The symbol theory of pseudodifferential operators via the Čech-Dolbeault cohomology (Čech-Dolbeault コホモロジーを用いた無限階擬微分作用素の表象理論)

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March. 2020.

Introduction

The aim of this paper is to construct the sheaf morphism from the sheaf $\mathscr{E}_X^{\mathbb{R}}$ of pseudodifferential operators to its symbol class $\mathfrak{S}/\mathfrak{N}$. Firstly we look back the historical background of pseudodiferential operators and mention the difference of two words "pseudo-differential" and "pseudodifferential".

The theory of pseudo-differential operators in real variables was done by Monvel [9] and Monvel-Krée [10], and that in complex variables was done by Sato-Kawai-Kashiwara [16]. In the case of real domain Monvel [9] introduced pseudo-differential operators of infinite order in the analytic category and Monvel-Krée [10] constructed the class of pseudo-differential operators of finite order which are continuous on the class of Gevrey functions. On the other hand, Sato-Kawai-Kashiwara [16] developed the theory of pseudo-differential operators and its symbol theory by employing the sheaf cohomology theory and they studied systems of differential equations in the complex domain. They defined the sheaf $\mathscr{P}_X$ of pseudo-differential operators, which is now denoted by $\mathscr{E}_X^{\infty}$ and is called the sheaf of microdifferential operators. Those two theories were finally linked up by Kataoka [7]. He realized the symbols of operators in $\mathscr{E}_X^{\mathbb{R}}$ by utilizing the Radon transformations and it is known that pseudo-differential operators in [9] can be obtained by the restriction of $\mathscr{E}_X^{\mathbb{R}}$ to the real domain.

While the essential idea of pseudodifferential operators had already been introduced in [16], the explicit definition was not given. After a few years, Kashiwara and Kawai gave the definition of the sheaf $\mathscr{P}_X^{\mathbb{R}}$ of pseudodifferential operators in [8]. However the sheaf $\mathscr{P}_X^{\mathbb{R}}$ had not been named even at that time. Finally $\mathscr{P}_X^{\mathbb{R}}$ had been denoted by $\mathscr{E}_X^{\mathbb{R}}$ after the work of Kashiwara-Schapira [11], and $\mathscr{E}_X^{\mathbb{R}}$ came to be called the sheaf of pseudodifferential operators after the work of Aoki [2].

Since $\mathscr{E}_X^{\mathbb{R}}$ is explicitly defined by using the sheaf cohomology, Kataoka [7] introduced symbols of pseudodifferential operators by the aid of the Radon transformations for the study of $\mathscr{E}_X^{\mathbb{R}}$ in analytic category, and Aoki [1],[2] developed the symbol theory of $\mathscr{E}_X^{\mathbb{R}}$. The sheaf $\mathscr{E}_X^{\mathbb{R}}$ is sufficiently large class of differential operators so that it contains indispensable operators like differential operators of fractional order, which are not contained in the class $\mathscr{E}_X^{\infty}$ of microdifferential operators.

In the foundation of Aoki's symbol theory, however, there remain some issues. In this paper we study one of those problems, which is the equivalence of the sheaf $\mathscr{E}_X^{\mathbb{R}}$ and its symbol class $\mathfrak{S}/\mathfrak{N}$. Aoki had already proved the equivalence of each stalks of $\mathscr{E}_X^{\mathbb{R}}$ and $\mathfrak{S}/\mathfrak{N}$ in the following method:

Thanks to SKK [16] the stalk $\mathscr{E}_{X,z^*}^{\mathbb{R}}$ has the following cohomological expression at a point $z^* \in T^*X$

$$\mathscr{E}_{X,z^*}^{\mathbb{R}} = \varinjlim_{r,\varepsilon} H^n_{G_{r,\varepsilon}}(U_r\,;\,\mathscr{O}_{X\times X}^{(0,n)}),$$

where U_r ranges through the family of open subsets of $X \times X$ and $G_{r,\varepsilon}$ through the family of closed subsets of $X \times X$ satisfying some appropriate conditions. By taking convenient Stein coverings we can identify the right-hand side cohomology with the Čech cohomology. Thus, by the Čech cohomology we can see that a pseudodifferential operator P is represented by some holomorphic n-form $\psi(z, z' - z)dz'$, i.e.,

$$P = [\psi(z, z' - z)dz'].$$

Via such a representation $P = [\psi(z, z' - z)dz']$, Aoki defined the map of stalks

$$\sigma : \mathscr{E}^{\mathbb{R}}_{X,z^*} \to \mathfrak{S}_{z^*}/\mathfrak{N}_{z^*}$$

by the formula

$$\sigma(P) = \left[\int_{\beta_0}^{\beta_1} \oint_{\gamma_2} \cdots \oint_{\gamma_n} \psi(z,w) e^{\langle w, \zeta \rangle} dw \right] \in \mathfrak{S}_{z^*}/\mathfrak{N}_{z^*},$$

where the path of the integration is taken suitably. (See [2].) Under this definition he gave the following theorem.

Theorem 0.1 ([2], Theorem 4.3 and Theorem 4.5)**.** There exists the homomorphism of stalks

$$\varpi : \mathfrak{S}_{z^*}/\mathfrak{N}_{z^*} \longrightarrow \mathscr{E}^{\mathbb{R}}_{X,z^*}.$$

Moreover $\varpi \circ \sigma = id$ and $\sigma \circ \varpi = id$.

However when we apply this argument to the set $\mathscr{E}^{\mathbb{R}}_X(V)$ of sections on an open cone V, we encounter several difficulties: There is no suitable Čech covering of Stein open sets, that is, we cannot take a good representation of a section of $\mathscr{E}^{\mathbb{R}}_X$. Furthermore we cannot choose an appropriate path of the above integration in this global case.

To overcome these difficulties we apply the theory of Čech-Dolbeault cohomology, which is introduced by Honda-Izawa-Suwa [4], to representation of $\mathscr{E}^{\mathbb{R}}_X(V)$. They introduce the Čech-Dolbeault complex and succeed in calculating local cohomology groups $H^n_M(X\,;\,\mathscr{O}^{(p)}_X)$ by applying the simple Čech covering which consists of two open subsets to the Dolbeault complex. As the Čech-Dolbeault complex consists of pairs (ω_1, ω_{01}) of a (p,q)-form ω_1 and a $(p, q-1)$-form ω_{01} with coefficients in C^∞ functions, we can control the support of Čech-Dolbeault representative of $H^n_M(X\,;\,\mathscr{O}^{(p)}_X)$ under some suitable conditions by the aid of a partition of unity. Such a modification of supports is not allowed if we calculate $H^n_M(X\,;\,\mathscr{O}^{(p)}_X)$ with the standard Čech coverings since the standard Čech representation of cohomology groups take holomorphic functions as their representatives. As important applications they construct several operations such as the boundary value morphism, external products, integrations along fibers and so on from the viewpoint of the Čech-Dolbeault cohomology. By making full use of these advantages of the Čech-Dolbeault cohomology theory, we construct globally a sheaf morphism from $\mathscr{E}^{\mathbb{R}}_X$ to $\mathfrak{S}^\infty/\mathfrak{N}^\infty$ of symbols as given in Theorem 4.1 of Section 4.

The plan of this paper is as follows. In Section 1 we prepare some notations and definitions used in this paper, and in Section 2 we briefly recall the definitions of the sheaf $\mathscr{E}^{\mathbb{R}}_X$ of pseudodifferential operators and the theory of Čech-Dolbeault cohomology. Due to a fiber formula by Kashiwara-Schapira [12] it is known that the group $\mathscr{E}^{\mathbb{R}}_X(V)$ of the sections on an open cone V is represented by the inductive limit of local cohomology groups. As important examples the Čech-Dolbeault representation of hyperfunctions $\mathscr{B}_M(M)$ and pseudodifferential operators $\mathscr{E}^{\mathbb{R}}_X(V)$ are given. In Section 3 we introduce a new symbol class $\mathfrak{S}^\infty/\mathfrak{N}^\infty$ called the symbols of C^∞-type and prove that the symbol class $\mathfrak{S}^\infty/\mathfrak{N}^\infty$ of C^∞-type is equivalent to the classical one $\mathfrak{S}/\mathfrak{N}$. The work of Hörmander [14] plays an important role in the proof. Section 5 gives the morphism ς from $\mathscr{E}^{\mathbb{R}}_X$ to $\mathfrak{S}^\infty/\mathfrak{N}^\infty$ via Čech-Dolbeault representation. By the results in Section 4 we finally obtain the sheaf morphism from $\mathscr{E}^{\mathbb{R}}_X$ to $\mathfrak{S}/\mathfrak{N}$. Well-definedness of ς is also given. Together with Aoki's results ς turns out to be an isomorphism of sheaves.

1 Preliminaries

Through this paper we shall follow the notations and definitions introduced below.

We denote by $\mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$ the sets of integers, of real numbers and of complex numbers, respectively. Moreover we set $\mathbb{R}_+ = \{r \in \mathbb{R} \mid r > 0\}$ and $\mathbb{C}^\times = \{c \in \mathbb{C} \mid c \neq 0\}$.

Let X be a complex manifold of dimension n and M a submanifold of X of codimension m. Set the diagonal set

$$\Delta_X = \{(z_1, z_2) \in X \times X \mid z_1 = z_2\}.$$

We write Δ instead of Δ_X if there is no risk of confusion. One denote by p_1 and p_2 the first and the second projections from $X \times X$ to X, respectively.

$$\begin{tikzcd} & X_1 \times X_2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X_1 & & X_2. \end{tikzcd}$$

One denotes by $\tau : TX \to X$ the canonical projection from the tangent bundle to X and $\pi : T^*X \to X$ that from the cotangent bundle to X.

Let ω be a (p,q)-form with coefficients in C^∞-functions, and ∂_z and $\bar{\partial}_z$ the Dolbeault operators with respect to the variable z, that is, for a local coordinate $z = (z_1, z_2, \ldots, z_n)$, the form ω can be written by

$$\omega = \sum_{|I|=p, |J|=q} f_{IJ}(z) dz^I \wedge d\bar{z}^J.$$

Moreover the Dolbeault operators are written by

$$\partial_z \omega = \sum_{i=1}^{n} \sum_{|I|=p,|J|=q} \frac{\partial}{\partial z_i} f_{IJ}(z) dz_i \wedge dz^I \wedge d\bar{z}^J,$$
$$\bar{\partial}_z \omega = \sum_{i=1}^{n} \sum_{|I|=p,|J|=q} \frac{\partial}{\partial \bar{z}_i} f_{IJ}(z) d\bar{z}_i \wedge dz^I \wedge d\bar{z}^J.$$

Definition 1.1. We define several sheaves:

1. Let $\mathscr{O}_X^{(p)}$ be the sheaf of holomorphic p-forms on X. In particular $\mathscr{O}_X^{(0)} = \mathscr{O}_X$ is the sheaf of holomorphic functions on X.
2. We denote by or_X and $or_{M/X} = \mathscr{H}_M^m(\mathbb{Z}_X)$ the orientation sheaf on X and the relative orientation sheaf on M, respectively.
3. Set $\Omega_X^{(n)} = \mathscr{O}_X^{(n)} \underset{\mathbb{C}_X}{\otimes} or_X$ and $\mathscr{O}_{X\times X}^{(0,n)} = \mathscr{O}_{X\times X} \underset{p_2^{-1}\mathscr{O}_X}{\otimes} p_2^{-1}\Omega_X^{(n)}$.
4. One denotes by $C_X^{\infty,(p,q)}$ the sheaf of (p,q)-forms with coefficients in C^∞ on X.
5. One denotes by $\mathscr{E}_X^{\mathbb{R}}$ the sheaf of pseudodifferential operators on T^*X.

Let $(z;\zeta)$ be a local coordinate system of T^*X. Set $\mathring{T}^*X = T^*X \setminus T^*_X X$ where $T^*_X X$ is the zero section. We identify $T^*_\Delta(X \times X)$ with T^*X by the map

$$(z, z; \zeta, -\zeta) \mapsto (z; \zeta), \tag{1.1}$$

which is induced from the first projection $p_1 : X \times X \to X$.

Definition 1.2. Let V be a set in $\mathring{T}^*X$. The set V is called a cone, or equivalently called a conic set in $\mathring{T}^*X$ if and only if

$$(z; \zeta) \in V \Rightarrow (z; t\zeta) \in V \text{ for any } t \in \mathbb{R}_+.$$

Remark 1.3. Let V be a set in T^*X. We say that V is convex (resp. conic, resp. proper) if for any $z \in \pi(V)$, the set $\pi^{-1}(z) \cap V$ is convex (resp. conic, resp. proper). Recall that a cone is said to be proper if its closure contains no lines.

Let V and V' be subsets in T^*X. We write $V' \Subset V$ if V' is a relatively compact set in V for the usual topology.

Definition 1.4. Let V be an open cone in $\mathring{T}^*X$. A set $W \subset V$ is an infinitesimal wedge of type V at infinity if for any $K \Subset V$ there exists $\delta > 0$ such that

$$K_\delta = \{(z; t\zeta) \mid (z; \zeta) \in K, \, t > \delta\} \subset W.$$

In what follows W is called the infinitesimal wedge of type V for short.

Definition 1.5. Let V and V' be cones in $\mathring{T}^*X$ with $V' \subset V$. The cone V' is a relatively compact cone in V if there exists a relatively compact set K of V such that

$$V' = \{(z; t\zeta) \mid t \in \mathbb{R}_+, (z; \zeta) \in K\}.$$

To clarify the differences, one is denoted by $V' \underset{\text{cone}}{\Subset} V$ if V' is a relatively compact cone in V.

2 Pseudodifferential operators via the Čech-Dolbeault cohomology

The aim of this section is to obtain the Čech-Dolbeault representation of pseudodifferential operators.

2.1 The sheaf $\mathscr{E}_X^{\mathbb{R}}$ of pseudodifferential operators

Let X be a complex manifold of dimension n. The sheaf $\mathscr{E}_X^{\mathbb{R}}$ of pseudodifferential operators on T^*X is defined by

$$\mathscr{E}_X^{\mathbb{R}} = \mathscr{H}^n(\mu_\Delta(\mathscr{O}_{X \times X}^{(0,n)})), \tag{2.1}$$

where $\mu_\Delta(\mathscr{O}_{X \times X}^{(0,n)})$ is the microlocalization of $\mathscr{O}_{X \times X}^{(0,n)}$ along the diagonal set Δ. (For the definition of μ, see [13].) One denotes by $\mathscr{E}_{X,z^*}^{\mathbb{R}}$ the stalk of $\mathscr{E}_X^{\mathbb{R}}$ at a point $z^* \in T^*X$.

Let us recall the normal deformation $\tilde{X}^2_\Delta$ of $X \times X$ along Δ. We denote by $t \in \mathbb{R}$ the deformation parameter. Set $\Omega = \{(z_1, z_2, t) \in \tilde{X}^2_\Delta \mid t > 0\}$. Then the following diagram commutes

$$\begin{tikzcd} T_\Delta(X \times X) \arrow[r, "s"] \arrow[d, "\tau"'] & \tilde{X}^2_\Delta \arrow[d, "p"] & \Omega \arrow[l, "j"'] \arrow[dl, "\tilde{p}"] \\ \Delta \arrow[r, "i"'] & X \times X, & \end{tikzcd}$$

where i and j are embeddings and $\tilde{p} = p \circ j$. (For the details, see [13].)

Definition 2.1. Let G be a subset of $X \times X$. The normal cone to G along Δ, denoted by $C_\Delta(G)$, is the set

$$C_\Delta(G) = T_\Delta(X \times X) \cap \overline{\tilde{p}^{-1}(G)}.$$

Definition 2.2. Let V be a subset of T^*X. The polar set V° of V is defined by

$$V^\circ = \{y \in TX \mid \tau(y) \in \pi(V) \text{ and } \operatorname{Re}\langle x, y\rangle \geq 0 \text{ for all } x \in \pi^{-1}\tau(y) \cap V\}.$$

Theorem 2.3 ([11], Theorem 4.3.2)**.** Let V be an open convex cone in T^*X. We have

$$\mathscr{E}^{\mathbb{R}}_X(V) = \varinjlim_{U,G} H^n_{G\cap U}(U; \mathscr{O}^{(0,n)}_{X\times X}), \tag{2.2}$$

where U ranges through the family of open subsets of $X \times X$ such that $U \cap \Delta = \pi(V)$ and G through the family of closed subsets of $X \times X$ such that $C_\Delta(G) \subset V^\circ$.

Remark 2.4. In the above argument, we can construct the polar set V° of the cone $V \subset T^*_\Delta(X\times X)$ by the identification (1.1). Hereafter we sometimes use this identification without notice.

2.2 The Čech-Dolbeault representation of $\mathscr{E}^{\mathbb{R}}_X$

In the previous subsection we obtain the cohomological expression (2.2) of $\mathscr{E}^{\mathbb{R}}_X$ due to a fiber formula by Kashiwara and Schapira. In this paragraph we briefly recall the theory of Čech-Dolbeault cohomology introduced in [4] and obtain the Čech-Dolbeault representation of the group $\mathscr{E}^{\mathbb{R}}_X(V)$ of sections on an open convex cone V.

Let S be a closed subset of X. Set $V_0 = X \setminus S$ and let V_1 be an open neighborhood of S in X. For a covering $\mathcal{V} = \{V_0, V_1\}$ of X we set

$$C^{\infty,(p,q)}_X(\mathcal{V}) = C^{\infty,(p,q)}_X(V_0) \oplus C^{\infty,(p,q)}_X(V_1) \oplus C^{\infty,(p,q-1)}_X(V_{01}), \tag{2.3}$$

where $V_{01} = V_0 \cap V_1$. We define the differential $\bar{\vartheta} : C^{\infty,(p,q)}_X \to C^{\infty,(p,q+1)}_X$ by

$$\bar{\vartheta}(\omega_0, \omega_1, \omega_{01}) = (\bar{\partial}\omega_0, \bar{\partial}\omega_1, \omega_1 - \omega_0 - \bar{\partial}\omega_{01}). \tag{2.4}$$

Then we can easily see that $\bar{\vartheta} \circ \bar{\vartheta} = 0$ and the pair $(C^{\infty,(p,\bullet)}_X(\mathcal{V}), \bar{\vartheta})$ is a complex.

Definition 2.5. The Čech-Dolbeault cohomology $H^{p,q}_{\bar{\vartheta}}(\mathcal{V})$ of $\mathcal{V}$ of type (p,q) is the q-th cohomology of the complex $(C^{\infty,(p,\bullet)}_X(\mathcal{V}), \bar{\vartheta})$.

Next we introduce the relative Čech-Dolbeault cohomology. Let $\mathcal{V}' = \{V_0\}$ be a covering of $X \setminus S$. We set

$$C_X^{\infty,(p,q)}(\mathcal{V},\mathcal{V}') = \{(\omega_0,\omega_1,\omega_{01}) \in C_X^{\infty,(p,q)}(\mathcal{V}) \mid \omega_0 = 0\} = C_X^{\infty,(p,q)}(V_1) \oplus C_X^{\infty,(p,q)}(V_{01}).$$

Then the pair $(C_X^{\infty,(p,\bullet)}(\mathcal{V},\mathcal{V}'),\bar{\vartheta})$ is a subcomplex of $(C_X^{\infty,(p,\bullet)}(\mathcal{V}),\bar{\vartheta})$ which is called the relative Čech-Dolbeault complex.

Definition 2.6. The relative Čech-Dolbeault cohomology $H_{\bar{\vartheta}}^{p,q}(\mathcal{V},\mathcal{V}')$ of $(\mathcal{V},\mathcal{V}')$ of type (p,q) is the q-th cohomology of the complex $(C_X^{\infty,(p,\bullet)}(\mathcal{V},\mathcal{V}'),\bar{\vartheta})$.

Here we have the following proposition.

Proposition 2.7 ([4], Proposition 4.6)**.** The relative Čech-Dolbeault cohomology $H_{\bar{\vartheta}}^{p,q}(\mathcal{V},\mathcal{V}')$ is independent of the choice of V_1 and determined uniquely up to isomorphism.

Therefore we can choose X as V_1, and hereafter $H_{\bar{\vartheta}}^{p,q}(\mathcal{V},\mathcal{V}')$ is also denoted by $H_{\bar{\vartheta}}^{p,q}(X,X\setminus S)$.

Theorem 2.8 ([4], Theorem 4.9)**.** There is a canonical isomorphism

$$H_{\bar{\vartheta}}^{p,q}(X,X\setminus S) \simeq H_S^q(X\,;\,\mathscr{O}_X^{(p)}). \tag{2.5}$$

Example 2.9. Let M be a real analytic manifold of dimension n and X the complexification of M. Assume M to be oriented, and set $\mathcal{V} = \{X, X\setminus M\}$ and $\mathcal{V}' = \{X\setminus M\}$. Since we have

$$\mathscr{B}_M(M) = H_M^n(X;\mathscr{O}_X) = H_{\bar{\vartheta}}^{0,n}(X,X\setminus M),$$

the representative $\omega = (\omega_1,\omega_{01})$ of a hyperfunction $u \in \mathscr{B}_M(M)$ is written in the forms

$$\omega_1 = f(z)d\bar{z}, \qquad \omega_{01} = \sum_{i=1}^n f_i(z)d\bar{z}_{\hat{i}},$$

satisfying the cocycle condition $\bar{\vartheta}\omega = 0$. Here $f(z)$ is a C^∞-function on X, $f_i(z)$ is a C^∞-function on $X\setminus M$, $d\bar{z} = d\bar{z}_1\wedge d\bar{z}_2\wedge\ldots\wedge d\bar{z}_n$ and $d\bar{z}_{\hat{i}} = d\bar{z}_1\wedge\ldots d\bar{z}_{i-1}\wedge d\bar{z}_{i+1}\ldots\wedge d\bar{z}_n$. Note that ω_1 consists of the only one term.

We apply Theorem 2.8 to the cohomology $H_{G\cap U}^n(U;\mathscr{O}_{X\times X}^{(0,n)})$ in Theorem 2.3.

Definition 2.10. The sheaf $C_{X\times X}^{\infty,(p,q,r)}$ is the sheaf of $(p+q,r)$-forms with coefficients in C^∞-functions which are holomorphic p-forms with respect to the first variables, holomorphic q-forms with respect to the second variables and antiholomorphic r-forms with respect to the first and the second variables. In other words, for a local coordinate $z = (z_1,z_2)$ of $X\times X$ and for an open subset V of $X\times X$, the form $f(z_1,z_2) \in C_{X\times X}^{\infty,(p,q,r)}(V)$ is written by

$$f(z_1,z_2) = \sum_{|I|=p,|J|=q,|K|=r} f_{IJK}(z_1,z_2)dz_1^I\wedge dz_2^J\wedge d\bar{z}^K,$$

where each $f_{IJK}(z_1,z_2)$ is a C^∞-function on V.

Set $V_0 = U \setminus G$ and let V_1 be an open neighborhood of $G \cap U$ in U. Moreover set $V_{01} = V_0 \cap V_1$. For coverings $\mathcal{V} = \{V_0, V_1\}$ of U and $\mathcal{V}' = \{V_0\}$ of $U \setminus G$, we define

$$C^{\infty,(p,q,r)}_{X\times X}(\mathcal{V}, \mathcal{V}') = C^{\infty,(p,q,r)}_{X\times X}(V_1) \oplus C^{\infty,(p,q,r-1)}_{X\times X}(V_{01}).$$

We can also define the differential $\bar{\vartheta} : C^{\infty,(p,q,r)}_{X\times X}(\mathcal{V}, \mathcal{V}') \to C^{\infty,(p,q,r+1)}_{X\times X}(\mathcal{V}, \mathcal{V}')$ as usual, and the pair $(C^{\infty,(p,q,\bullet)}_{X\times X}(\mathcal{V}, \mathcal{V}'), \bar{\vartheta})$ becomes a complex.

Definition 2.11. The r-th relative Čech-Dolbeault cohomology $H^{p,q,r}_{\bar{\vartheta}}(\mathcal{V}, \mathcal{V}')$ is the r-th cohomology of the complex $(C^{\infty,(p,q,\bullet)}_{X\times X}(\mathcal{V}, \mathcal{V}'), \bar{\vartheta})$.

By Proposition 2.7 we can choose U as V_1 and obtain the following theorem.

Theorem 2.12. There is a canonical isomorphism

$$H^{0,n,n}_{\bar{\vartheta}}(U, U \setminus G) \simeq H^n_{G\cap U}(U\,;\, \mathscr{O}^{(0,n)}_{X\times X}). \tag{2.6}$$

Thus the group of sections of the sheaf $\mathscr{E}^{\mathbb{R}}_X$ on an open convex cone V is expressed by

$$\mathscr{E}^{\mathbb{R}}_X(V) = \varinjlim_{U,G} H^n_{G\cap U}(U\,;\, \mathscr{O}^{(0,n)}_{X\times X}) = \varinjlim_{U,G} H^{0,n,n}_{\bar{\vartheta}}(U, U \setminus G),$$

where U and G run in the same sets as those in Theorem 2.3.

Remark 2.13. The case of pseudodifferential operators is quite different from the one of hyperfunctions. In this case the length of the relative Čech-Dolbeault complex

$$\begin{aligned} 0 \to C^{\infty,(0,n,0)}_{X\times X}(\mathcal{V}, \mathcal{V}') \to C^{\infty,(0,n,1)}_{X\times X}(\mathcal{V}, \mathcal{V}') \to \dots \\ \to C^{\infty,(0,n,2n)}_{X\times X}(\mathcal{V}, \mathcal{V}') \to C^{\infty,(0,n,2n+1)}_{X\times X}(\mathcal{V}, \mathcal{V}') \to 0 \end{aligned}$$

is equal to $2n + 1$. Hence for the representative $\omega = (\omega_1, \omega_{01})$ of an element $u \in H^{0,n,n}_{\bar{\vartheta}}(U, U \setminus G)$, ω_1 and ω_{01} consist of several terms satisfying the cocycle condition $\bar{\vartheta}\omega = 0$.

3 Classical symbols and symbols of C^∞-type

While the classical symbol theory $\mathfrak{S}/\mathfrak{N}$ of $\mathscr{E}^{\mathbb{R}}_X$ is based on holomorphic functions, the Čech-Dolbeault expression is based on C^∞-functions, and hence it is difficult to construct the map from Čech-Dolbeault expression to the classical symbol class directly. In this section we construct a new symbol class which is of C^∞-type and show that the new symbol class is isomorphic to the classical symbol class.

3.1 The sheaf $\mathfrak{S}/\mathfrak{N}$ of classical symbols

First we review the classical symbol theory. Let $z^* = (z; \zeta)$ be a local coordinate system of T^*X. First we construct two conic sheaves $\mathfrak{S}$ and $\mathfrak{N}$ on $\mathring{T}^*X$.

Definition 3.1. Let $V \subset \mathring{T}^*X$ be an open cone.

1. A function $f(z,\zeta)$ is called a symbol on V if the following conditions hold.

   (i) There exists an infinitesimal wedge W of type V such that
$$f(z,\zeta) \in \mathscr{O}_{T^*X}(W).$$

   (ii) For any open cone $V' \underset{\text{cone}}{\Subset} V$ there exists an infinitesimal wedge $W' \subset W$ of type V' such that $f(z,\zeta)$ satisfies the following condition:
   For any constant $h > 0$, there exists a constant $C > 0$ such that
$$|f(z,\zeta)| \le C \cdot e^{h|\zeta|} \quad \text{on } W'. \tag{3.1}$$

2. A symbol $f(z,\zeta)$ on V is called a null-symbol if for any open cone $V' \underset{\text{cone}}{\Subset} V$ there exist an infinitesimal wedge $W' \subset W$ of type V' and constants $h > 0$ and $C > 0$ such that
$$|f(z,\zeta)| \le C \cdot e^{-h|\zeta|} \quad \text{on } W'. \tag{3.2}$$

3. We denote by $\mathfrak{S}(V)$ and $\mathfrak{N}(V)$ the set of all the symbols on V and the set of all the null-symbols on V, respectively. Moreover we set
$$\mathfrak{S}_{z^*} = \varinjlim_{V \ni z^*} \mathfrak{S}(V),$$
$$\mathfrak{N}_{z^*} = \varinjlim_{V \ni z^*} \mathfrak{N}(V),$$
   where V runs through the family of open conic neighborhoods of $z^* \in \mathring{T}^*X$.

Remark 3.2. A function $f(z,\zeta)$ satisfying the estimate (3.1) is said to be an infra-exponential function. Similarly a function $g(z,\zeta)$ satisfying the estimate (3.2) is said to be an exponentially small function.

Next we extend the sheaves $\mathfrak{S}$ and $\mathfrak{N}$ to the sheaves on T^*X. We define the sheaves $\mathfrak{S}|_{T^*_X X}$ and $\mathfrak{N}|_{T^*_X X}$ on the zero section $T^*_X X = X$ in the following way.

1. Let U be an open set in X. The group $\mathfrak{S}|_{T^*_X X}(U)$ of sections is a family of $f(z,\zeta) \in \mathscr{O}_{T^*X}(\pi^{-1}(U))$ where for any compact set $K \Subset U$ and for any constant $h > 0$ there exists a constant $C > 0$ such that
$$|f(z,\zeta)| \le C \cdot e^{h|\zeta|} \quad \text{on } \pi^{-1}(K).$$

2. Set $\mathfrak{N}|_{T^*_X X} = 0$.

Then the sheaves $\mathfrak{S}$ and $\mathfrak{N}$ are naturally extended to T^*X.

Remark 3.3. Well-definedness of the sheaves $\mathfrak{S}$ and $\mathfrak{N}$ can be shown as follows. Let $\mathscr{V} = \{V_\alpha\}_{\alpha \in A}$ be a conic basis of $\mathring{T}^*X$ which consists of open cones and $\mathscr{U} = \{U_\beta\}_{\beta \in B}$ a basis of X. Then we can easily see that $\mathscr{W} = \{V_\alpha, \pi^{-1}(U_\beta)\}_{\alpha \in A, \beta \in B}$ is a basis of T^*X. For any subsets $V \in \mathscr{V}$ and $U \in \mathscr{U}$ with $\pi(V) \subset U$, the restriction $\rho_{V\pi^{-1}(U)}$: $\mathfrak{S}(\pi^{-1}(U)) \to \mathfrak{S}(V)$ is just given by $f \mapsto f|_V$.

Next we construct the quotient sheaf $\mathfrak{S}/\mathfrak{N}$.

Proposition 3.4. Given an open cone V in $\mathring{T}^*X$, the group $\mathfrak{N}(V)$ of null-symbols is an ideal of $\mathfrak{S}(V)$.

Proof. Let $f(z,\zeta) \in \mathfrak{N}(V)$ and $g(z,\zeta) \in \mathfrak{S}(V)$. We assume that $f(z,\zeta)$ and $g(z,\zeta)$ are holomorphic on some common infinitesimal wedge W of type V. The definition of $\mathfrak{N}$ implies that for any $V' \underset{\text{cone}}{\Subset} V$ there exist an infinitesimal wedge W' of type V' and the constants $h > 0$ and $C > 0$ such that

$$|f(z,\zeta)| \leq C \cdot e^{-h|\zeta|}.$$

Similarly definition of $\mathfrak{S}$ implies that for V', W' and $h > 0$ which are the same ones as above, there exists a constant $C' > 0$ such that

$$|g(z,\zeta)| \leq C' \cdot e^{\frac{1}{2}h|\zeta|}.$$

Hence we obtain

$$|f(z,\zeta) \cdot g(z,\zeta)| \leq C \cdot e^{-h|\zeta|} \cdot C' \cdot e^{\frac{1}{2}h|\zeta|} \leq CC' \cdot e^{-\frac{1}{2}h|\zeta|}.$$

□

One denotes by $\widehat{\mathfrak{S}/\mathfrak{N}}$ the presheaf defined by the correspondence

$$V \mapsto \mathfrak{S}(V)/\mathfrak{N}(V),$$

where V is an open cone in $\mathring{T}^*X$, and let $\mathfrak{S}/\mathfrak{N}$ be an associated sheaf to $\widehat{\mathfrak{S}/\mathfrak{N}}$. We have the following exact sequence of sheaves

$$0 \longrightarrow \mathfrak{N} \longrightarrow \mathfrak{S} \xrightarrow{\kappa_1} \mathfrak{S}/\mathfrak{N} \longrightarrow 0. \tag{3.3}$$

Here κ_1 is the composition of the canonical morphisms $\mathfrak{S} \to \widehat{\mathfrak{S}/\mathfrak{N}} \to \mathfrak{S}/\mathfrak{N}$, and this exact sequence induces the long exact sequence

$$0 \to \mathfrak{N}(V) \to \mathfrak{S}(V) \to \mathfrak{S}/\mathfrak{N}(V) \to H^1(V;\mathfrak{N}) \to \cdots.$$

To treat $\mathfrak{S}/\mathfrak{N}(V)$ as it is a quotient group $\mathfrak{S}(V)/\mathfrak{N}(V)$, we claim $H^1(V;\mathfrak{N}) = 0$ for a suitable V.

Theorem 3.5. Assume X to be a complex vector space and let $\widetilde{V}$ be a closed cone in $\mathring{T}^*X$. Moreover assume that $\widetilde{V}$ satisfies the following three conditions.

1. A family of conic open neighborhoods of $\widetilde{V}$ has a cofinal family which consists of Stein open cones in $\mathring{T}^*X$.
2. The projection $\pi(\widetilde{V})$ is compact in X.
3. There exists $\zeta_0 \in \mathbb{C}^n \setminus \{0\}$ such that

$$\widetilde{V} \subset \{(z;\zeta) \in \mathring{T}^*X \mid z \in \pi(\widetilde{V}), \operatorname{Re}\langle\zeta,\zeta_0\rangle > 0\}.$$

Then $H^k(\widetilde{V};\mathfrak{N}) = 0$ holds for any $k > 0$.

Example 3.6. We can construct a closed cone $\widetilde{V}$ satisfying the above three conditions as follows. Let N be a natural number and $f_1(z), f_2(z), \ldots, f_N(z)$ holomorphic functions on X. Set

$$B = \bigcap_{i=1}^{N} \{|f_i(z)| \le 1\},$$

and assume B to be compact, and let Γ be a closed proper convex cone. Then $\widetilde{V} = B \times \Gamma$ satisfies the second and the third conditions in Theorem 3.5. A cofinal family of $B \times \Gamma$ is given in the following way. We can take a family $\{B_\varepsilon\}_{\varepsilon\in\mathbb{R}_+}$ of open neighborhoods of B as follows

$$B_\varepsilon = \bigcap_{1\le i\le N} \{|f_i(z)| < 1 + \varepsilon\}.$$

Since Γ is a closed proper convex cone we can take a cofinal family $\{\Gamma_\lambda\}_{\lambda\in\Lambda}$ which consists of open convex conic neighborhoods of Γ. Then the family $\{B_\varepsilon \times \Gamma_\lambda\}_{(\varepsilon,\lambda)\in\mathbb{R}_+\times\Lambda}$ is what we want.

For the proof of Proposition 3.5, we construct a soft resolution of $\mathfrak{N}$. First we introduce the radial compactification $\hat{T}^*X$ of T^*X with respect to the fibers.

Definition 3.7. The radial compactification $\mathbb{D}_{\mathbb{C}^n}$ of $\mathbb{C}^n$ is defined by

$$\mathbb{D}_{\mathbb{C}^n} = \mathbb{C}^n \sqcup S^{2n-1}\infty.$$

We define the fundamental system of neighborhoods. If z_0 belongs to $\mathbb{C}^n$ a family of fundamental neighborhoods of z_0 consists of open sets

$$B_\varepsilon(z_0) = \{z \in \mathbb{C}^n \mid |z - z_0| < \varepsilon\}$$

for $\varepsilon > 0$, otherwise that of $z_0\infty$ consists of open sets

$$G_r(\Gamma) = \left\{ z \in \mathbb{C}^n \;\middle|\; |z| > r, \frac{z}{|z|} \in \Gamma \right\} \sqcup \Gamma,$$

where $r > 0$ and Γ is an open neighborhood of $z_0\infty$ in $S^{2n-1}\infty$.

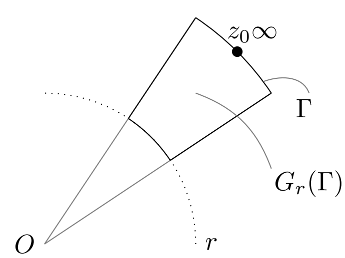

Figure 1: $G_r(\Gamma)$

We denote by $\overline{\overline{V}}$ the closure of V taken in $\mathbb{D}_{\mathbb{C}^n}$.

Definition 3.8. The radial compactification $\hat{T}^*X$ of T^*X with respect to the fiber is

$$\hat{T}^*X = \bigsqcup_{z \in X} \overline{\overline{T_z^*X}}.$$

Here $\overline{\overline{T_z^*X}} \simeq \overline{\overline{\mathbb{C}_\zeta^n}} = \mathbb{C}_\zeta^n \sqcup S^{2n-1}\infty$.

The topology of $\hat{T}^*X$ is induced from that of $\mathbb{D}_{\mathbb{C}^n}$. Let V be an open set in $\hat{T}^*X$. We define several sheaves on $\hat{T}^*X$.

Definition 3.9.

1. Let $\tilde{L}_{2,loc}$ be the sheaf of rapidly decreasing locally L^2-functions. Namely, for an open set $V \subset \hat{T}^*X$, a function $f(z,\zeta)$ belongs to $\tilde{L}_{2,loc}(V)$ if and only if for any compact set W in V there exists a constant $h > 0$ such that

$$f(z,\zeta) \cdot e^{h|\zeta|} \in L^2(W \cap T^*X).$$

2. Let $\tilde{L}_{2,loc}^{(p,q)}$ be the sheaf of (p,q)-forms with coefficients in $\tilde{L}_{2,loc}$

3. The sheaf $\tilde{\mathscr{L}}_{2,loc}^{(p,q)}$ is the subsheaf of $\tilde{L}_{2,loc}^{p,q)}$ defined below:

   A (p,q)-form $f \in \tilde{L}_{2,loc}^{(p,q)}(V)$ belongs to $\tilde{\mathscr{L}}_{2,loc}^{(p,q)}(V)$ if and only if $\bar{\partial} f(z,\zeta) \in \tilde{L}_{2,loc}^{(p,q+1)}(V)$.

Lemma 3.10. The resolution of $\mathfrak{N}$

$$0 \to \mathfrak{N} \overset{\bar{\partial}}{\to} \tilde{\mathscr{L}}_{2,loc}^{(0,0)} \overset{\bar{\partial}}{\to} \tilde{\mathscr{L}}_{2,loc}^{(0,1)} \overset{\bar{\partial}}{\to} \cdots \tilde{\mathscr{L}}_{2,loc}^{(0,2n)} \to 0 \tag{3.4}$$

is a soft resolution.

In order to prove Lemma 3.10, we show that the sequence

$$0 \to \tilde{\mathscr{L}}_{2,loc}^{(0,0)}(\widetilde{V}) \to \tilde{\mathscr{L}}_{2,loc}^{(0,1)}(\widetilde{V}) \to \cdots \tilde{\mathscr{L}}_{2,loc}^{(0,2n)}(\widetilde{V}) \to 0 \tag{3.5}$$

is exact for any $\widetilde{V}$ which satisfies the assumptions of Theorem 3.5. Actually we obtain Lemma 3.10 by applying the inductive limit $\varinjlim_{\mathrm{Int}\,(\widetilde{V}) \ni z^*}$ to (3.5).

We recall the Hörmander's existence theorem for the $\bar{\partial}$ operator, which is crucial in the proof of the exactness of (3.5).

Theorem 3.11 ([14], Theorem 4.4.2)**.** Let Ω be a pseudoconvex open set in $\mathbb{C}^n$ and φ any plurisubharmonic function in Ω. For every $g \in L_2^{(p,q)}(\Omega,\varphi)$ with $\bar{\partial} g = 0$ there is a solution $u \in L_{2,loc}^{(p,q)}(\Omega)$ of the equation $\bar{\partial} u = g$ such that

$$\int_\Omega |u|^2 e^{-\varphi}(1+|z|^2)^{-2} d\lambda \leq \int_\Omega |g|^2 e^{-\varphi} d\lambda.$$

Remark 3.12. In Theorem 3.11 we adopt Hörmander's notation. A form $g \in L_2^{(p,q)}(\Omega,\varphi)$ is a (p,q)-form on Ω with coefficients in square integrable functions with respect to the measure $e^{-\varphi} d\lambda$ where $d\lambda$ is the Lebesgue measure.

Now let us prove the exactness of (3.5). Let $\widetilde{V}$ be a closed cone which satisfies the assumptions of Theorem 3.5 and set $\tilde{f}(z,\zeta) \in \tilde{\mathscr{L}}^{(0,q+1)}_{2,loc}(\widetilde{V})$. Then there exist a Stein open cone V_1 with $\widetilde{V} \underset{\text{cone}}{\Subset} V_1$ and $f(z,\zeta) \in \tilde{\mathscr{L}}^{(0,q+1)}_{2,loc}(V_1)$ such that $\tilde{f}(z,\zeta) = f(z,\zeta)$ on $\widetilde{V}$. Fix $\zeta_0 \in \mathbb{C}^n \setminus \{0\}$ whose existence is guaranteed by the third assumption in Theorem 3.5. We can assume $|\zeta_0| = 1$ without loss of generality. By the definition of $\tilde{\mathscr{L}}^{(0,q+1)}_{2,loc}$ there exist an infinitesimal wedge W_1 of type V_1 and a constant $h > 0$ such that

$$f(z,\zeta) \cdot e^{h|\zeta|} \in L_2^{(0,q+1)}(W_1).$$

Let V_2 be a Stein open cone with $\widetilde{V} \underset{\text{cone}}{\Subset} V_2 \underset{\text{cone}}{\Subset} V_1$. From Definition 1.4 there exists a compact set K of V_1 such that

$$V_2 = \{(z;t\zeta) \mid t \in \mathbb{R}_+, (z,\zeta) \in K\}.$$

Then by the definition of an infinitesimal wedge W_1, for such a compact set K there exists a constant $\delta > 0$ such that

$$K_\delta = \{(z;t\zeta) \mid (z;\zeta) \in K,\, t > \delta\} \subset W.$$

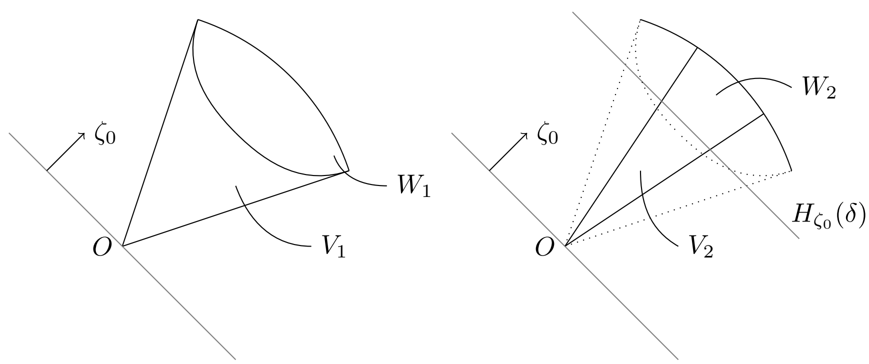

Figure 2: W_1 and W_2

Hence for an open half space

$$H_{\zeta_0}(\delta) = \{(z;\zeta) \in T^*X \mid z \in \pi(V_1),\ \operatorname{Re}\langle \zeta - \delta\zeta_0, \zeta_0\rangle > 0\},$$

the set $W_2 = V_2 \cap H_{\zeta_0}(\delta)$ is a convex set. Then W_2 is relatively compact set of V_1 and is a Stein infinitesimal wedge of type V_2. Thus we can fix a small constant $h' < h$ so that a function $f(z,\zeta) \cdot e^{h'\langle\zeta_0,\zeta\rangle}$ satisfies the assumption of Theorem 3.11. Here we set a plurisubharmonic function φ by

$$\varphi = -\log(d(z,\zeta)),$$

where $d(z,\zeta) = \operatorname{dist}((z;\zeta), \partial W_2)$ is the distance function. Setting $F(z,\zeta) = f(z,\zeta) \cdot e^{h'\langle\zeta_0,\zeta\rangle}$, by Theorem 3.11 we can find $u(z,\zeta) \in L^{(0,q)}_{2,loc}(W_2)$ such that $\bar{\partial}u = F$ and

$$\int_{W_2} (|\zeta|^2+1)^{-2}|u(z,\zeta)|^2 \cdot e^{-\varphi} d\lambda \le \int_{W_2} |F(z,\zeta)|^2 \cdot e^{-\varphi} d\lambda < \infty. \tag{3.6}$$

Hence setting $g(z,\zeta) = u(z,\zeta) \cdot e^{-h'\langle \zeta_0, \zeta\rangle}$ we have

$$\bar{\partial} g = e^{-h'\langle \zeta_0,\zeta\rangle} \cdot \bar{\partial} u(z,\zeta) = e^{-h'\langle \zeta_0,\zeta\rangle} \cdot F(z,\zeta) = f(z,\zeta).$$

In particular such a $g(z,\zeta)$ belongs to $\tilde{\mathscr{L}}^{(0,q)}_{2,loc}(\widetilde{V})$ because of (3.6) and the exactness of (3.5) has been proved.

Finally we can calculate the global cohomology $H^k(\widetilde{V};\mathfrak{N})$ by using the resolution $\Gamma(\widetilde{V};\tilde{\mathscr{L}}^{(0,\bullet)}_{2,loc})$ and get the vanishing $H^k(\widetilde{V};\mathfrak{N}) = 0$ of the higher global cohomology for an arbitrary natural number k.

Corollary 3.13. Let $\widetilde{V}$ be a closed cone satisfying the assumptions in Theorem 3.5. Then an arbitrary element $f(z,\zeta) \in \mathfrak{S}/\mathfrak{N}(\widetilde{V})$ is represented by some symbol $f'(z,\zeta) \in \mathfrak{S}(\widetilde{V})$.

The claim follows immediately from the exactness of the sequence

$$0 \to \mathfrak{N}(\widetilde{V}) \to \mathfrak{S}(\widetilde{V}) \to \mathfrak{S}/\mathfrak{N}(\widetilde{V}) \to 0.$$

3.2 The sheaf $\mathfrak{S}^\infty/\mathfrak{N}^\infty$ of symbols of C^∞-type

Next we introduce symbols of C^∞-type. Let V be an open cone in $\mathring{T}^*X$ and $z^* = (z;\zeta)$ a local coordinate system of T^*X. We construct conic sheaves $\mathfrak{S}^\infty$ and $\mathfrak{N}^\infty$ on $\mathring{T}^*X$.

Definition 3.14. We define the sheaf $C^\infty_z\mathscr{O}_\zeta$ as follows.

A function $f(z,\zeta) \in C^\infty_z\mathscr{O}_\zeta(V) \Leftrightarrow$ A function $f(z,\zeta)$ is a C^∞-function on V and a holomorphic on V in the second variables.

Remark 3.15. The sheaf $C^\infty_z\mathscr{O}_\zeta$ is invariant under the coordinate transformation of X.

Definition 3.16. 1. A function $f(z,\zeta)$ is said to be a null-symbol of C^∞-type on V if it satisfies the following conditions.

$N1$. There exists an infinitesimal wedge W of type V such that

$$f(z,\zeta) \in C^\infty_z\mathscr{O}_\zeta(W).$$

$N2$. For any open cone $V' \underset{\text{cone}}{\Subset} V$ and any multi-indices $\alpha = (\alpha_1,\alpha_2,\dots,\alpha_n) \in \mathbb{Z}^n_{\geq 0}$ and $\beta = (\beta_1,\beta_2,\dots,\beta_n) \in \mathbb{Z}^n_{\geq 0}$, there exist an infinitesimal wedge $W' \subset W$ of type V' and constants $h > 0, C > 0$ such that

$$\left|\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta} f(z,\zeta)\right| \leq C \cdot e^{-h|\zeta|} \quad \text{on } W'.$$

2. A function $f(z,\zeta)$ is said to be a symbol of C^∞-type on V if it satisfies the following conditions.

$S1$. There exists an infinitesimal wedge W of type V such that

$$f(z,\zeta) \in C^\infty_z\mathscr{O}_\zeta(W).$$

$S2$. For any open cone $V' \underset{\text{cone}}{\Subset} V$ and any multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{Z}^n_{\geq 0}$, there exists an infinitesimal wedge $W' \subset W$ of type V' such that $f(z, \zeta)$ satisfies the following condition:
For any $h > 0$ there exists a constant $C > 0$ such that

$$\left| \frac{\partial^\alpha}{\partial z^\alpha} \frac{\partial^\beta}{\partial \bar{z}^\beta} f(z, \zeta) \right| \leq C \cdot e^{h|\zeta|} \quad \text{on } W'.$$

$S3$. The derivative $\dfrac{\partial}{\partial \bar{z}_i} f(z, \zeta)$ is a null-symbol on V for any $i = 1, 2, \ldots, n$.

3. We denote by $\mathfrak{S}^\infty(V)$ and $\mathfrak{N}^\infty(V)$ the set of all the symbols of C^∞-type on V and the set of all the null-symbols of C^∞-type on V, respectively. Moreover we set

$$\mathfrak{S}^\infty z^* = \varinjlim_{V \ni z^*} \mathfrak{S}^\infty(V),$$
$$\mathfrak{N}^\infty_{z^*} = \varinjlim_{V \ni z^*} \mathfrak{N}^\infty(V),$$

where V runs through the family of open conic neighborhoods of z^*.

As we showed in Remark 3.3, we can also extend the sheaf $\mathfrak{S}^\infty$ and $\mathfrak{N}^\infty$ to the sheaves on T^*X. Set $\mathfrak{S}^\infty|_{T^*_X X} = \mathfrak{S}|_{T^*_X X}$ and $\mathfrak{N}^\infty|_{T^*_X X} = 0$. Then the sheaves $\mathfrak{S}^\infty$ and $\mathfrak{N}^\infty$ are well-defined on T^*X.

Proposition 3.17. Let V be an open cone in $\mathring{T}^*X$. Then $\mathfrak{N}^\infty(V)$ is an ideal of $\mathfrak{S}^\infty(V)$.

Proof. Let $f(z, \zeta) \in \mathfrak{N}^\infty(V)$ and $g(z, \zeta) \in \mathfrak{S}^\infty(V)$. Then we can take an infinitesimal wedge W of type V such that $f(z, \zeta)$ and $g(z, \zeta)$ are in $C^\infty_z \mathscr{O}_\zeta(W)$. By the product rule we have

$$\frac{\partial^\alpha}{\partial z^\alpha}(f(z, \zeta) \cdot g(z, \zeta)) = \sum_{0 \leq \alpha_1 \leq \alpha} \frac{\partial^{\alpha_1}}{\partial z^{\alpha_1}} f(z, \zeta) \cdot \frac{\partial^{\alpha - \alpha_1}}{\partial z^{\alpha - \alpha_1}} g(z, \zeta).$$

By the assumption for any $V' \underset{\text{cone}}{\Subset} V$ there exists an infinitesimal wedge $W' \subset W$ of type V' such that $f(z, \zeta)$ satisfies the following condition: For any $h_{\alpha_1} > 0$ there exists a positive constant C_{α_1} such that

$$\left| \frac{\partial^{\alpha_1}}{\partial z^{\alpha_1}} f(z, \zeta) \right| \leq C_{\alpha_1} \cdot e^{h_{\alpha_1}|\zeta|}.$$

Similarly for the same $V' \underset{\text{cone}}{\Subset} V$ and the same $W' \subset W$ there exist constants $h'_{\alpha_1} > 0$ and $C'_{\alpha_1} > 0$ such that

$$\left| \frac{\partial^{\alpha - \alpha_1}}{\partial z^{\alpha - \alpha_1}} g(z, \zeta) \right| \leq C'_{\alpha_1} \cdot e^{-h'_{\alpha_1}|\zeta|}.$$

Thus by taking $h_{\alpha_1} = \dfrac{1}{2} h'_{\alpha_1}$ we obtain

$$\begin{aligned} \left| \frac{\partial^\alpha}{\partial z^\alpha}(f(z, \zeta) \cdot g(z, \zeta)) \right| &\leq \sum_{0 \leq \alpha_1 \leq \alpha} \left| \frac{\partial^{\alpha_1}}{\partial z^{\alpha_1}} f(z, \zeta) \right| \cdot \left| \frac{\partial^{\alpha - \alpha_1}}{\partial z^{\alpha - \alpha_1}} g(z, \zeta) \right| \\ &\leq \sum_{0 \leq \alpha_1 \leq \alpha} C_{\alpha_1} C'_{\alpha_1} \cdot e^{\frac{1}{2} h'_{\alpha_1}|\zeta|} \cdot e^{-h'_{\alpha_1}|\zeta|} \leq mCe^{-\frac{1}{2} h|\zeta|}, \end{aligned}$$

where $h = \min_{0 \le \alpha_1 \le \alpha} \{h'_{\alpha_1}\}$, $C = \max_{0 \le \alpha_1 \le \alpha} \{C_{\alpha_1} C'_{\alpha_1}\}$ and m is the number of α_1 satisfying $0 \le \alpha_1 \le \alpha$. By the similar argument

$$\left| \frac{\partial^\beta}{\partial \bar{z}^\beta} (f(z,\zeta) \cdot g(z,\zeta)) \right| \le C_\beta^{-h_\beta |\zeta|}$$

holds for some $C_\beta > 0$ and $h_\beta > 0$, and these complete the proof. $\square$

One denotes by $\widehat{\mathfrak{S}^\infty / \mathfrak{N}^\infty}$ the presheaf defined by the correspondence

$$V \mapsto \mathfrak{S}^\infty(V)/\mathfrak{N}^\infty(V),$$

where V is an open cone in $\mathring{T}^* X$, and let $\mathfrak{S}^\infty/\mathfrak{N}^\infty$ be an associated sheaf to $\widehat{\mathfrak{S}^\infty/\mathfrak{N}^\infty}$. We have the following exact sequence of sheaves.

$$0 \longrightarrow \mathfrak{N}^\infty \longrightarrow \mathfrak{S}^\infty \xrightarrow{\kappa_2} \mathfrak{S}^\infty/\mathfrak{N}^\infty \longrightarrow 0. \tag{3.7}$$

Here κ_2 is the composition of the canonical morphisms $\mathfrak{S}^\infty \to \widehat{\mathfrak{S}^\infty/\mathfrak{N}^\infty} \to \mathfrak{S}^\infty/\mathfrak{N}^\infty$. As is in the previous subsection, we want the exactness of the sequence on $\widetilde{V}$

$$0 \to \mathfrak{N}^\infty(\widetilde{V}) \to \mathfrak{S}^\infty(\widetilde{V}) \to \mathfrak{S}^\infty/\mathfrak{N}^\infty(\widetilde{V}) \to 0,$$

where $\widetilde{V}$ satisfies the assumptions in Theorem 3.5, and this exactness is guaranteed by the following argument. We have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{N}(\widetilde{V}) \arrow[r] \arrow[d, "\iota_2(\widetilde{V})"] & \mathfrak{S}(\widetilde{V}) \arrow[r, "\kappa_1(\widetilde{V})"] \arrow[d, "\iota_1(\widetilde{V})"] & \mathfrak{S}/\mathfrak{N}(\widetilde{V}) \arrow[r] \arrow[d, "\iota(\widetilde{V})"] & 0 \\
0 \arrow[r] & \mathfrak{N}^\infty(\widetilde{V}) \arrow[r] & \mathfrak{S}^\infty(\widetilde{V}) \arrow[r, "\kappa_2(\widetilde{V})"'] & \mathfrak{S}^\infty/\mathfrak{N}^\infty(\widetilde{V}), &
\end{tikzcd}$$

where $\iota_1(\widetilde{V})$ and $\iota_2(\widetilde{V})$ are canonical inclusions and horizontal sequences are exact. Assuming $\iota(\widetilde{V})$ is an isomorphism, the surjectivity of $\kappa_2(\widetilde{V})$ follows from the fact that the composition $\iota(\widetilde{V}) \circ \kappa_1(\widetilde{V})$ is a surjective, and hence, we have the exact sequence.

It shall be proved in the next subsection that ι is an isomorphism between $\mathfrak{S}/\mathfrak{N}$ and $\mathfrak{S}^\infty/\mathfrak{N}^\infty$.

Corollary 3.18. Let $\widetilde{V}$ be a closed cone satisfying the assumptions in Theorem 3.5. Then an arbitrary element $f(z,\zeta) \in \mathfrak{S}^\infty/\mathfrak{N}^\infty(\widetilde{V})$ is represented by some symbol $f'(z,\zeta) \in \mathfrak{S}^\infty(\widetilde{V})$.

3.3 The equivalence of two symbol classes

The aim of this subsection is to prove the equivalence of $\mathfrak{S}/\mathfrak{N}$ and $\mathfrak{S}^\infty/\mathfrak{N}^\infty$.

By the definitions of classical symbols and symbols of C^∞-type, there exist canonical inclusions

$$\iota_1 : \mathfrak{S} \hookrightarrow \mathfrak{S}^\infty, \quad \iota_2 : \mathfrak{N} \hookrightarrow \mathfrak{N}^\infty,$$

which induce the morphism

$$\iota : \mathfrak{S}/\mathfrak{N} \longrightarrow \mathfrak{S}^\infty/\mathfrak{N}^\infty.$$

Theorem 3.19. The induced morphism

$$\iota : \mathfrak{S}/\mathfrak{N} \longrightarrow \mathfrak{S}^\infty/\mathfrak{N}^\infty$$

is an isomorphism of sheaves.

By the definitions we have $\mathfrak{S}/\mathfrak{N}|_{T^*_X X} = \mathfrak{S}^\infty/\mathfrak{N}^\infty|_{T^*_X X} = \mathscr{D}^\infty_X$. Hence Theorem 3.19 holds obviously on the zero section $T^*_X X$ and it is sufficient to prove Theorem 3.19 on $\mathring{T}^*X$.

Since we have already had the map

$$\iota : \mathfrak{S}/\mathfrak{N} \to \mathfrak{S}^\infty/\mathfrak{N}^\infty,$$

it suffices to show ι_{z^*} to be an isomorphism of stalks at $z^* \in \mathring{T}^*X$.

For this purpose we prepare the following proposition. As the problem is local, we may assume that $T^*X \simeq \mathbb{C}^n_z \times \mathbb{C}^n_\zeta$ until the end of this subsection. In addition we can take $z^* = z^*_0 = (0\,; 1, 0, \ldots, 0)$ without loss of generality. Let $D = D_1(r_1, 0) \times D_2(r_2, 0) \times \cdots \times D_n(r_n, 0)$ be a polydisc in $\mathbb{C}^n_z$ where $D_i(r_i, 0)$ is an open disc in $\mathbb{C}$ whose radius is r_i and the center is at the origin. Set

$$V = D \times \Gamma,$$

where Γ is an open convex cone containing $(1, 0, \ldots, 0) \in \mathbb{C}^n_\zeta$.

We denote by $\mathfrak{N}^{\infty,(p,q)}$ the sheaf of (p,q)-forms on T^*X with respect to the variables z with coefficients in $\mathfrak{N}^\infty$.

Proposition 3.20. Let $V = D \times \Gamma$ be an open set defined above and let $f \in \mathfrak{N}^{\infty,(p,q)}(V)$ satisfy $\bar\partial_z f = 0$. For any polydisc $D' \Subset D$ we can find $u \in \mathfrak{N}^{\infty,(p,q+1)}(V')$ where $V' = D' \times \Gamma$ such that $\bar\partial_z u = f$ on V'.

Proof. By the induction with respect to k, we prove that the lemma is true if f does not contain $d\bar z_{k+1}, \ldots, d\bar z_n$. If $k = 0$, it is obvious that $f = 0$. Assuming that it has been proved when k is replaced by $k-1$, we write

$$\begin{aligned} f &= d\bar z_k \wedge g + h, \\ g &= {\sum_{|I|=p}}' {\sum_{|J|=q}}' g_{IJ}\, dz^I \wedge d\bar z^J. \end{aligned}$$

Here g is a sum of (p,q)-forms on V with coefficients in $C^\infty_z\mathscr{O}_\zeta$ and h is a sum of $(p,q+1)$-forms on V with coefficients in $C^\infty_z\mathscr{O}_\zeta$. Moreover g and h do not contain $d\bar z_k, \ldots, d\bar z_n$ and $\sum'$ means that we sum only over increasing multi-indices. Since $\bar\partial f = 0$, we have

$$\frac{\partial g_{IJ}}{\partial \bar z_j} = 0,$$

for $j > k$ so that g_{IJ} is analytic in these variables.
We want to construct the solution G_{IJ} of the equation

$$\frac{\partial G_{IJ}}{\partial \bar z_k} = g_{IJ}.$$

For this purpose we fix a C^∞-function ψ on $D_k(r_k, 0)$ with compact support such that $\psi(z_k) = 1$ on a neighborhood $D'' \subset D$ of $\overline{D'}$, and set

$$\begin{aligned} G_{IJ} &= (2\pi\sqrt{-1})^{-1} \iint (\tau - z_k)^{-1}\psi(\tau)g_{IJ}(z_1, \dots, z_{k-1}, \tau, z_{k+1}, \dots, z_n)d\tau \wedge d\bar{\tau} \\ &= -(2\pi\sqrt{-1})^{-1} \iint \tau^{-1}\psi(z_k - \tau)g_{IJ}(z_1, \dots, z_{k-1}, z_k - \tau, z_{k+1}, \dots, z_n)d\tau \wedge d\bar{\tau}. \end{aligned}$$

The last integral representation show that G_{IJ} is a $C^\infty_z\mathscr{O}_\zeta$ function. We confirm that G_{IJ} satisfies the condition $N2$. in Definition 3.16.

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{Z}^n_{\geq 0}$ be multi-indices. Hereafter $g_{IJ}(z_1, \dots, z_{k-1}, z_k - \tau, z_{k+1}, \dots, z_n)$ is also denoted by $g_{IJ}(z_k - \tau)$ for short. Then we have

$$\begin{aligned} \left|\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}G_{IJ}\right| =&(2\pi)^{-1}\left|\iint \tau^{-1}\cdot\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}(\psi(z_k - \tau)g_{IJ}(z_k - \tau))d\tau \wedge d\bar{\tau}\right| \\ =&(2\pi)^{-1}\left|\iint \tau^{-1}\cdot\frac{\partial^\alpha}{\partial \tau^\alpha}\frac{\partial^\beta}{\partial \bar{\tau}^\beta}(\psi(z_k - \tau)g_{IJ}(z_k - \tau))d\tau \wedge d\bar{\tau}\right|. \end{aligned}$$

Here we can calculate the integrand as follows.

$$\begin{aligned} &\tau^{-1}\frac{\partial^\alpha}{\partial \tau^\alpha}\frac{\partial^\beta}{\partial \bar{\tau}^\beta}(\psi(z_k - \tau)g_{IJ}(z_k - \tau)) \\ &= \sum_{0\leq\alpha'\leq\alpha}\sum_{0\leq\beta'\leq\beta}\tau^{-1}\frac{\partial^{\alpha'}}{\partial \tau^{\alpha'}}\frac{\partial^{\beta'}}{\partial \bar{\tau}^{\beta'}}\psi(z_k - \tau)\cdot\frac{\partial^{\alpha-\alpha'}}{\partial \tau^{\alpha-\alpha'}}\frac{\partial^{\beta-\beta'}}{\partial \bar{z}^{\beta-\beta'}}g_{IJ}(z_k - \tau). \end{aligned}$$

Since $\psi(z_k - \tau)$ has a compact support and g_{IJ} is of $\mathfrak{N}^\infty$-type,

$$\left|\iint \tau^{-1}\cdot\frac{\partial^{\alpha'}}{\partial \tau^{\alpha'}}\frac{\partial^{\beta'}}{\partial \bar{\tau}^{\beta'}}\psi(z_k - \tau)\cdot\frac{\partial^{\alpha-\alpha'}}{\partial \tau^{\alpha-\alpha'}}\frac{\partial^{\beta-\beta'}}{\partial \bar{z}^{\beta-\beta'}}g_{IJ}(z_k - \tau)d\tau \wedge \bar{\tau}\right| \leq C_{\alpha'\beta'}e^{-h_{\alpha'\beta'}|\zeta|}$$

holds for some $C_{\alpha'\beta'} > 0$ and $h_{\alpha'\beta'} > 0$. As the sets $\{\alpha \in \mathbb{Z}^n_{\geq 0} \mid 0 \leq \alpha' \leq \alpha\}$ and $\{\beta \in \mathbb{Z}^n_{\geq 0} \mid 0 \leq \beta' \leq \beta\}$ are finite we obtain

$$\left|\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}G_{IJ}\right| \leq Ce^{-h|\zeta|}$$

for some $C > 0$ and $h > 0$.

We construct the solution u with the family $\{G_{IJ}\}$ of functions. Set

$$G = {\sum_{I,J}}' G_{IJ}dz^I \wedge d\bar{z}^J.$$

It follows that

$$\bar{\partial}G = {\sum_{I,J}}'{\sum_j}'\frac{\partial G}{\partial \bar{z}_j}d\bar{z}_j \wedge dz^I \wedge d\bar{z}^J = d\bar{z}_k \wedge g + h_1,$$

where h_1 is the sum when j runs from 1 to $k-1$ and does not involve $d\bar{z}_k, \dots, d\bar{z}_n$. Thus by the inductive hypothesis we can find v so that $\bar{\partial}v = f - \bar{\partial}G$ and $u = v + G$ satisfies the equation $\bar{\partial}u = f$. The proof has been completed. □

Now we are ready to prove Theorem 3.19.

Proof. It is sufficient to prove that the stalks of sheaves are isomorphic to each other. One denotes by

$$\iota_{z^*} : \mathfrak{S}_{z^*}/\mathfrak{N}_{z^*} \longrightarrow \mathfrak{S}^{\infty}_{z^*}/\mathfrak{N}^{\infty}_{z^*}$$

the induced morphism from $\iota : \mathfrak{S}/\mathfrak{N} \to \mathfrak{S}^{\infty}/\mathfrak{N}^{\infty}$. The injectivity of ι_{z^*} is obvious. We prove the surjectivity.

Set $F \in \mathfrak{S}^{\infty}_{z^*}$ and let $f \in \mathfrak{S}(V)$ be a representative of F on some open cone $V = D \times \Gamma$. Then f satisfies $\bar{\partial} f \in \mathfrak{N}^{\infty}_{(0,1)}(V)$ and $\bar{\partial}^2 f = 0$. By Proposition 3.20 there exist $D' \Subset D$ and $g \in \mathfrak{N}^{\infty}(V')$ where $V' = D' \times \Gamma$ such that $\bar{\partial} g = \bar{\partial} f$. This implies that $f - g \in \mathfrak{S}(V')$. Set $F' = (f-g)_{z^*}$. Then we have $F' = \iota^{-1}_{z^*}(F)$ and the proof has been completed. □

4 The equivalence of $\mathscr{E}^{\mathbb{R}}_X$ and $\mathfrak{S}^{\infty}/\mathfrak{N}^{\infty}$

In this section X is assumed to be a complex vector space of dimension n. We identify $X \times X$ with TX by the map

$$\varrho : X \times X \ni (z, w) \mapsto (z, w - z) \in TX, \tag{4.1}$$

then we can easily see that the following diagram commutes

$$\begin{tikzcd} X \times X \arrow[rr, "\varrho"] \arrow[dr, "p_1"'] & & TX \arrow[dl, "\tau"] \\ & X. & \end{tikzcd}$$

The aim of this section is to construct the sheaf morphism $\varsigma : \mathscr{E}^{\mathbb{R}}_X \longrightarrow \mathfrak{S}^{\infty}/\mathfrak{N}^{\infty}$ and prove the following theorem.

Theorem 4.1. The sheaf $\mathscr{E}^{\mathbb{R}}_X$ of pseudodifferential operators is isomorphic to the sheaf $\mathfrak{S}/\mathfrak{N}$ of the classical symbol class.

4.1 The map ς from $\mathscr{E}^{\mathbb{R}}_X$ to $\mathfrak{S}^{\infty}/\mathfrak{N}^{\infty}$

Let $\widetilde{V}$ be a closed convex proper cone in $\mathring{T}^*X$, and let V and V' be open convex proper cones with $\widetilde{V} \underset{\text{cone}}{\Subset} V' \underset{\text{cone}}{\Subset} V$. Assume $\pi(\widetilde{V})$ is compact, and $\pi(V')$ and $\pi(V)$ are relatively compact sets. Recall that we have the cohomological expression

$$\mathscr{E}^{\mathbb{R}}_X(V) = \varinjlim_{U,G} H^n_{G\cap U}(U; \mathscr{O}^{(0,n)}_{X\times X})$$

under suitable conditions. If we have already obtained the map

$$\widetilde{\varsigma} : H^n_{G\cap U}(U\,;\,\mathscr{O}^{(0,n)}_{X\times X}) \to \mathfrak{S}^{\infty}/\mathfrak{N}^{\infty}(V'),$$

by taking inductive limits $\varinjlim\limits_{U,G}$ and $\varinjlim\limits_{\widetilde{V} \underset{\text{cone}}{\Subset} V' \underset{\text{cone}}{\Subset} V}$ to $\widetilde{\varsigma}$ in this order, we have

$$\varsigma_{\widetilde{V}} : \mathscr{E}^{\mathbb{R}}_X(\widetilde{V}) \longrightarrow \mathfrak{S}^{\infty}/\mathfrak{N}^{\infty}(\widetilde{V}).$$

Hence our aim is to construct the map $\widetilde{\varsigma}$ concretely.

We recall the γ-topology on TX. Let γ be a closed convex cone in TX.

Definition 4.2. The γ-topology on TX is the topology for which the open sets Ω satisfy:

1. Ω is open for the usual topology.
2. $\Omega \mathring{+} \gamma = \Omega$.

Here $\mathring{+}$ is defined by

$$\Omega \mathring{+} \gamma = \bigsqcup_{z \in \tau(\Omega)} (\Omega_z + \gamma_z),$$

where $\Omega_z = \Omega \cap \tau^{-1}(z)$ and $\gamma_z = \gamma \cap \tau^{-1}(z)$. In particular if $\gamma_z = \emptyset$, set $\Omega_z + \gamma_z = \Omega_z$.

An open set V of TX is called γ-open if V is open in the sense of γ-topology.

Now let us construct the map

$$\widetilde{\varsigma} : H^n_{G \cap U}(U\,; \mathscr{O}^{(0,n)}_{X \times X}) \longrightarrow \mathfrak{S}^\infty / \mathfrak{N}^\infty(V').$$

Let Γ_1 and Γ_2 be open convex proper cones in T^*X such that

1. $V' \underset{\text{cone}}{\Subset} \Gamma_2 \underset{\text{cone}}{\Subset} \Gamma_1 \underset{\text{cone}}{\Subset} V$.
2. $G \cap U \subset \varrho^{-1}(\mathrm{Int}\,(\Gamma_1^\circ)) \cup \Delta$ in $p_1^{-1}(\pi(V'))$.

Remark 4.3. By taking U sufficiently small, we can guarantee the existence of Γ_1 since G is tangent to $\varrho^{-1}(V^\circ)$ near the edge.

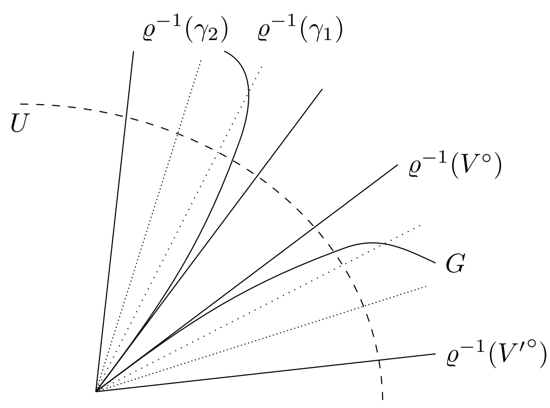

Figure 3: Geometrical relations in $X \times X$

We construct the paths of the integrations in the following way. Set $\gamma_i = \Gamma_i^\circ$ for $i = 1, 2$. Let D_1 and D_2 be open domains in $X \times X$ with C^∞-smooth boundaries such that

D1. $\varrho(D_i)$ is a γ_i-open set for $i = 1, 2$.

D2. $\Delta(\pi(V')) \subset D_1$, where $\Delta : X \to X \times X$ is a diagonal embedding.

D3. $\overline{D}_2 \cap p_1^{-1}(\pi(V')) \subset \mathrm{Int}\,(\gamma_2)$.

D4. $\overline{D} \cap p_1^{-1}(\pi(V')) \subset U$ where $D = D_1 \setminus D_2$.

D5. $\overline{E} \cap p_1^{-1}(\pi(V')) \subset U \setminus G$ where $E = \partial D_1 \setminus D_2$.

D6. ∂D_1 and ∂D_2 intersect transversally in $p_1^{-1}(\pi(V'))$

D7. $p_1^{-1}(z)$ and ∂D_1 (resp. ∂D_2) intersect transversally for any $z \in \pi(V')$.

Example 4.4. Except the conditions $D6$. and $D7$., such D_1 and D_2 can be easily constructed in the following way. Set

$$\widehat{D}_1 = \bigsqcup_{z \in \pi(V')} \{(z, v) \in T_z X \mid \mathrm{dist}_{T_z X}(v, \gamma_1 \cap \tau^{-1}(z)) < \varepsilon_1\},$$
$$\widehat{D}_2 = \bigsqcup_{z \in \pi(V')} \{(z, v) \in T_z X \mid \mathrm{dist}_{T_z X}(v, \tau^{-1}(z) \setminus \gamma_2)) > \varepsilon_2\},$$

for $\varepsilon_1, \varepsilon_2 > 0$. Then $D_1 = \varrho^{-1}(\widehat{D}_1)$ and $D_2 = \varrho^{-1}(\widehat{D}_2)$ satisfy the conditions D1.$\sim$D5. if we take sufficiently small ε_1 and ε_2.

Moreover slightly modifying D_1 and D_2 we can obtain D_1 and D_2 with C^∞-boundaries which satisfy all the above conditions.

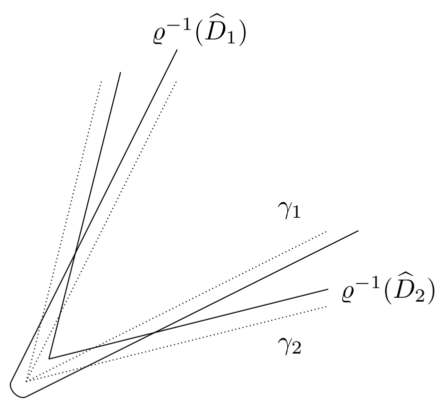

Figure 4: The figure of $\widehat{D}_1$ and $\widehat{D}_2$

Let D_1 and D_2 be open domains in $X \times X$ satisfying the above conditions. Set $D = D_1 \setminus D_2$, $E = \partial D_1 \setminus D_2$, $D_z = D \cap p_1^{-1}(z)$ and $E_z = E \cap p_1^{-1}(z)$.

Definition 4.5. Let u belong to $H^{0,n,n}_{\bar\vartheta}(U, U\backslash G)$ and let $\omega = (\omega_1, \omega_{01})$ be a representative of u. The map

$$\widetilde{\varsigma} : H^n_{G\cap U}(U\,;\,\mathscr{O}^{(0,n)}_{X\times X}) = H^{0,n,n}_{\bar\vartheta}(U, U\setminus G) \to \mathfrak{S}^\infty/\mathfrak{N}^\infty(V') \tag{4.2}$$

is defined by

$$\widetilde{\varsigma}(\omega)(z,\zeta) = \int_{D_z} \omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle} - \int_{E_z} \omega_{01}(z,w)\cdot e^{\langle w-z,\zeta\rangle}. \tag{4.3}$$

In the next paragraph we will show well-definedness of $\widetilde{\varsigma}$. More precisely we shall prove that $\widetilde{\varsigma}$ is independent of the choices of the domains D_1, D_2 and a representative ω of u.

4.2 Well-definedness of the map $\widetilde{\varsigma}$

Well-definedness of $\widetilde{\varsigma}$ follows from the proposition below. Let V and V' be the same as in the previous subsection.

Proposition 4.6. Let $\omega = (\omega_1, \omega_{01})$ be a representative of $u \in H^{0,n,n}_{\bar\vartheta}(U, U\setminus G)$. The map $\widetilde{\varsigma}$ has the following properties.

1. The image $\widetilde{\varsigma}(\omega)$ belongs to $\mathfrak{S}^\infty(V')$.
2. The image $\widetilde{\varsigma}(\omega)$ belongs to $\mathfrak{N}^\infty(V')$ if ω is equal to 0 as an element of the relative Čech-Dolbeault cohomology.
3. The image $\widetilde{\varsigma}(\omega)$ does not depend on the choices of D_1 and D_2.

For the proof of Proposition 4.6 we expect $\widetilde{\varsigma}$ and $\frac{\partial}{\partial z}$ (resp. $\widetilde{\varsigma}$ and $\frac{\partial}{\partial \bar z}$) to be commutative. However $\widetilde{\varsigma}$ and $\frac{\partial}{\partial z}$ (resp. $\widetilde{\varsigma}$ and $\frac{\partial}{\partial \bar z}$) do not commute in general since the paths D_z and E_z of the integrations $\widetilde{\varsigma}$ depend on the variables z. We start with surmounting this difficulty.

For a fixed point $z_0 \in X$ and a constant $\varepsilon > 0$, set

$$B(z_0,\varepsilon) = \{z \in X \mid |z - z_0| < \varepsilon\}.$$

We denote by $\widetilde{D}(z_0,\varepsilon)$ and $\widetilde{E}(z_0,\varepsilon)$ subsets in $X \times X$ defined by

$$\widetilde{D}(z_0,\varepsilon) = B(z_0,\varepsilon) \times p_2(D_{z_0}) \subset X \times X,$$
$$\widetilde{E}(z_0,\varepsilon) = B(z_0,\varepsilon) \times p_2(E_{z_0}) \subset X \times X.$$

We also set

$$\widehat{\varsigma}_{z_0}(\omega) = \int_{\widetilde{D}(z_0,\varepsilon)\cap p_1^{-1}(z_0)} \omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle} - \int_{\widetilde{E}(z_0,\varepsilon)\cap p_1^{-1}(z_0)} \omega_{01}(z,w)\cdot e^{\langle w-z,\zeta\rangle}.$$

Lemma 4.7. Let $z_0 \in \pi(V')$ and let $\omega = (\omega_1, \omega_{01})$ be a representative of $u \in H^{0,n,n}_{\bar\vartheta}(U, U\setminus G)$. Then there exists a constant $\varepsilon > 0$ such that the difference on $\pi^{-1}(B(z_0,\varepsilon))$ of the integrations

$$\widehat{\varsigma}_{z_0}(\omega) = \int_{\widetilde{D}(z_0,\varepsilon)\cap p_1^{-1}(z)} \omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle} - \int_{\widetilde{E}(z_0,\varepsilon)\cap p_1^{-1}(z)} \omega_{01}(z,w)\cdot e^{\langle w-z,\zeta\rangle}$$

and

$$\widetilde{\varsigma}(\omega) = \int_{D_z} \omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle} - \int_{E_z} \omega_{01}(z,w)\cdot e^{\langle w-z,\zeta\rangle}$$

belongs to the null class $\mathfrak{N}^\infty(V' \cap \pi^{-1}(B(z_0,\varepsilon)))$.

Proof. It suffices to prove that $\widehat{\varsigma}_{z_0}(\omega) - \widetilde{\varsigma}(\omega)$ satisfies the condition $N2.$ in Definition 3.16. Since the domains D_1 and D_2 are smooth we can assume the following properties by taking ε sufficiently small.

(i) $(z,z) \in \widetilde{D}(z_0,\varepsilon)$ for $z \in B(z_0,\varepsilon)$.

(ii) $\partial\widetilde{D}(z_0,\varepsilon) \setminus \widetilde{E}(z_0,\varepsilon) \subset \varrho^{-1}(\gamma_2)$.

Here we take domains D' and $\widetilde{D}'(z_0,\varepsilon)$ in $X \times X$ such that

1. $D' \subset D$ and $\widetilde{D}'(z_0,\varepsilon) \subset \widetilde{D}(z_0,\varepsilon)$.
2. $\partial D \setminus E \subset \partial D'$ and $\partial\widetilde{D}(z_0,\varepsilon) \setminus \widetilde{E}(z_0,\varepsilon) \subset \widetilde{D}'(z_0,\varepsilon)$.
3. For any $z \in B(z_0,\varepsilon)$, $D' \cap p_1^{-1}(z) = \widetilde{D}'(z_0,\varepsilon) \cap p_1^{-1}(z)$ on $U \setminus \varrho^{-1}(\gamma_2)$.

Set

$$\begin{aligned} E' &= \partial D' \setminus (\partial D \setminus E), \\ \widetilde{E}'(z_0,\varepsilon) &= \partial\widetilde{D}'(z_0,\varepsilon) \setminus (\partial\widetilde{D}(z_0,\varepsilon) \setminus \widetilde{E}(z_0,\varepsilon)). \end{aligned}$$

By the Stoke's formula we have

$$\begin{aligned} \widehat{\varsigma}_{z_0}(\omega) &= \int_{\widetilde{D}(z_0,\varepsilon)\cap p_1^{-1}(z)} \omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle} - \int_{\widetilde{E}(z_0,\varepsilon)\cap p_1^{-1}(z)} \omega_{01}(z,w)\cdot e^{\langle w-z,\zeta\rangle} \\ &= \int_{\widetilde{D}'(z_0,\varepsilon)\cap p_1^{-1}(z)} \omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle} - \int_{\widetilde{E}'(z_0,\varepsilon)\cap p_1^{-1}(z)} \omega_{01}(z,w)\cdot e^{\langle w-z,\zeta\rangle}, \end{aligned}$$

and

$$\begin{aligned} \widetilde{\varsigma}(\omega) &= \int_{D_z} \omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle} - \int_{E_z} \omega_{01}(z,w)\cdot e^{\langle w-z,\zeta\rangle} \\ &= \int_{D'_z} \omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle} - \int_{E'_z} \omega_{01}(z,w)\cdot e^{\langle w-z,\zeta\rangle}. \end{aligned}$$

Here we write D'_z and E'_z instead of $D' \cap p_1^{-1}(z)$ and $E' \cap p_1^{-1}(z)$ for short, respectively.

Therefore we obtain

$$\begin{aligned}
&\widehat{\varsigma}_{z_0}(\omega) - \widetilde{\varsigma}(\omega) \\
&= \left(\int_{\widetilde{D}(z_0,\varepsilon)\cap p_1^{-1}(z)} \omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle} - \int_{\widetilde{E}(z_0,\varepsilon)\cap p_1^{-1}(z)} \omega_{01}(z,w)\cdot e^{\langle w-z,\zeta\rangle}\right) \\
&\qquad\qquad - \left(\int_{D_z} \omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle} - \int_{E_z} \omega_{01}(z,w)\cdot e^{\langle w-z,\zeta\rangle}\right) \\
&= \left(\int_{\widetilde{D}'(z_0,\varepsilon)\cap p_1^{-1}(z)} \omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle} - \int_{D'_z} \omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle}\right) \\
&\qquad\qquad - \left(\int_{\widetilde{E}'(z_0,\varepsilon)\cap p_1^{-1}(z)} \omega_{01}(z,w)\cdot e^{\langle w-z,\zeta\rangle} - \int_{E'_z} \omega_{01}(z,w)\cdot e^{\langle w-z,\zeta\rangle}\right).
\end{aligned}$$

As $\widetilde{D}'(z_0,\varepsilon)\cap p_1^{-1}(z_0)$ and D'_z (resp. $\widetilde{E}'(z_0,\varepsilon)\cap p_1^{-1}(z_0)$ and E'_z) coincide in the domain $U\setminus\varrho^{-1}(\gamma_2)$ by the definitions of $\widetilde{D}'(z_0,\varepsilon)$ and D' (resp. $\widetilde{E}'(z_0,\varepsilon)$ and E'), we obtain

$$\begin{aligned}
&\int_{\widetilde{D}'(z_0,\varepsilon)\cap p_1^{-1}(z_0)} \omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle} - \int_{D'_z} \omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle} \\
&= \int_{\widetilde{D}'(z_0,\varepsilon)\cap p_1^{-1}(z_0)\cap\varrho^{-1}(\gamma_2)} \omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle} - \int_{D'_z\cap\varrho^{-1}(\gamma_2)} \omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle}.
\end{aligned}$$

We estimate the former integration. Let $\alpha\in\mathbb{Z}^n_{\geq 0}$ and $\beta\in\mathbb{Z}^n_{\geq 0}$ be multi-indices. Since the path of the integration does not depend on z, we have

$$\begin{aligned}
&\left|\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial\bar{z}^\beta}\int_{\widetilde{D}'(z_0,\varepsilon)\cap p_1^{-1}(z_0)\cap\varrho^{-1}(\gamma_2)} \omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle}\right| \\
&= \left|\int_{\widetilde{D}'(z_0,\varepsilon)\cap p_1^{-1}(z_0)\cap\varrho^{-1}(\gamma_2)} \frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial\bar{z}^\beta}(\omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle})\right| \\
&= \left|\int_{\widetilde{D}'(z_0,\varepsilon)\cap p_1^{-1}(z_0)\cap\varrho^{-1}(\gamma_2)} \sum_{0\leq\alpha'\leq\alpha}\frac{\partial^{\alpha'}}{\partial z^{\alpha'}}\frac{\partial^\beta}{\partial z^\beta}\omega_1(z,w)\cdot\frac{\partial^{\alpha-\alpha'}}{\partial z^{\alpha-\alpha'}}e^{\langle w-z,\zeta\rangle}\right| \\
&\leq \sum_{0\leq\alpha'\leq\alpha}\int_{\widetilde{D}'(z_0,\varepsilon)\cap p_1^{-1}(z_0)\cap\varrho^{-1}(\gamma_2)} \left|\frac{\partial^{\alpha'}}{\partial z^{\alpha'}}\frac{\partial^\beta}{\partial z^\beta}\omega_1(z,w)\cdot\frac{\partial^{\alpha-\alpha'}}{\partial z^{\alpha-\alpha'}}e^{\langle w-z,\zeta\rangle}\right| \\
&= \sum_{0\leq\alpha'\leq\alpha}\int_{\widetilde{D}'(z_0,\varepsilon)\cap p_1^{-1}(z_0)\cap\varrho^{-1}(\gamma_2)} \left|\frac{\partial^{\alpha'}}{\partial z^{\alpha'}}\frac{\partial^\beta}{\partial z^\beta}\omega_1(z,w)\cdot(-\zeta)^{\alpha-\alpha'}e^{\langle w-z,\zeta\rangle}\right| \leq Ce^{-h|\zeta|}
\end{aligned}$$

for some $C>0$ and $h>0$.

Next we estimate the latter integration. Let $D'_z\cap\varrho^{-1}(\gamma_2)=\sqcup_{i=1}^N K_i$ be a partition such that each K_i is bounded measurable subset in $p_1^{-1}(z)$. Then we have

$$\left|\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}\int_{D'_z\cap\varrho^{-1}(\gamma_2)}\omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle}\right| = \left|\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}\int_{\sqcup_{i=1}^N K_i}\omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle}\right|$$
$$\leq \sum_{i=1}^N \left|\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}\int_{K_i}\omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle}\right|.$$

Give the local coordinate $(z,w)=(z_1,\dots,z_n,w_1,\dots,w_n)$ for an open neighborhood U_i of K_i and consider the coordinate transformation

$$\Phi_i : (z,w) \longrightarrow (z,\tilde{w}^i)$$

such that $L_i=\Phi_i(K_i)$ is independent of the variables z. Then we have

$$\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}\int_{K_i}\omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle} = \frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}\int_{L_i}\widetilde{\omega}_1(z,\tilde{w}^i)\cdot e^{\langle w-z,\zeta\rangle}\cdot|J_{\Phi_i}|.$$

Here $\widetilde{\omega}_1(z,\tilde{w}^i)=\omega_1(z,w)$ holds under the coordinate transform Φ_i, and J_{Φ_i} is the Jacobian.

Remark 4.8. The existence of the coordinate transformation Φ_i is guaranteed by the condition $D7$.

As the domain K_i is independent of the variables z we obtain

$$\begin{aligned}
&\left|\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}\int_{K_i}\omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle}\right| \\
=&\left|\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}\int_{L_i}\widetilde{\omega}_1(z,\tilde{w}^i)\cdot e^{\langle w-z,\zeta\rangle}\cdot|J_{\Phi_i}|\right| \\
\leq&\int_{L_i}\left|\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}(\widetilde{\omega}_1(z,\tilde{w}^i)\cdot|J_{\Phi_i}|\cdot e^{\langle w-z,\zeta\rangle})\right| \\
\leq&\int_{L_i}\sum_{0\leq\alpha'\leq\alpha}\left|\frac{\partial^{\alpha'}}{\partial z^{\alpha'}}\frac{\partial^\beta}{\partial \bar{z}^\beta}(\widetilde{\omega}_1(z,\tilde{w}^i)\cdot|J_{\Phi_i}|)\cdot\frac{\partial^{\alpha-\alpha'}}{\partial z^{\alpha-\alpha'}}e^{\langle w-z,\zeta\rangle}\right| \\
\leq&\sum_{0\leq\alpha'\leq\alpha}\int_{L_i}\left|\frac{\partial^{\alpha'}}{\partial z^{\alpha'}}\frac{\partial^\beta}{\partial \bar{z}^\beta}(\widetilde{\omega}_1(z,\tilde{w}^i)\cdot|J_{\Phi_i}|)\cdot(-\zeta)^{\alpha-\alpha'}e^{\langle w-z,\zeta\rangle}\right|.
\end{aligned}$$

Since $\widetilde{\omega}_1(z,\tilde{w}^i)\cdot|J_{\Phi_i}|$ is a C^∞-function, the absolute value of its derivative is bounded on L_i. Hence there exist $C>0$ and $h>0$ such that

$$\sum_{i=1}^N\left|\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}\int_{K_i}\omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle}\right| \leq \sum_{i=1}^N Ce^{-h|\zeta|} \leq MCe^{-h|\zeta|}$$

hold. We can apply the same argument to

$$\int_{\widetilde{E}'(z_0,\varepsilon)\cap p_1^{-1}(z)}\omega_{01}(z,w)\cdot e^{\langle w-z,\zeta\rangle} - \int_{E'_z}\omega_{01}(z,w)\cdot e^{\langle w-z,\zeta\rangle}$$

and these complete the proof. □

The corollary below follows immediately from Lemma 4.7.

Corollary 4.9. Let $\omega = (\omega_1, \omega_{01})$ be a representative of $u \in H^{0,n,n}_{\bar{\vartheta}}(U, U \setminus G)$. For any $i = 1, \dots, n$ the symbols

$$\frac{\partial}{\partial z_i}\widetilde{\varsigma}(\omega_1, \omega_{01}), \quad \frac{\partial}{\partial \bar{z}_i}\widetilde{\varsigma}(\omega_1, \omega_{01}) \tag{4.4}$$

are equal to

$$\widetilde{\varsigma}\left(\frac{\partial}{\partial z_i}\omega_1, \frac{\partial}{\partial z_i}\omega_{01}\right), \quad \widetilde{\varsigma}\left(\frac{\partial}{\partial \bar{z}_i}\omega_1, \frac{\partial}{\partial \bar{z}_i}\omega_{01}\right) \tag{4.5}$$

in $\mathfrak{S}^\infty/\mathfrak{N}^\infty(V')$, respectively.

Proof. Due to Lemma 4.7 we can identify

$$\widetilde{\varsigma}(\omega_1, \omega_{01})(z, \zeta) = \int_{D_z} \omega_1(z, w) \cdot e^{\langle w - z, \zeta\rangle} - \int_{E_z} \omega_{01}(z, w) \cdot e^{\langle w - z, \zeta\rangle}$$

with

$$\int_{D_{z_0}} \omega_1(z, w) \cdot e^{\langle w - z, \zeta\rangle} - \int_{E_{z_0}} \omega_{01}(z, w) \cdot e^{\langle w - z, \zeta\rangle}$$

as in the symbol class $\mathfrak{S}^\infty/\mathfrak{N}^\infty(V')$. On the other hands, since the domains D_{z_0} and E_{z_0} do not depend on z we have

$$\begin{aligned}
&\frac{\partial}{\partial z}\left(\int_{D_{z_0}} \omega_1(z, w) \cdot e^{\langle w - z, \zeta\rangle} - \int_{E_{z_0}} \omega_{01}(z, w) \cdot e^{\langle w - z, \zeta\rangle}\right) \\
=&\int_{D_{z_0}} \frac{\partial}{\partial z}\omega_1(z, w) \cdot e^{\langle w - z, \zeta\rangle} - \int_{E_{z_0}} \frac{\partial}{\partial z}\omega_{01}(z, w) \cdot e^{\langle w - z, \zeta\rangle} \\
=&\widetilde{\varsigma}\left(\frac{\partial}{\partial z}\omega_1, \frac{\partial}{\partial z}\omega_{01}\right).
\end{aligned}$$

By the same argument the second claim holds and these complete the proof. $\square$

Now we start the proof of Proposition 4.6. In the following proof the Dolbeault operator $\bar{\partial}_z + \bar{\partial}_w$ is denoted by $\bar{\partial}$ without notice.

Proof. 1. We can divide D_z into two subsets $D_{z,1}(\varepsilon)$ and $D_{z,2}(\varepsilon)$ in $p_1^{-1}(z)$ for sufficiently small $\varepsilon > 0$ which have piecewise smooth boundaries such that

(a) $G \cap D_{z,1}(\varepsilon) = \emptyset$.

(b) $\partial E_z(\varepsilon) = \partial(\partial D_z \setminus E_z)$.

(c) $D_{z,2}(\varepsilon) \subset (\varrho^{-1}(\gamma_2) \cup B((z, z), \varepsilon)) \cap p_1^{-1}(z)$ where $B((z, z), \varepsilon)$ is an open ball in $X \times X$ with radius ε whose center is at (z, z).

Here we set $E_z(\varepsilon) = \partial D_{z,1}(\varepsilon) \cap \partial D_{z,2}(\varepsilon)$.

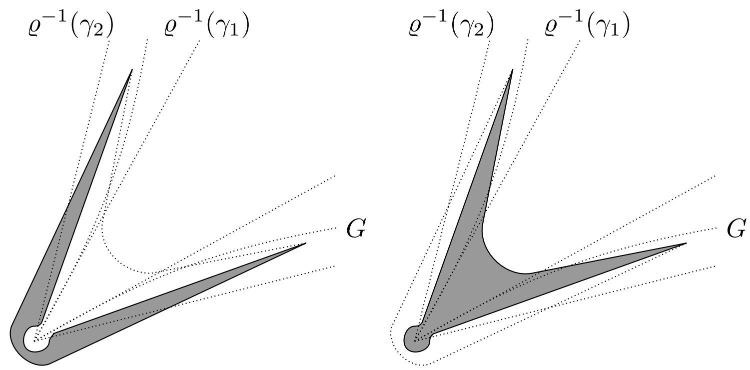

Figure 5: The subsets $D_{1,\varepsilon}$ and $D_{2,\varepsilon}$

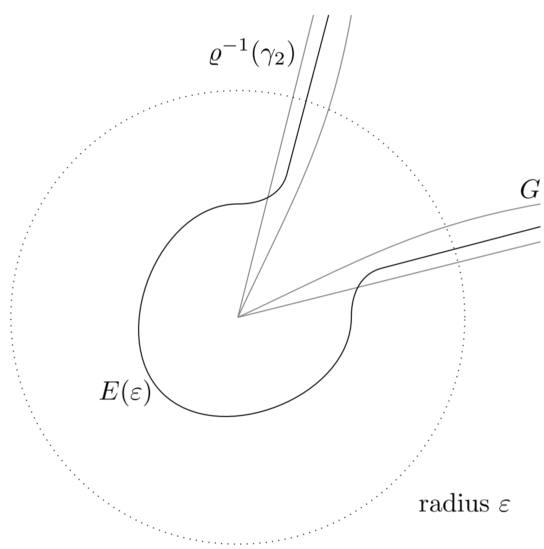

Figure 6: E_ε near the vertex

By the Stoke's formula we have

$$\begin{aligned}\int_{D_{z,1}(\varepsilon)} \omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle} &= \int_{D_{z,1}(\varepsilon)} \bar{\partial}\omega_{01}(z,w)\cdot e^{\langle w-z,\zeta\rangle} \\ &= \int_{E_z+E(\varepsilon)} \omega_{01}(z,w)\cdot e^{\langle w-z,\zeta\rangle}.\end{aligned}$$

For any $h>0$, by retaking ε small enough to satisfy $\operatorname{Re}\langle w-z,\zeta\rangle \le h|\zeta|$ in

$B((z,z),\varepsilon)\cap p_1^{-1}(z)$, we can see that

$$
\begin{aligned}
&\left|\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}\widetilde{\varsigma}(\omega)\right| \\
=&\left|\int_{D_{z,2}(\varepsilon)}\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}\omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle}+\int_{E(\varepsilon)}\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}\omega_{01}(z,w)\cdot e^{\langle w-z,\zeta\rangle}\right| \\
\le&\int_{D_{z,2}(\varepsilon)}\left|\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}\omega_1(z,w)\right|\cdot e^{\mathrm{Re}\,\langle w-z,\zeta\rangle}+\int_{E(\varepsilon)}\left|\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta}\omega_{01}(z,w)\right|\cdot e^{\mathrm{Re}\langle w-z,\zeta\rangle} \\
\le& C\cdot e^{h|\zeta|}.
\end{aligned}
$$

Next we check that $\dfrac{\partial}{\partial \bar{z}_i}\widetilde{\varsigma}(\omega)$ belongs to $\mathfrak{N}^\infty(V')$ for any $i=1,2,\ldots,n$. By the Stoke's formula and the facts that $(\bar{\partial}_z+\bar{\partial}_w)\omega_{01}=\omega_1$ and $\bar{\partial}_z\omega_1=-\bar{\partial}_w\omega_1$, we obtain

$$
\begin{aligned}
\left|\bar{\partial}_z\widetilde{\varsigma}(\omega)\right|=&\left|\int_{D_{z,2}(\varepsilon)}\bar{\partial}_z\omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle}+\int_{E(\varepsilon)}\bar{\partial}_z\omega_{01}(z,w)\cdot e^{\langle w-z,\zeta\rangle}\right| \\
=&\left|-\int_{D_{z,2}(\varepsilon)}\bar{\partial}_w\omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle}\right. \\
&\qquad\left.+\int_{E(\varepsilon)}(\omega_1(z,w)-\bar{\partial}_w\omega_{01}(z,w))\cdot e^{\langle w-z,\zeta\rangle}\right| \\
=&\left|\int_{\partial D_{z,2}(\varepsilon)}\omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle}\right. \\
&\qquad\left.-\int_{E(\varepsilon)}\omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle}+\int_{\partial E(\varepsilon)}\omega_{01}(z,w)\cdot e^{\langle v,\zeta\rangle}\right| \\
=&\left|\int_{\partial D_{z,2}(\varepsilon)\setminus E(\varepsilon)}\omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle}+\int_{\partial E(\varepsilon)}\omega_{01}(z,w)\cdot e^{\langle w-z,\zeta\rangle}\right|.
\end{aligned}
$$

By the construction of the partitions $D_{z,1}$ and $D_{z,2}$ we obtain $\partial D_{z,2}(\varepsilon)\setminus E(\varepsilon)\subset \mathrm{Int}\,(\varrho^{-1}(\gamma_2))$ and $\partial E(\varepsilon)\subset \mathrm{Int}\,(\varrho^{-1}(\gamma_2))$, and hence there exist positive constants h and C such that

$$
\begin{aligned}
&\left|\int_{\partial D_{z,2}(\varepsilon)\setminus E(\varepsilon)}\omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle}+\int_{\partial E(\varepsilon)}\omega_{01}(z,w)\cdot e^{\langle w-z,\zeta\rangle}\right| \\
\le&\int_{\partial D_{z,2}(\varepsilon)\setminus E(\varepsilon)}|\omega_1(z,w)|\cdot e^{-h|\zeta|}+\int_{\partial E(\varepsilon)}|\omega_{01}(z,w)|\cdot e^{-h|\zeta|} \\
\le& C\cdot e^{-h|\zeta|},
\end{aligned}
$$

and we obtain the claim.

2. In addition to the assumption in the above proof we assume that ω is equal to 0 as an element in the relative Čech-Dolbeault cohomology. Then there exists

$\tau = (\tau_1, \tau_{01}) \in C^{\infty,(0,n,n-1)}_{X\times X}(\mathcal{V}, \mathcal{V}')$ with $\bar{\vartheta}\tau = \omega$. By substituting $(\omega_1, \omega_{01}) = (\bar{\partial}\tau_1, \tau_1 - \bar{\partial}\tau_{01})$ we have

$$\begin{aligned}\widetilde{\varsigma}(\omega) &= \int_{D_{z,2}(\varepsilon)} \omega_1(z,w)\cdot e^{\langle w-z,\zeta\rangle} + \int_{E(\varepsilon)} \omega_{01}(z,w)\cdot e^{\langle w-z,\zeta\rangle} \\ &= \int_{D_{z,2}(\varepsilon)} \bar{\partial}\tau_1(z,w)\cdot e^{\langle w-z,\zeta\rangle} + \int_{E(\varepsilon)} (\tau_1(z,w) - \bar{\partial}\tau_{01}(z,w))\cdot e^{\langle w-z,\zeta\rangle}.\end{aligned}$$

Since the integrations

$$\int_{D_{z,2}(\varepsilon)} \bar{\partial}_z\tau_1(z,w)\cdot e^{\langle w-z,\zeta\rangle}, \quad \int_{E(\varepsilon)} \bar{\partial}_z\tau_{01}(z,w)\cdot e^{\langle w-z,\zeta\rangle}$$

vanish, we have

$$\begin{aligned}&\int_{D_{z,2}(\varepsilon)} \bar{\partial}\tau_1(z,w)\cdot e^{\langle w-z,\zeta\rangle} + \int_{E(\varepsilon)} (\tau_1(z,w) - \bar{\partial}\tau_{01}(z,w))\cdot e^{\langle w-z,\zeta\rangle} \\ &= \int_{\partial D_{z,2}(\varepsilon)\setminus E(\varepsilon)} \tau_1(z,w)\cdot e^{\langle w-z,\zeta\rangle} + \int_{\partial E(\varepsilon)} \tau_{01}(z,w)\cdot e^{\langle w-z,\zeta\rangle}.\end{aligned}$$

By the same argument as in the proof of 1, we can find constants $h > 0$ and $C > 0$ such that

$$\begin{aligned}&\left|\int_{\partial D_{z,2}(\varepsilon)\setminus E(\varepsilon)} \tau_1(z,w)\cdot e^{\langle w-z,\zeta\rangle} + \int_{\partial E(\varepsilon)} \tau_{01}(z,w)\cdot e^{\langle w-z,\zeta\rangle}\right| \\ &\le \int_{\partial D_{z,2}(\varepsilon)\setminus E(\varepsilon)} |\tau_1(z,w)|\cdot e^{-h|\zeta|} + \int_{\partial E(\varepsilon)} |\tau_{01}(z,w)|\cdot e^{-h|\zeta|} \le C\cdot e^{-h|\zeta|}.\end{aligned}$$

3. The claim follows immediately from the argument in the above proofs.

□

The corollary below follows from Proposition 4.6.

Corollary 4.10. The map $\widetilde{\varsigma}$ is well-defined.

In the subsequent section we prove the main theorem.

4.3 The proof of the main Theorem 4.1

First we briefly recall the classical symbol theory introduced by Aoki [2].

Let $U_c, Z_{c,\varepsilon}$ and $V^{(\nu)}$ subsets in $X \times X$ defined in [2], Page 184. Then we have the following cohomological expression

$$\mathscr{E}_{X,z^*} = \varinjlim_{c,\varepsilon} H^n_{Z_{c,\varepsilon}}(U_c\,;\,\mathscr{O}^{(0,n)}_{X\times X}).$$

Here we take $z^* = (0; \lambda, 0, \ldots, 0)$ with $\lambda \in \mathbb{C}^\times$. Moreover we set

$$V = \bigcap_{\nu=1}^{n} V^{(\nu)}, \quad \hat{V}^{(\nu)} = \bigcap_{i\neq\nu} V^{(i)}.$$

Then we have the following exact sequence

$$\bigoplus_{\nu=1}^{n}\Gamma(\hat{V}^{(\nu)}\,;\,\mathscr{O}^{(0,n)}_{X\times X})\longrightarrow\Gamma(V\,;\,\mathscr{O}^{(0,n)}_{X\times X})\longrightarrow H^n_{Z_{c,\varepsilon}}(U_c\,;\,\mathscr{O}^{(0,n)}_{X\times X})\longrightarrow 0.$$

Hence any pseudodifferential operator $P\in\mathscr{E}^{\mathbb{R}}_{X,z^*}$ can be represented as an equivalence of a holomorphic form $\psi(z,z')dz'\in\Gamma(V\,;\,\mathscr{O}^{(0,n)}_{X\times X})$. By the aid of Radon transformations Aoki defined the symbol mapping σ at point z^* by

$$\sigma:\mathscr{E}^{\mathbb{R}}_{X,z^*}\simeq\varinjlim_{c,\varepsilon}H^n(\mathcal{W},\mathcal{W}'\,;\,\mathscr{O}^{(0,n)}_{X\times X})\longrightarrow(\mathfrak{S}/\mathfrak{N})_{z^*}.$$

The following theorem established by Aoki is crucial.

Theorem 4.11 ([2], Theorem 4.3 and Theorem 4.5)**.** The symbol mapping σ is an isomorphism of stalks.

For the proof of this theorem Aoki constructed the inverse map

$$\varpi:(\mathfrak{S}/\mathfrak{N})_{z^*}\longrightarrow\mathscr{E}^{\mathbb{R}}_{X,z^*},$$

and had confirmed that $\sigma\circ\varpi=id$ and $\varpi\circ\sigma=id$.

Now let us prove Theorem 4.1. As a consequence of Subsections 4.1 and 4.2, we have a well-defined morphism

$$\varsigma_{\widetilde{V}}:\mathscr{E}^{\mathbb{R}}_X(\widetilde{V})\longrightarrow\mathfrak{S}^\infty/\mathfrak{N}^\infty(\widetilde{V})$$

for any closed convex proper cone $\widetilde{V}$ in $\mathring{T}^*X$ with $\pi(\widetilde{V})$ being compact.

Let $\widetilde{V}'$ be a closed convex proper cone contained in $\widetilde{V}$. Then, it follows from Proposition 4.6 that we have the following commutative diagram:

$$\begin{tikzcd}
\mathscr{E}^{\mathbb{R}}_X(\widetilde{V}) \arrow[r, "\varsigma_{\widetilde{V}}"] \arrow[d] & \mathfrak{S}^\infty/\mathfrak{N}^\infty(\widetilde{V}) \arrow[d] \\
\mathscr{E}^{\mathbb{R}}_X(\widetilde{V}') \arrow[r, "\varsigma_{\widetilde{V}'}"] & \mathfrak{S}^\infty/\mathfrak{N}^\infty(\widetilde{V}').
\end{tikzcd}$$

Since the family of closed convex proper cones in $\mathring{T}^*X$ is a basis of sets on which a conic sheaf can be defined, the family $\{\varsigma_{\widetilde{V}}\}_{\widetilde{V}}$ of morphisms gives a sheaf morphism on $\mathring{T}^*X$

$$\varsigma:\mathscr{E}^{\mathbb{R}}_X\longrightarrow\mathfrak{S}^\infty/\mathfrak{N}^\infty.$$

Thus we have obtained a sheaf morphism from $\mathscr{E}^{\mathbb{R}}_X$ to the symbol space of C^∞-type. The rest of the problem is to show ς to be an isomorphism, and it suffices to show the morphism $\varsigma_{z^*}:\mathscr{E}^{\mathbb{R}}_{X,z^*}\to(\mathfrak{S}^\infty/\mathfrak{N}^\infty)_{z^*}$ of stalks is isomorphic. It is easy to see that the diagram

$$\begin{tikzcd}
 & (\mathfrak{S}/\mathfrak{N})_{z^*} \arrow[dd, "\iota_{z^*}"] \\
\mathscr{E}^{\mathbb{R}}_{X,z^*} \arrow[ur, "\sigma"] \arrow[dr, "\varsigma_{z^*}"'] & \\
 & (\mathfrak{S}^\infty/\mathfrak{N}^\infty)_{z^*}
\end{tikzcd}$$

commutes at each $z^* \in \mathring{T}^*X$, where σ is a symbol map described above. The vertical arrow in the diagram is isomorphic by Theorem 3.19 and σ is also isomorphic by Theorem 4.11. Hence ς_{z^*} is also isomorphic, which completes the proof of Theorem 4.1.

Acknowledgment

First of all I would like to express my gratitude to my supervisor, Professor Naofumi Honda. Under his support and guidance I have been studying mathematics for six years. Without his advice and suggestions, I would not have achieved the work. In addition, I learned not only how to study mathematics but also the ideal way of a researcher. I thank him from the bottom of my heart.

Moreover I would like to thank all the staffs in Department of Mathematics, Hokkaido University. I could concentrate on my research thanks to their kind and sincere help.

Finally I would like to heartily thank my parents. They had understood me and helped me a lot in my daily life. I am deeply grateful to them.

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