q(a) = \frac{(k/H_0)^2}{g(a)}. \tag{5.1}$$

As a result, we obtain the following equation:

$$\left[\frac{d^3}{da^3} + \left(\frac{3g'(a)}{2g(a)} + \frac{6}{a}\right)\frac{d^2}{da^2} + \left(\frac{g''(a)}{2g(a)} + \frac{4g'(a)}{ag(a)} + \frac{6}{a^2} + \frac{4(k/H_0)^2}{g(a)}\right)\frac{d}{da} + \frac{8(k/H_0)^2}{ag(a)}\right] y = 0. \tag{5.2}$$

Because the coefficients of this equation are all rational function of a, we search for solutions which are rational in a. Specifically, we consider the following ansatz:

$$y(a) = \frac{P(a)}{Q(a)}, \tag{5.3}$$

where $P(a)$ and $Q(a)$ are polynomials. Rational functions in this form has in general several poles at the zeros of $Q(a)$. If we demand this expression to be a solution to the differential equation, The locations of these poles must coincide with the singular points of that equation because differentiation of a finite order doesn't change the position of poles. Thus, we can decompose the above ansatz in partial fractions of $1/a$ and $1/(a-a_n)$, where a_n ($n = 1 \sim 4$) are the roots of $g(a) = 0$. Furthermore, we assume the order of the polynomial $P(a)$ is less than or equal to that of $Q(a)$, which restrict the solution y to decaying (or possibly constant) modes in the far future $a \to \infty$. In order to determine the possible order of these singularities, we assume $y \sim 1/(a-a_n)^{N_n}$ in the limit $a \to a_n$. Because $g(a) \propto (a - a_n)$ when a_n is not a degenerate root, only the first and second term in the above differential equation contribute to the leading order. As a result, we obtain the determination equation for the index N_n,

$$0 = N_n(N_n + 1)(N_n + 1/2),$$

which implies that actually y cannot be singular except for $a = 0$ when the roots of $g(a)$ are all distinct. On the other hand, if any two of four roots coincide, i.e. $g(a) = 0$ has a degenerate solution, the behavior of $g(a)$ around that point becomes $g(a) \propto (a-a_n)^2$ and the determination equation for N_n would be altered. However, in such a case it turns out that the determination equation contains the wave number k and N_n becomes complex and dependent on k in general, which means that the ansatz are no longer adequate. Therefore we only consider the solutions y that is non singular at the roots of $g(a) = 0$ hereafter.

Effectively, this assumption with the ansatz above is equivalent to restrict ourselves to polynomial solutions in $1/a$. For that purpose, at first we expand y as

$$y = \sum_{n=0}^{\infty} \frac{y_n}{a^n} \tag{5.4}$$

and derive the recurrence relation for the coefficients y_n. In doing so, it is useful to rewrite the equation for y, namely (5.2), by using the differential operator $\theta = ad/da$ instead of d/da. This can be easily done by noting the identity $a^j(d/da)^j = \theta(\theta-1)\cdots(\theta-j+1)$ and the result is given below,

$$\left[\sum_{j=0}^{4} b_j a^j \theta(\theta+j+2)\left(\theta+1+\frac{j}{2}\right) + \frac{4k^2}{H_0^2}a^2(\theta+2)\right] y = 0, \tag{5.5}$$

where we define for notational simplicity

$$b_0 = \Omega_r,\ b_1 = \Omega_m,\ b_2 = \Omega_K,\ b_3 = 0,\ b_4 = \Omega_\Lambda. \tag{5.6}$$

As we have mentioned in the previous section, this is equivalent to search for doubly periodic solutions when we take the conformal time η as the independent variable because the solution to the Friedmann equation (2.28) is in general elliptic functions.

Although $b_3 = 0$ in the standard ΛCDM cosmology, from now on we include this term in our consideration of the recurrence relation. This term corresponds to some exotic matter whose energy density is proportional to $1/a$ and physical relevance of such a matter will be discussed later.

Now substituting the expansion (5.4) into the third order equation (5.5), we obtain the recurrence relation for y_n. As one can see from Eq.(5.5), y_n obeys a five term recurrence in general, of which general solutions cannot be obtained. However, recalling that a^{-n} is the eigenfunction of the operator θ with the eigenvalue $-n$, the first few recurrenc relations gives relations only among less terms. The most notable restriction comes from the coefficients of a^3 and a^{-2}, which gives

$$b_4 y_1 = (k/H_0)^2 y_4 = 0. \tag{5.7}$$

The latter equation leads to $y_4 = 0$ unless $k = 0$, namely zero mode. Since the zero mode solution for χ can be easily obtained directly from Eq.(3.37), which will be discussed later, we assume $k = 0$ and so $y_4 = 0$ in this section.

From the coefficient of a^{-n} with $n \geq 3$, we obtain the following recurrence:

$$\begin{aligned}
0 = {} & b_4(n+4)(n+1)(n-2)y_{n+4} + b_3(n+3)(n+1/2)(n-2)y_{n+3} \\
& + \left[b_2(n+2)(n-2) + 4(k/H_0)^2\right] n y_{n+2} \\
& + b_1(n+1)(n-1/2)(n-2)y_{n+1} + b_0 n(n-1)(n-2)y_n.
\end{aligned} \tag{5.8}$$

We can regard this recurrence as a specific example of the following general five term homogeneous recurrence relation:

$$0 = A_n y_{n+4} + B_n y_{n+3} + C_n y_{n+2} + D_n y_{n+1} + E_n y_n. \tag{5.9}$$

Let us consider the condition under which the recurrence relation allows a finite term solution. We denote the index of the first term and the number of terms of such a solution by n_0 and $N(\geq 1)$ respectively. In other words, we assume $y_n = 0$ for $n < n_0$ and $n > n_0 + N - 1$. Then, the recurrence relation (5.8) gives non trivial constraint only for from $n = n_0 - 4$ to $n = n_0 + N - 1$,

$$\begin{aligned}
0 &= A_{n_0-4} y_{n_0}, \\
0 &= A_{n_0-3} y_{n_0+1} + B_{n_0-3} y_{n_0}, \\
&\vdots \\
0 &= D_{n_0+N-2} y_{n_0+N-1} + E_{n_0+N-2} y_{n_0+N-2}, \\
0 &= E_{n_0+N-1} y_{n_0+N-1}.
\end{aligned}$$

In general, this is a over determined system because there are $N+4$ equations for the N variables $y_{n_0}, \cdots, y_{n_0+N-1}$. In order to allow a non trivial solution, the rank of the $(N+4) \times N$ matrix of the coefficients must be less than N. The unknown number n_0 and N can be determined from the necessary conditions $0 = A_{n_0-4} = E_{n_0+N-1}$ because $y_{n_0} y_{n_0+N-1} \neq 0$ by definition. After that, we can consider the $(N+2) \times N$ matrix that consists of the remaining equations for the given value of n_0 and N, and can (in principle) write down the condition for the existence of a non trivial finite term solution.

We apply this argument to the specific case of Eq.(5.8). In this case we have

$$A_n = b_4(n+4)(n+1)(n-2), \quad E_n = b_0 n(n-1)(n-2). \tag{5.10}$$

Note that if $b_4 = 0$, A_n vanishes identically and so we have to require $B_{n_0-3} = 0$ instead of $A_{n_0-4} = 0$. Similarly, we take $D_{n_0+N-2} = 0$ in place of $E_{n_0+N-1} = 0$ when $b_0 = 0$. Such cases are discussed later. Assuming $b_4 b_0 \neq 0$, possible pairs of (n_0, N) turn out to be $(0,1), (0,2)$, and $(0,3)$. Furthermore, as we have mentioned earlier, Eq.(5.7) is required in any case. Since we are now assuming $b_4 \neq 0$, this leads to $y_1 = 0$. Then, one of the possible pairs, namely $(n_0, N) = (0,2)$ is identical to $(0,1)$. We consider the remaining three possibilities in the following.

$(n_0, N) = (0,1)$ In this case, only non trivial equation is given by Eq.(5.8) with $n = -2$,

$$0 = (k/H_0)^2 y_0,$$

which, provided that $k \neq 0$, has only trivial solution $y_0 = 0$.

$(n_0, N) = (0,3)$ In this case, there are three constraints among two variables y_0 and y_2, which are given by

$$0 = b_1 y_2 = b_3 y_2 = b_4 y_2 - (k/H_0)^2 y_0.$$

Thus, if and only if $b_1 = b_3 = 0$, there is a polynomial solution,

$$y = b_4 + \frac{k^2}{H_0^2 a^2}. \tag{5.11}$$

Now let us turn to the case where either $b_0 = 0$ or $b_4 = 0$. When $b_4 = 0$ but still $b_0 \neq 0$, the condition at the boundary becomes $B_{n_0-3} = E_{n_0+N-1} = 0$. For our equation, $B_n = b_3(n+3)(n+1/2)(n-2)$ and the possible pairs are found to be again $(n_0, N) = (0,1), (0,2)$, and $(0,3)$, but in this case we don't have to require $y_1 = 0$. However, we can see that the first case is actually impossible because the equation for y_0 doesn't change from the previous case. Therefore we examine the latter two cases.

$(n_0, N) = (0,2)$ There are three constraints for y_0 and y_1,

$$0 = 3b_3 y_2 + (3b_2 - 4(k/H_0)^2) y_1 = (3b_2 - 4(k/H_0)^2) y_1 = b_1 y_1.$$

We can immediately see that there is only the trivial solution $y_0 = y_1 = 0$.

$(n_0, N) = (0,3)$ In this case, there are for equations for y_0, y_1, and y_2,

$$0 = 3b_3 y_1 - 4(k/H_0)^2 y_0 = 3b_3 y_2 + (3b_2 - 4(k/H_0)^2) y_1 = b_1 y_1 = b_1 y_2,$$

from the last two of which imply $b_1 = 0$ for non zero solution. Then the other two equation gives y_1 and y_2 interms of y_0, and we obtain the following polynomial solution:

$$y = (3b_3)^2 + \frac{12k^2 b_3}{H_0^2 a} + \frac{4(k/H_0)^2(4(k/H_0)^2 - 3b_2)}{a^2}. \tag{5.12}$$

The next possibility is $b_0 = 0$ but $b_4 \neq 0$, in which case we require $A_{n_0-4} = D_{n_0+N-2} = 0$ as necessary conditions. From the expression $D_n = b_1(n+1)(n-1/2)(n-2)$, we obtain $(n_0, N) = (0,1), (0,4)$, and $(3,1)$. Since the procedure is identical to the previous cases, we only state the result. A non trivial polynomial solution can possible only for the case $(0,4)$ with the additional constraint $b_3=0$, and the solution is given by

$$y=b_4(4(k/H_0)^2-3b_2)+\frac{k^2(4(k/H_0)^2-3b_2)}{H_0^2a^2}+\frac{b_1k^2}{H_0^2a^3}. \tag{5.13}$$

Finally, we consider the case where $b_0=b_4=0$. In this case, we demand $B_{n_0-3}=D_{n_0+N-2}=0$ and only non trivial possibility turns out to be $(n_0,N)=(0,4)$. There are four equations for the four variables $y_0,\cdots,y_3$,

$$\begin{pmatrix}-8(k/H_0)^2 & 6b_3 & 0 & 0\\ 0 & 3b_2-4(k/H_0)^2 & 3b_3 & 0\\ 0 & b_1 & 0 & -3b_3\\ 0 & 0 & -b_1 & -3b_2+4(k/H_0)^2\end{pmatrix}\begin{pmatrix}y_0\\ y_1\\ y_2\\ y_3\end{pmatrix}=0$$

Fortunately, the determinant of the coefficient matrix is identically zero, which allows the following polynomial solution:

$$y=(3b_3)^2+\frac{12k^2b_3}{H_0^2a}+\frac{4k^2(4(k/H_0)^2-3b_2)}{H_0^2a^2}+\frac{4k^2b_1}{H_0^2a^3}. \tag{5.14}$$

Now let us summarize the result obtained so far. We have examined the solutions for the third order equation (5.5), in particular solutions that are rational functions of a. It turned out that only possible singularities (poles, in the case of rational functions) are at $a=0$, under the physical assumption that y is bounded in the limit $a\to\infty$. Then, by expanding y as a series of $1/a$, the expansion coefficients are determined by the five term recurrence relation (5.8). We found that there are four non trivial cases where the recurrence allows a finite term solution. These solutions are listed below:

$b_1=b_3=0$

$$y=b_4+\frac{k^2}{H_0^2a^2}. \tag{5.15}$$

$b_1=b_4=0$

$$y=(3b_3)^2+\frac{12k^2b_3}{H_0^2a}+\frac{4(k/H_0)^2(4(k/H_0)^2-3b_2)}{a^2}. \tag{5.16}$$

$b_0=b_3=0$

$$y=b_4(4-3b_2(H_0/k)^2)+\frac{4(k/H_0)^2-3b_2}{a^2}+\frac{b_1}{a^3}. \tag{5.17}$$

$b_0=b_4=0$

$$y=\frac{9H_0^2b_3^2}{4k^2}+\frac{3b_3}{a}+\frac{4(k/H_0)^2-3b_2}{a^2}+\frac{b_1}{a^3}. \tag{5.18}$$

These solutions can be used to construct the amplitude of tensor mode χ by performing the integration in the general formula (4.11). In the following sections we perform this integral in each of these cases and give the explicit solutions. Also, physical relevance of these solutions will be discussed.

6 Gravitational waves in $\Omega_m\Omega_K\Omega_\Lambda \neq 0$

In this section, we consider the FLRW spacetime with the non relativistic matter, cosmological constant, and spatial curvature. From the view point of the real universe, this spacetime can be regarded as a model for expansion of our universe after the radiation component becomes negligible. The observations of CMB so far have revealed that our universe is almost spatially flat. In this sence, it's enough to consider the case where $\Omega_k = 0$. Nevertheless, we derive the explicit expression for the scale factor and the gravitational amplitude with non zero curvature term for completeness.

Let us begin with the Friedmann equation (2.28), which in this case reduces to

$$\left(\frac{da}{d\eta}\right)^2 = H_0^2 \left(b_1 a + b_2 a^2 + b_4 a^4\right). \tag{6.1}$$

In order to solve for a in terms of the Weierstrass's elliptic function, we need some changes variables to convert this equation into the "standard form" (A.9). Such a transformation can be found in, for example (HTF vol.2). In our case, we introduce the new variable $\tilde{x}$ and $\tilde{\eta}$ by

$$a = \frac{1}{\tilde{x} - b_2/3b_1}, \quad \tilde{\eta} = \frac{1}{2}\sqrt{b_1} H_0 \eta. \tag{6.2}$$

Then, we can arrive at the following equation,

$$\left(\frac{d\tilde{x}}{d\tilde{\eta}}\right)^2 = 4\tilde{x}^3 - g_2\tilde{x} - g_3, \tag{6.3}$$

where the constants are defined by

$$g_2 = 3\left(\frac{2b_2}{3b_1}\right)^2, \quad g_3 = -\left(\frac{2b_2}{3b_1}\right)^3 - \frac{4b_4}{b_1}. \tag{6.4}$$

This is exactly the differential equation for the Weierstrass's function and the general solution is given by $\tilde{x} = \wp(\tilde{\eta} - \tilde{\eta}_0)$ with a integration constant $\tilde{\eta}_0$. We choose the origin of the conformal time so that $a = 0$ at $\tilde{\eta} = 0$, which leads to $\tilde{\eta}_0 = 0$. Thus, the solution for the Friedmann equation is given by

$$a(\eta) = \frac{1}{\wp(\tilde{\eta}) - b_2/3b_1}. \tag{6.5}$$

Since $\wp(z)$ has a second order pole at the origin, in the early epoch ($\tilde{\eta} \sim 0$) the scale factor behaves as $a \sim \tilde{\eta}^2$, which shows the evolution in the matter dominant era as we have seen in ... As $\tilde{\eta}$ gradually increases along the positive real axis, the value of $\wp(\tilde{\eta})$ decreases and eventually becomes equal to $b_2/3b_1$, where the scale factor diverges. This means that physical domain of the conformal time $\tilde{\eta}$ is an interval $\tilde{\eta} \in (0, \tilde{\eta}_f)$, where the endpoint $\tilde{\eta}_f$ is defined through $\wp(\tilde{\eta}_f) = b_2/3b_1$. The value of the derivative $\wp'(\tilde{\eta}_f)$ can be obtained by using the differential equation (A.9). Although there is an ambiguity of sign, the above discussion shows $\wp'(\tilde{\eta}_f)$ must be negative. Then, by considering the expansion of $\wp(\tilde{\eta})$ around $\tilde{\eta}_f$, we obtain the asymptotic form of the scale factor,

$$a(\eta) \sim \frac{1}{2}\sqrt{\frac{b_1}{b_4}} \frac{1}{\tilde{\eta}_f - \tilde{\eta}}, \quad (\tilde{\eta} \sim \tilde{\eta}_f) \tag{6.6}$$

which clearly shows the de Sitter expansion.

To completely understand the behavior of the scale factor, we have to determine the fundamental periods of the elliptic function, ω_1 and ω_2, and the value of the end point $\tilde{\eta}_f$. The inverse relation for $\wp(\tilde{\eta})$ and $\tilde{\eta}$ are given by (A.10), or equivalently (A.11), which in this case we write

$$\tilde{\eta} = \int_{\wp(\tilde{\eta})}^{\infty} \frac{d\tilde{x}}{\sqrt{4\tilde{x}^3 - g_2\tilde{x} - g_3}} = \int_{\wp(\tilde{\eta})}^{\infty} \frac{d\tilde{x}}{\sqrt{4(\tilde{x} - e_1)(\tilde{x} - e_2)(\tilde{x} - e_3)}}. \tag{6.7}$$

Since the discriminant D for the polynomial in the square root becomes negative, $D = -27b_4(b_4+4b_2^3) < 0$, one of the branch point of the integrand is real and the other two are complex conjugate each other. We take $e_1 \in \mathbb{R}$ and the imaginary part of e_2 to be positive. As is discussed in the appendix A, we can always take one of the half periods, say ω_1, to be real positive. Because only e_1 is the branch point on the real axis, ω_1 can be now evaluated by the following integral:

$$\omega_1 = \int_{e_1}^{\infty} \frac{d\tilde{x}}{\sqrt{4(\tilde{x}-e_1)(\tilde{x}-e_2)(\tilde{x}-e_3)}} (> 0). \tag{6.8}$$

By a suitable change of the integration variable, we can express this integral by using the complete elliptic integral of the 1st kind,

$$\omega_1 = \frac{1}{[(e_1-e_2)(e_1-e_3)]^{1/4}} K(q), \tag{6.9}$$

where

$$K(q) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-q^2\sin^2\theta}} \tag{6.10}$$

and the modulus q is given by

$$q = \sqrt{\frac{1}{2} + \frac{e_2+e_3-2e_1}{4\sqrt{(e_1-e_2)(e_1-e_3)}}}. \tag{6.11}$$

Note that the real root e_1 can be explicitly expressed by using g_2 and g_3 as

$$e_1 = \frac{1}{2}\left(\sqrt{g_3^2-(g_2/3)^3}+g_3\right)^{1/3} + \frac{g_2}{6}\left(\sqrt{g_3^2-(g_2/3)^3}+g_3\right)^{-1/3} (< 0). \tag{6.12}$$

Also, the real part e_2+e_3 and the squared absolute value e_2e_3 can be determined through

$$e_1+e_2+e_3 = 0, \quad e_1e_2e_3 = \frac{1}{4}g_3. \tag{6.13}$$

Thus we can now easily evaluate the half period ω_1 directly from the cosmological parameters b_1, b_2 and b_3. Another half period, say ω_2, can be defined by for example

$$\omega_2 = \int_{e_2}^{\infty} \frac{d\tilde{x}}{\sqrt{4(\tilde{x}-e_1)(\tilde{x}-e_2)(\tilde{x}-e_3)}}, \tag{6.14}$$

where in this case we have to be careful about the path of integration. When the path goes along the two line segments one of which connects e_2 and e_1 and the other connects e_1 and $+\infty$, with a suitable choice of the branch, we can obtain

$$\omega_2 = \frac{1}{2}\omega_1 - \frac{i}{2}\frac{1}{[(e_1-e_2)(e_1-e_3)]^{1/4}} K(\bar{q}), \tag{6.15}$$

where the mudulus $\bar{q}$ is given by

$$\bar{q} = \sqrt{\frac{1}{2} - \frac{e_2+e_3-2e_1}{4\sqrt{(e_1-e_2)(e_1-e_3)}}}. \tag{6.16}$$

In order to evaluate $\tilde{\eta}_f$, it is convenient to use the scale factor itself as a integration variable because $\tilde{\eta} = \tilde{\eta}_f$ corresponds to $a = \infty$. Then, the integral expression for $\tilde{\eta}_f$ is given by

$$\tilde{\eta}_f = \frac{1}{2}\int_0^{\infty} \frac{da}{\sqrt{a+(b_2/b_1)a^2+(b_4/b_1)a^4}}, \tag{6.17}$$

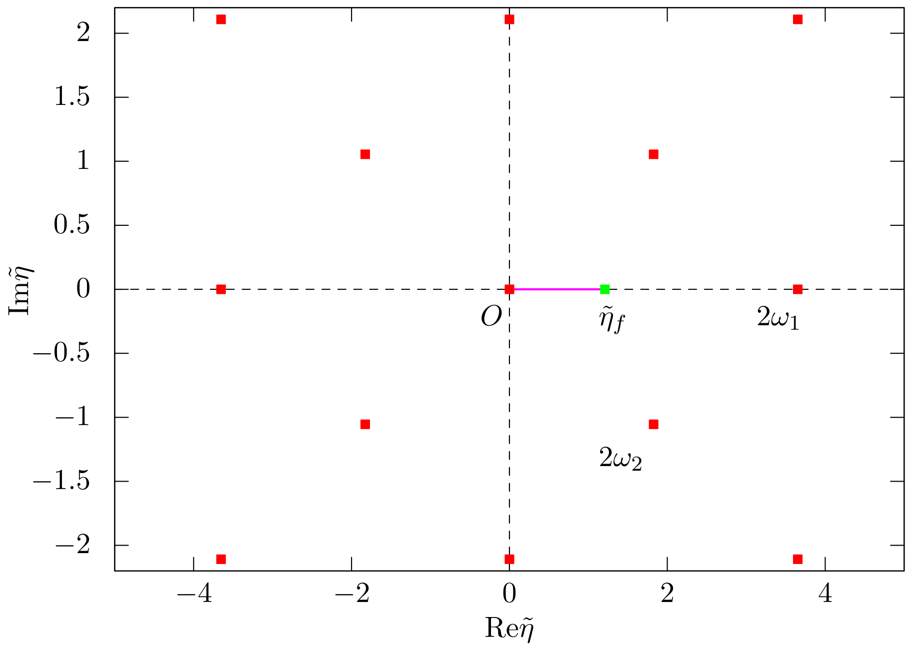


Figure 1: Fundamental lattice points of $\wp(\tilde{\eta})$ when $b_1 = 0.3, b_2 = 0, b_4 = 0.7$. A magenta line segment corresponds to the physical interval.

which can be converted into the standard form of the elliptic integral as

Now let us consider the gravitational waves on this background spacetime. As we have seen, the time evolution of the squared amplitude y of the gravitational waves on this spacetime can be represented by the polynomial (5.17) of $1/a$. Another factor of the waves, namely the time evolution of the phase, can be obtained from the general formula (4.11) by performing the integral including the amplitude $\sqrt{y}$. Before that, we need the constant C associated with y through Eq.(4.9). For the solution (5.17), we obtain

$$C^2 = -(1 - b_2/\tilde{k}^2)\left[4\tilde{k}^2(4\tilde{k}^2 - 3b_2)^2 + 27b_1^2 b_4\right] =: (i\tilde{C})^2, \tag{6.18}$$

where we define $\tilde{k} = k/H_0$. Now we write the solution $\chi_\pm$ as

$$\chi_\pm = \sqrt{y}\exp\left(\pm\frac{i\tilde{C}}{2}\int\frac{da}{a^2 y\sqrt{g(a)}}\right). \tag{6.19}$$

We can use the Friedmann equation to convert the integral variable from a to $\tilde{\eta}$,

$$I := \frac{\sqrt{b_1}}{2}\int\frac{da}{a^2 y\sqrt{g(a)}} = \int\frac{d\tilde{\eta}}{a^2 y}. \tag{6.20}$$

By recalling that the scale factor as a function of $\tilde{\eta}$ is given by (6.5), the integrand can be written as a rational function of the elliptic function $\wp(\tilde{\eta})$. Actually, y is a third order polynomial of $\wp(\tilde{\eta})$, and the integral I can be written as

$$I = \frac{1}{b_1}\int\frac{(\wp(\tilde{\eta}) - b_2/3b_1)^2 d\tilde{\eta}}{(\wp(\tilde{\eta}) - \wp(c_1))(\wp(\tilde{\eta}) - \wp(c_2))(\wp(\tilde{\eta}) - \wp(c_3))}, \tag{6.21}$$

where $\wp(c_i)$ $(i=1,2,3)$ are the roots of $y=0$ as a polynomial of $\wp(\tilde{\eta})$. More specifically, they are the solutions for the following equations:

$$\begin{aligned}\wp_1+\wp_2+\wp_3&=-\frac{4(\tilde{k}^2-b_2)}{b_1},\\ \wp_1\wp_2+\wp_2\wp_3+\wp_3\wp_1&=\frac{b_2(7b_2-8\tilde{k}^2)}{3b_1^2},\\ \wp_1\wp_2\wp_3&=-\frac{b_4(4\tilde{k}^2-3b_2)}{b_1\tilde{k}^2}-\frac{4\tilde{k}^2-10b_2/3}{b_1}\left(\frac{b_2}{3b_1}\right)^2,\end{aligned} \tag{6.22}$$

where we use simplified notation $\wp_i=\wp(c_i)$. The integral can be decomposed into partial fractions,

$$I=\frac{1}{b_1}\sum_{i=1}^{3}A_i\int\frac{d\tilde{\eta}}{\wp(\tilde{\eta})-\wp_i}, \tag{6.23}$$

where the constants A_i are given by

$$A_1=\frac{(\wp_1-b_2/3b_1)^2}{(\wp_2-\wp_1)(\wp_3-\wp_1)},\quad A_2=\frac{(\wp_2-b_2/3b_1)^2}{(\wp_1-\wp_2)(\wp_3-\wp_2)},\quad A_3=\frac{(\wp_3-b_2/3b_1)^2}{(\wp_2-\wp_3)(\wp_1-\wp_3)}. \tag{6.24}$$

Note that this decomposition can work only when the polynomial y has three distinct roots, which is assured because the discriminant D is always negative

$$D=-b_4(4-3b_2/\tilde{k}^2)^2\left[4\tilde{k}^2(4\tilde{k}^2-3b_2)^2+27b_1^2b_4\right]<0. \tag{6.25}$$

Also, this fact shows one of $\wp_i$ is real.

In order to evaluate the remaining integral, we utilize the addition theorem for the function $\zeta(z)$, Eq.(A.29). Since the function $\zeta(z)$ is the logarithmic derivative of $\sigma(z)$, we can integrate Eq.(A.29) with $z=\tilde{\eta}, w=c_i$,

$$\frac{1}{2}\wp'(c_i)\int\frac{d\tilde{\eta}}{\wp(\tilde{\eta})-\wp(c_i)}=\frac{1}{2}\log\left(\wp(\tilde{\eta})-\wp(c_i)\right)-\log\frac{\sigma(\tilde{\eta}+c_i)}{\sigma(\tilde{\eta})\sigma(c_i)}+\zeta(c_i)\tilde{\eta}. \tag{6.26}$$

Thus, by inserting this result into Eq.(6.19), we obtain the following expression:

$$\chi_{\pm}=\sqrt{y}e^{\pm i\kappa\tilde{\eta}}\prod_{i=1}^{3}\left[\left(\wp(\tilde{\eta})-\wp(c_i)\right)\left(\frac{\sigma(\tilde{\eta})\sigma(c_i)}{\sigma(\tilde{\eta}+c_i)}\right)^2\right]^{\pm i\Theta_i}, \tag{6.27}$$

where the exponents q and Θ_i $(i=1,2,3)$ are defined by

$$\kappa=2\sum_{i=1}^{3}\zeta(c_i)\Theta_i,\quad \Theta_i=\frac{\tilde{C}A_i}{(b_1)^{3/2}\wp'(c_i)}. \tag{6.28}$$

Another addition theorem (A.28) enables to further simplify this expression,

$$\chi_{\pm}=\sqrt{y}e^{\pm i\kappa\tilde{\eta}}\prod_{i=1}^{3}\left(\frac{\sigma(c_i-\tilde{\eta})}{\sigma(c_i+\tilde{\eta})}\right)^{\pm i\Theta_i}. \tag{6.29}$$

The constants $\wp(c_j)$ are the roots of the third order equation, of which explicit forms are given by

$$\begin{aligned}\wp(c_j)=&-\frac{4\tilde{k}^2-3b_2}{3b_1}-\frac{\alpha_j}{3b_1}\left(1-\frac{3b_2}{4\tilde{k}^2}\right)^{1/3}\left(\boldsymbol{C}-\sqrt{27b_1^2b_4}\right)^{2/3}\\ &-\frac{(4\tilde{k}^2-3b_2)^2}{3b_1}\alpha_j^*\left(1-\frac{3b_2}{4\tilde{k}^2}\right)^{-1/3}\left(\boldsymbol{C}-\sqrt{27b_1^2b_4}\right)^{-2/3},\end{aligned} \tag{6.30}$$

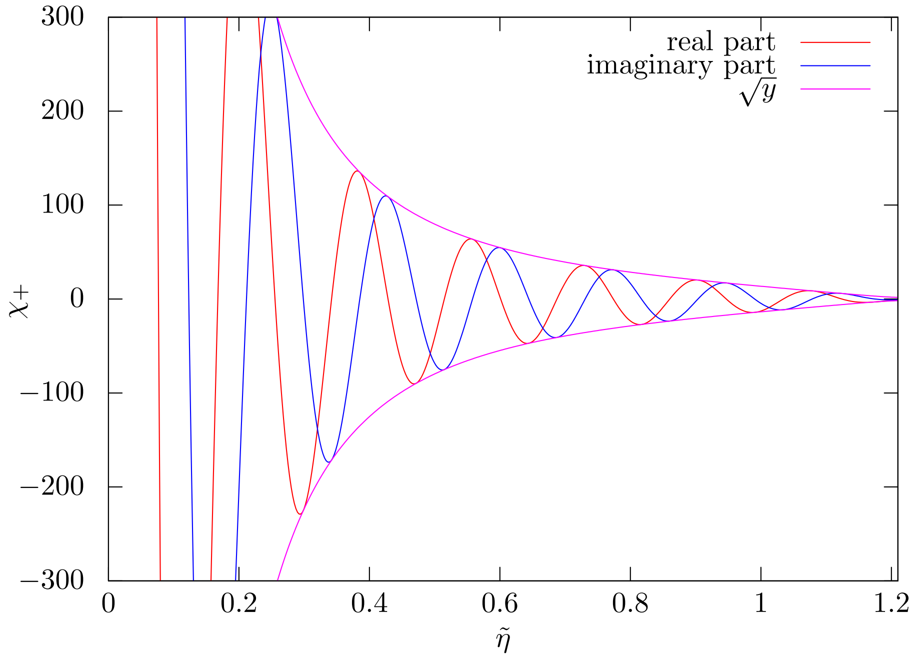


Figure 2: Numerical plot of $\chi_+(\tilde{\eta})$ with $b_1 = 0.3, b_2 = 0, b_4 = 0.7$ and $\tilde{k} = 10.0$.

where $\alpha_j = e^{2i\pi(j-1)/3}$ $(j = 1, 2, 3)$ are square roots of 1 and

$$\boldsymbol{C} = \sqrt{4\tilde{k}^2(4\tilde{k}^2 - 3b_2)^2 + 27b_1^2 b_4}. \tag{6.31}$$

Note that although this expression uniquely determine the value $\wp(c_j)$, c_j itself is not determined uniquely because of the periodicity and even parity of $\wp(z)$. However, the solution (6.29) is invariant under both the change of the sign $c_j \to -c_j$ and the translation by the periods $c_j \to c_j + 2\omega_i$, so we can arbitrary choose each c_j.

In the following, we restrict ourselves to the case of $b_2 = 0$, i.e. spatially flat case. Since cosmological observations so far indicates our space is almost flat, the solution obtained here can be considered as a model for gravitational waves propagating in the cosmological scale after the radiation energy density becomes negligible. In the early epoch in this model, the energy density of non relativistic matter $b_1 = \Omega_m$ dominates the universe while after the matter-Λ equality the universe begin to accelerated expansion. As we have seen in Sec.3, in each epoch where the energy density consists of only single term, the gravitational wave equation can be solved by using elementary functions. Although our exact solution (6.29) seems rather non trivial form even when $b_2 = 0$, it should include these elementary function solution as limiting cases. To explicitly confirm these limits is important not only for verifying Eq.(6.29) is indeed the solution but also for giving the matching condition for the two limiting solutions.

First we take $b_1 \to 0$, which corresponds to Λ-dominant limit. Note that in this limit the physical interval of the normalized conformal time $\tilde{\eta}$, namely $(0, \tilde{\eta}_f)$, shrinks to a single point as you can see from the integral expression (6.17). Instead of $\tilde{\eta}$, we have to use the original conformal time η. We change the origin of η from (6.2) as

$$\tilde{\eta} - \tilde{\eta}_f = \frac{1}{2}\sqrt{b_1} H_0 \eta, \tag{6.32}$$

In order the limit to be finite. More precisely, since the asymptotic behavior of $\tilde{\eta}_f$ is proportional to $(b_1)^{1/6}$, matter epoch $\tilde{\eta} \sim 0$ corresponds to $\eta \sim -\infty$. As the relevant region in this limit is

$\tilde{\eta} \sim \tilde{\eta}_f$, we expand the elliptic function around this point, which can be obtained by repeated use of the differential equation (A.9),

$$\wp(\tilde{\eta}) = -\sqrt{\frac{4b_4}{b_1}}(\tilde{\eta} - \tilde{\eta}_f) + \frac{48b_4}{b_1}(\tilde{\eta} - \tilde{\eta}_f)^4 + O((\tilde{\eta} - \tilde{\eta}_f)^7). \tag{6.33}$$

By keeping the variable η finite, this expansion shows that in the limit $b_1 \to 0$ the elliptic function converges to $\wp(\tilde{\eta}) \to -H_0\eta$. Then, the scale factor is given by

$$a(\eta) \to -\frac{1}{H_0\eta}, \ (b_1 \to 0) \tag{6.34}$$

which is exactly the scale factor for the de Sitter expansion. Note that we have used the constraint $b_1 + b_4 = 1$.

In order to obtain the asymptotic form of the gravitational wave amplitude (6.29), we have to consider the position of c_i in this limit. From the exact expression of $\wp(c_j)$ (6.30), the leading behavior can be found to be

$$\wp(c_1) \to -\frac{4\tilde{k}^2}{b_1}, \quad \wp(c_2) \to \frac{i}{\tilde{k}}, \quad \wp(c_3) \to -\frac{i}{\tilde{k}}. \tag{6.35}$$

Recalling the fact that only the singularities of $\wp(\tilde{\eta})$ is the origin $\tilde{\eta} = 0$ and the congruent points, we take c_1 so that $c_1 \to 0$ in this limit. Then, we can use the leading behavior $\wp(\tilde{\eta}) = 1/\tilde{\eta}^2 + O(\tilde{\eta}^2)$ to deduce the following:

$$c_1 \to \pm i\frac{\sqrt{b_1}}{2\tilde{k}}. \tag{6.36}$$

On the other hand, because $\wp(c_{2,3})$ are finite we can use $\wp(\tilde{\eta}) \to -H_0\eta$, which gives

$$c_2 \to \tilde{\eta}_f - \frac{i\sqrt{b_1}}{2\tilde{k}}, \quad c_3 \to \tilde{\eta}_f + \frac{i\sqrt{b_1}}{2\tilde{k}}. \tag{6.37}$$

The leading behavior of the constants appearing in the exponents, κ and Θ_i, turns out to be

$$\Theta_1 \to \pm\frac{i}{2}, \ \Theta_2 \to -\frac{i}{2}, \ \Theta_3 \to \frac{i}{2}, \ \kappa \to \frac{2\tilde{k}}{\sqrt{b_1}}. \tag{6.38}$$

We here observe that the exponent $\kappa\tilde{\eta}$ is indefinite in this limit because $\tilde{\eta} \to \tilde{\eta}_f = O((b_1)^{1/6})$ with η kept finite. In order to obtain the definite limit we normalize the amplitude as

$$\frac{\chi_\pm(\tilde{\eta})}{\chi_\pm(\tilde{\eta}_f)} = \sqrt{1 + \frac{\tilde{k}^2}{b_4 a^2} + \frac{b_1}{4b_4 a^3}} e^{\pm i\kappa(\tilde{\eta} - \tilde{\eta}_f)} \prod_{i=1}^{3} \left[\frac{\sigma(c_i - \tilde{\eta})\sigma(c_i + \tilde{\eta}_f)}{\sigma(c_i - \tilde{\eta}_f)\sigma(c_i + \tilde{\eta})}\right]^{\pm i\Theta_i}. \tag{6.39}$$

Then we can see that the exponential factor converges to $e^{\pm i\kappa(\tilde{\eta} - \tilde{\eta}_f)} \to e^{\pm ik\eta}$. The remaining factor is the sigma function part. Since c_1 vanishes faster than $\tilde{\eta}$ and $\tilde{\eta}_f$, $i = 1$ factor in the above expression just gives 1. On the other hand $i = 2$ part gives non trivial factor as follows:

$$\frac{\sigma(c_2 - \tilde{\eta})}{\sigma(c_2 - \tilde{\eta}_f)} \to \frac{\sigma(-i\sqrt{b_1}/2\tilde{k} - \sqrt{b_1}H_0\eta/2)}{\sigma(-i\sqrt{b_2}/2\tilde{k})} \to 1 - ik\eta, \tag{6.40}$$

and $i = 3$ part are just the complex conjugate. Finally, combining all the results we obtain

$$\frac{\chi_\pm(\tilde{\eta})}{\chi_\pm(\tilde{\eta}_f)} \to (1 \mp ik\eta)\, e^{\pm ik\eta}. \tag{6.41}$$

Next let us take the other limit, $b_4 \to 0$. In this case, we don't have to change the origin of the conformal time and the two coordinates $\tilde{\eta}$ and $\tilde{\eta}$ are connected as

$$\tilde{\eta} = \frac{1}{2}H_0\eta. \tag{6.42}$$

Furthermore, since $g_2 = g_3 = 0$ in this limit we can directly integrate the differential equation (A.9) to obtain the scale factor in matter dominant era,

$$\wp(\tilde{\eta}) \to \frac{1}{\tilde{\eta}^2} \;\Rightarrow\; a(\eta) \to \left(\frac{H_0 \eta}{2}\right)^2 . \tag{6.43}$$

Also, similar calculation to the previous case shows

$$\chi_\pm(\eta) \to \left(\frac{2}{H_0\eta}\right)^3 (1 \mp ik\eta)e^{\pm ik\eta}. \tag{6.44}$$

As a model for relic gravitational waves generated in the early universe, we have to consider the initial condition for $\chi(\eta)$. Because the solution obtained in this section is applicable only after the radiation component of the energy density becomes negligible, we need some boundary condition at the transition from radiation-dominated era to matter-dominated era. Although we don't know the analytic solution there, the relevant wave modes for future direct observations should be almost constant mode around the transition, otherwise the amplitude is suppress with the expansion and it becomes impossible to directly observe. Thus, we look for the specific superposition of the basis solutions $\chi_\pm(\eta)$ that behaves as almost constant mode at $\eta \sim 0$ in the super-horizon limit. Let us first consider the expansion of Eq.(6.29). The absolute value $\sqrt{y}$ can be easily expanded with the help of the expansion of the elliptic function (A.6). To expand the phase factor, we note that the n-th logarithmic derivative of $\sigma(z)$ is given by

$$\frac{d^n}{dz^n}\log\sigma(z) = -\wp^{(n-2)}(z).\ (n \geq 2) \tag{6.45}$$

By using this identity, we obtain

$$\log\frac{\sigma(c_i - \tilde{\eta})}{\sigma(c_i + \tilde{\eta})} = -2\zeta(c_i)\tilde{\eta} + \sum_{n=1}^{\infty}\frac{2\tilde{\eta}^{2n+1}}{(2n+1)!}\wp^{(2n-1)}(c_i), \tag{6.46}$$

which then gives the following expansion around $\tilde{\eta} = 0$ of the phase factor:

$$e^{\pm i\kappa\tilde{\eta}}\prod_{j=1}^{3}\left(\frac{\sigma(c_j - \tilde{\eta})}{\sigma(c_j + \tilde{\eta})}\right)^{\pm i\Theta_j} = \exp\left(\pm i\sum_{n=1}^{\infty}K_n\tilde{\eta}^{2n+1}\right), \tag{6.47}$$

where the coefficients K_n are defined by

$$K_n = \frac{2}{(2n+1)!}\sum_{j=1}^{3}\Theta_j\wp^{(2n-1)}(c_j).\ (n \geq 1) \tag{6.48}$$

In principle we can calcualte the explicit expression of K_n for arbitrary order n by using the value of $\wp(c_j)$ given in Eq.(6.30) and the differential equation (A.9). Actually, because K_n is symmetric under any permutation of c_j, we don't need the explicit expression (6.30) but only Eqs.(6.22) are enough. For our purpose, we take only the leading term in $\tilde{\eta} \to 0$, namely K_1, which turns out to be

$$K_1 = \frac{\tilde{C}}{3(b_1)^{3/2}} \to \frac{1}{3}\sqrt{\left(\frac{4\tilde{k}^2}{b_1}\right)^3 + \frac{27b_4}{b_1}}\ (b_2 \to 0) \tag{6.49}$$

Note that the first expression holds even when $b_2 \neq 0$. Then, we obtain the expansion of $\chi_\pm$ around $\tilde{\eta} = 0$,

$$\chi_\pm(\eta) = \frac{\sqrt{b_1}}{\tilde{\eta}^3}\left(1 + \frac{2\tilde{k}^2}{b_1}\tilde{\eta}^2 \pm iK_1\tilde{\eta}^3 + O(\tilde{\eta}^4)\right). \tag{6.50}$$

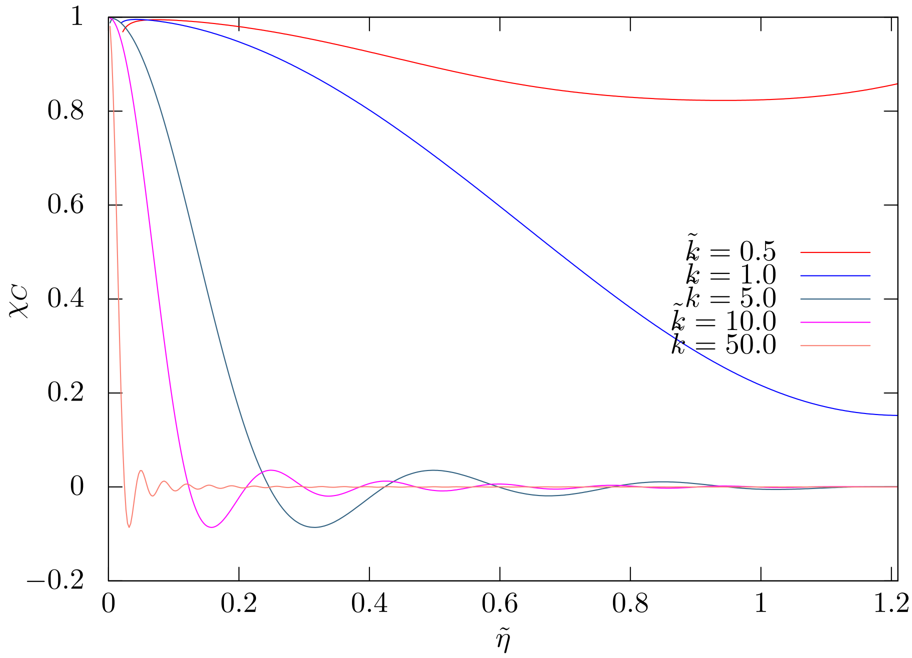


Figure 3: χ_C with several values of $\tilde{k}$.

We can immediately see that the specific combination $\chi_C(\eta) := (\chi_+(\eta) - \chi_-(\eta))/(2i)$ is the only non singular solution around $\tilde{\eta} = 0$. Furthermore, one can easily verify, though we don't show explicitly, that the next-leading order coefficient of the power series expansion is proportional to $\tilde{k}^2$, which means in the super-horizon limit $\tilde{k} \to 0$, $\chi_C(\eta)$ really behaves as almost constant.

To understand the super-horizon limit of the solutions, we next consider low frequency expansion. Taking the limit $\tilde{k} \to 0$ directly in the expression (6.29) is not so straightforward because we have to examine the behavior of c_j. Instead, we directly solve the evolution equation (3.37) with the assumption that $\chi_\pm(\eta) = f_\pm(\eta) + \tilde{k}^2 g_\pm(\eta) + O(\tilde{k}^4)$, where $f_\pm$ and $g_\pm$ are $\tilde{k}$ independent functions. As we have mentioned in Sec.3, the leading term, namely zero modes, are in general given by the constant mode and decaying mode (3.38). In our case, the integral expression for the decaying mode can be performed by using the second order differential equation satisfied by the elliptic function,

$$\wp''(\tilde{\eta}) = 6\wp^2(\tilde{\eta}) - \frac{1}{2}g_2. \tag{6.51}$$

Together with the condition $b_2 = 0$, we conclude that the general solution of zero mode is given by

$$A + B\wp'(\tilde{\eta}) = -\frac{2B}{\tilde{\eta}^3} + A + O(\tilde{\eta}^6). \tag{6.52}$$

By comparing the last expansion with Eq.(6.50), the leading term $f_\pm(\eta)$ turns out to be

$$f_\pm(\eta) = -\frac{\sqrt{b_1}}{2}\left(\wp'(\tilde{\eta}) \mp 2i\sqrt{\frac{3b_4}{b_1}}\right). \tag{6.53}$$

The next order term $g_\pm$ can be obtained by solving the zero mode equation with a inhomogeneous term,

$$\left[\frac{d^2}{d\tilde{\eta}^2} - \frac{2\wp'(\tilde{\eta})}{\wp(\tilde{\eta})}\frac{d}{d\tilde{\eta}}\right] g_\pm(\tilde{\eta}) + \frac{4}{b_1} f_\pm(\tilde{\eta}) = 0, \tag{6.54}$$

which is derived from the evolution equation (3.37). We can integrate this equation for $g_\pm$ and the integration constants are determined again by comparing the result with Eq.(6.50). The result is

$$g_\pm(\eta) = \frac{2}{\sqrt{b_1}}\zeta(\tilde{\eta}) \mp \frac{4i}{\sqrt{b_1}}\sqrt{\frac{3b_4}{b_1}}\int_0^{\tilde{\eta}} dz\wp^2(z)\int_0^z \frac{dw}{\wp^2(w)}. \tag{6.55}$$

Let us consider the energy density spectrum of this gravitational wave solution. In the general formula (3.56), we need to take average over several periods. To perform this, it is useful to rewrite the basis solution (6.29) as $\chi_\pm = \sqrt{y}e^{\pm i\theta}$, where the phase factor θ as wel as the absolute value of the amplitude $\sqrt{y}$ are functions of η. As we can see from Eq.(5.17), the amplitude $\sqrt{y}$ is given by a polynomial of the reciprocal of the scale factor while the phase factor $e^{\pm i\theta}$ shows oscillatory behavior in the sub-horizon regime. Thus, we approximate the averaging by simply keeping terms which don't contain this phase factor and ignoring the other terms. As an illustration, consider the averaging for the purely positive frequency solution $\chi_+ = \sqrt{y}e^{i\theta}$. The squared absolute value of the derivative is given by

$$\left|\frac{d\chi_+}{d\tilde{\eta}}\right|^2 = \frac{d\chi_+}{d\tilde{\eta}}\frac{d\chi_-}{d\tilde{\eta}} = \frac{1}{4y}\left(\frac{dy}{d\tilde{\eta}}\right)^2 + y\left(\frac{d\theta}{d\tilde{\eta}}\right)^2, \tag{6.56}$$

which doesn't contain any $e^{\pm i\theta}$, hence the averaging for this solution just gives

$$\left\langle \left|\frac{d\chi_+}{d\tilde{\eta}}\right|^2 \right\rangle = \left|\frac{d\chi_+}{d\tilde{\eta}}\right|^2. \tag{6.57}$$

One important note is that though this expression includes both y and θ, in fact, what is really necessary is the amplitude y. As we have mentioned (see the discussion above Eq.(4.5)), the symmetric products of two solutions χ_1, χ_2 and their first derivatives can be expressed in terms of y and its derivatives. In particular, we have in this case

$$\frac{d\chi_+}{d\tilde{\eta}}\frac{d\chi_-}{d\tilde{\eta}} = \frac{1}{2}\frac{d^2y}{d\tilde{\eta}^2} + \frac{1}{a}\frac{da}{d\tilde{\eta}}\frac{dy}{d\tilde{\eta}} + \frac{4\tilde{k}^2}{b_1}y. \tag{6.58}$$

Using the polynomial form of y, Eq.(5.17) and the Friedmann equation (6.1), we obtain

$$\left|\frac{d\chi_+}{d\tilde{\eta}}\right|^2 = \frac{4\tilde{k}^2(4\tilde{k}^2 - 3b_2)}{b_1 a^2} + \frac{12\tilde{k}^2}{a^3} + \frac{9b_1}{a^4}. \tag{6.59}$$

Then, from the general formula (3.56), the spectrum for the specific solution χ_+ is given by a rational function of the scale factor

$$\Omega_{\rm GW} = \frac{P(k)a^2}{96\pi^2 g(a)}\left[\frac{4\tilde{k}^2(4\tilde{k}^2 - 3b_2)}{a^2} + \frac{12b_1\tilde{k}^2}{a^3} + \frac{9b_1^2}{a^4}\right]. \tag{6.60}$$

We have to move on to the case of cosmologically relevant solution χ_C. In this case, we need to consider the averaging of the following expression:

$$\left|\frac{d\chi_C}{d\tilde{\eta}}\right|^2 = \frac{1}{2}\frac{d\chi_+}{d\tilde{\eta}}\frac{d\chi_-}{d\tilde{\eta}} - \frac{1}{4}\left[\left(\frac{d\chi_+}{d\tilde{\eta}}\right)^2 + \left(\frac{d\chi_-}{d\tilde{\eta}}\right)^2\right]. \tag{6.61}$$

As we have noted, we ignore the last term in the averaging because they contain the oscillatory factor $e^{\pm 2i\theta}$. Then, we can see the remaining term gives just a half contribution of the spectrum for χ_+ given above. Also, we have to care about the normalization of the solution. So far, our convention is such that $\chi_C(\eta) \to \sqrt{b_1}K_1$ as $\eta \to 0$ and this gives a non trivial k dependent normalization. We now normalize the solution so as to $\chi_C(\eta) \to 1$ as $\eta \to 0$, which finally gives the following spectrum:

$$\Omega_{\mathrm{GW}} = \frac{3b_1^2 a^2 P(k)}{64\pi^2 g(a)} \left(1 - \frac{b_2}{\tilde{k}^2}\right)^{-1} \left[4\tilde{k}^2(4\tilde{k}^2 - 3b_2)^2 + 27b_1^2 b_4\right]^{-1} \times \left[\frac{4\tilde{k}^2(4\tilde{k}^2 - 3b_2)}{a^2} + \frac{12b_1\tilde{k}^2}{a^3} + \frac{9b_1^2}{a^4}\right]. \tag{6.62}$$

This treatment of averaging is valid only when the mode comes into the sub-horizon regime, $k \gg aH$. To examine this condition, we first identify the horizon crossing time $\tilde{\eta}_k$ for a given mode with frequency k. By definition, $\tilde{\eta}_k$ satisfies

$$\frac{2\tilde{k}}{\sqrt{b_1}} = \frac{1}{a}\frac{da}{d\tilde{\eta}}(\tilde{\eta}_k). \tag{6.63}$$

Before solving this equation, we must be aware that the right hand side, which is the comoving Hubble scale, has the minimum value because the comoving Hubble scale increases in the matter dominated era while decreases in the cosmological constant dominated era. To find the minimum, we take the square of the comoving Hubble scale and use the Friedmann equation to obtain

$$\frac{1}{a^2}\left(\frac{da}{d\tilde{\eta}}\right)^2 = 4\left(\frac{1}{a} + \frac{b_4}{b_1}a^2\right). \tag{6.64}$$

We can easily find the minimum value $6(2b_4/b_1)^{1/3}$ at $a = (b_1/2b_4)^{1/3}$, which is slightly earlier than the matter-Λ equality. Thus, the horizon crossing can occur only for the mode with frequency larger than $\tilde{k}_{\min}$ given by

$$\tilde{k}_{\min} = \sqrt{\frac{3}{2}}\left(2b_1^2 b_4\right)^{1/6}. \tag{6.65}$$

By inserting the standard value $b_1 = 0.3, b_4 = 0.7$, and $H_0 = 70$ km/s Mpc at $a = 1$, we obtain the minimum frequency $\nu_{\min} = k_{\min}/2\pi \sim 0.3 \times 10^{-18}$Hz, or equivalently the maximum wave length $\lambda_{\max} \sim 300$Gpc. Taking the square of the crossing equation and using the Friedmann equation, we obtain the 3rd order equation which determines the scale factor at the crossing,

$$a^3(\tilde{\eta}_k) - \frac{\tilde{k}^2}{b_4}a(\tilde{\eta}_k) + \frac{b_1}{b_4} = 0. \tag{6.66}$$

For sufficiently large value of $\tilde{k}$, namely $\tilde{k} \geq \tilde{k}_{\min}$, this equation have two positive solutions, which correspond to the horizon entering and exit respectively. In order the formula for the spectrum above to be valid, $\tilde{k}$ must be much larger than $\tilde{k}_{\min}$ and at the same time, a should not be so large or small. More precisely, denoting the scale factor at the horizon enter and exit by a_1 and a_2 respectively, $a_1 \ll a \ll a_2$ must hold. Note that necessary condition $a_1 \ll a_2$ is assured by $\tilde{k} \gg \tilde{k}_{\min}$.

7 Relic gravitational waves in quintessence universe

In this section, we construct the exact solution $\chi(\eta)$ corresponding to the polynomial solution (5.18). Although, as far as we know, there is no matter whose energy density dilute as $\rho \propto 1/a$ with the expansion of the universe, we consider this background spacetime as an effective model of the quintessence scenario for the late time acceleration.

As in the previous section, we first consider the Friedmann equation

$$\left(\frac{da}{d\eta}\right)^2 = H_0^2\left(b_1 a + b_2 a^2 + b_3 a^3\right). \tag{7.1}$$

Because the right hand side is already a 3rd order polynomial, the standard form of Weierstrass's elliptic function can be achieved by shifting the scale factor $a = x - b_2/(3b_3)$ and normalizing the conformal time $\tilde{\eta} = \sqrt{b_3}H_0\eta/2$,

$$\left(\frac{dx}{d\tilde{\eta}}\right)^2 = 4x^3 - g_2 x - g_3, \tag{7.2}$$

where the constants g_2 and g_3 are given by

$$g_2 = 3\left(\frac{2b_2}{3b_3}\right)^2 - \frac{4b_1}{b_3}, \quad g_3 = -\left(\frac{2b_2}{3b_3}\right)^3 + \frac{4b_1 b_2}{3b_3^2}. \tag{7.3}$$

Again, the general solution is given by $x = \wp(\tilde{\eta} - \tilde{\eta}_f)$ with an integration constant $\tilde{\eta}_f$. Physically relevant region is from $a = 0$ to $a = \infty$. We set the origin of the conformal time as the initial time, namely $a(0) = 0$. Then, the constant $\tilde{\eta}_f$ is determined by the equation $\wp(\tilde{\eta}_f) = b_2/(3b_3)$. The other endpoint of the conformal time is $\tilde{\eta} = \tilde{\eta}_f$, where the scale factor diverges as

$$a(\eta) \to \frac{1}{(\tilde{\eta}_f - \tilde{\eta})^2}. \tag{7.4}$$

In order to determine the periods of this elliptic function, we have to see the locations of the roots of $4x^3 - g_2 x - g_3 = 0$. The discriminant D for this equation is found to be

$$D = \left(\frac{b_1}{b_3}\right)^2 \left[\left(\frac{b_2}{b_3}\right)^2 - \frac{4b_1}{b_3}\right], \tag{7.5}$$

whose sign is indefinite depending on the density parameters. We first assume $D < 0$, where the solutions for the 3rd order equation consists of one real solution e_1 and two complex solutions e_2, e_3. Actually, the real solution is given by $e_1 = b_2/(3b_3)$, which can be easily verified from the original form of the Friedmann equation (7.1). Regarding the result of observations, this case including $b_2 = 0$ is physically more attractive. The half periods for this case are formally given by the same expressions as in the previous section, namely Eqs.(6.9) and (6.15), because the configuration of the roots e_1, e_2, e_3 is the same. Similarly, the endpoint $\tilde{\eta}_f$ is evaluated by the following integral:

$$\tilde{\eta}_f = \frac{1}{2}\int_{b_2/(3b_3)}^{\infty} \frac{dx}{\sqrt{(x-e_1)(x-e_2)(x-e_3)}} = \frac{1}{2}\int_0^{\infty} \frac{da}{\sqrt{a^3 + (b_2/b_3)a^2 + (b_1/b_3)a}}. \tag{7.6}$$

From the first expression, we can see that this value is actually equal to the real half period ω_1. Thus the scale factor can be written as

$$a(\eta) = \wp(\omega_1 - \tilde{\eta}) - e_1. \tag{7.7}$$

Let us now turn to the construction of the gravitational wave amplitude on this background by using the general formula (4.11) with the squared amplitude (5.18). First of all, we obtain the Wronskian constant C as

$$C^2 = -(4\tilde{k}^2 - 3b_2)\left[4(\tilde{k}^2 - b_2)(4\tilde{k}^2 - 3b_2) - 18b_1 b_3\right] - \frac{b_1 b_3}{4\tilde{k}^2}\left(64\tilde{k}^4 + 81 b_1 b_3\right), \tag{7.8}$$

where $\tilde{k} = k/H_0$.

The integration of the phase factor,

$$I := \int \frac{d\tilde{\eta}}{a^2 y} \tag{7.9}$$

is very similar to the case in the previous section. We first factorize the integrand,

$$I=\frac{4\tilde{k}^2}{9b_3^2}\int\frac{(\wp-b_2/3b_3)d\tilde{\eta}}{(\wp-\wp_1)(\wp-\wp_2)(\wp-\wp_3)},\tag{7.10}$$

where $\wp=\wp(\tilde{\eta}-\tilde{\eta}_f)$ and $\wp_j=\wp(c_j-\tilde{\eta}_f)$ $(j=1,2,3)$. The constants c_j are defined so that $\wp(c_j-\tilde{\eta}_f)=z_j+b_2/3b_3$ with the solutions z_j of the following equation:

$$z^3+\frac{4\tilde{k}^2}{3b_3}z^2+\frac{4\tilde{k}^2}{9b_3^2}(4\tilde{k}^2-3b_2)z+\frac{4\tilde{k}^2b_1}{9b_3^2}=0.\tag{7.11}$$

Next we decompose in a partial fraction as

$$I=\frac{4\tilde{k}^2}{9b_3^2}\sum_{j=1}^{3}B_j\int\frac{d\tilde{\eta}}{\wp-\wp_j},\tag{7.12}$$

where the constants B_j are given by

$$B_1=\frac{\wp_1-b_2/3b_3}{(\wp_1-\wp_2)(\wp_1-\wp_3)},\ B_2=\frac{\wp_2-b_2/3b_3}{(\wp_2-\wp_1)(\wp_2-\wp_3)},\ B_3=\frac{\wp_3-b_2/3b_3}{(\wp_3-\wp_2)(\wp_3-\wp_1)}.\tag{7.13}$$

Then the remaining integral can be evaluated by using the addition theorem as

$$\frac{1}{2}\wp'(c_j-\tilde{\eta}_f)\int\frac{d\tilde{\eta}}{\wp-\wp_j}=\frac{1}{2}\log\frac{\sigma(c_j-\tilde{\eta})}{\sigma(\tilde{\eta}+c_j-2\tilde{\eta}_f)}+\zeta(c_j-\tilde{\eta}_f)\tilde{\eta}.\tag{7.14}$$

Finally, we arrive at the following exact solution:

$$\chi_\pm(\eta)=\sqrt{y}e^{\pm i\kappa\tilde{\eta}}\prod_{j=1}^{3}\left(\frac{\sigma(c_j-2\tilde{\eta}_f)\sigma(c_j-\tilde{\eta})}{\sigma(\tilde{\eta}+c_j-2\tilde{\eta}_f)\sigma(c_j)}\right)^{\pm i\Theta_j},\tag{7.15}$$

where

$$\kappa=2\sum_{j=1}^{3}\Theta_j\zeta(c_j-\tilde{\eta}_f),\quad \Theta_j=\frac{\tilde{C}}{\sqrt{b_3}}\frac{4\tilde{k}^2}{9b_3^2}\frac{B_j}{\wp'(c_j-\tilde{\eta}_f)},\tag{7.16}$$

and we adjusted the integration constant so that the phase factor takes the value 1 at $\tilde{\eta}=0$.

From now on, we set $b_2=0$ and investigate the properties of this solution. Let us begin with takeing the limit of single component, namely $b_1\to 0$ and $b_3\to 0$ and confirm our solution really converges to the elementary function solutions in each case. In the flat space case, the conformal time at the endpoint $\tilde{\eta}=\omega_1$ can be expressed as a simple power of b_3/b_1,

$$\tilde{\eta}_f=\omega_1=\frac{4\Gamma^2(5/4)}{\sqrt{\pi}}\left(\frac{b_3}{b_1}\right)^{1/4}.\tag{7.17}$$

In the matter-dominant limit $b_3\to 0$, because $\tilde{\eta}=O((b_3)^{1/2})$ vanishes faster than $\tilde{\eta}_f=((b_3)^{1/4})$, we expand the scale factor at $\tilde{\eta}=0$, of which only the leading term remains non vanishing in this limit,

$$a(\eta)\to\frac{1}{4}(H_0\eta)^2.\quad(b_3\to 0)\tag{7.18}$$

We next consider the initial condition for this solution. In particular, we are interested in the solution that is regular at $\tilde{\eta}=0$. We expect such a solution is realized by the superposition $\chi_C=(\chi_+-\chi_-)/(2i)$, as in the case of Sec.6. Note that this expectation comes from the specific normalization of the solution (7.15) where the phase factor becomes unity at $\tilde{\eta}=0$. In other words, if we write the solution as $\chi_\pm=\sqrt{y}e^{\pm i\theta}$, the phase θ vanishes at $\tilde{\eta}=0$, which then implies $\chi_C=\sqrt{y}\sin\theta$ can become finite at $\tilde{\eta}=0$. Although it is rather natural that this specific

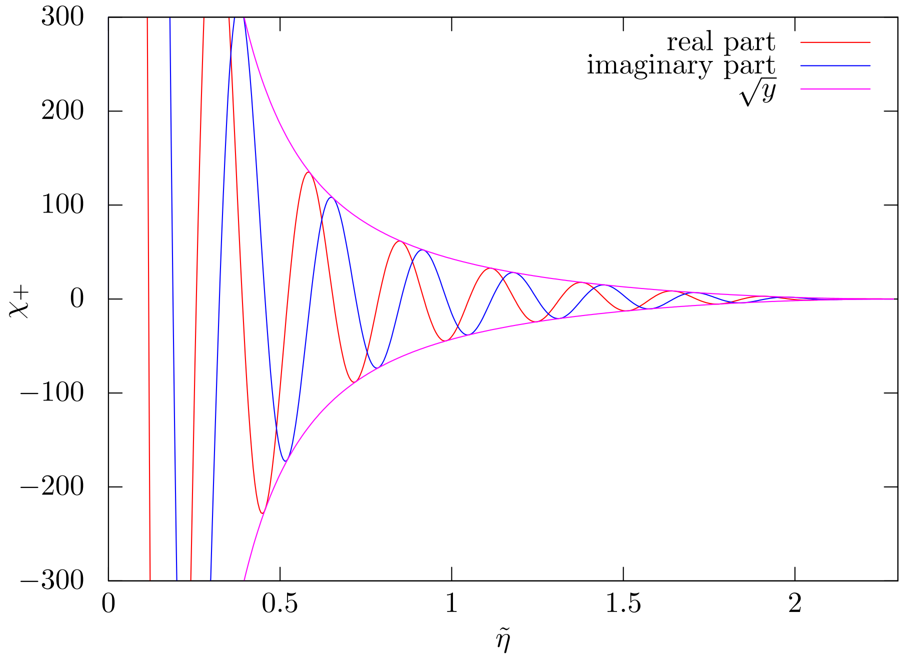


Figure 4: Numerical plot of $\chi_+(\tilde{\eta})$ with $b_1 = 0.3, b_2 = 0, b_3 = 0.7$, and $\tilde{k} = 10.0$.

combination χ_C is regular there, we show that this is indeed the case below. The reason is that without explicit calculation we cannot extract the normalization factor of χ_C at $\tilde{\eta} = 0$. The calculation is similar to the case in Sec.6 but it is not so trivial in this case because of the choice of the origin of the conformal time. The taylor expansion of the σ functions are obtained in the same manner,

$$\log \frac{\sigma(c_j - 2\tilde{\eta}_f)\sigma(c_j - \tilde{\eta})}{\sigma(\tilde{\eta} + c_j - 2\tilde{\eta}_f)\sigma(c_j)} = -\left[\zeta(c_j - 2\tilde{\eta}_f) + \zeta(c_j)\right]\tilde{\eta} + \sum_{n=2}^{\infty} \frac{\tilde{\eta}^n}{n!}\left[\wp^{(n-2)}(c_j - 2\tilde{\eta}_f) - (-1)^n \wp^{(n-2)}(c_j)\right]. \tag{7.19}$$

Recalling $\tilde{\eta}_f = \omega_1$ is one of the half period, the last line of the right hand side is formally equal to the expansion obtained in Eq.(6.46). However, the linear part doesn't seem cancel out with the exponent of $e^{\pm i\kappa\tilde{\eta}}$. The possibly problematic part of the exponent is given by

$$\sum_{j=1}^{3} \left(2\zeta(c_j - \omega_1) - \zeta(c_j - 2\omega_1) - \zeta(c_j)\right)\Theta_j. \tag{7.20}$$

(Recall the definition of κ.) We prove by performing the sum this expression is actually vanishing below. By using the quasi periodicity of the zeta function (A.22) and the addition theorem (A.29), we can manipulate as

$$2\zeta(c_j - \omega_1) - \zeta(c_j - 2\omega_j) - \zeta(c_j) = 2\left[\zeta(c_j - \omega_1) - \zeta(c_j) + \zeta(\omega_1)\right] = \frac{\wp'(c_j)}{\wp(c_j)}. \tag{7.21}$$

Note that in applying the addition theorem we used the assumption $b_2 = 0$ and the differential equation for $\wp(z)$. Because $g_3 = 0$ now, another addition theorem (A.27) gives the following "half periodicity":

$$\wp(z-\omega_1) = -\frac{g_2}{4}\frac{1}{\wp(z)}, \quad \wp'(z-\omega_1) = \frac{g_2}{4}\frac{\wp'(z)}{\wp^2(z)}. \tag{7.22}$$

Then, equation (7.20) can be reduced as follows

$$\begin{aligned}
&\sum_{j=1}^{3}\left(2\zeta(c_j-\omega_1)-\zeta(c_j-2\omega_1)-\zeta(c_j)\right)\Theta_j \\
&\propto \sum_{j=1}^{3}\frac{B_j}{\wp'(c_j-\omega_1)}\frac{\wp'(c_j)}{\wp(c_j)} \\
&= \frac{4}{g_2}\sum_{j=1}^{3}\wp(c_j)B_j \\
&= -\left[\frac{1}{(\wp_1-\wp_2)(\wp_1-\wp_3)}+\frac{1}{(\wp_2-\wp_1)(\wp_2-\wp_3)}+\frac{1}{(\wp_3-\wp_2)(\wp_3-\wp_1)}\right]=0.
\end{aligned}$$

Finally we obtain the expansion around $\tilde{\eta}=0$,

$$\chi_\pm(\eta) = \sqrt{y}\exp\left(\pm i\sum_{n=1}^{\infty}K_n\tilde{\eta}^{2n+1}\right), \tag{7.23}$$

where

$$K_n = \frac{2}{(2n+1)!}\sum_{j=1}^{3}\Theta_j\wp^{(2n-1)}(c_j). \tag{7.24}$$

From this, the leading term of χ_C in $\tilde{\eta}\to 0$ can be obtained as

$$\chi_C(\eta) = \frac{b_3^{3/2}}{b_1}K_1 + O(\tilde{\eta}^2), \tag{7.25}$$

which shows χ_C is regular as expected. For later use, we show the explicit form of K_1 below:

$$K_1 = \frac{\tilde{C}}{3b_3^{3/2}}. \tag{7.26}$$

Let us examine the energy density spectrum. Because the structure of the solution is very similar to that in Sec.6, the argument to derive Eq. (6.62) can be applied here. Thus, we only show the final result,

$$\Omega_{\rm GW} = \frac{b_3P(k)a^2}{96\pi^2 g(a)}\left(\frac{3b_1}{\tilde{C}}\right)^2\left[\frac{2\tilde{k}^2}{a}+\frac{8\tilde{k}^4}{b_3a^2}+\frac{6\tilde{k}^2b_1}{b_3a^3}+\frac{9b_1^2}{2b_3a^4}\right], \tag{7.27}$$

where $\tilde{C}$ is given by

$$\tilde{C}^2 = (4\tilde{k}^2)^3 - 56b_1b_3\tilde{k}^2 + \frac{(9b_1b_3)^2}{4\tilde{k}^2}. \tag{7.28}$$

If we put the matter density parameter at $b_1=0.3$ as in the standard ΛCDM model, numerical plot of $\Omega_{\rm GW}$ at $a=1$ divided by the primordial spectrum $P(k)$ is given in figure 6, where for comparison the spectrum for the matter and cosmological constant background is shown as well. As one can see, for large value of $\tilde{k}$ they are almost the same because such high frequency modes enter the sub-horizon regime deep in the matter dominated era. However, the assumption that the matter density parameter b_1 takes the same value for both cases may not be adequate. Although a recent work[32] on observational constraints for various quintessential

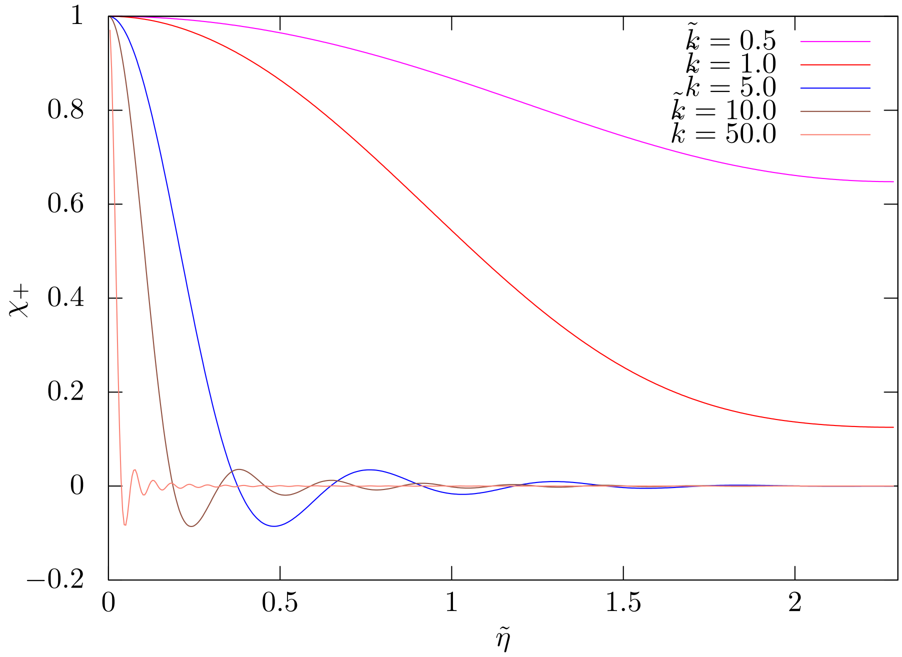


Figure 5: χ_C with several values of $\tilde{k}$. $b_1 = 0.3, b_2 = 0, b_3 = 0.7$.

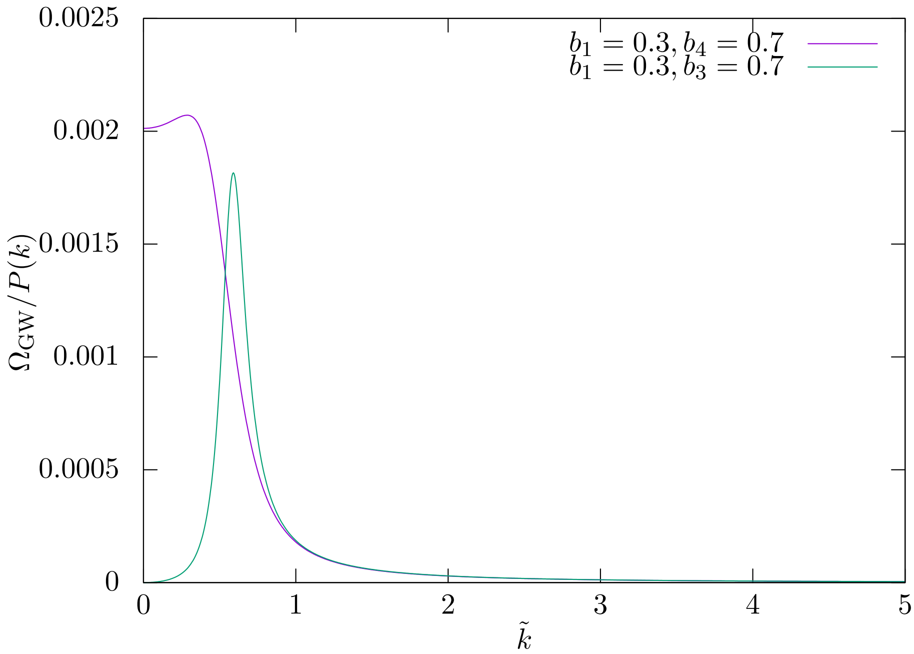


Figure 6: Comparison of the energy density spectra of gravitational wave solutions at $a = 1$. Large deviation appears at small $\tilde{k}$ region while they are almost identical in $\tilde{k} \geq 1$.

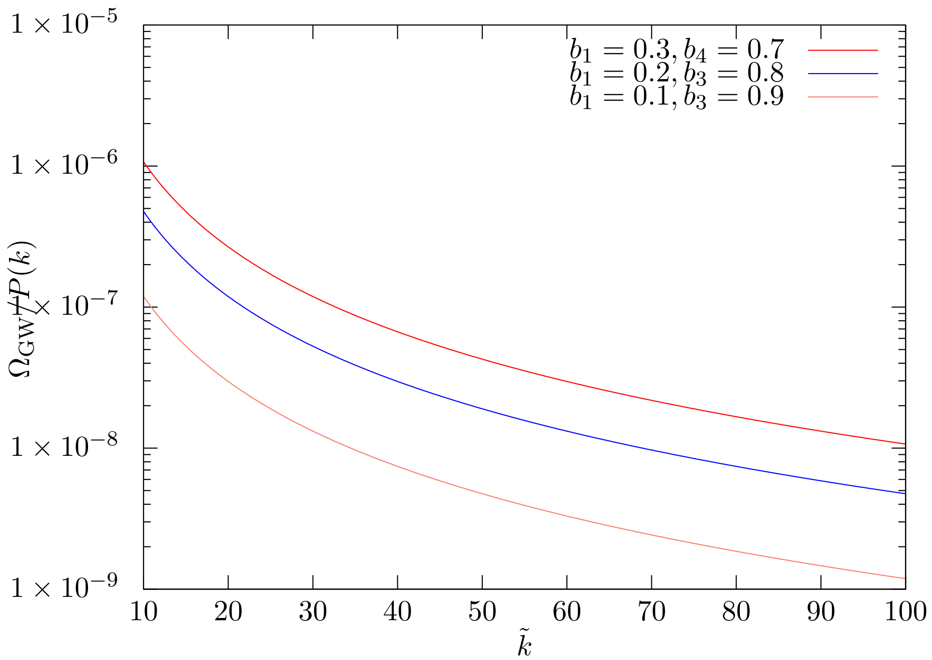


Figure 7: Gravitational spectrum at $a = 1$ divided by $P(k)$ with smaller values of matter energy parameter b_1. The spectrum with $b_1 = 0.3, b_3 = 0.7$ is not displayed because it is almost indistinguishable from the case of standard model $b_1 = 0.3, b_4 = 0.7$.

models reported almost identical value with the standard ΛCDM model, namely $b_1 \sim 0.3$, our model doesn't treat dynamical quintessece scalar field but have an effective energy density proportional to $1/a$ so we have to be careful to compare. In doing so, we consider the equation of state parameter $w = p/\rho$. In our background spacetime, w takes 0 in early matter dominated era and gradually decreses to $w = -2/3$. On the other hand, in typical models of quintessence universe, including ones discussed in Ref.[32], the parameter w goes to -1 at late time and the expansion becomes almost exponential type one. In their analysis, the parameter w at present ($a = 1$) ranges from $w \sim -0.7$ to $w \sim -0.6$ depending on the detail of model parameters. Since the minimum value of w in our model is $w = -2/3 \sim -0.67$ at $a \to \infty$, in order to match the value to the above models, we have to take smaller value of matter density $b_1 = \Omega_m$. From the spectrum Eq.(7.27), we can see that $\Omega_{\rm GW}$ for large value of $\tilde{k}$ is proportional to $(b_1)^2$, thus the spectrum is suppressed by taking smaller b_1 (Fig.7). Although our model doesn't include the radiation component, this approximation would valid up to $\tilde{k} \sim O(10^3)$, modes above which enter the sub-horizon regime when the radiation energy, and the change of effective degrees of freedom of relativistic particles as well[33], cannot be ignored. Then, the range where our exact formula (7.27) is valid is roughly $1 \ll \tilde{k} \ll 10^3 \sim 10^{-15}$Hz, which is out of the range in near future direct detection experiments. Nevertheless, our result here implies that in existence of quintessence field the spectrum of the primordial gravitational waves may be suppressed in higher frequency region as well.

8 Primordial perturbations in a pre-inflationary era

In this section, we consider an application of the solution (5.15). The background spacetime in this case contains radiation, spatial curvature, and vacuum energy as energy density. Because non relativistic matter component is not included, we cannot consider this as a model of our universe after the hot Big Bang. Nevertheless, it can be considered as a model that describes a pre-inflationary era, which include certain radiation component at initial time and after some expansion the universe starts the de Sitter expansion. As we have demonstrated in Sec.3, in the standard inflationary scenarios, the seed fluctuations that is considered as the origin of the structures in our universe at present is generated by quantum fluctuations of some scalar field, called inflaton field. In typical models of inflation, the predicted power spectrums of the fluctuation take the almost scale invariant form. However, this result relies on the assumption that the quantum states of the inflaton fluctuation is in the so-called the Bunch-Davies vacuum state[3], which is defined as Minkowski-like vacuum state at the infinite past in de Sitter spacetime. Although this assumption seems plausible and natural, there has been a lot of arguments and speculations about the possibility of non-Bunch-Davies vacuum states or even excited states[17, 18, 19].

Furthermore, if some pre-inflationary era where the expansion is not de Sitter like precedes the standard de Sitter phase, then the natural choice of vacuum state would be altered, and the power spectrum may be no more scale invariant. For example, several works (see [20] and references therein) considered a scenario where the slow-roll inflation is preceded by a fast-roll phase. Although in these models there exists only inflaton field in the pre-inflationary era, it is possible that some radiation component dominate the spacetime. Also, as one of the motivations to consider the inflationary epoch is to solve the flatness problem, it is natural to take into account non zero spatial curvature in such a pre-inflationary era. Note that if our universe started with the quantum to classical transition, the potential energy caused by the spatial curvature is expected to be the same order of magnitude as the kinetic energy. A pre-inflationary era dominated by radiation component and its effect on primordial spectrum has been considered in several works[21, 22, 23]. Also, the effect of the spatial curvature on density perturbations in inflation has been analyzed in several papers [24, 25, 26, 27]. They showed that the power spectrum for low l region, namely large length scale, deviates from flat spectrum. The analysis comparing to Planck results has been done in Refs.[28, 29]. However, as far as we know, there is no simultaneous accounts of these two effects.

In these point of views, our background evolution can be regarded as a model of a pre-inflationary era that contains both radiation and spatial curvature before the de Sitter expansion starts. Then, the solution (5.15) can be regarded as the squared gravitational wave amplitude on generated on this background. At the same time, we can consider the solution as a fluctuation of the inflaton field because the scalar field equation on the FRW background spacetime is the same as the evolution equation for the tensor mode. Explicitly, the action for a minimally coupled massless free scalar field is given by

$$S_\varphi = \int d^4x\sqrt{-g}\left[-\frac{1}{2}\nabla_\mu\varphi\nabla^\mu\varphi\right], \tag{8.1}$$

from which the equation of motion can be derived as

$$0 = \partial_t^2\varphi + \frac{3\dot{a}}{a}\partial_t\varphi - \frac{1}{a^2}D^2\varphi. \tag{8.2}$$

Comparing this equation with the evolution equation for tensor modes (3.35), by a suitable redefinition of the eigenvalue of the Laplacian, namely the wave number k, we can see that these two equations coincide.

8.1 Background evolution

Before constructing the exact solution for this equation from the solution for the associated 3rd order equation (5.15), we summarize fundamental properties of the background spacetime. By

[3] In the literature, it is also called Euclidean or thermal vacuum.

defining the following constants,

$$H_V^2 = b_4 H_0^2, \quad A = \frac{b_2}{b_4}, \quad B = \frac{b_0}{b_4}, \tag{8.3}$$

the Friedmann equation reduces to

$$\left(\frac{da}{d\eta}\right)^2 = H_V^2 \left(a^4 + Aa^2 + B\right). \tag{8.4}$$

We can transform this equation into the Weierstrass's standard form by $x = a^2 + A/3$ and $\tilde{\eta} = H_V \eta$,

$$\left(\frac{dx}{d\tilde{\eta}}\right)^2 = 4x^3 - g_2 x - g_3 = 4(x - e_1)(x - e_2)(x - e_3), \tag{8.5}$$

where

$$g_2 = \frac{4}{3}\left(A^2 - 3B\right), \quad g_3 = \frac{4A}{3}\left(B - \frac{2A^2}{9}\right). \tag{8.6}$$

The roots e_j are easily obtained from the biquadratic equation in terms of a as

$$e_1 = \frac{A}{3}, \; e_2 = -\frac{A}{6} + \sqrt{\frac{A^2}{4} - B}, \; e_3 = -\frac{A}{6} - \sqrt{\frac{A^2}{4} - B}. \tag{8.7}$$

There are two cases: (i) $A > -2\sqrt{B}$ and (ii) $A \leq -2\sqrt{B}$. In the former case, the roots e_2 and e_3 correspond to complex value of a so that we can take the origin of the conformal time as $a = 0$ at $\tilde{\eta} = 0$. Taking into account the fact that the $\wp(z)$ is equal to the roots e_1 at the half period $z = \omega_1$, the scale factor can be written as

$$a(\eta) = \left[\wp(\omega_1 - \tilde{\eta}) - e_1\right]^{1/2}. \tag{8.8}$$

The half period ω_1 is positive and given by

$$\omega_1 = \frac{1}{2}\int_{e_1}^{\infty} \frac{dx}{\sqrt{(x - e_1)(x - e_2)(x - e_3)}}. \tag{8.9}$$

Physical interval is from $\tilde{\eta} = 0$ to $\tilde{\eta} = \omega_1$, around which the scale factor behaves as

$$a(\eta) \sim \sqrt{B}\tilde{\eta}, \; (\tilde{\eta} \sim 0) \tag{8.10}$$

$$a(\eta) \sim \frac{1}{\omega_1 - \tilde{\eta}}. \; (\tilde{\eta} \sim \omega_1) \tag{8.11}$$

These behaviors are consistent with the initial radiation-dominated era and the final de Sitter expansion era.

In the case (ii), all the three roots e_j $(j = 1, 2, 3)$ correspond to real and positive value of a. The interval from $a = \sqrt{e_3 - A/3}$ to $a = \sqrt{e_2 - A/3}$ is unacceptable because $da/d\tilde{\eta}$ becomes imaginary there from the Friedmann equation. Another branch from $a = 0$ to $a = \sqrt{e_3 - A/3}$ is physically possible but this solution corresponds to the Big Crunch. In this case the spacetime cannot sufficiently expand to start the inflation, therefore we discard this possibility. Eventually, we have to consider the spacetime starting with the finite scale factor $a = a_0 := \sqrt{e_2 - A/3}$. By setting the integration constant so that $a = a_0$ at $\tilde{\eta} = 0$, the time evolution of the scale factor is given by

$$a(\eta) = \left[\wp(\omega_2 - \tilde{\eta}) - e_1\right]^{1/2}. \tag{8.12}$$

In contrast to the case (i), another half period ω_2 defined by

$$\omega_2 = \frac{1}{2}\int_{e_2}^{\infty} \frac{dx}{\sqrt{(x - e_1)(x - e_2)(x - e_3)}} \tag{8.13}$$

is real and positive. Around the endpoints $\tilde{\eta} = 0$ and $\tilde{\eta} = \omega_2$ the scale factor behaves as

$$a(\eta) \sim a_0 \left(1 + \frac{e_2 - e_3}{2}\tilde{\eta}^2\right), \ (\tilde{\eta} \sim 0) \tag{8.14}$$

$$a(\eta) \sim \frac{1}{\omega_2 - \tilde{\eta}}. \ (\tilde{\eta} \sim \omega_2) \tag{8.15}$$

Expansion rate at the initial stage is different from that in the usual radiation dominanted era because of the large value of the curvature energy (compare with the case (i) above), while the late time de Sitter phase is present.

Finally, we briefly discuss the comoving Hubble scale $\tilde{\mathcal{H}} = (da/d\tilde{\eta})/a$, which can be read off from the Friedmann equation as a function of the scale factor,

$$\tilde{\mathcal{H}} = \sqrt{a^2 + A + \frac{B}{a^2}}. \tag{8.16}$$

In the large a limit, namely well after the radiation and the curvature components becomes negligible, we see $\tilde{\mathcal{H}} \sim a$. In other words the comoving Hubble scale increases with time, which is a typical behavior in accelerated expansion era. Conversely, if we go back to the past direction in the pure de Sitter phase, the comoving Hubble can be arbitrarily small, or the Hubble radius can be infinitely large. Thus, in the de Sitter inflation any long wave length modes can be sub-horizon regime in sufficiently early time, which is the reason one typically assumes the Bunch-Davies vacuum in this situation. However, if there is a decelerate expansion era before the inflation as in our model, the comoving Hubble scale generally has the minimum at some point around the transition epoch from the decelerate expansion to the accelerated expansion. In our case, this minimum is given by

$$\tilde{\mathcal{H}}_{\text{min}} = \sqrt{2\sqrt{B} + A} \quad \text{at} \quad a = B^{1/4}. \tag{8.17}$$

Therefore, we expect that in this kind of pre-inflationary models, various physical quantities like the power spectrum of quantum fluctuations would be modified for modes with this or more larger length scale. In the following, we will see this is indeed the case by exactly calculating the energy density and the fluctuation of a test scalar field.

8.2 Exact solution for the evolution equation

We now move on to the scalar field equation (8.2) on this background. Using the conformal time $\tilde{\eta}$ as a independent variable and taking the spatial coordinate dependence as the eigenfunction of the Laplacian, the equation reduces to

$$\frac{d^2\phi_k}{d\tilde{\eta}^2} + \frac{2}{a}\frac{da}{d\tilde{\eta}}\frac{d\phi_k}{d\tilde{\eta}} + \tilde{k}^2\phi_k = 0, \tag{8.18}$$

where the comoving momentum is normalized by the Hubble scale of the inflation $\tilde{k} = k/H_V$.

Before solving this equation by using the exact solution (5.15) for the associated 3rd order equation, we note that this equation is actually equivalent to the Lamé equation (3.58). It can be seen by the rescaling of the solution as $\phi_k = v_k/a$ so that the first order derivative term is removed,

$$\begin{aligned} \frac{d^2 v_k}{d\tilde{\eta}^2} &= \left[2\wp(\omega_1 - \tilde{\eta}) + e_1 - \tilde{k}^2\right] v_k, \quad \text{case} \ \ \text{(i)} \\ \frac{d^2 v_k}{d\tilde{\eta}^2} &= \left[2\wp(\omega_2 - \tilde{\eta}) + e_1 - \tilde{k}^2\right] v_k. \quad \text{case} \ \ \text{(ii)} \end{aligned} \tag{8.19}$$

This is nothing but the Lamé equation with $n = 1$ and $B = e_1 - \tilde{k}^2$, whose basis solutions are known in the literature [14] as

$$v_k^+ = \frac{\sigma(\omega_j - \tilde{\eta} + c)}{\sigma(\omega_j - \tilde{\eta})\sigma(c)} e^{-(\omega_j - \tilde{\eta})\zeta(c)}, \quad v_k^- = \frac{\sigma(\omega_j - \tilde{\eta} - c)}{\sigma(\omega_j - \tilde{\eta})\sigma(-c)} e^{-(\omega_j - \tilde{\eta})\zeta(-c)}, \tag{8.20}$$

where $j = 1, 2$ indicates the case (i) and (ii) respectively, and the constant c is defined in both cases through the equation

$$\wp(c) = e_1 - \tilde{k}^2. \tag{8.21}$$

Note that although this equation determines c up to the shift of the periods $c \sim c + 2n\omega_1 + 2m\omega_2$ ($^\forall n, m \in \mathbb{Z}$), these solutions are invariant under the shift, i.e. periodic in c (but not in $\tilde{\eta}$). In addition, because the elliptic function $\wp(z)$ is even function, c has sign ambiguity. Recalling that the sigma $\sigma(z)$ and zeta $\zeta(z)$ functions are odd functions, we observe that the sign change of c is equivalent to the interchange of v_k^+ and v_k^-. We will fix this ambiguity by requiring $v_k^\pm$ represents the positive/negative frequency solutions respectively.

We derive this solution by applying the general formula (4.11) from the exact squared amplitude (5.15). First of all, we need the Wronskian constant C evaluated from Eq.(4.9). We divide Eq.(5.15) by b_4 to rescale y,

$$y = 1 + \frac{\tilde{k}^2}{a^2}, \tag{8.22}$$

for which the formula (4.9) gives

$$\tilde{C}^2 = -C^2 = 4\tilde{k}^2(b_4\tilde{k}^4 - b_2\tilde{k}^2 + b_0). \tag{8.23}$$

We realize that by definition of $\wp(c)$ and the differential equation for $\wp(z)$ leads to the following relationship:

$$\left[\wp'(c)\right]^2 = -\frac{\tilde{C}^2}{b_4}. \tag{8.24}$$

For definiteness we take $\wp'(c) = +i\tilde{C}/\sqrt{b_4}$. With this convention, the general solution constructed by the formula (4.11) can be written as

$$\chi_\pm = \sqrt{y} \exp\left(\pm\frac{1}{2}\int \frac{\wp'(c)d\tilde{\eta}}{\wp(\omega_j - \tilde{\eta}) - \wp(c)}\right). \tag{8.25}$$

By multiplying a and exponentiating the factor $a\sqrt{y}$ to write it as an integration of logarithmic derivative of $\wp(\omega_j - \tilde{\eta}) - \wp(c)$, we can manipulate as follows:

$$v_k^\pm = \sqrt{\wp(\omega_j - \tilde{\eta}) - \wp(c)} \exp\left(\pm\frac{1}{2}\int \frac{\wp'(c)d\tilde{\eta}}{\wp(\omega_j - \tilde{\eta}) - \wp(c)}\right) \tag{8.26}$$

$$= \exp\left(-\frac{1}{2}\int \frac{\wp'(\omega_j - \tilde{\eta}) - \wp'(\pm c)}{\wp(\omega_j - \tilde{\eta}) - \wp(\pm c)} d\tilde{\eta}\right). \tag{8.27}$$

Finally we can apply the addition theorem (A.29) to obtain the expression (8.20).

Let us examine fundamental properties of this solution and determine the normalization. One important task would be to clarify the relationship between v_k^+ and v_k^-. Although natural expectation is they are complex conjugate each other, it is non trivial because we introduce $\sigma(c)$, which is in general complex valued, as an integration constant in deriving the above solution. We have to consider complex conjugate properties of elliptic and related functions $\wp(z), \zeta(z)$, and $\sigma(z)$ as well as the constant c. As for the elliptic function, we can easily conclude the simple behavior $(\wp(z))^* = \wp(z^*)$, because the coefficients of the differential equation for $\wp(z)$ are all real. (Strictly speaking, only the reality of g_2 and g_3 is not enough but the requirement that the position of the pole is fixed at $z = 0$ is also needed.) Similarly we can verify $(\zeta(z))^* = \zeta(z^*)$ and

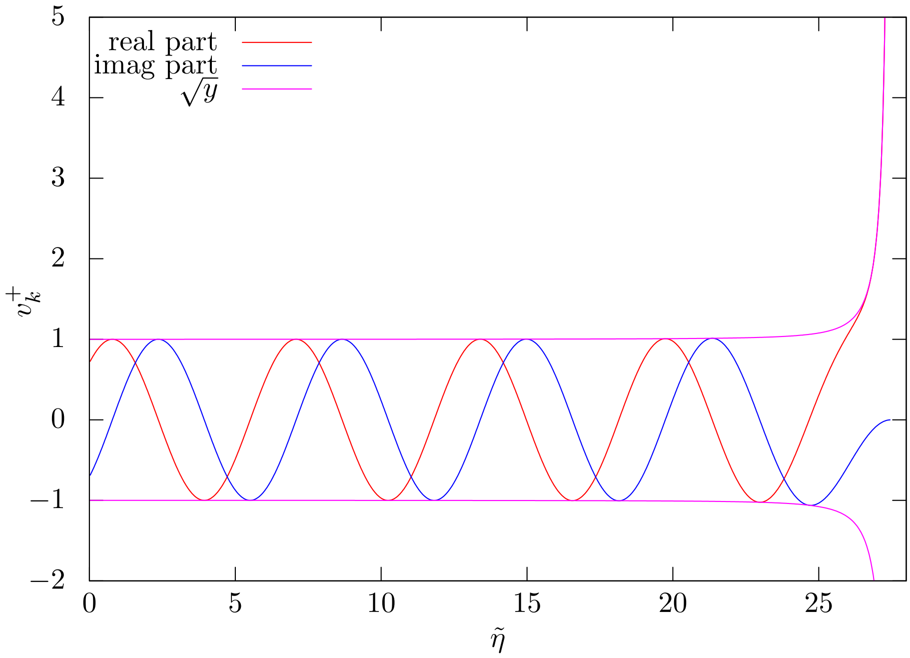


Figure 8: Numerical plot of $v_k^+(\tilde{\eta})$ with parameters $A = 5 \times 10^{-3}, B = 1.25 \times 10^{-5}$, and $\tilde{k} = 1.0$.

$(\sigma(z))^* = \sigma(z^*)$. Next we consider the complex conjugate of the constant c. From the definition of c (8.21), we can formally obtain the integral expression,

$$c = \frac{1}{2}\int_{e_1-\tilde{k}^2}^{\infty} \frac{dz}{\sqrt{(z-e_1)(z-e_2)(z-e_3)}} = \omega_1 + \frac{1}{2}\int_{e_1-\tilde{k}^2}^{e_1} \frac{dz}{\sqrt{(z-e_1)(z-e_2)(z-e_3)}}. \tag{8.28}$$

For real value of $\tilde{k}$, the integrand in the last expression is always pure imaginary hence we can deduce

$$(c-\omega_1)^* = -(c-\omega_1), \tag{8.29}$$

in other words, $c-\omega_1$ is pure imaginary. When $B > A^2/4$ holds (included in case (i)), because the half period ω_1 is real, this gives

$$c^* = -c + 2\omega_1 \sim -c, \tag{8.30}$$

where we denote the equivalence up to the periods by '$\sim$'. Then, we can see from this result and the fact that solutions (8.20) is periodic in c, that v_k^+ and v_k^- are indeed complex conjugate in this case. On the other hand, in the remaining part in the case (i), namely $A > 2\sqrt{B}$, the situation changes drastically because the integrand in the integral expression of c has other branch point on the real axis at $z = e_2$ and $z = e_3$. As a function of $\tilde{k}$, complex conjugate of c is given by

$$(c-\omega_1)^* = -(c-\omega_1), \quad (\tilde{k}^2 \leq A/2 - \sqrt{A^2/4 - B}) \tag{8.31}$$

$$(c-\omega_2)^* = c-\omega_2, \quad (A/2 - \sqrt{A^2/4 - B} < \tilde{k}^2 \leq A/2 + \sqrt{A^2/4 - B}) \tag{8.32}$$

$$(c-\omega_3)^* = -(c-\omega_3), \quad (A/2 + \sqrt{A^2/4 - B} < \tilde{k}^2). \tag{8.33}$$

Observing that the imaginary part of ω_2 and the real part of ω_3 are both half periods, we can summarize as follows:

$$c^* \sim \begin{cases} c & (\tilde{k}^4 - A\tilde{k}^2 + B < 0) \\ -c & (\tilde{k}^4 - A\tilde{k}^2 + B \geq 0) \end{cases} \tag{8.34}$$

When c is real, the basis solutions $v_k^{\pm}$ are also real $(v_k^{\pm})^* = v_k^{\pm}$, which means that in this region of $\tilde{k}$ these solutions cannot be interpreted as usual plane wave like solutions, but exponential type solutions. For case (ii), similar consideration shows $v_k^{\pm}$ are complex conjugate for all real value of $\tilde{k}$. We also note that for the anomalous case where $c^* \sim c$, the Wronskian constant $\tilde{C}$ becomes pure imaginary. However, the wave number k^2 must be larger than the curvature scale $|K|^{1/2}$ to make the spatial part to be oscillatory function. If we restrict to that case, namely $k^2 \geq |K|$, then $\tilde{k}^2 \geq A \geq A/2 - \sqrt{A^2/4 - B}$ holds for $A > 2\sqrt{B}$, in which case $c^* \sim -c$ hence $(v_k^{\pm})^* = v_k^{\mp}$.

8.3 Quantization of the scalar field

In this subsection, we promote the scalar field to a quantum field operator. For that purpose we have to normalize the solution obtined in the previous subsection. Normalization of the basis solutions $v_k^{\pm}$ can be defined through the Wronskian

$$W(v_k^+, v_k^-) = v_k^+ \frac{dv_k^-}{d\eta} - \frac{dv_k^+}{d\eta} v_k^- . \tag{8.35}$$

Since the differential equation for v_k doesn't have the first order derivative term, $W(v_k^+, v_k^-)$ is independent of time η. We can make use of the addition theorems (A.28) and (A.29) to calculate the Wronskian,

$$W(v_k^+, v_k^-) = -H_V \wp'(c) = -\frac{i\tilde{C} H_V}{\sqrt{b_4}}. \tag{8.36}$$

Because $\tilde{C}$ is positive real for $A \leq 2\sqrt{B}$ as we have mentioned, we can normalize the solution $v_k^{\pm}$ so that $W(V_k^+, V_k^-) = -i$ by defining

$$V_k^- := \sqrt{\frac{\sqrt{b_4}}{H_V \tilde{C}}} v_k^- , \quad V_k^+ = (V_k^-)^*, \tag{8.37}$$

the former of which is shown to be the positive frequency solution at the initial time below. The expansion of v_k^- at $\tilde{\eta} = 0$ turns out to be

$$\begin{aligned} &\frac{v_k^-(\tilde{\eta})}{v_k^-(0)} \\ &= \exp\left[-\left(\zeta(\omega_j - c) - \zeta(-c) - \zeta(\omega_j)\right)\tilde{\eta} + \sum_{n=2}^{\infty} \frac{(-\tilde{\eta})^n}{n!} \left(\wp^{(n-2)}(\omega_j - c) - \wp^{(n-2)}(\omega_j) \right) \right]. \end{aligned} \tag{8.38}$$

The coefficient of the leading order $O(\tilde{\eta})$ can be simplified by using the addition theorem (A.29) as

$$\zeta(\omega_j - c) - \zeta(-c) - \zeta(\omega_j) = \frac{i\tilde{C}}{2\sqrt{b_4}} \frac{1}{e_j - e_1 + \tilde{k}^2}. \tag{8.39}$$

In deriving this expression we have used $\wp(\omega_j) = e_j, \wp'(\omega_j) = 0$ and the definition of c (8.21). Thus, the asymptotic behavior of v_k^- is given by

$$\frac{v_k^-(\tilde{\eta})}{v_k^-(0)} \to \begin{cases} \exp\left(-i\tilde{k} \left(1 - A/\tilde{k}^2 + B/\tilde{k}^4\right)^{1/2} \tilde{\eta} \right) & (\text{case} \quad (\text{i})) \\ \exp\left(-i\tilde{k} \left(1 + a_0^2/\tilde{k}^2\right)^{-1} \left(1 - A/\tilde{k}^2 + B/\tilde{k}^4\right)^{1/2} \tilde{\eta} \right) & (\text{case} \quad (\text{ii})) \end{cases} \tag{8.40}$$

We can see that v_k^- is the positive frequency plane wave solution $e^{-ik\eta}$ at the initial time for sufficiently large value of wave number $\tilde{k}$. However, for smaller value of $\tilde{k}$, although the plane

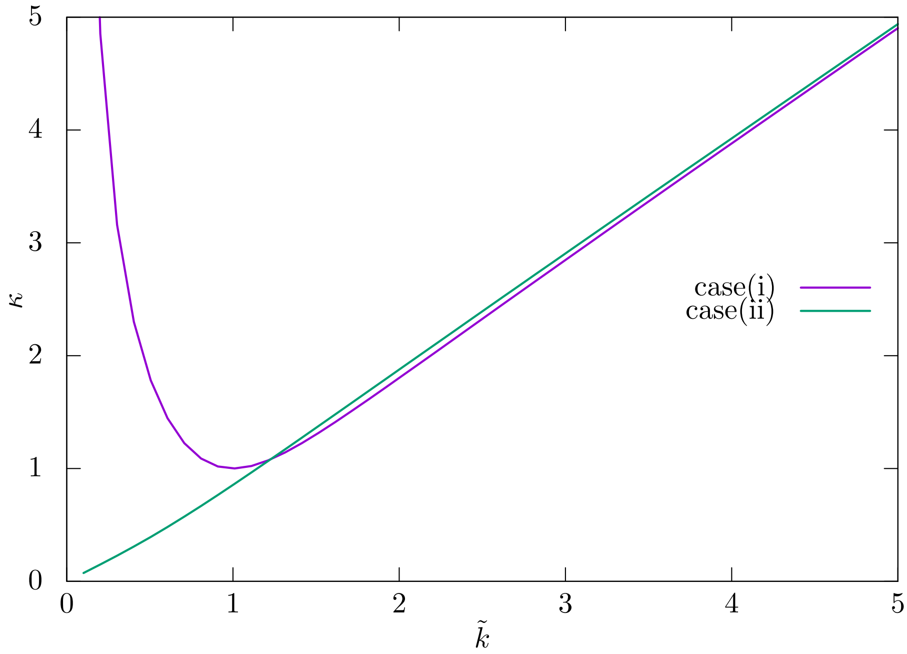


Figure 9: Numerical plot of the effective number $\tilde{\kappa}$. The density parameters are taken $A = 5.0 \times 10^{-3}, B = 1.25 \times 10^{-5}$ for case (i) and $A = -0.01, B = 1.25 \times 10^{-5}$ for case (ii).

wave form is preserved the wave number is deformed by virtue both of the spatial curvature and the radiation energy. Therefore we introduce the effective wave number $\tilde{\kappa}(\tilde{k})$,

$$\tilde{\kappa} = \tilde{k}\left(1 - A/\tilde{k}^2 + B/\tilde{k}^4\right)^{1/2} \tag{8.41}$$

for case (i) and

$$\tilde{\kappa} = \tilde{k}\left(1 + a_0^2/\tilde{k}^2\right)^{-1}\left(1 - A/\tilde{k}^2 + B/\tilde{k}^4\right)^{1/2} \tag{8.42}$$

for case (ii) to write the asymptotic behavior as $v_k^-(\tilde{\eta})/v_k^-(0) \to e^{-i\tilde{\kappa}\tilde{\eta}}$. Although $\tilde{\kappa}$ coincides with $\tilde{k}$ for large $\tilde{k}$ in both cases, small $\tilde{k}$ behaviors are totally different as can be seen from Fig.(9). For case (i), $\tilde{\kappa}(\tilde{k})$ has the minimum value $\tilde{\kappa}_{\min} = \sqrt{-A + 2\sqrt{B}}$ at $\tilde{k} = B^{1/4}$ and diverges as $\tilde{\kappa} \to \sqrt{B}/\tilde{k}$ in the infrared limit $\tilde{k} \to 0$. Note that the existence of the minimum for this case is due to the radiation energy density, which means that the radiation component acts as an effective infrared cutoff. We discuss this point again later. On the other hand for case (ii), $\tilde{\kappa}$ is monotonically increasing but slightly modified at small $\tilde{k}$ limit as $\tilde{\kappa} \to (\sqrt{B}/a_0^2)\tilde{k}$.

It is also of interest to investigate the behavior of the solutions at late time $\tilde{\eta} \sim \omega_j$ $(j = 1, 2)$. Recalling the relationship $\zeta'(z) = -\wp(z)$ and $(\log \sigma(z))' = \zeta(z)$, we can find

$$v_k^{\pm}(\tilde{\eta}) = \frac{1}{\omega_j - \tilde{\eta}}\left(1 + \frac{g_2}{240}(\omega_j - \tilde{\eta})^4 + O((\omega_j - \tilde{\eta})^6)\right)\exp\left(-\sum_{n=2}^{\infty}\frac{(\omega_j - \tilde{\eta})^n}{n!}\wp^{(n-2)}(\pm c)\right). \tag{8.43}$$

The second factor comes from the expansion formula (A.26). The exponential factor can be explicitly obtained by using the definition of $\wp(c)$ and the differential equation for $\wp(z)$ repeatedly. Inserting the leading and the next leading terms, we obtain

$$v_k^{\pm}(\tilde{\eta}) = \frac{1}{\omega_j - \tilde{\eta}}\left[1 + \frac{1}{2}(\tilde{k}^2 - e_1)(\omega_j - \tilde{\eta})^2 \mp \frac{i\tilde{C}}{6\sqrt{b_4}}(\omega_j - \tilde{\eta})^3 + O((\omega_j - \tilde{\eta})^4)\right]. \tag{8.44}$$

We compare this expression to the normalized solution in the pure de Sitter expansion case (see Eq.(3.45)),

$$v^{\pm}_{\mathrm{dS},k} = \left(1 \pm \frac{i}{k\eta}\right) \frac{e^{\pm ik\eta}}{\sqrt{2k}}, \quad (\eta < 0) \tag{8.45}$$

where the subscript "dS" stands for "de Sitter". Up to the order given in the above expansion, the asymptotic behavior turns out to be

$$\begin{aligned} V^{-}_{k} \to\, & i \cosh \log \left(\frac{\sqrt{\tilde{k}^2 \tilde{\kappa}}}{(\tilde{k^2} - e_1)^{3/4}} \right) v^{-}_{\mathrm{dS},\sqrt{k^2 - H_V^2 e_1}} \\ & + i \sinh \log \left(\frac{\sqrt{\tilde{k}^2 \tilde{\kappa}}}{(\tilde{k^2} - e_1)^{3/4}} \right) v^{+}_{\mathrm{dS},\sqrt{k^2 - H_V^2 e_1}} \qquad (8.46) \\ =: & \sum_{k'} \left(\alpha_{k,k'} v^{-}_{\mathrm{dS},k'} + \beta_{k,k'} v^{+}_{\mathrm{dS},k'} \right), \qquad (8.47) \end{aligned}$$

where κ is the effective wave number for case (i) defined in Eq.(8.41) though this asymptotic relation holds both for case (i) and (ii). In the last equality, we have defined the Bogoliubov coefficients by

$$\begin{aligned} \alpha_{k,k'} &= i \cosh \log \left(\frac{\sqrt{\tilde{k}^2 \tilde{\kappa}}}{(\tilde{k^2} - e_1)^{3/4}} \right) \delta_{k',\sqrt{k^2 - H_V^2 e_1}}, \\ \beta_{k,k'} &= i \sinh \log \left(\frac{\sqrt{\tilde{k}^2 \tilde{\kappa}}}{(\tilde{k^2} - e_1)^{3/4}} \right) \delta_{k',\sqrt{k^2 - H_V^2 e_1}}, \end{aligned} \tag{8.48}$$

and the summation over k' represents integration with suitable mearsures for negative and zero spatial curvature cases. Although this relation is asymptotic one because $v^{\pm}_{\mathrm{dS},k}$ are not the solutions to the full evolution equation, we can define the exact solutions $V^{\pm}_{\mathrm{dS},k}$ that behaves as these de Sitter solutions by using the inverse Bogoliubov transformation,

$$V^{-}_{\mathrm{dS},k} := \sum_{k'} \left(\alpha^{*}_{k',k} V^{-}_{k'} - \beta_{k',k} V^{+}_{k'} \right), \quad V^{+}_{\mathrm{dS},k} := (V^{-}_{\mathrm{dS},k})^{*}. \tag{8.49}$$

One can verify this is indeed the inverse transformation of (8.46) by observing that the Bogoliubov coefficients (8.48) satisfy the following:

$$\begin{aligned} \sum_{k''} \left(\alpha_{k,k''} \alpha^{*}_{k',k''} - \beta_{k,k''} \beta^{*}_{k',k''} \right) &= \delta_{k,k'}, \\ \sum_{k''} \left(\alpha_{k,k''} \beta_{k',k''} - \beta_{k,k''} \alpha_{k',k''} \right) &= 0. \end{aligned}$$

As in the case of initial state, the wave number is deformed by existence of radiation and curvature. In addition, we observe that $V^{\pm}_{k}$ is in general a mixture of the positive and negative frequency wave functions $V^{\pm}_{\mathrm{dS},k}$. This kind of mixing of positive and negative frequency modes can happen when the background spacetime undergoes non stationary eras, as in the cases of expanding universe which we are now considering[30]. In particular, because the coefficient $\beta_{k,k'}$ is non vanishing, there happens particle creation if the vacuum state defined through the wave functions $V^{\pm}_{k}$ is chosen, compared to the vacuum state defined by the pure de Sitter wave functions $V^{\pm}_{\mathrm{dS},k}$, which is the typical assumption in considering the quantum fluctuation in inflationary era.

8.4 Particle creation and simple estimate of the energy density of created particles

We are now going to evaluate the spectrum and the total number of created particles. We first promote the field ϕ to quantum field in the Heisenberg picture. Then, from the Heisenberg field equation for the quantum field operator ϕ, we can expand it by using either the basis solutions of $V_k^{\pm}$ or $V_{\mathrm{dS},k}^{\pm}$ as

$$V(\eta,\boldsymbol{x}) := a(\eta)\phi(\eta,\boldsymbol{x}) = \sum_{\boldsymbol{k}} \left(A_{\boldsymbol{k}} V_k^{-}(\eta)\psi_{\boldsymbol{k}}(\boldsymbol{x}) + A_{\boldsymbol{k}}^{\dagger} V_k^{+}(\eta)\psi_{\boldsymbol{k}}^{*}(\boldsymbol{x}) \right) \tag{8.50}$$

$$= \sum_{\boldsymbol{k}} \left(B_{\boldsymbol{k}} V_{\mathrm{dS},k}^{-}(\eta)\psi_{\boldsymbol{k}}(\boldsymbol{x}) + B_{\boldsymbol{k}}^{\dagger} V_{\mathrm{dS},k}^{+}(\eta)\psi_{\boldsymbol{k}}^{*}(\boldsymbol{x}) \right). \tag{8.51}$$

Precise forms of the measure $\sum_{\boldsymbol{k}}$ depends on the spatial curvature and the normalization of the basis solution $\psi_{\boldsymbol{k}}(\boldsymbol{x})$. We set the normalization through the completeness relation by using the invariant Dirac's delta function as

$$\sum_{\boldsymbol{k}} \psi_{\boldsymbol{k}}(\boldsymbol{x})\psi_{\boldsymbol{k}}^{*}(\boldsymbol{y}) = \frac{1}{\sqrt{\gamma}}\delta^3(\boldsymbol{x}-\boldsymbol{y}). \tag{8.52}$$

Then, the expansion coefficients $A_{\boldsymbol{k}}$, $B_{\boldsymbol{k}}$, and their conjugates are annihilation and creation operators obeying the canonical commutation relations,

$$\left[A_{\boldsymbol{k}}, A_{\boldsymbol{k}'}^{\dagger}\right] = \left[B_{\boldsymbol{k}}, B_{\boldsymbol{k}'}^{\dagger}\right] = \delta_{\boldsymbol{k},\boldsymbol{k}'}, \tag{8.53}$$

$$[A_{\boldsymbol{k}}, A_{\boldsymbol{k}'}] = [B_{\boldsymbol{k}}, B_{\boldsymbol{k}'}] = 0. \tag{8.54}$$

The spatial function $\psi_{\boldsymbol{k}}(\boldsymbol{x})$ is the eigenfunction of the spatial Laplacian D^2 (see Eq.(8.2)) with the eigenvalue $-k^2$, whose functional form depends on the spatial curvature K. From the Bogoliubov transformation (8.49), the annihilation operator associated with the de Sitter mode can be expanded as

$$B_{\boldsymbol{k}} = \sum_{\boldsymbol{k}'} \left(A_{\boldsymbol{k}'}\alpha_{k',k} + A_{\boldsymbol{k}'}^{\dagger}\beta_{k',k}^{*} \right), \tag{8.55}$$

and vice versa. We define the initial vacuum state $|0_A\rangle$ by using the annihilation operator $A_{\boldsymbol{k}}$ as

$$A_{\boldsymbol{k}}|0_A\rangle = 0, \quad {}^{\forall}\boldsymbol{k}. \tag{8.56}$$

Then, the number of created particle of mode $\boldsymbol{k}$ associated with the operator $B_{\boldsymbol{k}}$ is given by

$$\langle 0_A|B_{\boldsymbol{k}}^{\dagger}B_{\boldsymbol{k}}|0_A\rangle = \sum_{\boldsymbol{k}'} |\beta_{k,k'}|^2. \tag{8.57}$$

To precisely determine the spectrum of the created particle, we have to define the integration measure, which depends on the spatial curvature. Furthermore, it isn't necessarily valid to consider within the concept of particles, especially for large wave length modes. In fact, the Bogoliubov coefficient β_k dacays at large k because both the initial and final positive frequency modes coincide at the shord wave length limit. In general, quantum particle creation is relevant for modes whose wave length or frequency is comparable to the curvature scale of the background spacetime.. In the cosmological setting there are two kinds of curvature scale in the background, namely the Hubble scale and the spatial curvature scale. Although the concept of particle is not clear for modes around the Hubble scale, in the following we estimate the energy density of created particles by using the flat space approximation where each free particle with frequency k has its energy k in the natural unit $\hbar = c = 1$. The result will be confirmed by more rigorous (though some ambiguity may exist in regularization procedure) treatment of energy momentum tensor.

We restrict to the most familiar and simplest case $K = 0$. In this case, the eigenfunction of the spatial Laplacian D^2 is given by the plane wave $\psi_k(\boldsymbol{x}) \sim e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ and the eigenvalue $-k^2 = -|\boldsymbol{k}|^2$ takes arbitrary negative values. In this case, the expansion of the field (8.50) becomes

$$V(\eta, \boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3} \left(A_{\boldsymbol{k}} V_k^-(\eta) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + A_{\boldsymbol{k}}^\dagger V_k^+(\eta) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \right), \tag{8.58}$$

and the nonzero canonical commutation relation becomes

$$\left[A_{\boldsymbol{k}}, A_{\boldsymbol{k}'}^\dagger \right] = (2\pi)^3 \delta^3(\boldsymbol{k} - \boldsymbol{k}'), \tag{8.59}$$

Noth that in the flat case the Bogoliubov transformation (8.49) doesn't change the wave number k, and the relation between two sets of annihilation/creation operators are given by

$$B_{\boldsymbol{k}} = A_{\boldsymbol{k}} \alpha_k + A_{-\boldsymbol{k}}^\dagger \beta_k^*, \quad B_{\boldsymbol{k}}^\dagger = A_{\boldsymbol{k}}^\dagger \alpha_k^* + A_{-\boldsymbol{k}} \beta_k, \tag{8.60}$$

where

$$\alpha_k = i \cosh \log \sqrt{\frac{\tilde{\kappa}}{\bar{\tilde{k}}}}, \quad \beta_k = i \sinh \log \sqrt{\frac{\tilde{\kappa}}{\bar{\tilde{k}}}}. \tag{8.61}$$

Then, the expectation value of the "$B_{\boldsymbol{k}}$" particle number operator reduces to

$$\langle 0_A | B_{\boldsymbol{k}}^\dagger B_{\boldsymbol{k}} | 0_A \rangle = (2\pi)^3 \delta^3(0) |\beta_k|^2. \tag{8.62}$$

Although this expression is divergent because of the Dirac's delta function, we can interpret this factor as the comoving spatial volume by considering the $\boldsymbol{k} \to 0$ limit in the following expression:

$$(2\pi)^3 \delta^3(\boldsymbol{k}) = \int d^3x e^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{8.63}$$

Thus, we define the number density $n_{\boldsymbol{k}}$ of particles of mode $\boldsymbol{k}$ by

$$n_{\boldsymbol{k}} := \frac{\langle 0_A | B_{\boldsymbol{k}}^\dagger B_{\boldsymbol{k}} | 0_A \rangle}{V} = \frac{|\beta_k|^2}{a^3}, \tag{8.64}$$

where $V = a^3 \int d^3x$ is the proper spatial volume. Then, the total number density of created particle n_B is given by integrating $n_{\boldsymbol{k}}$ as

$$n_B = \int \frac{d^3k}{(2\pi)^3} n_{\boldsymbol{k}} = \frac{1}{2\pi^2 a^3} \int_0^\infty k^2 dk |\beta_k|^2. \tag{8.65}$$

In the second equality, we have integrated out the angular part. Inserting the coefficient (8.61), the above integral can be evaluated in terms of various elliptic integrals and gives

$$n_B = \frac{\Gamma^2(1/4)}{96\pi^{5/2}} \frac{(b_0 b_4)^{3/4} H_0^3}{a^3} \sim 7.8 \times 10^{-3} \times (b_0 b_4)^{3/4} \frac{H_0^3}{a^3}. \tag{8.66}$$

Similarly, taking the effect of gravitational redshift, we can calculate the energy density ρ_B by

$$\rho_B = \int \frac{d^3k}{(2\pi)^3} \frac{k}{a} n_{\boldsymbol{k}} = \frac{1}{2\pi^2 a^4} \int_0^\infty k^3 dk |\beta_k|^2. \tag{8.67}$$

In this case, the integral can be easily performed in terms of elementary functions and gives

$$\rho_B = \frac{B H_V^4}{32\pi^2 a^4} = \frac{b_0 b_4 H_0^4}{32\pi^2 a^4}. \tag{8.68}$$

We also consider the energy density spectrum

$$\frac{d\rho_B}{dk} := \frac{k^3 |\beta_k|^2}{2\pi^2 a^4} = \frac{H_V^3 \tilde{k}}{8\pi^2 a^4} \left(\frac{2\tilde{k}^4 + B}{\sqrt{\tilde{k}^4 + B}} - 2\tilde{k}^2 \right). \tag{8.69}$$

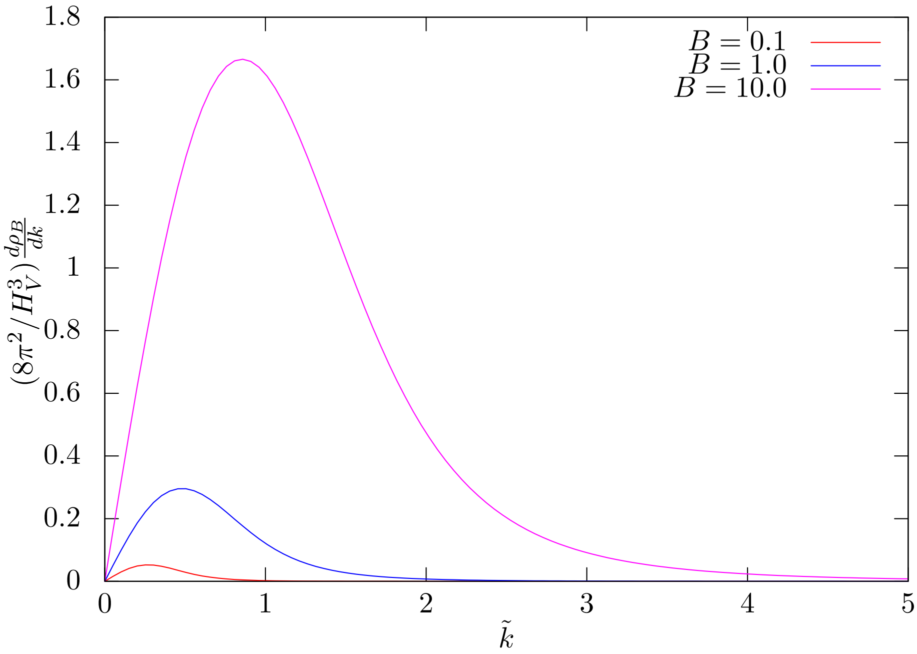


Figure 10: Energy density spectrum of created particles for spatially flat case ($K = 0$).

This distribution has a single peak,

$$\left.\frac{d\rho_B}{dk}\right|_{\max} = \frac{H_V^3}{8\pi^2 a^4}\left(\frac{4\sqrt{6}-9}{15}\right)^{3/4}\frac{3\sqrt{6}-2}{2}B^{3/4} \quad \text{at} \quad \tilde{k} = \left(\frac{4\sqrt{6}-9}{15}\right)^{1/4} B^{1/4}. \tag{8.70}$$

In the infrared and ultraviolet limit, the distribution obeys simple power law behavior,

$$\frac{d\rho_B}{dk} \to \begin{cases} \dfrac{\sqrt{B}H_V^3}{8\pi^2 a^4}\tilde{k}, & (\tilde{k} \to 0) \\[2ex] \dfrac{B^2 H_V^3}{32\pi^2 a^4}\dfrac{1}{\tilde{k}^5}. & (\tilde{k} \to \infty) \end{cases} \tag{8.71}$$

As can be immediately seen from the numerical plot Fig.(10), the peak of the spectrum lies at $\tilde{k} \lesssim 1$, namely below the Hubble scale.

Let us compare this distribution with the Planck distribution of temperature T,

$$\frac{d\rho_{\text{Planck}}}{dk} = \frac{1}{\pi^2}\frac{k^3}{e^{k/k_B T} - 1}, \tag{8.72}$$

where k_B is the Boltzmann constant. The integrated energy density ρ_{Planck} is given by the well known Stefan-Boltzmann law

$$\rho_{\text{Planck}} = \frac{\pi^2}{15}(k_B T)^4. \tag{8.73}$$

For a local observer, the expansion of the universe can be negligible so we can set $a = 1$. Then, one can regard the expression ρ_B as the Stefan-Boltzmann law with the effective temperature $k_B T_{\text{eff}}$ proportional to $B^{1/4}H_V = (b_0 b_4)^{1/4} H_0$.

Although the shape of the spectrum $d\rho_B/dk$ in the figure 10 seems similar to the Planck distribution at first sight, they are not the same at all. The difference can be seen both in the infrared region ($k \to 0$) and the ultraviolet region ($k \to \infty$), where the asymptotic behavior for $d\rho_B/dk$ is given in (8.71) while that for the Planck distribution is given by the famous Rayleigh-Jeans law and the Wien's law,

$$\frac{d\rho_{\text{Planck}}}{dk} \to \begin{cases} \dfrac{k_B T}{\pi^2} k^2 & (k \to 0) \\ \\ \dfrac{k^3}{\pi^2} e^{-k/k_B T} & (k \to \infty) \end{cases} . \tag{8.74}$$

We can see that in the both limits the obtained spectrum $d\rho_B/dk$ decays slower than the Planck distribution with any temperature.

8.5 Vacuum expectation value of the energy density

Although our treatment above is simple and the result seems acceptable, in order to take account of the effect of curvature we need more rigorous treatment. The most canonical quantity which doesn't rely on the concept of particle to calculate the energy density is the energy momentum tensor. The energy density is given by the tt-component of the energy momentum tensor

$$\rho_\phi = T_{tt} = \frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2a^2} D_i \phi D^i \phi. \tag{8.75}$$

We are interested in the expectation value of ρ_ϕ in the vacuum state $|0_A\rangle$. Using the mode expansion (8.50), the expectation value reduces to

$$\langle 0_A | \rho_\phi | 0_A \rangle = \frac{1}{2} \sum_{\boldsymbol{k}} |\psi_{\boldsymbol{k}}(\boldsymbol{x})|^2 \left[\partial_t \left(\frac{V_k^-}{a} \right) \partial_t \left(\frac{V_k^+}{a} \right) + \frac{k^2}{a^2} \frac{V_k^-}{a} \frac{V_k^+}{a} \right]. \tag{8.76}$$

This expectation value is divergent if we take arbitrarily large value of k (ultraviolet divergence) as in the case of flat spacetime. In the flat spacetime usually such UV divergences are renormalized by redefinition of various coupling constants or the strength of the field itself. If a quantum field in the flat spacetime is free, namely without any interaction, only the divergence comes from the vacuum energy but it is just subtracted by hand arguing what we can observe is only the difference in energy. This procedure is valid when we can ignore gravitational effect of the quantum field. However, when we take account of gravitational interaction with quantum fields, we have to consider the absolute value of the energy (or more generally, energy momentum tensor) of the fields because it can affect the dynamics of geometry through the Einstein equation. Thus, how to obtain a finite energy density of quantum fields on a curved sapcetime is an intriguing problem. See [30] as a classical reference. There are several methods of regularization and renormalization in this situation, among which the adiabatic regularization is a method to directly obtain a finite expectation value for the energy momentum tensor[34]. The term "adiabatic" comes from the adiabatic expansion of physical quantities such as the field itself or the energy momentum tensor. The adiabatic expansion is a inverse power series expansion in terms of the so-called adiabatic parameter denoted by T, which represents the slowness of the expansion in our case, or more generally the time scale of the background spacetime. In the adiabatic limit $T \to \infty$ the spacetime becomes stationary, where there is no ambiguity in the concept of particles since we can make use of the time-like Killing vector field. Even if the spacetime is not stationary at all, gravitational effects for sufficiently large frequency modes can be considered as small. In other words, we can assume large k expansion, in which case the adiabatic parameter T is identified with k itself. Large k expansion for the solutions to the equations like (8.19) can be implemented by the WKB expansion with the ansatz

$$V_k^{\pm} = \frac{1}{\sqrt{2W_k}} \exp\left(\pm i \int W_k d\eta \right), \tag{8.77}$$

where W_k is assumed to have a power series expansion in terms of $1/k^2$ and the leading behavior is $W_k \to k$. As far as the leading term has dominant contribution to $V_k^{\pm}$, this set of solutions can be considered as a reasonable basis to define the vacuum state. In the FRW spacetime, this condition is fulfilled for sufficiently large value of k given that the time variation of the scale factor a is smooth while such a mode may not be available at the singular points of a. Once the WKB solution is obtained up to some finite order of the adiabatic parameter (or $1/k$), we can expand the energy momentum tensor as well. The divergence in the expectation value of the energy density (8.76) comes from the first few orders (up to 4th order in 4-dimensional spacetime) of this adiabatic expansion, which can be verified by dimensional arguments. Then, by subtracting the contributions up to the 4th order the resulting expectation value becomes finite.

We here note that our exact solution $V_k^{\pm}$ is nothing but the full WKB solution, which can be seen from Eq.(8.26). Explicitly, the normalized solution $V_k^{\pm}$ can be written as the WKB form (8.77) with

$$W_k = \frac{k\left(\tilde{k}^4 - A\tilde{k}^2 + B\right)^{1/2}}{\wp(\omega_j - \tilde{\eta}) - \wp(c)}. \quad (j = 1, 2) \tag{8.78}$$

For a fixed finite value of $\tilde{\eta}$, this WKB factor W_k is exactly the adiabatic mode, namely in the large $\tilde{k}$ limit it behaves as

$$W_k = k\left(1 + O(\tilde{k}^{-2})\right).$$

This expansion is valid only when the $\tilde{k}$ is sufficiently large compared to $\wp(\omega_j - \tilde{\eta}) - e_1 = a^2$. As this expression tells, the lower bound of the adiabatic region gradually increases as the universe expands, which finally diverges. This behavior is well known in the standard de Sitter inflations, where any modes becomes after sufficient number of e-folds super-horizon modes. On the other hand, at the early time $\tilde{\eta} \sim 0$, W_k is almost constant, more precisely, it satisfies the following initial condition:

$$W_k(0) = \frac{H_V(\tilde{k}^4 - A\tilde{k}^2 + B)^{1/2}}{\tilde{k}}, \quad W_k'(0) = 0. \tag{8.79}$$

Note that this initial condition is consistent with the plane wave behavior at the initial time described earlier. Since the time evolution equation for W_k is a 2nd order differential equation, this W_k is the unique exact solution with this initial condition. However, higer order derivatives of W_k is not vanishing at the initial time reflecting the fact that the background spacetime is not stationary. For example, the initial condition for the second derivative is given by $W_k''(0) = -2BH_V(\tilde{k}^4 - A\tilde{k}^2 + B)^{1/2}/\tilde{k}^3$. In the WKB approximation, one usually assumes the leading behavior $W_k \sim \sqrt{k^2 - |K|}$ by neglecting the derivative term in the differential equation for v_k, but our exact solution shows a further contribution from the radiation energy, which can be seen in the effective wave number (8.41). This modification of the initial wavenumber comes from the non-vanishing second derivative. But this deviation from the usual WKB leading order is small as far as the wavenumber is sufficiently large, i.e. $\tilde{k} \gg B^{1/4}$. This condition says that for such modes the initial expansion rate of the universe is negligible compared with their frequency.

Let us examine the energy density (8.76) with this exact solution. The summation over all $\boldsymbol{k}$ in this formula can be divided into the radial part and the angular part. Since the time evolution $V_k^{\pm}$ depends only on the norm of $\boldsymbol{k}$, what is relevant to the summation over the angular part is

$|\psi_{\boldsymbol{k}}(\boldsymbol{x})|^2$, which has been performed in Ref.[34] and the result is summarized as

$$\int d\mu(k) := (2\pi^2)\sum_{\boldsymbol{k}}|\psi_{\boldsymbol{k}}(\boldsymbol{x})|^2 = \begin{cases} K^{3/2}\sum_{l=0}^{\infty}(l+1)^2 & (K>0) \\ \int_0^\infty k^2 dk & (K=0) \\ (-K)^{3/2}\int_0^\infty p^2 dp & (K<0) \end{cases}, \tag{8.80}$$

where the norm k is given by $k^2 = Kl(l+2)$ and $k^2 = |K|(p^2+1)$ for $K>0$ and $K<0$ respectively. Inserting the WKB form of $V_k^{\pm}$, the expectation value (8.76) now reduces to

$$\langle 0_A|\rho_\phi|0_A\rangle = \frac{1}{4\pi^2 a^4}\int d\mu(k)\left[\frac{(W_k')^2}{8W_k^3} + \frac{1}{2}W_k + \frac{1}{2W_k}\left(k^2 + \left(\frac{a'}{a}\right)^2\right) + \frac{a'}{2a}\frac{W_k'}{W_k^2}\right], \tag{8.81}$$

where prime indicates the derivative with respect to the conformal time η. Recalling the leading behavior $W_k \to k$ in the UV limit, this integral contains divergent terms except for the first term in the bracket, and the order of divergence is at most quartic, followed by quadratic and logarithmic divergence.

At the same time we have to care about small k region, where usually infrared divergence is expected for minimally coupled massless scalar fields[35]. Remarkably, the above expectation value does not suffer from such infrared divergence, except for the zero mode contribution ($l=0$) in the closed space case ($K>0$). The reasons for the safe behavior in IR limit are different for $K=0$ and $K<0$ cases. For the latter case, this is due to the cut off $k_{\text{min}} = |K|^{1/2}$ caused by the spatial curvature, while for the flat space case the radiation component B gives another cut off as we have mentioned earlier. One can see this fact from the exact WKB solution (8.78) for $A=0$, which behaves as $W_k = O(k)$ if $B\neq 0$ exists, otherwise $W_k = O(k^3)$ causing the infrared divergence.

The IR divergence in the case of closed universe comes from the zero mode ($l=0$) contribution. In the cases of flat and open space, zero mode contribution is suppressed by the integration measure $\int d\mu(k)$ but in the closed space case such suppression doesn't happen, so we have to care about the zero mode in this case. We note that in de Sitter space proper treatment of the zero mode is necessary to construct de Sitter invariant vacuum state for minimally coupled massless scalar field[36]. The peculiarity of the zero mode in our solution (8.20) can be seen from their Wronskian (8.36), which vanishes when $k\to 0$. This fact shows the basis solutions $v_k^{\pm}$ (or normalized ones $V_k^{\pm}$) are linearly dependent in this limit. In fact, the constant c defined through (8.21) becomes one of the half periods when $k=0$, namely $c(k=0)=\omega_1$, which then implies $v_0^+ = v_0^-$. (Recall that $v_k^{\pm}$ is periodic in c.) To remedy this singularity, we have to go back to the evolution equation (8.2), from which two independent solutions for zero mode can be easily obtained as

$$v_0^{(1)} = a, \quad v_0^{(2)} = a\int\frac{d\tilde{\eta}}{a^2}. \tag{8.82}$$

As $k\to 0$ limit $v_0^{\pm}$ is non singular at $a=0$ (or $\tilde{\eta}=0$), we can conclude that $v_0^{\pm}\propto v_0^{(1)}$. For quantization, we have to choose "positive frequency" mode as a specific linear combination of $v_0^{(1)}$ and $v_0^{(2)}$. Here, the term "positive frequency" just means the wave function with which the annihilation operator is associated. Although it contains considerable amount of possibilities, we here take $V_0^-\propto v_0^{(2)}$ because $v_0^{(2)}$ is singular at the initial time $a\to 0$ and the vacuume state should not have such singular mode. Another basis solution is taken as $V_0^+\propto v_0^{(1)}$. With this choice, we can see that zero mode doesn't contribute to the energy density expectation value

(8.76). Therefore in the closed space case we merely remove $l = 0$ contribution to evaluate the energy density.

Now we consider the problem of UV divergence. We employ the adiabatic regularization but naive expansion in terms of $1/k$ and subtraction up to the 4th adiabatic order causes infrared divergences. In order to preserve the infrared safety, instead, we expand in terms of $\mathcal{K} = (\tilde{k}^4 - A\tilde{k}^2 + B)^{1/4}$. Before restricting to the specific solution (8.78), we formally expand W_k as

$$\frac{W_k}{k} = 1 + \frac{W_k^{(2)}}{\mathcal{K}^2} + \frac{W_k^{(4)}}{\mathcal{K}^4} + \frac{W_k^{(6)}}{\mathcal{K}^6} + O(\mathcal{K}^{-8}), \tag{8.83}$$

Then, the energy density (8.76) can also be expanded as

$$\begin{aligned}\frac{4\pi^2 a^4}{H_V^4}\langle 0_A|\rho_\phi|0_A\rangle &= \int d\tilde{\mu}(k)\tilde{k}\left[1 + \frac{1}{2\mathcal{K}^2}\left(\frac{a'}{a}\right)^2\right.\\ &\left.\quad + \frac{1}{2\mathcal{K}^4}\left\{\left(W_k^{(2)}\right)^2 + \frac{a'}{a}W_k^{(2)\prime} - \left(\frac{a'}{a}\right)^2\left(W_k^{(2)} + \frac{A}{2}\right)\right\} + O(\mathcal{K}^{-6})\right] \quad (8.84)\\ &=: \frac{4\pi^2 a^4}{H_V^4}\left(\langle\rho\rangle_{\rm div} + \langle\rho\rangle_{\rm reg}\right). \quad (8.85)\end{aligned}$$

Here $H_V^3 d\tilde{\mu}(k) = d\mu(k)$ and we divided the energy density into divergent term $\langle\rho\rangle_{\rm div}$ which consists of the terms up to 4th order and regular term $\langle\rho\rangle_{\rm reg}$ which consists of the terms of higher order. The leading term k in the bracket represents the energy for the mode k in the flat spacetime as expected. Terms of order $\mathcal{K}^{-2}$ and $\mathcal{K}^{-4}$, which doesn't exist in the flat space free scalar field case, are also divergent. We subtract these terms from the expectation value to obtain regularized energy density,

$$\langle\rho\rangle_{\rm reg} = \langle\rho_\phi\rangle - \langle\rho\rangle_{\rm div}. \tag{8.86}$$

We apply this procedure to the exact solution (8.78). Unregularized density is given by

$$\frac{4\pi^2 a^4}{H_V^4}\frac{d\langle\rho_\phi\rangle}{d\tilde{\mu}(k)} = \frac{\tilde{k}}{2\mathcal{K}^2}a^2 + \frac{\tilde{k}^3}{\mathcal{K}^2} + \frac{B\tilde{k}}{2\mathcal{K}^2 a^2}, \tag{8.87}$$

where we defined the spectrum density $d\langle\rho_\phi\rangle/d\tilde{\mu}(k)$ by

$$\langle 0_A|\rho_\phi|0_A\rangle = \int d\tilde{\mu}(k)\frac{d\langle\rho_\phi\rangle}{d\tilde{\mu}(k)}. \tag{8.88}$$

The adiabatic expansion of (8.78) is given by [4]

$$\frac{W_k}{k} = 1 - \frac{1}{\mathcal{K}^2}\left(a^2 + \frac{A}{2}\right) + \frac{1}{\mathcal{K}^4}\left(a^4 + Aa^2 + \frac{B}{2} + \frac{A^2}{8}\right) + O(\mathcal{K}^{-6}), \tag{8.89}$$

where we have used the Friedmann equation to express the derivatives of a as a function of a. which then enables to calculate the subtraction term as

$$\begin{aligned}\frac{4\pi^2 a^4}{H_V}\frac{d\langle\rho_{\rm div}\rangle}{d\mu(k)} &= \left(\frac{3}{2\tilde{k}\mathcal{K}^2} + \frac{\tilde{k}}{2\mathcal{K}^4}\right)a^4 + \left\{\left(\frac{\tilde{k}}{2\mathcal{K}^4} + \frac{7}{4\tilde{k}\mathcal{K}^2}\right)A + \frac{1}{2\tilde{k}}\right\}a^2\\ &\quad + \tilde{k} + \frac{A}{2\tilde{k}} + \left(\frac{\tilde{k}}{8\mathcal{K}^4} + \frac{1}{4\tilde{k}\mathcal{K}^2}\right)A^2 + \frac{3B}{2\tilde{k}\mathcal{K}^2} + \left(\frac{1}{2\tilde{k}} + \frac{A}{4\tilde{k}\mathcal{K}^2}\right)\frac{B}{a^2}. \quad (8.90)\end{aligned}$$

[4] Although only up to the 2nd order is needed to compute the regularized energy density $\langle\rho\rangle_{\rm reg}$, the 4th order term is shown here because it is necessary to regularize the pressure later.

We finally obtain the regularized density spectrum

$$\frac{4\pi^2 a^4}{H_V}\frac{d\langle\rho_{\rm reg}\rangle}{d\mu(k)} = \left(\frac{1}{2\tilde{k}\mathcal{K}^2}-\frac{\tilde{k}}{2\mathcal{K}^4}\right)a^4+\left\{\frac{\tilde{k}}{2\mathcal{K}^2}-\frac{1}{2\tilde{k}}+A\left(\frac{1}{4\tilde{k}\mathcal{K}^2}-\frac{\tilde{k}}{2\mathcal{K}^4}\right)\right\}a^2$$
$$+\frac{\tilde{k}^3}{\mathcal{K}^2}-\tilde{k}-\frac{A}{2\tilde{k}}+\frac{B}{2\tilde{k}\mathcal{K}^2}-\left(\frac{\tilde{k}}{8\mathcal{K}^4}+\frac{1}{4\tilde{k}\mathcal{K}^2}\right)A^2+\left(\frac{\tilde{k}}{2\mathcal{K}^2}-\frac{1}{2\tilde{k}}-\frac{A}{4\tilde{k}\mathcal{K}^2}\right)\frac{B}{a^2}. \tag{8.91}$$

We can see that regularized energy consists of four terms which have distinct behavior under the expansion of the space. We thus separate these terms as

$$\frac{d\langle\rho_{\rm reg}\rangle}{d\mu(k)} = \frac{\omega_6(k)}{a^6}+\frac{\omega_4(k)}{a^4}+\frac{\omega_2(k)}{a^2}+\omega_0(k). \tag{8.92}$$

While ω_0, ω_2, and ω_4 gives quantum correction to the background energy density, there appears a new contribution ω_6 that doesn't exsit in the background matter. To get an insight into this result, we first consider the flat case $A = 0$. We expect that the radiation-like component ω_4 corresponds to $d\rho_B/dk$ calculated before using the Bogoliubov coefficient. Recalling the measure in the flat space $d\mu(k) = k^2 dk$, this expectation turns out to be indeed true,

$$k^2\omega_4(k)\big|_{K=0} = \frac{H_V^3\tilde{k}}{8\pi^2}\left(\frac{2\tilde{k}^2+B}{\sqrt{\tilde{k}^4+B}}-2\tilde{k}^2\right) = a^4\frac{d\rho_B}{dk} \tag{8.93}$$

However, we have now other contribution that are not directly derived from the Bogoliubov coefficients, among which the most important component is ω_0 because the others are more or less "washed away" after sufficiently long period of inflation, otherwise the flatness problem would arise again. ω_0 gives a quantum correction to the cosmological constant,

$$k^2\omega_0(k) = \frac{H_V^4\tilde{k}}{8\pi^2}\left(\frac{1}{\sqrt{\tilde{k}^4+B}}-\frac{\tilde{k}^2}{\tilde{k}^4+B}\right). \tag{8.94}$$

A quite remarkable result appears when we integrate it out to obtain the total energy density from ω_0,

$$\int_0^\infty k^2\omega_0(k)dk = \frac{H_V^4}{16\pi^2}\log 2. \tag{8.95}$$

Although the density spectrum itself depends on the radiation energy B, the total amount of energy density depends only on the Hubble parameter in the inflationary era.

For the negative curvature case ($A > 0 \Leftrightarrow K < 0$), the total energy density of ω_0 is given by the following integral:

$$\int d\mu(k)\omega_0(k) = \frac{H_V^4}{8\pi^2}\int_0^\infty p^2dp\left[\frac{1}{\sqrt{(p^2+1)(p^2(p^2+1)+B/A^2)}}-\frac{\sqrt{p^2+1}}{p^2(p^2+1)+B/A^2}\right]. \tag{8.96}$$

Although the first term contains a square root of 6th order polynomial of p, a variable change $x = p^2$ reduces to an elliptic integral,

$$\int d\mu(k)\omega_0(k) = \frac{H_V^4}{16\pi^2}\int_0^\infty\left[\frac{\sqrt{x}}{\sqrt{(x+1)(x(x+1)+B/A^2)}}-\frac{\sqrt{x(x+1)}}{x(x+1)+B/A^2}\right]dx. \tag{8.97}$$

Instead of convert it into the standard forms of elliptic integrals, we consider two limiting cases $A/\sqrt{B} \to 0$ and $A/\sqrt{B} \to \infty$, after which exhibit a numerical result. Large curvature limit $A/\sqrt{B} \to \infty$ is easily obtained since it reduces to an elementary integral,

$$\int d\mu(k)\omega_0(k) \to \frac{H_V^4}{16\pi^2}\int_0^\infty\left(\frac{1}{x+1}-\frac{1}{\sqrt{x(x+1)}}\right)dx = -\frac{H_V^4}{8\pi^2}\log 2. \tag{8.98}$$

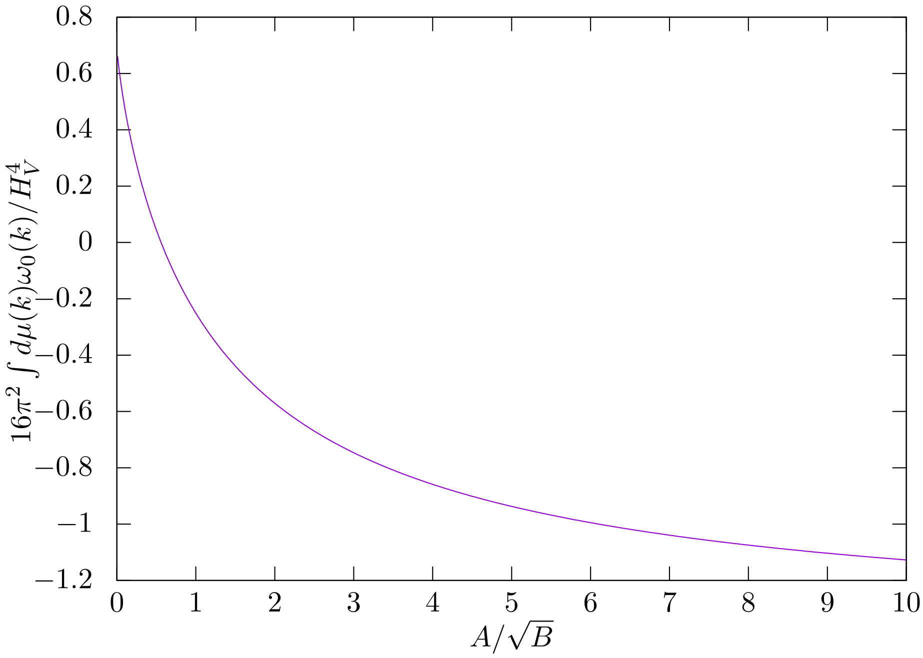


Figure 11: Integrated energy density of $\omega_0(k)$ in the negative curvature space normalized by the inflation Hubble scale H_V^4 as a function of $A/\sqrt{B}$.

In the other limit $A/\sqrt{B} \to 0$ this integral should converges to the result of flat space. We can confirm this result as

$$\int d\mu(k)\omega_0(k) \to \frac{H_V^4}{16\pi^2}\int_0^\infty \left(\frac{1}{x(x+1)+B/A^2} - \frac{x}{x(x+1)+B/A^2}\right)dx \to \frac{H_V^4}{16\pi^2}\log 2. \quad (8.99)$$

A numerical plot Fig.11 shows that the energy density is monotonically decreasing from the flat value.

We finally consider the positive curvature case. In this case, the total energy density is evaluated not by an integration but the following infinite sum:

$$\int d\mu(k)\omega_0(k) = \frac{H_V^4}{8\pi^2}\sum_{l=1}^{\infty}(l+1)^2\left[\frac{1}{\sqrt{l(l+2)(l(l+1)^2(l+2)+B/A^2)}} - \frac{\sqrt{l(l+2)}}{l(l+1)^2(l+2)+B/A^2}\right]. \quad (8.100)$$

Although it cannot be expressed in a closed form even in the limiting cases considered in the open space case, we expect essentially similar behavior as a function of $A/\sqrt{B}$ since the summand is the same as the integrand in the open space case and only the difference comes from the measure. However, a numerical result (Fig.12) shows different behavior in the large curvature region, while the small curvature limit is consistent with the flat space result.

8.6 Density fluctuation

In this section, we consider the density fluctuation that arise from the quantum fluctuation during the inflationary era. Strictly speaking, we have to consider both the scalar part of linear

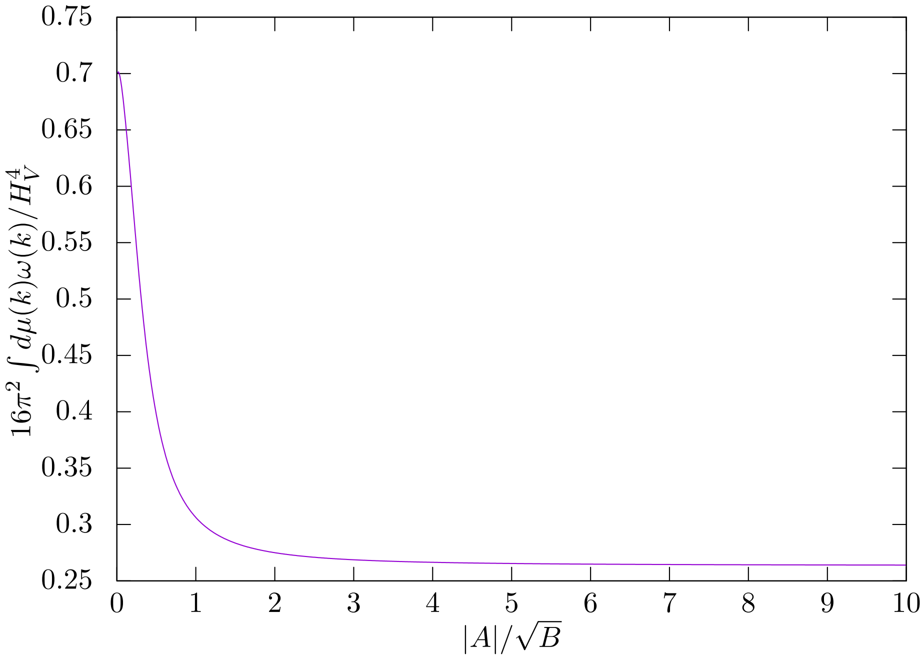


Figure 12: Total energy density of $\omega_0(k)$ in the closed curvature space ($A < 0$) normalized by the inflation Hubble scale H_V^4 as a function of $|A|/\sqrt{B}$.

perturbation to the metric tensor and the perturbation in the matter fields, where the main dynamical equation, called the Mukhanov-Sasaki equation, is given by[5]

$$v_k'' + \left(k^2 c_s^2 - \frac{z''}{z}\right) v_k = 0. \tag{8.101}$$

In this equation, v_k is related to the curvature perturbation, c_s is the speed of sound of the scalar perturbation, and $z^2 = 2a^2\epsilon/c_s^2$. ϵ is called the first slow roll parameter defined by $\epsilon = -\dot{H}/H^2$, which represents the variation rate of the Hubble parameter. In order the inflation to last for sufficiently long period the slow roll parameter should be small.

However, these effects appear only when one deals with the dynamics of the inflaton field. Since the inflationary Hubble parameter H_V in our treatment is constant and our purpose here is to consider the effects of the pre-inflationary era on the power spectrum, we ignore those effects mentioned above. One can show that in the limit $\epsilon \to 0$ the Mukhanov-Sasaki equation reduces to just the massless minimally coupled scalar field equation, therefore we can use the solution obtained in the previous section to investigate the modification to the density fluctuation caused by the pre-inflationary era. Actually, in the linear order of the slow roll parameter, the dynamics of the background inflaton field in general gives slight violation of the scale invariance in the power spectrum. Although one can take into account that dynamics simultaneously with the pre-inflationary effects, we don't go into such a detail.

Essentially, the density fluctuation which is considered as the origin of the large scale structure at present, is caused by the vacuum fluctuation of the scalar field $\langle 0||\phi(x)|^2|0\rangle$. As we have seen in Sec.3, after the horizon exit this expectation value is frozen to a constant and gives the scale invariant power spectrum in the de Sitter space.

We take the same vacuum state as in the previous sections, namely $|0_A\rangle$. Then using the mode expansion (8.50) and recalling the definition of the measure (8.80), the vacuum fluctuation of $V(\eta, \boldsymbol{x})$ can be given as a mode sum

$$\langle 0_A || V(\eta, \boldsymbol{x})|^2 | 0_A \rangle = \frac{1}{2\pi^2} \int d\mu(k) |V_k^-(\eta)|^2. \tag{8.102}$$

Recall that our construction of the exact solution for $V_k^\pm$ relies on the simple polynomial solution for the squared amplitude $|V_k^\pm|^2 \sim a^2 y$, where y is given in Eq.(8.22). Taking account of the normalization constant as well, the fluctuation for the field ϕ is given by

$$\langle 0_A || \phi(\eta, \boldsymbol{x})|^2 | 0_A \rangle = \frac{1}{4\pi^2 H_V} \int \frac{d\mu(k)}{\tilde{k}\mathcal{K}^2} \left(1 + \frac{\tilde{k}^2}{a^2} \right), \tag{8.103}$$

where $\mathcal{K} = (\tilde{k}^4 - A\tilde{k}^2 + B)^{1/4}$ is already defined in the previous section. We can see that the fluctuation is frozen after the inflation $a \to \infty$ as

$$\lim_{a\to\infty} \langle 0_A || \phi|^2 | 0_A \rangle = \frac{1}{4\pi^2 H_V} \int \frac{d\mu(k)}{\tilde{k}\mathcal{K}^2}. \tag{8.104}$$

As usual, we define the power spectrum $P(k)$ for $K = 0$ as

$$\lim_{a\to\infty} \langle 0_A || \phi|^2 | 0_A \rangle =: \int \frac{dk}{k} P(k), \tag{8.105}$$

which gives

$$P(k) = \left(\frac{H_V}{2\pi} \right)^2 \frac{\tilde{k}^2}{\mathcal{K}^2}. \tag{8.106}$$

One can see that in the large k limit this power spectrum recovers the scale invariance while the effect of the pre-inflationary matters become manifest in the small k region. Figure 13 explicitly shows the deviation from scale invariance at very low value of $\tilde{k}$. For the flat space case, the large scale suppression due to the radiation component can be seen, which was reported in Ref.[22]. The setting in their work is completely the same as ours except for the full account of the spatial curvature, thus the result should be consistent. Indeed, it can be seen that FIG.1 in their paper is completely identical to our result (top of Fig.13) except for the rapid oscillatory behavior in their result. They mentioned such a oscillation is caused by an improper choice of the initial condition for the numerical calculations. As our result is mathematically exact in the linearized level, the oscillatory behavior appearing in the previous work is confirmed to be totally due to the numerical calculations.

To define the power spectrum for curved cases, we have to care about the integration or summation measure. In general, they don't affect the large k behavior but it modifies significantly in the small k region, where the deviation of the power spectrum is also important. To compare the power spectrum for different spatial curvature cases, we define it by Eq.(8.106) for the curved cases as well.

Figure 14 shows the comparison of the power spectrum for different values of the spatial curvature. The top figure represents a case where the curvature contribution is small compared to the background vacuum energy of the inflaton. In this case, as one expect, the three cases shown in the figure is almost identical. In contrast, if the curvature component becomes larger (bottom of Fig.14) the deviation from the flat one is apparent and the difference between the positive and negative curvature cases seems rather nontrivial. For the closed case, the effect of curvature is just an additional suppression of the spectrum. On the other hand, for the open space the power spectrum is enhanced at the largest length scale. We can see this behavior analytically. The spectrum (8.106) has the maximum at $\tilde{k}_{\rm peak} = \sqrt{2B/A}$, which exists only when the curvature is negative ($\Leftrightarrow A > 0$). The maximum value relative to the value of the standard spectrum $P(\infty)$ is given by

$$\frac{P(k_{\rm peak})}{P(\infty)} = \frac{2\sqrt{B}/A}{\sqrt{(2\sqrt{B}/A)^2 - 1}}. \tag{8.107}$$

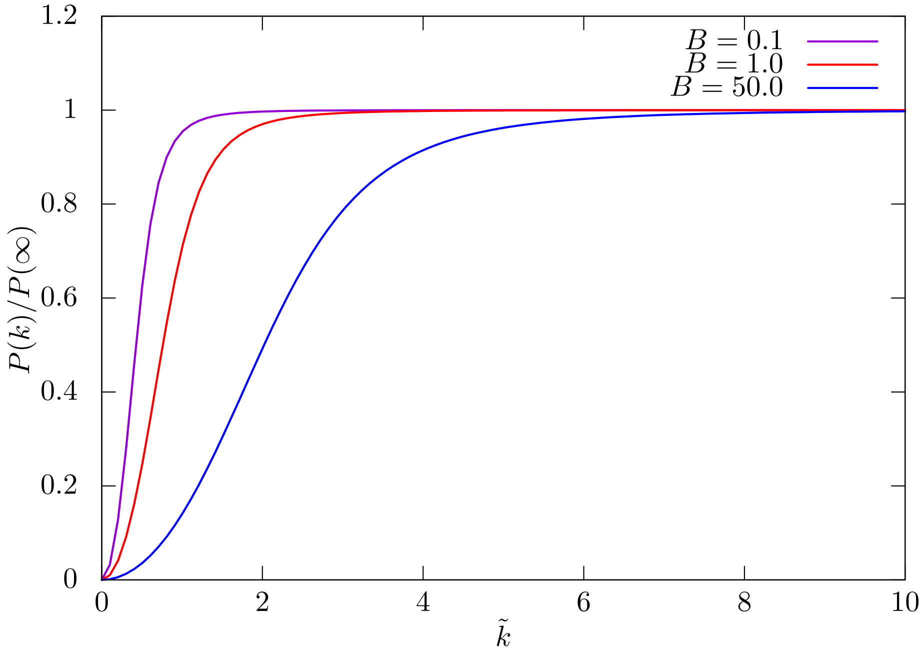


Figure 13: Numerical plot of vacuum fluctuation for the flat space, normalized by the value at the UV limit $P(\infty) = (H_V/2\pi)^2$, which is the spectrum in the pure de Sitter expansion. The parameters are set to $A = 0, B = 1.25 \times 10^{-5}$

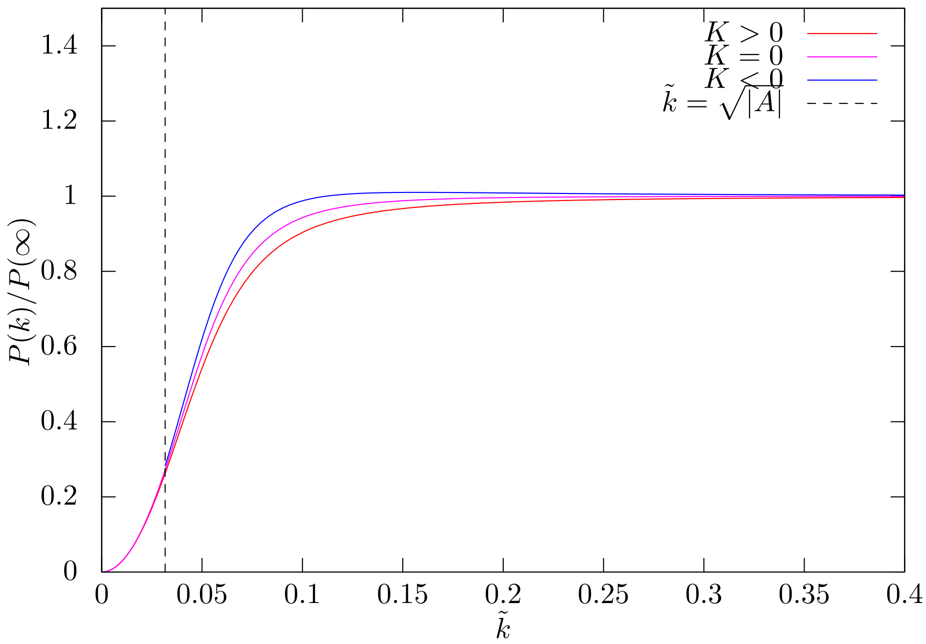
The relative peak value converges to 1 in the limit $2\sqrt{B}/A \to \infty$, namely the flat limit, which is consistent with the absence of the peak in the flat case. In the large curvature limit, the peak value apparently diverges at $2\sqrt{B}/A = 1$ but this divergence is actually out of the physical relevance because in this case the peak position $\tilde{k}_{\rm peak}$ is smaller than the spatial curvature scale $A^{1/2}$. When the negative curvature is not so large, i.e. $A < \sqrt{2B}$, the peak is inside the physical region $\tilde{k}_{\rm peak} > A^{1/2}$ but in this case the relative peak value has an upper bound

$$\frac{P(k_{\rm peak})}{P(\infty)} < \sqrt{2} \sim 1.41. \tag{8.108}$$

This bound can be achieved in the degenerate case $A = \sqrt{2B}$, although in this case the peak position is equal to the spatial curvature scale $\tilde{k}_{\rm peak} = A^{1/2}$. In fact, the bottom of Fig.14 corresponds to this degenerate case, as one can see from the coincidence of the peak position and the cut off indicated by the vertical line.

Note that in the figure the deviation appears around very large length scale compared to the inflationaly Hubble scale (recall the normalization $\tilde{k} = k/H_V$). However, this is due to the choice of small value of the parameter B, which represents the radiation energy compared to the vacuum energy of the background inflaton. If we take $B = O(1)$ then the deviation from the scale invariant spectrum can be seen in the Hubble scale of the inflation, although we don't explicitly list the figure because the qualitative behavior is not unchanged from Fig.14.

Let us consider the connection to the observational result of the CMB[37, 38]. As noted so far, non-standard expansion history before the inflation only affects the large scale fluctuation after the inflationary epoch. Therefore, we are interested in the low multipole moment (l) components of the CMB anisotropy. It is well known that there is a large scale anomaly in the CMB power spectrum, namely the suppression of the power at low l modes compared to the prediction from the almost scale invariant primordial spectrum with the standard ΛCDM model. Although in



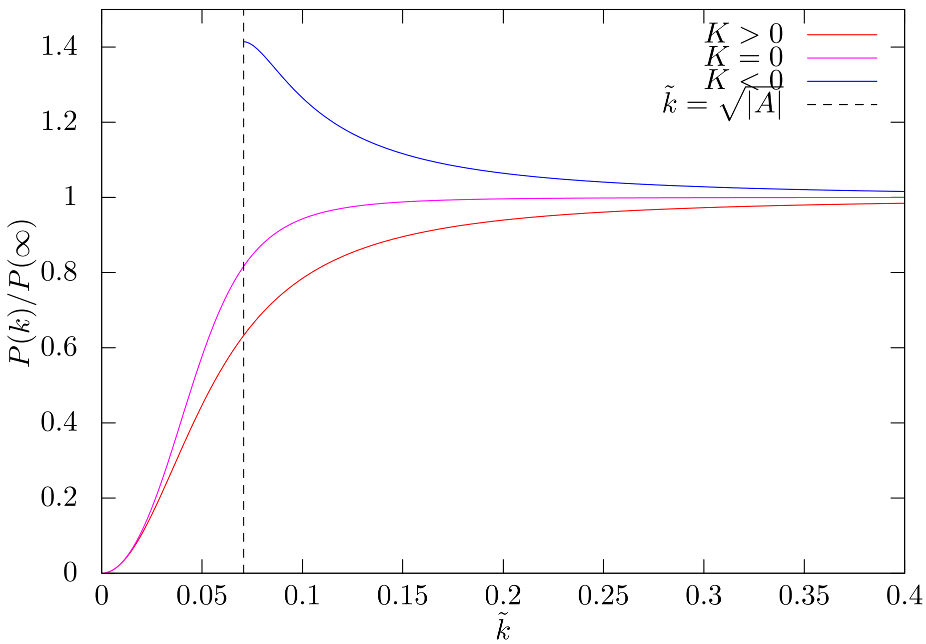


Figure 14: Comparison of the power spectrum normalized by the value in the UV limit, with different values of the spatial curvature. The radiation component is fixed to $B = 1.25 \times 10^{-5}$, while the curvature is (top) $|A| = 1.0 \times 10^{-3}$ and (bottom) $|A| = 5.0 \times 10^{-3}$ for curved cases ($A = 0$ for flat case of course). The vertical dashed line represents the momentum cut off $\tilde{k}_{\min} = \sqrt{A}$ for the negative curvature case.

such a small l region there exists a difficulty of the cosmic variance[5], it might be a signal of the violation of the scale invariance of the primordial power spectrum. We note here that because of this suppression, our result above seems to indicate that the large negative curvature should be excluded from the consideration. However, if the amount of the radiation component in the pre-inflationary era is comparable to the curvature component, then the resultant spectrum is rather simillar to each other because the radiation component B can compensate for the negative curvature ($A > 0$) effect as one can see from the expression $\mathcal{K} = (\tilde{k}^4 - A\tilde{k}^2 + B)^{1/4}$. In this sence there is a degeneracy in the parameter space of our model if we only consider the primordial power spectrum. It is possible that this degeneracy would be resolved by taking the dynamics of the inflaton field into account and consider more precise (of course model dependent) time evolution in the inflationary era.

Considering possible effects on other obaservable quantities is also important. Recently, it has been argued that the enhancement of the lensing amplitude in the Planck results can be naturally explained by assuming the positive curvature, though in this case discrepancies between various local observational results including BAO (baryon acoustic oscillation) becomes larger[39].

9 Conclusion

We considered the linear perturbation theory, especially time evolution equation on the FRW spacetime with various combinations of background matters. Surprisingly, although the solutions to the evolution equations have rather complicated expression in terms of Weierstrass's elliptic and related functions, the associated 3rd order equations satisfied by the symmetric power of the wave amplitude allow simple solutions that are polynimials of the resiprocal of the scale factor a. The method utilized there was originally employed in mathematical literature, but it is also physically motivated because the associated 3rd order differential equation directly describes the time development of the squared amplitude, which is related to the energy density of the propagating waves. Although we have not succeeded in applying this method to other physical systems, it is worth searching for such examples.

On mathematical side, there is a generalization of the idea, where in general k-th powers of the solutions of original second order differential equation are used[40]. In this work, the authors the condition that the condition where the Lamé equation allows algebraic solutions in terms of the monodromy group of the equation. These kind of studies may be useful to understand why there are polynomial solutions in the cosmological setting treated in this thesis only in the four cases, although such an understanding (if possible) is physically relevant or not is unclear.

After obtaining the exact solutions, we considered applications of these solutions. In the first part (Sec.6,7), we considered propagation of gravitational waves in the late time universe. In the former section, the exact solution after the radiation component becomes negligible in the standard cosmological model was studied. Note that our solution can afford to include the spatial curvature as well, although the observable effect would not exist. The solution in terms of elliptic functions are found to reduce to the known solutions when the limit is appropriately taken, which confirms the validity of the exact solution. We also obtained the exact expression for the energy spectrum of the gravitational waves in terms of a rational function of the scale factor a and the comoving wave number k. In the latter section, we treated the similar problem but with the cosmological constant replaced by a non standard energy content which is proportional to $1/a^3$. We regarded this as a simplified model for the quintessence scenario in the late time acceleration and obtained the energy spectrum in the similar manner. The result shows that the spectrum of these two cases differ significantly in the large scale. Although the spectrum is almost identical to each other in the physically relevant, meaning observable in the near future, region if the density parameter Ω_m of the non relativistic matter is set to the same value, it may be possible that Ω_m differs from the standard value if some quintessence scenario is true.

In the second part, we considered a model of a pre-inflationary era filled with radiation and spatial curvature components, followed by the de Sitter expansion. We obtained the exact solution for the massless minimally coupled scalar field in this background, which obeys the Lamé equation of the order $n = 1$. The obtained exact solution was found to be positive and negative frequency modes at the initial time when the radiation energy is dominant, and the Bogoiubov transformation that connects the late time positive/negative frequency solutions which are usually utilized to define the Bunch-Davies vacuum state in the inflation. Using the Bogoliubov transformation, we calculated the number and energy density of particles created due to non trivial time evolution of the background gravitational field. The resultant energy distribution turned out to be non thermal one, differently from the typical examples of the gravitational particle creation, namely the Hawking radiation from black holes. We further calculated the vacuum expectation value of the energy density of the scalar field. In contrast to the case of the de Sitter spacetime, there is no infrared divergence observed, while the usual ultraviolet divergence exists. To regularize the divergence, we use the adiabatic regularization but slightly modified so as to keep the infrared finiteness. The regularized energy density contains the spectrum obtained by using the Bogoliubov coefficients as a radiation like component, and other components with different behaviors under the cosmic expansion. Among them, we pick up the term that is independent of the scale factor a, namely the quantum correction to the cosmological constant. We found the sign of the correction can change with the value of the spatial curvature. Lastly, we calculated the primordial power spectrum, which is given by the simple function of the comoving wave number and the density parameters normalized by the Hubble parameter of the inflation. As in the case of energy density correction, the positive and negative curvature gives the opposite contribution to the power spectrum. Except for the sufficiently large negative curvature cases, the pre-inflationary era suppresses the power spectrum at the very large length scale, which may be reponsible for the observed anomaly in the CMB temperature anisotropy at low multipole l.

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A Preliminary on Weierstrass elliptic function and related functions

In this appendix, we briefly summarize basic facts about (especially Weierstrass's) elliptic functions. The readers who would like to know more on this topic can refer to ref [1].

A.1 Fundamentals on elliptic functions

A complex function $f(z)$ which is meromorphic and doubly periodic is called an elliptic function. A meromorphic function is a function whose singularities are only isolated poles. Double periodicity means that there exist two complex parameters ω_1 and ω_2 such that for arbitrary integers n and m

$$f(z + 2m\omega_1 + 2n\omega_2) = f(z), \ {}^{\forall}z \in \mathbb{C} \setminus \{\text{all singular points}\}. \tag{A.1}$$

Note that $\omega_1/\omega_2 \notin \mathbb{R}$ is assumed. Because of double periodicity, the entire properties of an elliptic function are determined within a parallelogram with vertices z_0, $z_0+2\omega_1$, $z_0+2\omega_1+2\omega_2$, and $z_0 + 2\omega_2$ for some point $z_0 \in \mathbb{C}$. We call such a parallelogram a (fundamental) cell. If an elliptic function $f(z)$ is analytic in a cell, then it is bounded in the entire complex plane. This implies that an analytic elliptic function must be a constant function. Conversely, any non-constant elliptic function should have at least one pole in a cell.

By using Cauchy's integral formula, we can derive several properties of elliptic functions. First we consider the following integral:

$$\oint \frac{dz}{2\pi i} f(z). \tag{A.2}$$

The contour is the boundary of a cell in the counterclockwise direction. Since $f(z)$ takes the same values on each pair of opposite sides and the integrations are in opposite direction, we can see this integral vanishes. Cauchy's theorem states that this integral is equal to the sum of the residues of $f(z)$ within the cell, therefore

$$\sum_{\text{in a cell}} \mathrm{Res} f(z) = 0. \tag{A.3}$$

Next we observe that $f'(z)/f(z)$ is also a elliptic function. Thus we can apply the above result, but in this case simple poles of $f'(z)/f(z)$ comes from every (including higher order) poles of $f(z)$ as well as every roots of $f(z)$. Therefore, we obtain

$$\sum_{\text{poles}} (\text{order of poles}) = \sum_{\text{roots}} (\text{multiplicity of roots}). \tag{A.4}$$

This value (though not completely) characterize an elliptic function and is called the order of $f(z)$. Because the sum of residues must vanish, the minimal value of the order is 2.

A.2 Weierstrass's elliptic function $\wp(z)$

Weierstrass's elliptic function is one of the simplest elliptic functions, namely order 2, which is defined by

$$\wp(z) := \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left[\frac{1}{(z - 2m\omega_1 - 2n\omega_2)^2} - \frac{1}{(2m\omega_1 + 2n\omega_2)^2} \right], \tag{A.5}$$

where the summation is taken over all integers m, n except for $m = n = 0$. We introduce a shorthand notation $\sum'_{m,n}$ for such a summation. Double periodicity $\wp(z + 2\omega_1) = \wp(z + 2\omega_2) =$

$\wp(z)$ and even parity $\wp(-z) = \wp(z)$ are evident from the definition. Also, singularities of $\wp(z)$ are only double poles at $z = 2m\omega_1 + 2n\omega_2$.

Taking into acount the facts that $\wp(z)$ is an even function and that the summation in the definition gives 0 at $z = 0$, we can conclude that the Laurent series of $\wp(z)$ around $z = 0$ is given as

$$\wp(z) = \frac{1}{z^2} + \frac{g_2}{20}z^2 + \frac{g_3}{28}z^4 + O(z^6), \tag{A.6}$$

where g_2 and g_3 are some constants. By differentiating this series term by term, we obtain

$$\wp'(z) = -\frac{2}{z^3} + \frac{g_2}{10}z + \frac{g_3}{7}z^3 + O(z^5). \tag{A.7}$$

Because the singular terms of three expressions $[\wp'(z)]^2$, $\wp^3(z)$, and $\wp(z)$ are only $1/z^6, 1/z^2$, we can arrange them to pick up a non singular function. By a simple manipulation, we can find the following result:

$$[\wp'(z)]^2 - 4\wp^3(z) + g_2\wp(z) + g_3 = O(z^2). \tag{A.8}$$

The left side of this equation is evidently an elliptic function, but as the right side shows it is non singular. Therefore, it should be a constant function, especially zero. Thus, $\wp(z)$ satisfies the following first order differential equation,

$$[\wp'(z)]^2 - 4\wp^3(z) + g_2\wp(z) + g_3 = 0. \tag{A.9}$$

The above differential equation is invariant under the translation $z \to z + z_0$ and is first order, so the general solution to this equation is given by the elliptic function $\wp(z + z_0)$. Formal integration of Eq.(A.9) gives an implicit expression for $\wp(z)$,

$$z = \int_{\wp(z)}^{\infty} \frac{d\wp}{[4\wp^3 - g_2\wp - g_3]^{1/2}}. \tag{A.10}$$

Note that the endpoints of integration is chosen so that $\wp(z)$ has a pole at $z = 0$. Because the integration contains a multivalued complex function, namely square root, branch cuts and integration path must be specified. The branch points are given by solutions to the algebraic equation $4\wp^3 - g_2\wp - g_3 = 0$, which, at most, has three solutions. We write the solutions as e_1, e_2, and e_3. When $(g_2)^3 = 27(g_3)^2$, at least two of e_i coincide and Eq.(A.10) can be integrated with elementary functions. In this case the solution to Eq.(A.9) is no longer an elliptic function. We assume $(g_2)^3 \neq 27(g_3)^2$ hereafter. Note that because Eq.(A.9) doesn't have quadratic term, $e_1 + e_2 + e_3 = 0$ holds.

Now we write Eq.(A.10) as

$$z = \int_{\wp(z)}^{\infty} \frac{dy}{[4(y - e_1)(y - e_2)(y - e_3)]^{1/2}}. \tag{A.11}$$

The configuration of three distinct roots e_i are divided into two cases: (i) three real roots, (ii) one real root and a pair of complex conjugate roots. For clarity, we assume $e_1 > e_2 > e_3$ in case (i) and $e_1 \in \mathbb{R}$ in case (ii). Since the integrand is a square root, we need a Riemann surface that consists of two sheets of complex plane. We fix the end point of the integral ($y = \infty$) as the infinity along the positive real axis on the first sheet. Then, for a given point in the Riemann surface, various integration paths in Eq.(A.11) are specified by the winding numbers around the branch points. Consider two paths with the same end points. If the difference of these paths is just a small circle around one of the branch points, the values of the two integrals are equal. On the other hand, if one of the paths differs from the other by a closed contour that encircles two branch points, then the two integrals give different values, the difference of which is one of the periods of this elliptic function.

Another important implication of $\wp(z)$ is related to the parity of the elliptic function, namely $\wp(-z) = \wp(z)$. Since the Riemann surface can be considered as two sheets of complex plane connected by branch cuts, there are two points in the surface that correspond to a given value of $\wp(z)$. Taking account of the fact that the difference of these choices is just the sign, we can conclude that the two integrals which start from different sheets give sign change up to the periods.

To illustrate these facts, let us consider the case (i). As an example, we fix the starting point $\wp(z)$ at the upper half plane of the same sheet with the endpoint $y = \infty$. One of the simplest paths, which we call C_1, would consist of two straight lines one of which connects $y = \wp(z)$ and $y = e_1$, and the other goes from $y = e_1$ to $y = \infty$ along the real axis,

$$\int_{C_1} \frac{dy}{f(y)} = \int_{\wp(z)}^{e_1} \frac{dy}{f(y)} + \int_{e_1}^{\infty} \frac{dx}{f(x)}. \tag{A.12}$$

Here $f(y)$ denotes the integrand given in Eq.(A.11) and we introduced a real valued variable x to indicate that the second integral is real. Now consider another path C_2; after the straight line from $y = \wp(z)$ to $y = e_1$, it encircles two branch points e_1 and e_2 in the counterclockwise direction, and comes back to the half real axis from $y = e_1$ to $y = \infty$. The difference of the two integrals with C_1 and C_2 is given by the following integral:

$$2i \int_{e_2}^{e_1} \frac{dx}{\sqrt{4(e_1 - x)(x - e_2)(x - e_3)}} =: 2\omega_2. \tag{A.13}$$

Next we consider the path C_3 that encircles e_2 and e_3 in the counterclockwise direction. Then the difference between the integrals with C_1 and that with C_3 is given by

$$2 \int_{e_3}^{e_2} \frac{dx}{\sqrt{4(e_1 - x)(e_2 - x)(x - e_3)}} =: 2\omega_1. \tag{A.14}$$

In conclusion, we find two fundamental periods can be chosen so that one is real and the other is pure imaginary in case (i). We note that the periods Eqs.(A.14) and (A.13) can be also expressed by the following integrals:

$$2\omega_1 = 2 \int_{e_1}^{\infty} \frac{dx}{\sqrt{4(x - e_1)(x - e_2)(x - e_3)}}, \tag{A.15}$$

$$2\omega_2 = 2i \int_{-\infty}^{e_3} \frac{dx}{\sqrt{4(e_1 - x)(e_2 - x)(e_3 - x)}}. \tag{A.16}$$

We can understand this result by observing that, for ω_1, integrals in Eqs.(A.14) and (A.15) can be combined to form a closed contour that doesn't encircle any singularity. The equality between Eqs.(A.13) and (A.16) can be established in the similar way. Of course, appropriate changes of variables convert the integrals of Eqs.(A.14) and (A.13) to those of Eqs.(A.15) and (A.16). In case (ii), although we cannot take ω_1 and ω_2 such that ω_2/ω_1 is pure imaginary, fundamental periods are given by integrals encircling two branch points, or equivalently one branch point and infinity.

Let us determine the positions z where $\wp(z) = e_i$. From the differential equation Eq.(A.9), we find $\wp'(z) = 0$ there. Because $\wp(z)$ is an order two elliptic function, $\wp(z) = e_i$ must has two solutions for z in a fundamental lattice for each value of $i(= 1, 2, 3)$. Then, since $\wp'(z)$ is an order three elliptic function, $\wp'(z) = 0$ has three distinct, simple roots. We can find these roots by observing the following result:

$$\wp'(\omega_i) = -\wp'(-\omega_i) = -\wp'(\omega_i) \;\Rightarrow\; \wp'(\omega_i) = 0, \tag{A.17}$$

where, in aadition to ω_1 and ω_2, it holds for $\omega_3 := \omega_1 + \omega_2$. Note that the choice of the fundamental half periods ω_1 and ω_2 is not unique although e_i are invariant up to the permutation of the index i.

A.3 Weierstrass's sigma $\sigma(z)$ and zeta $\zeta(z)$

We introduce two complex functions $\sigma(z)$ and $\zeta(z)$ related to $\wp(z)$,

$$\sigma(z) = z \prod_{(m,n)\neq(0,0)} \left(1 - \frac{z}{2m\omega_1 + 2n\omega_2}\right) \exp\left(\frac{z}{2m\omega_1 + 2n\omega_2} + \frac{z^2}{2(2m\omega_1 + 2n\omega_2)^2}\right), \tag{A.18}$$

$$\zeta(z) = \frac{1}{z} + \sum_{(m,n)\neq(0,0)} \left[\frac{1}{z - 2m\omega_1 - 2n\omega_2} + \frac{1}{2m\omega_1 + 2n\omega_2} + \frac{z}{(2m\omega_1 + 2n\omega_2)^2}\right]. \tag{A.19}$$

These definitions immediately lead to the following relations:

$$\wp(z) = -\zeta'(z), \quad \zeta(z) = \frac{\sigma'(z)}{\sigma(z)}, \quad \zeta(-z) = -\zeta(z), \quad \sigma(-z) = -\sigma(z). \tag{A.20}$$

These functions are "quasi" periodic, meaning

$$\sigma(z + 2\omega_i)/\sigma(z) = -\exp\left(2\eta_i(z + \omega_i)\right), \tag{A.21}$$

$$\zeta(z + 2\omega_i) - \zeta(z) = 2\eta_i, \tag{A.22}$$

where $\eta_i = \zeta(\omega_i)$ $(i = 1, 2)$ are constants. These relations can be verified as follows. By definition, it is easy to see that $\zeta(z + 2\omega_i) - \zeta(z)$ is independent of z, so we write them as $2\eta_i$. Then, because $\zeta(z)$ is an odd function, we obtain

$$\zeta(\omega_i) = -\zeta(-\omega_i) = -\zeta(\omega_i) + 2\eta_i \;\Rightarrow\; \eta_i = \zeta(\omega_i). \tag{A.23}$$

By integrating $\zeta(z)$ along the boundary of the fundamental lattice with vertices $z = \pm\omega_1 \pm \omega_2$, we can derive a constraint on η_i,

$$2\pi i = \oint \zeta(z)dz = 4(\omega_2\eta_1 - \omega_1\eta_2). \tag{A.24}$$

Now integrating Eq.(A.22) and taking $\sigma(-\omega_i) = -\sigma(\omega_i)$ into acount, we can obtain Eq.(A.21).

The power series expansion around the origin can be easily derived by using the expansion of $\wp(z)$ and integrating it, of which first several terms are given by

$$\zeta(z) = \frac{1}{z} - \frac{g_2}{60}z^3 - \frac{g_3}{140}z^5 + O(z^7), \tag{A.25}$$

$$\sigma(z) = z - \frac{g_2}{240}z^5 - \frac{g_3}{840}z^7 + O(z^9). \tag{A.26}$$

A.4 Addition theorems

Weierstrass functions defined in the previous sections satisfy the following addition theorems:

$$\wp(z + w) = -\wp(z) - \wp(w) + \frac{1}{4}\left(\frac{\wp'(z) - \wp'(w)}{\wp(z) - \wp(w)}\right)^2, \tag{A.27}$$

$$\wp(z) - \wp(w) = -\frac{\sigma(z - w)\sigma(z + w)}{\sigma^2(z)\sigma^2(w)}, \tag{A.28}$$

$$\zeta(z + w) = \zeta(z) + \zeta(w) + \frac{1}{2}\frac{\wp'(z) - \wp'(w)}{\wp(z) - \wp(w)}. \tag{A.29}$$

B Laplacian on 3-dimensional homogeneous and isotropic space and its eigenfunctions

Here we summarize the fundamental results about the Laplacian D^2 on the 3-dimensional homogeneous and isotropic space, namely spatial section of the FRW spacetime. D^2 can be defined by

$$D^2 = \frac{1}{\sqrt{\gamma}}\partial_i \left(\sqrt{\gamma}\gamma^{ij}\partial_j\right), \tag{B.1}$$

where γ_{ij} is the metric tensor of the space,

$$ds^2_{(3)} = \gamma_{ij}dx^i dx^j = \frac{dr^2}{1-Kr^2} + r^2 d\Omega^2, \tag{B.2}$$

and γ is the determinant of γ_{ij}. The eigenvalues and the eigenfunctions can be found in the literature[34], but we show the result with some derivations. For flat section ($K = 0$), the Cartesian coordinate system is available so that the eigenvalues and eigenfunctions are well known,

$$D^2 e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = -k^2 e^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{B.3}$$

When $\mathrm{Im}k \neq 0$, the eigenfunction diverges exponentially at infinity, so we assume $k \in \mathbb{R}$, or the eigenvalue is negative $-k^2 < 0$.

In the cases of non zero curvature $K \neq 0$, we take a dimensionless radial coordinate χ defined by

$$|K|^{1/2}r = R(\chi) := \begin{cases}\sin\chi \ (K>0) \\ \sinh\chi \ (K<0)\end{cases} \quad \Rightarrow \ d\chi = \frac{|K|^{1/2}dr}{\sqrt{1-Kr^2}}. \tag{B.4}$$

Using this coordinate, the Laplacian normalized by the curvature $\mathcal{D}^2 = |K|^{-1}D^2$ is explicitly given by

$$\mathcal{D}^2 = \frac{1}{R^2(\chi)}\partial_\chi \left(R^2(\chi)\partial_\chi\right) + \frac{1}{R^2(\chi)}\boldsymbol{D}^2, \tag{B.5}$$

where $\boldsymbol{D}^2$ is the Laplacian on the 2-dimensional sphere of unit radius. Recalling that the eigenvalue of $\boldsymbol{D}^2$ is $l(l+1)$ with a non negative integer l and its eigenfunction is the spherical harmonics $Y_{l,m}$ ($|m| = 0, 1, \cdots, l$), the eigenfunction of $\mathcal{D}^2$ with the eigenvalue $-k^2/|K|$ can be written as $\psi_{k,l,m}(\chi,\theta,\phi) = f_{k,l}(\chi)Y_{l,m}(\theta,\phi)$, where $f_{k,l}$ is the solution to the following ordinaly differential equation:

$$\frac{1}{R^2(\chi)}\frac{d}{d\chi}\left(R^2(\chi)\frac{df_{k,l}}{d\chi}\right) - \frac{l(l+1)}{R^2(\chi)}f_{k,l} + \frac{k^2}{|K|}f_{k,l} = 0. \tag{B.6}$$

We solve the negative curvature case first. The solution for the positive curvature case can be easily obtained by $\chi \to i\chi$ as one can see from the definition of $R(\chi)$. It is useful to introduce the coordinate $z = \cosh\chi$. Then, appropriate rescaling $g_{k,l} = (z^2-1)^{1/4}f_{k,l}$ leads to the Legendre's differential equation for $g_{k,l}$,

$$0 = \left[(z^2-1)\frac{d^2}{dz^2} + 2z\frac{d}{dz} - \left(\frac{k^2}{|K|} - \frac{3}{4}\right) - \frac{(l+1/2)^2}{z^2-1}\right]g_{k,l}, \tag{B.7}$$

of which basis solutions are the famous Legendre functions $P^{l+1/2}_{-1/2+ip}(z)$ and $Q^{l+1/2}_{-1/2+ip}(z)$ of the 1st and 2nd kind, respectively. Here we defined $p = \sqrt{k^2/|K| - 1}$. Since $l + 1/2$ is half an odd integer, we can express these solutions as finite term series by using suitable transformations of the Legendre functions (see [15] for such transformations). Consequently we can arrive at the following explicit forms,

$$f_{p,l}^{(1)}(\chi) = \frac{e^{ip\chi}}{\sinh\chi}\sum_{n=0}^{l}\frac{(l+1)_n(-l)_n}{n!(1-ip)_n}\left(\frac{-e^{-\chi}}{2\sinh\chi}\right)^n, \tag{B.8}$$

$$f_{p,l}^{(2)}(\chi) = \frac{e^{-ip\chi}}{\sinh\chi}\sum_{n=0}^{l}\frac{(l+1)_n(-l)_n}{n!(1+ip)_n}\left(\frac{-e^{-\chi}}{2\sinh\chi}\right)^n, \tag{B.9}$$

where $(x)_n := \Gamma(x+n)/\Gamma(x)$ is the Pochhammer symbol. Another possible expression for these solutions are

$$f_{p,l}^{(1)}(\chi) = \frac{(-1)^{l+1}}{(-ip)_{l+1}}\sinh^l\chi\left(\frac{d}{d\cosh\chi}\right)^{l+1}e^{ip\chi}, \tag{B.10}$$

and its complex conjugate for $f_{p,l}^{(2)}$. One can confirm this expression indeed solves Eq.(B.6) by direct substitution and the coefficient can be fixed by taking the limit $\chi\to\infty$ for example.

We define the basis solution to the full equation $\mathcal{D}^2\psi_{p,l,m} = -(p^2+1)\psi_{p,l,m}$ by $\psi_{p,l,m} = N_{p,l,m}(f_{p,l}^{(1)}+f_{p,l}^{(2)})Y_{l,m}$ with a normalization constant $N_{p,l,m}$. This specific combination is chosen in order to make $\psi_{p,l,m}$ regular at the coordinate origin $\chi=0$. Orthogonality of $\psi_{p,l,m}$ can be readily verified from that of $Y_{l,m}$ and $f_{p,l}^{(1)}$, the latter of which can be seen from the differential equation for $f_{p,l}$. More precisely, from Eq.(B.6) we can derive

$$(p^2-p'^2)\int_0^\infty d\chi R^2 f_{p,l}f_{p',l} = \int_0^\infty d\chi\frac{d}{d\chi}\left[R^2\left(f_{p,l}\frac{df_{p',l}}{d\chi}-\frac{df_{p,l}}{d\chi}f_{p',l}\right)\right] = 0. \tag{B.11}$$

Strictly speaking the boundary value at $\chi\to\infty$ doesn't vanish but we can insert an infinitesimal imaginary part to the eigenvalue p so that the integral really vanish, as in the case of flat space. We set the normalization of $\psi_{p,l,m}$ by the following condition:

$$\int d^3x\sqrt{\gamma}\psi_{p,l,m}(x)\psi^*_{p',l',m'}(x) = \delta(p-p')\delta_{l,l'}\delta_{m,m'}. \tag{B.12}$$

Then, if we expand an arbitrary field $\phi(x)$ as

$$\phi(x) = \int_0^\infty dp\sum_{l,m}\tilde{\phi}_{p,l,m}\psi_{p,l,m}(x), \tag{B.13}$$

the coefficients $\tilde{\phi}_{p,l,m}$ can be obtained from $\phi(x)$ by

$$\tilde{\phi}_{p,l,m} = \int d^3x\sqrt{\gamma}\phi(x)\psi^*_{p,l,m}(x). \tag{B.14}$$

Equivalently, the basis solutions $\psi_{p,l,m}(x)$ satisfies the completeness,

$$\int_0^\infty dp\sum_{l,m}\psi^*_{p,l,m}(x)\psi_{p',l',m'}(x') = \sqrt{\gamma}\delta^3(x-x'). \tag{B.15}$$

C Adiabatic expansion of a scalar field in FRW spacetime

In this appendix we describe the adiabatic expansion of a scalar field ϕ in a 4-dimensional homogeneous and isotropic spacetime [5]. In the main part of this paper we only consider a

[5] Although our argument is restricted to the case of four spacetime dimensions, it is straightforward to extend to other dimensionality.

massless field with minimal coupling to the curvature, but here we consider a massive field and take massless limit at the end. The scalar field equation is given by

$$0 = \left[\partial_t^2 + 3H\partial_t - \frac{D^2}{a^2} + m^2\right]\phi, \tag{C.1}$$

where $H = \dot{a}/a$ is the Hubble parameter and D^2 is the spatial Laplacian. Changes of both the dependent and independent variables $\phi \to v = a\phi$ and $t \to \eta = \int dt/a$ respectively removes the first order derivative term. Also, we assume the separated form $v(\eta, \vec{x}) = v_k(\eta)\psi_{\vec{k}}(\vec{x})$ with the eigenfunction of the spatial Laplacian $D^2\psi_{\vec{k}} = -k^2\psi_{\vec{k}}$. Then, the ordinary differential equation for v_k is given by

$$0 = v_k'' + (\omega_k^2 - a''/a)v_k, \quad \omega_k^2 = k^2 + m^2a^2. \tag{C.2}$$

Thus, the scalar field equation in the FRW spacetime is equivalent to a set of harmonic oscillators with time-dependent frequencsies. If the background spacetime is static, i.e. a =const. then ω_k is constant for all mode k and the solution is given by just a plane wave $e^{\pm i\omega_k\eta}/\sqrt{2\omega_k}$. The adiabatic expansion assumes that the spacetime is "close" to a static one. When the spacetime is dynamical, the plane wave solution is affected by the derivatives of ω_k and a. Therefore the adiabatic expansion is essentially a derivative expansion and implemented by the following ansatz:

$$v_k(\eta) = \frac{1}{\sqrt{2W_k(\eta)}}\exp\left(\pm i\int W_k(\eta)d\eta\right), \tag{C.3}$$

where the unknown function W_k satisfies a nonlinear 2nd order differential equation

$$W_k^2 = \omega_k^2 - \frac{a''}{a} + \frac{3}{4}\left(\frac{W_k'}{W_k}\right)^2 - \frac{1}{2}\frac{W_k''}{W_k} \tag{C.4}$$

and is expanded as

$$W_k = \omega_k\left(1 + W_k^{(2)} + W_k^{(4)} + \cdots\right). \tag{C.5}$$

In general $W_k^{(2A)}$ consists of terms with $2A$ derivatives. The second and fourth adiabatic order terms are explicitly given by

$$W_k^{(2)} = \frac{3}{8\omega_k^2}\left(\frac{\omega_k'}{\omega_k}\right)^2 - \frac{1}{4\omega_k^2}\frac{\omega_k''}{\omega_k} - \frac{1}{2\omega_k^2}\frac{a''}{a}, \tag{C.6}$$

$$W_k^{(4)} = \frac{W_k^{(2)'}}{4\omega_k^2}\frac{\omega_k'}{\omega_k} - \frac{W_k^{(2)''}}{4\omega_k^2} - \frac{1}{2}(W_k^{(2)})^2. \tag{C.7}$$

Similarly the energy momentum tensor $T_{\mu\nu}$ can be expanded with respect to the number of derivatives. As one can immediately verify, large-k behavior of $W_k^{(2A)}$ is of order $O(k^{-2A})$. Thus we can regard the adiabatic expansion as large-k expansion as well. Even for free scalar fields, the vacuum expectation value of $T_{\mu\nu}$ has UV divergence as in the flat spacetime. Since large-k limit of $W_k^{(2A)}$ is suppressed as the number of adiabatic order $2A$ increases, the UV divergence of the expectation value of $T_{\mu\nu}$ comes only from the first few terms in the adiabatic expansion. This observation leads to the adiabatic regularization scheme to obtain a UV-finite expectation value, where the first several terms that cause the UV divergences are subtracted from the expectation value.

In the FRW spacetimes, because of the spatial homogeneity and isotropy, the vacuum expectation value of $T_{\mu\nu}$ takes the perfect fluid form as long as the vacuum state $|0\rangle$ is chosen so that it respect the spatial symmetry. In this case, $\langle 0|T_{\mu\nu}|0\rangle$ can be characterized by the energy density $\langle\rho\rangle = \langle 0|T_{tt}|0\rangle$ and the isotropic pressure $\langle p\rangle = \langle 0|T^i{}_i|0\rangle$. When the vacuum state is determined by the mode function (C.3), $\langle\rho\rangle$ and $\langle p\rangle$ are written as mode sums,

$$\langle\rho\rangle = \frac{1}{8\pi^2 a^4}\int d\mu(k)\left[W_k + \frac{\omega_k^2}{W_k} + \frac{1}{W_k}\left(\frac{W_k'}{2W_k} + \frac{a'}{a}\right)^2\right],$$
$$\langle p\rangle = \frac{1}{8\pi^2 a^4}\int d\mu(k)\left[W_k - \frac{\omega_k^2 + 2m^2a^2}{3W_k} + \frac{1}{W_k}\left(\frac{W_k'}{2W_k} + \frac{a'}{a}\right)^2\right].$$

Note that although the mode sums $\int d\mu(k)$ both for $\langle\rho\rangle$ and $\langle p\rangle$ are divergent, these expectation values obey the conservation law $\nabla_\mu T^\mu{}_\nu = 0$, which now reads

$$0 = \langle\rho\rangle' + \frac{3a'}{a}\left(\langle\rho\rangle + \langle p\rangle\right). \tag{C.8}$$

This equality can be verified by observing the following formal differentiation:

$$\langle\rho\rangle' + \frac{3a'}{a}\left(\langle\rho\rangle + \langle p\rangle\right) = \int \frac{d\mu(k)}{8\pi^2 a^4}\frac{(a^2W_k)'}{a^2W_k}\left[\frac{W_k''}{2W_k} + W_k^2 - \frac{3}{4}\left(\frac{W_k'}{W_k}\right)^2 - \omega_k^2 + \frac{a''}{a}\right]. \tag{C.9}$$

The expression in the bracket is precisely the equation of motion (C.4). This result shows that although $\langle\rho\rangle$ and $\langle p\rangle$ are divergent if we integrate arbitrarily high momentum modes, they satisfy the conservation law as long as the mode function is a solution to the equation of motion. Also, if we expand W_k in terms of a certain parameter and the equation of motion is satisfied by each order, then the conservation law holds order by order, which in turn implies that any subtraction of the terms with a specific order both from $\langle\rho\rangle$ and $\langle p\rangle$ doesn't break the conservation law. In particular, if the expansion parameter is chosen so that the expansion is identical to the $1/k$ expansion in the large-k limit, then by subtracting the divergent terms from $\langle\rho\rangle$ and $\langle p\rangle$ we can construct a UV-finite expression for these expectation values that satisfy the conservation law. This procedure is called the adiabatic regularization of the energy-momentum tensor.

Now we specialize to the WKB expansion (C.5). The adiabatic expansion in terms of the number of derivatives for the "bare" expectation values $\langle\rho\rangle$ and $\langle p\rangle$ are given by

$$\langle\rho\rangle = \langle\rho^{(0)}\rangle + \langle\rho^{(2)}\rangle + \langle\rho^{(4)}\rangle + \cdots, \tag{C.10}$$
$$\langle p\rangle = \langle p^{(0)}\rangle + \langle p^{(2)}\rangle + \langle p^{(4)}\rangle + \cdots, \tag{C.11}$$

where

$$\langle\rho^{(0)}\rangle = \frac{1}{4\pi^2a^4}\int d\mu(k)\omega_k, \tag{C.12}$$
$$\langle\rho^{(2)}\rangle = \frac{1}{4\pi^2a^4}\int \frac{d\mu(k)}{2\omega_k}\left(\frac{a'}{a} + \frac{\omega_k'}{2\omega_k}\right)^2, \tag{C.13}$$
$$\langle\rho^{(4)}\rangle = \frac{1}{4\pi^2a^4}\int \frac{d\mu(k)}{2\omega_k}\left[\left(\omega_k W_k^{(2)}\right)^2 - W_k^{(2)}\left(\frac{a'}{a} + \frac{\omega_k'}{2\omega_k}\right)^2 + W_k^{(2)\prime}\left(\frac{a'}{a} + \frac{\omega_k'}{2\omega_k}\right)\right], \tag{C.14}$$

and

$$\langle p^{(0)}\rangle = \frac{1}{4\pi^2a^4}\int d\mu(k)\frac{\omega_k}{3}\left(1 - \frac{m^2a^2}{\omega_k^2}\right), \tag{C.15}$$
$$\langle p^{(2)}\rangle = \frac{1}{4\pi^2a^4}\int \frac{d\mu(k)}{2\omega_k}\left[\frac{2\omega_k^2}{3}\left(2 + \frac{m^2a^2}{\omega_k^2}\right)W_k^{(2)} + \left(\frac{a'}{a} + \frac{\omega_k'}{2\omega_k}\right)^2\right], \tag{C.16}$$
$$\begin{aligned}\langle p^{(4)}\rangle = \frac{1}{4\pi^2a^4}\int \frac{d\mu(k)}{2\omega_k}&\left[\frac{2\omega_k^2}{3}\left(2 + \frac{m^2a^2}{\omega_k^2}\right)W_k^{(4)} - \frac{\omega_k^2}{3}\left(1 + \frac{2m^2a^2}{\omega_k^2}\right)\left(W_k^{(2)}\right)^2\right.\\ &\left.-W_k^{(2)}\left(\frac{a'}{a} + \frac{\omega_k'}{2\omega_k}\right)^2 + W_k^{(2)\prime}\left(\frac{a'}{a} + \frac{\omega_k'}{2\omega_k}\right)\right].\end{aligned} \tag{C.17}$$

By simple order counting we see that the terms of higher adiabatic order than that are shown above are UV-finite.

Note that the 4-th order terms have logarithmic singularities when we take the massless limit $m \to 0$ in the flat case ($K = 0$). These singularities are manifestation of the breakdown of the WKB expansion for the zero mode ($k = 0$) in the massless case, i.e. $\omega_k = 0$ at $k = m = 0$. Thus, we introduce both UV and IR cutoff parameter Λ and μ respectively. With this regularization, we denote the expectation values with the subscript reg. The adiabatic expansion of energy density become

$$\langle \rho^{(0)} \rangle_{\rm reg} = \frac{1}{32\pi^2 a^4} \left[\Lambda (2\omega_\Lambda^2 - m^2 a^2)\omega_\Lambda - \mu(2\omega_\mu^2 - m^2 a^2)\omega_\mu - m^4 a^4 \log\left(\frac{\Lambda + \omega_\Lambda}{\mu + \omega_\mu}\right)\right], \tag{C.18}$$

$$\begin{aligned}\langle \rho^{(2)} \rangle_{\rm reg} = \frac{1}{16\pi^2 a^4} \left(\frac{a'}{a}\right)^2 \Bigg[& \Lambda\omega_\Lambda - \mu\omega_\mu - 2m^2 a^2 \left(\frac{\Lambda}{\omega_\Lambda} - \frac{\mu}{\omega_\mu}\right) \\ & + \frac{m^2 a^2}{6}\left(\frac{\Lambda^3}{\omega_\Lambda^3} - \frac{\mu^3}{\omega_\mu^3}\right) + \frac{m^2 a^2}{2} \log\left(\frac{\Lambda + \omega_\Lambda}{\mu + \omega_\mu}\right)\Bigg],\end{aligned} \tag{C.19}$$

$$\begin{aligned}\langle \rho^{(4)} \rangle_{\rm reg} = \frac{1}{32\pi^2 a^4} \Bigg[& \left\{ \left(\frac{a''}{a}\right)^2 - 2\frac{a'}{a}\frac{a'''}{a} + 4\left(\frac{a'}{a}\right)^2 \frac{a''}{a}\right\} \left\{ \log(k + \omega_k) - \frac{k}{\omega_k}\right\} \\ & + \frac{k^3}{3\omega_k^3}\left\{ \left(\frac{a'}{a}\right)^4 + 6\left(\frac{a'}{a}\right)^2 \frac{a''}{a} - 2\frac{a'}{a}\frac{a'''}{a} + \left(\frac{a''}{a}\right)^2 \right\} \\ & + \frac{k^3}{30\omega_k^3}\left(1 + \frac{3m^2a^2}{2\omega_k^2}\right)\left\{ 31\left(\frac{a'}{a}\right)^4 + 36\left(\frac{a'}{a}\right)^2\frac{a''}{a} + \left(\frac{a''}{a}\right)^2 - 2\frac{a'}{a}\frac{a'''}{a}\right\} \\ & + \frac{k^3}{15\omega_k^3}\left(1 + \frac{3m^2a^2}{2\omega_k^2} + \frac{15m^4a^4}{8\omega_k^4}\right)\left\{ \left(\frac{a'}{a}\right)^2 \frac{a''}{a} - \left(\frac{a'}{a}\right)^4\right\} \\ & - \frac{k^3}{12\omega_k^3}\left(\frac{a'}{a}\right)^4 \left(1 + \frac{3m^2a^2}{2\omega_k^2} + \frac{15m^4a^4}{8\omega_k^4} + \frac{35m^6a^6}{16\omega_k^6}\right)\Bigg]_{k=\mu}^{k=\Lambda}.\end{aligned} \tag{C.20}$$

In the massless limit ($m \to 0$) keeping the cutoff finite, these expectation values reduce to

$$\langle \rho^{(0)} \rangle_{\rm reg} \to \frac{1}{16\pi^2 a^4}\left(\Lambda^4 - \mu^4\right), \tag{C.21}$$

$$\langle \rho^{(2)} \rangle_{\rm reg} \to \frac{1}{16\pi^2 a^4}\left(\frac{a'}{a}\right)^2 \left(\Lambda^2 - \mu^2\right), \tag{C.22}$$

$$\langle \rho^{(4)} \rangle_{\rm reg} \to \frac{1}{32\pi^2 a^4}\left[\left\{ \left(\frac{a''}{a}\right)^2 - 2\frac{a'}{a}\frac{a'''}{a} + 4\left(\frac{a'}{a}\right)^2\frac{a''}{a}\right\} \log\left(\frac{\Lambda}{\mu}\right)\right]. \tag{C.23}$$

Although we don't discuss in detail in this thesis, it should be noted that the aidabatic regularization is in fact a specific description of the renormalization. For this point of view, see [41] for example.

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