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G-TUTTE POLYNOMIALS VIA COMBINATORICS, TOPOLOGY AND MATROID THEORY

(G-TUTTE多項式の組み合わせ論的、位相的、マトロイド理論的研究)

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ABSTRACT

We introduce and study the notion of G-Tutte polynomial for a list of elements in a finitely generated abelian group and an abelian group G through combinatorial, topological and matroid theoretical aspects. The G-Tutte polynomial establishes a common generalization of several "Tutte-like" polynomials appearing in the literature such as the (arithmetic) Tutte polynomial of realizable (arithmetic) matroid, the characteristic quasi-polynomial of integral arrangement, the Brändén-Moci's arithmetic version of the partition function of an abelian group-valued Potts model, and the modified Tutte-Krushkal-Renhardy polynomial of a finite CW-complex.

Through combinatorial viewpoint, we generalize the characteristic polynomials of hyperplane and toric arrangements to that of abelian Lie group arrangements and in turn give two arrangement theoretic interpretations for every constituent of the chromatic quasi-polynomial. Passing from general to particular consideration, we give several results on the characteristic quasi-polynomials of arrangements arising from root systems in connection with Ehrhart theory, Eulerian polynomial and signed graph. From topological viewpoint, we prove that the semialgebraic and topological Euler characteristics and Poincaré polynomial of a certain abelian Lie group arrangement can be expressed in terms of the associated G-Tutte polynomial, which generalizes many classical formulas. From matroid theoretical viewpoint, we prove that the G-Tutte polynomial, like many of its specializations, possesses deletion-contraction and convolution formulas, but unlike them, the G-Tutte polynomial may have negative coefficients. We propose some ideas and partial answers for finding under what conditions the G-Tutte polynomial has positive coefficients.

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INTRODUCTION

Background

The Tutte polynomial, due to Tutte (e.g., [Tut54]), is one of the most-loved invariants of graphs, and surely the most studied. This two-variable polynomial encodes a substantial amount of the combinatorial information of a graph, and specializes to several important graph polynomials (including the chromatic, flow and reliability polynomials). Significant features of the Tutte polynomial have also been shown in diverse areas of mathematics/sciences, for instances, it appears as the Jones and homfly polynomials in knot theory (e.g., [Jae88]), as the Ising and Potts model partition functions in statistical mechanics (e.g., [FK72]), and especially as the polynomial invariant of many objects in arrangement theory which is the main topic of discussion in this thesis.

From the arrangement theoretical viewpoint, the Tutte polynomial is unquestionably important because of the pervasiveness of its extensions from graphs to other objects that have richer combinatorial and topological properties. These extensions find applications to three broad types of arrangements. A hyperplane arrangement is a finite collection of 1-codimensional subspaces in a vector space, which is one of the most classical and appreciated types in the theory (e.g., [OT92]). Every undirected graph gives rise to a matroid, and one can define for any matroid a Tutte polynomial. To a given hyperplane arrangement $\mathcal{H}$, there is a matroid naturally associated whose Tutte polynomial specializes to the characteristic polynomial $\chi_{\mathcal{H}}(t)$, the fundamental invariant carrying the combinatorial and topological information of $\mathcal{H}$. Beyond hyperplane arrangements, many attempts were made in order to deal with arrangements of submanifolds inside a manifold. For example, arrangements of 1-codimensional subtori in a torus, or toric arrangements due to De Concini-

Procesi (e.g., [DCP05]). Toric arrangements have generated increasing interest recently. Among the others, the matroids get generalized to arithmetic matroids in pursuit of toric arrangements, with the arithmetic Tutte polynomials answering to the Tutte polynomials (see [Moc12], [DM13]). In yet another consideration, a finite list $\mathcal{A}$ of vectors in $\mathbb{Z}^\ell$ determines a hyperplane arrangement $\mathcal{A}(\mathbb{R})$, a toric arrangement $\mathcal{A}(\mathbb{S}^1)$, and especially a q-reduced arrangement $\mathcal{A}(\mathbb{Z}_q)$ of subgroups in the finite abelian group $\mathbb{Z}_q^\ell$ (e.g., [KTT08]). Evaluating the cardinality of the complement of $\mathcal{A}(\mathbb{Z}_q)$ produces a quasi-polynomial invariant, the characteristic quasi-polynomial $\chi_{\mathcal{A}}^{\text{quasi}}(q)$. The name is justified by the fact that the first constituent of this quasi-polynomial is identical with the characteristic polynomial $\chi_{\mathcal{A}(\mathbb{R})}(t)$ of $\mathcal{A}(\mathbb{R})$. Furthermore, a coincidence has been observed from a large number of particular calculations (e.g., [Sut98], [KTT07], [ACH15]), that the last constituent of $\chi_{\mathcal{A}}^{\text{quasi}}(q)$ coincides with the characteristic polynomial $\chi_{\mathcal{A}(\mathbb{S}^1)}(t)$ of $\mathcal{A}(\mathbb{S}^1)$.

Thus we arrive at the natural and essential problem that how we can build a general framework to study the arrangements and their Tutte-like polynomials en masse, rather than individually. More precisely, we are looking for a "useful" framework that (i) unifies concepts from the literature, and (ii) explains the "coincidences" among them. The problem will be given an answer in this thesis.

Objective

The key observation that enables us to build such a framework is that the arrangements and their polynomials mentioned above all are defined by means of counting homomorphisms between abelian groups. As an answer to the problem, we propose the notion of G-plexification defined for a finite list $\mathcal{A}$ of elements in a finitely generated abelian group Γ, and a torsion-wise finite abelian group G, with the associated G-Tutte polynomial. The notion of (F,p,q)-arrangement (i.e., G-plexification with $G = F \times (\mathbb{S}^1)^p \times \mathbb{R}^q$, $p, q \geq 0$, F is a finite abelian group) provides important examples of abelian Lie group arrangements and is a unification of a great number of concepts.

In particular, when we specify $\Gamma = \mathbb{Z}^\ell$, and let G be $\mathbb{R}$, $\mathbb{S}^1$, or $\mathbb{Z}_q$, we obtain the G-plexification as the hyperplane, toric or q-reduced arrangement, respectively, and the G-Tutte polynomial as the Tutte polynomial, arithmetic Tutte polynomial, or (a generalization of) characteristic quasi-polynomial, respectively.

Having introduced the notion of G-Tutte polynomial, it is again natural and essential to ask for what properties of the Tutte, arithmetic Tutte polynomials and characteristic quasi-polynomial, there is an analogous result related to the G-Tutte polynomial. It turns out that there are many properties that are preserved, but some properties are not. In the present thesis, we study these properties through three aspects: combinatorics, topology and matroid theory.

Organization

The structure of the thesis is as follows:

- In Chapter 1, we present a general theory of the G-Tutte polynomials.

  In §1.1, we recall the definitions of typical arrangements and the associated Tutte-like polynomials and their combinatorial and topological properties appearing in the literature: hyperplane arrangement and Tutte polynomial, toric arrangement and arithmetic Tutte polynomial, q-reduced arrangement and characteristic quasi-polynomial.

  In §1.2, we define the concepts of G-plexifications of finitely generated abelian groups and G-Tutte polynomials over torsion-wise finite abelian groups. We prove that the G-plexifications and G-Tutte polynomials possess deletion-contraction formulas (Proposition 1.2.1.9 and Corollary 1.2.2.13). Aside from the mentioned specializations, we further give some other "unexpected" specializations of the G-Tutte polynomial (Examples 1.2.2.20 and 1.2.2.21).

  The results of this chapter are found in [LTY], a joint work with Y. Liu and M. Yoshinaga.

- In Chapter 2, we study the G-Tutte polynomials via combinatorics.

  In §2.1, we define two different intersection posets of a (F, p, q)-arrangement and prove that the corresponding characteristic polynomials can be expressed in terms of the G-characteristic polynomials (Theorem 2.1.2.7 or Corollary 2.1.2.8). The results of this section are found in [TY19], a joint work with M. Yoshinaga.

  In §2.2, we explain two "coincidences": the first one is the equivalence between the chromatic quasi-polynomial and the Chen-Wang's quasi-polynomial (Theorem 2.2.1.5), the second one is the identity between the last constituent of the chromatic quasi-polynomial and the characteristic polynomial of the generalized toric arrangement (Corollary 2.2.1.9) which was mentioned in **Motivation**. The results of this section are found in [Tra18] and [LTY].

  In §2.3, we give two natural but nontrivial interpretations for every constituent of the chromatic quasi-polynomial via subspace and toric viewpoints (Corollaries 2.3.1.1 and 2.3.2.5). The results of this section are found in [TY19].

  In §2.4, we study the characteristic quasi-polynomials of the arrangements arising from irreducible root systems. We introduce the notion of $\mathcal{A}$-Eulerian polynomial and prove that this polynomial together with shift operator express the characteristic quasi-polynomial of certain Weyl subarrangements in terms of the Ehrhart quasi-polynomial of the fundamental alcove (Definition 2.4.3.2 and Theorem 2.4.3.11). Finally, we give a computational result on the characteristic quasi-polynomials of ideals of classical root systems with respect to the integer and root lattices via a connection with signed graphs (§2.4.5). The results of this section are found in [ATY19], a joint work with A. U. Ashraf and M. Yoshinaga and [Tra19].

- In Chapter 3, we study the G-Tutte polynomials via topology.

  In §3.1, we show that the topological and semialgebraic Euler characteristics of the complement of any (F, p, q)-arrangement can obtained as an evaluation of the associated G-characteristic polynomial (Theorem 3.1.2.2).

In §3.2, we prove that the Poincaré polynomial of any non-compact (F, p, q)-arrangement (i.e., $q > 0$) can be expressed in terms of the associated G-characteristic polynomial (Theorem 3.2.2.2). We emphasize that the non-compactness plays a crucial role in our proof, without it many arguments may not work (Remark 3.2.2.4 and Example 3.2.2.5).

The results of this chapter are found in [LTY].

- In Chapter 4, we study the G-Tutte polynomials via topology.

  In §4.1, we show that the G-multiplicities satisfy only four over five arithmetic matroid axioms, this is the first property of the arithmetic Tutte polynomial that the G-Tutte polynomial does not preserve (Theorem 4.1.1.13 and Remark 4.1.1.14). However, the G-Tutte polynomial does possess a convolution formula (Theorem 4.1.2.1).

  In §4.2, we show another "non-preserving" property of the G-Tutte polynomial, that is the G-Tutte polynomial may have negative coefficient (Example 4.2.1.1). We give some ideas and partial answers to a question that under what conditions the coefficients of the G-Tutte polynomial are positive (Theorem 4.2.1.2 and Propositions 4.2.1.3, 4.2.2.2).

  The results of this chapter are found in [LTY].

TABLE OF CONTENTS

1. A GENERAL THEORY OF G-TUTTE POLYNOMIALS

1.1 Known arrangements and their Tutte-like polynomials

In this section, we recall a setting that has been used to define and study many types of arrangements, including hyperplane, toric and q-reduced arrangements. associated with each arrangement, there is a polynomial often named after Tutte which encodes combinatorial, topological and enumerative information of the arrangement. We will see that the polynomials arise from very similar manners and share many common properties, giving us an evidence that there is a common framework to unify them and suitable to the mentioned setting.

In this thesis, the term *list* is synonymous with multiset. For example, the list $\mathcal{A} = \{\alpha, \alpha\}$ has 4 distinct sublists: $\mathcal{S}_1 = \emptyset, \mathcal{S}_2 = \{\alpha\}, \mathcal{S}_3 = \{\alpha\}, \mathcal{S}_4 = \{\alpha, \alpha\} = \mathcal{A}$. We distinguish $\mathcal{S}_2$ and $\mathcal{S}_3$, and hence $\mathcal{A} \smallsetminus \mathcal{S}_2 = \mathcal{S}_3$. If $\mathcal{A}$ is a list, then $\mathcal{S} \subseteq \mathcal{A}$ indicates that $\mathcal{S}$ is a sublist of $\mathcal{A}$.

1.1.1 Hyperplane, toric and q-reduced arrangements

Let $\Gamma := \bigoplus_{i=1}^{\ell} \mathbb{Z}\beta_i \simeq \mathbb{Z}^\ell$ be a free abelian group. Let $\mathcal{A}$ be a finite list (multiset) of elements in Γ. Let $(G, +)$ be an abelian group with the unit $e \in G$. For $\alpha = \sum_{i=1}^{\ell} a_i\beta_i \in \mathcal{A}$ and $m_\alpha \in G$, define a subset $H_{\alpha,m_\alpha,G}$ of G^ℓ by

$$H_{\alpha,m_\alpha,G} := \left\{ \mathbf{z} \in G^\ell \mid \sum_{i=1}^{\ell} a_i z_i \equiv m_\alpha \right\}.$$

Note that $H_{\alpha,m_\alpha,G}$ is a subgroup G^ℓ when $m_\alpha = e$. Given a vector $m = (m_\alpha)_{\alpha\in\mathcal{A}} \in G^{\mathcal{A}}$, the *arrangement of* $(\mathcal{A}, m)$ *with respect to* Γ is defined by

$$(\mathcal{A}, m)(G) := \{H_{\alpha,m_\alpha,G} \mid \alpha \in \mathcal{A}\}.$$

The *complement* of $(\mathcal{A}, m)(G)$ is defined by

$$\mathcal{M}((\mathcal{A}, m); \Gamma, G) := G^\ell \setminus \bigcup_{\alpha \in \mathcal{A}} H_{\alpha, m_\alpha, G}.$$

The arrangement $(\mathcal{A}, m)(G)$ is said to be *central* if $m_\alpha = e$ for all $\alpha \in \mathcal{A}$. In this case, we simply write $\mathcal{A}$ and $H_{\alpha,G}$ instead of $(\mathcal{A}, m)$ and $H_{\alpha,e,G}$, respectively.

The arrangement $(\mathcal{A}, m)(G)$ and its complement $\mathcal{M}((\mathcal{A}, m); \Gamma, G)$ are important objects of study in many contexts. We list some typical examples appearing in the literature.

(i) When G is a field (e.g., $G = \mathbb{C}, \mathbb{R}, \mathbb{F}_q$), $\mathcal{A}(G)$ is called the *(affine) hyperplane (or integral)* arrangement of $\mathcal{A}$ w.r.t. Γ (e.g., [OT92]).

(ii) When G is $\mathbb{C}^\times$ or $\mathbb{S}^1$, $\mathcal{A}(G)$ is called the *(affine) toric* arrangement of $\mathcal{A}$ w.r.t. Γ (e.g., [DCP05, Moc12]).

(iii) When G is a finite cyclic group $\mathbb{Z}_q := \mathbb{Z}/q\mathbb{Z}$, $\mathcal{A}(G)$ is called the *(affine) q-reduced* arrangement of $\mathcal{A}$ w.r.t. Γ (e.g., [KTT08, KTT11]).

(iv) When $G = \mathbb{R}^c$ with $c > 0$, $m_\alpha = e$ for all $\alpha \in \mathcal{A}$, $\mathcal{A}(G)$ is called the *c-plexification* of $\mathcal{A}$ w.r.t. Γ (e.g., [Bjö94, §5.2]).

(v) When $G = \mathbb{S}^1 \times \mathbb{S}^1$ (viewed as an elliptic curve), $m_\alpha = e$ for all $\alpha \in \mathcal{A}$, $\mathcal{A}(G)$ is called the *elliptic (or abelian)* arrangement of $\mathcal{A}$ w.r.t. Γ (e.g., [Bib16]).

In general, many properties of the arrangements depend on the choice of the free abelian group Γ (e.g., see Proposition 1.2.2.22, §2.4.5).

1.1.2 (Arithmetic) Tutte and characteristic (quasi-)polynomials

For $\mathcal{S} \subseteq \mathcal{A}$, let $M_\mathcal{S}$ denote the matrix of size $\#\mathcal{S} \times \ell$ whose each column is represented by $\alpha \in \mathcal{S}$. Denote by $r_\mathcal{S}$ the *rank* of $M_\mathcal{S}$, in other words, $r_\mathcal{S}$ is the rank of the subgroup $\langle \mathcal{S} \rangle \leq \Gamma$ generated by $\mathcal{S}$. The list $\mathcal{A}$ gives rise to a *realizable* *(or representable) matroid* represented by $M_{\mathcal{A}}$, and the (classical) *Tutte polynomial* $T_{\mathcal{A}}(x,y)$ of $\mathcal{A}$ (e.g., see §4.1.1) is defined by

$$T_{\mathcal{A}}(x,y) := \sum_{\mathcal{S}\subseteq\mathcal{A}} (x-1)^{r_{\mathcal{A}}-r_{\mathcal{S}}}(y-1)^{\#\mathcal{S}-r_{\mathcal{S}}}.$$

Let $d_{\mathcal{S},1},\dots,d_{\mathcal{S},r_{\mathcal{S}}}$ be the invariant factors of $M_{\mathcal{S}}$. Thus $1 \le d_{\mathcal{S},i}$ is a positive integer and $d_{\mathcal{S},i}$ divides $d_{\mathcal{S},i+1}$ $(1 \le i < r_{\mathcal{S}})$. The *(arithmetic) multiplicity* $m(\mathcal{S})$ of $\mathcal{S}$ is defined to be $\prod_{i=1}^{r_{\mathcal{S}}} d_{\mathcal{S},i}$. In other words, $m(\mathcal{S})$ equals the cardinality of the torsion subgroup of $\mathbb{Z}^{\ell}$ quotient by the subgroup generated by the column vectors of $M_{\mathcal{S}}$. The *arithmetic Tutte polynomial* $T^{\text{arith}}_{\mathcal{A}}(x,y)$ of $\mathcal{A}$ is defined in [Moc12] as follows:

$$T^{\text{arith}}_{\mathcal{A}}(x,y) := \sum_{\mathcal{S}\subseteq\mathcal{A}} m(\mathcal{S})(x-1)^{r_{\mathcal{A}}-r_{\mathcal{S}}}(y-1)^{\#\mathcal{S}-r_{\mathcal{S}}}.$$

Remark 1.1.2.1. The definitions above make sense in a more general setting, namely the free abelian group Γ is being replaced by any finitely generated abelian group in a natural way. This is what Moci actually used to define the arithmetic Tutte polynomial in [Moc12]. We use the same names, Tutte and arithmetic Tutte polynomials, to call the polynomials in this setting.

It should be also noted that the above polynomials can be defined for more general objects: matroids, arithmetic matroids [DM13] (see §4.1.1 for more details) and matroids over $\mathbb{Z}$ [FM16]. From an arrangement theoretical viewpoint, we will introduce the notion of *G-Tutte polynomials* in §1.2.2, giving another way for generalizing those polynomials. Now let us restrict our discussion to the *central* arrangements, although some of the results (e.g., those related to characteristic quasi-polynomials) hold true for the non-central ones. The polynomials mentioned above encode combinatorial and topological information of the arrangements. The *characteristic polynomial* of the ranked poset of flats (e.g., [OT92, Definition 2.52]) is often known as "the combinatorics" of a hyperplane arrangement. For instance, Whitney showed that the characteristic polynomial $\chi_{\mathcal{A}(\mathbb{R})}(t)$ of the hyperplane arrangement $\mathcal{A}(\mathbb{R})$ is given by (e.g., [Sta07, Theorem 2.4])

$$\chi_{\mathcal{A}(\mathbb{R})}(t) = (-1)^{r_{\mathcal{A}}} t^{\ell-r_{\mathcal{A}}} T_{\mathcal{A}}(1-t,0). \tag{1.1.1}$$

On the other hand, "the topology" of a hyperplane arrangement is often referred to the *Poincaré polynomial* of its complement. For instance, the Poincaré polynomial of $\mathcal{M}(\mathcal{A};\mathbb{Z}^\ell,\mathbb{R}^c)$ is given by (e.g., [GM88, Bjö94])

$$P_{\mathcal{M}(\mathcal{A};\mathbb{Z}^\ell,\mathbb{R}^c)}(t) = t^{r_{\mathcal{A}}\cdot(c-1)} \cdot T_{\mathcal{A}}\left(\frac{1+t}{t^{c-1}}, 0\right). \tag{1.1.2}$$

Note that the special cases $c = 1$ and $c = 2$ correspond to the famous formulas by Zaslavsky [Zas75] and Orlik-Solomon [OS80], respectively. Similarly, as proved by De Concini-Procesi [DCP05] and Moci [Moc12], the characteristic polynomial $\chi_{\mathcal{A}(\mathbb{S}^1)}(t)$ of the poset of layers (see, e.g., [Moc12, §5], §2.1.1) of the toric arrangement $\mathcal{A}(\mathbb{S}^1)$ is

$$\chi_{\mathcal{A}(\mathbb{S}^1)}(t) = (-1)^{r_{\mathcal{A}}} t^{\ell-r_{\mathcal{A}}} T^{\text{arith}}_{\mathcal{A}}(1-t, 0),$$

and the Poincaré polynomial of $\mathcal{M}(\mathcal{A};\mathbb{Z}^\ell,\mathbb{C}^\times)$ is

$$P_{\mathcal{M}(\mathcal{A};\mathbb{Z}^\ell,\mathbb{C}^\times)}(t) = (1+t)^{\ell-r_{\mathcal{A}}} \cdot t^{r_{\mathcal{A}}} \cdot T^{\text{arith}}_{\mathcal{A}}\left(\frac{1+2t}{t}, 0\right). \tag{1.1.3}$$

When G is an arbitrary finite cyclic group $\mathbb{Z}_q$ ($q \in \mathbb{Z}_{>0}$), the dimension no longer makes sense. However, $\mathcal{M}(\mathcal{A};\mathbb{Z}^\ell,\mathbb{Z}_q)$ still contains a number of interesting properties. Kamiya-Takemura-Terao [KTT08, KTT11] proved that $\#\mathcal{M}(\mathcal{A};\mathbb{Z}^\ell,\mathbb{Z}_q)$ is a quasi-polynomial in q, denoted by $\chi^{\text{quasi}}_{\mathcal{A}}(q)$, with period

$$\rho_{\mathcal{A}} := \operatorname{lcm}(d_{\mathcal{S},r_{\mathcal{S}}} \mid \mathcal{S} \subseteq \mathcal{A}). \tag{1.1.4}$$

This means that there exist polynomials $f^k_{\mathcal{A}}(t) \in \mathbb{Z}[t]$ $(1 \le k \le \rho_{\mathcal{A}})$ such that for any $q \in \mathbb{Z}_{>0}$ with $q \equiv k \bmod \rho_{\mathcal{A}}$,

$$\chi^{\text{quasi}}_{\mathcal{A}}(q) = f^k_{\mathcal{A}}(q).$$

The polynomial $f^k_{\mathcal{A}}(t)$ is called the k-constituent of $\chi^{\text{quasi}}_{\mathcal{A}}(q)$. The authors called the quasi-polynomial $\chi^{\text{quasi}}_{\mathcal{A}}(q)$ the *characteristic quasi-polynomial* of $\mathcal{A}$ as a result of the fact that the first constituent $f^1_{\mathcal{A}}(t)$ equals the characteristic polynomial $\chi_{\mathcal{A}(\mathbb{R})}(t)$ (e.g., [Ath96], [KTT08]). Furthermore, we will prove in Corollary 2.2.1.9 that the last constituent $f^{\rho_{\mathcal{A}}}_{\mathcal{A}}(t)$ coincides with the characteristic polynomial $\chi_{\mathcal{A}(\mathbb{S}^1)}(t)$. The mentioning properties open a new direction for studying the combinatorics and topology of hyperplane and toric arrangements in one single quasi-polynomial. The characteristic quasi-polynomial also has many enumerative properties related to Ehrhart theory, which we will in discuss in §2.4.

1.2 G-plexifications and G-Tutte polynomials

We observe that the concept of the arrangement $(\mathcal{A}, m)(G)$ in §1.1.1 can be redefined in a more general and abstract setting by viewing Γ as a finitely generated abelian group (e.g., in [Moc12, §5] where G is $\mathbb{S}^1$ or $\mathbb{C}^\times$). We shall call this process the *G-plexification*. Moreover, associated with a G-plexification, we hope to find a suitable notion of Tutte polynomial in a similar manner to how the (arithmetic) Tutte polynomials were defined. As also pointed out by Moci [Moc12, §3.2], working with finitely generated abelian groups is useful and somewhat essential so that the wanted Tutte polynomial satisfy a deletion-contraction formula as the classical Tutte polynomial does. For the sake of convenience, in this thesis, we choose to develop the theory almost only on the *central* arrangements, and leave the affine case for future research (some results on the non-central case will be mentioned in §2.4).

1.2.1 G-plexifications of finitely generated abelian groups

Let $(G, +)$ be an abelian group with the unit $e \in G$. Let Γ be a finitely generated abelian group, and let $\mathcal{A} \subseteq \Gamma$ be a finite list (multiset) of elements in Γ. Following the procedure in §1.1.1, we now define the "arrangement" associated with the list $\mathcal{A}$ over G.

The total space is $\mathrm{Hom}(\Gamma, G)$, the abelian group of all group homomorphisms from Γ to G under point-wise addition. For each $\alpha \in \mathcal{A}$, we define the *G-hyperplane* with respect to α as follows:

$$H_{\alpha,G} := \{\varphi \in \mathrm{Hom}(\Gamma, G) \mid \varphi(\alpha) = e\}.$$

In other words, $H_{\alpha,G}$ is the kernel of the following homomorphism naturally defined by α

$$\alpha : \mathrm{Hom}(\Gamma, G) \longrightarrow G, \varphi \longmapsto \varphi(\alpha).$$

Then the *G-plexification (or G-arrangement)* $\mathcal{A}(G)$ of $\mathcal{A}$ is the collection of the subgroups $H_{\alpha,G}$

$$\mathcal{A}(G) := \{H_{\alpha,G} \mid \alpha \in \mathcal{A}\}.$$

The *G-complement* $\mathcal{M}(\mathcal{A}; \Gamma, G)$ of $\mathcal{A}(G)$ is defined by

$$\mathcal{M}(\mathcal{A}; \Gamma, G) := \mathrm{Hom}(\Gamma, G) \setminus \bigcup_{\alpha \in \mathcal{A}} H_{\alpha,G}.$$

Remark 1.2.1.1. Suppose that $\Gamma \simeq \mathbb{Z}^\ell$ and fix a basis $\{\beta_1, \ldots, \beta_\ell\}$ for Γ. Thus the arrangement $\mathcal{A}(G)$ defined in §1.1.1 is indeed a G-plexification via the isomorphism $\mathrm{Hom}(\Gamma, G) \simeq G^\ell$, $\varphi \longmapsto (\varphi(\beta_1), \ldots, \varphi(\beta_\ell))$. In particular, the hyperplane, toric, q-reduced, elliptic arrangements and c-plexification are G-plexifications.

Example 1.2.1.2. Suppose that $\Gamma = \mathbb{Z}_d$. Then

$$\mathrm{Hom}(\Gamma, G) \simeq G[d] := \{x \in G \mid d \cdot x = e\}$$

is the *subgroup of G of d-torsion points.*

Example 1.2.1.3. Suppose that $G = \mathbb{Z}_d$. Then $\#\mathcal{M}(\mathcal{A}; \Gamma, \mathbb{Z}_q)$ is proved to be a quasi-polynomial and is called the *chromatic quasi-polynomial* of $\mathcal{A}$ [BM14, §9]. Thus any characteristic quasi-polynomial (see §1.1.2) is indeed a chromatic quasi-polynomial. A period $\rho_{\mathcal{A}}$ of the chromatic quasi-polynomial will be determined in Definition 1.2.2.17 and Theorem 1.2.2.19. When no confusion arises, we use the notations $\chi^{\text{quasi}}_{\mathcal{A}}(q)$, $f^k_{\mathcal{A}}(t)$ $(1 \le k \le \rho_{\mathcal{A}})$ for both characteristic and chromatic quasi-polynomials.

Example 1.2.1.4. Let $\Gamma = \mathbb{Z} \oplus \mathbb{Z}_4$ and $G = \mathbb{C}^\times$. Then $\mathrm{Hom}(\Gamma, G) \simeq \mathbb{C}^\times \times \{\pm 1, \pm i\}$, which is a (real 2-dimensional) Lie group with 4 connected components. If $\alpha_1 = (2, 2) \in \Gamma$, then $H_{\alpha_1,G} = \{(\pm 1, \pm 1), (\pm i, \pm i)\}$ consists of 8 points. If $\alpha_2 = (0, 2) \in \Gamma$, then $H_{\alpha_2,G} = \mathbb{C}^\times \times \{\pm 1\}$ is a union of two copies of $\mathbb{C}^\times$.

For each sublist $\mathcal{S} \subseteq \mathcal{A}$, we denote by $r_{\mathcal{S}}$ the *rank* (as an abelian group) of the subgroup $\langle \mathcal{S} \rangle \leq \Gamma$ generated by $\mathcal{S}$. Additionally, r_Γ denotes the rank of Γ. Given a group K, denote by K_{tor} the torsion subgroup of K.

Definition 1.2.1.5. Following D'Adderio-Moci [DM13], we define the *deletion* $\mathcal{A} \setminus \mathcal{S}$ as a list of elements in the same group Γ, and the *contraction* $\mathcal{A}/\mathcal{S}$ as the list of cosets $\{\overline{\alpha} \mid \alpha \in \mathcal{A} \setminus \mathcal{S}\}$ in the group $\Gamma/\langle \mathcal{S} \rangle$.

Proposition 1.2.1.6. *For each $\mathcal{S} \subseteq \mathcal{A}$, set*

$$H_{\mathcal{S},G} := \bigcap_{\alpha \in \mathcal{S}} H_{\alpha,G} = \{\varphi \in \mathrm{Hom}(\Gamma, G) \mid \varphi(\alpha) = e, \forall \alpha \in \mathcal{S}\}.$$

Then

$$H_{\mathcal{S},G} \simeq \mathrm{Hom}(\Gamma/\langle \mathcal{S} \rangle, G) \simeq \mathrm{Hom}((\Gamma/\langle \mathcal{S} \rangle)_{\mathrm{tor}}, G) \times G^{r_\Gamma - r_{\mathcal{S}}}.$$

Proof. The first isomorphism follows from the exact sequence

$$0 \longrightarrow \mathrm{Hom}(\Gamma/\langle \mathcal{S} \rangle, G) \longrightarrow \mathrm{Hom}(\Gamma, G) \longrightarrow \mathrm{Hom}(\langle \mathcal{S} \rangle, G).$$

From the structure theorem for finitely generated abelian groups, we may assume that $\Gamma/\langle \mathcal{S} \rangle \simeq (\Gamma/\langle \mathcal{S} \rangle)_{\mathrm{tor}} \oplus \mathbb{Z}^{r_\Gamma - r_{\mathcal{S}}}$. The second isomorphism follows automatically. $\square$

Corollary 1.2.1.7. *As sets,*

$$\mathcal{M}(\mathcal{A}/\mathcal{S}; \Gamma/\langle \mathcal{S} \rangle, G) \simeq \left\{ \varphi \in \mathrm{Hom}(\Gamma, G) \;\middle|\; \begin{array}{ll} \varphi(\alpha) = e, & \textit{for } \alpha \in \mathcal{S} \\ \varphi(\alpha) \neq e, & \textit{for } \alpha \in \mathcal{A} \setminus \mathcal{S} \end{array} \right\}.$$

Proof. By Proposition 1.2.1.6, $\mathrm{Hom}(\Gamma/\langle \mathcal{S} \rangle, G) \simeq \{\varphi \in \mathrm{Hom}(\Gamma, G) \mid \varphi(\alpha) = e, \forall \alpha \in \mathcal{S}\}$. The rest is clear. $\square$

Proposition 1.2.1.8.

$$\mathrm{Hom}(\Gamma, G) \simeq \bigsqcup_{\mathcal{S} \subseteq \mathcal{A}} \mathcal{M}(\mathcal{A}/\mathcal{S}; \Gamma/\langle \mathcal{S} \rangle, G).$$

Proof. It is easily seen that for any $\varphi \in \mathrm{Hom}(\Gamma, G)$, $\mathcal{S} = \mathcal{A}_\varphi := \{\alpha \in \mathcal{A} \mid \varphi(\alpha) = e\}$ is the unique sublist $\mathcal{S} \subseteq \mathcal{A}$ that satisfies $\varphi \in \mathcal{M}(\mathcal{A}/\mathcal{S}; \Gamma/\langle \mathcal{S} \rangle, G)$ (here we viewed the isomorphism in Corollary 1.2.1.7 as the identification). $\square$

Fix $\alpha \in \mathcal{A}$, define $\mathcal{A}' := \mathcal{A} \setminus \{\alpha\}$, $\Gamma' := \Gamma$ and $\mathcal{A}'' := \mathcal{A}/\{\alpha\}$, $\Gamma'' := \Gamma/\langle\alpha\rangle$. We call $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ the *triple (of lists)* in Γ with the distinguished element α. By Corollary 1.2.1.7, we can consider both $\mathcal{M}(\mathcal{A}; \Gamma, G)$ and $\mathcal{M}(\mathcal{A}''; \Gamma'', G)$ as subsets of $\mathrm{Hom}(\Gamma, G)$ (they are in fact subsets of $\mathcal{M}(\mathcal{A}'; \Gamma', G)$). These three sets are related by the following set-theoretic deletion-contraction formula.

Proposition 1.2.1.9.

$$\mathcal{M}(\mathcal{A}'; \Gamma', G) \simeq \mathcal{M}(\mathcal{A}''; \Gamma'', G) \sqcup \mathcal{M}(\mathcal{A}; \Gamma, G).$$

Proof. The set $\mathcal{M}(\mathcal{A}'; \Gamma', G)$ can be decomposed as $\{\varphi \in \mathcal{M}(\mathcal{A}'; \Gamma', G) \mid \varphi(\alpha) = e\} \sqcup \{\varphi \in \mathcal{M}(\mathcal{A}'; \Gamma', G) \mid \varphi(\alpha) \neq e\}$. The first term on the right-hand side is isomorphic to $\mathcal{M}(\mathcal{A}''; \Gamma'', G)$, and the second term is equal to $\mathcal{M}(\mathcal{A}; \Gamma, G)$. $\square$

1.2.2 G-Tutte polynomials over torsion-wise finite abelian groups

Recall that Γ is a finitely generated abelian group, and $\mathcal{A}$ is a finite list of elements in Γ. Inspired by the concept of (multivariate) arithmetic Tutte polynomial by [Moc12, BM14], we define the (multivariate) G-Tutte polynomial and the G-characteristic polynomial for the list $\mathcal{A}$ and abelian group G. The main ingredient is a refinement of the multiplicity $m(\mathcal{S})$, and we require a condition on G so that the new multiplicity remains finite.

Definition 1.2.2.1. An abelian group G is said to be *torsion-wise finite* if the subgroup of d-torsion points $G[d]$ (Example 1.2.1.2) is finite for all $d > 0$.

The class of torsion-wise finite groups is closed under taking subgroups and finite direct products.

Example 1.2.2.2. The following are examples of torsion-wise finite abelian groups.

- Every torsion-free abelian group (e.g., $\{0\}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$) is torsion-wise finite.

- Every finitely generated abelian group is torsion-wise finite.

- Every subgroup of the multiplicative group $\mathbb{K}^\times$ for any field $\mathbb{K}$ is torsion-wise finite (e.g., $(\mathbb{S}^1, \times)$ and $(\mathbb{C}^\times, \times)$).

Example 1.2.2.3. $(\mathbb{Z}/2\mathbb{Z})^\infty$ is not a torsion-wise finite group.

Remark 1.2.2.4. In this thesis, we are mainly interested in the torsion-wise finite groups of the form

$$G \simeq F \times (\mathbb{S}^1)^p \times \mathbb{R}^q,$$

where F is a finite abelian group and $p, q \geq 0$. Note that every abelian Lie group G with finitely many connected components is of this form (see, e.g., [HN12, Exercise 9.3.7]). We call the associated G-plexification $\mathcal{A}(G)$ the (F, p, q)*-arrangement*. Combinatorial and topological properties of this family of G-plexifications will be described in Chapters 2 and 3.

Proposition 1.2.2.5. *Let G be a torsion-wise finite abelian group. Let F be a finite abelian group. Then* $\mathrm{Hom}(F, G)$ *is finite.*

Proof. By the structure theorem, we may assume that $F \simeq \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_k}$. Then

$$\mathrm{Hom}(F, G) \simeq G[d_1] \times \cdots \times G[d_k],$$

which is finite by definition. □

Proposition 1.2.2.6.

(1) Let G_1, G_2 and Γ be groups. Then $\mathrm{Hom}(\Gamma, G_1 \times G_2) \simeq \mathrm{Hom}(\Gamma, G_1) \times \mathrm{Hom}(\Gamma, G_2)$. *In particular, if* $\mathrm{Hom}(\Gamma, G_1 \times G_2)$ *is finite, then* $\# \mathrm{Hom}(\Gamma, G_1 \times G_2) = \# \mathrm{Hom}(\Gamma, G_1) \times \# \mathrm{Hom}(\Gamma, G_2)$.

(2) Let d_1, d_2 be positive integers. Then

$$\mathrm{Hom}(\mathbb{Z}_{d_1}, \mathbb{Z}_{d_2}) \simeq \mathrm{Hom}(\mathbb{Z}_{d_2}, \mathbb{Z}_{d_1}) \simeq \mathbb{Z}_{\gcd(d_1, d_2)}.$$

In particular, $\# \mathrm{Hom}(\mathbb{Z}_{d_1}, \mathbb{Z}_{d_2}) = \gcd(d_1, d_2)$.

Proof. Straightforward. □

Unless otherwise stated, we assume that G is torsion-wise finite.

Definition 1.2.2.7. The *G-multiplicity* $m(\mathcal{S};G)$ for each $\mathcal{S}\subseteq\mathcal{A}$ is defined by

$$m(\mathcal{S};G):=\#\operatorname{Hom}\left((\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{tor}},G\right).$$

By Proposition 1.2.2.5, it is clear that $m(\mathcal{S};G)$ is a finite number. Let us describe $m(\mathcal{S};G)$ more explicitly. Because $\Gamma/\langle\mathcal{S}\rangle$ is a finitely generated abelian group, it is isomorphic to a group of the form $\mathbb{Z}_{d_1}\oplus\cdots\oplus\mathbb{Z}_{d_r}\oplus\mathbb{Z}^{r_\Gamma-r_\mathcal{S}}$. Thus $(\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{tor}}\simeq\bigoplus_{i=1}^{r}\mathbb{Z}_{d_i}$, and

$$\operatorname{Hom}\left((\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{tor}},G\right)\simeq\bigoplus_{i=1}^{r}G[d_i].$$

Therefore, $m(\mathcal{S};G)=\prod_{i=1}^{r}\#G[d_i]$.

Remark 1.2.2.8. It is also easily seen that $\operatorname{Hom}\left((\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{tor}},G\right)$ is (non-canonically) isomorphic to $\operatorname{Tor}_1^{\mathbb{Z}}(\Gamma/\langle\mathcal{S}\rangle,G)$. Hence $m(\mathcal{S};G)=\#\operatorname{Tor}_1^{\mathbb{Z}}(\Gamma/\langle\mathcal{S}\rangle,G)$.

Definition 1.2.2.9. Assume that $\mathcal{A}=\{\alpha_1,\dots,\alpha_n\}$ is a finite list of elements in a finitely generated abelian group Γ, and G a torsion-wise finite group. Recall that r_Γ and $r_\mathcal{S}$ denote the ranks of Γ and $\langle\mathcal{S}\rangle$, respectively.

(1) The *multivariate G-Tutte polynomial* $Z_\mathcal{A}^G(q,v_1,\dots,v_n)$ of $\mathcal{A}$ is defined by

$$Z_\mathcal{A}^G(q,v_1,\dots,v_n):=\sum_{\mathcal{S}\subseteq\mathcal{A}}m(\mathcal{S};G)q^{-r_\mathcal{S}}\prod_{\alpha_i\in\mathcal{S}}v_i.$$

(2) The *G-Tutte polynomial* $T_\mathcal{A}^G(x,y)$ of $\mathcal{A}$ is defined by

$$T_\mathcal{A}^G(x,y):=\sum_{\mathcal{S}\subseteq\mathcal{A}}m(\mathcal{S};G)(x-1)^{r_\mathcal{A}-r_\mathcal{S}}(y-1)^{\#\mathcal{S}-r_\mathcal{S}}.$$

(3) The *G-characteristic polynomial* $\chi_\mathcal{A}^G(t)$ of $\mathcal{A}$ is defined by

$$\chi_\mathcal{A}^G(t):=\sum_{\mathcal{S}\subseteq\mathcal{A}}(-1)^{\#\mathcal{S}}m(\mathcal{S};G)\cdot t^{r_\Gamma-r_\mathcal{S}}.$$

Proposition 1.2.2.10. *The leading coefficient of $\chi_\mathcal{A}^G(t)$ equals $\#\mathcal{M}(\mathcal{A}\cap\Gamma_{\mathrm{tor}};\Gamma_{\mathrm{tor}},G)$.*

Proof. Straightforward. $\square$

Like the (multivariate) Tutte and arithmetic Tutte polynomials [BM14, Moc12, Sok05], three polynomials above are related by the following formulas:

$$T^G_{\mathcal{A}}(x,y) = (x-1)^{r_{\mathcal{A}}} \cdot Z^G_{\mathcal{A}}((x-1)(y-1), y-1, \ldots, y-1), \tag{1.2.1}$$

$$\chi^G_{\mathcal{A}}(t) = (-1)^{r_{\mathcal{A}}} \cdot t^{r_\Gamma - r_{\mathcal{A}}} \cdot T^G_{\mathcal{A}}(1-t, 0). \tag{1.2.2}$$

Definition 1.2.2.11. Following [DM13, §4.4], we call $\alpha \in \mathcal{A}$ a *loop* (resp., *coloop*) if $\alpha \in \Gamma_{\mathrm{tor}}$ (resp., $r_{\mathcal{A}} = r_{\mathcal{A}\setminus\{\alpha\}} + 1$). An element α that is neither a loop nor a coloop is said to be *proper*.

Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be the triple with the distinguished element $\alpha_i \in \mathcal{A}$.

Lemma 1.2.2.12. *The multivariate G-Tutte polynomials satisfy*

$$Z^G_{\mathcal{A}}(q, \boldsymbol{v}) = \begin{cases} Z^G_{\mathcal{A}'}(q, \boldsymbol{v}) + v_i \cdot Z^G_{\mathcal{A}''}(q, \boldsymbol{v}), & \text{if } \alpha_i \text{ is a loop,} \\ Z^G_{\mathcal{A}'}(q, \boldsymbol{v}) + v_i \cdot q^{-1} \cdot Z^G_{\mathcal{A}''}(q, \boldsymbol{v}), & \text{otherwise.} \end{cases}$$

Proof. Proof follows along the lines of [BM14, Proof of Lemma 3.2]. ☐

Corollary 1.2.2.13. *The G-Tutte polynomials satisfy*

$$T^G_{\mathcal{A}}(x,y) = \begin{cases} T^G_{\mathcal{A}'}(x,y) + (y-1)T^G_{\mathcal{A}''}(x,y), & \text{if } \alpha_i \text{ is a loop,} \\ (x-1)T^G_{\mathcal{A}'}(x,y) + T^G_{\mathcal{A}''}(x,y), & \text{if } \alpha_i \text{ is a coloop,} \\ T^G_{\mathcal{A}'}(x,y) + T^G_{\mathcal{A}''}(x,y), & \text{if } \alpha_i \text{ is proper.} \end{cases}$$

Proof. It follows from Lemma 1.2.2.12 and formula (1.2.1). ☐

Corollary 1.2.2.14. *The G-characteristic polynomials satisfy*

$$\chi^G_{\mathcal{A}}(t) = \chi^G_{\mathcal{A}'}(t) - \chi^G_{\mathcal{A}''}(t).$$

Proof. It follows from Corollary 1.2.2.13 and formula (1.2.2). ☐

Recall the definitions of the Tutte polynomial $T_{\mathcal{A}}(x,y)$ and the arithmetic Tutte polynomial $T^{\mathrm{arith}}_{\mathcal{A}}(x,y)$ of $\mathcal{A}$ in Remark 1.1.2.1. The G-Tutte and G-characteristic polynomials have several specializations. First, we mention the expected ones.

Proposition 1.2.2.15. *Suppose that G is a torsion-free abelian group. Then $T_{\mathcal{A}}^{G}(x,y) = T_{\mathcal{A}}(x,y)$.*

Proof. This follows from the fact that $\mathrm{Hom}((\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{tor}}, G)$ is the trivial group for all $\mathcal{S} \subseteq \mathcal{A}$. □

Proposition 1.2.2.16. *Suppose that G is $\mathbb{S}^1$ or $\mathbb{C}^\times$. Then $T_{\mathcal{A}}^{G}(x,y) = T_{\mathcal{A}}^{\mathrm{arith}}(x,y)$.*

Proof. Note that $\#\,\mathrm{Hom}((\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{tor}}, G) = \#(\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{tor}}$, which is equal to the multiplicity $m(\mathcal{S})$ in the definition of the arithmetic Tutte polynomial (§1.1.2). □

Definition 1.2.2.17. Suppose that $\Gamma/\langle\mathcal{S}\rangle \simeq \bigoplus_{i=1}^{n_{\mathcal{S}}} \mathbb{Z}_{d_{\mathcal{S},i}} \oplus \mathbb{Z}^{r_\Gamma - r_{\mathcal{S}}}$ where $n_{\mathcal{S}} \geq 0$ and $1 < d_{\mathcal{S},i} | d_{\mathcal{S},i+1}$. The *LCM-period* $\rho_{\mathcal{A}}$ of $\mathcal{A}$ is defined by

$$\rho_{\mathcal{A}} := \mathrm{lcm}(d_{\mathcal{S},n_{\mathcal{S}}} \mid \mathcal{S} \subseteq \mathcal{A}).$$

Note that the LCM-period $\rho_{\mathcal{A}}$ defined above coincides with the number defined in (1.1.4) when $\Gamma \simeq \mathbb{Z}^\ell$. The arithmetic Tutte polynomial can also be seen as the G-Tutte polynomial in terms of other group G.

Proposition 1.2.2.18. $T_{\mathcal{A}}^{\mathbb{Z}_{\rho_{\mathcal{A}}}}(x,y) = T_{\mathcal{A}}^{\mathrm{arith}}(x,y)$.

Proof. Let $G = \mathbb{Z}_{\rho_{\mathcal{A}}}$. Since $d_{\mathcal{S},i} \mid \rho_{\mathcal{A}}$, we have $\#\,\mathrm{Hom}(\mathbb{Z}/d_{\mathcal{S},i}\mathbb{Z}, G) = d_{\mathcal{S},i}$ for all $\mathcal{S} \subseteq \mathcal{A}$ and $1 \leq i \leq n_{\mathcal{S}}$ which follows from Proposition 1.2.2.6. Thus, $m(\mathcal{S}; G) = m(\mathcal{S})$ for all $\mathcal{S}$. □

Theorem 1.2.2.19. *The chromatic quasi-polynomial of $\mathcal{A}$ (Example 1.2.1.3) is identical with the $\mathbb{Z}_q$-characteristic polynomial of $\mathcal{A}$, i.e.,*

$$\chi_{\mathcal{A}}^{\mathrm{quasi}}(q) = \chi_{\mathcal{A}}^{\mathbb{Z}_q}(q).$$

As a result, the LCM-period $\rho_{\mathcal{A}}$ is a period of $\chi_{\mathcal{A}}^{\mathrm{quasi}}(q)$.

Proof. It is proved in [BM14, §9]. Alternatively, it is a special case of Theorem 3.1.2.5. □

The G-Tutte polynomial also has some other specializations.

Example 1.2.2.20. In [CS04, Sok05], partition functions of abelian group valued Potts models were studied, and later were generalized to the arithmetic matroid setting by Brändén-Moci [BM14, Theorem 7.4]. Brändén-Moci's polynomial $Z_{\mathcal{L}}(\Gamma, H, \mathbf{v})$ is, in our terminology, equal to $(\#H)^{r_\Gamma} \cdot Z_{\mathcal{L}}^{H}(\#H, v_1, \dots, v_n)$, where $\mathcal{L}$ is a list of n elements in Γ, and H is a finite abelian group. In the same paper, Brändén-Moci also defined the Tutte quasi-polynomial $Q_{\mathcal{L}}(x, y)$, which is equal to $T_{\mathcal{L}}^{\mathbb{Z}_{(x-1)(y-1)}}(x, y)$ for any fixed integers x and y.

Example 1.2.2.21. Both the modified Tutte-Krushkal-Renhardy polynomial for a finite CW-complex (see [BBC14, §3], [DM18, §4] for details) and Bibby's Tutte polynomial for an elliptic arrangement [Bib16, Remark 4.4] can be expressed by $T_{\mathcal{A}}^{\mathbb{S}^1 \times \mathbb{S}^1}(x, y)$.

Let $\sigma : \Gamma_1 \longrightarrow \Gamma_2$ be a homomorphism between finitely generated abelian groups. The map σ induces a homomorphism

$$\sigma^* : \mathrm{Hom}(\Gamma_2, G) \longrightarrow \mathrm{Hom}(\Gamma_1, G). \tag{1.2.3}$$

Let $\alpha \in \Gamma_1$. It is easily seen that $(\sigma^*)^{-1}(H_{\alpha,G}) = H_{\sigma(\alpha),G}$. Hence (1.2.3) induces a map between the complements

$$\sigma^*|_{\mathcal{M}(\sigma(\mathcal{A});\Gamma_2,G)} : \mathcal{M}(\sigma(\mathcal{A}); \Gamma_2, G) \longrightarrow \mathcal{M}(\mathcal{A}; \Gamma_1, G).$$

A natural question is to compare $T_{\mathcal{A}}^{G}(x, y)$ and $T_{\sigma(\mathcal{A})}^{G}(x, y)$. This comparison is in general difficult. However, in the case where $\Gamma_1 = \Gamma_2 = \mathbb{Z}^\ell$ and G is a connected Lie group, the constant terms of the G-characteristic polynomials can be controlled by $\det(\sigma)$.

Proposition 1.2.2.22. *Let $\Gamma = \mathbb{Z}^\ell$, $\sigma : \Gamma \longrightarrow \Gamma$ be a homomorphism, $\mathcal{A}$ be a finite list of elements in Γ, and $G = (\mathbb{S}^1)^p \times \mathbb{R}^q$ with $p > 0$. Then*

$$\chi_{\sigma(\mathcal{A})}^{G}(0) = |\det(\sigma)|^p \cdot \chi_{\mathcal{A}}^{G}(0).$$

Proof. By formula (1.2.2), $\chi_{\mathcal{A}}^{G}(t)$ is divisible by $t^{r_\Gamma - r_{\mathcal{A}}}$. If $\det(\sigma) = 0$, then $r_{\sigma(\mathcal{A})} < \ell = r_\Gamma$, and $\chi_{\sigma(\mathcal{A})}^{G}(t)$ is divisible by t. Therefore the left-hand side vanishes, and the assertion holds trivially.

We assume instead that $\det(\sigma) \neq 0$. Note that for a sublattice $L \subseteq \Gamma$ of rank ℓ, we have $(\Gamma : \sigma(L)) = |\det(\sigma)| \cdot (\Gamma : L)$. Second, if $r_{\mathcal{S}} = \ell$, then $(\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{tor}} = \Gamma/\langle\mathcal{S}\rangle$, and we have $m(\mathcal{S}; G) = m(\mathcal{S}; \mathbb{S}^1)^p = \#(\Gamma/\langle\mathcal{S}\rangle)^p$. Third, because $\sigma : \Gamma \longrightarrow \Gamma$ is injective, $r_{\sigma(\mathcal{S})} = r_{\mathcal{S}}$ and $\#\sigma(\mathcal{S}) = \#\mathcal{S}$ for every sublist $\mathcal{S} \subseteq \mathcal{A}$. Therefore,

$$\begin{aligned}
\chi^G_{\sigma(\mathcal{A})}(0) &= \sum_{\substack{\sigma(\mathcal{S}) \subseteq \sigma(\mathcal{A}) \\ r_{\sigma(\mathcal{S})} = \ell}} (-1)^{\#\sigma(\mathcal{S})} m(\sigma(\mathcal{S}); G) \\
&= \sum_{\substack{\mathcal{S} \subseteq \mathcal{A} \\ r_{\mathcal{S}} = \ell}} (-1)^{\#\mathcal{S}} m(\sigma(\mathcal{S}); G) \\
&= \sum_{\substack{\mathcal{S} \subseteq \mathcal{A} \\ r_{\mathcal{S}} = \ell}} (-1)^{\#\mathcal{S}} |\det(\sigma)|^p m(\mathcal{S}; G) \\
&= |\det(\sigma)|^p \cdot \chi^G_{\mathcal{A}}(0).
\end{aligned}$$

$\square$

2. G-TUTTE POLYNOMIALS VIA COMBINATORICS

2.1 Combinatorics of (F, p, q)-arrangements

The notions of hyperplane, toric and q-reduced arrangements are unified by that of *G-plexifications* (Remark 1.2.1.1), and more specifically, by that of *(F, p, q)-arrangements* (Remark 1.2.2.4). Inspired by the pioneered work of Moci on combinatorics of generalized toric arrangements, we will associate with any (F, p, q)-arrangement the *partial* and *total* intersection posets and prove that the corresponding characteristic polynomials can be expressed in terms of the G-characteristic polynomials.

2.1.1 Generalized toric arrangements

Let us first fix some definitions and notations throughout this section. Let Γ be a finitely generated abelian group, and let $\mathcal{A} \subseteq \Gamma$ be a finite list (multiset) of elements in Γ. The ranks of $\langle \mathcal{S} \rangle$ for $\mathcal{S} \subseteq \Gamma$ and Γ are denoted by $r_{\mathcal{S}}$, r_Γ, respectively. Given a group K, denote by K_{tor} the torsion subgroup of K. Denote $\mathcal{S}^{\mathrm{tor}} := \mathcal{S} \cap \Gamma_{\mathrm{tor}}$.

Let $(\mathcal{P}, \leq_{\mathcal{P}})$ be a finite poset. The *Möbius function* $\mu_{\mathcal{P}}$ of $\mathcal{P}$ is the function $\mu_{\mathcal{P}} : \mathcal{P} \times \mathcal{P} \longrightarrow \mathbb{Z}$ defined by

$$\mu_{\mathcal{P}}(a, b) := \begin{cases} 0 & \text{if } a \not\leq_{\mathcal{P}} b, \\ 1 & \text{if } a =_{\mathcal{P}} b, \\ -\sum_{a \leq c < b} \mu_{\mathcal{P}}(a, c) & \text{if } a <_{\mathcal{P}} b. \end{cases}$$

A poset $\mathcal{P}$ is said to be *ranked* if for every $a \in \mathcal{P}$, all maximal chains among those with a as greatest element have the same length, denoted this common number by $\mathrm{rk}_{\mathcal{P}}(a)$.

Now we briefly recall what has been known on combinatorics of generalized toric arrangements following [Moc12, §5]. Set $T := \mathrm{Hom}(\Gamma, G)$ where G is either $\mathbb{S}^1$ or

$\mathbb{C}^\times$. The G-plexification $\mathcal{A}(G)$ is called the *generalized toric arrangement* defined by $\mathcal{A}$ on T. In particular, when $\Gamma \simeq \mathbb{Z}^\ell$, T is a torus and $\mathcal{A}(G)$ is called the *toric arrangement* (see §1.1.1). To describe the combinatorics of $\mathcal{A}(G)$, we associate with it an *intersection poset* $L_{\mathcal{A}(G)}$, which is the set of all connected components of all intersections of the G-hyperplanes (subvarieties) $H_{\alpha,G}$. The poset $L_{\mathcal{A}(G)}$ is ranked by the dimension of its elements (*layers*). Note that the dimension is defined over $\mathbb{R}$ (resp., $\mathbb{C}$) when $G = \mathbb{S}^1$ (resp., $G = \mathbb{C}^\times$). The *characteristic polynomial* often regarded as "the combinatorics" of $\mathcal{A}(G)$ is defined by

$$\chi^{\mathrm{toric}}_{\mathcal{A}}(t) := \sum_{\mathcal{C} \in L_{\mathcal{A}(G)}} \mu(T^{\mathcal{C}}, \mathcal{C}) t^{\dim(\mathcal{C})},$$

where $T^{\mathcal{C}}$ is the connected component of T that contains $\mathcal{C}$. In comparison with the notation in §1.1.2, the notation $\chi^{\mathrm{toric}}_{\mathcal{A}}(t)$ here means either $\chi_{\mathcal{A}(\mathbb{S}^1)}(t)$ or $\chi_{\mathcal{A}(\mathbb{C}^\times)}(t)$ depending on whether G is $\mathbb{S}^1$ or $\mathbb{C}^\times$.

Moci showed that $\chi^{\mathrm{toric}}_{\mathcal{A}}(t)$ can be computed by the arithmetic Tutte polynomial $T^{\mathrm{arith}}_{\mathcal{A}}(x, y)$ in the same way as the Whitney's theorem (formula (1.1.1)) showing how the characteristic polynomial of a hyperplane arrangement is computed by the Tutte polynomial.

Theorem 2.1.1.1. *Assume that G is either $\mathbb{S}^1$ or $\mathbb{C}^\times$. If Γ is a free abelian group and $0_\Gamma \notin \mathcal{A}$ (or even if Γ is an arbitrary finitely generated abelian group with $\mathcal{A}^{\mathrm{tor}} = \emptyset$), then*

$$\chi^{\mathrm{toric}}_{\mathcal{A}}(t) = (-1)^{r_{\mathcal{A}}} \cdot t^{r_\Gamma - r_{\mathcal{A}}} \cdot T^{\mathrm{arith}}_{\mathcal{A}}(1-t, 0) = \chi^G_{\mathcal{A}}(t).$$

Proof. The first equality follows from [Moc12, Theorem 5.6]. The second follows from Proposition 1.2.2.16. ∎

2.1.2 Characteristic polynomials of (F, p, q)-arrangements

In the remainder of this subsection, we assume that G is an abelian Lie group with finitely many connected components, i.e., $G \simeq (\mathbb{S}^1)^p \times \mathbb{R}^q \times F$ and F is a finite abelian group. In addition, we assume that the dimension is defined over $\mathbb{R}$. Thus $g := \dim_{\mathbb{R}}(G) = p + q \geq 0$. For each $\mathcal{S} \subseteq \mathcal{A}$, by Proposition 1.2.1.6, we have

$$\begin{aligned} H_{\mathcal{S},G} &= \bigcap_{\alpha \in \mathcal{S}} H_{\alpha,G} \\ &\simeq \mathrm{Hom}((\Gamma/\langle \mathcal{S} \rangle)_{\mathrm{tor}}, G) \times F^{r_\Gamma - r_{\mathcal{S}}} \times \left((\mathbb{S}^1)^p \times \mathbb{R}^q\right)^{r_\Gamma - r_{\mathcal{S}}}. \end{aligned} \tag{2.1.1}$$

We agree that $T = \mathrm{Hom}(\Gamma, G) := H_{\emptyset,G}$. Each connected component of $H_{\mathcal{S},G}$ is isomorphic to $((\mathbb{S}^1)^p \times \mathbb{R}^q)^{r_\Gamma - r_{\mathcal{S}}}$. If either $r_\Gamma = 0$ or $g = 0$, it can be identified with a point. The set of the connected components of $H_{\mathcal{S},G}$ is denoted by $\mathrm{cc}(H_{\mathcal{S},G})$. The following lemma is somewhat more general than [Moc12, Lemma 5.4].

Lemma 2.1.2.1. $\#\mathrm{cc}(H_{\mathcal{S},G}) = m(\mathcal{S}; G) \cdot (\#F)^{r_\Gamma - r_{\mathcal{S}}}$.

Proof. It follows directly from Definition 1.2.2.7 and (2.1.1). □

Definition 2.1.2.2.

(1) The *total* intersection poset of $\mathcal{A}(G)$ is defined by

$$L = L^{\mathrm{tot}}_{\mathcal{A}(G)} := \{\text{connected components of nonempty } H_{\mathcal{S},G} \mid \mathcal{S} \subseteq \mathcal{A}\},$$

whose elements, called *layers*, are ordered by reverse inclusion ($\mathcal{D} \leq_L \mathcal{C}$ if $\mathcal{D} \supseteq \mathcal{C}$).

(2) The *total* characteristic polynomial of $\mathcal{A}(G)$ is defined by

$$\chi^{\mathrm{tot}}_{\mathcal{A}(G)}(t) := \sum_{\mathcal{C} \in L} \mu(T^{\mathcal{C}}, \mathcal{C}) t^{\dim(\mathcal{C})}.$$

Here $T^{\mathcal{C}}$ is the connected component of T that contains $\mathcal{C}$ and $\mu := \mu_L$.

The set of minimal elements of L is exactly $\mathrm{cc}(T)$. The connected components of $H_{\mathcal{A},G}$ are maximal elements of L but the converse is not necessarily true. For each $\mathcal{C} \in L$, set

$$\mathcal{R}(\mathcal{C}) := \{\mathcal{S} \subseteq \mathcal{A} \mid \mathcal{C} \in \mathrm{cc}(H_{\mathcal{S},G})\}.$$

One observes that $\dim(\mathcal{C}) = \dim(H_{\mathcal{S},G}) = g(r_\Gamma - r_{\mathcal{S}})$ for every $\mathcal{S} \in \mathcal{R}(\mathcal{C})$. The *localization* of $\mathcal{A}$ with respect to $\mathcal{C}$ is defined by

$$\mathcal{A}_{\mathcal{C}} := \{\alpha \in \mathcal{A} \mid \mathcal{C} \subseteq H_{\alpha,G}\}. \tag{2.1.2}$$

Stated differently, $\mathcal{A}_{\mathcal{C}}$ is the unique maximal element of $\mathcal{R}(\mathcal{C})$ in the sense that $\mathcal{S} \subseteq \mathcal{A}_{\mathcal{C}}$ for every $\mathcal{S} \in \mathcal{R}(\mathcal{C})$. We also can write

$$\mathcal{R}(\mathcal{C}) = \{\mathcal{S} \subseteq \mathcal{A}_{\mathcal{C}} \mid r_{\mathcal{S}} = r_{\mathcal{A}_{\mathcal{C}}}\}. \tag{2.1.3}$$

Thus L is a ranked poset with a rank function given by

$$\mathrm{rk}_L(\mathcal{C}) := r_{\mathcal{A}_{\mathcal{C}}} = \mathrm{codim}(\mathcal{C})/g \quad (\mathcal{C} \in L).$$

We are interested in a particular subset $\mathrm{scc}(T)$ of $\mathrm{cc}(T)$,

$$\begin{aligned} \mathrm{scc}(T) &:= \{T_i \in \mathrm{cc}(T) \mid (\mathcal{A}_{T_i})^{\mathrm{tor}} = \emptyset\} \\ &= \mathrm{cc}(T) \setminus \bigcup_{\alpha \in \mathcal{A}^{\mathrm{tor}}} \mathrm{cc}(H_{\alpha,G}). \end{aligned}$$

By using the Inclusion-Exclusion principle,

$$\#\mathrm{scc}(T) = \#\mathcal{M}(\mathcal{A}^{\mathrm{tor}}; \Gamma_{\mathrm{tor}}, G) \cdot (\#F)^{r_\Gamma}. \tag{2.1.4}$$

Definition 2.1.2.3.

(1) The *partial* intersection poset of $\mathcal{A}(G)$ is defined by

$$L^{\mathrm{par}} := \{\mathcal{C} \in L \mid T^{\mathcal{C}} \in \mathrm{scc}(T)\},$$

and the Möbius function of L^{par} is the restriction of μ i.e., $\mu_{L^{\mathrm{par}}} = \mu|_{L^{\mathrm{par}} \times L^{\mathrm{par}}}$.

(2) The *partial* characteristic polynomial of $\mathcal{A}(G)$ is defined by

$$\chi^{\mathrm{par}}_{\mathcal{A}(G)}(t) := \sum_{\mathcal{C} \in L^{\mathrm{par}}} \mu(T^{\mathcal{C}}, \mathcal{C}) t^{\dim(\mathcal{C})}.$$

In other words, L^{par} is the dual order ideal (e.g., [Sta11, §3.1]) of L generated by $\mathrm{scc}(T)$. It follows from the definition above that $\chi^{\mathrm{par}}_{\mathcal{A}(G)}(t) = 0$ if $\mathrm{scc}(T) = \emptyset$.

Remark 2.1.2.4. Removing from $\mathcal{A}(G)$ the hyperplanes $H_{\alpha,G}$ with $\alpha \in \mathcal{A}^{\mathrm{tor}}$ does not affect the structure of the poset, i.e.,

$$L_{\mathcal{A}(G)} = L_{(\mathcal{A} \setminus \mathcal{A}^{\mathrm{tor}})(G)} = L^{\mathrm{par}}_{(\mathcal{A} \setminus \mathcal{A}^{\mathrm{tor}})(G)}.$$

As a consequence,

$$\chi^{\mathrm{tot}}_{\mathcal{A}(G)}(t) = \chi^{\mathrm{tot}}_{(\mathcal{A}\setminus\mathcal{A}^{\mathrm{tor}})(G)}(t) = \chi^{\mathrm{par}}_{(\mathcal{A}\setminus\mathcal{A}^{\mathrm{tor}})(G)}(t).$$

In particular, $\chi^{\mathrm{tot}}_{\mathcal{A}(G)}(t) = \chi^{\mathrm{par}}_{\mathcal{A}(G)}(t)$ if $\mathcal{A}^{\mathrm{tor}} = \emptyset$.

In the lemma below, we generalize the result in [Moc12, Lemma 5.5] as we include the possibility $\mathcal{A}^{\mathrm{tor}} \neq \emptyset$.

Lemma 2.1.2.5. *If $\mathcal{C} \in L$, then*

$$\sum_{\mathcal{S}\in\mathcal{R}(\mathcal{C})} (-1)^{\#\mathcal{S}} = \begin{cases} \mu(T^{\mathcal{C}}, \mathcal{C}) & \text{if } \mathcal{C} \in L^{\mathrm{par}}, \\ 0 & \text{if } \mathcal{C} \notin L^{\mathrm{par}}. \end{cases}$$

Proof. The proof of the formula when $\mathcal{C} \in L^{\mathrm{par}}$ is processed by induction on $\mathrm{rk}_L(\mathcal{C})$, which runs essentially the same as that of [Moc12, Lemma 5.5]. Note that $\mathcal{A}_{\mathcal{D}} \subseteq \mathcal{A}_{\mathcal{C}}$ whenever $\mathcal{C} \subseteq \mathcal{D} \subseteq T^{\mathcal{C}}$. If $(\mathcal{A}_{T^{\mathcal{C}}})^{\mathrm{tor}} \neq \emptyset$, then $(\mathcal{A}_{\mathcal{C}})^{\mathrm{tor}} \neq \emptyset$. The remaining part of the formula follows from Proposition 2.2.2.5. Indeed, by (2.1.3)

$$\sum_{\mathcal{S}\in\mathcal{R}(\mathcal{C})} (-1)^{\#\mathcal{S}} = \sum_{\substack{\mathcal{S}\subseteq\mathcal{A}_{\mathcal{C}} \\ r_{\mathcal{S}} = r_{\mathcal{A}_{\mathcal{C}}}}} (-1)^{\#\mathcal{S}}$$

equals the coefficient of $t^{r_\Gamma - r_{\mathcal{A}_{\mathcal{C}}}}$ in $f^1_{\mathcal{A}_{\mathcal{C}}}(t)$, which is 0. ∎

Corollary 2.1.2.6. *The Möbius function of L strictly alternates in sign. That is, for all $\mathcal{C} \in L$,*

$$(-1)^{\mathrm{rk}_L(\mathcal{C})} \mu(T^{\mathcal{C}}, \mathcal{C}) > 0.$$

Proof. Consider $\mathcal{C} \in L^{\mathrm{par}}$. Note that $f^1_{\mathcal{A}_{\mathcal{C}}}(t) = \chi_{(\mathcal{A}_{\mathcal{C}})(\mathbb{R})}(t) = \sum_{j=r_\Gamma - r_{\mathcal{A}_{\mathcal{C}}}}^{r_\Gamma} b_j t^j$ with $(-1)^{r_\Gamma - j} b_j > 0$ for all j (e.g., [Sta07, Corollary 3.5]). By Proof of Lemma 2.1.2.5, $\mu(T^{\mathcal{C}}, \mathcal{C})$ is equal to the coefficient of $t^{r_\Gamma - r_{\mathcal{A}_{\mathcal{C}}}}$ in $f^1_{\mathcal{A}_{\mathcal{C}}}(t)$, which strictly alternates in sign, i.e., $(-1)^{r_{\mathcal{A}_{\mathcal{C}}}} \mu(T^{\mathcal{C}}, \mathcal{C}) > 0$. If $\mathcal{C} \notin L^{\mathrm{par}}$, we consider $\mathcal{A} \setminus \mathcal{A}^{\mathrm{tor}}$ instead of $\mathcal{A}$ as argued in Remark 2.1.2.4. ∎

The main idea of the proof below is very similar to the one used in [Moc12, Theorem 5.6]. We include it with a detailed proof for the sake of completeness.

Theorem 2.1.2.7. *Let $G \simeq (\mathbb{S}^1)^p \times \mathbb{R}^q \times F$ with $g = p + q \in \mathbb{Z}_{>0}$. Then*

$$\chi^{\mathrm{par}}_{\mathcal{A}(G)}(t) = \chi^G_{\mathcal{A}}\left(\#F \cdot t^g\right).$$

Proof. We must prove that

$$\sum_{\mathcal{C}\in L^{\mathrm{par}}} \mu(T^{\mathcal{C}}, \mathcal{C}) t^{\dim(\mathcal{C})} = \sum_{\mathcal{S}\subseteq\mathcal{A}} (-1)^{\#\mathcal{S}} m(\mathcal{S}; G) \cdot (\#F)^{r_\Gamma - r_{\mathcal{S}}} \cdot t^{g(r_\Gamma - r_{\mathcal{S}})}.$$

It is equivalent to proving that for all $k = r_\Gamma - r_{\mathcal{A}}, \ldots, r_\Gamma$,

$$\sum_{\substack{\mathcal{C}\in L^{\mathrm{par}} \\ gk=\dim(\mathcal{C})}} \mu(T^{\mathcal{C}}, \mathcal{C}) = \sum_{\substack{\mathcal{S}\subseteq\mathcal{A} \\ k=r_\Gamma - r_{\mathcal{S}}}} (-1)^{\#\mathcal{S}} m(\mathcal{S}; G) \cdot (\#F)^{r_\Gamma - r_{\mathcal{S}}}.$$

We have

$$\begin{aligned}
\sum_{\substack{\mathcal{C}\in L^{\mathrm{par}} \\ gk=\dim(\mathcal{C})}} \mu(T^{\mathcal{C}}, \mathcal{C}) &= \sum_{\substack{\mathcal{C}\in L \\ gk=\dim(\mathcal{C})}} \sum_{\mathcal{S}\in\mathcal{R}(\mathcal{C})} (-1)^{\#\mathcal{S}} \\
&= \sum_{\substack{\mathcal{S}\subseteq\mathcal{A} \\ r_{\mathcal{S}}=r_\Gamma - k}} \left(\sum_{\mathcal{C}\in\mathrm{cc}(H_{\mathcal{S},G})} 1 \right) (-1)^{\#\mathcal{S}} \\
&= \sum_{\substack{\mathcal{S}\subseteq\mathcal{A} \\ k=r_\Gamma - r_{\mathcal{S}}}} (-1)^{\#\mathcal{S}} m(\mathcal{S}; G) \cdot (\#F)^{r_\Gamma - r_{\mathcal{S}}}.
\end{aligned}$$

We have applied Lemma 2.1.2.5 in the first equality, switched roles of sums in the second equality, and used Lemma 2.1.2.1 in the last equality. □

Corollary 2.1.2.8. *Let $G \simeq (\mathbb{S}^1)^p \times \mathbb{R}^q \times F$ with $g = p + q \in \mathbb{Z}_{>0}$. Then*

$$\chi^{\mathrm{tot}}_{\mathcal{A}(G)}(t) = \chi^G_{\mathcal{A}\setminus\mathcal{A}^{\mathrm{tor}}}\left(\#F \cdot t^g\right).$$

Proof. It follows from Theorem 2.1.2.7 and Remark 2.1.2.4. □

Remark 2.1.2.9. Although either Theorem 2.1.2.7 or Corollary 2.1.2.8 may not be valid when $g = 0$, there is no loss of information in these formulations. Namely, $\chi^{\mathrm{par}}_{\mathcal{A}(G)}(t) = \#\mathrm{scc}(T)$, and by equality (2.1.4) and Proposition 1.2.2.10, this equals the "leading part" of $\chi^G_{\mathcal{A}}(\#F)$ (the value of the leading term of $\chi^G_{\mathcal{A}}(t)$ evaluated at $\#F$). Similarly, $\chi^{\mathrm{tot}}_{\mathcal{A}(G)}(t) = \#\mathrm{cc}(T)$, which is equal to the leading part of $\chi^G_{\mathcal{A}\setminus\mathcal{A}^{\mathrm{tor}}}(\#F)$.

Remark 2.1.2.10. Note that when $G = \mathbb{S}^1$ (or $G = \mathbb{C}^\times$) and $\mathcal{A}^{\mathrm{tor}} = \emptyset$, $\chi^{\mathrm{par}}_{\mathcal{A}(G)}(t) = \chi^{\mathrm{tot}}_{\mathcal{A}(G)}(t) = \chi^{\mathrm{toric}}_{\mathcal{A}}(t)$. The result of Moci (Theorem 2.1.1.1) is a special case of Theorem 2.1.2.7.

2.2 An equivalent formulation of chromatic quasi-polynomials

Our main focus in this section is the Brändén-Moci's chromatic quasi-polynomial (see Example 1.2.1.3) and we prove that this quasi-polynomial can be formulated in a different way. Recall that the Kamiya-Takemura-Terao's characteristic quasi-polynomial (§1.1.2) is defined for a finite list of elements in $\mathbb{Z}^\ell$, and the chromatic quasi-polynomial extends the definition to any finitely generated abelian group. In a different setting involving two lists of elements in $\mathbb{Z}^\ell$, and by the elementary divisor method which was initially used by Kamiya-Takemura-Terao, Chen-Wang [CW12] found another generalization of the characteristic quasi-polynomial. We prove that the Chen-Wang's quasi-polynomial and the Brändén-Moci's chromatic quasi-polynomial are equivalent in the sense that the quasi-polynomials enumerate the cardinalities of isomorphic sets. Several applications including the periodicity of intersection posets of $\mathbb{Z}_q$-plexifications, an answer to a problem of Chen-Wang, and computation of the characteristic polynomials of $\mathbb{R}$-plexifications will also be discussed.

2.2.1 Unify the quasi-polynomials

Since we will be greatly involved with quasi-polynomials, let us recall a precise definition of them. A function $g : \mathbb{Z} \to \mathbb{C}$ is called a *quasi-polynomial* if there exist $\rho \in \mathbb{Z}_{>0}$ and polynomials $f^k(t) \in \mathbb{Z}[t]$ $(1 \le k \le \rho)$ such that for any $q \in \mathbb{Z}_{>0}$ with $q \equiv k \bmod \rho$,

$$g(q) = f^k(q).$$

The number ρ is called a *period* and the polynomial $f^k(t)$ is called the *k-constituent* of the quasi-polynomial g.

Recall that for a finite list $\mathcal{A}$ of elements in a finitely generated abelian group Γ, $\#\mathcal{M}(\mathcal{A};\Gamma,\mathbb{Z}_q)$ is the chromatic quasi-polynomial of $\mathcal{A}$. In particular, $\#\mathcal{M}(\mathcal{A};\mathbb{Z}^\ell,\mathbb{Z}_q)$ is the characteristic quasi-polynomial of $\mathcal{A}$.

To define the Chen-Wang's quasi-polynomial, in the following paragraph, we specify $\Gamma=\mathbb{Z}^\ell$. Let $q\in\mathbb{Z}_{>0}$, and set $\mathbb{Z}_q^\times := \mathbb{Z}_q\setminus\{\overline{0}\}$. For simplicity of notation, we use the same symbols $\mathcal{A}$ and $\mathbf{z}$ for the realizations of the list $\mathcal{A}\subseteq\mathbb{Z}^\ell$ and the element $\mathbf{z}\in\mathbb{Z}_q^\ell$ as matrices of size $\ell\times\#\mathcal{A}$ and $1\times\ell$, respectively. Let $\mathcal{B}$ be another finite list in $\mathbb{Z}^\ell$. Chen-Wang [CW12] defined

$$\mathrm{CW}(\mathcal{A},\mathcal{B},\mathbb{Z}^\ell;q) := \left\{ \mathbf{z}\in\mathbb{Z}_q^\ell \;\middle|\; \begin{array}{c} \mathbf{z}\cdot\mathcal{A}\in(\mathbb{Z}_q^\times)^{\#\mathcal{A}} \\ \mathbf{z}\cdot\mathcal{B}=(\overline{0})^{\#\mathcal{B}} \end{array} \right\},$$

and applied the elementary divisor method of [KTT08] to show that the cardinality $\#\mathrm{CW}(\mathcal{A},\mathcal{B},\mathbb{Z}^\ell;q)$ is a quasi-polynomial in q. The notion of Chen-Wang's quasi-polynomials strictly generalizes that of characteristic quasi-polynomials because $\mathcal{M}(\mathcal{A};\mathbb{Z}^\ell,\mathbb{Z}_q)=\mathrm{CW}(\mathcal{A},\mathcal{B},\mathbb{Z}^\ell;q)$ when $\mathcal{B}$ is the zero matrix, and $\#\mathcal{M}(\emptyset;\mathbb{Z}^\ell,\mathbb{Z}_q)=q^\ell$ while $\#\mathrm{CW}(\emptyset,\mathcal{B},\mathbb{Z}^\ell;q)$ still depends on $\mathcal{B}$.

Proposition 2.2.1.1. *Any Chen-Wang's quasi-polynomial is a chromatic quasi-polynomial.*

Proof. By Corollary 1.2.1.7 together with the isomorphism $\mathrm{Hom}(\mathbb{Z}^\ell,\mathbb{Z}_q)\simeq\mathbb{Z}_q^\ell$,

$$\mathrm{CW}(\mathcal{A},\mathcal{B},\mathbb{Z}^\ell;q)\simeq\mathcal{M}((\mathcal{A}\sqcup\mathcal{B})/\mathcal{B};\mathbb{Z}^\ell/\langle\mathcal{B}\rangle,\mathbb{Z}_q).$$

$\square$

The converse of Proposition 2.2.1.1 is also true as we will see in the lemma below.

Lemma 2.2.1.2. *If $\Gamma\simeq\mathbb{Z}^r\oplus\mathbb{Z}_{d_1}\oplus\cdots\oplus\mathbb{Z}_{d_s}$, then we can find two lists $Q\subseteq\mathcal{L}\subseteq\mathbb{Z}^{r+s}$ with $r_Q=s$ such that $\mathcal{A}=\mathcal{L}/Q$.*

Proof. We can view $\Gamma\simeq\mathbb{Z}^{r+s}/\langle Q\rangle$, where $Q=\{q_1,\dots,q_s\}\subseteq\mathbb{Z}^{r+s}$, q_i has d_i in the $(r+i)$-th coordinate and 0 elsewhere. Thus $\mathcal{A}$ can be identified with a list of cosets $\mathcal{A}=\{\overline{a}_1,\dots,\overline{a}_k\}$ with $a_i\in\mathbb{Z}^{r+s}$. We choose a representative $a_i\in\mathbb{Z}^{r+s}$ for each coset, which is determined up to a linear combination of elements from Q. Define $\tilde{\mathcal{A}}:=\{a_1,\dots,a_k\}\subseteq\mathbb{Z}^{r+s}$, and $\mathcal{L}:=\tilde{\mathcal{A}}\sqcup Q\subseteq\mathbb{Z}^{r+s}$. Thus $\mathcal{A}=\mathcal{L}/Q$. $\square$

Remark 2.2.1.3. The construction of the lists $\tilde{\mathcal{A}}$ and $\mathcal{L}$ presented in Lemma 2.2.1.2 is probably well-known among experts, for instance [DM13, §3.4], wherein it plays a crucial role in proving the representability of the duals of arithmetic matroids.

Proposition 2.2.1.4. *Any chromatic quasi-polynomial is a Chen-Wang's quasi-polynomial.*

Proof. With the notation as in Lemma 2.2.1.2, we can write

$$\mathcal{M}(\mathcal{A};\Gamma,\mathbb{Z}_q) = \mathrm{CW}(\tilde{\mathcal{A}}, Q, \mathbb{Z}^{r+s}; q).$$

$\square$

Theorem 2.2.1.5. *The Chen-Wang's and the Brändén-Moci's chromatic quasi-polynomials are equivalent in the sense that they enumerate the cardinalities of isomorphic sets.*

Proof. It follows directly from Propositions 2.2.1.1 and 2.2.1.4. $\square$

Lemma 2.2.1.2 is also useful to prove the periodicity of intersection posets of the $\mathbb{Z}_q$-plexification $\mathcal{A}(G)$. Following the idea of [KTT08, §3], we define the poset

$$L_{\mathcal{A}(\mathbb{Z}_q)} := \left\{ H_{\mathcal{S},\mathbb{Z}_q} \neq \emptyset \mid \mathcal{S} \subseteq \mathcal{A} \right\},$$

ordered by reverse inclusion.

Theorem 2.2.1.6. *There exist positive integers $q_0, \rho_{\mathcal{L}}$ such that the intersection poset $L_{\mathcal{A}(\mathbb{Z}_q)}$ is periodic in $q > q_0$ with a period $\rho_{\mathcal{L}}$, i.e.,*

$$L_{\mathcal{A}(\mathbb{Z}_{q+n\rho_{\mathcal{L}}})} \simeq L_{\mathcal{A}(\mathbb{Z}_q)} \text{ (as posets) for all } q > q_0 \text{ and } n \in \mathbb{Z}_{>0}.$$

Proof. Using the notation as in Lemma 2.2.1.2, for each $\mathcal{S} = \{\overline{a}_{i_1}, \ldots, \overline{a}_{i_s}\} \subseteq \mathcal{A}$, we can idenfity $H_{\mathcal{S},\mathbb{Z}_q}$ with $\bigcap_{j=1}^{s} H_{a_{i_j},\mathbb{Z}_q} \cap H_{Q,\mathbb{Z}_q}$, which is clearly an element of $L_{\mathcal{L}(\mathbb{Z}_q)}$. Thus the Hasse diagram[1] of $L_{\mathcal{A}(\mathbb{Z}_q)}$ is isomorphic (as directed graphs) to that of $L' := \{Y \in L_{\mathcal{L}(\mathbb{Z}_q)} \mid Y \subseteq H_{Q,\mathbb{Z}_q}\}$. By [KTT08, Corollary 3.3], there exists $q_0 \in \mathbb{Z}_{>0}$ such that $L_{\mathcal{L}(\mathbb{Z}_{q+n\rho_{\mathcal{L}}})} \simeq L_{\mathcal{L}(\mathbb{Z}_q)}$ for all $q > q_0$ and $n \in \mathbb{Z}_{>0}$. Here $\rho_{\mathcal{L}}$ is the LCM-period of the list $\mathcal{L}$. In particular, the isomorphism induces the periodicity of L'. This completes the proof. $\square$

[1]Hasse diagram of a poset $\mathcal{P}$ is the directed graph whose vertices are the elements of $\mathcal{P}$ and whose edges are the pairs (a, b) with the edge going upward from a to b whenever b covers a.

Now from Theorems 2.2.1.5 and 1.2.2.19 we know that the following concepts are equivalent: the chromatic, Chen-Wang's quasi-polynomials and $\mathbb{Z}_q$-characteristic polynomial of $\mathcal{A}$. We use the common notation $\chi^{\text{quasi}}_{\mathcal{A}}(q)$ to denote these quasi-polynomials, and also write $f^k_{\mathcal{A}}(t)$ for the k-constituents ($1 \le k \le \rho_{\mathcal{A}}$).

Proposition 2.2.1.7.

(1) For any k with $1 \le k \le \rho_{\mathcal{A}}$, $f^k_{\mathcal{A}}(t) = \chi^{\mathbb{Z}_k}_{\mathcal{A}}(t)$.

(2) $\chi^{\text{quasi}}_{\mathcal{A}}(q)$ satisfies the GCD-property, *i.e., $f^a_{\mathcal{A}}(t) = f^b_{\mathcal{A}}(t)$ if $\gcd(a, \rho_{\mathcal{A}}) = \gcd(b, \rho_{\mathcal{A}})$.*

(3) For any k with $1 \le k \le \rho_{\mathcal{A}}$, if $\gcd(q, \rho_{\mathcal{A}}) = k$, then $\chi^{\text{quasi}}_{\mathcal{A}}(q) = f^k_{\mathcal{A}}(q)$.

Proof. See [CW12, Theorem 2.3]. □

Corollary 2.2.1.8. $\chi^{\mathbb{S}^1}_{\mathcal{A}}(t) = \chi^{\mathbb{Z}_{\rho_{\mathcal{A}}}}_{\mathcal{A}}(t) = f^{\rho_{\mathcal{A}}}_{\mathcal{A}}(t)$.

Proof. The first equality is clear from Propositions 1.2.2.16 and 1.2.2.18. The second equality follows from Proposition 2.2.1.7(1). □

Corollary 2.2.1.9. *The last constituent $f^{\rho_{\mathcal{A}}}_{\mathcal{A}}(t)$ (resp., $f^{\rho_{\mathcal{A}}}_{\mathcal{A}\setminus\mathcal{A}^{\text{tor}}}(t)$) of the chromatic quasi-polynomial of $\mathcal{A}$ (resp., $\mathcal{A} \setminus \mathcal{A}^{\text{tor}}$) coincides with the partial (resp., total) characteristic polynomial of the generalized toric arrangement $\mathcal{A}(\mathbb{S}^1)$ of $\mathcal{A}$, i.e.,*

$$\begin{aligned}\chi^{\text{par}}_{\mathcal{A}(\mathbb{S}^1)}(q) =& f^{\rho_{\mathcal{A}}}_{\mathcal{A}}(t),\\ \chi^{\text{tot}}_{\mathcal{A}(\mathbb{S}^1)}(t) =& f^{\rho_{\mathcal{A}}}_{\mathcal{A}\setminus\mathcal{A}^{\text{tor}}}(t).\end{aligned}$$

In particular, when $\mathcal{A}^{\text{tor}} = \emptyset$,

$$\chi^{\text{toric}}_{\mathcal{A}}(t) = \chi^{\text{tot}}_{\mathcal{A}(\mathbb{S}^1)}(t) = \chi^{\text{par}}_{\mathcal{A}(\mathbb{S}^1)}(q) = f^{\rho_{\mathcal{A}}}_{\mathcal{A}}(t).$$

Proof. By definitions, $\chi^{\text{toric}}_{\mathcal{A}}(t) = \chi^{\text{tot}}_{\mathcal{A}(\mathbb{S}^1)}(t)$ (see §2.1). The rest follows from Theorem 2.1.2.7 and Corollaries 2.1.2.8, 2.2.1.8. □

Fix $\alpha \in \mathcal{A}$. Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be the triple in Γ with the distinguished element α.

Theorem 2.2.1.10 (Deletion-Contraction formula)**.**

$$\chi^{\text{quasi}}_{\mathcal{A}}(q) = \chi^{\text{quasi}}_{\mathcal{A}'}(q) - \chi^{\text{quasi}}_{\mathcal{A}''}(q).$$

Proof. This follows directly from Corollary 1.2.2.14 and Theorem 1.2.2.19 by letting $G = \mathbb{Z}_q$. It can also be proved by Proposition 1.2.1.9. □

Remark 2.2.1.11. Using Theorem 2.2.1.10, the Deletion-Restriction formula in [CW12, Lemma 3.3] can be exhibited by setting $\mathcal{A}$ as the contraction list $(A \sqcup B)/B$, where $A \neq \emptyset$ and B are finite lists in $\mathbb{Z}^\ell$.

Corollary 2.2.1.12. *If $k \leq \min\{\rho_{\mathcal{A}'}, \rho_{\mathcal{A}''}\}$, then the k-constituents satisfy*

$$f^k_{\mathcal{A}}(t) = f^k_{\mathcal{A}'}(t) - f^k_{\mathcal{A}''}(t).$$

Proof. Note that the LCM-period of any deletion/contract list is a divisor of the LCM-period of the parent list. The formula follows from Corollary 1.2.2.14 and Proposition 2.2.1.7(1). □

Remark 2.2.1.13. The LCM-period of $\chi^{\text{quasi}}_{\mathcal{A}}(q)$ is not necessarily the minimum period. We clarify it by an example. Let $\Gamma = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\mathcal{A} = \{\alpha, \beta\} \subsetneq \Gamma$ with $\alpha = (\overline{0}, \overline{0})$ and $\beta = (\overline{1}, \overline{0})$. Then $\rho_{\mathcal{A}} = 2$, while the minimum period is actually 1 and $\chi^{\text{quasi}}_{\mathcal{A}}(q) = 0$ for every q. Note that this fact can also be clarified by another class of examples originated from [CW12, Example 4.2].

We close this subsection by giving an answer to a problem asked by Chen-Wang in [CW12, Problem 2].

Problem 2.2.1.14. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be finite lists in $\mathbb{Z}^\ell$ with $r_{\mathcal{A}_2} = \ell$. Assume that $\#\mathrm{CW}(\mathcal{A}_1, \mathcal{A}_2, \mathbb{Z}^\ell; q) = 0$ for every $q \in \mathbb{Z}_{>0}$. Then there exists $\alpha \in \mathcal{A}_1$ such that $\alpha \in \langle \mathcal{A}_2 \rangle$.

Answer. The statement is true if and only if $\ell = 1$. By Proposition 2.2.1.1, we rewrite the assumption as $\#\mathcal{M}(\mathcal{A}; \Gamma, \mathbb{Z}_q) = 0$ where $\mathcal{A} = (\mathcal{A}_1 \sqcup \mathcal{A}_2)/\mathcal{A}_2$, and $\Gamma = \mathbb{Z}/\langle \mathcal{A}_2 \rangle$ is a finite group. Assume that $\ell = 1$. Then $\Gamma \simeq \mathbb{Z}_d$ for some $d \in \mathbb{Z}_{>0}$. Suppose to the contrary that for every $\alpha \in \mathcal{A}_1$, $\alpha \notin \langle \mathcal{A}_2 \rangle$. It is equivalent to saying that $\overline{a} \neq \overline{0}$ for all $\overline{a} \in \mathcal{A}$. Set $T := \{z \in \mathbb{C} \mid z^d = 1\}$. For each $\overline{a} \in \mathcal{A}$ with $1 \leq a \leq d-1$, set $T_a := \{z \in T \mid z^a = 1\}$. By Theorem 1.2.2.19, the degree of $\chi_{\mathcal{A}}^{\text{quasi}}(q)$ is 0. Moreover, the last constituent of $\chi_{\mathcal{A}}^{\text{quasi}}(q)$ can be computed by using Proposition 2.2.1.7 as follows:

$$f_{\mathcal{A}}^{\rho_{\mathcal{A}}}(t) = \chi_{\mathcal{A}}^{\mathbb{Z}/\rho_{\mathcal{A}}\mathbb{Z}}(t) = \#\left(T \setminus \bigcup_{\overline{a} \in \mathcal{A}} T_a\right) > 0,$$

which is a contradiction. For $\ell \geq 2$, we show that the statement is not true by providing a counterexample. Let us first prove the following fact: if $\Gamma = \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_\ell}$ is a finite abelian group containing at least two distinct nonidentity elements of order 2, say β_1, β_2, and $\mathcal{A} = \{\alpha \in \Gamma \mid \alpha \neq 0_\Gamma\}$, then $\#\mathcal{M}(\mathcal{A}; \Gamma, \mathbb{Z}_q) = 0$ for every $q \in \mathbb{Z}_{>0}$. Indeed by definition,

$$\begin{aligned}\mathcal{M}(\mathcal{A}; \Gamma, \mathbb{Z}_q) &= \{\varphi \in \operatorname{Hom}(\Gamma, \mathbb{Z}_q) \mid \varphi(\alpha) \neq \overline{0}, \text{ for all } \alpha \in \mathcal{A}\},\\ &= \{\varphi \in \operatorname{Hom}(\Gamma, \mathbb{Z}_q) \mid \varphi \text{ is injective}\}.\end{aligned}$$

If the above set is nonempty, then $\varphi(\alpha)$, $\varphi(\beta)$ are distinct and both have order 2 in $\mathbb{Z}_q$. This contradiction implies that $\#\mathcal{M}(\mathcal{A}; \Gamma, \mathbb{Z}_q) = 0$. By Proposition 2.2.1.4, $\#\mathrm{CW}(\tilde{\mathcal{A}}, Q, \mathbb{Z}^\ell; q) = 0$. Now choose $\Gamma = \mathbb{Z}_2^\ell$ with $\ell \geq 2$, and let $\mathcal{A}_1 = \tilde{\mathcal{A}}$, $\mathcal{A}_2 = Q$.

$\square$

2.2.2 Application to real hyperplane arrangements

As mentioned in §1.1.2, if $\Gamma = \mathbb{Z}^\ell$, then $f_{\mathcal{A}}^1(t) = \chi_{\mathcal{A}(\mathbb{R})}(t)$. However, this formula may fail if Γ is any finitely generated abelian group. It is because the list $\mathcal{A}$ may contain torsion elements of Γ, and $f_{\mathcal{A}}^1(t)$ vanishes while by definition $\chi_{\mathcal{A}(\mathbb{R})}(t)$ is never 0. Thus it is natural to ask of which real arrangement, the characteristic polynomial agrees with $f_{\mathcal{A}}^1(t)$; and of which list, the first constituent of the characteristic quasi-polynomial agrees with $\chi_{\mathcal{A}(\mathbb{R})}(t)$. These questions will be answered in this subsection. More generally, we will give two interpretations for every constituent through subspace and toric viewpoints in §2.3.

In the following setting and until before Proposition 2.2.2.3, we restrict our attention to the case $\Gamma = \mathbb{Z}^\ell$, and view $\mathcal{A}$ as a finite list of *nonzero* vectors in $\mathbb{Z}^\ell$. We regard $\{\epsilon_1, \ldots, \epsilon_\ell\}$ as the standard basis for $\mathbb{R}^\ell$, and equip to it the standard inner product $(\cdot, \cdot)$. Then the $\mathbb{R}$-plexification $\mathcal{A}(\mathbb{R})$ is an arrangement of (possibly repeated) hyperplanes in $\mathbb{R}^\ell$ with each hyperplane $H_{\alpha,\mathbb{R}}$ can be identified with $H_\alpha = \{x \in \mathbb{R}^\ell \mid (\alpha, x) = 0\}$. Let $L_{\mathcal{A}(\mathbb{R})}$ be the intersection poset of $\mathcal{A}(\mathbb{R})$ (see §2.1.2). Here the partial and total intersection posets are the same. Note that we require the intersection poset to be a set, not multiset. Also, the ambient space $\mathbb{R}^\ell$ can be added to the arrangement without affecting the arrangement's intersection poset. For each $X \in L_{\mathcal{A}(\mathbb{R})}$, the *localization* of $\mathcal{A}(\mathbb{R})$ on X is defined by

$$\mathcal{A}(\mathbb{R})_X := \{H \in \mathcal{A}(\mathbb{R}) \mid X \subseteq H\},$$

and the *restriction* $\mathcal{A}(\mathbb{R})^X$ of $\mathcal{A}(\mathbb{R})$ to X is defined by

$$\mathcal{A}(\mathbb{R})^X := \{H \cap X \mid H \in \mathcal{A}(\mathbb{R}) \setminus \mathcal{A}(\mathbb{R})_X\}.$$

Denote by $X^\perp$ the orthogonal complement of X in $\mathbb{R}^\ell$. Set

$$\mathcal{A}_X := \mathcal{A} \cap X^\perp \subseteq \mathcal{A}.$$

Proposition 2.2.2.1. *The following formulas are valid at level of multisets:*

(1) $\mathcal{A}(\mathbb{R})_X = (\mathcal{A}_X)(\mathbb{R})$.

(2) $\mathcal{A}(\mathbb{R})^X = (\mathcal{A}/\mathcal{A}_X)(\mathbb{R})$.

Proof. The proof of (1) is straightforward. To prove (2), for every $X \in L_{\mathcal{A}(\mathbb{R})}$ with $X \neq \mathbb{R}^\ell$, we use $X = \bigcap_{H \in \mathcal{A}(\mathbb{R})_X} H$, the longest expression of X in terms of intersection of the hyperplanes in $\mathcal{A}(\mathbb{R})$. To see $\mathcal{A}(\mathbb{R})^X = (\mathcal{A}/\mathcal{A}_X)(\mathbb{R})$ as multisets, note that the number of occurrences of each element $H_{\beta,\mathbb{R}} \cap X$ in these multisets is equal to $\#\{\gamma \in \mathcal{A} \setminus \mathcal{A}_X \mid \gamma \in \operatorname{span}_{\mathbb{R}}\{\beta, \mathcal{A}_X\}\}$. $\square$

Lemma 2.2.2.2. $\chi_{\mathcal{A}(\mathbb{R})^X}(t) = f^1_{\mathcal{A}/\mathcal{A}_X}(t)$.

Proof. It is essentially proved in [CW12, Corollary 2.4] (see also [Ath96, Corollary 6.1]). The idea is to use Whitney's theorem (see (1.1.1)) and Proposition 2.2.2.1. □

Unless otherwise stated, in the remainder of this subsection, we assume that Γ is an arbitrary finitely generated abelian group. Now we give an arrangement theoretic realization for $\mathcal{A}(\mathbb{R})$.

Proposition 2.2.2.3. *If $\mathcal{A}^{\mathrm{tor}} = \emptyset$, then $\mathcal{A}(\mathbb{R})$ is an integral arrangement, and also can be realized as a restriction of $\mathcal{L}(\mathbb{R})$ where $\mathcal{L}$ is a finite list in some free abelian group.*

Proof. We use the notation as in Lemma 2.2.1.2. Set $X := H_{Q,\mathbb{R}} \in L_{\mathcal{L}(\mathbb{R})}$, then $Q = \mathcal{L} \cap X^{\perp} = \mathcal{L}_X$. The condition $\mathcal{A}^{\mathrm{tor}} = \emptyset$ is crucial, otherwise it may happen that $Q \subsetneq \mathcal{L}_X$. By Proposition 2.2.2.1, $\mathcal{A}(\mathbb{R}) = (\mathcal{L}/\mathcal{L}_X)(\mathbb{R}) = \mathcal{L}(\mathbb{R})^X$. This means that $\mathcal{A}(\mathbb{R})$ is the restriction of $\mathcal{L}(\mathbb{R})$ to X, and also can be identified with an integral arrangement in $\mathbb{R}^{r_\Gamma}$. □

Next, we prove an important property of $\chi^{\mathrm{quasi}}_{\mathcal{A}}(q)$, which is the main theorem of this subsection.

Theorem 2.2.2.4. $\chi_{\mathcal{A}(\mathbb{R})}(t) = f^1_{\mathcal{A}\setminus\mathcal{A}^{\mathrm{tor}}}(t)$.

Proof. If $\mathcal{A}^{\mathrm{tor}} = \emptyset$, we apply Proposition 2.2.2.3 and Lemma 2.2.2.2. For the case $\mathcal{A}^{\mathrm{tor}} \neq \emptyset$, note that $\mathcal{A}(\mathbb{R})$ and $(\mathcal{A} \setminus \mathcal{A}^{\mathrm{tor}})(\mathbb{R})$ have the same intersection poset. □

The 1-constituent $f^1_{\mathcal{A}}(t)$ sometimes can be regarded as the chromatic polynomial defined on a graph, for example, via connection with *graphic arrangements* (e.g., [OT92, §2.4]). It is well-known (and easy to show) that the graphical chromatic polynomial is identical to 0 if the graph contains some (graph theoretic) loop. Recall from Definition 1.2.2.11 that an element $\alpha \in \mathcal{A}$ is called a loop if $\alpha \in \Gamma_{\mathrm{tor}}$. We prove in the proposition below that a similar result holds for $f^1_{\mathcal{A}}(t)$.

Proposition 2.2.2.5. *If $\mathcal{A}^{\mathrm{tor}} \neq \emptyset$, then $f^1_{\mathcal{A}}(t) = 0$.*

Proof. Use Corollary 2.2.1.12 (viewing as $k = 1$) to reduce the problem to the case that $\mathcal{A} = \mathcal{F} \sqcup \mathcal{T}$ with $\mathcal{F}$ and $\mathcal{T} \neq \emptyset$ consist of only coloops and loops, respectively. Then apply Proposition 2.2.1.7(1). □

Remark 2.2.2.6. There is a neater proof: fix $\alpha \in \mathcal{A}^{\mathrm{tor}}$ and break $f^1_{\mathcal{A}}(t)$ into two summations with one of them is taken over $\mathcal{B} \subseteq \mathcal{A}, \alpha \in \mathcal{B}$.

Corollary 2.2.2.7.

$$f^1_{\mathcal{A}}(t) = \begin{cases} 0 & \text{if } \mathcal{A}^{\mathrm{tor}} \neq \emptyset, \\ \chi_{\mathcal{A}(\mathbb{R})}(t) & \text{if } \mathcal{A}^{\mathrm{tor}} = \emptyset. \end{cases}$$

Proof. It follows directly from Theorem 2.2.2.4 and Proposition 2.2.2.5. □

Example 2.2.2.8. Let $\Gamma = \mathbb{Z}^2 \oplus \mathbb{Z}_4$, $\mathcal{A} = \{\alpha, \beta, \gamma\} \subsetneq \Gamma$ with $\alpha = (2, 2, \overline{1})$, $\beta = (0, 2, \overline{3})$ and $\gamma = (0, 0, \overline{3})$. Then $\rho_{\mathcal{A}} = \rho_{\mathcal{A}\setminus\{\gamma\}} = 8$, and

$$\chi^{\mathrm{quasi}}_{\mathcal{A}}(q) = \begin{cases} 0 & \text{if } \gcd(q,8) = 1, \\ q^2 & \text{if } \gcd(q,8) = 2, \\ 3q^2 - 4q + 4 & \text{if } \gcd(q,8) = 4, \\ 3q^2 - 12q + 12 & \text{if } \gcd(q,8) = 8. \end{cases}$$

$$\chi^{\mathrm{quasi}}_{\mathcal{A}\setminus\{\gamma\}}(q) = \begin{cases} q^2 - 2q + 1 & \text{if } \gcd(q,8) = 1, \\ 2q^2 - 4q + 4 & \text{if } \gcd(q,8) = 2, \\ 4q^2 - 8q + 8 & \text{if } \gcd(q,8) = 4, \\ 4q^2 - 16q + 16 & \text{if } \gcd(q,8) = 8. \end{cases}$$

Note that $(\mathcal{A} \setminus \{\gamma\})(\mathbb{R}) = \mathcal{L}(\mathbb{R})^X$, where $\mathcal{L}(\mathbb{R}) = \{\{2x + 2y + z = 0\}, \{2y + 3z = 0\}, \{z = 0\}\} \subseteq \mathbb{R}^3$ and $X = \{z = 0\}$, which can also be identified with the integral arrangement $\{\{x + y = 0\}, \{y = 0\}\}$ in $\mathbb{R}^2$. In either way,

$$\chi_{(\mathcal{A}\setminus\{\gamma\})(\mathbb{R})}(t) = f^1_{\mathcal{A}\setminus\{\gamma\}}(t) = t^2 - 2t + 1.$$

Corollary 2.2.2.9. *For $q \in \mathbb{Z}_{>0}$, we have*

$$\sum_{\mathcal{S} \subseteq \mathcal{A}} \chi^{\text{quasi}}_{\mathcal{A}/\mathcal{S}}(q) = \#\operatorname{Hom}(\Gamma, \mathbb{Z}_q). \tag{2.2.1}$$

As a result, if $\Gamma \simeq \mathbb{Z}^r \oplus \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_s}$, then

$$\sum_{\mathcal{S} \subseteq \mathcal{A}} f^1_{\mathcal{A}/\mathcal{S}}(t) = t^r, \tag{2.2.2}$$

$$\sum_{\mathcal{S} \subseteq \mathcal{A}} f^{\rho_{\mathcal{A}/\mathcal{S}}}_{\mathcal{A}/\mathcal{S}}(t) = d_1 \cdots d_r t^r. \tag{2.2.3}$$

Proof. Formula (2.2.1) follows from the definition of chromatic quasi-polynomials (Example 1.2.1.3) and Proposition 1.2.1.8. Formulas (2.2.2), (2.2.3) are immediate consequences of Formula (2.2.1) and Proposition 2.2.1.7(3). □

From Corollary 2.2.2.9, we see that the following well-known formula (e.g., [OS83, (3.2)]) for integral arrangments is a consequence of Formula (2.2.2).

Corollary 2.2.2.10. *Assume $\Gamma = \mathbb{Z}^\ell$ and $0_\Gamma \notin \mathcal{A}$. For every $X \in L_{\mathcal{A}(\mathbb{R})}$, we have*

$$\sum_{\substack{Y \in L_{\mathcal{A}(\mathbb{R})} \\ Y \subseteq X}} \chi_{\mathcal{A}(\mathbb{R})^Y}(t) = t^{\dim(X)}. \tag{2.2.4}$$

Proof. Given $X \in L_{\mathcal{A}(\mathbb{R})}$, replacing $\mathcal{A}, \Gamma$ in the Formula (2.2.2) by $\mathcal{A}/\mathcal{A}_X$, $\mathbb{Z}^\ell/\langle \mathcal{A}_X \rangle$, respectively, we obtain

$$\sum_{\mathcal{A}_X \subseteq \mathcal{S} \subseteq \mathcal{A}} f^1_{(\mathcal{A}/\mathcal{A}_X)/(\mathcal{S}/\mathcal{A}_X)}(t) = t^{\ell - r_{\mathcal{A}_X}}. \tag{2.2.5}$$

Notice that $\dim(X) = \ell - r_{\mathcal{A}_X}$, and $(\mathcal{A}/\mathcal{A}_X)/(\mathcal{S}/\mathcal{A}_X)$ can be identified with the list $\mathcal{A}/\mathcal{S}$ in $\mathbb{Z}^\ell/\langle \mathcal{S} \rangle$. In addition, by Proposition 2.2.2.5 we can write (2.2.5) as

$$\sum_{\substack{Y \in L_{\mathcal{A}(\mathbb{R})} \\ \mathcal{A}_X \subseteq \mathcal{A}_Y \subseteq \mathcal{A}}} f^1_{\mathcal{A}/\mathcal{A}_Y}(t) = t^{\dim(X)}. \tag{2.2.6}$$

Thus Formula (2.2.4) follows from Lemma 2.2.2.2 and Identity (2.2.6) above. □

2.3 Arrangement theoretic interpretations of the constituents

As mentioned in the previous subsection, we are interested in a question that given a constituent of a chromatic polynomial how we can describe it in connection with arrangement characteristic polynomials? Less is known, except for the first and the last (Corollaries 2.2.1.8 and 2.2.2.7). An attempt was made to describe certain classes of the constituents appeared in [DFM18, §10.3.3]. In this section, we give two complete interpretations for the constituents through subspace and toric viewpoints. The subspace interpretation is obtained from the combinatorics of $\mathcal{A}(\mathbb{R}^{\dim(G)} \times \mathbb{Z}_k)$, while the toric interpretation is obtained from the arithmetics of $\mathcal{A}(\mathbb{S}^1)$ (or $\mathcal{A}(\mathbb{C}^\times)$) by appropriately extracting its intersection poset.

2.3.1 Via subspace viewpoint

Corollary 2.3.1.1. *Let $G = \mathbb{R}^g \times \mathbb{Z}_k$ with $g > 0$ and $1 \le k \le \rho_{\mathcal{A}}$. Then*

$$\chi^{\mathrm{par}}_{\mathcal{A}(G)}(t) = f^k_{\mathcal{A}}(k \cdot t^g).$$

Proof. It follows from Theorem 2.1.2.7 and Proposition 2.2.1.7(1) that

$$\chi^{\mathrm{par}}_{\mathcal{A}(G)}(t) = \chi^{\mathbb{R}^g \times \mathbb{Z}_k}_{\mathcal{A}}(k \cdot t^g) = \chi^{\mathbb{Z}_k}_{\mathcal{A}}(k \cdot t^g) = f^k_{\mathcal{A}}(k \cdot t^g).$$

□

Let us explain Corollary 2.3.1.1 in more details. For nontriviality, we assume that $\mathrm{scc}(T) \ne \emptyset$ (e.g., when $\mathcal{A}^{\mathrm{tor}} = \emptyset$), and $r_\Gamma > 0$. Each connected component of $T = \mathrm{Hom}(\Gamma, \mathbb{R}^g \times \mathbb{Z}_k)$ is isomorphic to $\mathbb{R}^{g r_\Gamma}$. For each $T_i \in \mathrm{scc}(T)$, the poset $L_i = \{\mathcal{C} \in L \mid \mathcal{C} \subseteq T_i\}$ is isomorphic to the total (or equivalently, partial) intersection poset of a $\mathbb{R}^g$-plexification $\mathcal{G}_i$ in $\mathbb{R}^{g r_\Gamma}$ (or *g-plexification* in the sense of [Bjö94, §5.2], see §1.1.1), with each $\mathcal{G}_i$ is possibly empty and defined over the integers. Thus after a rescaling of variable, each constituent records the summation of the total characteristic polynomials of the $\mathcal{G}_i$'s, i.e.,

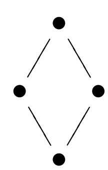
$$f^k_{\mathcal{A}}(kt^q) = \sum_{T_i \in \mathrm{scc}(T)} \chi^{\mathrm{tot}}_{\mathcal{G}_i}(t). \tag{2.3.1}$$

Remark 2.3.1.2. In particular, when $g = 1$, each $\mathcal{G}_i$ becomes an integral hyperplane arrangement $\mathcal{H}_i$ (Proposition 2.2.2.3). The conclusion related to the first constituent ($k = 1$) in Corollary 2.3.1.1 is the same as that stated in Corollary 2.2.2.7. In particular, if $\Gamma = \mathbb{Z}^\ell$, each hyperplane $H_{\alpha,\mathbb{R}\times\mathbb{Z}_k}$ in T can be identified with $H_{\alpha,\mathbb{R}} \times H_{\alpha,\mathbb{Z}_k}$ in $\mathbb{R}^\ell \times \mathbb{Z}_k^\ell$. Each arrangement $\mathcal{H}_i$ turns out to be a subarrangement of $\mathcal{A}(\mathbb{R})$, and in which components of T that the components of $H_{\alpha,\mathbb{R}\times\mathbb{Z}_k}$ locate depends on the arithmetics of the list $\mathcal{A}$.

Example 2.3.1.3. Let $\Gamma = \mathbb{Z}^2$, $\mathcal{A} = \{\alpha, \beta, \gamma\} \subsetneq \mathbb{Z}^2$ with $\alpha = (-1, 1)$, $\beta = (0, 2)$, and $\gamma = (0, 4)$. Then

$$\chi_{\mathcal{A}}^{\text{quasi}}(q) = \begin{cases} q^2 - 2q + 1 & \text{if } \gcd(q, 4) = 1, \\ q^2 - 3q + 2 & \text{if } \gcd(q, 4) = 2, \\ q^2 - 5q + 4 & \text{if } \gcd(q, 4) = 4. \end{cases}$$

Set $G_k := \mathbb{R} \times \mathbb{Z}_k$ with $k \in \{1, 2, 4\}$. The Hasse diagrams (arrow omitted) of $L_{\mathcal{A}(G_k)}$ are drawn in Figures 2.1, 2.2, 2.3. The total characteristic polynomials $\chi_{\mathcal{A}(G_k)}^{\text{tot}}(t)$ are computed according to the "$\times n$", indicator of the number of isomorphic Hasse diagrams of L_i's.



$\times 1$

Figure 2.1: $\chi_{\mathcal{A}(G_1)}^{\text{tot}}(t) = t^2 - 2t + 1 = f_{\mathcal{A}}^1(t)$.

Now we give a discussion on *reciprocity laws* for every constituent $f_{\mathcal{A}}^k(t)$. By [CW12, Theorem 1.2], $(-1)^\ell \mathrm{CW}(\mathcal{A}, \mathcal{B}, \mathbb{Z}^\ell; -q) \geq 0$ for all $q \in \mathbb{Z}_{>0}$. Thus by Theorem 2.2.1.5, $(-1)^{r_\Gamma} f_{\mathcal{A}}^k(-t) \geq 0$ for all $t \in \mathbb{Z}_{>0}$. Also, this fact can be derived from formula

Figure 2.2: $\chi^{\mathrm{tot}}_{\mathcal{A}(G_2)}(t) = 4t^2 - 6t + 2 = f^2_{\mathcal{A}}(2t)$.

Figure 2.3: $\chi^{\mathrm{tot}}_{\mathcal{A}(G_4)}(t) = 16t^2 - 20t + 4 = f^4_{\mathcal{A}}(4t)$.

(2.3.4) in this paper. It is natural to ask whether the evaluations $(-1)^{r_\Gamma} f^k_{\mathcal{A}}(-t)$ have any combinatorial meaning. A partial answer is probably well-known when $\Gamma = \mathbb{Z}^\ell$ that $(-1)^{r_\Gamma} f^k_{\mathcal{A}}(-t)$ can be expressed in terms of the Ehrhart quasi-polynomial of an "inside-out" polytope [BZ06]. The construction of the polytope and the hyperplanes cutting through it can be found in [KTT08, §2.2] (with some modification).

Owing to equality (2.3.1) and Remark 2.3.1.2, we can give an answer to the aforementioned question. For nontriviallity, we assume that $r_\Gamma > 0$. The reciprocity laws for the characteristic polynomial of an integral arrangement have been formulated by several methods, e.g., [Ath10], [BS18, §7], [Wan15, §4]. Thus the reciprocity laws for any (nonzero) constituent $f^k_{\mathcal{A}}(t)$ can be obtained from the reciprocity laws of the polynomials $\chi_{\mathcal{H}_i}(t)$ (at least the method from [Ath10] is applicable here) as follows:

$$(-1)^{r_\Gamma} f^k_{\mathcal{A}}(-t) = \sum_i (-1)^{r_\Gamma} \chi_{\mathcal{H}_i}\left(\frac{-t}{k}\right). \tag{2.3.2}$$

2.3.2 Via toric viewpoint

We may expect if there exists a "nicer" expression to describe every constituent without making any rescaling of variable. It turns out that such expression can be obtained from the toric arrangement by appropriately extracting its poset of layers. Now let us turn to the second interpretation via toric arrangement viewpoint. In the remainder of this subsection, we assume that G is either $\mathbb{S}^1$ or $\mathbb{C}^\times$. We retain the notation of the total group $T = \mathrm{Hom}(\Gamma, G)$, and its identity is denoted by $\mathbf{1}$. For each $k \in \mathbb{Z}$, consider the homomorphism

$$E_k : T \longrightarrow T \quad \text{via} \quad \varphi \mapsto \varphi^k := \varphi \cdots \varphi.$$

Definition 2.3.2.1.

(1) For each $k \in \mathbb{Z}$, the k-*total* intersection poset of $\mathcal{A}(G)$ is defined by

$$L[k] = \{\mathcal{C} \in L \mid \mathbf{1} \in E_k(\mathcal{C})\}.$$

(2) The k-*total* characteristic polynomial of $\mathcal{A}(G)$ is defined by

$$\chi^{k\text{-tot}}_{\mathcal{A}(G)}(t) := \sum_{\mathcal{C} \in L[k]} \mu(T^{\mathcal{C}}, \mathcal{C}) t^{\dim(\mathcal{C})}.$$

The cover relation in L is preserved in $L[k]$ i.e., if $\mathcal{C}$ covers $\mathcal{D}$ in L and $\mathcal{C} \in L[k]$ then $\mathcal{D} \in L[k]$, which implies that $L[k]$ is an *order ideal* (e.g., [Sta11, §3.1]). For each $\mathcal{S} \subseteq \mathcal{A}$, note that $H_{\mathcal{S},G}$ is a subtorus of T whose each connected component is isomorphic to the torus $G^{r_\Gamma - r_\mathcal{S}}$. Let $\mathcal{C}^{\mathbf{1}}_{\mathcal{S}} \in \mathrm{cc}(H_{\mathcal{S},G})$ be the *identity component* of $H_{\mathcal{S},G}$, that is, the connected component that contains $\mathbf{1}$. Thus $\mathrm{cc}(H_{\mathcal{S},G})$ can be identified with the quotient group $H_{\mathcal{S},G}/\mathcal{C}^{\mathbf{1}}_{\mathcal{S}}$. In the lemma below, we generalize [Moc12, Lemma 5.4] in an arithmetical manner.

Lemma 2.3.2.2. *Fix $k \in \mathbb{Z}_{>0}$. For each $\mathcal{S} \subseteq \mathcal{A}$, we have*

$$\#\left(\mathrm{cc}(H_{\mathcal{S},G}) \cap L[k]\right) = m(\mathcal{S}; \mathbb{Z}_k).$$

Proof. For each $\mathcal{S} \subseteq \mathcal{A}$, the homomorphism E_k induces the endomorphism $\overline{E_k}$ of $H_{\mathcal{S},G}/\mathcal{C}_{\mathcal{S}}^{\mathbf{1}}$ with

$$\ker(\overline{E_k}) = \mathrm{cc}(H_{\mathcal{S},G}) \cap L[k]. \tag{2.3.1}$$

Using the identification $H_{\mathcal{S},G} = \mathrm{Hom}(\Gamma/\langle\mathcal{S}\rangle, G)$ and a decomposition $\Gamma/\langle\mathcal{S}\rangle = (\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{tor}} \oplus (\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{free}}$, we can write

$$\mathcal{C}_{\mathcal{S}}^{\mathbf{1}} = \{\varphi \in H_{\mathcal{S},G} \mid \varphi(x) = 1,\ \forall x \in (\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{tor}}\}.$$

Applying the left exact functor $\mathrm{Hom}(-, G)$ to the following exact sequence

$$0 \longrightarrow \left(\frac{\Gamma}{\langle\mathcal{S}\rangle}\right)_{\mathrm{tor}} \longrightarrow \frac{\Gamma}{\langle\mathcal{S}\rangle} \longrightarrow \frac{\Gamma}{\langle\mathcal{S}\rangle} \Big/ \left(\frac{\Gamma}{\langle\mathcal{S}\rangle}\right)_{\mathrm{tor}} \longrightarrow 0,$$

we obtain

$$H_{\mathcal{S},G}/\mathcal{C}_{\mathcal{S}}^{\mathbf{1}} \simeq \mathrm{Hom}((\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{tor}}, G). \tag{2.3.2}$$

Furthermore, E_k induces the endomorphism $\widetilde{E_k}$ of $\mathrm{Hom}((\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{tor}}, G)$ with

$$\ker(\widetilde{E_k}) = \mathrm{Hom}((\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{tor}}, G[k]). \tag{2.3.3}$$

Here $G[k] = \{x \in G \mid x^k = 1\} \simeq \mathbb{Z}_k$. Combining (2.3.1), (2.3.2) and (2.3.3) we get $\#\left(\mathrm{cc}(H_{\mathcal{S},G}) \cap L[k]\right) = m(\mathcal{S}; \mathbb{Z}_k)$, as desired. □

Definition 2.3.2.3.

(1) For each $k \in \mathbb{Z}$, the *k-partial* intersection poset of $\mathcal{A}(G)$ is defined by

$$L^{\mathrm{par}}[k] := \{\mathcal{C} \in L^{\mathrm{par}} \mid \mathbf{1} \in E_k(\mathcal{C})\}.$$

(2) The *k-partial* characteristic polynomial of $\mathcal{A}(G)$ is defined by

$$\chi_{\mathcal{A}(G)}^{k\text{-par}}(t) := \sum_{\mathcal{C} \in L^{\mathrm{par}}[k]} \mu(T^{\mathcal{C}}, \mathcal{C}) t^{\dim(\mathcal{C})}.$$

Theorem 2.3.2.4. *If $q \in \mathbb{Z}_{>0}$, then*

$$\chi_{\mathcal{A}(G)}^{q\text{-par}}(q) = \chi_{\mathcal{A}}^{\mathrm{quasi}}(q).$$

Proof. Very similar to Proof of Theorem 2.1.2.7 including the use of Lemma 2.3.2.2. ☐

Corollary 2.3.2.5. *If* $1 \le k \le \rho_{\mathcal{A}}$, *then*

$$\chi^{k\text{-par}}_{\mathcal{A}(G)}(t) = f^k_{\mathcal{A}}(t).$$

Corollary 2.3.2.6. *If* $q \in \mathbb{Z}_{>0}$ *and* $1 \le k \le \rho_{\mathcal{A}}$, *then*

$$\begin{aligned}\chi^{q\text{-tot}}_{\mathcal{A}(G)}(q) =& \chi^{\text{quasi}}_{\mathcal{A}\setminus\mathcal{A}^{\text{tor}}}(q),\\ \chi^{k\text{-tot}}_{\mathcal{A}(G)}(t) =& f^{k}_{\mathcal{A}\setminus\mathcal{A}^{\text{tor}}}(t).\end{aligned}$$

Proof. It follows from Theorem 2.3.2.4 and Corollary 2.3.2.5. ☐

Remark 2.3.2.7. Note that when $G = \mathbb{S}^1$ (or $G = \mathbb{C}^\times$) and $\mathcal{A}^{\text{tor}} = \emptyset$, we have $f^{\rho_{\mathcal{A}}}_{\mathcal{A}}(t) = \chi^{\rho_{\mathcal{A}}\text{-par}}_{\mathcal{A}(G)}(t) = \chi^{\rho_{\mathcal{A}}\text{-tot}}_{\mathcal{A}(G)}(t) = \chi^{\text{toric}}_{\mathcal{A}}(t)$. The result of Moci (Theorem 2.1.1.1) is a special case of Corollary 2.3.2.5 because by Corollary 2.2.1.8, $\chi^{\mathbb{S}^1}_{\mathcal{A}}(t) = \chi^{\mathbb{C}^\times}_{\mathcal{A}}(t) = f^{\rho_{\mathcal{A}}}_{\mathcal{A}}(t)$.

By Theorem 2.3.2.4, we can write

$$\chi^{\text{quasi}}_{\mathcal{A}}(q) = \sum_{j=r_\Gamma - r_{\mathcal{A}}}^{r_\Gamma} (-1)^{r_\Gamma - j}\beta_j(q)q^j, \tag{2.3.4}$$

with each coefficient $\beta_j(q)$ is a periodic function given by

$$\beta_j(q) = (-1)^{r_\Gamma - j} \sum_{\substack{\mathcal{C}\in L^{\text{par}}(q)\\ j=\dim(\mathcal{C})}} \mu(T^{\mathcal{C}}, \mathcal{C}) \ge 0.$$

It is easily seen that if $a, b \in \mathbb{Z}_{>0}$ and $a \mid b$, then $L^{\text{par}}[a] \subseteq L^{\text{par}}[b]$. This obvious inclusion between the subposets implies the result in [CW12, Theorem 1.2] about the inequality of the constituent coefficients.

Corollary 2.3.2.8 (Theorem 1.2, [CW12])**.** *If* a, b *are positive integers and* a *divides* b, *then for all* j *with* $r_\Gamma - r_{\mathcal{A}} \le j \le r_\Gamma$,

$$0 \le \beta_j(a) \le \beta_j(b).$$

Proof. Note that μ strictly alternates in sign (Corollary 2.1.2.6). ☐

Example 2.3.2.9. Let $\Gamma = \mathbb{Z}^2$, $\mathcal{A} = \{\alpha, \beta, \gamma\} \subsetneq \mathbb{Z}^2$ with $\alpha = (-1, 1)$, $\beta = (0, 2)$, and $\gamma = (0, 4)$ as in Example 2.3.1.3. The Hasse diagrams (arrow omitted) of $L^{\mathrm{par}}[k]$ are drawn in Figures 2.4, 2.5, 2.6. The k-partial characteristic polynomials $\chi^{k\text{-par}}_{\mathcal{A}(G)}(t)$ are computed according to the subposets extracted from L.

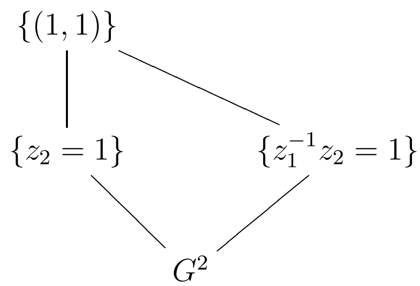


Figure 2.4: $\chi^{1\text{-par}}_{\mathcal{A}(G)}(t) = t^2 - 2t + 1 = f^1_{\mathcal{A}}(t)$.

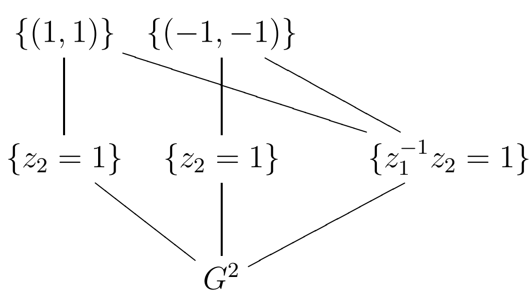


Figure 2.5: $\chi^{2\text{-par}}_{\mathcal{A}(G)}(t) = t^2 - 3t + 2 = f^2_{\mathcal{A}}(t)$.

2.4 Characteristic quasi-polynomials and root systems

In this section, we study the characteristic quasi-polynomials through a very well-behaved class of arrangements, the class of Weyl arrangements arising from irreducible root systems. Let $\mathcal{H}$ be a Weyl arrangement. We introduce the notion of $\mathcal{A}$-Eulerian polynomial producing an Eulerian-like polynomial for any subarrangement of $\mathcal{H}$. This polynomial together with shift operator describe how the characteris-

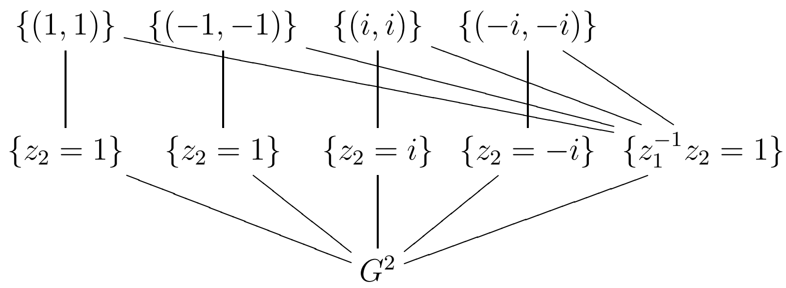


Figure 2.6: $\chi^{\text{4-par}}_{\mathcal{A}(G)}(t) = t^2 - 5t + 4 = f^4_{\mathcal{A}}(t)$.

tic quasi-polynomial of a certain class of subarrangements of $\mathcal{H}$ can be expressed in terms of the Ehrhart quasi-polynomial of the fundamental alcove. The method can also be extended to define two types of deformed Weyl subarrangements containing the families of the extended Shi, Catalan, Linial arrangements and to compute their characteristic quasi-polynomials. We obtain several known results in the literature as specializations, including the formula of the characteristic polynomial of $\mathcal{H}$ via Ehrhart theory due to Athanasiadis [Ath96], Blass-Sagan [BS98], Suter [Sut98] and Kamiya-Takemura-Terao [KTT10]; and the formula relating the number of coweight lattice points in the fundamental parallelepiped with the Lam-Postnikov's Eulerian polynomial [LP18] due to Yoshinaga [Yos18b]. Finally, using information from signed graphs, we give a numerical description of the characteristic quasi-polynomials of ideals of classical root systems with respect to the integer and root lattices.

2.4.1 Root systems and Worpitzky partition

Our standard references for root systems and their Weyl groups is [Hum90]. Assume that $V = \mathbb{R}^\ell$ with the standard inner product $(\cdot,\cdot)$. Let Φ be an irreducible (crystallographic) root system in V with the Coxeter number h and the Weyl group W. Fix a positive system $\Phi^+ \subseteq \Phi$ and let $\Delta := \{\alpha_1, \ldots, \alpha_\ell\}$ be the set of simple roots (base) of Φ associated with Φ^+. Define the partial order $\succeq$ on Φ^+ such that $\beta_1 \succeq \beta_2$ if and only if $\beta_1 - \beta_2 = \sum_{i=1}^{\ell} n_i\alpha_i$ with all $n_i \in \mathbb{Z}_{\ge 0}$. A subset I of Φ^+ is called an *ideal* if, for $\beta_1, \beta_2 \in \Phi^+$, $\beta_1 \succeq \beta_2, \beta_1 \in I$ then $\beta_2 \in I$. For $\beta = \sum_{i=1}^{\ell} c_i\alpha_i \in \Phi^+$, the *height* of β is defined by $\mathrm{ht}(\beta) := \sum_{i=1}^{\ell} c_i$. The highest root (w.r.t. $\succeq$), denoted by $\tilde{\alpha} \in \Phi^+$, can be expressed uniquely as a linear combination $\tilde{\alpha} = \sum_{i=1}^{\ell} c_i\alpha_i$ ($c_i \in \mathbb{Z}_{>0}$). Set $\alpha_0 := -\tilde{\alpha}$, $c_0 := 1$, and $\widetilde{\Delta} := \Delta \cup \{\alpha_0\}$. Then we have the linear relation

$$c_0\alpha_0 + c_1\alpha_1 + \cdots + c_\ell\alpha_\ell = 0.$$

The coefficients c_i are important in our study and will appear frequently throughout the section.

For $\alpha \in V \setminus \{0\}$, denote $\alpha^\vee := \frac{2\alpha}{(\alpha,\alpha)}$. The *root lattice* $Q(\Phi)$, *coroot lattice* $Q(\Phi^\vee)$, *weight lattice* $Z(\Phi)$, and *coweight lattice* $Z(\Phi^\vee)$ are defined as follows:

$$\begin{aligned} Q(\Phi) &:= \bigoplus_{i=1}^{\ell} \mathbb{Z}\alpha_i, \\ Q(\Phi^\vee) &:= \bigoplus_{i=1}^{\ell} \mathbb{Z}\alpha_i^\vee, \\ Z(\Phi) &:= \{x \in V \mid (\alpha_i^\vee, x) \in \mathbb{Z}\, (1 \le i \le \ell)\}, \\ Z(\Phi^\vee) &:= \{x \in V \mid (\alpha_i, x) \in \mathbb{Z}\, (1 \le i \le \ell)\}. \end{aligned}$$

Then $Q(\Phi)$ is a subgroup of $Z(\Phi)$ of finite index f, and similarly $Q(\Phi^\vee)$ is a subgroup of $Z(\Phi^\vee)$ of index f. The number f is called the *index of connection*. Let $\{\varpi_1^\vee, \dots, \varpi_\ell^\vee\} \subseteq Z(\Phi^\vee)$ be the dual basis of the base Δ, namely, $(\alpha_i, \varpi_j^\vee) = \delta_{ij}$. Then $Z(\Phi^\vee) = \bigoplus_{i=1}^{\ell} \mathbb{Z}\varpi_i^\vee$.

For $m \in \mathbb{Z}$ and $\alpha \in \Phi$, the *affine hyperplane* $H_{\alpha,m}$ is defined by

$$H_{\alpha,m} := \{x \in V \mid (\alpha, x) = m\}.$$

A connected component of $V \setminus \bigcup_{\alpha\in\Phi^+, m\in\mathbb{Z}} H_{\alpha,m}$ is called an *alcove*. The *fundamental alcove* A° is defined by

$$A^\circ := \left\{ x \in V \;\middle|\; \begin{array}{c} (\alpha_i, x) > 0\, (1 \le i \le \ell), \\ (\alpha_0, x) > -1 \end{array} \right\}.$$

The closure $\overline{A^\circ} = \{x \in V \mid (\alpha_i, x) \ge 0\, (1 \le i \le \ell), (\alpha_0, x) \ge -1\}$ is a simplex, which is the convex hull of $0, \varpi_1^\vee/c_1, \dots, \varpi_\ell^\vee/c_\ell \in V$. The supporting hyperplanes of the facets of $\overline{A^\circ}$ are $H_{\alpha_1,0}, \dots, H_{\alpha_\ell,0}, H_{\alpha_0,-1}$. The affine Weyl group $W_{\mathrm{aff}} := W \ltimes Q(\Phi^\vee)$ acts simply transitively on the set of alcoves and admits $\overline{A^\circ}$ as a fundamental domain for its action on V.

The fundamental domain $P^\diamondsuit$ of the coweight lattice $Z(\Phi^\vee)$, called the *fundamental parallelepiped*, is defined by

$$\begin{aligned} P^\diamondsuit &:= \sum_{i=1}^{\ell} (0,1]\varpi_i^\vee \\ &= \{x \in V \mid 0 < (\alpha_i, x) \leq 1 \, (1 \leq i \leq \ell)\}. \end{aligned}$$

Let $N := \frac{\#W}{f}$, and denote $[N] := \{1, 2, \dots, N\}$. Then the set Σ of all alcoves contained in $P^\diamondsuit$ has cardinality N (see, e.g., [Hum90, Theorem 4.9]). Let us write

$$\Sigma = \{A_i^\circ \subseteq P^\diamondsuit \mid i \in [N]\},$$

where each A_i° is written uniquely as

$$A_i^\circ = \left\{ x \in V \,\middle|\, \begin{array}{l} (\alpha, x) > k_\alpha \, (\alpha \in I), \\ (\beta, x) < k_\beta \, (\beta \in J) \end{array} \right\},$$

where $k_\alpha \in \mathbb{Z}_{\geq 0}$, $k_\beta \in \mathbb{Z}_{>0}$ and the sets $I, J \subseteq \Phi^+$ with $\#(I \sqcup J) = \ell + 1$ indicate the constraints on $x \in V$ according to the inequality symbols $>$, $<$, respectively.

Definition 2.4.1.1. For each $A_i^\circ \in \Sigma$, the *partial closure* $A_i^\diamondsuit$ of A_i° is defined by

$$A_i^\diamondsuit := \left\{ x \in V \,\middle|\, \begin{array}{l} (\alpha, x) > k_\alpha \, (\alpha \in I), \\ (\beta, x) \leq k_\beta \, (\beta \in J) \end{array} \right\}.$$

Theorem 2.4.1.2 (Worpitzky partition)**.**

$$P^\diamondsuit = \bigsqcup_{i \in [N]} A_i^\diamondsuit.$$

Proof. See, e.g., [Yos18b, Proposition 2.5], [Hum90, Exercise 4.3]. □

2.4.2 Characteristic and Ehrhart quasi-polynomials

Let us recall the notion of affine q-reduced arrangement (or $\mathbb{Z}_q$-plexification) in §1.1.1. Let $\Gamma = \bigoplus_{i=1}^{\ell} \mathbb{Z}\beta_i \simeq \mathbb{Z}^\ell$ be a free abelian group. Let $\mathcal{L}$ be a finite list (multiset) of elements in Γ. For $\alpha = \sum_{i=1}^{\ell} a_i\beta_i \in \mathcal{L}$ and $m_\alpha \in \mathbb{Z}$, set

$$H_{\alpha,m_\alpha,\mathbb{Z}_q} = \left\{\mathbf{z} \in \mathbb{Z}_q^\ell \mid \sum_{i=1}^{\ell} \overline{a_i} z_i \equiv \overline{m_\alpha}\right\}.$$

Given a vector $m = (m_\alpha)_{\alpha\in\mathcal{L}} \in \mathbb{Z}^{\mathcal{L}}$, the affine q-reduced arrangement of $(\mathcal{L}, m)$ w.r.t. Γ is

$$(\mathcal{L}, m)(\mathbb{Z}_q) = \{H_{\alpha,m_\alpha,\mathbb{Z}_q} \mid \alpha \in \mathcal{L}\}.$$

The complement of $(\mathcal{L}, m)(\mathbb{Z}_q)$ is

$$\mathcal{M}((\mathcal{L}, m); \Gamma, \mathbb{Z}_q) = \mathbb{Z}_q^\ell \setminus \bigcup_{\alpha\in\mathcal{L}} H_{\alpha,m_\alpha,\mathbb{Z}_q}.$$

For each $\mathcal{S} \subseteq \mathcal{L}$, write $\Gamma/\langle\mathcal{S}\rangle \simeq \bigoplus_{i=1}^{n_\mathcal{S}} \mathbb{Z}_{d_{\mathcal{S},i}} \oplus \mathbb{Z}^{r_\Gamma - r_\mathcal{S}}$ where $n_\mathcal{S} \geq 0$ and $1 < d_{\mathcal{S},i} | d_{\mathcal{S},i+1}$. The LCM-period $\rho_\mathcal{L}$ of $\mathcal{L}$ is

$$\rho_\mathcal{L} = \operatorname{lcm}(d_{\mathcal{S},n_\mathcal{S}} \mid \mathcal{S} \subseteq \mathcal{L}).$$

It is not hard to see that $\rho_\mathcal{B}$ divides $\rho_\mathcal{L}$ for any $\mathcal{B} \subseteq \mathcal{L}$.

Theorem 2.4.2.1. *$\#\mathcal{M}((\mathcal{L}, m); \Gamma, \mathbb{Z}_q)$ is a monic quasi-polynomial in q for which $\rho_\mathcal{L}$ is a period. The quasi-polynomial is called the* characteristic quasi-polynomial *of $(\mathcal{L}, m)$ w.r.t. Γ, and denoted by $\chi^{\text{quasi}}_{(\mathcal{L},m)}(q)$.*

Proof. See [KTT08, Theorem 2.4] and [KTT11, Theorem 3.1]. □

For a real affine hyperplane arrangement $\mathcal{H}$, denote by $\chi_\mathcal{H}(t)$ the characteristic polynomial (e.g., see [OT92, Definition 2.52]) of $\mathcal{H}$.

Theorem 2.4.2.2. *The first constituent of $\chi^{\text{quasi}}_{(\mathcal{L},m)}(q)$ coincides with the characteristic polynomial of the $\mathbb{R}$-plexification $(\mathcal{L}, m)(\mathbb{R})$, i.e.,*

$$f^1_{(\mathcal{L},m)}(t) = \chi_{(\mathcal{L},m)(\mathbb{R})}(t).$$

Proof. See, e.g., [KTT08, Theorem 2.5] and [KTT11, Remark 3.3]. □

Convention: From §2.4.2 to §2.4.4, for every $\Psi \subseteq \Phi^+ (\subseteq Q(\Phi))$, the characteristic quasi-polynomial $\chi^{\mathrm{quasi}}_{\Psi}(q)$ is always defined with respect to the root lattice $\Gamma = Q(\Phi)$. In §2.4.5, our computation will be involved with another choice of lattice, the integer lattice.

For each $\Psi \subseteq \Phi^+$, define the *Weyl arrangement* of Ψ by $\mathcal{H}_\Psi := \{H_\alpha \mid \alpha \in \Psi\}$, where $H_\alpha = \{x \in V \mid (\alpha, x) = 0\}$ is the hyperplane orthogonal to α. It is not hard to see that $H_\alpha \simeq H_{\alpha,\mathbb{R}}$ (as vector spaces), thus we can view $\mathcal{H}_\Psi$ as the $\mathbb{R}$-plexification of Ψ, i.e., $\mathcal{H}_\Psi = \Psi(\mathbb{R})$. In standard terminology, $\mathcal{H}_{\Phi^+}$ is known with the name *Weyl (or Coxeter) arrangement*, and clearly $\mathcal{H}_\Psi$ is a (Weyl) subarrangement of $\mathcal{H}_{\Phi^+}$.

Remark 2.4.2.3. Sometimes, when we say "the" characteristic quasi-polynomial of $(\mathcal{L}, m)(\mathbb{Z}_q)$ or of $(\mathcal{L}, m)(\mathbb{R})$, we literally mean $\chi^{\mathrm{quasi}}_{(\mathcal{L},m)}(q)$. In particular, $\chi^{\mathrm{quasi}}_{\Psi}(q)$ will be referred to the characteristic quasi-polynomial $\chi^{\mathrm{quasi}}_{\mathcal{H}_\Psi}(q)$ of $\mathcal{H}_\Psi$. We will use this notation later in §2.4.4 when we deal with deformed Weyl arrangements of Ψ.

Let Γ be a lattice. For a polytope $\mathcal{P}$ with vertices in the rational vector space generated by Γ, the *Ehrhart quasi-polynomial* $\mathrm{L}_{\mathcal{P}}(q)$ of $\mathcal{P}$ with respect to Γ is defined by

$$\mathrm{L}_{\mathcal{P}}(q) := \#(q\mathcal{P} \cap \Gamma).$$

We denote by $\mathcal{P}^\circ$ the relative interior of $\mathcal{P}$. Similarly, we can define

$$\mathrm{L}_{\mathcal{P}^\circ}(q) := \#(q\mathcal{P}^\circ \cap \Gamma).$$

For $q > 0$, the following Ehrhart reciprocity law holds:

$$\mathrm{L}_{\mathcal{P}}(-q) = (-1)^{\dim \mathcal{P}} \mathrm{L}_{\mathcal{P}^\circ}(q).$$

Convention: Throughout the section, the Ehrhart quasi-polynomials $\mathrm{L}_{\overline{A^\circ}}(q)$, $\mathrm{L}_{A^\circ}(q)$ are defined with respect to the coweight lattice $\Gamma = Z(\Phi^\vee)$.

Let $F_0 := H_{\alpha_0,-q}$, $F_i := H_{\alpha_i,0}$ $(1 \le i \le \ell)$ denote the supporting hyperplanes of the facets of $q\overline{A^\circ}$. Then the number of coweight lattice points in $q\overline{A^\circ}$ after removing some facets can be computed by $\mathrm{L}_{\overline{A^\circ}}$ with the scale factor of dilation being reduced.

Proposition 2.4.2.4. *Let* $\{i_1, \ldots, i_k\} \subseteq \{0, 1, \ldots, \ell\}$. *Suppose that* $q > c_{i_1} + \cdots + c_{i_k}$. *Then*

$$\#(q\overline{A^\circ} \cap Z(\Phi^\vee) \setminus (F_{i_1} \cup \cdots \cup F_{i_k})) = \mathrm{L}_{\overline{A^\circ}}(q - (c_{i_1} + \cdots + c_{i_k})).$$

In particular, for $q \in \mathbb{Z}$,

$$\mathrm{L}_{A^\circ}(q) = \mathrm{L}_{\overline{A^\circ}}(q - \mathrm{h}).$$

Proof. See [Yos18b, Corollaries 3.4 and 3.5]. □

In general, it is not easy to find explicit formulas which involve both characteristic and Ehrhart quasi-polynomials. With regards to root systems, there is an interesting relation between these quasi-polynomials.

Theorem 2.4.2.5.

$$\chi^{\mathrm{quasi}}_{\Phi^+}(q) = \frac{\#W}{f}\mathrm{L}_{A^\circ}(q).$$

Proof. See, e.g., [KTT10], [Yos18b, Proposition 3.7]. □

Theorem 2.4.2.6. *The minimum period of* $\chi^{\mathrm{quasi}}_{\Phi^+}(q)$ *is equal to* $\mathrm{lcm}(c_1, \ldots, c_\ell)$. *Furthermore, by a case-by-case argument, we can verify that the minimum period coincides with the LCM-period* ρ_{Φ^+}.

Proof. See [KTT10, Corollary 3.2 and Remark 3.3]. □

Corollary 2.4.2.7. $\chi^{\mathrm{quasi}}_{\Phi^+}(q) > 0$ *(equivalently,* $\mathrm{L}_{A^\circ}(q) > 0$*) if and only if* $q \geq \mathrm{h}$.

Proof. See, e.g., [KTT10, Corollary 3.4]. □

We can generalize the result above to have a formula for $\chi^{\mathrm{quasi}}_{\Psi}(q)$ for any $\Psi \subseteq \Phi^+$ in terms of lattice point counting functions. It will also explain why the choice of lattices for the characteristic and Ehrhart quasi-polynomials is important. In the proposition below, we view Ψ as a list with possible repetitions of elements.

Proposition 2.4.2.8. *Let* $m = (m_\alpha)_{\alpha \in \Psi}$ *be a vector in* $\mathbb{Z}^\Psi$. *Set*

$$X_{(\Psi,m)}(q) := qP^\diamondsuit \cap Z(\Phi^\vee) \setminus \bigcup_{\alpha \in \Psi, k \in \mathbb{Z}} H_{\alpha, kq+m_\alpha},$$

$$Y_{(\Psi,m)}(q) := \{\overline{x} \in Z(\Phi^\vee)/qZ(\Phi^\vee) \mid (\alpha, x) \not\equiv m_\alpha \bmod q, \forall \alpha \in \Psi\}.$$

We have bijections between sets

$$X_{(\Psi,m)}(q) \simeq Y_{(\Psi,m)}(q) \simeq \mathcal{M}((\Psi, m); \mathbb{Z}^\ell, \mathbb{Z}_q).$$

As a result,

$$\chi^{\mathrm{quasi}}_{(\Psi,m)}(q) = \#X_{(\Psi,m)}(q) = \#Y_{(\Psi,m)}(q).$$

Proof. The bijection $X_{(\Phi^+,m)}(q) \simeq Y_{(\Phi^+,m)}(q)$ is proved in [Yos18b, §3.3]. We can use exactly the same argument applied to any Ψ. The proof of $Y_\Psi(q) \simeq \mathcal{M}(\Psi; \mathbb{Z}^\ell, \mathbb{Z}_q)$ for an arbitrary $\Psi \subseteq \Phi^+$ runs as follows:

$$\begin{aligned} Y_{(\Psi,m)}(q) &= \{\overline{x} = \sum_{i=1}^{\ell} \overline{z_i}\varpi_i^\vee \mid (\alpha, x) \not\equiv m_\alpha \bmod q, \forall \alpha \in \Psi\} \\ &= \{\overline{x} = \sum_{i=1}^{\ell} \overline{z_i}\varpi_i^\vee \mid (\sum_{i=1}^{\ell} S_{ij}\alpha_i, \sum_{i=1}^{\ell} z_i\varpi_i^\vee) \not\equiv m_\alpha \bmod q,\ (1 \le j \le \#\Psi)\} \\ &\simeq \{\mathbf{z} = (\overline{z_1}, \dots, \overline{z_\ell}) \in \mathbb{Z}_q^\ell \mid \sum_{i=1}^{\ell} z_i S_{ij} \not\equiv m_\alpha \bmod q,\ (1 \le j \le \#\Psi)\} \\ &= \mathcal{M}((\Psi, m); \mathbb{Z}^\ell, \mathbb{Z}_q). \end{aligned}$$

□

Remark 2.4.2.9. The bijection $X_{\Phi^+}(q) \simeq \mathcal{M}(\Phi^+; \mathbb{Z}^\ell, \mathbb{Z}_q)$ appeared (without proof) in [KTT10, Proof of Theorem 3.1]. Theorem 2.4.2.5 is a special case of Proposition 2.4.2.8 because $\chi^{\mathrm{quasi}}_{\Phi^+}(q) = \#X_{\Phi^+}(q) = \frac{\#W}{f}\mathrm{L}_{A^\circ}(q)$ [Yos18b, §3.3].

2.4.3 Eulerian polynomials for Weyl subarrangements

Let $\Psi \subseteq \Phi^+$, and set $\Psi^c := \Phi^+ \setminus \Psi$.

Definition 2.4.3.1. The *descent* dsc_Ψ *with respect to* Ψ is the function $\mathrm{dsc}_\Psi : W \to \mathbb{Z}_{\ge 0}$ defined by

$$\mathrm{dsc}_\Psi(w) := \sum_{0 \le i \le \ell,\, w(\alpha_i) \in -\Psi^c} c_i.$$

Definition 2.4.3.2. The *(arrangement theoretical Eulerian or) $\mathcal{A}$-Eulerian polynomial* of Ψ is defined by

$$E_\Psi(t) := \frac{1}{f}\sum_{w\in W} t^{\mathrm{h}-\mathrm{dsc}_\Psi(w)}.$$

Remark 2.4.3.3.

(a) If $\Psi = \Phi^+$, then $\mathrm{dsc}_{\Phi^+}(w) = 0$ for all $w \in W$, and $E_{\Phi^+}(t) = \frac{\#W}{f}t^{\mathrm{h}}$.

(b) If $\Psi = \emptyset$, then $\mathrm{dsc}_\emptyset = \mathrm{dsc} = \mathrm{cdes}$, the descent statistic (see, e.g., [LP18, Definition 6.2], [Yos18b, Definition 4.1]). Then $E_\emptyset(t) = R_\Phi(t)$, the *generalized Eulerian polynomial* (e.g., [Yos18b, Definition 4.4]). Note that if Φ is of type A_ℓ, then $R_\Phi(t) = A_\ell(t)$, the classical *ℓ-th Eulerian polynomial* [LP18, Theorem 10.1].

Lemma 2.4.3.4. *For all $w \in W$, $0 \le \mathrm{dsc}_\Psi(w) < \mathrm{h}$. In particular, $E_\Psi(0) = 0$.*

Proof. If $\Psi = \Phi^+$, then the statements are clearly true by Remark 2.4.3.3(a). Assume that $\Psi \neq \Phi^+$. If $w(\alpha_i) \notin -\Psi^c$ for some $1 \le i \le \ell$, then $\mathrm{dsc}_\Psi(w) < \mathrm{h}$. Otherwise, we have $w(\alpha_0) = -\sum_{i=1}^{\ell} c_i w(\alpha_i) \in \Phi^+$. Thus $w(\alpha_0) \notin -\Psi^c$, and hence $\mathrm{dsc}_\Psi(w) < \mathrm{h}$. $\square$

Lemma 2.4.3.5.

(i) Let $w \in W$. Suppose that w induces a permutation on $\widetilde{\Delta}$. If $w(\alpha_i) = \alpha_{p_i}$, then $c_i = c_{p_i}$.

(ii) Let $w_1, w_2 \in W$. If there exists $\gamma \in V$ such that $w_1(A^\circ) = w_2(A^\circ) + \gamma$, then $\mathrm{dsc}_\Psi(w_1) = \mathrm{dsc}_\Psi(w_2)$.

Proof. (i) is exactly [Yos18b, Lemma 4.3(1)], and (ii) is similar to [Yos18b, Lemma 4.3(2)]. $\square$

Let A' be an arbitrary alcove. We can write $A' = w(A^\circ) + \gamma$ for some $w \in W$ and $\gamma \in Q(\Phi^\vee)$. By Lemma 2.4.3.5, we can extend dsc_Ψ to a function on the set of all alcoves (in particular, on the set Σ of alcoves contained in $P^\diamondsuit$) as follows:

Definition 2.4.3.6.

$$\mathrm{dsc}_\Psi(A') := \mathrm{dsc}_\Psi(w).$$

Theorem 2.4.3.7.

$$E_\Psi(t) = \sum_{i\in[N]} t^{\mathrm{h}-\mathrm{dsc}_\Psi(A_i^\circ)}.$$

Proof. Similar to [Yos18b, Theorem 4.7]. $\square$

In what follows, we shall sometimes abuse terminology and call a face of $\overline{A_i^\circ}$ (resp., $\overline{P^\diamondsuit}$) that is contained in the partial closure $A_i^\diamondsuit$ (resp., $P^\diamondsuit$), a face of $A_i^\diamondsuit$ (resp., $P^\diamondsuit$).

Definition 2.4.3.8. A subset $\Psi \subseteq \Phi^+$ is said to be *compatible* (with the Worpitzky partition) if for each $A_i^\diamondsuit \subseteq P^\diamondsuit$ the following condition holds: if $A_i^\diamondsuit \cap H_{\alpha,m_\alpha} \neq \emptyset$ for $\alpha \in \Psi, m_\alpha \in \mathbb{Z}$, then there exist $\beta \in \Psi, m_\beta \in \mathbb{Z}$ such that $A_i^\diamondsuit \cap H_{\alpha,m_\alpha} \subseteq A_i^\diamondsuit \cap H_{\beta,m_\beta}$ and $A_i^\diamondsuit \cap H_{\beta,m_\beta}$ is a facet of $A_i^\diamondsuit$.

Example 2.4.3.9. Clearly, Ψ is compatible if $\Psi = \emptyset$, or $\Psi = \Phi^+$. If $\Psi = \{\tilde{\alpha}\}$, where $\tilde{\alpha}$ is the highest root, then Ψ is not compatible because $H_{\tilde{\alpha},\mathrm{ht}(\tilde{\alpha})}$ intersects with $P^\diamondsuit$ only at $x = \sum_{i=1}^{\ell} \varpi_i^\vee$.

Definition 2.4.3.10. Let $f : \mathbb{Z} \to \mathbb{C}$ be a function and let $P(S) = a_0 + a_1 S + \cdots + a_n S^n$ be a polynomial in S. The *shift operator* via $P(S)$ acting on f is defined by

$$(P(S)f)(t) := \sum_{k=0}^{n} a_k f(t-k).$$

Theorem 2.4.3.11. *If Ψ is compatible, then*

$$\chi_\Psi^{\mathrm{quasi}}(q) = (E_\Psi(S)\mathrm{L}_{\overline{A^\circ}})(q).$$

Proof. The proof is similar in spirit to [Yos18b, Proof of Theorem 4.8]. Since both sides are quasi-polynomials, it is sufficient to prove the formula for $q \gg 0$ (actually, $q > \mathrm{h}$ is sufficient). For $i \in [N]$, we have

$$\begin{aligned}
&\#\left(qA_i^\diamondsuit \cap Z(\Phi^\vee) \setminus \bigcup_{\mu\in\Psi, k\in\mathbb{Z}} H_{\mu,kq} \right) \\
=&\#\left\{ x \in Z(\Phi^\vee) \;\middle|\; \begin{array}{c} (\alpha, x) > qk_\alpha \, (\alpha \in I) \\ (\beta, x) < qk_\beta \, (\beta \in J \cap \Psi) \\ (\delta, x) \leq qk_\delta \, (\delta \in J \cap \Psi^c) \end{array} \right\}
\end{aligned}$$

$$=\mathrm{L}_{\overline{A^\circ}}(q-\mathrm{h}+\mathrm{dsc}_\Psi(A_i^\circ)).$$

Note that we used Definition 2.4.1.1 of $A_i^\diamondsuit$ in the first equality. We applied Proposition 2.4.2.4 and Definition 2.4.3.6 in the second equality. More precisely, if $A_i^\circ = w(A^\circ)+\gamma$ for some $w \in W$ and $\gamma \in Q(\Phi^\vee)$, then qA_i° can be written as

$$qA_i^\circ = \left\{ x \in V \,\middle|\, \begin{array}{c} (-w(\alpha_0), x) < q((-w(\alpha_0),\gamma)+1), \\ (-w(\alpha_i), x) < q(-w(\alpha_i),\gamma), \ (1 \le i \le \ell) \end{array} \right\}.$$

Thus the half-spaces defined by $(\delta, x) \le qk_\delta \, (\delta \in J \cap \Psi^c)$ correspond exactly to the roots $\alpha_i \in \widetilde{\Delta}$ satisfying $w(\alpha_i) \in -\Psi^c$. The compatibility of Ψ ensures that if a lattice point in a face $qA_i^\diamondsuit \cap H_{\beta,qk_\beta}$ $(\beta \in \Psi)$ is removed, then all the lattice points in any facet $qA_i^\diamondsuit \cap H_{\gamma,qk_\gamma}$ $(\gamma \in \Psi)$ containing $qA_i^\diamondsuit \cap H_{\beta,qk_\beta}$ also get removed.

By the Worpitzky partition (Theorem 2.4.1.2) and Proposition 2.4.2.8, we have

$$\begin{aligned}\chi_\Psi^{\mathrm{quasi}}(q) &= \sum_{i\in[N]} \# \left(qA_i^\diamondsuit \cap Z(\Phi^\vee) \setminus \bigcup_{\mu\in\Psi, k\in\mathbb{Z}} H_{\mu,kq} \right) \\ &= \sum_{i\in[N]} \mathrm{L}_{\overline{A^\circ}}(q-\mathrm{h}+\mathrm{dsc}_\Psi(A_i^\circ)).\end{aligned}$$

Now using Theorem 2.4.3.7 together with the shift operator (Definition 2.4.3.10), we conclude that

$$\chi_\Psi^{\mathrm{quasi}}(q) = (E_\Psi(S)\mathrm{L}_{\overline{A^\circ}})(q).$$

□

Proposition 2.4.3.12.

$$\sum_{q\ge1} \mathrm{L}_{A^\circ}(q)t^q = \frac{t^{\mathrm{h}}}{\prod_{i=0}^{\ell}(1-t^{c_i})}.$$

Proof. See, e.g., [KTT10, Proof of Theorem 3.1]. □

Theorem 2.4.3.13.

(i)

$$\sum_{q\ge1} \chi_{\Phi^+}^{\mathrm{quasi}}(q)t^q = \frac{\frac{\#W}{f}t^{\mathrm{h}}}{\prod_{i=0}^{\ell}(1-t^{c_i})}.$$

(ii) Let $R_\Phi(t)$ be the generalized Eulerian polynomial (see Remark 2.4.3.3(b)). Then

$$\sum_{q\geq 1} q^\ell t^q = \frac{R_\Phi(t)}{\prod_{i=0}^{\ell}(1-t^{c_i})}.$$

Proof. For a proof of (i), see, e.g., [Ath96], [BS98, Theorem 4.1], [KTT10, Theorem 3.1]. (ii) follows from [LP18, Theorem 10.1]. □

Theorem 2.4.3.14. *If Ψ is compatible, then*

$$\sum_{q\geq 1} \chi_\Psi^{\mathrm{quasi}}(q) t^q = \frac{E_\Psi(t)}{\prod_{i=0}^{\ell}(1-t^{c_i})}.$$

Proof. By Theorem 2.4.3.11, Corollary 2.4.2.7 and Proposition 2.4.3.12, we have

$$\begin{aligned}
\sum_{q\geq 1} \chi_\Psi^{\mathrm{quasi}}(q) t^q &= \sum_{q\geq 1}\sum_{i\in[N]} \mathrm{L}_{A^\circ}(q+\mathrm{dsc}_\Psi(A_i^\circ))t^q \\
&= \sum_{i\in[N]} t^{-\mathrm{dsc}_\Psi(A_i^\circ)} \sum_{q\geq 1} \mathrm{L}_{A^\circ}(q+\mathrm{dsc}_\Psi(A_i^\circ))t^{q+\mathrm{dsc}_\Psi(A_i^\circ)} \\
&= \sum_{i\in[N]} t^{-\mathrm{dsc}_\Psi(A_i^\circ)} \sum_{n_\xi\geq \mathrm{h}} \mathrm{L}_{A^\circ}(n_\xi)t^{n_\xi} \\
&= \frac{\sum_{i\in[N]} t^{\mathrm{h}-\mathrm{dsc}_\Psi(A_i^\circ)}}{\prod_{i=0}^{\ell}(1-t^{c_i})} = \frac{E_\Psi(t)}{\prod_{i=0}^{\ell}(1-t^{c_i})}.
\end{aligned}$$

□

Remark 2.4.3.15. By Remark 2.4.3.3, Theorems 2.4.3.11 and 2.4.3.14, if

(a) $\Psi = \Phi^+$, then we recover Theorems 2.4.2.5 and 2.4.3.13(i).

(b) $\Psi = \emptyset$, then we recover [Yos18b, Theorem 4.8] and Theorem 2.4.3.13(ii).

2.4.4 Deformations of Weyl subarrangements

Let $\Psi \subseteq \Phi^+$, and recall the notation $\Psi^c = \Phi^+ \setminus \Psi$. Let $a \leq b$ be integers, and denote $[a,b] := \{m \in \mathbb{Z} \mid a \leq m \leq b\}$. Also, if $b \geq 1$, then write $[b]$ instead of $[1,b]$.

Definition 2.4.4.1. Let $a \leq b$, $c \leq d$ be integers. Define two types of the *deformed Weyl arrangements* of Ψ as follows:

(Type I) $\mathrm{I}_{\Psi}^{[a,b]} := \{H_{\alpha,m} \mid \alpha \in \Psi, m \in [a,b]\}$.

(Type II) $\mathrm{II}_{\Psi,\Psi^c}^{[a,b],[c,d]} := \mathrm{I}_{\Psi}^{[a,b]} \cup \mathrm{I}_{\Psi^c}^{[c,d]}$.

Remark 2.4.4.2.

(a) There is an obvious duality $\mathrm{II}_{\Psi,\Psi^c}^{[a,b],[c,d]} = \mathrm{II}_{\Psi^c,\Psi}^{[c,d],[a,b]}$. We can list some specializations: $\mathrm{I}_{\emptyset}^{[a,b]} = \varnothing$ the empty arrangement, $\mathrm{II}_{\emptyset,\Phi^+}^{[a,b],[c,d]} = \mathcal{H}_{\Phi^+}^{[c,d]}$ the *truncated affine Weyl arrangement*, including the extended Shi, Catalan, Linial arrangements, see, e.g., [SP00, §9]. In addition, $\mathrm{I}_{\Phi^+}^{[a,b]} = \mathrm{II}_{\Psi,\Psi^c}^{[a,b],[a,b]} = \mathcal{H}_{\Phi^+}^{[a,b]}$. We refer the reader to [Ath99], [Ath04], [Yos18b] and [Yos18a] for more details on the characteristic (quasi-)polynomials of $\mathcal{H}_{\Phi^+}^{[a,b]}$.

(b) The deformed Weyl arrangements of an arbitrary Ψ are less well-known. When Φ is of type A, the *deleted (or graphical) Shi arrangement*, see, e.g., [Ath96, §3] or [AR12], is the product [OT92, Definition 2.13] of the one dimensional empty arrangement and $\mathrm{II}_{\Psi,\Psi^c}^{[0,1],[0,0]}$.

Definition 2.4.4.3. For $w \in W$, define

$$\overline{\mathrm{dsc}}_{\Psi}(w) := \sum_{0 \le i \le \ell,\, w(\alpha_i) \in -\Psi} c_i,$$

$$\mathrm{asc}_{\Psi}(w) := \sum_{0 \le i \le \ell,\, w(\alpha_i) \in \Psi^c} c_i,$$

$$\overline{\mathrm{asc}}_{\Psi}(w) := \sum_{0 \le i \le \ell,\, w(\alpha_i) \in \Psi} c_i.$$

Obviously, $\mathrm{asc}_{\Psi}(w) + \overline{\mathrm{asc}}_{\Psi}(w) + \mathrm{dsc}_{\Psi}(w) + \overline{\mathrm{dsc}}_{\Psi}(w) = \mathrm{h}$ for all $w \in W$. Similar to Lemma 2.4.3.4, each function defined above takes values in $[0, \mathrm{h}-1]$. Furthermore,

$$\mathrm{asc}_{\emptyset}(w) = \overline{\mathrm{asc}}_{\Phi^+}(w) = \mathrm{asc}_{\Psi}(w) + \overline{\mathrm{asc}}_{\Psi}(w),$$

$$\mathrm{dsc}_{\emptyset}(w) = \overline{\mathrm{dsc}}_{\Phi^+}(w) = \mathrm{dsc}_{\Psi}(w) + \overline{\mathrm{dsc}}_{\Psi}(w).$$

Similar to Definition 2.4.3.6, we can extend the functions above to functions on the set of all alcoves.

Now let us formulate a deformed version of Proposition 2.4.2.4. Set

$$F_i^{[a_i,b_i]} := \bigcup_{m\in[a_i,b_i]} H_{\alpha_i,m} \ (1 \le i \le \ell),$$
$$F_0^{[a_0,b_0]} := \bigcup_{m\in[a_0,b_0]} H_{\alpha_0,-q+m}.$$

Proposition 2.4.4.4. *Let $\{i_1,\dots,i_k\} \subseteq [0,\ell]$, and let $b_{i_j} \ge 0$ for all $1 \le j \le k$. Suppose that $q > \sum_{j\in[k]}(b_{i_j}+1)c_{i_j}$. Then*

$$\#\left(q\overline{A^\circ} \cap Z(\Phi^\vee) \setminus \bigcup_{j\in[k]} F_{i_j}^{[0,b_{i_j}]}\right) = \mathrm{L}_{\overline{A^\circ}}(q - \sum_{j\in[k]}(b_{i_j}+1)c_{i_j}).$$

Proof. The formula was implicitly used in [Yos18b, §5] and its proof is very similar to the non-deformed case. Note that if $i \in [\ell]$, then

$$\begin{aligned}\#\left(q\overline{A^\circ} \cap Z(\Phi^\vee) \setminus F_i^{[0,b_i]}\right) &= \#\left\{ x \in Z(\Phi^\vee) \;\middle|\; \begin{array}{c} (\alpha_i, x) \ge b_i + 1 \\ (\alpha_j, x) \ge 0 \,(j \in [\ell]\setminus\{i\}) \\ (\alpha_0, x) \ge -q \end{array} \right\} \\ &= \mathrm{L}_{\overline{A^\circ}}(q - (b_i+1)c_i).\end{aligned}$$

Here the last equality follows from the bijection $x \mapsto x + (b_i+1)\varpi_1^\vee$. The proof for $i = 0$ is similar. Then apply the formula above repeatedly. □

Remark 2.4.4.5. If we replace the interval $[0, b_{i_j}]$ in Proposition 2.4.4.4 by $[a, b_{i_j}]$ with $a \ge 1$, there might be a large change in the right-hand side of the formula above. For example, if $1 \le a_1 \le b_1$, then

$$\#\left(q\overline{A^\circ} \cap Z(\Phi^\vee) \setminus F_1^{[a_1,b_1]}\right) = \mathrm{L}_{\overline{A^\circ}}(q - (b_1+1)c_1) + \mathrm{L}_{\overline{A^\circ}}(q) - \mathrm{L}_{\overline{A^\circ}}(q - a_1c_1).$$

Theorem 2.4.4.6. *Let Ψ be a compatible subset of Φ^+.*

(i) If $a, b \ge 0$, then

$$\chi^{\mathrm{quasi}}_{\mathrm{I}_\Psi^{[-a,b]}}(q) = \sum_{i\in[N]} \mathrm{L}_{\overline{A^\circ}}(q - (b+1)\overline{\mathrm{asc}}_\Psi(A_i^\circ) - \mathrm{asc}_\Psi(A_i^\circ) - (a+1)\overline{\mathrm{dsc}}_\Psi(A_i^\circ)).$$

(ii) If $b \geq 1$, then

$$\chi^{\mathrm{quasi}}_{\mathrm{I}_\Psi^{[1,b]}}(q) = \sum_{i \in [N]} \mathrm{L}_{\overline{A^\circ}}(q - (b+1)\overline{\mathrm{asc}}_\Psi(A_i^\circ) - \mathrm{asc}_\Psi(A_i^\circ)).$$

Proof. Proofs of (i) and (ii) are similar, and both are similar in spirit to the proof of [Yos18b, Theorem 5.1]. See also Proof of Theorem 2.4.3.11 in this paper. First, we give a proof for (i). Since both sides are quasi-polynomials, it is sufficient to prove the equality for $q \gg 0$ (actually, $q > (a+b+3)\mathrm{h}$ is sufficient). By Proposition 2.4.4.4, for $i \in [N]$,

$$\begin{aligned}
&\#\left(qA_i^\diamondsuit \cap Z(\Phi^\vee) \setminus \bigcup_{\mu \in \Psi, k \in \mathbb{Z}, m \in [-a,b]} H_{\mu,kq+m}\right) \\
=&\#\left\{ x \in Z(\Phi^\vee) \;\middle|\; \begin{array}{c} (\alpha, x) \geq qk_\alpha + b + 1 \,(\alpha \in I \cap \Psi) \\ (\eta, x) > qk_\eta \,(\eta \in I \cap \Psi^c) \\ (\beta, x) \leq qk_\beta - a - 1 \,(\beta \in J \cap \Psi) \\ (\delta, x) \leq qk_\delta \,(\delta \in J \cap \Psi^c) \end{array} \right\} \\
=&\mathrm{L}_{\overline{A^\circ}}(q - (b+1)\overline{\mathrm{asc}}_\Psi(A_i^\circ) - \mathrm{asc}_\Psi(A_i^\circ) - (a+1)\overline{\mathrm{dsc}}_\Psi(A_i^\circ)).
\end{aligned}$$

By Proposition 2.4.2.8, we have

$$\chi^{\mathrm{quasi}}_{\mathrm{I}_\Psi^{[-a,b]}}(q) = \sum_{i \in [N]} \mathrm{L}_{\overline{A^\circ}}(q - (b+1)\overline{\mathrm{asc}}_\Psi(A_i^\circ) - \mathrm{asc}_\Psi(A_i^\circ) - (a+1)\overline{\mathrm{dsc}}_\Psi(A_i^\circ)).$$

For (ii), note that for $i \in [N]$,

$$\begin{aligned}
&\#\left(qA_i^\diamondsuit \cap Z(\Phi^\vee) \setminus \bigcup_{\mu \in \Psi, k \in \mathbb{Z}, m \in [1,b]} H_{\mu,kq+m}\right) \\
=&\#\left\{ x \in Z(\Phi^\vee) \;\middle|\; \begin{array}{c} (\alpha, x) \geq qk_\alpha + b + 1 \,(\alpha \in I \cap \Psi) \\ (\eta, x) > qk_\eta \,(\eta \in I \cap \Psi^c) \\ (\beta, x) \leq qk_\beta \,(\beta \in J \cap \Psi) \\ (\delta, x) \leq qk_\delta \,(\delta \in J \cap \Psi^c) \end{array} \right\} \\
=&\mathrm{L}_{\overline{A^\circ}}(q - (b+1)\overline{\mathrm{asc}}_\Psi(A_i^\circ) - \mathrm{asc}_\Psi(A_i^\circ)).
\end{aligned}$$

$\square$

Remark 2.4.4.7. Theorem 2.4.4.6 is a generalization of several known results.

(a) When $\Psi = \Phi^+$, we obtain [Ath04, Theorem 1.2] ($a = b \geq 0$), and Theorem 5.1 ($a = b-1 \geq 0$), Theorem 5.2 ($b \geq 1$), Theorem 5.3 ($b = n+k$, $a = k-1$, $n, k \geq 1$) in [Yos18b]. When $\Psi = \emptyset$, we obtain [Yos18b, Theorem 4.8].

(b) When $\Psi \subseteq \Phi^+$ is compatible and $a = b = 0$, we obtain Theorem 2.4.3.11.

By using the same method, we have the following result for the arrangements of type II.

Theorem 2.4.4.8. *Let Ψ be a compatible subset of Φ^+.*

(i) If $a, b, c, d \geq 0$, then

$$\chi^{\mathrm{quasi}}_{\mathrm{II}^{[a,b],[c,d]}_{\Psi,\Psi^c}}(q) = \sum_{i\in[N]} \mathrm{L}_{\overline{A^\circ}}(q - (b+1)\overline{\mathrm{asc}}_\Psi(A_i^\circ) - (d+1)\mathrm{asc}_\Psi(A_i^\circ)$$
$$- (a+1)\overline{\mathrm{dsc}}_\Psi(A_i^\circ) - (c+1)\mathrm{dsc}_\Psi(A_i^\circ)).$$

(ii) If $a, b \geq 0, d \geq 1$, then

$$\chi^{\mathrm{quasi}}_{\mathrm{II}^{[a,b],[1,d]}_{\Psi,\Psi^c}}(q) = \sum_{i\in[N]} \mathrm{L}_{\overline{A^\circ}}(q - (b+1)\overline{\mathrm{asc}}_\Psi(A_i^\circ) - (d+1)\mathrm{asc}_\Psi(A_i^\circ)$$
$$- (a+1)\overline{\mathrm{dsc}}_\Psi(A_i^\circ)).$$

(iii) If $b, d \geq 1$, then

$$\chi^{\mathrm{quasi}}_{\mathrm{II}^{[1,b],[1,d]}_{\Psi,\Psi^c}}(q) = \sum_{i\in[N]} \mathrm{L}_{\overline{A^\circ}}(q - (b+1)\overline{\mathrm{asc}}_\Psi(A_i^\circ) - (d+1)\mathrm{asc}_\Psi(A_i^\circ)).$$

Remark 2.4.4.9.

(a) One can work with other intervals $[a, b]$ but the computation may become more complicated (see Remark 2.4.4.5).

(b) One can define and study the arrangement $\bigsqcup_{k=1}^n \mathrm{I}^{[a_k,b_k]}_{\Psi_k}$ where $\Phi^+ = \bigsqcup_{k=1}^n \Psi_k$ with $n \geq 3$. See, e.g., [Ath96, Theorem 3.11] for an example when $n = 3$. We choose not to develop this direction here.

It is interesting to compare the following result with [AR12, Theorem 3.2] and [Ath96, Theorem 3.9].

Corollary 2.4.4.10. *Define* $M_\Psi(t) := \frac{1}{f}\sum_{w\in W} t^{\mathrm{h}+\overline{\mathrm{asc}}_\Psi(w)}$*. If* Ψ *is compatible, then*

$$\chi^{\mathrm{quasi}}_{\mathrm{II}^{[0,1],[0,0]}_{\Psi,\Psi^c}}(q) = (M_\Psi(S)\mathrm{L}_{\overline{A^\circ}})(q).$$

2.4.5 Classical root systems, ideals and signed graphs

In this subsection, with a combinatorial ingredient of signed graphs, we compute the characteristic quasi-polynomial of every ideal of a given classical root system with respect to the integer and root lattices. The method used here is numerical and greatly different to that of §2.4.3, and that would be interesting to compare them.

Let us recall some notations on root systems from §2.4.1. Let Φ be an irreducible (crystallographic) root system in $V = \mathbb{R}^\ell$. Fix a positive system $\Phi^+ \subseteq \Phi$ and the associated base $\Delta = \{\alpha_1, \dots, \alpha_\ell\} \subseteq \Phi^+$.

Let us recall the recent advance towards the study of the ideals. Let $\Theta^{(k)} \subseteq \Phi^+$ be the set consisting of positive roots of height k. Let I be an ideal of Φ^+ and set $M := \max\{\mathrm{ht}(\beta) \mid \beta \in I\}$. The *height distribution* of I is defined as a sequence of positive integers:

$$(i_1, \dots, i_k, \dots, i_M),$$

where $i_k := \#\Theta^{(k)}$ for $1 \le k \le M$. The *dual partition* $\mathcal{DP}(I)$ of (the height distribution of) I is given by a sequence of nonnegative integers:

$$\mathcal{DP}(I) := \left((0)^{\ell-i_1}, (1)^{i_1-i_2}, \dots, (M-1)^{i_{M-1}-i_M}, (M)^{i_M}\right),$$

where notation $(a)^b$ means the integer a appears exactly b times. Although the definition of the dual partition seems to esteem the (increasing) order of components in the sequence, this requirement is not important in this subsection. Two dual partitions of an ideal are conventionally identical if the partitions differ only by a re-ordering of the components.

Theorem 2.4.5.1. *Any ideal subarrangement $\mathcal{H}_I$ is free (in the sense of Terao) and the set of exponents coincides with $\mathcal{DP}(I)$.*

Proof. See [ABC+16]. □

Corollary 2.4.5.2. *For any ideal $I \subseteq \Phi^+$, the characteristic polynomial $\chi_{\mathcal{H}_I}(\Phi, t)$ of $\mathcal{H}_I$ factors as follows:*

$$\chi_{\mathcal{H}_I}(\Phi, t) = \prod_{i=1}^{\ell}(t - d_i),$$

where $\mathcal{DP}(I) = (d_1, \ldots, d_\ell)$.

Proof. See [ABC+16]. □

Let us recall some notions on characteristic quasi-polynomials from §2.4.2. For simplicity of notation, an integral matrix M and the list of its column vectors will be denoted by the same symbol M. For each $\Psi \subseteq \Phi^+$, we assume that an $\ell \times \#\Psi$ integral matrix $S_\Psi = [S_{ij}]$ satisfies

$$\Psi = \Big\{ \sum_{i=1}^{\ell} S_{ij}\alpha_i \mid 1 \le j \le \#\Psi \Big\}.$$

In other words, S_Ψ is the coefficient matrix of Ψ with respect to the base Δ. Denote $\mathbb{Z}_q^\times = \mathbb{Z}_q \setminus \{\overline{0}\}$. Then $\chi^{\mathrm{quasi}}_{S_\Psi}(\Phi, q)$ is called the characteristic quasi-polynomial of Ψ with respect to the *root lattice* $\Gamma = \bigoplus_{i=1}^{\ell} \mathbb{Z}\alpha_i$, and interpreted by the formula

$$\chi^{\mathrm{quasi}}_{S_\Psi}(\Phi, q) = \#\{\mathbf{z} \in \mathbb{Z}_q^\ell \mid \mathbf{z} \cdot S_\Psi \in (\mathbb{Z}_q^\times)^{\#\Psi}\}.$$

In the remainder of this subsection, we are mainly interested in the root system Φ of classical type $(ABCD)$. Let us recall briefly the constructions of these root systems[2] following [Bou68, Chapter VI, §4]. Let $\{\epsilon_1, \ldots, \epsilon_\ell\}$ be an orthonormal basis for V. If $\ell \ge 2$ then

$$\Phi(B_\ell) = \{\pm\epsilon_i \, (1 \le i \le \ell), \pm(\epsilon_i \pm \epsilon_j) \, (1 \le i < j \le \ell)\},$$

[2]We decided to omit the construction of type A root systems as the calculation on this type follows from those on the other types (e.g., see formula (2.4.1)).

with $\#\Phi(B_\ell) = 2\ell^2$ is an irreducible root system in V of type B_ℓ. We may choose a positive system

$$\Phi^+(B_\ell) = \{\epsilon_i\,(1 \le i \le \ell), \epsilon_i \pm \epsilon_j\,(1 \le i < j \le \ell)\}.$$

Define $\alpha_i := \epsilon_i - \epsilon_{i+1}$, for $1 \le i \le \ell - 1$, and $\alpha_\ell := \epsilon_\ell$. Then $\Delta(B_\ell) = \{\alpha_1, \ldots, \alpha_\ell\}$ is the base associated with $\Phi^+(B_\ell)$. We may express

$$\begin{aligned}\Phi^+(B_\ell) = \{\epsilon_i = \sum_{i \le k \le \ell} \alpha_k\,(1 \le i \le \ell), \epsilon_i - \epsilon_j = \sum_{i \le k < j} \alpha_k\,(1 \le i < j \le \ell),\\ \epsilon_i + \epsilon_j = \sum_{i \le k < j} \alpha_k + 2\sum_{j \le k \le \ell} \alpha_k\,(1 \le i < j \le \ell)\}.\end{aligned}$$

For any $\Psi \subseteq \Phi^+(B_\ell)$, we write $T_\Psi = [T_{ij}]$ for the coefficient matrix of Ψ with respect to the orthonormal basis. Then $\chi^{\text{quasi}}_{T_\Psi}(\Phi, q)$ is called the characteristic quasi-polynomial of Ψ with respect to the *integer lattice* $\Gamma = \bigoplus_{i=1}^{\ell} \mathbb{Z}\epsilon_i$. The matrices T_Ψ and S_Ψ are related by $T_\Psi = P(B_\ell) \cdot S_\Psi$, where $P(B_\ell)$ is an unimodular matrix of size $\ell \times \ell$ given by

$$P(B_\ell) = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & -1 & 1 \end{bmatrix}.$$

Similarly, let $\ell \ge 2$, an irreducible root system of type C_ℓ is given by

$$\begin{aligned}\Phi(C_\ell) &= \{\pm 2\epsilon_i\,(1 \le i \le \ell), \pm(\epsilon_i \pm \epsilon_j)\,(1 \le i < j \le \ell)\},\\ \Phi^+(C_\ell) &= \{2\epsilon_i\,(1 \le i \le \ell), \epsilon_i \pm \epsilon_j\,(1 \le i < j \le \ell)\},\\ \Delta(C_\ell) &= \{\alpha_i = \epsilon_i - \epsilon_{i+1}\,(1 \le i \le \ell - 1),\ \alpha_\ell = 2\epsilon_\ell\},\\ \Phi^+(C_\ell) &= \{2\epsilon_i = 2\sum_{i \le k < \ell} \alpha_k + \alpha_\ell\,(1 \le i \le \ell), \epsilon_i - \epsilon_j = \sum_{i \le k < j} \alpha_k\,(1 \le i < j \le \ell),\\ &\qquad \epsilon_i + \epsilon_j = \sum_{i \le k < j} \alpha_k + 2\sum_{j \le k < \ell} \alpha_k + \alpha_\ell\,(1 \le i < j \le \ell)\}.\end{aligned}$$

Finally, let $\ell \geq 3$, an irreducible root system of type D_ℓ is given by

$$\begin{aligned}
\Phi(D_\ell) &= \{\pm(\epsilon_i \pm \epsilon_j)\,(1 \leq i < j \leq \ell)\},\\
\Phi^+(D_\ell) &= \{\epsilon_i \pm \epsilon_j\,(1 \leq i < j \leq \ell)\},\\
\Delta(D_\ell) &= \{\alpha_i = \epsilon_i - \epsilon_{i+1}\,(1 \leq i \leq \ell-1),\, \alpha_\ell = \epsilon_{\ell-1} + \epsilon_\ell\},\\
\Phi^+(D_\ell) &= \{\epsilon_i + \epsilon_\ell = \sum_{i \leq k \leq \ell-2} \alpha_k + \alpha_\ell\,(1 \leq i < \ell),\\
&\qquad \epsilon_i - \epsilon_j = \sum_{i<k<j} \alpha_k\,(1 \leq i < j \leq \ell),\\
&\qquad \epsilon_i + \epsilon_j = \sum_{i \leq k < j} \alpha_k + 2\sum_{j \leq k < \ell-1} \alpha_k + \alpha_{\ell-1} + \alpha_\ell\,(1 \leq i < j < \ell)\}.
\end{aligned}$$

From the constructions above, we obtain the comparison of the height placements of positive roots in $\Phi(B_\ell)$, $\Phi(C_\ell)$ and $\Phi(D_\ell)$ as in Table 2.1.

Remark 2.4.5.3. The matrix S_Ψ is independent of the choice of base for Φ, while T_Ψ is not. In addition, the characteristic quasi-polynomials with respect to the root lattice have a nice expression in terms of the Ehrhart quasi-polynomials of the closed fundamental alcoves with respect to the coweight lattice (see Theorem 2.4.3.11). We shall see that the computation related to the integer lattice is simpler, and the relation $T = P \cdot S$ enables us to carry out the computation in the root lattice case.

Root	Height in B_ℓ	Height in C_ℓ	Height in D_ℓ
ϵ_i $(1 \leq i \leq \ell)$	$\ell - i + 1$	None	None
$2\epsilon_i$ $(1 \leq i \leq \ell)$	None	$2(\ell - i) + 1$	None
$\epsilon_i + \epsilon_j$ $(1 \leq i < j \leq \ell)$	$2\ell - i - j + 2$	$2\ell - i - j + 1$	$2\ell - i - j$
$\epsilon_i - \epsilon_j$ $(1 \leq i < j \leq \ell)$	$j - i$	$j - i$	$j - i$

Table 2.1: Height placements in $\Phi(B_\ell)$, $\Phi(C_\ell)$ and $\Phi(D_\ell)$.

In the language of *signed graphs* following [Zas81, §5], we can associate with each subset $\Psi \subseteq \Phi^+(B_\ell)$ a signed graph $G := G(\Psi) = (V_G, E_{G^+}, E_{G^-}, L_G)$ on the vertex set

$$V_G := \{v_i, v_j \mid \epsilon_i \in \Psi \text{ or } \epsilon_i - \epsilon_j \in \Psi \text{ or } \epsilon_i + \epsilon_j \in \Psi\},$$

with the set of positive edges $E_{G^+} := \{e^+_{ij} \mid \epsilon_i + \epsilon_j \in \Psi\}$, the set of negative edges $E_{G^-} := \{e^-_{ij} \mid \epsilon_i - \epsilon_j \in \Psi\}$, and the set of loops $L_G := \{\ell_i \mid \epsilon_i \in \Psi\}$. Alternatively, if $\Psi \subseteq \Phi^+(C_\ell)$, we can define $L_G := \{\ell_i \mid 2\epsilon_i \in \Psi\}$. To extract information from Ψ by using $G(\Psi)$, we associate with it an unordered sequence of nonnegative integers, denoted

$$\mathcal{SG}(\Psi) := (p_1, \ldots, p_\ell),$$

where for each i $(1 \le i \le \ell)$

$$p_i := \#\{e^+_{ij} \mid e^+_{ij} \in E_{G^+}\} + \#\{e^-_{ij} \mid e^-_{ij} \in E_{G^-}\} + \#\{\ell_i \mid \ell_i \in L_G\}.$$

Notice that in general, p_i is not the degree of the vertex v_i in $G(\Psi)$.

Remark 2.4.5.4. Strictly speaking, following the general theory of signed graphs in [Zas82], one would prefer to say an element of $L_{G(\Psi)}$ is a halfedge and a negative loop when Ψ is a subset of $\Phi^+(B_\ell)$ and $\Phi^+(C_\ell)$, respectively. Since we treat the root systems of type B, C separately, there will be no signed graphs containing both halfedges and negative loops. In this subsection, we simply call them loops.

Let I be an ideal of Φ^+. When Φ is of type B_ℓ or C_ℓ, $\mathcal{DP}(I) = \mathcal{SG}(I)$, while $\mathcal{DP}(I) \neq \mathcal{SG}(I)$ if Φ is of type D_ℓ. Also, $\mathcal{SG}(I)$ does not determine I, for instance, $I_1 = \{\epsilon_4 - \epsilon_5\}$ and $I_2 = \{\epsilon_4 + \epsilon_5\}$ are distinct ideals of $\Phi^+(D_5)$, but $\mathcal{SG}(I_1) = \mathcal{SG}(I_2) = (0, 0, 0, 1, 0)$.

Computation on ideals

In the remainder of this subsection, we assume that Φ is of classical type. We summarize some easy cases that the computation of the characteristic quasi-polynomials is manageable thanks to Corollary 2.4.5.2. The minimum period coincides with the

LCM-period (see Theorem 2.4.2.6). So the minimum period of $\chi^{\text{quasi}}_{S_I}(A_\ell, q)$ is 1 for every $I \subseteq \Phi^+$; hence $\chi^{\text{quasi}}_{S_I}(A_\ell, q) = \chi_{\mathcal{H}_I}(A_\ell, q)$. For other cases, the minimum period of $\chi^{\text{quasi}}_{S_I}(\Phi, q)$ is at most 2; hence we know the 1-constituents: $f^1_{S_I}(\Phi, t) = \chi_{\mathcal{H}_I}(\Phi, t)$. We are left with the task of determining $f^2_{S_I}(\Phi, t)$, or equivalently, $\chi^{\text{quasi}}_{S_I}(\Phi, q)$ when q is even, and Φ is of type B, C or D. Turning the problem around, we would like to verify Corollary 2.4.5.2 by using the information of ideals via signed graphs without relying on the freeness, which we will do in Theorem 2.4.5.16.

Type B root systems

By [KTT07, Theorem 4.1][3], if $\Psi \subseteq \Phi^+(B_\ell)$,

$$\chi^{\text{quasi}}_{S_\Psi}(B_\ell, q) = \chi^{\text{quasi}}_{T_\Psi}(B_\ell, q).$$

Let I be an ideal of $\Phi^+(B_\ell)$. Assume that $\mathcal{DP}(I) = \mathcal{SG}(I) = (d_1, \dots, d_\ell)$. For each k ($1 \le k \le \ell$), write $d_k = d_k^{(+)} + d_k^{(-)} + d_k^{(0)}$ for a partition of d_k with

$$d_k^{(0)} := \begin{cases} 0 & \text{if } \epsilon_k \notin I, \\ 1 & \text{if } \epsilon_k \in I. \end{cases}$$

$$d_k^{(\pm)} := \#\{\epsilon_k \pm \epsilon_j \mid \epsilon_k \pm \epsilon_j \in I\}.$$

The partitions give rise to a partition of I which we call it the B-*partition*, as follows: $I = I^0 \sqcup I^- \sqcup I^+$, where

$$I^0 := \{\epsilon_i \mid \epsilon_i \in I\}$$

$$I^\pm := \{\epsilon_i \pm \epsilon_j \mid \epsilon_i \pm \epsilon_j \in I\}.$$

We work out the computation on type B root systems slowly and painstakingly so that it can serve as a template for the computations on type C, D root systems to come.

[3] Note that the number $b(q)$ in [KTT07, Theorem 4.1] should be read as $b(q) = \prod_{i=1}^m \gcd\{q, d_i\}$, where d_i's are invariant factors of the matrix P. In particular, if $\det(P)$ takes the value in $\{1, 2\}$, then $b(q) = \gcd\{q, \det(P)\}$. Hence [KTT07, Theorem 4.1] is valid if Φ is a classical root system.

If $\epsilon_i + \epsilon_j \notin I$ for all i, j, then the computation is considered on type A root systems. For all $q \in \mathbb{Z}_{>0}$,

$$\chi^{\text{quasi}}_{T_I}(B_\ell, q) = \#\left\{ \mathbf{z} \in \mathbb{Z}_q^\ell \mid \overline{z}_i \neq \overline{z}_j \, (\epsilon_i - \epsilon_j \in I) \right\} = \prod_{i=1}^{\ell} (q - d_i). \tag{2.4.1}$$

This is because there are $(q - d_\ell)$ ways to choose $\overline{z}_\ell$, $(q - d_{\ell-1})$ ways to choose $\overline{z}_{\ell-1}$, etc.

Now assume that some $\epsilon_i + \epsilon_j \in I$ with $1 \le i < j \le \ell$. In particular, $\epsilon_i \in I$ because $\epsilon_i + \epsilon_j \succeq \epsilon_i + \epsilon_\ell \succeq \epsilon_i$. Set $s := \min\{1 \le k \le \ell \mid \epsilon_k \in I\}$. Denote $R := I \setminus R'$ where $R' := \{\epsilon_i - \epsilon_j \in I \mid 1 \le i < s, i < j \le \ell\}$. For any $\beta_1, \beta_2 \in R$ with $\beta_1 \succeq \beta_2$, the difference $\beta_1 - \beta_2$ can be written as a linear combination with the positive coefficients of elements in $R \cap \Delta(B_\ell)$. Thus R is an ideal of the root subsystem of $\Phi(B_\ell)$ of type $B_{\ell-s+1}$ with a base given by $\Delta(B_{\ell-s+1}) = \{\alpha_s, \ldots, \alpha_\ell\}$. In addition, for all $q \in \mathbb{Z}_{>0}$,

$$\chi^{\text{quasi}}_{T_I}(B_\ell, q) = \chi^{\text{quasi}}_{T_R}(B_{\ell-s+1}, q) \cdot \prod_{i=1}^{s-1} (q - d_i).$$

Then it suffices to consider $s = 1$, i.e., $\epsilon_1 \in I$. For such ideals, $d_k^{(-)} = \ell - k$, $d_k^{(0)} = 1$ for $1 \le k \le \ell$.

Lemma 2.4.5.5. *Let I be an ideal of $\Phi^+(B_\ell)$ with $\epsilon_1 \in I$. Set $J := I \setminus I^0$.*

(a) J is an ideal of $\Phi^+(D_\ell)$.

(b) $\mathcal{DP}(J) = (p_1, \ldots, p_\ell)$ with $p_k = d_k^{(-)} + d_{k-1}^{(+)}$ for all $1 \le k \le \ell$. Here we agree that $d_0^{(2)} \equiv 0$.

Proof. The proof of (a) is straightforward by the definition of ideals. The proof of (b) follows from the height placements in Table 2.1. $\square$

Theorem 2.4.5.6. *Under the Lemma 2.4.5.5's assumptions, if $q \in \mathbb{Z}_{>0}$ is even,*

$$\chi^{\text{quasi}}_{T_I}(B_\ell, q) = \chi^{\text{quasi}}_{T_J}(D_\ell, q - 1) = \prod_{i=1}^{\ell} (q - p_i - 1).$$

Proof. The proof of the first equality is similar to (but more general than) that of [KTT07, Lemma 4.4(11)].

$$
\begin{aligned}
\chi^{\mathrm{quasi}}_{T_I}(B_\ell,q) &= \#\left\{ \mathbf{z}\in\mathbb{Z}_q^\ell \;\middle|\; \begin{array}{c} \overline{z}_i\neq\overline{z}_j\,(1\le i<j\le\ell),\\ \overline{z}_i+\overline{z}_j\neq\overline{0}\,(\epsilon_i+\epsilon_j\in I),\\ \overline{z}_i\neq\overline{0}\,(1\le i\le\ell)\end{array}\right\}\\
&= \#\left\{ (z_1,\dots,z_\ell)\in\mathbb{Z}^\ell \;\middle|\; \begin{array}{c} z_i\neq z_j\,(1\le i<j\le\ell),\\ z_i+z_j\neq q\,(\epsilon_i+\epsilon_j\in I),\\ 1\le z_i\le q-1\,(1\le i\le\ell)\end{array}\right\}\\
&= \#\left\{ (v_1,\dots,v_\ell)\in\mathbb{Z}^\ell \;\middle|\; \begin{array}{c} v_i\neq v_j\,(1\le i<j\le\ell),\\ v_i+v_j\neq 0\,(\epsilon_i+\epsilon_j\in I),\\ -\frac{q-2}{2}\le v_i\le\frac{q-2}{2}\,(1\le i\le\ell)\end{array}\right\}\\
&= \#\left\{ (t_1,\dots,t_\ell)\in\mathbb{Z}^\ell \;\middle|\; \begin{array}{c} t_i\neq t_j\,(1\le i<j\le\ell),\\ t_i+t_j\neq q-1\,(\epsilon_i+\epsilon_j\in I),\\ 0\le t_i\le q-2\,(1\le i\le\ell)\end{array}\right\}\\
&= \#\left\{ \mathbf{u}\in\mathbb{Z}_{q-1}^\ell \;\middle|\; \begin{array}{c} \overline{u}_i\neq\overline{u}_j\,(1\le i<j\le\ell),\\ \overline{u}_i+\overline{u}_j\neq\overline{0}\,(\epsilon_i+\epsilon_j\in J)\end{array}\right\}\\
&= \chi^{\mathrm{quasi}}_{T_J}(D_\ell,q-1).
\end{aligned}
$$

We have used the following changes of variables

$$
v_i = z_i-\frac{q}{2} \quad\text{and}\quad t_i=\begin{cases} v_i \text{ if } v_i\ge 0,\\ v_i+q-1 \text{ if } v_i<0.\end{cases}
$$

The second equality follows from Lemma 2.4.5.5 and Corollary 2.4.5.2. □

Example 2.4.5.7. Table 2.2 shows the B-partition of an ideal $I=\{\alpha\in\Phi^+(B_5)\mid \mathrm{ht}(\alpha)\le 7\}$ (in colored region), with I^0,I^-,I^+ are colored red, green, blue, respectively. Table 2.3 shows the corresponding partition of the ideal $J=I\setminus I^0$ in $\Phi^+(D_5)$. In this case, $\mathcal{DP}(I)=(7,7,5,3,1)$ and $\mathcal{DP}(J)=(4,5,5,3,1)$. Hence for even $q\in\mathbb{Z}_{>0}$, we have

$$
\chi^{\mathrm{quasi}}_{T_I}(B_\ell,q)=(q-2)(q-4)(q-5)(q-6)^2.
$$

Height						
9	$\epsilon_1+\epsilon_2$					
8	$\epsilon_1+\epsilon_3$					
7	$\epsilon_1+\epsilon_4$	$\epsilon_2+\epsilon_3$				
6	$\epsilon_1+\epsilon_5$	$\epsilon_2+\epsilon_4$				
5	ϵ_1	$\epsilon_2+\epsilon_5$	$\epsilon_3+\epsilon_4$			
4	$\epsilon_1-\epsilon_5$	ϵ_2	$\epsilon_3+\epsilon_5$			
3	$\epsilon_1-\epsilon_4$	$\epsilon_2-\epsilon_5$	ϵ_3	$\epsilon_4+\epsilon_5$		
2	$\epsilon_1-\epsilon_3$	$\epsilon_2-\epsilon_4$	$\epsilon_3-\epsilon_5$	ϵ_4		
1	$\epsilon_1-\epsilon_2$	$\epsilon_2-\epsilon_3$	$\epsilon_3-\epsilon_4$	$\epsilon_4-\epsilon_5$	ϵ_5	
	7	7	5	3	1	$\mathcal{DP}(I)$

Table 2.2: The B-partition of an ideal I in $\Phi^+(B_5)$.

Height						
7		$\epsilon_1+\epsilon_2$				
6		$\epsilon_1+\epsilon_3$				
5		$\epsilon_1+\epsilon_4$	$\epsilon_2+\epsilon_3$			
4	$\epsilon_1-\epsilon_5$	$\epsilon_1+\epsilon_5$	$\epsilon_2+\epsilon_4$			
3	$\epsilon_1-\epsilon_4$	$\epsilon_2-\epsilon_5$	$\epsilon_2+\epsilon_5$	$\epsilon_3+\epsilon_4$		
2	$\epsilon_1-\epsilon_3$	$\epsilon_2-\epsilon_4$	$\epsilon_3-\epsilon_5$	$\epsilon_3+\epsilon_5$		
1	$\epsilon_1-\epsilon_2$	$\epsilon_2-\epsilon_3$	$\epsilon_3-\epsilon_4$	$\epsilon_4-\epsilon_5$	$\epsilon_4+\epsilon_5$	
	4	5	5	3	1	$\mathcal{DP}(J)$

Table 2.3: The resulting partition of $J = I \setminus I^0$ in $\Phi^+(D_5)$.

Remark 2.4.5.8. When $I = \Phi^+(B_\ell)$, $\mathcal{DP}(I) = (2\ell-1, 2\ell-3, \dots, 3, 1)$ and $\mathcal{DP}(J) = (\ell-1, 2\ell-3, \dots, 3, 1)$. Thus for even $q \in \mathbb{Z}_{>0}$, we have

$$\chi^{\text{quasi}}_{S_{\Phi^+}}(B_\ell, q) = \chi^{\text{quasi}}_{T_{\Phi^+}}(B_\ell, q) = (q-2)(q-4)\dots(q-(2\ell-2))(q-\ell),$$

which recovers the result of [KTT07, Theorem 4.8] for type B root systems.

Type C root systems

By [KTT07, Theorem 4.1], for any $\Psi \subseteq \Phi^+(C_\ell)$ and for even $q \in \mathbb{Z}_{>0}$, we have

$$\chi^{\text{quasi}}_{S_\Psi}(C_\ell, q) = \frac{1}{2}\left(\chi^{\text{quasi}}_{T_\Psi}(C_\ell, q) + F_\Psi(C_\ell, q)\right),$$

$$F_\Psi(C_\ell, q) := \#\{\mathbf{z} \in \mathbb{Z}_q^\ell \mid \mathbf{z} \cdot T_\Psi + \mathbf{g} \cdot S_\Psi \in (\mathbb{Z}_q^\times)^{\#\Psi}\},$$
$$\mathbf{g} := (\overline{0}, \overline{0}, \dots, \overline{1}) \in \mathbb{Z}_q^\ell.$$

Let I be an ideal of $\Phi^+(C_\ell)$ with $\mathcal{DP}(I) = \mathcal{SG}(I) = (d_1, \dots, d_\ell)$. We need only consider $\epsilon_i + \epsilon_j \in I$ for some $1 \le i < j \le \ell$. In particular, $2\epsilon_k \in I$ for all $j \le k \le \ell$. Set $s := \min\{1 \le k \le \ell \mid 2\epsilon_k \in I\}$. Define $R := I \setminus \{\epsilon_i \pm \epsilon_j \in I \mid 1 \le i < s, i < j \le \ell\}$. Thus R itself is the positive system of a root system of type $C_{\ell-s+1}$ with a base given by $\Delta(C_{\ell-s+1}) = \{\alpha_s, \dots, \alpha_\ell\}$. Furthermore, for all $q \in \mathbb{Z}_{>0}$, we have

$$\chi^{\text{quasi}}_{T_I}(C_\ell, q) = \chi^{\text{quasi}}_{T_R}(C_{\ell-s+1}, q) \cdot \prod_{i=1}^{s-1}(q - d_i),$$
$$F_I(C_\ell, q) = F_R(C_{\ell-s+1}, q) \cdot \prod_{i=1}^{s-1}(q - d_i).$$

Then it suffices to consider $s = 1$ or equivalently, $I = \Phi^+(C_\ell)$. The computations of $\chi^{\text{quasi}}_{T_{\Phi^+}}(C_\ell, q)$, $F_{\Phi^+}(C_\ell, q)$ and $\chi^{\text{quasi}}_{S_{\Phi^+}}(C_\ell, q)$ were already done in [KTT07, Theorem 4.7 and §4.3]. More direct computations are also obtainable. For instance, when q is even, we have

$$\chi^{\text{quasi}}_{T_{\Phi^+}}(C_\ell, q) = \#\{\mathbf{z} \in \mathbb{Z}_q^\ell \mid \overline{z}_i \notin \{\overline{0}, \overline{q/2}, \pm\overline{z}_j\}, 1 \le i < j \le \ell\}$$
$$= \prod_{i=1}^{\ell}(q - (d_i + 1)).$$

Example 2.4.5.9. Table 2.4 shows an example of an ideal $I \subsetneq \Phi^+(C_5)$ (in enclosed region). In this case, $\mathcal{DP}(I) = (4, 6, 5, 3, 1)$. Hence for even $q \in \mathbb{Z}_{>0}$, we have

$$\begin{aligned}\chi^{\text{quasi}}_{T_I}(C_\ell, q) &= (q-6)^2(q-4)^2(q-2),\\ F_{\Phi^+}(C_\ell, q) &= (q-6)(q-4)^2(q-2)q,\\ \chi^{\text{quasi}}_{S_I}(C_\ell, q) &= (q-6)(q-4)^2(q-3)(q-2).\end{aligned}$$

Height						
9	$2\epsilon_1$					
8	$\epsilon_1+\epsilon_2$					
7	$\epsilon_1+\epsilon_3$	$2\epsilon_2$				
6	$\epsilon_1+\epsilon_4$	$\epsilon_2+\epsilon_3$				
5	$\epsilon_1+\epsilon_5$	$\epsilon_2+\epsilon_4$	$2\epsilon_3$			
4	$\epsilon_1-\epsilon_5$	$\epsilon_2+\epsilon_5$	$\epsilon_3+\epsilon_4$			
3	$\epsilon_1-\epsilon_4$	$\epsilon_2-\epsilon_5$	$\epsilon_3+\epsilon_5$	$2\epsilon_4$		
2	$\epsilon_1-\epsilon_3$	$\epsilon_2-\epsilon_4$	$\epsilon_3-\epsilon_5$	$\epsilon_4+\epsilon_5$		
1	$\epsilon_1-\epsilon_2$	$\epsilon_2-\epsilon_3$	$\epsilon_3-\epsilon_4$	$\epsilon_4-\epsilon_5$	$2\epsilon_5$	
	4	6	5	3	1	$\mathcal{DP}(I)$

Table 2.4: An ideal I in $\Phi^+(C_5)$.

Type D root systems

The computation on this type requires a bit more effort. By [KTT07, Theorem 4.1], if $\Psi \subseteq \Phi^+(D_\ell)$ and q is even,

$$\chi^{\text{quasi}}_{S_\Psi}(D_\ell, q) = \frac{1}{2}\left(\chi^{\text{quasi}}_{T_\Psi}(D_\ell, q) + F_\Psi(D_\ell, q)\right), \tag{2.4.2}$$

$$F_\Psi(D_\ell, q) := \#\{\mathbf{z} \in \mathbb{Z}_q^\ell \mid \mathbf{z}\cdot T_\Psi + \mathbf{g}\cdot S_\Psi \in (\mathbb{Z}_q^\times)^{\#\Psi}\},$$

$$\mathbf{g} := (\overline{0}, \overline{0}, \ldots, \overline{1}) \in \mathbb{Z}_q^{\ell}.$$

Let I be an ideal of $\Phi^+(D_\ell)$. We need only consider $\epsilon_i + \epsilon_j \in I$ for some $1 \leq i < j \leq \ell$. In particular, $\epsilon_{\ell-1} + \epsilon_\ell \in I$. Define

$$s := \min\{2 \leq k \leq \ell \mid \epsilon_{k-1} + \epsilon_k \in I\}. \tag{2.4.3}$$

If $\epsilon_{\ell-1} - \epsilon_\ell \notin I$, we must have $s = \ell$. Then the computation can be reduced (up to a bijection) to that on the type A root systems, which can be done easily. Suppose henceforth that $\epsilon_{\ell-1} - \epsilon_\ell \in I$, and set

$$r := \min\{1 \leq k \leq \ell \mid \epsilon_k + \epsilon_\ell \in I \text{ and } \epsilon_k - \epsilon_\ell \in I\}.$$

Obviously, $r \leq s - 1$. Assume that $\mathcal{SG}(I) = (p_1, \ldots, p_\ell)$ with for each i

$$p_i = \#\{\epsilon_i - \mu_{i,j}\epsilon_j \in I \mid \mu_{i,j} \in \{\pm 1\}\}. \tag{2.4.4}$$

Define $R := I \setminus \{\epsilon_i \pm \epsilon_j \in I \mid 1 \leq i < r, i < j \leq \ell\}$. Thus R is an ideal of the root subsystem of $\Phi(D_\ell)$ of type $D_{\ell-r+1}$ with a base given by $\Delta(D_{\ell-r+1}) = \{\alpha_r, \ldots, \alpha_\ell\}$. Furthermore, for all $q \in \mathbb{Z}_{>0}$, we have

$$\chi^{\text{quasi}}_{T_I}(D_\ell, q) = \chi^{\text{quasi}}_{T_R}(D_{\ell-r+1}, q) \cdot \prod_{i=1}^{r-1}(q - p_i),$$

$$F_I(D_\ell, q) = F_R(D_{\ell-r+1}, q) \cdot \prod_{i=1}^{r-1}(q - p_i).$$

Then it suffices to consider $r = 1$, i.e., $\epsilon_1 \pm \epsilon_\ell \in I$. For such ideals, $p_i^{(-)} = p_{i+1}^{(-)} + 1 = \ell - i$ for $1 \leq i \leq \ell - 1$. Moreover, the subset $\{\epsilon_i \pm \epsilon_j \mid s-1 \leq i < j \leq \ell\} \subseteq I$ is the positive system of a root subsystem of $\Phi(D_\ell)$ of type $D_{\ell-s+2}$. Thus $p_i \leq p_{i+1} + 1$, $p_i^{(+)} \leq p_{i+1}^{(+)}$ for all $1 \leq i \leq s - 3$, and $p_i + 2 = p_{i-1}$ for $s \leq i \leq \ell$. We will need the following lemma.

Lemma 2.4.5.10. *Let Ψ be a subset of $\Phi^+(B_\ell)$ such that $\{\epsilon_i \pm \epsilon_j \mid s - 1 \leq i < j \leq \ell\} \subseteq \Psi$ for some $2 \leq s \leq \ell$. Assume that $\mathcal{SG}(\Psi) = (p_1, \ldots, p_\ell)$ with $p_i \leq p_{i+1} + 1$ for all $1 \leq i \leq s - 3$. Then Ψ is an ideal of $\Phi^+(B_\ell)$.*

Proof. For $\beta_1, \beta_2 \in \Phi^+(B_\ell)$, $\beta_1 \succeq \beta_2$, $\beta_1 \in \Psi$, we will prove that $\beta_2 \in \Psi$. Note that for each $\beta \in \Phi^+(B_\ell)$, we have $\#\{\gamma \in \Phi^+(B_\ell) \mid \beta - \gamma \in \Delta(B_\ell)\} \leq 2$. Since $\beta_1 \succeq \beta_2$, there exists a path in the Hasse diagram of $\Phi^+(B_\ell)$ connecting β_1 and β_2.[4] It follows that this path must lie entirely within Ψ, yielding $\beta_2 \in \Psi$. $\square$

Let Π be an irreducible root system of type $B_{\ell-1}$ with a base given by $\Delta = \{\alpha_i = \epsilon_i - \epsilon_{i+1}\,(1 \leq i \leq \ell-2),\ \alpha_{\ell-1} = \epsilon_{\ell-1}\}$. We define a sequence of subsets $\{U_k\}_{k=1}^{\ell}$ (depending on I) of $\Pi^+(B_{\ell-1})$ classified into two types as follows:

(i) Type I,

$$\mathcal{SG}(U_k) = (p_1, \ldots, p_{k-1}, \widehat{p_k}, p_{k+1}+1, \ldots, p_\ell + 1), \tag{2.4.5}$$

for $1 \leq k \leq s-2$. Here $\widehat{p_k}$ means omission.

(ii) Type II,

$$\mathcal{SG}(U_k) = (p_1 - \tau_{1,k}, \ldots, p_{s-2} - \tau_{s-2,k}, p_{s-1} - 1, \ldots, p_{\ell-1} - 1), \tag{2.4.6}$$

for $s-1 \leq k \leq \ell$, $1 \leq n \leq s-2$, with

$$\tau_{n,k} := \begin{cases} 0 & \text{if } \epsilon_n + \epsilon_k \notin I \\ 1 & \text{if } \epsilon_n + \epsilon_k \in I. \end{cases}$$

It is easily seen that $\tau_{n,k} \leq \tau_{n+1,k}$ (as well as $\tau_{n,k} \leq \tau_{n,k+1}$), hence $p_n - \tau_{n,k} \leq p_{n+1} - \tau_{n+1,k} + 1$ for all $1 \leq n \leq s-3$. By Lemma 2.4.5.10, the subsets $\{U_k\}_{k=1}^{\ell}$ are indeed ideals of $\Pi^+(B_{\ell-1})$. We also define

$$K := I \sqcup \{\epsilon_k \mid 1 \leq k \leq \ell\}. \tag{2.4.7}$$

Then again by Lemma 2.4.5.10, K is an ideal of $\Phi^+(B_\ell)$ with

$$\mathcal{SG}(K) = (p_1 + 1, p_2 + 1, \ldots, p_\ell + 1).$$

The following result is a generalization of [KTT07, Lemma 4.4(12)].

[4]This fact is true for any root system, which is a consequence of, e.g., [Som05, Lemma 3.2].

Lemma 2.4.5.11. *Let I be an ideal of $\Phi^+(D_\ell)$ so that $\epsilon_1 \pm \epsilon_\ell \in I$. For all $q \in \mathbb{Z}_{>0}$, we have*

$$\chi_{T_I}^{\text{quasi}}(D_\ell, q) = \sum_{k=1}^{\ell} \chi_{T_{U_k}}^{\text{quasi}}(B_{\ell-1}, q) + \chi_{T_K}^{\text{quasi}}(B_\ell, q), \tag{2.4.8}$$

where U_k and $K$are defined in (2.4.5), (2.4.6), (2.4.7).

Proof. With the notion of contraction lists (see Definition 1.2.1.5), we can write $\chi_{T_{U_k}}^{\text{quasi}}(B_{\ell-1}, q) = \chi_{\mathcal{A}_k}^{\text{quasi}}(B_\ell, q)$ with $\mathcal{A}_k := T_{I\cup\{\epsilon_\ell,\ldots,\epsilon_k\}}/T_{\{\epsilon_k\}}$ for $1 \le k \le \ell$. For all $q \in \mathbb{Z}_{>0}$, by applying the Deletion-Contraction formula (Theorem 2.2.1.10) recursively, we get

$$\begin{aligned}
\chi_{T_I}^{\text{quasi}}(D_\ell, q) &= \chi_{T_{U_\ell}}^{\text{quasi}}(B_{\ell-1}, q) + \chi_{T_{I\cup\{\epsilon_\ell\}}}^{\text{quasi}}(B_\ell, q) \\
&= \chi_{T_{U_\ell}}^{\text{quasi}}(B_{\ell-1}, q) + \chi_{T_{U_{\ell-1}}}^{\text{quasi}}(B_{\ell-1}, q) + \chi_{T_{I\cup\{\epsilon_\ell,\epsilon_{\ell-1}\}}}^{\text{quasi}}(B_\ell, q) \\
&= \cdots \\
&= \sum_{k=1}^{\ell} \chi_{T_{U_k}}^{\text{quasi}}(B_{\ell-1}, q) + \chi_{T_K}^{\text{quasi}}(B_\ell, q).
\end{aligned}$$

□

In Lemma 2.4.5.12 and Theorem 2.4.5.13 below, we use the same assumption and notation as in Lemma 2.4.5.11.

Lemma 2.4.5.12. *For even $q \in \mathbb{Z}_{>0}$, we have*

$$F_I(D_\ell, q) = F_{I\cup\{2\epsilon_s,\ldots,2\epsilon_\ell\}}(C_\ell, q) = \prod_{i=1}^{\ell}(q - p_i).$$

Proof. This follows from the height placements in Table 2.1. □

Theorem 2.4.5.13. *For even $q \in \mathbb{Z}_{>0}$, we have*

$$\chi_{S_I}^{\text{quasi}}(D_\ell, q) = \frac{1}{2}\left(\sum_{k=1}^{\ell} \chi_{T_{U_k}}^{\text{quasi}}(B_{\ell-1}, q) + \chi_{T_K}^{\text{quasi}}(B_\ell, q) + \prod_{i=1}^{\ell}(q - p_i)\right).$$

Proof. This follows from formula (2.4.2), and Lemmas 2.4.5.11, 2.4.5.12. □

Example 2.4.5.14. Table 2.6 shows an example of the ideal $I = \{\alpha \in \Phi^+(D_5) \mid \mathrm{ht}(\alpha) \leq 6\}$ (in colored region), with positive roots contributing to p_1, p_2, p_3, p_4 are colored in green, pink, blue, red, respectively. In this case, $s = 2$ since $\epsilon_2 + \epsilon_3 \in I$, but $\epsilon_1 + \epsilon_2 \notin I$. We have $\mathcal{SG}(I) = (7, 6, 4, 2, 0)$, and the computation on the ideals K and U_k for even $q \in \mathbb{Z}_{>0}$ is given in Table 2.5. By Theorem 2.4.5.13, for even $q \in \mathbb{Z}_{>0}$, we have

$$\chi^{\text{quasi}}_{S_I}(D_\ell, q) = (q-2)(q-4)(q^3 - 13q^2 + 51q - 51).$$

Ideals	$\mathcal{DP}$	Roots of χ^{quasi}_T
K	$(8, 7, 5, 3, 1)$	$7, 6, 5, 4, 2$
U_1, U_2	$(7, 5, 3, 1)$	$6, 4, 4, 2$
U_3, U_4, U_5	$(6, 5, 3, 1)$	$5, 4, 4, 2$

Table 2.5: Computation of Example 2.4.5.14.

Remark 2.4.5.15. When $I = \Phi^+(D_\ell)$, $\mathcal{SG}(I) = (2\ell-2, 2\ell-4, \ldots, 2, 0)$, $\mathcal{DP}(I) = (\ell-1, 2\ell-3, \ldots, 3, 1)$, $\mathcal{DP}(K) = (2\ell-1, 2\ell-3, \ldots, 3, 1)$, and $\mathcal{DP}(U_k) = (2\ell-3, \ldots, 3, 1)$ for all $1 \leq k \leq \ell$. Note that $s = 2$, so there is no ideal U_k of type I. Then by Lemma 2.4.5.11, for odd $q \in \mathbb{Z}_{>0}$

$$\chi^{\text{quasi}}_{S_{\Phi^+}}(D_\ell, q) = \chi^{\text{quasi}}_{T_{\Phi^+}}(D_\ell, q) = (q-1)(q-3)\ldots(q-(2\ell-3))(q-(\ell-1)),$$

which agrees with Corollary 2.4.5.2. Moreover, for even $q \in \mathbb{Z}_{>0}$

$$\chi^{\text{quasi}}_{S_{\Phi^+}}(D_\ell, q) = (q-2)(q-4)\ldots(q-(2\ell-4))\left(q^2 - 2(\ell-1)q + \frac{\ell(\ell-1)}{2}\right),$$

which recovers the result of [KTT07, Theorem 4.8] for type D root systems.

Height						
7		$\epsilon_1+\epsilon_2$				
6		$\epsilon_1+\epsilon_3$				
5		$\epsilon_1+\epsilon_4$	$\epsilon_2+\epsilon_3$			
4	$\epsilon_1-\epsilon_5$	$\epsilon_1+\epsilon_5$	$\epsilon_2+\epsilon_4$			
3	$\epsilon_1-\epsilon_4$	$\epsilon_2-\epsilon_5$	$\epsilon_2+\epsilon_5$	$\epsilon_3+\epsilon_4$		
2	$\epsilon_1-\epsilon_3$	$\epsilon_2-\epsilon_4$	$\epsilon_3-\epsilon_5$	$\epsilon_3+\epsilon_5$		
1	$\epsilon_1-\epsilon_2$	$\epsilon_2-\epsilon_3$	$\epsilon_3-\epsilon_4$	$\epsilon_4-\epsilon_5$	$\epsilon_4+\epsilon_5$	
	7	6	4	2	0	$\mathcal{SG}(I)$

Table 2.6: $I = \{\alpha \in \Phi^+(D_5) \mid \mathrm{ht}(\alpha) \leq 6\}$ in $\Phi^+(D_5)$.

Applications

The last constituent of every characteristic quasi-polynomial is proved to be identical with the characteristic polynomial of the corresponding toric arrangement (Corollary 2.2.1.8). The first application that we obtain automatically from our computations is a full description of the toric arrangement characteristic polynomials defined by the ideals in terms of the signed graphs.

For the second application, we give a direct verification of Corollary 2.4.5.2 when Φ is any classical root system. We restrict the discussion to type D root systems as the other cases are easy. For any ideal $I \subseteq \Phi^+(D_\ell)$ with $\mathcal{SG}(I) = (p_1, \dots, p_\ell)$ defined in (2.4.4), we write

$$p_i = p_i^{(+)} + p_i^{(-)}, \text{where, } p_i^{(\pm)} := \#\{\epsilon_i \pm \epsilon_j \mid \epsilon_i \pm \epsilon_j \in I\},$$

for each $1 \leq i \leq \ell$. It is easily seen that $\mathcal{DP}(I) = (d_1, \dots, d_\ell)$ with

$$d_i = p_i^{(-)} + p_{i-1}^{(+)}.$$

Here we agree that $p_0^{(+)} = 0$.

Theorem 2.4.5.16. *Let I be an ideal of $\Phi^+(D_\ell)$. For odd $q \in \mathbb{Z}_{>0}$, we have*

$$\chi^{\text{quasi}}_{T_I}(D_\ell, q) = \prod_{i=1}^{\ell} (q - d_i)\,. \tag{2.4.9}$$

Proof. It suffices to prove Theorem 2.4.5.16 when $\epsilon_1 \pm \epsilon_\ell \in I$, as the other cases are straightforward. For such ideals, $d_1 = \ell - 1$, $d_i = p_i^{(-)} + p_{i-1}^{(+)} = p_{i-1} - 1$ for all $2 \le i \le \ell$. We recall the notation of the parameter s defined in (2.4.3) that $s = \min\{2 \le k \le \ell \mid \epsilon_{k-1} + \epsilon_k \in I\}$. It follows from Lemma 2.4.5.11 and Remark 2.4.5.15 that both sides of (2.4.9) are divisible by $\prod_{i=s}^{\ell} (q - p_{i-1} + 1)$. Hence we need only prove the following:

$$A + B + C = (q - \ell + 1) \prod_{i=2}^{s-1} (q - p_{i-1} + 1)\,, \tag{2.4.10}$$

where

$$\begin{aligned}
A &:= \prod_{i=1}^{s-1} (q - p_i - 1)\,,\\
B &:= \sum_{k=1}^{s-2} (q - p_1) \ldots (q - p_{k-1})(q - p_{k+1} - 1) \ldots (q - p_{s-1} - 1),\\
C &= \sum_{k=s-1}^{\ell} C_k, \text{ with } C_k := \prod_{n=1}^{s-2} (q - p_n + \tau_{n,k})\,,
\end{aligned}$$

and $\tau_{n,k}$ is defined in (2.4.6). Since $\tau_{n,s-1} = 0$ for all $1 \le n \le s-2$, $C_{s-1} = \prod_{i=1}^{s-2} (q - p_i)$. It is routine to check that

$$A + B + C_{s-1} = \prod_{i=1}^{s-1} (q - p_i)\,. \tag{2.4.11}$$

Write $M_\tau = [\tau_{n,k}]$ for a matrix of size $(s-2) \times (\ell - s + 1)$ whose entries are the $\tau_{n,k}$'s (the columns indexed by the set $\{s, \ldots, \ell\}$). Then

$$M_\tau = \begin{bmatrix}
0 & \cdots & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1\\
0 & \cdots & 0 & \cdots & 1 & \cdots & 1 & \cdots & 1\\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots\\
0 & \cdots & 0 & 1 & \cdots & \cdots & 1 & \cdots & 1
\end{bmatrix},$$

with the number of 1's on the n-th row is exactly $p_n^{(+)}$, and the entries on the k-th column contribute to the evaluation of C_k. Thus

$$\sum_{k=s}^{\ell} C_k = \sum_{n=0}^{s-2} \left(p_{n+1}^{(+)} - p_n^{(+)}\right) \prod_{i=1}^{n} (q - p_i) \prod_{i=n+1}^{s-2} (q - p_i + 1)\,. \tag{2.4.12}$$

Now combining (2.4.11) and (2.4.12) with a rigorous check, we obtain (2.4.10).

$\square$

3. G-TUTTE POLYNOMIALS VIA TOPOLOGY

3.1 Euler characteristics of (F, p, q)-arrangements

In this section, we show a first topological property of the G-Tutte polynomial. We prove that the topological and semialgebraic Euler characteristics of the complement of any (F, p, q)-arrangement can obtained as an evaluation of the associated G-characteristic polynomial. In particular, in the case of arrangement of finite groups (i.e., $(F, 0, 0)$-arrangement), we obtain a generalization of Theorem 1.2.2.19.

3.1.1 Topological and semialgebraic Euler characteristics

We first recall the notion of Euler characteristic for semialgebraic sets (see [Cos07, BPR06] for further details). A subset $X \subseteq \mathbb{R}^n$ is said to be a *semialgebraic* set if it is expressed as a finite union of sets of the form:

$$\{x \in \mathbb{R}^n \mid p(x) = 0, q_1(x) > 0, \ldots, q_m(x) > 0\},$$

where $m \geq 0$, and $p, q_1, \ldots, q_m \in \mathbb{R}[x_1, \ldots, x_n]$. Every semialgebraic set X has a decomposition $X = \bigsqcup_{i=1}^N X_i$ such that each semialgebraic subset X_i is *semialgebraically homeomorphic* to the open simplex $\sigma_{d_i} = \{(x_1, \ldots, x_{d_i}) \in \mathbb{R}^{d_i} \mid x_i > 0, \sum x_i < 1\}$ for some $d_i = \dim X_i$. The *semialgebraic Euler characteristic* of X is defined by

$$e_{\mathrm{semi}}(X) = \sum_{i=1}^{N} (-1)^{d_i}.$$

The Euler characteristic $e_{\mathrm{semi}}(X)$ is independent of the choice of decomposition. Furthermore, it satisfies the following additivity and multiplicativity (e.g., [Cos07])

$$e_{\mathrm{semi}}(X \sqcup Y) = e_{\mathrm{semi}}(X) + e_{\mathrm{semi}}(Y),$$

$$e_{\mathrm{semi}}(X \times Y) = e_{\mathrm{semi}}(X) \times e_{\mathrm{semi}}(Y).$$

Remark 3.1.1.1. If X is compact, then $e_{\text{semi}}(X)$ is equal to the *topological* Euler characteristic $e_{\text{top}}(X) := \sum_{i\geq 0}(-1)^i b_i(X)$ ($b_i(X)$ is the i-th Betti number). However, unlike the topological Euler characteristic, the semialgebraic Euler characteristic $e_{\text{semi}}(X)$ is not homotopy invariant. Even contractible semialgebraic sets have different semialgebraic Euler characteristics, e.g., $e_{\text{semi}}([0,1]) = 1$, $e_{\text{semi}}(\mathbb{R}_{\geq 0}) = 0$, $e_{\text{semi}}(\mathbb{R}) = -1$.

For a locally compact semialgebraic set X, the semialgebraic Euler characteristic is known to be equal to the Euler characteristic of the Borel-Moore homology $H_i^{BM}(X)$ (see [Cos07, Chapter 1] for details). If X is a manifold (without boundary), we have an isomorphism $H_i^{BM}(X) \simeq H^{\dim X - i}(X)$ (see [Ive86, Theorem 4.7, Chapter IX]). Thus $e_{\text{semi}}(X)$ and $e_{\text{top}}(X)$ are related by the following formula:

$$e_{\text{semi}}(X) = (-1)^{\dim X} \cdot e_{\text{top}}(X). \tag{3.1.1}$$

3.1.2 Euler characteristics of (F, p, q)-arrangements

In the remainder of this subsection, we assume that $\mathcal{A}$ is a finite list of elements in a finitely generated abelian group Γ. We also assume that G is an abelian Lie group with finitely many connected components, i.e., $G = (\mathbb{S}^1)^p \times \mathbb{R}^q \times F$ with $g = \dim_{\mathbb{R}}(G) = p + q \geq 0$ and F is a finite abelian group. Such a group G can be realized as a semialgebraic set, where the group operations are defined by C^∞ semialgebraic maps. Hence subsets defined by using group operations are always semialgebraic sets.

Proposition 3.1.2.1.

$$e_{\text{semi}}(G) = \begin{cases} 0, & \textit{if } p > 0, \\ (-1)^g \cdot \#F, & \textit{if } p = 0, \end{cases}$$

$$e_{\text{top}}(G) = \begin{cases} 0, & \textit{if } p > 0, \\ \#F, & \textit{if } p = 0. \end{cases}$$

Proof. Straightforward. ∎

The space $\mathcal{M}(\mathcal{A};\Gamma,G)$ is a semialgebraic set, and, if it is not empty, it is also a manifold (without boundary) of $\dim \mathcal{M}(\mathcal{A};\Gamma,G) = gr_\Gamma$. We prove that the topological and semialgebraic Euler characteristics of $\mathcal{M}(\mathcal{A};\Gamma,G)$ can be computed by the G-Tutte polynomial (in fact, by the G-characteristic polynomial) of $\mathcal{A}$.

Theorem 3.1.2.2.

$$\begin{aligned} e_{\mathrm{semi}}(\mathcal{M}(\mathcal{A};\Gamma,G)) &= \chi^G_{\mathcal{A}}(e_{\mathrm{semi}}(G)),\\ e_{\mathrm{top}}(\mathcal{M}(\mathcal{A};\Gamma,G)) &= (-1)^{g\cdot r_\Gamma}\cdot \chi^G_{\mathcal{A}}\left((-1)^g\cdot e_{\mathrm{top}}(G)\right). \end{aligned}$$

Proof. We can compute $e_{\mathrm{semi}}(\mathcal{M}(\mathcal{A};\Gamma,G))$ by using the additivity of $e_{\mathrm{semi}}(-)$, the Inclusion-Exclusion principle and Proposition 1.2.1.6 as follows:

$$\begin{aligned} e_{\mathrm{semi}}(\mathcal{M}(\mathcal{A};\Gamma,G)) &= \sum_{\mathcal{S}\subseteq\mathcal{A}}(-1)^{\#\mathcal{S}}\cdot e_{\mathrm{semi}}\left(\bigcap_{\alpha\in\mathcal{S}} H_{\alpha,G}\right)\\ &= \sum_{\mathcal{S}\subseteq\mathcal{A}}(-1)^{\#\mathcal{S}}\cdot m(\mathcal{S};G)\cdot e_{\mathrm{semi}}(G)^{r_\Gamma - r_\mathcal{S}}\\ &= \chi^G_{\mathcal{A}}(e_{\mathrm{semi}}(G)). \end{aligned}$$

□

Remark 3.1.2.3. We can also prove Theorem 3.1.2.2 by using the deletion-contraction formula (Proposition 1.2.1.9). Note that if $\mathcal{A}=\emptyset$, then $\chi^G_{\mathcal{A}}(t) = \#\operatorname{Hom}(\Gamma_{\mathrm{tor}},G)\cdot t^{r_\Gamma}$. Hence $\chi^G_{\mathcal{A}}(e_{\mathrm{semi}}(G)) = \#\operatorname{Hom}(\Gamma_{\mathrm{tor}},G)\cdot e_{\mathrm{semi}}(G)^{r_\Gamma} = e_{\mathrm{semi}}(\operatorname{Hom}(\Gamma,G))$. Theorem 3.1.2.2 then follows easily from Proposition 1.2.1.9 and Corollary 1.2.2.14 by induction on $\#\mathcal{A}$.

Remark 3.1.2.4. Theorem 3.1.2.2 is a generalization of [Moc12, Theorem 5.15] by viewing $G=\mathbb{S}^1, \Gamma=\mathbb{Z}^\ell$.

Now we prove a generalization of Theorem 1.2.2.19 by replacing $\mathbb{Z}_q$ by any finite abelian group.

Theorem 3.1.2.5. *If G is a finite abelian group (i.e., $g=0$), then*

$$\#\mathcal{M}(\mathcal{A};\Gamma,G) = \chi^G_{\mathcal{A}}(\#G).$$

Proof. If G is finite, then $\mathcal{M}(\mathcal{A};\Gamma,G)$ is also a finite set. Proposition 3.1.2.1 and Theorem 3.1.2.2 complete the proof. $\square$

Let Φ be an irreducible root system of rank ℓ, and let $\Gamma := \mathbb{Z}\Phi$ be the root lattice of Φ. Consider the list $\mathcal{A} := \Phi^+ \subseteq \Gamma$ of positive roots. Let W be the Weyl group of Φ, and let f be the index of connection.

Proposition 3.1.2.6. *The constant term of the characteristic polynomial of the toric arrangement of Φ^+ can be computed as follows:*

$$\chi^{\mathrm{toric}}_{\Phi^+}(0) = \frac{(-1)^\ell \# W}{f}.$$

Proof. Recall from §2.4.2 the notation $L_{\overline{A^\circ}}(q)$ of the Ehrhart quasi-polynomial of the closed fundamental alcove $\overline{A^\circ}$ w.r.t. the coweight lattice. By Theorem 2.4.2.5, the Ehrhart reciprocity law, and the fact that $L_{\overline{A^\circ}}(0) = 1$ (e.g., [BS18, Exercise 5.15]), we have

$$f^{\rho_{\Phi^+}}_{\Phi^+}(0) = \frac{(-1)^\ell \# W}{f} L_{\overline{A^\circ}}(0) = \frac{(-1)^\ell \# W}{f}.$$

By Corollary 2.2.1.9,

$$\chi^{\mathrm{toric}}_{\Phi^+}(0) = f^{\rho_{\Phi^+}}_{\Phi^+}(0) = \frac{(-1)^\ell \# W}{f}.$$

$\square$

Remark 3.1.2.7. The Cartan matrix of Φ whose determinant is the index of connection f expresses the change of basis between the root lattice and the weight lattice. It follows from Propositions 1.2.2.22 and 3.1.2.6 that the constant term of the characteristic polynomial of the toric arrangement w.r.t. the weight lattice equals $(-1)^\ell \# W$. This gives a new proof for [Moc12, Corollary 7.4].

Corollary 3.1.2.8 (Theorem 7.3, [Moc12])**.**

$$e_{\mathrm{semi}}(\mathcal{M}(\Phi^+;\Gamma,\mathbb{C}^\times)) = e_{\mathrm{top}}(\mathcal{M}(\Phi^+;\Gamma,\mathbb{C}^\times)) = \frac{(-1)^\ell \# W}{f}.$$

Proof. Note that $e_{\mathrm{top}}(\mathbb{C}^\times) = e_{\mathrm{semi}}(\mathbb{C}^\times) = 0$ (Proposition 3.1.2.1). Theorem 3.1.2.2 and Proposition 3.1.2.6 complete the proof. $\square$

3.2 Poincaré polynomials of non-compact (F, p, q)-arrangements

As a next step after the formulas of the Euler characteristics in the previous section, we are interested in finding the relation between the Poincaré polynomial of a (F, p, q)-arrangement and the G-Tutte polynomial. In this section, we prove that the Poincaré polynomial of any non-compact (F, p, q)-arrangement (i.e., $q > 0$) can be expressed in terms of the associated G-characteristic polynomial. To this end, we introduce two special classes of homology cycles called the torus and meridian cycles in $H_*(\mathcal{M}(\mathcal{A}; \Gamma, G), \mathbb{Z})$. The former provides a version of lifts of cycles in a compact torus, while the latter generates the entire homology group $H_*(\mathcal{M}(\mathcal{A}; \Gamma, G), \mathbb{Z})$. Then we prove that the Poincaré polynomials satisfy a recursive formula, which in turn prove the desired formula. It turns out that the non-compactness plays a crucial role in our proof, without it many arguments may not work.

3.2.1 Torus and meridian cycles

Assume that $G \simeq (\mathbb{S}^1)^p \times \mathbb{R}^q \times F$ with $g = \dim_{\mathbb{R}}(G) = p + q \geq 0$ and F is a finite abelian group. Write $G_c = F \times (\mathbb{S}^1)^p$ (compact part) and $V = \mathbb{R}^q$ (non-compact part). Let Γ be a finitely generated abelian group. Fix a decomposition $\Gamma = \Gamma_{\text{tor}} \oplus \Gamma_{\text{free}}$, where $\Gamma_{\text{free}} \simeq \mathbb{Z}^{r_\Gamma}$. Note that $\text{Hom}(\Gamma_{\text{tor}}, V) \simeq \{0\}$. Thus

$$\begin{aligned} \text{Hom}(\Gamma, G) &\simeq \text{Hom}(\Gamma, G_c) \times \text{Hom}(\Gamma_{\text{free}}, V), && (3.2.1)\\ &\simeq \text{Hom}(\Gamma_{\text{tor}}, G_c) \times \text{Hom}(\Gamma_{\text{free}}, G_c) \times \text{Hom}(\Gamma_{\text{free}}, V). && (3.2.2) \end{aligned}$$

The first component $\text{Hom}(\Gamma_{\text{tor}}, G_c)$ of (3.2.2) is a finite abelian group, the second component $\text{Hom}(\Gamma_{\text{free}}, G_c)$ is a compact abelian Lie group (not necessarily connected), and the third component is $\text{Hom}(\Gamma_{\text{free}}, V) \simeq V^{r_\Gamma} \simeq \mathbb{R}^{q \cdot r_\Gamma}$.

Let $\alpha = (\beta, \eta) \in \Gamma_{\text{tor}} \oplus \Gamma_{\text{free}}$. According to decomposition (3.2.1), the subgroup $H_{\alpha,G} \subseteq \text{Hom}(\Gamma, G)$ can be expressed as

$$H_{\alpha,G} = H_{\alpha,G_c} \times H_{\eta,V},$$

where $H_{\alpha,G_c} \subseteq \mathrm{Hom}(\Gamma, G_c)$ and $H_{\eta,V} \subseteq \mathrm{Hom}(\Gamma_{\text{free}}, V)$. If $\alpha \in \Gamma_{\text{tor}}$, or equivalently $\alpha = (\beta, 0)$, then using (3.2.2) gives

$$H_{\alpha,G} = H_{\beta,G_c} \times \mathrm{Hom}(\Gamma_{\text{free}}, G_c) \times \mathrm{Hom}(\Gamma_{\text{free}}, V),$$

where H_{β,G_c} is a subgroup of the finite abelian group $\mathrm{Hom}(\Gamma_{\text{tor}}, G_c)$. In this case, $H_{\alpha,G}$ is a collection of connected components of $\mathrm{Hom}(\Gamma, G)$.

Similarly, the complement can be expressed as

$$\begin{aligned} \mathcal{M}(\{\alpha\}; \Gamma, G) &= \mathrm{Hom}(\Gamma, G) \setminus H_{\alpha,G} \\ &= (\mathrm{Hom}(\Gamma_{\text{tor}}, G_c) \setminus H_{\beta,G_c}) \times \mathrm{Hom}(\Gamma_{\text{free}}, G_c) \times \mathrm{Hom}(\Gamma_{\text{free}}, V). \end{aligned}$$

More generally, if $\mathcal{A} \subseteq \Gamma_{\text{tor}} \subseteq \Gamma$, then

$$\begin{aligned} \mathcal{M}(\mathcal{A}; \Gamma, G) &= \left(\mathrm{Hom}(\Gamma_{\text{tor}}, G_c) \setminus \bigcup_{\alpha \in \mathcal{A}} H_{\alpha,G_c} \right) \times \mathrm{Hom}(\Gamma_{\text{free}}, G_c) \times \mathrm{Hom}(\Gamma_{\text{free}}, V) \\ &= \mathcal{M}(\mathcal{A}; \Gamma_{\text{tor}}, G_c) \times \mathrm{Hom}(\Gamma_{\text{free}}, G_c) \times \mathrm{Hom}(\Gamma_{\text{free}}, V). \end{aligned} \tag{3.2.3}$$

Therefore, $\mathcal{M}(\mathcal{A}; \Gamma, G)$ is a collection of some of the connected components of $\mathrm{Hom}(\Gamma, G)$.

Let $\mathcal{A} \subseteq \Gamma$ be a finite list of elements. Recall the notation $\mathcal{A}^{\text{tor}} = \mathcal{A} \cap \Gamma_{\text{tor}}$. As mentioned above, $\mathcal{M}(\mathcal{A}^{\text{tor}}; \Gamma, G)$ is a collection of components of $\mathrm{Hom}(\Gamma, G)$. Consider the following diagram:

$$\begin{tikzcd} \mathcal{M}(\mathcal{A}; \Gamma, G) \arrow[r, "\subseteq"] & \mathcal{M}(\mathcal{A}^{\text{tor}}; \Gamma, G) \arrow[r, "\subseteq"] \arrow[d] & \mathrm{Hom}(\Gamma, G) \ni (f, t, v) \arrow[d, "\pi"] \\ & \mathcal{M}(\mathcal{A}^{\text{tor}}; \Gamma, G_c) \arrow[r, "\subseteq"] & \mathrm{Hom}(\Gamma, G_c) \ni (f, t), \end{tikzcd} \tag{3.2.4}$$

where $\pi : \mathrm{Hom}(\Gamma, G) \longrightarrow \mathrm{Hom}(\Gamma, G_c)$ is the projection defined by $\pi(f, t, v) = (f, t)$ for $(f, t, v) \in \mathrm{Hom}(\Gamma_{\text{tor}}, G_c) \times \mathrm{Hom}(\Gamma_{\text{free}}, G_c) \times \mathrm{Hom}(\Gamma_{\text{free}}, V) \simeq \mathrm{Hom}(\Gamma, G)$.

Now assume that $q > 0$. The fiber of the projection π is isomorphic to $\mathrm{Hom}(\Gamma, V) \simeq V^{r_\Gamma} \simeq \mathbb{R}^{q \cdot r_\Gamma}$. Then

$$\mathcal{M}(\mathcal{A} \setminus \mathcal{A}^{\text{tor}}; \Gamma, V) = \mathrm{Hom}(\Gamma, V) \setminus \bigcup_{\alpha \in \mathcal{A} \setminus \mathcal{A}^{\text{tor}}} H_{\alpha,V}$$

is the complement of a collection of proper subspaces. Hence it is non-empty. Fix an element $v_0 \in \mathcal{M}(\mathcal{A} \setminus \mathcal{A}^{\mathrm{tor}}; \Gamma, V)$. For a given $(f,t) \in \mathrm{Hom}(\Gamma, G_{\mathrm{c}})$, define $i_{v_0}(f,t) := (f,t,v_0)$. This induces a map

$$i_{v_0} : \mathcal{M}(\mathcal{A}^{\mathrm{tor}}; \Gamma, G_{\mathrm{c}}) \longrightarrow \mathcal{M}(\mathcal{A}; \Gamma, G),$$

which is a section of the projection $\pi|_{\mathcal{M}(\mathcal{A};\Gamma,G)} : \mathcal{M}(\mathcal{A}; \Gamma, G) \longrightarrow \mathcal{M}(\mathcal{A}^{\mathrm{tor}}; \Gamma, G_{\mathrm{c}})$ in (3.2.4).

Definition 3.2.1.1. Assume that $q > 0$. A cycle $\gamma \in H_*(\mathcal{M}(\mathcal{A}; \Gamma, G), \mathbb{Z})$ is said to be a *torus cycle* if there exist a connected component $T \subseteq \mathcal{M}(\mathcal{A}^{\mathrm{tor}}; \Gamma, G_{\mathrm{c}})$, a cycle $\widetilde{\gamma} \in H_*(T, \mathbb{Z}) \subseteq H_*(\mathcal{M}(\mathcal{A}^{\mathrm{tor}}; \Gamma, G_{\mathrm{c}}), \mathbb{Z})$ and $v_0 \in \mathcal{M}(\mathcal{A} \setminus \mathcal{A}^{\mathrm{tor}}; \Gamma, V)$ such that

$$\gamma = (i_{v_0})_*(\widetilde{\gamma}).$$

The subgroup of $H_*(\mathcal{M}(\mathcal{A}; \Gamma, G), \mathbb{Z})$ generated by torus cycles is denoted by $H_*^{\mathrm{torus}}(\mathcal{A}(G))$.

Remark 3.2.1.2. If $q > 1$, then the homology class $(i_{v_0})_*(\widetilde{\gamma})$ is independent of the choice of $v_0 \in \mathcal{M}(\mathcal{A} \setminus \mathcal{A}^{\mathrm{tor}}; \Gamma, V)$, because $\mathcal{M}(\mathcal{A} \setminus \mathcal{A}^{\mathrm{tor}}; \Gamma, V)$ is connected. On the other hand, if $q = 1$, then the subspace $H_{\alpha,V}$ is a real hyperplane in $\mathrm{Hom}(\Gamma, V) \simeq V^{r_\Gamma}$. Hence the homology class $(i_{v_0})_*(\widetilde{\gamma})$ may depend on the connected component of $V^{r_\Gamma} \setminus \bigcup H_{\alpha,V}$ which contains v_0.

Lemma 3.2.1.3. *Assume that $q > 0$. Let $\alpha \in \mathcal{A} \setminus \mathcal{A}^{\mathrm{tor}}$, and $\mathcal{A}' = \mathcal{A} \setminus \{\alpha\}$. Then the map $\iota : H_*^{\mathrm{torus}}(\mathcal{A}(G)) \longrightarrow H_*^{\mathrm{torus}}(\mathcal{A}'(G))$ induced by the inclusion $\mathcal{M}(\mathcal{A}; \Gamma, G) \hookrightarrow \mathcal{M}(\mathcal{A}'; \Gamma, G)$ is surjective.*

Proof. Let $(i_{v_0})_*(\widetilde{\gamma}) \in H_*(\mathcal{M}(\mathcal{A}'; \Gamma, G), \mathbb{Z})$ be a torus cycle. If $v_0 \notin H_{\alpha,V}$, then $(i_{v_0})_*(\widetilde{\gamma})$ is clearly contained in the image of the map ι. If $v_0 \in H_{\alpha,V}$, since $\mathcal{M}(\mathcal{A} \setminus \mathcal{A}^{\mathrm{tor}}; \Gamma, V)$ is nonempty, there exists a small perturbation v_0' of v_0 such that $v_0' \in \mathcal{M}(\mathcal{A} \setminus \mathcal{A}^{\mathrm{tor}}; \Gamma, V)$ (see Remark 3.2.1.2). Then $H_*^{\mathrm{torus}}(\mathcal{A}(G)) \ni (i_{v_0'})_*(\widetilde{\gamma}) \longmapsto (i_{v_0})_*(\widetilde{\gamma}) \in H_*^{\mathrm{torus}}(\mathcal{A}'(G))$. □

The torus cycles defined previously are not enough to generate the homology group $H_*(\mathcal{M}(\mathcal{A}; \Gamma, G), \mathbb{Z})$. We also need to consider meridians of $H_{\alpha,G}$ to generate

$H_*(\mathcal{M}(\mathcal{A};\Gamma,G),\mathbb{Z})$. Let us first recall the notion of layers (Definition 2.1.2.2). A layer of $\mathcal{A}(G)$ is a connected component of a non-empty intersection of elements of $\mathcal{A}(G)$. Let $\mathcal{S}\subseteq\mathcal{A}$. By Proposition 1.2.1.6 (see also (2.1.1)), every connected component of $H_{\mathcal{S},G}=\bigcap_{\alpha\in\mathcal{S}}H_{\alpha,G}$ is isomorphic to

$$\left((\mathbb{S}^1)^p\times\mathbb{R}^q\right)^{r_\Gamma-r_\mathcal{S}}.$$

We sometimes call the number $r_\mathcal{S}$ the rank of the layer. Since $H_{\emptyset,G}=\mathrm{Hom}(\Gamma,G)$, a connected component of $\mathrm{Hom}(\Gamma,G)$ is a layer of rank 0. Similarly, a connected component of $H_{\alpha,G}$ for $\alpha\in\mathcal{A}\setminus\mathcal{A}^{\mathrm{tor}}$ is a layer of rank 1.

Let L be a layer. Recall from (2.1.2) the notion of localization of $\mathcal{A}$ w.r.t L by $\mathcal{A}_L=\{\alpha\in\mathcal{A}\mid L\subseteq H_{\alpha,G}\}$, and denote $\mathcal{A}^L:=\mathcal{A}/\mathcal{A}_L$. Note that L can be considered as a rank 0 layer of $\mathcal{A}^L(G)$. Define

$$\begin{aligned}\mathcal{M}^L(\mathcal{A}):=&L\setminus\bigcup_{H_{\alpha,G}\not\supseteq L}H_{\alpha,G}\\ =&L\cap\mathcal{M}(\mathcal{A}^L;\Gamma/\langle\mathcal{A}_L\rangle,G).\end{aligned}$$

Note that we considered $\mathcal{M}(\mathcal{A}^L;\Gamma/\langle\mathcal{A}_L\rangle,G)$ as a subset of $\mathrm{Hom}(\Gamma,G)$ as in Proposition 1.2.1.8.

Let $L_1\subseteq\mathrm{Hom}(\Gamma,G)$ be a rank 1 layer of $\mathcal{A}(G)$, and let L_0 be the rank 0 layer that contains L_1. We wish to define the meridian homomorphism

$$\mu^\varepsilon_{L_0/L_1}:H_*(\mathcal{M}^{L_1}(\mathcal{A}),\mathbb{Z})\longrightarrow H_{*+\varepsilon\cdot(g-1)}(\mathcal{M}^{L_0}(\mathcal{A}),\mathbb{Z}),$$

where $g=\dim G=p+q>0$ and $\varepsilon\in\{0,1\}$. Since the normal bundle of L_1 in L_0 is trivial, there is a tubular neighborhood U of $\mathcal{M}^{L_1}(\mathcal{A})$ in L_0 such that $U\simeq\mathcal{M}^{L_1}(\mathcal{A})\times D^g$ with the identification $\mathcal{M}^{L_1}(\mathcal{A})=\mathcal{M}^{L_1}(\mathcal{A})\times\{0\}$, where D^g is the g-dimensional disk. Then $U\cap\mathcal{M}^{L_0}(\mathcal{A})\simeq\mathcal{M}^{L_1}(\mathcal{A})\times D^{g*}$, where $D^{g*}:=D^g\setminus\{0\}$. We denote the corresponding inclusion by $i:\mathcal{M}^{L_1}(\mathcal{A})\times D^{g*}\hookrightarrow M^{L_0}(\mathcal{A})$. For a given $\gamma\in H_*(\mathcal{M}^{L_1}(\mathcal{A}),\mathbb{Z})$, define the element $\mu^\varepsilon_{L_0/L_1}(\gamma)\in H_{*+\varepsilon\cdot(g-1)}(\mathcal{M}^{L_0}(\mathcal{A}),\mathbb{Z})$ as follows:

(0) When $\varepsilon=0$, let $p_0\in D^{g*}$. Then $\gamma\times[p_0]\in H_*(\mathcal{M}^{L_1}(\mathcal{A}))\otimes H_0(D^{g*})\subseteq H_*(\mathcal{M}^{L_1}(\mathcal{A})\times D^{g*})$, and $\mu^0_{L_0/L_1}(\gamma):=i_*(\gamma\times[p_0])$.

(1) When $\varepsilon = 1$, let $\mathbb{S}^{g-1} \subseteq D^{g*}$ be a sphere of small radius. Then $\gamma \times [\mathbb{S}^{g-1}] \in H_*(\mathcal{M}^{L_1}(\mathcal{A})) \otimes H_{g-1}(D^{g*}) \subseteq H_{*+g-1}(\mathcal{M}^{L_1}(\mathcal{A}) \times D^{g*})$ (this part is essentially the Gysin homomorphism). Now define $\mu^1_{L_0/L_1}(\gamma) := i_*(\gamma \times [\mathbb{S}^{g-1}])$.

Similarly, we can define the meridian map

$$\mu^{\varepsilon}_{L_j/L_{j+1}} : H_*(\mathcal{M}^{L_{j+1}}(\mathcal{A}), \mathbb{Z}) \longrightarrow H_{*+\varepsilon\cdot(g-1)}(\mathcal{M}^{L_j}(\mathcal{A}), \mathbb{Z})$$

between layers $L_j \supseteq L_{j+1}$ with consecutive ranks.

Definition 3.2.1.4. A cycle $\gamma \in H_d(\mathcal{M}(\mathcal{A}; \Gamma, G), \mathbb{Z})$ is called a *meridian cycle* if there exists some $k \geq 0$ and

(a) a flag $L_0 \supseteq L_1 \supseteq \cdots \supseteq L_k$ of layers with $\operatorname{rank} L_j = j$, such that $L_0 \cap \mathcal{M}(\mathcal{A}; \Gamma, G) \neq \emptyset$ (or equivalently, $L_0 \subseteq \mathcal{M}(\mathcal{A}^{\mathrm{tor}}; \Gamma, G)$),

(b) a sequence $\varepsilon_1, \ldots, \varepsilon_k \in \{0, 1\}$, and

(c) a torus cycle $\tau \in H_{d-(g-1)\sum_{i=1}^k \varepsilon_i}(\mathcal{M}^{L_k}(\mathcal{A}), \mathbb{Z})$,

such that

$$\gamma = \mu^{\varepsilon_1}_{L_0/L_1} \circ \mu^{\varepsilon_2}_{L_1/L_2} \circ \cdots \circ \mu^{\varepsilon_k}_{L_{k-1}/L_k}(\tau).$$

We call the minimum such k the depth of γ.

By definition, a meridian cycle of depth 0 is a torus cycle of a connected component L_0. Furthermore, a cycle $\gamma \in H_*(\mathcal{M}(\mathcal{A}; \Gamma, G), \mathbb{Z})$ is a meridian cycle of depth $k > 0$ if and only if there exist layers $L_0 \supseteq L_1$ of rank 0 and 1 respectively, with $L_0 \cap \mathcal{M}(\mathcal{A}; \Gamma, G) \neq \emptyset$, $\varepsilon \in \{0, 1\}$ and a meridian cycle $\gamma' \in H_{*-\varepsilon\cdot(g-1)}(\mathcal{M}^{L_1}(\mathcal{A}), \mathbb{Z})$ of depth $(k-1)$ such that $\gamma = \mu^{\varepsilon}_{L_0/L_1}(\gamma')$.

Note that in Definition 3.2.1.4, $\mathcal{M}^{L_0}(\mathcal{A})$ is a non-empty open subset of $\mathcal{M}(\mathcal{A}; \Gamma, G)$. Hence we have the induced injection $H_*(\mathcal{M}^{L_0}(\mathcal{A}), \mathbb{Z}) \hookrightarrow H_*(\mathcal{M}(\mathcal{A}; \Gamma, G), \mathbb{Z})$. We denote by $H_*^{\mathrm{merid}}(\mathcal{A}(G))$ the submodule of $H_*(\mathcal{M}(\mathcal{A}; \Gamma, G), \mathbb{Z})$ generated by the images of meridian cycles. It is clear that

$$H_*^{\mathrm{torus}}(\mathcal{A}(G)) \subseteq H_*^{\mathrm{merid}}(\mathcal{A}(G)) \subseteq H_*(\mathcal{M}(\mathcal{A}; \Gamma, G), \mathbb{Z}).$$

Lemma 3.2.1.5. *Assume that $q > 0$. Let $\alpha \in \mathcal{A} \setminus \mathcal{A}^{\mathrm{tor}}$, and let $\mathcal{A}' = \mathcal{A} \setminus \{\alpha\}$. Then*

$$H_*^{\mathrm{merid}}(\mathcal{A}(G)) \longrightarrow H_*^{\mathrm{merid}}(\mathcal{A}'(G)) \tag{3.2.5}$$

is surjective.

Proof. We prove this by induction on $\#(\mathcal{A} \setminus \mathcal{A}^{\mathrm{tor}})$ and the depth k of the meridian cycle γ. If $\#(\mathcal{A} \setminus \mathcal{A}^{\mathrm{tor}}) = 1$, then $\mathcal{A}' = \mathcal{A}^{\mathrm{tor}}$. In this case, the meridian cycles of $\mathcal{M}(\mathcal{A}'; \Gamma, G)$ are torus cycles, and the statement follows from Lemma 3.2.1.3. Now assume that $\#(\mathcal{A} \setminus \mathcal{A}^{\mathrm{tor}}) > 1$. Let $\gamma \in H_*(\mathcal{M}(\mathcal{A}'; \Gamma, G), \mathbb{Z})$ be a meridian cycle of $\mathcal{A}'$. Suppose that γ can be expressed as $\gamma = \mu^{\varepsilon_1}_{L_0/L_1} \circ \cdots \circ \mu^{\varepsilon_k}_{L_{k-1}/L_k}(\tau)$, as in Definition 3.2.1.4. If $k = 0$, then $\gamma = \tau$ is a torus cycle. Hence, again by Lemma 3.2.1.3, γ is contained in the image of the map (3.2.5). We may therefore assume that $k > 0$. Let $\gamma' = \mu^{\varepsilon_2}_{L_1/L_2} \circ \cdots \circ \mu^{\varepsilon_k}_{L_{k-1}/L_k}(\tau)$. Then $\gamma' \in H^{\mathrm{merid}}_{*-\varepsilon_1\cdot(g-1)}((\mathcal{A}')^{L_1}(G))$ is a meridian cycle of depth $(k-1)$. We separate the proof into two cases depending on whether $\overline{\alpha}$ is a loop in $\mathcal{A}^{L_1}$.

Suppose that $\overline{\alpha}$ is a loop in $\mathcal{A}^{L_1}$. Then $H_{\alpha,G}$ either contains L_1 or does not intersect L_1. In either case, $\mathcal{M}^{L_1}(\mathcal{A}) = \mathcal{M}^{L_1}(\mathcal{A}')$. Hence the tubular neighborhood U of $\mathcal{M}^{L_1}(\mathcal{A}')$ satisfies $U \cap \mathcal{M}^{L_0}(\mathcal{A}') = U \cap \mathcal{M}^{L_0}(\mathcal{A}) = U \setminus L_1$, and the meridian cycle $\gamma = \mu^{\varepsilon}_{L_0/L_1}(\gamma')$ can be constructed in $\mathcal{M}^{L_0}(\mathcal{A}) \subseteq \mathcal{M}^{L_0}(\mathcal{A}')$. Consequently, γ is contained in the image $H_*^{\mathrm{merid}}(\mathcal{A}(G)) \longrightarrow H_*^{\mathrm{merid}}(\mathcal{A}'(G))$.

Suppose that $\overline{\alpha}$ is not a loop in $\mathcal{A}^{L_1}$. Then, by the induction hypothesis, there exists a meridian cycle $\widetilde{\gamma}' \in H^{\mathrm{merid}}_{*-\varepsilon_1\cdot(g-1)}(\mathcal{A}^{L_1}(G))$ that is sent to γ' by the induced map

$$H^{\mathrm{merid}}_{*-\varepsilon_1\cdot(g-1)}(\mathcal{A}^{L_1}(G)) \longrightarrow H^{\mathrm{merid}}_{*-\varepsilon_1\cdot(g-1)}((\mathcal{A}')^{L_1}(G)), \widetilde{\gamma}' \longmapsto \gamma'.$$

Using the following commutative diagram, we can conclude that γ is also contained in the image:

$$\begin{array}{ccc}
\widetilde{\gamma}' \in H^{\mathrm{merid}}_{*-\varepsilon_1\cdot(g-1)}(\mathcal{A}^{L_1}(G)) & \longrightarrow & H^{\mathrm{merid}}_{*-\varepsilon_1\cdot(g-1)}((\mathcal{A}')^{L_1}(G)) \ni \gamma' \\
{\scriptstyle \mu^{\varepsilon_1}_{L_0/L_1}}\big\downarrow & & \big\downarrow{\scriptstyle \mu^{\varepsilon_1}_{L_0/L_1}} \\
H_*^{\mathrm{merid}}(\mathcal{A}(G)) & \longrightarrow & H_*^{\mathrm{merid}}(\mathcal{A}'(G)) \ni \gamma.
\end{array}$$

□

3.2.2 Mayer-Vietoris sequences and Poincaré polynomials

For simplicity of notation, in this subsection, we denote $\mathcal{M}(\mathcal{A}) := \mathcal{M}(\mathcal{A};\Gamma,G)$, $\mathcal{M}(\mathcal{A}') := \mathcal{M}(\mathcal{A}';\Gamma,G)$, and $\mathcal{M}(\mathcal{A}'') := \mathcal{M}(\mathcal{A}'';\Gamma'',G)$.

Theorem 3.2.2.1. *Let $\mathcal{A}$ be a finite list of elements in a finitely generated abelian group Γ, and let $G = (\mathbb{S}^1)^p \times \mathbb{R}^q \times F$, where F is a finite abelian group. Assume that $q > 0$, and let $g = \dim G = p + q$. Then the following hold:*

(i) $H_*(\mathcal{M}(\mathcal{A}),\mathbb{Z})$ is generated by the meridian cycles. That is $H_*(\mathcal{M}(\mathcal{A}),\mathbb{Z}) = H_*^{\mathrm{merid}}(\mathcal{A}(G))$, and furthermore it is torsion free.

(ii) If α is not a loop, then $H_*(\mathcal{M}(\mathcal{A}),\mathbb{Z}) \longrightarrow H_*(\mathcal{M}(\mathcal{A}'),\mathbb{Z})$ is surjective.

(iii) Let $\alpha \in \mathcal{A}$. Then

$$P_{\mathcal{M}(\mathcal{A})}(t) = \begin{cases} P_{\mathcal{M}(\mathcal{A}')}(t) - P_{\mathcal{M}(\mathcal{A}'')}(t), & \textit{if } \alpha \textit{ is a loop,} \\ P_{\mathcal{M}(\mathcal{A}')}(t) + t^{g-1}\cdot P_{\mathcal{M}(\mathcal{A}'')}(t), & \textit{if } \alpha \textit{ is not a loop.} \end{cases}$$

Proof. We first note that when α is a loop, $\mathcal{M}(\mathcal{A}') = \mathcal{M}(\mathcal{A}) \sqcup \mathcal{M}(\mathcal{A}'')$ is a decomposition into disjoint open subsets. Thus (iii) is obvious when α is a loop.

We prove the other results by induction on $\#(\mathcal{A}\setminus\mathcal{A}^{\mathrm{tor}})$. If $\mathcal{A} = \mathcal{A}^{\mathrm{tor}}$, then (i) follows from

$$H_*(\mathcal{M}(\mathcal{A}),\mathbb{Z}) = H_*^{\mathrm{merid}}(\mathcal{A}(G)) = H_*^{\mathrm{torus}}(\mathcal{A}(G))$$

(see (3.2.3)), and there is nothing to prove for (ii) and (iii).

Assume that $\mathcal{A}\setminus\mathcal{A}^{\mathrm{tor}} \neq \emptyset$, and suppose that $\alpha \in \mathcal{A}\setminus\mathcal{A}^{\mathrm{tor}}$. Let U be a tubular neighborhood of $\mathcal{M}(\mathcal{A}'')$ in $\mathcal{M}(\mathcal{A}')$, as in §3.2.1. Set $U^* := U \cap \mathcal{M}(\mathcal{A}) \simeq \mathcal{M}(\mathcal{A}'') \times D^{g*}$. Consider the Mayer-Vietoris sequence associated with the covering $\mathcal{M}(\mathcal{A}') = U \cup \mathcal{M}(\mathcal{A})$. We have the following diagram:

$$\begin{tikzcd} \arrow[r] & H_k(U^*) \arrow[r, "f_k"] & H_k(U)\oplus H_k(\mathcal{M}(\mathcal{A})) \arrow[r, "g_k"] & H_k(\mathcal{M}(\mathcal{A}')) \arrow[r] & {} \\ & H_k^{\mathrm{merid}}(U^*) \arrow[u, "h_1"] \arrow[r, "f_k'"] & H_k^{\mathrm{merid}}(U)\oplus H_k^{\mathrm{merid}}(\mathcal{A}(G)) \arrow[u, "h_2"] \arrow[r, "g_k'"] & H_k^{\mathrm{merid}}(\mathcal{A}'(G)), \arrow[u, "h_3"] & \end{tikzcd}$$

where $H_*^{\text{merid}}(U^*) = H_*^{\text{merid}}(\mathcal{A}''(G)) \otimes H_*(D^{g*})$ and $H_k^{\text{merid}}(U) \simeq H_k^{\text{merid}}(\mathcal{A}''(G))$. The first line is a part of the Mayer-Vietoris long exact sequence. The vertical arrows h_1, h_2 and h_3 are the inclusions of the subgroups generated by the meridian cycles. By the induction hypothesis, h_1 and h_3 are isomorphic. Lemma 3.2.1.5 implies that g'_k is surjective. Hence, g_k is also surjective. The surjectivity of g_{k+1} implies that f_k is injective. Therefore, the long exact sequence breaks into short exact sequences. The torsion freeness follows immediately. Thus

$$\operatorname{rank} H_k(U) + \operatorname{rank} H_k(\mathcal{M}(\mathcal{A})) = \operatorname{rank} H_k(U^*) + \operatorname{rank} H_k(\mathcal{M}(\mathcal{A}')),$$

which implies the inductive formula (iii). A diagram chase shows that h_2 is also surjective. Hence $H_*(\mathcal{M}(\mathcal{A}), \mathbb{Z}) = H_*^{\text{merid}}(\mathcal{A}(G))$. □

If $G = (\mathbb{S}^1)^p \times \mathbb{R}^q \times F$ as in the previous theorem, the Poincaré polynomial of G is

$$P_G(t) = (1+t)^p \times \#F.$$

We can compute the Poincaré polynomial of the complement $\mathcal{M}(\mathcal{A})$ using $P_G(t)$ and the G-Tutte/characteristic polynomial.

Theorem 3.2.2.2. *Let G be a non-compact abelian Lie group with finitely many connected components. Set $g = \dim G$. Then*

$$\begin{aligned} P_{\mathcal{M}(\mathcal{A})}(t) &= P_G(t)^{r_\Gamma - r_\mathcal{A}} \cdot t^{r_\mathcal{A}(g-1)} \cdot T_\mathcal{A}^G\left(\frac{P_G(t)}{t^{g-1}} + 1, 0\right) \\ &= (-t^{g-1})^{r_\Gamma} \cdot \chi_\mathcal{A}^G\left(-\frac{P_G(t)}{t^{g-1}}\right). \end{aligned}$$

Proof. We prove the result by induction on $\#\mathcal{A}$. Suppose that $\mathcal{A} = \emptyset$. Then $\mathcal{M}(\mathcal{A}) = \operatorname{Hom}(\Gamma, G) \simeq \operatorname{Hom}(\Gamma_{\text{tor}}, G) \times G^{r_\Gamma}$, and $\chi_\mathcal{A}^G(t) = \# \operatorname{Hom}(\Gamma_{\text{tor}}, G) \times t^{r_\Gamma}$. Theorem 3.2.2.2 follows immediately.

Suppose $\mathcal{A} \neq \emptyset$. Then, using Corollary 1.2.2.14 and Theorem 3.2.2.1 (iii), Theorem 3.2.2.2 can be proved by induction. □

Remark 3.2.2.3. Theorem 3.2.2.2 recovers the known formulas (1.1.2) and (1.1.3).

Remark 3.2.2.4. If G is a compact group, then Theorem 3.2.2.2 is not valid unless $\mathcal{A} = \emptyset$ (see Example 3.2.2.5 for the simplest example). There are several steps that fail for compact groups. For example the surjectivity of torus cycles (Lemma 3.2.1.3) fails, so the proof of the surjectivity of meridian cycles (Lemma 3.2.1.5) does not work. Furthermore, the existence of the fundamental class is an obstruction for breaking the Mayer-Vietoris sequence into short exact sequences.

Example 3.2.2.5. Let $G = \mathbb{S}^1, \Gamma = \mathbb{Z}$, and $\mathcal{A} = \{\alpha\}$ with $\alpha = 1 \in \mathbb{Z}$. Then $r_\Gamma = r_\mathcal{A} = 1$ and $g = \dim G = 1$. By definitions, $T_\mathcal{A}^{\mathbb{S}^1}(x, y) = x$ and $\chi_\mathcal{A}^{\mathbb{S}^1}(t) = t - 1$. The right-hand side of the second formula in Theorem 3.2.2.2 is equal to $2 + t$. However, since $\mathcal{M}(\mathcal{A}) = \mathbb{S}^1 \setminus \{pt\}$ is homeomorphic to $\mathbb{R}$, we have $P_{\mathcal{M}(\mathcal{A})}(t) = 1$, and the formula fails.

4. G-TUTTE POLYNOMIALS VIA MATROID THEORY

4.1 Relationship with arithmetic matroids

The G-Tutte polynomial is defined by inspiration of the arithmetic Tutte polynomial, and we have seen that they share a number of combinatorial and topological properties. Associated with an arithmetic Tutte polynomial, there is a matroidal structure defined through a system of axioms on the arithmetic multiplicities. It is natural to investigate the relationship between the G-multiplicity and the axioms, that is the main objective of this section. In addition, we present a G-Tutte polynomial version of convolution formula, a feature known to be possessed by several well-known polynomials including Tutte and arithmetic Tutte polynomials.

4.1.1 G-multiplicities and arithmetic matroid axioms

Definition 4.1.1.1. A *matroid* $\mathcal{M} = (E, r)$ is a finite list E with a *rank function* $r : 2^E \to \mathbb{Z}_{\geq 0}$ such that:

(1) if $A \subseteq E$, then $r(A) \leq |A|$,

(2) if $A \subseteq B \subseteq E$, then $r(A) \leq r(B)$,

(3) if $A, B \subseteq E$, then $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$.

Example 4.1.1.2. Let E be a finite list of elements in a finitely generated abelian group Γ. For $S \subseteq E$, denote $r(S) := r_S$ ($= \operatorname{rank}(\langle S\rangle_{\mathbb{Z}})$). Then (E, r) defines a matroid. Matroids of this form are called *realizable (or representable).*

Definition 4.1.1.3. The *Tutte polynomial* of a matroid $\mathcal{M} = (E, r)$ is defined by

$$T_{\mathcal{M}}(x, y) := \sum_{A \subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}.$$

More generally, let m be a function $m : 2^E \to \mathbb{R}$, called the *multiplicity* function. The *Tutte polynomial* of $(\mathcal{M}, m)$ is defined by

$$T_{(\mathcal{M},m)}(x,y) := \sum_{A \subseteq E} m(A) \cdot (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}.$$

Definition 4.1.1.4. Let $\mathcal{M} = (E, r)$ be a matroid. If $R \subseteq S \subseteq E$, let $[R, S] := \{A \subseteq E \mid R \subseteq A \subseteq S\}$. The set $[R, S]$ is called a *molecule* if $S = R \sqcup F \sqcup T$, and for each $A \in [R, S]$,

$$r(A) = r(R) + |A \cap F|.$$

A molecule $[R, S]$ is said to be *extreme* if either $F = \emptyset$, or $T = \emptyset$.

Proposition 4.1.1.5.

(i) If $[R, S]$ is a molecule and $[R', S'] \subseteq [R, S]$, then $[R', S']$ is a molecule.

(ii) A molecule $[R, S]$ is extreme if either $r(R) = r(S)$ (i.e., $F = \emptyset$), or $r(S) = r(R) + |S \setminus R|$ (i.e., $T = \emptyset$).

Proof. Straightforward. □

Let $\mathcal{M} = (E, r)$ be a matroid. Let m be an integral multiplicity function, i.e., $m : 2^E \to \mathbb{Z}$.

Definition 4.1.1.6. The triple (E, r, m) is called a *quasi-arithmetic matroid* if the multiplicities m satisfy

(Q1) for all $A \subseteq E$ and $a \in E$, if $r(A \cup \{a\}) = r(A)$, then $m(A \cup \{a\}) \mid m(A)$; otherwise $m(A) \mid m(A \cup \{a\})$,

(Q2) if $[R, S]$ is a molecule with $S = R \sqcup F \sqcup T$, then

$$m(R) \cdot m(S) = m(R \sqcup F) \cdot m(R \sqcup T).$$

Definition 4.1.1.7. The triple (E, r, m) is called a

(P) *pseudo-arithmetic* matroid if for every molecule $[R, S]$,

$$\rho(R,S) := (-1)^{|T|} \sum_{A \in [R,S]} (-1)^{|S|-|A|} m(A) \geq 0.$$

(NP) *nearly pseudo-arithmetic* matroid if for every extreme molecule $[R, S]$,

$$\rho(R,S) \geq 0.$$

(RP) *relatively pseudo-arithmetic* matroid if for every non-extreme molecule $[R, S]$,

$$\rho(R,S) \geq 0.$$

Clearly, a matroid together with a multiplicity function satisfy (P) if and only if they satisfy (NP) and (RP).

Proposition 4.1.1.8. *The multiplicities m satisfy (Q2) and (NP) if and only if they satisfy (Q2) and (P).*

Proof. See [BM14, §2]. □

Definition 4.1.1.9 (§2.3, [DM13])**.** The triple (E, r, m) is called an *arithmetic matroid* if the multiplicities m satisfy (Q1), (Q2) and (P).

Definition 4.1.1.10 (§4.2, [DM13])**.** Let $\mathcal{AM} = (E, r, m)$ be an arithmetic matroid. The Tutte polynomial $T_{\mathcal{AM}}(x, y)$ of $\mathcal{AM}$ is called the *arithmetic Tutte polynomial*.

Clearly, any matroid is an arithmetic matroid with *trivial* multiplicity function, i.e., $m(A) = 1$ for all $A \subseteq E$. Thus, the Tutte polynomial is a specialization of the arithmetic Tutte polynomial.

Theorem 4.1.1.11. *Let $\mathcal{PAM} = (E, r, m)$ be a pseudo-arithmetic matroid. Then the coefficients of the Tutte polynomial $T_{\mathcal{PAM}}(x, y)$ are nonnegative integers.*

Proof. See [BM14, Theorem 4.5] □

Example 4.1.1.12 (§2.4, [DM13], §5, [BM14])**.** Let (E, r) be a realizable matroid (see Example 4.1.1.2). For $A \subseteq E$, let $m(A) = \#(\Gamma/\langle A\rangle)_{\mathrm{tor}}$. Then (E, r, m) is an arithmetic matroid. Arithmetic matroids of this form are called *realizable (or representable).*

Unless otherwise stated, we assume that $\mathcal{A} = \{\alpha_1, \ldots, \alpha_n\}$ is a finite list of elements in a finitely generated abelian group Γ. Let G be a torsion-wise finite abelian group (Definition 1.2.2.1). We can define for the realizable matroid $(\mathcal{A}, r)$ a multiplicity function by using the definition of G-multiplicity in Definition 1.2.2.7, more precisely, $m(\mathcal{S}; G) = \#\operatorname{Hom}((\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{tor}}, G)$ for $\mathcal{S} \subseteq \mathcal{A}$. We wish to investigate the relationship between the G-multiplicities and arithmetic matroid axioms mentioned previously.

To do that, we need the construction of the dual of a representable arithmetic matroid described in [DM13, §3.4]. Let us recall the construction briefly (actually, the construction has been described in Lemma 2.2.1.2 for serving other purpose). Assume that Γ can be expressed as $\Gamma = \mathbb{Z}^m/\langle \mathbf{v}_1, \ldots, \mathbf{v}_h\rangle$. Choose a representative $\widetilde{\alpha}_i \in \mathbb{Z}^m$ of $\alpha_i \in \Gamma$ for each $i \in [n]$. Define

$$\Gamma^\dagger := \mathbb{Z}^{n+h}/\langle {}^t(\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_n, \mathbf{v}_1, \ldots, \mathbf{v}_h)\rangle,$$

where the denominator is the subgroup generated by m columns of the $(n+h) \times m$ matrix ${}^t(\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_n, \mathbf{v}_1, \ldots, \mathbf{v}_h)$. Let $\mathbf{e}_i$ be the standard basis of $\mathbb{Z}^{n+h}$. Set $\alpha_i^\dagger := \overline{\mathbf{e}_i} \in \Gamma^\dagger$ for $i \in [n]$. Now we have the list $\mathcal{A}^\dagger = \{\alpha_1^\dagger, \ldots, \alpha_n^\dagger\}$. It is proved in [DM13, §3.4] that for a subset $S \subseteq [n]$, we have

$$\begin{gathered} r_S^\dagger = \#S - r_{[n]} + r_{S^c}, \\ (\Gamma^\dagger/\langle \alpha_i^\dagger \mid i \in S\rangle)_{\mathrm{tor}} \simeq (\Gamma/\langle \alpha_i \mid i \in S^c\rangle)_{\mathrm{tor}}, \end{gathered} \tag{4.1.1}$$

where $S^c = [n] \setminus S$, $r_S = \operatorname{rank}\langle \alpha_i \mid i \in S\rangle$ and $r_S^\dagger = \operatorname{rank}\langle \alpha_i^\dagger \mid i \in S\rangle$ (the second relation in (4.1.1) is not a canonical isomorphism). Note that $\mathcal{A}^\dagger$ has rank $r_{\mathcal{A}^\dagger} = \#\mathcal{A} - r_{\mathcal{A}}$.

Denote the G-multiplicity of $(\Gamma^\dagger, \mathcal{A}^\dagger)$ by

$$m^\dagger(S; G) := \#\operatorname{Hom}\Big((\Gamma^\dagger/\langle \alpha_i^\dagger \mid i \in S\rangle)_{\mathrm{tor}}, G\Big).$$

The operation $(-)^\dagger$ is reflexive in the sense that

$$\begin{aligned} r_S &= \#S - r^\dagger_{[n]} + r^\dagger_{S^c}, \\ m(S;G) &= m^\dagger(S^c;G), \end{aligned} \tag{4.1.2}$$

and the G-Tutte polynomials satisfy

$$T^G_{\mathcal{A}^\dagger}(x,y) = T^G_{\mathcal{A}}(y,x). \tag{4.1.3}$$

Convention: In what follows, a dot under a letter indicates the parameter in the summation. For instance, $\sum_{\mathcal{S}\subseteq \underset{\bullet}{\mathcal{B}} \subseteq \mathcal{T}}$ indicates that $\mathcal{S}$ and $\mathcal{T}$ are fixed, and $\mathcal{B}$ is running between them.

Theorem 4.1.1.13. *Let G be a torsion-wise abelian finite group. Then the G-multiplicities satisfy (Q1) and (NP). In other words, they satisfy the following four properties (we borrow the numbering from [DM13, §2.3]).*

(1) If $\mathcal{S} \subseteq \mathcal{A}$ and $\alpha \in \mathcal{A}$ satisfy $r_{\mathcal{S}\cup\{\alpha\}} = r_{\mathcal{S}}$, then $m(\mathcal{S}\cup\{\alpha\};G)$ divides $m(\mathcal{S};G)$.

(2) If $\mathcal{S} \subseteq \mathcal{A}$ and $\alpha \in \mathcal{A}$ satisfy $r_{\mathcal{S}\cup\{\alpha\}} = r_{\mathcal{S}}+1$, then $m(\mathcal{S};G)$ divides $m(\mathcal{S}\cup\{\alpha\};G)$.

(4) If $\mathcal{S} \subseteq \mathcal{T} \subseteq \mathcal{A}$ and $r_{\mathcal{S}} = r_{\mathcal{T}}$, then

$$\rho(\mathcal{S},\mathcal{T};G) := \sum_{\mathcal{S}\subseteq \underset{\bullet}{\mathcal{B}} \subseteq \mathcal{T}} (-1)^{\#\mathcal{B}-\#\mathcal{S}} m(\mathcal{B};G) \geq 0.$$

(5) If $\mathcal{S} \subseteq \mathcal{T} \subseteq \mathcal{A}$ and $r_{\mathcal{T}} = r_{\mathcal{S}} + \#(\mathcal{T}\setminus\mathcal{S})$, then

$$\rho^*(\mathcal{S},\mathcal{T};G) := \sum_{\mathcal{S}\subseteq \underset{\bullet}{\mathcal{B}} \subseteq \mathcal{T}} (-1)^{\#\mathcal{T}-\#\mathcal{B}} m(\mathcal{B};G) \geq 0.$$

Additionally, if G is a (torsion-wise finite) divisible abelian group, that is, the multiplication-by-k map $k : G \longrightarrow G$ is surjective for any positive integer k, then the G-multiplicities satisfy (Q2), i.e., they satisfy the following.

(3) If $\mathcal{S} \subseteq \mathcal{T} \subseteq \mathcal{A}$ and $\mathcal{T}$ is a disjoint union $\mathcal{T} = \mathcal{S} \sqcup \mathcal{B} \sqcup \mathcal{C}$ such that for all $\mathcal{S} \subseteq \mathcal{R} \subseteq \mathcal{T}$, we have $r_{\mathcal{R}} = r_{\mathcal{S}} + \#(\mathcal{R}\cap\mathcal{B})$, then

$$m(\mathcal{S};G)\cdot m(\mathcal{T};G) = m(\mathcal{S}\sqcup\mathcal{B};G)\cdot m(\mathcal{S}\sqcup\mathcal{C};G).$$

Proof. By [BM14, Lemma 5.2], there exists a group epimorphism $(\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{tor}} \longrightarrow (\Gamma/\langle\mathcal{S}\cup\{\alpha\}\rangle)_{\mathrm{tor}}$. Applying the functor $\mathrm{Hom}(-, G)$ to this epimorphism, Property (1) follows. By the above construction, $(r^{\dagger}, m^{\dagger})$ satisfies Property (1), which is equivalent to Property (2) for (r, m) by the duality (4.1.2).

We prove Property (4) by showing that $\rho(\mathcal{S}, \mathcal{T}; G)$ is the cardinality of a certain finite set. Property (4) is clearly true if $\mathcal{S} = \mathcal{T}$, so assume that $\mathcal{S} \subsetneq \mathcal{T}$. Let us define Γ' by

$$\Gamma' := \{g \in \Gamma \mid \exists n > 0 \text{ such that } n \cdot g \in \langle\mathcal{S}\rangle\}.$$

It is also characterized by $(\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{tor}} = \Gamma'/\langle\mathcal{S}\rangle$. By the assumption $r_{\mathcal{S}} = r_{\mathcal{T}}$, we have $\mathcal{S} \subseteq \mathcal{T} \subseteq \Gamma'$. If $\mathcal{S} \subseteq \mathcal{B} \subseteq \mathcal{T}$, then $(\Gamma/\langle\mathcal{B}\rangle)_{\mathrm{tor}} = \Gamma'/\langle\mathcal{B}\rangle$. Therefore, $\mathrm{Hom}((\Gamma/\langle\mathcal{B}\rangle)_{\mathrm{tor}}, G) = \mathrm{Hom}(\Gamma'/\langle\mathcal{B}\rangle, G)$ can be considered as a subset of $\mathrm{Hom}((\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{tor}}, G) = \mathrm{Hom}(\Gamma'/\langle\mathcal{S}\rangle, G)$. By the Inclusion-Exclusion principle and Proposition 1.2.1.6, we have

$$\begin{aligned}
\rho(\mathcal{S}, \mathcal{T}; G) &= \sum_{\mathcal{S}\subseteq\underset{\bullet}{\mathcal{B}}\subseteq\mathcal{T}} (-1)^{\#\mathcal{B}-\#\mathcal{S}} \cdot m(\mathcal{B}; G) \\
&= \sum_{\mathcal{S}\subseteq\underset{\bullet}{\mathcal{B}}\subseteq\mathcal{T}} (-1)^{\#\mathcal{B}-\#\mathcal{S}} \cdot \#\,\mathrm{Hom}(\Gamma'/\langle\mathcal{B}\rangle, G) \\
&= \#\mathcal{M}(\mathcal{T}/\mathcal{S}; \Gamma'/\langle\mathcal{S}\rangle, G),
\end{aligned}$$

which is clearly non-negative. We can prove Property (5) by an argument similar to that for Property (2) by using the duality.

Finally, to prove Property (3) we generalize the argument used in [DM13, Lemma 2.6]. We consider the following diagram composing of two short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \left(\frac{\Gamma}{\langle\mathcal{S}\sqcup\mathcal{C}\rangle}\right)_{\mathrm{tor}} & \longrightarrow & \left(\frac{\Gamma}{\langle\mathcal{T}\rangle}\right)_{\mathrm{tor}} & \longrightarrow & \left(\frac{\Gamma}{\langle\mathcal{T}\rangle}\right)_{\mathrm{tor}} \Big/ \left(\frac{\Gamma}{\langle\mathcal{S}\sqcup\mathcal{C}\rangle}\right)_{\mathrm{tor}} & \longrightarrow & 0 \\
 & & & & & & \Big\uparrow \simeq & & \\
0 & \longrightarrow & \left(\frac{\Gamma}{\langle\mathcal{S}\rangle}\right)_{\mathrm{tor}} & \longrightarrow & \left(\frac{\Gamma}{\langle\mathcal{S}\sqcup\mathcal{B}\rangle}\right)_{\mathrm{tor}} & \longrightarrow & \left(\frac{\Gamma}{\langle\mathcal{S}\sqcup\mathcal{B}\rangle}\right)_{\mathrm{tor}} \Big/ \left(\frac{\Gamma}{\langle\mathcal{S}\rangle}\right)_{\mathrm{tor}} & \longrightarrow & 0.
\end{array}$$

Note that the isomorphism indicated by the vertical arrow is proved in [BM14, Lemma 5.3]. Since G is divisible, G is an injective $\mathbb{Z}$-module and the functor $\mathrm{Hom}(-, G)$ is exact. Applying the functor $\mathrm{Hom}(-, G)$ to the diagram we obtain Property (3). $\square$

Remark 4.1.1.14. When G is a connected abelian Lie group, that is, $G = (\mathbb{S}^1)^p \times \mathbb{R}^q$, G is a torsion-wise finite and divisible group. Theorem 4.1.1.13 is valid. We can see that Property (3) fails in many cases. For example, let $\Gamma = \mathbb{Z}^2$, $\mathcal{S} = \{(0,2)\}$, $\mathcal{B} = \{(2,1)\}$, $\mathcal{C} = \{(0,1)\}$ and $G = \mathbb{Z}_2$. Then $(\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{tor}} \simeq \mathbb{Z}_2$, $(\Gamma/\langle\mathcal{S}\cup\mathcal{B}\rangle)_{\mathrm{tor}} \simeq \mathbb{Z}_4$, $(\Gamma/\langle\mathcal{S}\cup\mathcal{C}\rangle)_{\mathrm{tor}} \simeq \{0\}$, $(\Gamma/\langle\mathcal{T}\rangle)_{\mathrm{tor}} \simeq \mathbb{Z}_2$, and $m(\mathcal{S};G)\cdot m(\mathcal{T};G) = 4 \neq 2 = m(\mathcal{S}\sqcup\mathcal{B};G)\cdot m(\mathcal{S}\sqcup\mathcal{C};G)$.

4.1.2 Convolution formula

As proved in Corollary 1.2.2.13, the G-Tutte polynomial possesses a deletion-contraction formula as the (arithmetic) Tutte polynomials do. We prove that the G-Tutte polynomials also satisfy a convolution formula, a feature known to be shared by many polynomials.

Theorem 4.1.2.1. *Let $\mathcal{A}$ be a finite list in a finitely generated group Γ, and let G_1 and G_2 be torsion-wise finite groups. Then*

$$T_{\mathcal{A}}^{G_1\times G_2}(x,y) = \sum_{\underset{\bullet}{\mathcal{B}}\subseteq\mathcal{A}} T_{\mathcal{B}}^{G_1}(0,y)\cdot T_{\mathcal{A}/\mathcal{B}}^{G_2}(x,0).$$

Proof. The right-hand side of the formula is equal to

$$\begin{aligned}
&\sum_{\underset{\bullet}{\mathcal{B}}\subseteq\mathcal{A}} \left\{ \sum_{\underset{\bullet}{\mathcal{S}}\subseteq\mathcal{B}} m(\mathcal{S};G_1)(-1)^{r_{\mathcal{B}}-r_{\mathcal{S}}}(y-1)^{\#\mathcal{S}-r_{\mathcal{S}}} \right\} \\
&\quad\times \left\{ \sum_{\mathcal{B}\subseteq\underset{\bullet}{\mathcal{T}}\subseteq\mathcal{A}} m(\mathcal{T};G_2)(x-1)^{r_{\mathcal{A}}-r_{\mathcal{B}}-(r_{\mathcal{T}}-r_{\mathcal{B}})}(-1)^{\#\mathcal{T}-\#\mathcal{B}-(r_{\mathcal{T}}-r_{\mathcal{B}})} \right\} \\
&= \sum_{\underset{\bullet}{\mathcal{S}}\subseteq\underset{\bullet}{\mathcal{B}}\subseteq\underset{\bullet}{\mathcal{T}}\subseteq\mathcal{A}} m(\mathcal{S};G_1)m(\mathcal{T};G_2)(x-1)^{r_{\mathcal{A}}-r_{\mathcal{T}}}(y-1)^{\#\mathcal{S}-r_{\mathcal{S}}}(-1)^{\#\mathcal{T}-\#\mathcal{B}-r_{\mathcal{T}}-r_{\mathcal{S}}} \\
&= \sum_{\underset{\bullet}{\mathcal{S}}=\underset{\bullet}{\mathcal{B}}=\underset{\bullet}{\mathcal{T}}\subseteq\mathcal{A}} m(\mathcal{S};G_1)m(\mathcal{S};G_2)(x-1)^{r_{\mathcal{A}}-r_{\mathcal{S}}}(y-1)^{\#\mathcal{S}-r_{\mathcal{S}}} \\
&\quad + \sum_{\underset{\bullet}{\mathcal{S}}\subsetneq\underset{\bullet}{\mathcal{T}}\subseteq\mathcal{A}} \left\{ m(\mathcal{S};G_1)m(\mathcal{T};G_2)(x-1)^{r_{\mathcal{A}}-r_{\mathcal{T}}}(y-1)^{\#\mathcal{S}-r_{\mathcal{S}}} \sum_{\mathcal{S}\subseteq\underset{\bullet}{\mathcal{B}}\subseteq\mathcal{T}} (-1)^{\#\mathcal{T}-\#\mathcal{B}-r_{\mathcal{T}}-r_{\mathcal{S}}} \right\}.
\end{aligned}$$

The first term is equal to $T_{\mathcal{A}}^{G_1\times G_2}(x,y)$ by the multiplicativity $m(\mathcal{S};G_1\times G_2) = m(\mathcal{S};G_1)\cdot m(\mathcal{S};G_2)$ (see Proposition 1.2.2.6). The second term vanishes because, when $\mathcal{S}\subsetneq\mathcal{T}$, we have $\sum_{\mathcal{S}\subseteq\mathcal{B}\subseteq\mathcal{T}}(-1)^{\#\mathcal{B}} = 0$. □

The classical convolution formula [ELV98, KRS99] for matroids representable over $\mathbb{Q}$ is obtained from Theorem 4.1.2.1 by replacing G_1 and G_2 by $\{0\}$. Theorem 4.1.2.1 can also be specialized to the Backman-Lenz's convolution formula [BL16] when $G_1\times G_2 = \mathbb{S}^1\times\{0\}$ or $\{0\}\times\mathbb{S}^1$.

4.2 Coefficients of G-Tutte polynomials

We have already seen that the G-Tutte polynomial can not be directly associated with a realizable arithmetic matroid as the G-multiplicities satisfy only four over five required axioms. Furthermore, although the arithmetic Tutte polynomial is a polynomial with positive coefficients, we will show that the positivity of the coefficients is not preserved for the G-Tutte polynomial. This leaves us with a question under what conditions the coefficients of the G-Tutte polynomial are positive. We propose some ideas and partial answers.

4.2.1 (Non-)positivity of coefficients

In general, the G-Tutte polynomial can have negative coefficients as in the next example.

Example 4.2.1.1. Let $\Gamma=\mathbb{Z}\oplus\mathbb{Z}_4$, let $\mathcal{A}=\{\alpha,\beta\}\subseteq\Gamma$ with $\alpha=(2,\overline{1})$ and $\beta=(0,\overline{2})$, and let $G=\mathbb{Z}_4$. Then by direct computation, we have

$$T_{\mathcal{A}}^{G}(x,y) = 2xy+2x+2y-2.$$

This also produces a counter-example to axiom (P) in Definition 4.1.1.7. Note that $[\emptyset,\mathcal{A}]$ is a molecule, and

$$\rho(\emptyset,\mathcal{A};G) = (-1)\cdot\sum_{\mathcal{B}\subseteq\mathcal{A}}(-1)^{2-\#\mathcal{B}}m(\mathcal{B};G) = -2 < 0.$$

As hinted in the example above, to obtain the positivity of the coefficients of G-Tutte polynomial, some condition either on the group G or the list $\mathcal{A}$ is needed. The first idea is to keep the generality of $\mathcal{A}$, and restrict G.

Theorem 4.2.1.2. *Let G be a torsion-wise finite divisible abelian group. Then the coefficients of the G-Tutte polynomial $T_{\mathcal{A}}^{G}(x, y)$ are nonnegative integers.*

Proof. It is proved in Theorem 4.1.1.13 that the pair $(\Gamma, \mathcal{A})$ together with the G-multiplicities form an arithmetic matroid. Theorem 4.1.1.11 completes the proof. □

Owing to the deletion-contraction formula (Corollary 1.2.2.13) and the convolution formula (Theorem 4.1.2.1) we propose some other ideas to approach the positivity problem.

Proposition 4.2.1.3. *Let G be a torsion-wise finite abelian group. The coefficients of the G-Tutte polynomial $T_{\mathcal{A}}^{G}(x, y)$ are nonnegative integers if the group G satisfies one of the following conditions:*

(i) for any finite list $\mathcal{D} \subseteq \Gamma$ containing no proper elements, the coefficients of the G-Tutte polynomial $T_{\mathcal{D}}^{G}(x, y)$ are nonnegative integers.

(ii) for any finite list $\mathcal{D} \subseteq \Gamma$,

$$T_{\mathcal{D}}^{G}(0,0) = \sum_{\mathcal{B} \subseteq \mathcal{D}} (-1)^{r_{\mathcal{D}} - \#\mathcal{B}} m(\mathcal{B}; G) \geq 0.$$

Proof. (i) follows from Corollary 1.2.2.13 because we can write $T_{\mathcal{A}}^{G}(x, y)$ as a sum of the G-Tutte polynomials of the lists having no proper elements. (ii) follows from Theorem 4.1.2.1 because

$$\begin{aligned} T_{\mathcal{A}}^{G}(x, y) &= \sum_{\mathcal{B} \underset{\bullet}{\subseteq} \mathcal{A}} T_{\mathcal{B}}^{G}(0, y) \cdot T_{\mathcal{A}/\mathcal{B}}^{\{0\}}(x, 0) \\ &= \sum_{\mathcal{B} \underset{\bullet}{\subseteq} \mathcal{A}} T_{\mathcal{A}/\mathcal{B}}^{\{0\}}(x, 0) \left(\sum_{\mathcal{S} \underset{\bullet}{\subseteq} \mathcal{B}} T_{\mathcal{S}}^{\{0\}}(0, y) \cdot T_{\mathcal{B}/\mathcal{S}}^{G}(0, 0) \right). \end{aligned}$$

Note also that the coefficients of the classical Tutte polynomial (i.e., the $\{0\}$-Tutte polynomial) are nonnegative integers. □

4.2.2 An interpretation of coefficients

As suggested previously, to obtain the positivity we may look at the second idea, that is to keep the generality of G, and put some condition on the list $\mathcal{A}$. We prove that if $\mathcal{A}$ consists of only torsion elements of Γ, then $T_{\mathcal{A}}^{G}(x,y)$ is a polynomial in y by definition, and it always has nonnegative coefficients. Moreover, the coefficients of the G-Tutte polynomial can be explicitly described.

Theorem 4.2.2.1. *Let G be a torsion-wise finite abelian group. Suppose that $\mathcal{A}$ is contained in Γ_{tor}. Then*

$$T_{\mathcal{A}}^{G}(x,y) = \sum_{k=0}^{\#\mathcal{A}} \left(\sum_{\substack{\mathcal{S}\subseteq\mathcal{A} \\ \#\mathcal{S}=k}} \#\mathcal{M}(\mathcal{A}/\mathcal{S};\Gamma_{\mathrm{tor}}/\langle\mathcal{S}\rangle, G) \right) y^{k}.$$

In particular, $T_{\mathcal{A}}^{G}(x,y)$ is a polynomial in y with nonnegative coefficients.

Proof. By assumption, $r_{\mathcal{A}} = r_{\mathcal{S}} = 0$ and $(\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{tor}} = \Gamma_{\mathrm{tor}}/\langle\mathcal{S}\rangle$ for every $\mathcal{S}\subseteq\mathcal{A}$. Using Proposition 1.2.1.8, we have

$$\begin{aligned}
T_{\mathcal{A}}^{G}(x,y) &= \sum_{\mathcal{S}\subseteq\mathcal{A}} \#\operatorname{Hom}(\Gamma_{\mathrm{tor}}/\langle\mathcal{S}\rangle, G)\cdot (y-1)^{\#\mathcal{S}} \\
&= \sum_{\mathcal{S}\subseteq\mathcal{A}} \#\operatorname{Hom}(\Gamma_{\mathrm{tor}}/\langle\mathcal{S}\rangle, G)\cdot \sum_{k=0}^{\#\mathcal{S}} y^{k}\cdot(-1)^{\#\mathcal{S}-k}\cdot\binom{\#\mathcal{S}}{k} \\
&= \sum_{k=0}^{\#\mathcal{A}} y^{k}\cdot \sum_{\substack{\mathcal{S}\subseteq\mathcal{A} \\ \#\mathcal{S}\geq k}} (-1)^{\#\mathcal{S}-k}\binom{\#\mathcal{S}}{k} \sum_{\mathcal{S}\subseteq\mathcal{T}\subseteq\mathcal{A}} \#\mathcal{M}(\mathcal{A}/\mathcal{T};\Gamma_{\mathrm{tor}}/\langle\mathcal{T}\rangle, G) \\
&= \sum_{k=0}^{\#\mathcal{A}} y^{k}\cdot \sum_{\substack{\mathcal{T}\subseteq\mathcal{A} \\ \#\mathcal{T}\geq k}} \#\mathcal{M}(\mathcal{A}/\mathcal{T};\Gamma_{\mathrm{tor}}/\langle\mathcal{T}\rangle, G)\cdot \sum_{\substack{\mathcal{S}\subseteq\mathcal{T} \\ \#\mathcal{S}\geq k}} (-1)^{\#\mathcal{S}-k}\binom{\#\mathcal{S}}{k} \\
&= \sum_{k=0}^{\#\mathcal{A}} y^{k}\cdot \sum_{\substack{\mathcal{T}\subseteq\mathcal{A} \\ \#\mathcal{T}\geq k}} \#\mathcal{M}(\mathcal{A}/\mathcal{T};\Gamma_{\mathrm{tor}}/\langle\mathcal{T}\rangle, G)\cdot \sum_{k\leq m\leq\#\mathcal{T}} (-1)^{m-k}\binom{m}{k}\cdot\binom{\#\mathcal{T}}{m}
\end{aligned}$$

$$= \sum_{k=0}^{\#\mathcal{A}} y^k \cdot \sum_{\substack{\mathcal{T} \subseteq \mathcal{A} \\ \#\mathcal{T} \geq k}} \#\mathcal{M}(\mathcal{A}/\mathcal{T}; \Gamma_{\mathrm{tor}}/\langle \mathcal{T} \rangle, G) \cdot \binom{\#\mathcal{T}}{k} \sum_{k \leq m \leq \#\mathcal{T}} (-1)^{m-k} \binom{\#\mathcal{T} - k}{m - k}$$

$$= \sum_{k=0}^{\#\mathcal{A}} y^k \cdot \sum_{\substack{\mathcal{T} \subseteq \mathcal{A} \\ \#\mathcal{T} = k}} \#\mathcal{M}(\mathcal{A}/\mathcal{T}; \Gamma_{\mathrm{tor}}/\langle \mathcal{T} \rangle, G).$$

□

Proposition 4.2.2.2. *Let G be a torsion-wise finite group.*

(i) If $\mathcal{A} \subseteq \Gamma$ consists of loops (i.e., $\mathcal{A} \subseteq \Gamma_{\mathrm{tor}}$), then $T_{\mathcal{A}}^G(x, y)$ has positive coefficients.

(ii) If $\mathcal{A} \subseteq \Gamma$ consists of coloops (i.e., $r_{\mathcal{A}} = \#\mathcal{A}$), then $T_{\mathcal{A}}^G(x, y)$ has positive coefficients.

Proof. (i) follows immediately from Theorem 4.2.2.1. (ii) follows immediately from (i) and (4.1.3) (note that if $\mathcal{A}$ consists of coloops, then $\mathcal{A}^\dagger$ consists of loops). □

LIST OF REFERENCES

[ABC+16] T. Abe, M. Barakat, M. Cuntz, T. Hoge, and H. Terao. The freeness of ideal subarrangements of Weyl arrangements. *J. Eur. Math. Soc.*, 18(6):1339–1348, 2016.

[ACH15] F. Ardila, F. Castillo, and M. Henley. The arithmetic Tutte polynomials of the classical root systems. *Internat. Math. Res. Notices*, 2015(12):3830–3877, 2015.

[AR12] D. Armstrong and B. Rhoades. The Shi arrangement and the Ish arrangement. *Trans. Amer. Math. Soc.*, 364(3):1509–1528, 2012.

[Ath96] C. A. Athanasiadis. Characteristic polynomials of subspace arrangements and finite fields. *Adv. Math.*, 122(2):193–233, 1996.

[Ath99] C. A. Athanasiadis. Extended Linial hyperplane arrangements for root systems and a conjecture of Postnikov and Stanley. *J. Algebr. Comb*, 10:207–225, 1999.

[Ath04] C. A. Athanasiadis. Generalized Catalan numbers, Weyl groups and arrangements of hyperplanes. *Bull. Lond. Math. Soc.*, 36:294–302, 2004.

[Ath10] C. A. Athanasiadis. A combinatorial reciprocity theorem for hyperplane arrangements. *Canad. Math. Bull.*, 53(1):3–10, 2010.

[ATY19] A. U. Ashraf, T. N. Tran, and M. Yoshinaga. Eulerian polynomials for subarrangements of Weyl arrangements. arXiv preprint, 2019. https://arxiv.org/abs/1911.01650.

[BBC14] C. Bajo, B. Burdick, and S. Chmutov. On the Tutte-Krushkal-Renardy polynomial for cell complexes. *J. Combin. Theory Ser. A*, 123:186–201, 2014.

[Bib16] C. Bibby. Cohomology of abelian arrangements. *Proc. Amer. Math. Soc.*, 144(7):3093–3104, 2016.

[Bjö94] A. Björner. *Subspace arrangements*. In: Joseph A., Mignot F., Murat F., Prum B., Rentschler R. (eds) First European Congress of Mathematics. Progress in Mathematics, vol 3. Birkhäuser Basel, 1994.

[BL16] S. Backman and M. Lenz. A convolution formula for Tutte polynomials of arithmetic matroids and other combinatorial structures. arXiv preprint, 2016. https://arxiv.org/abs/1602.02664.

[BM14] P. Brändén and L. Moci. The multivariate arithmetic Tutte polynomial. *Trans. Amer. Math. Soc.*, 366(10):5523–5540, 2014.

[Bou68] N. Bourbaki. *Groupes et Algèbres de Lie*. Chapitres 4,5 et 6, Hermann, Paris, 1968.

[BPR06] B. Basu, R. Pollack, and M. F. Roy. *Algorithms in real algebraic geometry*. Second edition. Algorithms and Computation in Mathematics, 10. *Springer-Verlag, Berlin*, x+662 pp, 2006.

[BS98] A. Blass and B. Sagan. Characteristic and Ehrhart polynomials. *J. Algebr. Comb.*, 7(2):115–126, 1998.

[BS18] M. Beck and R. Sanyal. *Combinatorial reciprocity theorems: An invitation to enumerative geometric combinatorics*. Volume 195, Graduate Studies in Mathematics, American Mathematical Society, 2018.

[BZ06] M. Beck and T. Zaslavsky. Inside-out polytopes. *Adv. Math.*, 205(1):134–162, 2006.

[Cos07] M. Coste. *Real Algebraic Sets. Arc spaces and additive invariants in real algebraic and analytic geometry*, 1-32, Panor. Synthèses, 24, Soc. Math. France, Paris, 2007.

[CS04] S. Caracciolo and A. Sportiello. General duality for abelian-group-valued statistical-mechanics models. *J. Phys. A*, 37(30):7407–7432, 2004.

[CW12] B. Chen and S. Wang. Comparison on the coefficients of characteristic quasi-polynomials of integral arrangements. *J. Combin. Theory Ser. A*, 119:271–281, 2012.

[DCP05] C. De Concini and C. Procesi. On the geometry of toric arrangements. *Transform. Groups*, 10(3-4):387–422, 2005.

[DFM18] C. Dupont, A. Fink, and L. Moci. Universal Tutte characters via combinatorial coalgebras. *Algebraic Combinatorics*, 1(5):603–651, 2018.

[DM13] M. D'Adderio and L. Moci. Arithmetic matroids, the Tutte polynomial and toric arrangements. *Adv. Math.*, 232(1):335–367, 2013.

[DM18] E. Delucchi and L. Moci. Products of arithmetic matroids and quasipolynomial invariants of CW-complexes. *J. Combin. Theory Ser. A*, 157:28–40, 2018.

[ELV98] G. Etienne and M. Las Vergnas. External and internal elements of a matroid basis. *Discrete Math.*, 179(1-3):111–119, 1998.

[FK72] C.M. Fortuin and P.W. Kasteleyn. On the random-cluster model: I. Introduction and relation to other models. *Physica*, 57(4):536–564, 1972.

[FM16] A. Fink and L. Moci. Matroids over a ring. *J. Eur. Math. Soc. (JEMS)*, 18(4):681–731, 2016.

[GM88] M. Goresky and R. MacPherson. *Stratified Morse Theory*. Ergeb. Math. Grenzgeb., Vol. 14, Springer-Verlag, Berlin, 1988.

[HN12] J. Hilgert and K-H. Neeb. *Structure and Geometry of Lie Groups*. Springer Monographs in Mathematics, Springer-Verlag New York, 2012.

[Hum90] J. E. Humphreys. *Reflection Groups and Coxeter Groups*. Cambridge University Press, 1990.

[Ive86] B. Iversen. *Cohomology of sheaves*. Universitext, Springer-Verlag, Berlin, xii+464, 1986.

[Jae88] F. Jaeger. Tutte polynomials and link polynomials. *Proc. Amer. Math. Soc.*, 103:647–654, 1988.

[KRS99] W. Kook, V. Reiner, and D. Stanton. A convolution formula for the Tutte polynomial. *J. Combin. Theory Ser. B*, 76(2):297–300, 1999.

[KTT07] H. Kamiya, A. Takemura, and H. Terao. The characteristic quasi-polynomials of the arrangements of root systems. arXiv preprint [v1], 2007. https://arxiv.org/abs/0707.1381v1.

[KTT08] H. Kamiya, A. Takemura, and H. Terao. Periodicity of hyperplane arrangements with integral coefficients modulo positive integers. *J. Algebr. Comb.*, 27(3):317–330, 2008.

[KTT10] H. Kamiya, A. Takemura, and H. Terao. The characteristic quasi-polynomials of the arrangements of root systems and mid-hyperplane arrangements. *Arrangements, local systems and singularities*, pages 177–190, Progr. Math., 283, Birkhäuser Verlag, Basel, 2010.

[KTT11] H. Kamiya, A. Takemura, and H. Terao. Periodicity of non-central integral arrangements modulo positive integers. *Ann. Comb.*, 15(3):449–464, 2011.

[LP18] T. Lam and A. Postnikov. Alcoved polytopes II. *In: Kac V., Popov V. (eds) Lie Groups, Geometry, and Representation Theory*, pages 253–272, Progr. Math., 326, Birkhäuser Verlag, Cham, 2018.

[LTY] Y. Liu, T. N. Tran, and M. Yoshinaga. G-Tutte polynomials and abelian Lie group arrangements. *Int. Math. Res. Not. IMRN*, to appear.

[Moc12] L. Moci. A Tutte polynomial for toric arrangements. *Trans. Amer. Math. Soc.*, 364(2):1067–1088, 2012.

[OS80] P. Orlik and L. Solomon. Combinatorics and topology of complements of hyperplane. *Invent. Math.*, 56(1):167–189, 1980.

[OS83] P. Orlik and L. Solomon. Coxeter arrangements. *Proc. Symp. Pure Math., Amer. Math. Soc*, 40(2):269–292, 1983.

[OT92] P. Orlik and H. Terao. *Arrangements of hyperplanes*. Grundlehren der Mathematischen Wissenschaften 300, Springer-Verlag, Berlin, 1992.

[Sok05] A. D. Sokal. *The multivariate Tutte polynomial (alias Potts model) for graphs and matroids*. Surveys in combinatorics, 173-226, London Math. Soc. Lecture Note Ser., 327, Cambridge Univ. Press, Cambridge, 2005.

[Som05] E. N. Sommers. B-stable ideals in the nilradical of a Borel subalgebra. *Canad. Math. Bull.*, 48(3):460–472, 2005.

[SP00] R.P. Stanley and A. Postnikov. Deformations of Coxeter hyperplane arrangements. *J. Combin. Theory Ser. A*, 91:544–597, 2000.

[Sta07] R.P. Stanley. *An introduction to hyperplane arrangements.* in: E. Miller, V. Reiner, B. Sturmfels (Eds.), Geometric Combinatorics, in: IAS/Park City Math. Ser., vol. 13, Amer. Math. Soc., Providence, RI, pp. 389-496, 2007.

[Sta11] R.P. Stanley. *Enumerative Combinatorics: Volume 1.* Cambridge University Press, New York, NY, USA, 2nd edition, 2011.

[Sut98] R. Suter. The number of lattice points in alcoves and the exponents of the finite Weyl groups. *Math. Comp.*, 67(222):751–758, 1998.

[Tra18] T. N. Tran. An equivalent formulation of chromatic quasi-polynomials. arXiv preprint, 2018. https://arxiv.org/abs/1803.08649.

[Tra19] T. N. Tran. Characteristic quasi-polynomials of ideals and signed graphs of classical root systems. *European J. Combin.*, 79:179–192, 2019.

[Tut54] W. T. Tutte. A contribution to the theory of chromatic polynomials. *Canad. J. Math.*, 6:80–91, 1954.

[TY19] T. N. Tran and M. Yoshinaga. Combinatorics of certain abelian Lie group arrangements and chromatic quasi-polynomials. *J. Combin. Theory Ser. A*, 165:258–272, 2019.

[Wan15] S. Wang. Möbius conjugation and convolution formulae. *J. Combin. Theory Ser. B*, 115:117–131, 2015.

[Yos18a] M. Yoshinaga. Characteristic polynomials of Linial arrangements for exceptional root systems. *J. Combin. Theory Ser. A*, 157:267–286, 2018.

[Yos18b] M. Yoshinaga. Worpitzky partitions for root systems and characteristic quasi-polynomials. *Tohoku Math. J.*, 70(1):39–63, 2018.

[Zas75] T. Zaslavsky. Facing up to arrangements: Face-count formulas for partitions of space by hyperplanes. *Memoirs Amer. Math. Soc.*, 1(154):vii+102 pp, 1975.

[Zas81] T. Zaslavsky. The geometry of root systems and signed graphs. *The American Mathematical Monthly*, 88(2):88–105, 1981.

[Zas82] T. Zaslavsky. Signed graphs. *Discrete Appl. Math.*, 4(1):47–74, 1982.