THE HYDROSTATIC APPROXIMATION FOR THE PRIMITIVE EQUATIONS BY THE SCALED NAVIER-STOKES EQUATIONS UNDER THE NO-SLIP BOUNDARY CONDITION

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Dedicated to Professor Matthias Hieber on the occasion of his 60th birthday

Abstract. In this paper we justify the hydrostatic approximation of the primitive equations in the maximal L^p-L^q-setting in the three-dimensional layer domain $\Omega = \mathbb{T}^2 \times (-1,1)$ under the no-slip (Dirichlet) boundary condition in any time interval $(0,T)$ for $T>0$. We show that the solution to the scaled Navier-Stokes equations with Besov initial data $u_0 \in B^s_{q,p}(\Omega)$ for $s > 2 - 2/p + 1/q$ converges to the solution to the primitive equations with the same initial data in $\mathbb{E}_1(T) = W^{1,p}(0,T; L^q(\Omega)) \cap L^p(0,T; W^{2,q}(\Omega))$ with order $O(\epsilon)$ where $(p,q) \in (1,\infty)^2$ satisfies $\frac{1}{p} \le \min(1-1/q, 3/2-2/q)$. The global well-posedness of the scaled Navier-Stokes equations in $\mathbb{E}_1(T)$ is also proved for sufficiently small $\epsilon > 0$. Note that $T = \infty$ is included.

1. Introduction

We consider the primitive equations of the form

$$(PE) \begin{cases} \partial_t u - \Delta u + u \cdot \nabla v + \nabla_H \pi &= 0 \quad \text{in } \Omega \times (0,\infty), \\ \partial_z \pi &= 0 \quad \text{in } \Omega \times (0,\infty), \\ \operatorname{div} u &= 0 \quad \text{in } \Omega \times (0,\infty), \\ u &= 0 \quad \text{on } \partial\Omega \times (0,\infty), \end{cases}$$

where $u = (v,w) \in \mathbb{R}^2 \times \mathbb{R}$ and π are the unknown velocity field and pressure field, respectively, $\nabla_H = (\partial_x, \partial_y)^T$, and $\Omega = \mathbb{T}^2 \times (-1,1)$ for $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. By divergence-free condition w is given by the formula

$$w(x', x_3, t) = -\int_{-1}^{x_3} \operatorname{div}_H v(x', \zeta, t) d\zeta d\zeta = \int_{x_3}^{1} \operatorname{div}_H v(x', \zeta, t) d\zeta;$$

here we invoked physically reasonable condition $w(\cdot,\cdot,\pm 1,\cdot) = 0$. The primitive equations are fundamental model for geographic flow. Existence of the global weak solution to the primitive equations on the sphere with L^2-initial data was proved by Lions, Temam and Wang [26]. Local-in-time well-posedness was proved by Guillén-González, Masmoudi and Rodríguez-Bellido [18]. Although global well-posedness of the 3-dimensional Navier-Stokes equations are the well-known open problem, for the primitive equations, this problem has been solved by Cao and Titi [3]. Hieber and Kashiwabara [20] extended this result to prove global well-posedness for the primitive equations in L^p-settings. In these papers boundary conditions are imposed no-slip (Dirichlet) on the bottom and slip (Neumann) on the top. Recently, the second and last authors together with Gries, Hieber and Hussein [13] obtained global-in-time well-posedness in the maximal regularity spaces (mixed Lebesgue-Sobolev spaces) $W^{1,p}(0,T; L^q(\Omega)) \cap L^p(0,T; W^{2,q}(\Omega))$ for $T > 0$ and appropriate $1 < p, q < \infty$ under various boundary conditions.

Our aim in this paper is to give a rigorous justification of the derivation of the primitive equations under the Dirichlet boundary condition. We begin by explaining its derivation. Let us consider the anisotropic viscous Navier-Stokes equations in a thin domain of the form

$$(ANS) \begin{cases} \partial_t u - (\Delta_H + \epsilon^2 \partial_z^2) u + u \cdot \nabla u + \nabla \pi &= 0 \quad \text{in } \Omega_\epsilon \times (0,\infty), \\ \operatorname{div} u &= 0 \quad \text{in } \Omega_\epsilon \times (0,\infty), \end{cases}$$

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where $\Omega_\epsilon = (-\epsilon, \epsilon) \times \mathbb{T}^2$. If $\epsilon = 1$, (ANS) is the usual Navier-Stokes equations. The equations (ANS) are considered as a good model to describe motion of incompressible viscous fluid filled in a thin domain. Actually, if we put the Reynolds number 1, since length and velocity are of ϵ-order, apparent viscosity for vertical direction must be of ϵ^2-order from the Reynolds number point of view. The primitive equations are formally derived from (ANS). We introduce new unknowns of (ANS) by rescaling as

- $u_\epsilon := (v_\epsilon, w_\epsilon)$
- $v_\epsilon(x, y, z, t) := v(x, y, \epsilon z, t)$
- $w_\epsilon(x, y, z, t) := w(x, y, \epsilon z, t)/\epsilon$
- $\pi_\epsilon(x, y, z, t) := \pi(x, y, \epsilon z, t)$,

where $x, y \in \mathbb{T}$, $z \in (-1, 1)$ and $t > 0$. Then, $(u_\epsilon, \pi_\epsilon)$ satisfy the scaled Navier-Stokes equations in a fixed domain

$$(SNS) \begin{cases} \partial_t v_\epsilon - \Delta v_\epsilon + u_\epsilon \cdot \nabla v_\epsilon + \nabla_H \pi_\epsilon &= 0 \quad \text{in} \quad \Omega \times (0, \infty), \\ \epsilon^2 \left(\partial_t w_\epsilon - \Delta w_\epsilon + u_\epsilon \cdot \nabla w_\epsilon\right) + \partial_z \pi_\epsilon &= 0 \quad \text{in} \quad \Omega \times (0, \infty), \\ \operatorname{div} u &= 0 \quad \text{in} \quad \Omega \times (0, \infty). \end{cases}$$

Taking formally $\epsilon \to 0$ for the above equations, we get the primitive equations.

The Navier-Stokes equations (SNS) are well-studied for $\epsilon = 1$ since the work of Leray [24], where a global weak solution is constructed in $\Omega = \mathbb{R}^3$. For a general domain see Farwig, Kozono, Sohr [7]. A local strong solution is constructed by Fujita and Kato [9] when initial data is in $H^{1/2}$. It is extended to various domains in various function spaces; see e.g. Ladyzenskaya [22], Kato [21], Giga and Miyakawa [15] for early development. The reader refers to a book of Lemarié-Rieusset [23] and review articles by Farwig, Kozono and Sohr [8] and Gallagher [11] for resent development. Many results can be extended for general $\epsilon > 0$ but it is not often written explicitly except in a book of Chemin, Desjardins, Gallagher and Grenier [4].

Rigorous justification of the primitive equations from the scaled Navier-Stokes equations was studied by Azérad and Guillén [2]. They obtained weak* convergence in the natural energy space $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ for $\Omega = \mathbb{T}^2 \times (-1, 1)$ and $T > 0$. Recently, Li and Titi [25] improved their result to get strong convergence by energy method with the aid of regularity of the solution to the primitive equations. The authors together with Hieber, Hussein and Wrona [10] extended Li and Titi's result in maximal-regularity spaces $W^{1,p}(0, T; L^q(\mathbb{T}^3)) \cap L^p(0, T; W^{2,q}(\mathbb{T}^3))$ with initial trace in the Besov space $B^{2-2/p}_{q,p}$ for $T > 0$ and $1/p \leq \min(1 - 1/q, 3/2 - 2/q)$ by an operator theoretic approach. The case of $p = q = 2$ is corresponding to Li and Titi's result. Note the case of the torus corresponding to Neumann boundary conditions on top and bottom part, moreover, the work of Azérad and Guillén treats mixed boundary conditions with Dirichlet boundary conditions on the bottom, while Li and Titi deal with Neumann boundary conditions only. As we already mentioned, the primitive equations are a model for geographic flow. Although it is more physically natural to consider the case of Dirichlet-Neumann and Dirichlet boundary conditions, there was no result of justification of derivation in a strong topology to the primitive equations from the Navier-Stokes equations.

Let

$$\begin{aligned} &\mathbb{E}_1(T) = \{u \in W^{1,p}(0, T; L^q(\Omega)) \cap L^p(0, T; W^{2,q}(\Omega))\, ;\, \operatorname{div} u = 0,\ u|_{x=\pm 1} = 0\}, \\ &\mathbb{E}_0(T) = \{u \in L^p(0, T; L^q(\Omega))\, ;\, \operatorname{div} u = 0,\ u|_{x=\pm 1} = 0\}, \\ &\mathbb{E}_1^\pi(T) = \{\pi \in L^p(0, T; W^{1,q}(\Omega))\, ;\, \int_\Omega \pi \, dx = 0\}, \end{aligned}$$

and

$$X_\gamma = \{u \in B^{2-2/p}_{q,p}(\Omega)\, ;\, \operatorname{div} u = 0,\ u|_{x=\pm 1} = 0\}$$

be the initial trace space of $\mathbb{E}_1(T)$, where $B^s_{q,p}$ denotes the L^q-Besov space of order s. In this paper, we frequently use $\|\cdot\|_{\mathbb{E}_0(T)}$ as the norm of $L^p(0, T; L^q(\Omega))$ and $\|\cdot\|_{\mathbb{E}_1(T)}$ as the norm of $W^{1,p}(0, T; L^q(\Omega)) \cap L^p(0, T; W^{2,q}(\Omega))$ to simplify the notation. Let us seek the solution $U_\epsilon = (V_\epsilon, W_\epsilon)$ to

$$\begin{cases} \partial_t V_\epsilon - \Delta V_\epsilon + \nabla_H P_\epsilon &= F_H \quad \text{in} \quad \Omega \times (0, T), \\ \partial_t(\epsilon W_\epsilon) - \Delta(\epsilon W_\epsilon) + \frac{\partial_z}{\epsilon} P_\epsilon &= \epsilon F_z + \epsilon F \quad \text{in} \quad \Omega \times (0, T), \\ \operatorname{div} U_\epsilon &= 0 \quad \text{in} \quad \Omega \times (0, T), \\ U_\epsilon &= 0 \quad \text{on} \quad \partial\Omega \times (0, T), \\ U_\epsilon(0) &= 0 \quad \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where

- $F_H = -\left(U_\epsilon \cdot \nabla V_\epsilon + u \cdot \nabla V_\epsilon + U_\epsilon \cdot \nabla v\right)$

- $F_z = -\left(U_\epsilon \cdot \nabla W_\epsilon + u \cdot \nabla W_\epsilon + U_\epsilon \cdot \nabla w\right)$
- $F = -\left(\partial_t w - \Delta w + u \cdot \nabla w\right)$.

The system (1.1) is the equation of the difference between the solution to the (PE) and (SNS).

Theorem 1.1. *Let $T > 0$ and $0 < \epsilon \le 1$. Suppose $(p,q) \in (1,\infty)^2$ satisfies $\frac{1}{p} \le \min(1-1/q, 3/2-2/q)$, $u_0 = (v_0, w_0) \in X_\gamma$ and $v_0 \in B^s_{q,p}(\Omega)$ for $s > 2 - 2/p + 1/q$. Let $u \in \mathbb{E}_1(T)$ be a solution of (PE) with initial data $u_0 \in X_\gamma$. Then there exists constant $C = C(p, q, \|u\|_{\mathbb{E}_1(T)})$ and a unique solution $U_\epsilon = (V_\epsilon, W_\epsilon)$ to (1.1) such that*

$$\| \left(V_\epsilon, \epsilon W_\epsilon\right) \|_{\mathbb{E}_1(T)} \le \epsilon C. \tag{1.2}$$

Moreover, $u_\epsilon = (v_\epsilon, w_\epsilon) := (v + V_\epsilon, w + W_\epsilon)$ is the unique solution to (SNS) in $\mathbb{E}_1(T)$.

This theorem implies the justification of the hydrostatic approximation.

Corollary 1.2. *Let $T > 0$ and $0 < \epsilon \le 1$. Suppose $(p,q) \in (1,\infty)^2$ satisfies $\frac{1}{p} \le \min(1-1/q, 3/2-2/q)$, $u_0 = (v_0, w_0) \in X_\gamma$ and $v_0 \in B^s_{q,p}(\Omega)$ for $s > 2 - 2/p + 1/q$. Let u and u_ϵ be a solution of (PE) and (SNS) in $\mathbb{E}_1(T)$ under the Dirichlet boundary condition with initial data u_0, respectively, such that*

$$||u||_{\mathbb{E}_1(T)} + ||(v_\epsilon, \epsilon w_\epsilon)||_{\mathbb{E}_1(T)} \le C_0 \tag{1.3}$$

for some $C_0 = C_0(u_0, p, q)$. Then there exists a positive $C = C(p, q, C_0)$ such that

$$||(v_\epsilon - v, \epsilon(w_\epsilon - w))||_{\mathbb{E}_1(T)} \le \epsilon C.$$

Our strategy to show Theorem 1.1 is based on the estimate for $(V_\epsilon, \epsilon W_\epsilon)$. It consists of two key steps: maximal regularity result of the anisotropic Stokes operator and improved regularity result for the vertical component of the solution to the primitive equations. We consider the non-linear term in (SNS) as an external force term f and set $u_\epsilon = (v_\epsilon, \epsilon w_\epsilon)$ to get

$$\left\{\begin{array}{rcll} \partial_t u_\epsilon - \Delta u_\epsilon + \nabla_\epsilon \pi_\epsilon & = & f & \text{in} \quad \Omega \times (0,T), \\ \operatorname{div}_\epsilon u_\epsilon & = & 0 & \text{in} \quad \Omega \times (0,T), \\ u_\epsilon & = & 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u_\epsilon(0) & = & u_0 & \text{in} \quad \Omega, \end{array}\right. \tag{1.4}$$

where $\nabla_\epsilon = (\partial_1, \partial_2, \partial_3/\epsilon)^T$ and $\operatorname{div}_\epsilon = \nabla_\epsilon \cdot$. We define the function space $\mathbb{E}_{\epsilon,j}(T)$ for $j = 0, 1$ similarly as $\mathbb{E}_j(T)$ by replacing div by $\operatorname{div}_\epsilon$. Although the space $\mathbb{E}_{\epsilon,j}(T)$ depends on ϵ, the norm is just the norm in $W^{1,p}(0,T; L^q(\Omega)) \cap L^p(0,T; W^{2,q}(\Omega))$, so we shall write the norm in $\mathbb{E}_{\epsilon,j}(T)$ simply by $\| \cdot \|_{\mathbb{E}_j(T)}$. The space $X_{\epsilon,\gamma}$ is the initial trace space of $\mathbb{E}_{\epsilon,j}(T)$ and it is almost the same as X_γ by replacing div by $\operatorname{div}_\epsilon$. Since the norm of X_γ is that of $B^{2-2/p}_{q,p}(\Omega)$ and is independent of ϵ, we shall write the norm in $X_{\epsilon,\gamma}$ simply by $\| \cdot \|_{X_\gamma}$. We recall some known results on maximal regularity of the Stokes operator, which is corresponding to the case $\epsilon = 1$. Solonnikov [28] first proved L^q-L^q maximal regularity for the Stokes operator by a potential-theoretic approach. The second author [12] established a bound for the pure imaginary power of the Stokes operator in a bounded domain. This type of property will be simply called a bounded imaginary power, shortly BIP. This BIP implies the maximal regularity L^p-L^q regularity via Dore-Venni theory [6]. Indeed, the second author and Sohr [16] established a global-in-time maximal regularity in an exterior domain by estimating BIP. Further studies on maximal regularity were done by many researchers, for instance, Dore and Venni [6] and Weis [29]. See Denk, Hieber and Prüss [5] for further comprehensive research. In our case, we have to clarify ϵ-dependence in estimates for maximal regularity, which is a key point. Our key maximal regularity result is

Lemma 1.3. *Let $1 < p, q < \infty$, $0 < \epsilon \le 1$ and $T > 0$. Let $f \in \mathbb{E}_{\epsilon,0}(T)$ and $u_0 \in X_{\epsilon,\gamma}$. Then there exist constants $C = C(p,q) > 0$ and $C' = C'(p,q) > 0$, which are independent of ϵ, and (u, π) satisfying (1.4) such that*

$$||\partial_t u||_{\mathbb{E}_0(T)} + \left|\left|\nabla^2 u\right|\right|_{\mathbb{E}_0(T)} + ||\nabla_\epsilon \pi||_{\mathbb{E}_0(T)} \le C||f||_{\mathbb{E}_0(T)} + C'||u_0||_{X_\gamma}. \tag{1.5}$$

Lemma 1.3 follows from a maximal regularity involving the Stokes operator, which follows from a bound for the pure imaginary power by Dore-Venni theory. However, we need to clarify that C and C' can be taken independent of ϵ. For $\epsilon = 1$, a necessary BIP estimate for the Stokes operator has been established by Abels [1], where a resolvent decomposition similar to [12] is used. Unfortunately, the ϵ-dependent case is not discussed here. However, the strategy in [1] works for our problem. We construct the anisotropic Stokes operator by the method in [1] and show the boundedness of imaginary power. Note that, in our previous paper [10], maximal regularity of the anisotropic Stokes operator is much easier since the corresponding Stokes operator is essentially the same as the Laplace operator on $\mathbb{T}^3$. In the case of the Dirichlet boundary condition, the corresponding Stokes operator becomes to be much more difficult by the effect of boundaries, which is substantially different from the case of the periodic boundary conditions. The maximal regularity was proved in a layer domain for the Stokes operator under various boundary conditions by Saito [27] by proving $\mathcal{R}$-boundedness of the resolvent operator when $\epsilon = 1$. Unfortunately, it seems very difficult to check the dependence of ϵ, so we do not take this approach,

The term $F = \partial_t w - \Delta w + u \cdot \nabla w$ appears in the right-hand side of (1.1). Thus, we need to improve the regularity of w and estimate this term in $L^p(0,T;L^q(\Omega))$.

Lemma 1.4. *Let $T > 0$ and $u_0 = (v_0, w_0) \in X_\gamma$ with $w_0 = -\int_{-1}^{x_3} \mathrm{div}_H\, v_0\, d\zeta$ and $v_0 \in B^s_{q,p}(\Omega)$ for $s > 2 - 2/p + 1/q$ and $u = (v,w)$ be the solution to (PE). Assume $v \in \mathbb{E}_1(T)$. Then there exists a constant $C > 0$ such that*

$$\|w\|_{\mathbb{E}_1(T)} \le C. \tag{1.6}$$

Since $v \in \mathbb{E}_1(T)$, which is the horizontal component of the solution to the primitive equations, has already proved, it follows $w(\cdot, x_3) = -\int_{-1}^{x_3} \mathrm{div}_H v(\cdot,\zeta) d\zeta \in W^{1,p}(0,T;W^{-1,q}(\Omega)) \cap L^p(0,T;W^{1,p}(\Omega))$. This derivative loss is due to the absence of the equation of time-evolution of w in the primitive equations. In our previous paper [10], which treats the periodic boundary condition, we recover the regularity of w by deriving the equation which w satisfies and applying maximal regularity of the Laplace operator to the equation. However, in the case of the Dirichlet boundary condition, this method is not applicable directly because of the presence of the second-order derivative term at the boundary, which vanishes in the case of periodic boundary condition. Thus, we are forced to impose additional regularity for initial data to get regularity for v. If $v_0 \in B^s_{q,p}(\Omega)$ for $s > 2 - 2/p + 1/q$, then we obtain $v \in L^p(0,T;W^{s+2/p,q}(\Omega))$ and the trace of the second derivative belongs to $\mathbb{E}_0(T)$. Let us explain our strategy to show Theorem 1.1. By Lemma 1.4, our main result Theorem 1.1, can be proved the same way as [10]. The proof we give here is slightly different from that of [10] in the sense of the constant C in Theorem 1.1 is clarified. We first show the boundedness of non-linear terms F_H and F_z in (1.1) in the space $\mathbb{E}_0(T)$. We know that F is also bounded in $\mathbb{E}_0(T)$ by Lemma 1.3. We next apply Lemma 1.4 to (1.1) to get a quadratic inequality, which leads to $||(V_\epsilon, \epsilon W_\epsilon)||_{\mathbb{E}_1(T^*)} \le C\epsilon$ for some short time $T^* > 0$ and ϵ-independent constant $C > 0$. Since C depends only on p, q, $\|u_0\|_{X_\gamma}$, $\|u\|_{\mathbb{E}_1(T)}$ and T, if we take ϵ small, we are able to extend the time to all finite time T by finite step.

This paper is organized as follows. In section 2, the boundedness of pure imaginary power is proved. The resolvent operator of the anisotropic Stokes operator is decomposed into three parts, and for each part uniform bound on ϵ is proved. In section 3, improved regularity for w is proved. In section 4, we give a proof of our main theorem by iteration.

In this paper, $||\cdot||_{X\to Y}$ denotes the operator norm from a Banach space X to a Banach space Y. We denote by $C_0^\infty(\Omega)$ the set of compactly supported smooth functions in Ω. We denote $L^q(\Omega)$ is the Lebesgue space for $1 \le q \le \infty$ equipped with the norm

$$\|f\|_{L^q(\Omega)} = \left(\int_\Omega |f(x)|^q\, dx\right)^{1/q}.$$

We use the usual modification when $q = \infty$. For $m \in \mathbb{Z}_{\ge 0}$ and $1 \le q \le \infty$ we denote by $W^{m,q}(\Omega)$ the m-th order Sobolev space equipped with the norm

$$\|f\|_{W^{m,q}(\Omega)} = \|\nabla^m f\|_{L^q(\Omega)}.$$

We define the fractional Sobolev spaces $W^{s,q}(\Omega)(= B^s_{qq}(\Omega))$ for $s \notin \mathbb{Z}$ and $1 < q < \infty$ by the real interpolation $\left(W^{[s],q}(\Omega), W^{[s]+1,q}(\Omega)\right)_{s-[s],q}$, where $[\cdot]$ denotes the Gauss symbol. We define the Fourier transform by

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx$$

and the Fourier inverse transform by

$$\mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi} f(\xi)\, d\xi.$$

The Fourier transform on the d-dimensional torus $\mathbb{T}^d$ and its inverse transform are defined by $[\mathcal{F}_d f](n) = \int_{\mathbb{T}^d} e^{-ix\cdot n} f(x) dx$ and $[\mathcal{F}_d^{-1} g](x) = \frac{1}{(2\pi)^d}\sum_n g_n e^{in\cdot x}$, respectively. We denote by $\mathcal{F}_{x'}$ the partial Fourier transform with respect to $x' \in \mathbb{R}^2$ and by $\mathcal{F}_{\xi'}^{-1}$ the partial Fourier inverse transform with respect to ξ'. We denote by $\mathcal{F}_{d,x'}$ the partial Fourier transform with respect to $x' \in \mathbb{T}^2$ and by $\mathcal{F}_{d,n'}^{-1}$ the partial Fourier inverse transform with respect to $n' \in \mathbb{Z}^2$. Define $\Sigma_\theta := \{\lambda \in \mathbb{C} : |\arg\lambda| < \pi - \theta\}$. For a Fourier multiplier operator $\mathcal{F}_\xi^{-1} m(\xi)\mathcal{F}_x$ in $\mathbb{R}^3$, we denote by $[m]_{\mathcal{M}}$ the Mikhlin constant. $\mathcal{F}_{\xi'}^{-1}m(\xi)\mathcal{F}_{x'}$ is a Fourier multiplier operator in $\mathbb{R}^2$ with Mikhlin constant $[m]_{\mathcal{M}'}$. For $0<\epsilon\le 1$, $\Delta_\epsilon=\partial_1^2+\partial_2^2+\partial_3^2/\epsilon^2$ denotes the anisotropic Laplace operator. We denote by E_0 the zero-extension operator with respect to the vertical variable from $(-1,1)$ to $\mathbb{R}$. We denote by R_0 the restriction operator with respect to the vertical variable from $\mathbb{R}$ to $(-1,1)$. For an integrable function f defined on Ω, we write its vertical average by $\overline{f}=\frac{1}{2}\int_{-1}^{1} f(\cdot,\cdot,\zeta)\,d\zeta$.

2. A uniform bound for pure imaginary power of the anisotropic Stokes operator and its maximal regularity

In this section, we first establish a uniform bound independent of ϵ for the pure imaginary power to the anisotropic Stokes operator along with [1]. Then we shall give the proof of Lemma 1.3.

2.1. **Boundedness of Fourier multipliers.** Although the case of infinite the layer $\mathbb{R}^2\times(-1.1)$ is considered in [1], his method also works in the case of the periodic layer $\Omega=\mathbb{T}^2\times(-1,1)$ thanks to Fourier multiplier theorem on the torus, e.g. Proposition 4.5 in [19] and Section 4 of Grafakos's book [17].

Proposition 2.1 ([19])**.** *Let $1<p<\infty$ and $m\in C^d(\mathbb{R}^d\setminus\{0\})$ satisfies the Mikhlin condition:*

$$[m]_{\mathcal{M}}:=\sup_{\alpha\in\{0,1\}^d}\sup_{\xi\in\mathbb{R}^d\setminus\{0\}}\left|\xi^\alpha\partial_\xi^\alpha m(\xi)\right|<\infty. \tag{2.1}$$

Let $a_k=m(k)$ for $k\in\mathbb{Z}^d\setminus\{0\}$ and $a_0\in\mathbb{C}$. For $f(x)=\sum_{n\in\mathbb{Z}^d}\widehat{f}_n e^{in\cdot x}\in L^q(\mathbb{T}^d)$ and a sequence $a=\{a_n\}_{n\in\mathbb{Z}^d}$, we set the Fourier multiplier operator of discrete type by

$$[Tf](x):=\mathcal{F}_d^{-1}a\mathcal{F}_d f=\sum_{n\in\mathbb{Z}^d}a_n\hat{f}_n e^{in\cdot x}. \tag{2.2}$$

Then there exits a constant $C=C(p,d)>0$ such that

$$\|Tf\|_{L^q(\mathbb{T}^d)}\le C\max([m]_{\mathcal{M}},a_0)\|f\|_{L^q(\mathbb{T}^d)}. \tag{2.3}$$

Let us consider the resolvent problem to (1.4) ;

$$\left\{\begin{array}{rll}\lambda u-\Delta u+\nabla_\epsilon\pi &=f & \text{in}\quad\Omega,\\ \operatorname{div}_\epsilon u&=0 & \text{in}\quad\Omega,\\ u&=0 & \text{on}\quad\partial\Omega,\end{array}\right. \tag{2.4}$$

for $\lambda\in\Sigma_\theta$ $(0<\theta<\pi/2)$ and $f\in L^q(\Omega)$. Let

$$H_\epsilon: L^q(\Omega)\to L^q_{\sigma,\epsilon}(\Omega)=\{u\in L^q(\Omega)\,;\,\operatorname{div}_\epsilon u=0, u|_{x_3=\pm1}=0\},\ (1<p<\infty)$$

be the anisotropic Helmholtz projection on Ω, its L^q-boundedness is proved later. Let $A_\epsilon=H_\epsilon(-\Delta)$ be the Stokes operator with the domain $D(A_\epsilon)=L^q_{\sigma,\epsilon}(\Omega)\cap W^{2,q}(\Omega)$. For $0<a<1/2$ and $-a<\operatorname{Re}z<0$, the fractional power of A_ϵ is defined via the Dunford calculus

$$A_\epsilon^z=\frac{1}{2\pi i}\int_{\Gamma_\theta}(-\lambda)^z(\lambda+A_\epsilon)^{-1}\,d\lambda,$$

where $0<\theta<\pi/2$ and $\Gamma_\epsilon=\mathbb{R}e^{i(-\pi+\theta)}\cup\mathbb{R}e^{i(\pi-\theta)}$. Our aim in this section is to prove

Lemma 2.2. *Let $1<q<\infty$, $0<\epsilon\le 1$, $0<a<1/2$, $z\in\mathbb{C}$ satisfying $-a<\operatorname{Re}z<0$ and $0<\theta<\pi/2$. Then there exists a constant $C=C(q,a,\theta)$ such that*

$$||A_\epsilon^z||_{L^q(\Omega)\to L^q(\Omega)}\le Ce^{\theta|\mathrm{Im}z|}. \tag{2.5}$$

Once the above lemma is proved, then we obtain the maximal regularity of the anisotropic Stokes operator via the formula

$$\left(\frac{d}{dt}+A_\epsilon\right)^{-1}=\int_{c+i\infty}^{c-i\infty}\frac{(d/dt)^z A_\epsilon^{1-z}}{\sin\pi z}dz \tag{2.6}$$

for $0<c<1$ and the Dore-Venni theory [6].

To show Lemma 2.2, we decompose the solution (u,π) to (2.4) into three parts;

$$u=R_0v_1-v_2+\nabla_\epsilon\pi_3, \tag{2.7}$$
$$\nabla_\epsilon\pi=\nabla_\epsilon\pi_1+\nabla_\epsilon\pi_2, \tag{2.8}$$

where v_j and π_j are solutions to

$$(I)\left\{\begin{array}{rcll}\lambda v_1-\Delta v_1+\nabla_\epsilon\pi_1&=&E_0f&\text{in}\quad\mathbb{T}^2\times\mathbb{R},\\ \operatorname{div}_\epsilon v_1&=&0&\text{in}\quad\mathbb{T}^2\times\mathbb{R},\end{array}\right.$$

$$(II)\begin{cases} \lambda v_2 - \Delta v_2 + \nabla_\epsilon \pi_2 &= 0 \quad \text{in } \Omega,\\ \operatorname{div}_\epsilon v_2 &= 0 \quad \text{in } \Omega,\\ v_2 &= \gamma v_1 - (\gamma v_1\cdot\nu)\nu \quad \text{on } \partial\Omega, \end{cases}$$

and

$$(III)\begin{cases} \Delta_\epsilon \pi_3 &= 0 \quad \text{in } \Omega,\\ \nabla_\epsilon \pi_3 \cdot \nu &= (\gamma v_1\cdot\nu)\nu \quad \text{on } \partial\Omega, \end{cases}$$

respectively, where $\gamma = \gamma_\pm$ is the trace operator to the upper and lower boundary, respectively, and ν is the unit outer normal. To show Lemma 2.2, we need to obtain

$$\left|\left|\frac{1}{2\pi i}\int_\Gamma (-\lambda)^z R_0 v_1 \, d\lambda\right|\right|_{L^q(\Omega)} + \left|\left|\frac{1}{2\pi i}\int_\Gamma (-\lambda)^z v_2 \, d\lambda\right|\right|_{L^q(\Omega)}$$
$$+ \left|\left|\frac{1}{2\pi i}\int_\Gamma (-\lambda)^z \nabla_\epsilon \pi_3 \, d\lambda\right|\right|_{L^q(\Omega)} \le Ce^{\theta|\operatorname{Im} z|}||f||_{L^q(\Omega)}$$

for some Constant $C > 0$, which is independent of ϵ.

Remark 2.3. For $f \in L^q(\Omega)$ and $\tilde{f} := \int_{\mathbb{T}^2} f dx'$, we can solve the resolvent problem (2.4) with external force $\tilde{f}$ to get $u' = \left((\lambda - \partial_3^2)^{-1}\tilde{f}_H, 0\right)^T$ and $\pi' = \epsilon\int_{-1}^{x_3}\tilde{f}_3 d\zeta/\mathbb{R}$, where $\tilde{f}_H$ is the horizontal component of $\tilde{f}$ and $/\mathbb{R}$ means average-free. Since $-\partial_3^2$ has BIP and the resolvent operator is linear, by taking the difference between the solution to (2.4) and (u', π'), we can always assume without loss of generality that f is horizontal average-free.

We define the space of horizontally average-free L^q-vector fields by

$$L^q_{\mathrm{af}}(\Omega) := \left\{ f \in L^q(\Omega) : \tilde{f} = 0 \right\}.$$

Similarly we define

$$W^{s,q}_{\mathrm{af}}(\Omega) := \left\{ f \in W^{s,q}(\Omega) : \tilde{f} = 0 \right\}.$$

Throughout this section we frequently use partial Fourier transform to construct solutions and estimate these partial Fourier multipliers.

Proposition 2.4 ([1]). *Let $1 < q < \infty$ and $a, b \in \{-1, 1\}$. Set a integral operator M by*

$$Mf(x', x_3) = \int_{-1}^{1} \frac{f(x',\zeta)}{|x_3 - a| + |\zeta - b|}\, d\zeta$$

for $f \in L^q(\Omega)$. Then there exists a constant $C > 0$ such that

$$||Mf||_{L^q} \le C||f||_{L^q}.$$

Proof. See Lemma 3.3 in [1]. □

Rescaled L^q-Fourier multipliers are also bounded L^q multiplier by the direct consequence of the Mikhlin theorem.

Proposition 2.5. *Let $1 < q < \infty$ and $0 < \epsilon \le 1$. Let $m \in C^d(\mathbb{R}^d \setminus \{0\})$ be a L^q-Fourier multiplier with the Mikhlin constant $[m]_{\mathcal{M}} \le C$ for some $C > 0$. Then rescaled one $m_\epsilon(\xi) := m(\epsilon\xi)$ is also bounded from L^q into itself such that*

$$[m_\epsilon]_{\mathcal{M}} \le C.$$

The above proposition is frequently used in this section to get ϵ-independent estimate for scaled multipliers. We show boundedness of some Fourier multiplier operators in advance. We set

$$s_\lambda = (\lambda + |\xi'|^2)^{1/2}$$

for $\xi' \in \mathbb{R}^2$. In this paper we use s_λ to denote $(\lambda + |n'|^2)^{1/2}$ for $n' \in \mathbb{Z}^2$ to simplify notation.

Proposition 2.6.

- *Let $0 < \theta < \pi/2$, $\lambda \in \Sigma_\theta$, $t > 0$ and α be a positive integer. Then there exist constants $c > 0$ and $C > 0$ such that*

$$\left[|\xi'|^\alpha e^{-ts_\lambda}\right]_{\mathcal{M}'} \le C\frac{e^{-ct|\lambda|^{1/2}}}{t^\alpha}, \quad \left[\frac{e^{-s_\lambda}}{s_\lambda}\right]_{\mathcal{M}'} \le C|\lambda|^{-1/2}e^{-c|\lambda|^{1/2}}. \tag{2.9}$$

- *Let $-1 \le x_3 \le 1$. Then there exists a constant $C > 0$ which is independent of ϵ, such that*
$$\left[\frac{\sinh(\epsilon|\xi'|x_3)}{\sinh(\epsilon|\xi'|)} \frac{\epsilon|\xi'|}{1+\epsilon|\xi'|} \right]_{\mathcal{M}'} \le C, \quad \left[\frac{\cosh(\epsilon|\xi'|x_3)}{\sinh(\epsilon|\xi'|)} \frac{\epsilon|\xi'|}{1+\epsilon|\xi'|} \right]_{\mathcal{M}'} \le C \tag{2.10}$$
for all $0 < \epsilon \le 1$.
- *Let $-1 \le x_3 \le 1$. Then there exists a constant $C > 0$, which is independent of ϵ, such that*
$$\left[\frac{\sinh(\epsilon|\xi'|x_3)}{\cosh(\epsilon|\xi'|)} \right]_{\mathcal{M}'} \le C, \quad \left[\frac{\sinh(\epsilon|\xi'|x_3)}{\cosh(\epsilon|\xi'|)} \right]_{\mathcal{M}'} \le C \tag{2.11}$$
for all $0 < \epsilon \le 1$.

Proof. The estimate (2.9) is a direct consequence of Lemma 3.5 in [1] and the Mikhlin theorem. By definition of sinh and cosh, we find the formula
$$\frac{\sinh(\epsilon|\xi'|x_3)}{\sinh(\epsilon|\xi'|)} = \frac{e^{\epsilon|\xi'|x_3} - e^{-\epsilon|\xi'|x_3}}{e^{\epsilon|\xi'|} - e^{-\epsilon|\xi'|}} = \frac{e^{-\epsilon|\xi'|(x_3-1)}}{1-e^{-2\epsilon|\xi'|}} - \frac{e^{-\epsilon|\xi'|(x_3+1)}}{1-e^{-2\epsilon|\xi'|}} \tag{2.12}$$
and
$$\frac{\cosh(\epsilon|\xi'|x_3)}{\sinh(\epsilon|\xi'|)} = \frac{e^{-\epsilon|\xi'|(x_3-1)}}{1-e^{-2\epsilon|\xi'|}} + \frac{e^{-\epsilon|\xi'|(x_3+1)}}{1-e^{-2\epsilon|\xi'|}}. \tag{2.13}$$
Thus, multiplying $\frac{\epsilon|\xi'|}{1+\epsilon|\xi'|}$ by both sides of (2.12), we find from Proposition 2.5 that
$$\begin{aligned} &\left[\frac{\sinh(\epsilon|\xi'|x_3)}{\sinh(\epsilon|\xi'|)} \frac{\epsilon|\xi'|}{1+\epsilon|\xi'|} \right]_{\mathcal{M}'} \\ &\le C \left[e^{-\epsilon|\xi'|(x_3-1)} \frac{\epsilon|\xi'|}{(1-e^{-2\epsilon|\xi'|})} \frac{1}{(1+\epsilon|\xi'|)} \right]_{\mathcal{M}'} \\ &+ \left[e^{-\epsilon|\xi'|(x_3+1)} \frac{\epsilon|\xi'|}{(1-e^{-2\epsilon|\xi'|})} \frac{1}{(1+\epsilon|\xi'|)} \right]_{\mathcal{M}'} \\ &\le C. \end{aligned} \tag{2.14}$$
The second inequality of (2.10) is proved by the same as above using (2.13). Similarly, by definition of sinh and cosh, the estimate (2.11) follows. □

2.2. **Estimate for v_1.** Let us consider the equations (I). For $a \in \mathbb{R}$ we denote by $\tau_a f = f(a\cdot)$ the rescaling operator by a. The anisotropic Helmholtz projection $\mathbb{P}_\epsilon^{\mathbb{R}^3}$ on $\mathbb{R}^3$ with symbols
$$\mathcal{F}\mathbb{P}_\epsilon^{\mathbb{R}^3} = I_3 - \xi_\epsilon \otimes \xi_\epsilon, \quad \xi_\epsilon = \left(\xi_1, \xi_2, \frac{\xi_3}{\epsilon} \right) \in \mathbb{R}^3,$$
is bounded in $L^q(\mathbb{R}^3)$ by boundedness of the Riesz operator and the formula
$$\mathcal{F}_\xi^{-1} m(a\xi) \mathcal{F}_x f = \tau_{a^{-1}} \left[\mathcal{F}_\xi^{-1} m(\xi) \mathcal{F}_x \tau_a f \right]. \tag{2.15}$$
Actually apply (2.15) with respect to the third variable, then, the symbol is no longer dependent on ϵ. Changing the variable with respect to and using boundedness of the Riesz operator, we find
$$\|\mathbb{P}_\epsilon^{\mathbb{R}^3} f\|_{L^q(\mathbb{R}^3)} = \epsilon^{-1} \|\mathbb{P}_1 \left[\tau_{1/\epsilon}^3 f \right] \|_{L^q(\mathbb{R}^3)} \le C \|f\|_{L^q(\mathbb{R}^3)},$$
where τ_a^3 is the rescaled operator with respect to the third variable for $a > 0$. We define the anisotropic Helmholtz projection $\mathbb{P}_\epsilon^{\mathbb{T}^2 \times \mathbb{R}}$ on $\mathbb{T}^2 \times \mathbb{R}$ with symbols by
$$\mathcal{F}_{x_3} \mathcal{F}_{d,x'} \mathbb{P}_\epsilon^{\mathbb{T}^2 \times \mathbb{R}} = I_3 - \begin{pmatrix} n_1 \\ n_2 \\ \xi_3/\epsilon \end{pmatrix} \otimes \begin{pmatrix} n_1 \\ n_2 \\ \xi_3/\epsilon \end{pmatrix}, \quad n_1, n_2 \in \mathbb{Z}, \ \xi_3 \in \mathbb{R}.$$
We find $\mathbb{P}_\epsilon^{\mathbb{T}^2 \times \mathbb{R}}$ is bounded from $L^q(\mathbb{T}^2 \times \mathbb{R})$ into itself by boundedness of $\mathbb{P}_\epsilon^{\mathbb{R}^3}$ and Proposition 2.1 uniformly in $\epsilon \in (-1, 1)$.

Proposition 2.7. *Let $1 < q < \infty$, $0 < a < 1/2$, $0 < \epsilon \le 1$, $z \in \mathbb{C}$ satisfying $-a < \mathrm{Re} z < 0$ and $0 < \theta < \pi/2$. Then there exists a constant $C = C(q, a, \theta)$ such that*
$$\begin{aligned} &\left|\left| \frac{1}{2\pi i} R_0 \int_{\Gamma_\theta} (-\lambda)^z (\lambda - \Delta_{\mathbb{T}^2 \times \mathbb{R}})^{-1} \mathbb{P}_\epsilon^{\mathbb{T}^2 \times \mathbb{R}} E_0 f \, d\lambda \right|\right|_{L^q(\mathbb{T}^2 \times \mathbb{R})} \\ &\le C e^{\theta |\mathrm{Im}\, z|} ||f||_{L^q(\Omega)} \end{aligned} \tag{2.16}$$
for all $f \in L^q(\Omega)$.

Proof. It is known that the Laplace operator on a cylinder $\mathbb{T}^2 \times \mathbb{R}$ has BIP. Combining this fact and L^q-boundedness of $\mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}}$, we have (2.16). $\square$

Proposition 2.8. *Let $1 < q < \infty$, $0 < \epsilon \leq 1$, $0 < \theta < \pi/2$ and $\lambda \in \Sigma_\theta$. Then there exists a constant $C = C(q) > 0$, which is independent of ϵ, such that*

$$\left\| \nabla^2 \left(\lambda - \Delta_{\mathbb{T}^2\times\mathbb{R}}\right)^{-1} \mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}} E_0 f \right\|_{L^q(\Omega)} \leq C \|f\|_{L^q(\Omega)}.$$

Proof. This follows from Propositions 2.1 and 2.5 since $P_\epsilon^{\mathbb{T}^2\times\mathbb{R}}$ is uniformly bounded from $L^q(\mathbb{T}^2 \times \mathbb{R})$ into itself. $\square$

Let us calculate the partial Fourier transform for v_1 with respect to the horizontal variable. This is needed to obtain representation formula for v_2 later. Let $g \in L^q(\mathbb{T}^2 \times \mathbb{R})$. The solution $\tilde{v}$ to the equation

$$\begin{cases} \lambda\tilde{v} - \Delta\tilde{v} + \nabla_\epsilon\tilde{\pi} &= g \quad \text{in} \quad \mathbb{T}^2 \times \mathbb{R}, \\ \operatorname{div}_\epsilon \tilde{v} &= 0 \quad \text{in} \quad \mathbb{T}^2 \times \mathbb{R}, \end{cases}$$

is given by

$$\tilde{v} = \left(\lambda - \Delta_{\mathbb{R}^3}\right)^{-1} \mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}} g.$$

Moreover,

$$\begin{aligned} K_{\lambda,\epsilon} g &:= \left(\lambda - \Delta_{\mathbb{T}^2\times\mathbb{R}}\right)^{-1} \mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}} g \\ &= \mathcal{F}^{-1}(\lambda + |n'|^2 + \xi_3^2)^{-1} \left(I_3 - \frac{\xi_\epsilon \otimes \xi_\epsilon}{|\xi_\epsilon|^2} \right) \mathcal{F} g \\ &= \mathcal{F}_{n'}^{-1} \int_{\mathbb{R}} k_{\lambda,\epsilon}(n', x_3 - \zeta) \mathcal{F}_{n'} g(n', \zeta) \, d\zeta, \end{aligned} \tag{2.17}$$

where $\xi_\epsilon = (n', \xi_3/\epsilon) \in \mathbb{Z}^2 \times \mathbb{R}$ and

$$\begin{aligned} & k'_{\lambda,\epsilon}(n', x_3) \\ &= \mathcal{F}_{\xi_3}^{-1} \left[(\lambda + |n'|^2 + \xi_3^2)^{-1} \left(I_3 - \frac{\xi_\epsilon \otimes \xi_\epsilon}{|\xi_\epsilon|^2} \right) \right] \\ &= \frac{e^{-s_\lambda}}{2s_\lambda} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \\ &\quad - \begin{pmatrix} n' \otimes n' \frac{\epsilon^2}{\lambda + (1-\epsilon^2)|n'|^2} \frac{-\epsilon|n'| e^{-|x_3| s_\lambda} + s_\lambda e^{-|x_3|\epsilon|n'|}}{2 s_\lambda \epsilon |n'|} & -in' \frac{\epsilon^2}{\lambda + (1-\epsilon^2)|n'|^2} \frac{e^{-|x_3| s_\lambda} - e^{-|x_3|\epsilon|n'|}}{2} \\ -in'^T \frac{\epsilon^2}{\lambda + (1-\epsilon^2)|n'|^2} \frac{e^{-|x_3| s_\lambda} - e^{-|x_3|\epsilon|n'|}}{2} & -|n'|^2 \frac{\epsilon^2}{\lambda + (1-\epsilon^2)|n'|^2} \frac{-\epsilon|n'| e^{-|x_3| s_\lambda} + s_\lambda e^{-|x_3|\epsilon|n'|}}{2 s_\lambda \epsilon |n'|} \end{pmatrix} \\ &=: \frac{e^{-s_\lambda}}{2s_\lambda} \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} n' \otimes n' \eta'_{\lambda,\epsilon}(n', x_3) & -in' \partial_3 \eta'_{\lambda,\epsilon}(n', x_3) \\ -in'^T \partial_3 \eta'_{\lambda,\epsilon}(n', x_3) & -|\xi'|^2 \eta'_{\lambda,\epsilon}(n', x_3) \end{pmatrix}. \end{aligned} \tag{2.18}$$

The kernel function $k_{\lambda,\epsilon}(n', x_3)$ is calculated by the residue theorem. Actually, since poles of $\left(\lambda + |n'|^2 + \xi_3^2\right)^{-1}$ are $\xi_3 = \pm i s_\lambda$, the residue theorem implies the partial Fourier inverse transform of $\left(\lambda + |n'|^2 + \xi_3^2\right)^{-1}$ with respect to ξ_3 is given by inserting $\xi_3 = i s_\lambda$ or $- i s_\lambda$ into $e^{i x_3 \xi_3}$ so that the real part become to be negative. Thus, we have

$$e'_\lambda(n', x_3) := \mathcal{F}_{\xi_3}^{-1} \left(\lambda + |n'|^2 + |x_3|^2\right)^{-1} = \frac{e^{-|x_3| s_\lambda}}{s_\lambda}. \tag{2.19}$$

Moreover, this formula leads to

$$\mathcal{F}_{\xi_3}^{-1} \left[|\xi_\epsilon|^2\right]^{-1} = \mathcal{F}_{\xi_3}^{-1} \left[\frac{\epsilon^2}{\epsilon^2 |n'|^2 + \xi_3^2} \right] = \frac{\epsilon e^{-|x_3|\epsilon|n'|}}{|n'|}.$$

Combining the above two calculations and the formula

$$I_3 - \frac{\xi_\epsilon \otimes \xi_\epsilon}{|\xi_\epsilon|^2} = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \frac{n' \otimes n'}{|\xi_\epsilon|^2} & \frac{\xi_3 n'/\epsilon}{|\xi_\epsilon|^2} \\ \frac{\xi_3 n'^T/\epsilon}{|\xi_\epsilon|^2} & -\frac{|n'_\epsilon|^2}{|\xi_\epsilon|^2} \end{pmatrix},$$

we obtain (2.18).

2.3. **Boundedness of the anisotropic Helmholtz projection.** Next, we consider the equation (III) with boundary data $\phi = (\phi_+, \phi_-) \in C_0^\infty(\Omega)$. Applying the partial Fourier transform to (III), we have

$$\begin{cases} \left(\frac{\partial_z^2}{\epsilon^2} - |n'|^2\right) \mathcal{F}_{d,x'}\pi_3(n', x_3) &= 0, \\ \frac{\partial_z}{\epsilon} \mathcal{F}_{d,x'}\pi_3(n', \pm 1) &= \mathcal{F}_{d,x'}\phi_\pm(n'), \end{cases} \tag{2.20}$$

for $n' \in \mathbb{Z}^2 \backslash \{0\}$ and $x_3 \in (-1, 1)$. The solution to (2.20) is of the form

$$\mathcal{F}_{d,x'}\pi_3(n', x_3) = C_1 e^{\epsilon x_3 |n'|} + C_2 e^{-\epsilon x_3 |n'|}$$

for some constant C_1 and C_2. Take the constants so that (2.20) satisfied, namely

$$C_1 = \frac{\mathcal{F}_{d,x'}\phi_+ + \mathcal{F}_{d,x'}\phi_-}{4|n'|\cosh(\epsilon|n'|)} + \frac{\mathcal{F}_{d,x'}\phi_+ - \mathcal{F}_{d,x'}\phi_-}{4|n'|\sinh(\epsilon|n'|)},$$
$$C_2 = -\frac{\mathcal{F}_{d,x'}\phi_+ + \mathcal{F}_{d,x'}\phi_-}{4|\xi'|\cosh(\epsilon|n'|)} + \frac{\mathcal{F}_{d,x'}\phi_+ - \mathcal{F}_{d,x'}\phi_-}{4|\xi'|\sinh(\epsilon|n'|)},$$

then the solution to (2.20) is given by

$$\begin{aligned} &\pi_3(x', x_3) \\ &= \mathcal{F}^{-1}_{d,n'} \left(\frac{\sinh(\epsilon x_3|n'|)}{|n'|\cosh(\epsilon|n'|)} \frac{\mathcal{F}_{d,x'}\phi_+ + \mathcal{F}_{x'}\phi_-}{2} + \frac{\cosh(\epsilon x_3|n'|)}{|n'|\sinh(\epsilon|n'|)} \frac{\mathcal{F}_{d,x'}\phi_+ - \mathcal{F}_{d,x'}\phi_-}{2} \right). \end{aligned}$$

Moreover, its anisotropic gradient given by

$$\begin{aligned} \nabla_\epsilon \pi_3 &= \mathcal{F}^{-1}_{d,n'} \begin{pmatrix} \frac{in' \sinh(\epsilon x_3|n'|)}{|n'|\cosh(\epsilon|n'|)} \frac{\mathcal{F}_{d,x'}\phi_+ + \mathcal{F}_{d,x'}\phi_-}{2} + \frac{in' \cosh(\epsilon x_3|n'|)}{|n'|\sinh(\epsilon|n'|)} \frac{\mathcal{F}_{d,x'}\phi_+ - \mathcal{F}_{d,x'}\phi_-}{2} \\ \frac{\cosh(\epsilon x_3|n'|)}{\cosh(\epsilon|n'|)} \frac{\mathcal{F}_{d,x'}\phi_+ + \mathcal{F}_{d,x'}\phi_-}{2} + \frac{\sinh(\epsilon x_3|n'|)}{\sinh(\epsilon|n'|)} \frac{\mathcal{F}_{d,x'}\phi_+ - \mathcal{F}_{d,x'}\phi_-}{2} \end{pmatrix} \\ &=: \mathcal{F}^{-1}_{d,n'}\alpha_{\epsilon,+}(n', x_3)\mathcal{F}_{d,x'}\phi_+ + \mathcal{F}^{-1}_{d,n'}\alpha_{\epsilon,-}(n', x_3)\mathcal{F}_{d,x'}\phi_-. \end{aligned} \tag{2.21}$$

We apply the trace to (2.21) to get

$$\gamma_\pm \nabla_\epsilon \pi_3 = \mathcal{F}^{-1}_{d,n'} \begin{pmatrix} \frac{\pm in' \sinh(\epsilon|n'|)}{|n'|\cosh(\epsilon|n'|)} \frac{\mathcal{F}_{d,x'}\phi_+ + \mathcal{F}_{d,x'}\phi_-}{2} + \frac{in' \cosh(\epsilon|n'|)}{|n'|\sinh(\epsilon|n'|)} \frac{\mathcal{F}_{d,x'}\phi_+ - \mathcal{F}_{d,x'}\phi_-}{2} \\ \frac{\mathcal{F}_{d,x'}\phi_+ + \mathcal{F}_{d,x'}\phi_-}{2} \pm \frac{\mathcal{F}_{d,x'}\phi_+ - \mathcal{F}_{d,x'}\phi_-}{2} \end{pmatrix}.$$

We insert $\phi_\pm = \gamma_\pm \mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}} f$ to (2.21) for $f \in C_0^\infty(\Omega)$ satisfying $\tilde{f} = 0$ and set

$$\begin{aligned} \Pi_\epsilon f := \mathcal{F}^{-1}_{d,n'} &\left[\alpha_{\epsilon,+}(n', x_3)\gamma_+ \mathcal{F}_{d,x'}\left(e_3 \cdot \mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}} E_0 f\right)\right] \\ + \mathcal{F}^{-1}_{d,n'} &\left[\alpha_{\epsilon,-}(n', x_3)\gamma_- \mathcal{F}_{d,x'}\left(e_3 \cdot \mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}} E_0 f\right)\right]. \end{aligned}$$

Lemma 2.9. *Let $1 < q < \infty$, $0 < \epsilon \leq 1$ and $s \geq 0$. Then there exists a constant $C = C(q)$, which is independent of ϵ, the operator Π_ϵ can be extended to a bounded operator from $W^{s,q}_{\mathrm{af}}(\Omega)$ into itself such that*

$$||\Pi_\epsilon f||_{W^{s,q}(\Omega)} \leq C||f||_{W^{s,q}(\Omega)} \tag{2.22}$$

for all $f \in W^{s,q}_{\mathrm{af}}(\Omega)$.

Proof. Let $f \in C_0^\infty(\Omega)$ satisfy $\tilde{f} = 0$. We seek the multiplier of Π_ϵ by a direct calculation. Recall that the symbol of $\mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}}$ is of the form

$$\mathcal{F}_{x_3}\mathcal{F}_{d,x'}\mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}} = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \frac{n'\otimes n'}{|n'_\epsilon|^2} & \frac{n'\xi_3/\epsilon}{|n'_\epsilon|^2} \\ \frac{n'^T\xi_3/\epsilon}{\epsilon|n'_\epsilon|^2} & -\frac{|n'|^2}{|n'_\epsilon|^2} \end{pmatrix}. \tag{2.23}$$

Since the symbol of $\mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}}$ have poles at $\xi_3 = \pm i\epsilon|n'|$, we apply $e_3\cdot$ to (2.23) by the left hand side and use the residue theorem so that the power of e is negative to get

$$\begin{aligned} \mathcal{F}_{d,x'}(e_3 \cdot \mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}} E_0 f) = &-\int_{-1}^1 \frac{ie^{-|x_3-\zeta|\epsilon|n'|}}{2}\epsilon n' \cdot \mathcal{F}_{d,x'}f'(n', \zeta)\, d\zeta \\ &+ \int_{-1}^1 \frac{e^{-|x_3-\zeta|\epsilon|n'|}}{2}\epsilon|n'|\mathcal{F}_{d,x'}f_3(n', \zeta)\, d\zeta. \end{aligned} \tag{2.24}$$

Note that the integration is due to the relationship between the Fourier transform and convolution. Applying trace operators $\gamma_\pm$ and $\alpha_{\epsilon,\pm}(n', x_3)$, respectively, and taking Fourier inverse transform with respect to n', we find

$$\begin{aligned}
&\Pi_\epsilon f(x',x_3)\\
&= -\mathcal{F}^{-1}_{d,n'} \int_{-1}^{1} \alpha_{\epsilon,+}(n',x_3) \frac{ie^{-|1-\zeta|\epsilon|n'|}}{2} \epsilon n' \cdot \mathcal{F}_{d,x'} f'(n',\zeta)\, d\zeta\\
&+ \mathcal{F}^{-1}_{d,n'} \int_{-1}^{1} \alpha_{\epsilon,+}(n',x_3) \frac{e^{-|1-\zeta|\epsilon|n'|}}{2} \epsilon |n'| \mathcal{F}_{d,x'} f_3(n',\zeta)\, d\zeta\\
&- \mathcal{F}^{-1}_{d,n'} \int_{-1}^{1} \alpha_{\epsilon,-}(n',x_3) \frac{ie^{-|-1-\zeta|\epsilon|n'|}}{2} \epsilon n' \cdot \mathcal{F}_{d,x'} f'(n',\zeta)\, d\zeta\\
&+ \mathcal{F}^{-1}_{d,n'} \int_{-1}^{1} \alpha_{\epsilon,-}(n',x_3) \frac{e^{-|-1-\zeta|\epsilon|n'|}}{2} \epsilon |n'| \mathcal{F}_{d,x'} f_3(n',\zeta)\, d\zeta\\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{2.25}$$

By the definition of $\alpha_{\pm,\epsilon}$,

$$I_1 = -\mathcal{F}^{-1}_{d,n'} \int_{-1}^{1} \frac{1}{2} \begin{pmatrix} \frac{in' \sinh(\epsilon x_3|n'|)}{|n'|\cosh(\epsilon|n'|)} + \frac{in' \cosh(\epsilon x_3|n'|)}{|n'|\sinh(\epsilon|n'|)} \\ \frac{\cosh(\epsilon x_3|n'|)}{\cosh(\epsilon|n'|)} + \frac{\sinh(\epsilon x_3|n'|)}{\sinh(\epsilon|n'|)} \end{pmatrix} \times \frac{ie^{-|1-\zeta|\epsilon|n'|}}{2} \epsilon n' \cdot \mathcal{F}_{d,x'} f'(n',\zeta) d\zeta.$$

Symbols in the integral can be written by $A(\epsilon n')(1+\epsilon|n'|)e^{-\epsilon|n'|(|x_3\pm1|+|\zeta\pm1|)}$ for a symbol A with an ϵ-independent Mikhlin constant by Proposition 2.6. The same argument is valid for I_j $(j=2,3,4)$. Thus, we find from Propositions 2.4 and 2.5 that

$$\begin{aligned}
||\Pi_\epsilon f||_{L^q(\Omega)} &\le C||f||_{L^q(\Omega)} + C\left|\left| \int_{-1}^{1} \frac{||f(\cdot,\zeta)||_{L^q(\mathbb{T}^2)}}{|x_3\pm1|+|\zeta\pm1|}\, d\zeta \right|\right|_{L^q(-1,1)}\\
&\le C||f||_{L^q(\Omega)}
\end{aligned}$$

for all $f \in C_0^\infty(\Omega)$ satisfying $\tilde{f}=0$, where the constant C is independent of ϵ. Thus the estimate (2.22) holds for $m=0$. We find from the formula (2.25) that ∂_j commutes with Π_ϵ for $j=1,2$. Moreover, the equation (2.20) implies

$$\begin{aligned}
\partial_3^2\, \Pi_\epsilon f &= -\epsilon^2(\partial_1^2+\partial_2^2)\, \Pi_\epsilon f\\
&= -\epsilon^2 \Pi_\epsilon\, (\partial_1^2+\partial_2^2) f
\end{aligned}$$

Thus we find (2.22) holds for all positive even number m. We can obtain (2.22) for all $s>0$ by interpolation. □

We set the operator

$$P_{N,\epsilon} := R_0\left(\mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}} - \Pi_\epsilon\right) f$$

for all $f \in C_0^\infty(\Omega)$ satisfying $\tilde{f}=0$. Then Lemma 2.9 implies

Corollary 2.10. *Let $1<q<\infty$, $0<\epsilon\le1$ and $s>0$. Then there exists a constant $C>0$, which is independent of ϵ, the operator $P_{N,\epsilon}$ can be extended to a bounded operator from $W^{s,q}_{\mathrm{af}}(\Omega)$ into itself such that*

$$||P_{N,\epsilon} f||_{W^{s,q}(\Omega)} \le C||f||_{W^{s,q}(\Omega)}$$

for all $f \in W^{s,q}_{\mathrm{af}}(\Omega)$.

Note that $P_{N,\epsilon}$ is not the anisotropic Helmholtz projection on Ω. $P_{N,\epsilon}$ is the operator which maps from the L^q-vector fields into L^q-divergence-free vector fields with tangential trace. However, we find that the anisotropic Helmholtz projection is bounded from $L^q(\Omega)$ into itself by the same method of Lemma 2.9. Let $u \in C_0^\infty(\Omega)$. Then, we obtain the solution π_ϵ to the Neumann problem

$$\begin{cases} \Delta_\epsilon \pi_\epsilon &= \mathrm{div}_\epsilon\, u \quad \text{in} \quad \Omega,\\ \gamma_\pm \frac{\partial_3 \pi_\epsilon}{\epsilon} &= u\cdot\nu_\pm \quad \text{on} \quad \partial\Omega. \end{cases} \tag{2.26}$$

The anisotropic Helmholtz projection H_ϵ is defined by

$$H_\epsilon u = u - \nabla_\epsilon \pi_\epsilon.$$

In the case of the Dirichlet boundary condition, i.e. $\gamma_\pm u = 0$, the right hand side of the second equality of (2.26) is zero. Let us consider the L^q-boundedness of $\nabla_\epsilon \pi_\epsilon$, which implies the boundedness of the anisotropic Helmholtz projection. For the solution π^0 to the equation

$$\begin{cases} \partial_3^2 \pi^0(x_3)/\epsilon^2 &= \partial_3 \tilde{u}_3(x_3)/\epsilon, \quad x_3 \in (-1,1), \\ \partial_3 \pi^0(\pm 1)/\epsilon &= 0, \end{cases}$$

where $\tilde{u}_3 = \int_{\mathbb{T}^2} u_3 \, dx'$, we have

$$\nabla_\epsilon \pi^0 = (0,0,\tilde{u}_3)^T. \tag{2.27}$$

Let π_ϵ^1 and π_ϵ^2 be the solutions to

$$\Delta_\epsilon \pi_\epsilon^1 = E_0 \mathrm{div}_\epsilon u \quad \text{in} \quad \mathbb{T}^2 \times \mathbb{R}, \tag{2.28}$$

and

$$\begin{cases} \Delta_\epsilon \pi_\epsilon^2 &= 0 & \text{in} \quad \Omega, \\ \gamma_\pm \partial_3 \pi_\epsilon^2/\epsilon &= -\gamma_\pm \nu_\pm \cdot \nabla_\epsilon \Delta_\epsilon^{-1} E_0 \mathrm{div}_\epsilon \, u & \text{on} \quad \partial\Omega, \end{cases}$$

respectively, for $u \in C_0^\infty(\Omega)$ satisfying $\tilde{u} = 0$. Let us first consider (2.28). It follows from integration by parts

$$\begin{aligned} \mathcal{F}_{x_3}\mathcal{F}_{d,x'} E_0 \mathrm{div}_\epsilon u &= \mathcal{F}_{d,x'} \int_{-1}^{1} e^{-ix_3\xi_3} \left(\mathrm{div}_H u'(\cdot, x_3) + \frac{\partial_3 u_3(\cdot, x_3)}{\epsilon} \right) dx_3 \\ &= i \begin{pmatrix} n' & \xi_3/\epsilon \end{pmatrix}^T \cdot \mathcal{F}_{d,x'}(E_0 u) \\ &= \mathcal{F}_{x_3}\mathcal{F}_{d,x'} \mathrm{div}_\epsilon (E_0 u). \end{aligned}$$

This formula, the Mikhlin theorem and Proposition 2.1 imply

$$\left|\left|\nabla_\epsilon \pi_\epsilon^1\right|\right|_{L^q(\Omega)} \leq C||u||_{L^q(\Omega)}, \tag{2.29}$$

where $C > 0$ is independent of ϵ. Moreover, since $e_3 \cdot \nabla_\epsilon \Delta_\epsilon^{-1} \mathrm{div}_\epsilon$ is given by the left-hand side of (2.24), we can use the same method as Lemma 2.9 to get

$$\left|\left|\nabla_\epsilon \pi_\epsilon^2\right|\right|_{L^q(\Omega)} \leq C||u||_{L^q(\Omega)}, \tag{2.30}$$

where $C > 0$ is also independent of ϵ. The formula (2.27) and estimates (2.29) and (2.30) imply L^p-boundedness of the anisotropic Helmholtz projection on Ω. Summing up the above argument, we have

Lemma 2.11. *Let $1 < q < \infty$ and $0 < \epsilon \leq 1$. Then there exists a constant $C > 0$, which is independent of ϵ, such that*

$$||H_\epsilon f||_{L^q(\Omega)} \leq C||f||_{L^q(\Omega)}$$

for all $f \in L^q(\Omega)$.

Proposition 2.12. *Let $0 < \epsilon \leq 1$, $1 < q < \infty$, $0 < a < 1/2$, $z \in \mathbb{C}$ satisfying $-a < \mathrm{Re} z < 0$ and $0 < \theta < \pi/2$. Then there exists a constant $C = C(q, a, \theta)$, which is independent of ϵ, the solution π_3 to (III) with boundary data $(\gamma K_{\lambda,\epsilon} f \cdot \nu)\nu$ satisfies*

$$\left|\left| \frac{1}{2\pi i} \int_{\Gamma_\theta} (-\lambda)^z \nabla_\epsilon \pi_3 \, d\lambda \right|\right|_{L^q(\Omega)} \leq C e^{\theta |\mathrm{Im}\, z|} ||f||_{L^q(\Omega)}$$

for all $f \in L^q(\Omega)$.

Proof. In view of Remark 2.3, we may assume $\tilde{f} = 0$ without loss of generality. Since

$$\nabla_\epsilon \pi_3 = \Pi_\epsilon K_{\lambda,\epsilon} E_0 f \tag{2.31}$$

and the Cauchy integral commutes with Π_ϵ, the conclusion is obtained from Proposition 2.7 and Lemma 2.9. □

Proposition 2.13. *Let $0 < \epsilon \leq 1$, $1 < q < \infty$, $0 < \theta < \pi/2$ and $\lambda \in \Sigma_\theta$. Then there exists a constant $C = C(q, \theta)$, which is independent of ϵ, the solution π_3 to (III) with boundary data $(\gamma K_{\lambda,\epsilon} f \cdot \nu)\nu$ satisfies*

$$\left|\left|\nabla^2 \nabla_\epsilon \pi_3\right|\right|_{L^q(\Omega)} \leq C||f||_{L^q(\Omega)} \tag{2.32}$$

for all $f \in L^q(\Omega)$.

Proof. The estimate (2.32) is a direct consequence of (2.17), (2.31), Lemma 2.9 and Proposition 2.8. □

2.4. **Estimate for v_2.** Let us consider the equation (II) with tangential boundary data $g = (g_+, g_-)$. Set

$$y'_{\lambda,\epsilon}(n') = 2s_\lambda \left(I_2 + \frac{\epsilon|n'|}{s_\lambda} \frac{n' \otimes n'}{|n'|^2} \right),$$
$$y_{\lambda,\epsilon}(n') = \begin{pmatrix} y'_{\lambda,\epsilon}(n') & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.33}$$

Then, $y_{\lambda,\epsilon}$ satisfies

$$k'_{\lambda,\epsilon}(n', 0) y_{\lambda,\epsilon}(n') = J_2 := \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.34}$$

where $k'_{\lambda,\epsilon}$ is defined by (2.18). Let us define a multiplier operator $L_{\lambda,\epsilon}$ as

$$L_{\lambda,\epsilon} g(n', x_3) = \mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}} \mathcal{F}^{-1}_{d,n'} \left[e'_\lambda(n', 1 - x_3) y_{\lambda,\epsilon}(n') \mathcal{F}_{d,x'} g_+(n') \right]$$
$$+ \mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}} \mathcal{F}^{-1}_{d,n'} \left[e'_\lambda(n', -1 - x_3) y_{\lambda,\epsilon}(n') \mathcal{F}_{d,x'} g_-(n') \right], \tag{2.35}$$

where e'_λ is defined by (2.19). Let $p'_\epsilon(n', x_3)$ be a partial Fourier transform of the symbol of $\mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}}$ with respect to ξ_3. Then

$$L_{\lambda,\epsilon} g(n', \cdot) = \mathcal{F}^{-1}_{d,n'} \left[p'_\epsilon(n', \cdot) *_3 e'_\lambda(n', 1 - \cdot) y'_{\lambda,\epsilon}(n') \mathcal{F}_{d,x'} g_+(n') \right]$$
$$+ \mathcal{F}^{-1}_{d,n'} \left[p'_\epsilon(n', \cdot) *_3 e'_\lambda(n', -1 - \cdot) y'_{\lambda,\epsilon}(n') \mathcal{F}_{d,x'} g_-(n') \right], \tag{2.36}$$

where $\cdot *_3 \cdot$ is convolution with respect to x_3. We set

$$W_{\lambda,\epsilon} = P_{N,\epsilon} L_{\lambda,\epsilon}. \tag{2.37}$$

Then, $W_{\lambda,\epsilon} g$ is a solution to (II) with boundary data $\gamma W_{\lambda,\epsilon} g$. We first get the Fourier multiplier of $\gamma W_{\lambda,\epsilon}$. Next, we show the map $S_{\lambda,\epsilon} : g \mapsto \gamma W_{\lambda,\epsilon} g$ has a bounded inverse for large λ. Put

$$V_{\lambda,\epsilon} g = W_{\lambda,\epsilon} S^{-1}_{\lambda,\epsilon} g, \tag{2.38}$$

then, $V_{\lambda,\epsilon} g$ gives the solution to (II) with boundary data g.

Proposition 2.14. *Let $r > 0$ be sufficiently large. Let $1 < q < \infty$, $0 < \epsilon \le 1$, $s > 0$, $0 < \theta < \pi/2$ and $\lambda \in \Sigma_\theta$. Then, for $|\lambda| > r$, there exists a bounded operator $R_{\lambda,\epsilon}$ from $L^q(\Omega)$ into itself satisfying*

$$||R_{\lambda,\epsilon}||_{W^{s,q}_{\mathrm{af}}(\mathbb{T}^2) \to W^{s,q}_{\mathrm{af}}(\mathbb{T}^2)} \le \frac{C}{|\lambda|^{1/2}}, \tag{2.39}$$

and

$$||R_{\lambda,\epsilon}||_{W^{s,q}_{\mathrm{af}}(\mathbb{T}^2) \to W^{s+1,q}_{\mathrm{af}}(\mathbb{T}^2)} \le C, \tag{2.40}$$

where $C > 0$ is independent of ϵ, such that

$$-S^{-1}_{\lambda,\epsilon} = I + R_{\lambda,\epsilon}. \tag{2.41}$$

Proof. Let $g \in C^\infty(\mathbb{T}^2)$ be horizontal average-free. Since e'_λ is an even function with respect to x_3, we find from the change of variable that

$$p'_\epsilon(n', \cdot) *_3 e'_\lambda(n', 1 - \cdot) = \int_{\mathbb{R}} p'_\epsilon(n', \cdot - \zeta) e'_\lambda(n', 1 - \zeta) \, d\zeta$$
$$= - \int_{\mathbb{R}} p'_\epsilon(n', \eta) e'_\lambda(n', -1 + \cdot - \eta) \, d\eta$$
$$= -k'_{\lambda,\epsilon}(n', -1 + \cdot),$$

and similarly

$$p'_\epsilon(n', \cdot) *_3 e'_\lambda(n', -1 - \cdot) = -k'_{\lambda,\epsilon}(n', 1 + \cdot).$$

Thus, we find from (2.36) that

$$L_{\lambda,\epsilon} g(n', x_3) = \mathcal{F}^{-1}_{d,n'} \left[-k'_{\lambda,\epsilon}(n', -1 + x_3) y'_{\lambda,\epsilon}(n') \mathcal{F}_{d,x'} g_+(n') \right]$$
$$+ \mathcal{F}^{-1}_{d,n'} \left[-k'_{\lambda,\epsilon}(n', 1 + x_3) y'_{\lambda,\epsilon}(n') \mathcal{F}_{d,x'} g_-(n') \right]. \tag{2.42}$$

We apply $P_{N,\epsilon}$ to (2.42) to get

$$\begin{aligned} S_{\lambda,\epsilon} g &= \gamma_\pm W_{\lambda,\epsilon} g \\ &= -\mathcal{F}^{-1}_{d,n'}\left[k'_{\lambda,\epsilon}(n',-1\pm 1) y_{\lambda,\epsilon}(\xi')\mathcal{F}_{d,x'} g_+(n')\right] \\ &- \mathcal{F}^{-1}_{d,n'}\left[\alpha_{+,\epsilon}(n',\pm 1) e_3 \cdot k'_{\lambda,\epsilon}(n',2) y_{\lambda,\epsilon}(n')\mathcal{F}_{d,x'} g_-(n')\right] \\ &- \mathcal{F}^{-1}_{d,n'}\left[k'_{\lambda,\epsilon}(n',1\pm 1) y_{\lambda,\epsilon}(\xi')\mathcal{F}_{d,x'} g_-(n')\right] \\ &- \mathcal{F}^{-1}_{d,n'}\left[\alpha_{-,\epsilon}(n',\pm 1) e_3 \cdot k'_{\lambda,\epsilon}(n',-2) y_{\lambda,\epsilon}(n')\mathcal{F}_{d,x'} g_+(n')\right] \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{2.43}$$

Let us estimate I_1 and I_3. The identity (2.34) implies

$$\mathcal{F}^{-1}_{d,n'}\left[k'_{\lambda,\epsilon}(n',0) y_{\lambda,\epsilon}(n')\mathcal{F}_{d,x'} g_\pm(n')\right] = g_\pm. \tag{2.44}$$

We need to show the other terms are $O(1/|\lambda|^{1/2})$. By (2.18) and (2.33), we have

$$\begin{aligned} &k'_{\lambda,\epsilon}(\xi',\pm 2) y_{\lambda,\epsilon}(n') \\ &= e^{-s_\lambda}\left(J_2 + \frac{\epsilon|n'|}{s_\lambda}\frac{J_2 n\otimes J_2 n}{|n'|^2}\right) \\ &- J_2 n\otimes J_2 n \frac{\epsilon^2}{\lambda + (1-\epsilon^2)|n'|^2} e^{-2s_\lambda}\left(J_2 + \frac{\epsilon|n'|}{s_\lambda}\frac{J_2 n\otimes J_2 n}{|n'|^2}\right) \\ &- J_2 n\otimes J_2 n \frac{\epsilon^2}{\lambda + (1-\epsilon^2)|n'|^2}\frac{e^{-2\epsilon|n'|}}{\epsilon|n'|} s_\lambda \left(J_2 + \frac{\epsilon|n'|}{s_\lambda}\frac{J_2 n\otimes J_2 n}{|n'|^2}\right) \\ &=: II_1 + II_2 + II_3. \end{aligned}$$

We find from (2.9) and (2.10) in Proposition 2.6 and

$$\left[\frac{|\xi'|}{s_\lambda}\right]_{\mathcal{M}'} + \left[\frac{J_2\xi\otimes J_2\xi}{|\xi'|^2}\right]_{\mathcal{M}'} \le C, \quad \xi = (\xi',\xi_3)\in\mathbb{R}^3, \tag{2.45}$$

that

$$[II_1]_{\mathcal{M}'} \le Ce^{-c|\lambda|^{\frac{1}{2}}}, \quad [\,|\xi'|\,II_1]_{\mathcal{M}'} \le Ce^{-c|\lambda|^{\frac{1}{2}}}, \tag{2.46}$$

where we interpret that the multiplier II_1 is extended from $\mathbb{Z}^2$ to $\mathbb{R}^2$ in the canonical way. Since

$$\left[\frac{1}{\lambda + (1-\epsilon^2)|\xi'|^2}\right]_{\mathcal{M}'} \le \frac{C}{|\lambda|}, \tag{2.47}$$

by the same way as above we find

$$[II_2]_{\mathcal{M}'} \le \frac{Ce^{-c|\lambda|^{1/2}}}{|\lambda|}, \quad [\,|\xi'|\,II_2]_{\mathcal{M}'} \le Ce^{-c|\lambda|^{\frac{1}{2}}}, \tag{2.48}$$

for $\lambda\in\Sigma_\theta$, where constants $c, C>0$ are independent of ϵ. Note that II_3 has a little bit problem near $\epsilon = 0$ since, at this point, we can not use the decay of $e^{-2\epsilon|\xi'|}$ to obtain uniform boundedness of the Mikhlin constant. However, we can use the decay of $1/(\lambda + (1-\epsilon^2)|\xi'|^2)$ around $\epsilon = 0$. On the other hand, when ϵ is away from 0, we have no problem to use decay of $e^{-2\epsilon|\xi'|}$. Thus, combining this observation with Proposition 2.5, (2.45) and (2.47), we conclude that

$$[II_3]_{\mathcal{M}'} \le \frac{C}{|\lambda|^{\frac{1}{2}}}, \quad [\,|\xi'|\,II_3]_{\mathcal{M}'} \le C, \tag{2.49}$$

where $C>0$ is independent of ϵ. Thus we find from (2.46), (2.48) and (2.49) that

$$\begin{aligned} &||I_1 + I_3 - g_+ - g_-||_{L^q(\mathbb{T}^2)} \\ &\le \left|\left|\mathcal{F}^{-1}_{d,n'}\left[k'_{\lambda,\epsilon}(n',2) y_{\lambda,\epsilon}(n')\mathcal{F}_{d,x'} g(n')\right]\right|\right|_{L^q(\Omega)} \\ &+ \left|\left|\mathcal{F}^{-1}_{d,n'}\left[k'_{\lambda,\epsilon}(n',-2) y_{\lambda,\epsilon}(n')\mathcal{F}_{d,x'} g(n')\right]\right|\right|_{L^q(\Omega)} \\ &\le \frac{C}{|\lambda|^{\frac{1}{2}}}||g||_{L^q(\mathbb{T}^2)}. \end{aligned} \tag{2.50}$$

Next, we estimate I_2 and I_4. It follows from (2.18) that

$$
\begin{aligned}
& e_3 \cdot k'_{\lambda,\epsilon}(n',\pm 2) y_{\lambda,\epsilon}(n') \mathcal{F}_{d,x'} g_\mp(n') \\
&= e_3 \cdot \left[\frac{e^{-s_\lambda}}{2 s_\lambda} y_{\lambda,\epsilon}(n') \mathcal{F}_{d,x'} g_\mp - \begin{pmatrix} \eta'_{\lambda,\epsilon}(n',\pm 2) n' \otimes n' y'_{\lambda,\epsilon}(n') & 0 \\ -\partial_3 \eta'_{\lambda,\epsilon}(n',\pm 2) i n'^T y'_{\lambda,\epsilon}(n') & 0 \end{pmatrix} \mathcal{F}_{d,x'} g_\mp \right] \\
&= - \begin{pmatrix} -\partial_3 \eta'_{\lambda,\epsilon}(n',\pm 2) i n'^T y'_{\lambda,\epsilon}(n') & 0 \end{pmatrix} \mathcal{F}_{d,x'} g_\mp . \qquad (2.51)
\end{aligned}
$$

Recall $\partial_3 \eta_{\lambda,\epsilon}(n',\pm 2) = \frac{\epsilon^2}{\lambda + (1-\epsilon^2)|n'|^2} \frac{e^{-2s_\lambda} - e^{-2\epsilon|n'|}}{2}$. Then, we find from the first inequality of (2.9) and (2.47) that

$$
\begin{aligned}
& [(1+\epsilon|\xi'|)\, \partial_3 \eta_{\lambda,\epsilon}(\xi',\pm 2) y'_\lambda(\xi')]_{\mathcal{M}'} \\
&= \left[\frac{\epsilon^2}{\lambda + (1-\epsilon^2)|\xi'|^2} (1+\epsilon|\xi'|)\, s_\lambda \left(e^{-2s_\lambda} - e^{-2\epsilon|\xi'|} \right) \left(I_2 + \frac{\epsilon|\xi'|}{s_\lambda} \frac{\xi' \otimes \xi'}{|\xi'|^2} \right) \right]_{\mathcal{M}'} \\
&\le \frac{C}{|\lambda|^{\frac{1}{2}}},
\end{aligned}
$$

and

$$
[|\xi'|\,(1+\epsilon|\xi'|)\, \partial_3 \eta_{\lambda,\epsilon}(\xi',\pm 2) y'_\lambda(\xi')]_{\mathcal{M}'} \le C, \qquad (2.52)
$$

where $C > 0$ is independent of ϵ. The formula (2.21), estimates (2.10) and (2.11) lead to

$$
\left[\frac{\alpha_{\pm,\epsilon}(\xi',\pm 1)\epsilon|\xi'|}{1+\epsilon|\xi'|} \right]_{\mathcal{M}'} < \infty,
$$

uniformly on ϵ. We find from Proposition 2.1 that

$$
\begin{aligned}
& ||I_2 + I_4||_{L^q(\mathbb{T}^2)} \\
\le & \left|\left| \mathcal{F}^{-1}_{d,n'} \alpha_{+,\epsilon}(n',\pm 1) e_3 \cdot k'_{\lambda,\epsilon}(n',2) y_{\lambda,\epsilon}(n') \mathcal{F}_{d,x'} g_- \right|\right|_{L^q(\mathbb{T}^2)} \\
& + \left|\left| \mathcal{F}^{-1}_{d,n'} \alpha_{-,\epsilon}(n',\pm 1) e_3 \cdot k'_{\lambda,\epsilon}(n',-2) y_{\lambda,\epsilon}(n') \mathcal{F}_{d,x'} g_+ \right|\right|_{L^q(\mathbb{T}^2)} \\
\le & \frac{C}{|\lambda|^{\frac{1}{2}}} ||g||_{L^q(\mathbb{T}^2)}, \qquad (2.53)
\end{aligned}
$$

where C is independent of ϵ. Thus, taking $|\lambda|$ sufficiently large, clearly the choice of λ is also independent of ϵ, we can conclude by (2.43), (2.44), (2.50) and (2.53) that

$$
-S_{\lambda,\epsilon} = I + O(|\lambda|^{-1/2}).
$$

By the Neumann series argument we obtain (2.39) for $s = 0$. Moreover, we find from (2.46), (2.48), (2.49) and (2.52) that (2.40) holds for $s = 0$. Since ∂_j ($j = 1, 2$) commutes Fourier multiplier operators, we obtain (2.39) and (2.40) for $s > 0$. □

Proposition 2.15. *Let $1 < q < \infty$, $0 < \epsilon \le 1$, $0 < \theta < \pi/2$ and $\lambda \in \Sigma_\theta$. Then there exist $r > 0$ and a constant $C > 0$, which is independent of ϵ and λ, if $|\lambda| \ge r$, $V_{\lambda,\epsilon}$ defined by (2.38) satisfies*

$$
||V_{\lambda,\epsilon} g||_{L^q(\Omega)} \le C|\lambda|^{-1/2q} ||g||_{L^q(\partial\Omega)} \qquad (2.54)
$$

for all $g \in L^q(\partial\Omega)$ satisfying $\tilde{g} = 0$.

Proof. We take $r > 0$ so that $R_{\lambda,\epsilon}$ exists. Then $S^{-1}_{\lambda,\epsilon}$ is bounded on $L^q(\Omega)$. We find from the resolvent estimate for the Dirichlet Laplacian on Ω, see Lemma 5.3 in [1], and Proposition 2.14 that

$$
\left|\left| L_{\lambda,\epsilon} S^{-1}_{\lambda,\epsilon} g \right|\right|_{L^q(\Omega)} \le C|\lambda|^{-1/2q} ||g||_{L^q(\Omega)},
$$

where $C > 0$ is independent of ϵ. By Corollary 2.10, we obtain (2.54).

□

Proposition 2.16. *Let $1 < q < \infty$ and $0 < \epsilon \le 1$. Then there exists a constant $C > 0$, which is independent of ϵ, such that*

$$
\left|\left| \mathcal{F}^{-1}_{d,n'} \frac{1+\epsilon|n'|}{\epsilon|n'|} \left(e_3 \cdot \mathcal{F}_{d,x'} \left(\mathbb{P}^{\mathbb{T}^2 \times \mathbb{R}}_\epsilon E_0 f \right) \right) \right|\right|_{L^q(\Omega)} \le C||f||_{L^q(\Omega)} \qquad (2.55)
$$

for all $f \in L^q_{\mathrm{af}}(\Omega)$.

Proof. Since the symbol have poles at $\xi_3 = \pm i\epsilon|n'|$, we obtain its partial Fourier transform with respect to ξ_3 by the residue theorem. Thus, we have

$$
\begin{aligned}
&\mathcal{F}^{-1}_{d,n'}\frac{1}{\epsilon|n'|}\left(e_3\cdot\mathcal{F}_{d,x'}\left(\mathbb{P}^{\mathbb{T}^2\times\mathbb{R}}_\epsilon E_0 f\right)\right)\\
&=\frac{1}{2}\mathcal{F}^{-1}_{d,n'}\int_{-1}^{1}\frac{1}{\epsilon|n'|}\Big[e^{-|x_3-\zeta|\epsilon|n'|}i\epsilon n'\cdot\mathcal{F}_{d,x'}f'(n',\zeta)\\
&\qquad\qquad +e^{-|x_3-\zeta|\epsilon|n'|}\epsilon|n'|\mathcal{F}_{d,x'}f_3(n',\zeta)\Big]\,d\zeta.
\end{aligned}
$$

This formula and Proposition 2.1 imply

$$
\left|\left|\mathcal{F}^{-1}_{d,n'}\frac{1}{\epsilon|n'|}\left(e_3\cdot\mathcal{F}_{d,x'}\left(\mathbb{P}^{\mathbb{T}^2\times\mathbb{R}}_\epsilon E_0 f\right)\right)\right|\right|_{L^q(\mathbb{T}^2)}\le C||f||_{L^q(\mathbb{R}^2)},
$$

where C is independent of ϵ. Combining this estimate with the boundedness of $\mathbb{P}^{\mathbb{T}^2\times\mathbb{R}}_\epsilon$, we obtain (2.55). □

Let us show BIP for the solution operator for the equation (II).

Proposition 2.17. *Let $1<q<\infty$, $0<\epsilon\le 1$, $0<\theta<\pi/2$, $\lambda\in\Sigma_\theta$, $0<a<1/2$, and z satisfying $-a<\mathrm{Re}z<0$. Then there exists a constant $C=C(q,a,\theta)$, it holds that*

$$
\left|\left|\frac{1}{2\pi i}\int_{\Gamma_\theta}(-\lambda)^z V_{\lambda,\epsilon}\left[\gamma v_1-(\gamma v_1\cdot\nu)\nu\right]\,d\lambda\right|\right|_{L^q(\Omega)}\le Ce^{|\mathrm{Im}z|\theta}||f||_{L^q(\Omega)} \tag{2.56}
$$

for all $f\in L^q(\Omega)$, where $v_1=K_{\lambda,\epsilon}f$.

Proof. In view of Remark 2.3, we may assume $\tilde{f}=0$ without loss of generality. It holds by (2.17) that

$$
\gamma v_1-(\gamma v_1\cdot\nu)\nu=\gamma K_{\lambda,\epsilon}E_0 f-\gamma\Pi_\epsilon K_{\lambda,\epsilon}E_0 f.
$$

We find from this formula, (2.21), (2.35), (2.37), (2.38) and (2.41) that the integrand of the left hand side of (2.56) can be essentially written as

$$
\begin{aligned}
&P_{N,\epsilon}\mathbb{P}^{\mathbb{T}^2\times\mathbb{R}}_\epsilon\mathcal{F}^{-1}_{d,n'}\int_{-1}^{1}e'_\lambda(\xi',\pm1-x_3)y_{\lambda,\epsilon}(n')e'_\lambda(\xi',\pm1-\zeta)\\
&\qquad\qquad\times\mathcal{F}_{d,x'}\left(\mathbb{P}^{\mathbb{T}^2\times\mathbb{R}}_\epsilon E_0 f\right)(n',\zeta)d\zeta\\
&+P_{N,\epsilon}\mathbb{P}^{\mathbb{T}^2\times\mathbb{R}}_\epsilon\mathcal{F}^{-1}_{d,n'}\int_{-1}^{1}e'_\lambda(n',\pm1-x_3)y_{\lambda,\epsilon}(n')\alpha_{\epsilon,\pm}(n',\pm1)e'_\lambda(n',\pm1-\zeta)\\
&\qquad\qquad\times\frac{\epsilon|n'|}{1+\epsilon|n'|}\frac{1+\epsilon|n'|}{\epsilon|n'|}\mathcal{F}_{d,x'}\left(\mathbb{P}^{\mathbb{T}^2\times\mathbb{R}}_\epsilon E_0 f\right)(n',\zeta)d\zeta\\
&+W_{\lambda,\epsilon}R_{\lambda,\epsilon}\left[\gamma K_{\lambda,\epsilon}E_0 f-\gamma\Pi_\epsilon K_{\lambda,\epsilon}E_0 f\right]\\
&=:I_1+I_2+I_3,
\end{aligned} \tag{2.57}
$$

where $\pm$ should be take properly. It follows from (2.9) and (2.45) that

$$
\begin{aligned}
&\left[e'_\lambda(\xi',\pm1-x_3)y_{\lambda,\epsilon}(\xi')e'_\lambda(\xi',\pm1-\zeta)\right]_{\mathcal{M}'}\\
&=2\left[e^{-|\pm1-x_3|s_\lambda}\left(I_2+\frac{\epsilon|\xi'|}{s_\lambda}\frac{\xi'\otimes\xi'}{|\xi'|^2}\right)\frac{e^{-|\pm1-\zeta|s_\lambda}}{s_\lambda}\right]_{\mathcal{M}'}\\
&\le C\frac{e^{-c|\lambda|^{1/2}(|\pm1-x_3|+|\pm1-\zeta|)}}{|\lambda|^{1/2}}.
\end{aligned} \tag{2.58}
$$

Let $R > 0$ be large enough so that $S_{\lambda,\epsilon}^{-1}$ in Proposition 2.14 exists. Then we find from the change of integral curve around the origin to ensure $|\lambda| > R$ and Proposition 2.1 that

$$\begin{aligned}
&\left|\left|\frac{1}{2\pi i}\int_\Gamma (-\lambda)^z I_1 \; d\lambda\right|\right|_{L^q(\Omega)} \\
&\leq C\left|\left|\int_{-1}^1\int_\Gamma |\lambda^z| \frac{e^{-c|\lambda|^{1/2}(|x_3-a|+|\zeta-b|)}}{|\lambda|^{1/2}}\left|\left|\mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}} E_0 f(\cdot,\zeta)\right|\right|_{L^q(\mathbb{T}^2)} d\lambda \; d\zeta\right|\right|_{L^q(-1,1)} \\
&\leq Ce^{\theta|\mathrm{Im} z|}||f||_{L^q(\Omega)} \\
&+ C\left|\left|\int_{-1}^1\int_R^\infty e^{\theta|\mathrm{Im} z|} r^{\mathrm{Re} z-1/2} e^{-cr^{1/2}(|x_3-a|+|\zeta-b|)}||f(\cdot,\zeta)||_{L^q(\mathbb{T}^2)} dr d\zeta\right|\right|_{L^q(-1,1)} \\
&\leq Ce^{\theta|\mathrm{Im} z|}||f||_{L^q(\Omega)} + C_R e^{\theta|\mathrm{Im} z|}\left|\left|\int_{-1}^1 \frac{||f(\cdot,\zeta)||_{L^q(\mathbb{T}^2)}}{|x_3-a|+|\zeta-b|} dr\right|\right|_{L^q(\Omega)}
\end{aligned}$$

for some $a, b \in \{-1, 1\}$, where C and C_R are independent of ϵ. Applying Proposition 2.4, we obtain

$$\left|\left|\int_\Gamma (-\lambda)^z I_1 \; d\lambda\right|\right|_{L^q(\Omega)} \leq Ce^{\theta|\mathrm{Im} z|}||f||_{L^q(\Omega)}. \tag{2.59}$$

It follows from (2.19), (2.21), (2.33) and Proposition 2.6 that

$$\begin{aligned}
&\left[e'_\lambda(\xi', \pm 1 - x_3) y_{\lambda,\epsilon}(\xi')\alpha_{\epsilon,\pm}(\xi', \pm 1) e'_\lambda(\xi', \pm 1 - \zeta)\frac{\epsilon|\xi'|}{1+\epsilon|\xi'|}\right]_{\mathcal{M}'} \\
&\leq C\frac{e^{-c|\lambda|^{1/2}(|\pm 1 - x_3|+|\pm 1-\zeta|)}}{|\lambda|^{1/2}}.
\end{aligned} \tag{2.60}$$

Thus we find from Proposition 2.16 that

$$\begin{aligned}
&\left|\left|\frac{1}{2\pi i}\int_\Gamma (-\lambda)^z I_2 \; d\lambda\right|\right|_{L^q(\Omega)} \\
&\leq Ce^{\theta|\mathrm{Im} z|}\left|\left|\mathcal{F}_{d,n'}^{-1}\frac{1+\epsilon|n'|}{\epsilon|n'|}\mathcal{F}_{d,x'}\left(\mathbb{P}_\epsilon^{\mathbb{T}^2\times\mathbb{R}} E_0 f\right)(n',\zeta)\right|\right|_{L^q(\Omega)} \\
&\leq Ce^{\theta|\mathrm{Im} z|}||f||_{L^q(\Omega)}.
\end{aligned}$$

By Proposition 2.14, the trace theorem, Lemma 2.9 and the resolvent estimate for the Laplace operator on $\mathbb{T}^2 \times \mathbb{R}$, we have

$$\begin{aligned}
&||R_{\lambda,\epsilon}\gamma\left[K_{\lambda,\epsilon}E_0 f - \gamma\Pi_\epsilon K_{\lambda,\epsilon}E_0 f\right]||_{L^q(\partial\Omega)} \\
&\leq C|\lambda|^{-3/2+1/2q+\delta}||f||_{L^q(\Omega)}
\end{aligned}$$

for some small $\delta > 0$. The resolvent estimate for the Dirichlet Laplacian on Ω, see Lemma 5.3 in [1], and Lemma 2.9 imply

$$||P_{N,\epsilon}L_{\lambda,\epsilon}||_{L^q(\partial\Omega)\to L^q(\Omega)} \leq C|\lambda|^{-1/2q}$$

for some small $\delta > 0$. We find from the above two inequalities

$$||I_3||_{L^q(\Omega)} \leq C|\lambda|^{-3/2+\delta}||f||_{L^q(\Omega)}. \tag{2.61}$$

Thus we find from the change of integral line around the origin that

$$\left|\left|\frac{1}{2\pi i}\int_\Gamma (-\lambda)^z I_3 \; d\lambda\right|\right|_{L^q(\Omega)} \leq C||f||_{L^q(\Omega)},$$

□

where $C > 0$ is independent of ϵ.

Proof of Lemma 2.2. Lemma 2.2 is a direct consequence of Propositions 2.7, 2.12 and 2.17. □

We next prove Lemma 1.3 from Lemma 2.2. For this purpose we need further uniform estimate for the resolvent to compare $\|\nabla^2 u\|_{L^q(\Omega)}$ and $\|A_\epsilon u\|_{L^q(\Omega)}$. For resolvent estimates we begin with

Proposition 2.18. *Let $1 < q < \infty$, $0 < \epsilon \leq 1$, $0 < \theta < \pi/2$. Let $\lambda \in \Sigma_\theta$ be sufficiently large so that $S_{\lambda,\epsilon}^{-1}$ exists in Proposition 2.14. Then there exists a constant $C = C(q,\theta)$, it holds that*

$$\left|\left|\nabla^2 V_{\lambda,\epsilon}\left[\gamma v_1 - (\gamma v_1\cdot\nu)\nu\right]\right|\right|_{L^q(\Omega)} \leq C||f||_{L^q(\Omega)}$$

for all $f \in L^q(\Omega)$, where $v_1 = K_{\lambda,\epsilon} f$.

Proof. In view of Remark 2.3, we may assume $\tilde{f} = 0$ without loss of generality. It is enough to estimate the second derivative of the left-hand side of (2.57) in $L^q(\Omega)$. We find from (2.9) and (2.58) that

$$\begin{aligned}
&\left[|n'|^2\, e'_\lambda(\xi', \pm 1 - x_3) y_{\lambda,\epsilon}(\xi') e'_\lambda(\xi', \pm 1 - \zeta)\right]_{\mathcal{M}'} \\
&= 2\left[|\xi'|^2\, e^{-|\pm 1 - x_3| s_\lambda} \left(I_2 + \frac{\epsilon|\xi'|}{s_\lambda}\frac{\xi'\otimes\xi'}{|\xi'|^2}\right)\frac{e^{-|\pm 1-\zeta| s_\lambda}}{s_\lambda}\right]_{\mathcal{M}'} \\
&\leq \frac{C}{|\pm 1 - x_3| + |\pm 1 - \zeta|},
\end{aligned}$$

where $C > 0$ is independent of ϵ. Similarly, it follows from (2.33), (2.21) and Proposition 2.6 that

$$\begin{aligned}
&\left[|\xi'|^2\, e'_\lambda(\xi', \pm 1 - x_3) y_{\lambda,\epsilon}(\xi') \alpha_{\epsilon,\pm}(\xi', \pm 1) e'_\lambda(\xi', \pm 1 - \zeta)\frac{\epsilon|\xi'|}{1+\epsilon|\xi'|}\right]_{\mathcal{M}'} \\
&\leq \frac{C}{|\pm 1 - x_3| + |\pm 1 - \zeta|}.
\end{aligned}$$

Thus we find from Corollary 2.10 and Proposition 2.4 that

$$||\nabla_H \otimes \nabla_H I_j||_{L^q(\Omega)} \leq C\|f\|_{L^q(\Omega)}, \quad j = 1, 2,$$

where $\nabla_H = (\partial_1, \partial_2)^T$, I_j is defined in (2.57) and $C > 0$ is independent of ϵ. Since

$$\begin{aligned}
\partial_3 e'_\lambda(n', \pm 1 - x_3) &= \frac{\pm e^{-(1\mp x_3)s_\lambda}}{2}, \\
\partial_3^2 e'_\lambda(n', \pm 1 - x_3) &= \frac{s_\lambda e^{-(1\mp x_3)s_\lambda}}{2},
\end{aligned}$$

we can use the same way as above to get

$$||\nabla_H \partial_3 I_j||_{L^q(\Omega)} + \left|\left|\partial_3^2 I_j\right|\right|_{L^q(\Omega)} \leq C\|f\|_{L^q(\Omega)}, \quad j = 1, 2,$$

where I_j is defined in (2.57) and $C > 0$ is independent of ϵ. Propositions 2.8 and 2.14, Lemma 2.9 and the trace theorem imply

$$R_{\lambda,\epsilon}\left[\gamma K_{\lambda,\epsilon} E_0 f - \gamma \Pi_\epsilon K_{\lambda,\epsilon} E_0 f\right] \in W^{3-1/q,q}(\mathbb{T}^2)$$

and its norm is bounded uniformly on ϵ. By the definition of the operator $L_{\lambda,\epsilon}$, see (2.35), we have

$$L_{\lambda,\epsilon} R_{\lambda,\epsilon}\left[\gamma K_{\lambda,\epsilon} E_0 f - \gamma \Pi_\epsilon K_{\lambda,\epsilon} E_0 f\right]$$

solves the elliptic equations $\lambda u - \Delta u = 0$. Moreover, the boundary data belongs to $W^{3-1/q,q}(\mathbb{T}^2)$ by (2.19), (2.33) and Proposition 2.1. Thus we find from (2.38), Corollary 2.10 and smoothing effect of the solution operator to the elliptic equation that

$$\begin{aligned}
&||W_{\lambda,\epsilon} R_{\lambda,\epsilon}\left[\gamma K_{\lambda,\epsilon} E_0 f - \gamma \Pi_\epsilon K_{\lambda,\epsilon} E_0 f\right]||_{W^{2,q}(\Omega)} \\
&\leq C||L_{\lambda,\epsilon} R_{\lambda,\epsilon}\left[\gamma K_{\lambda,\epsilon} E_0 f - \gamma \Pi_\epsilon K_{\lambda,\epsilon} E_0 f\right]||_{W^{2,q}(\Omega)} \\
&\leq C||R_{\lambda,\epsilon}\left[\gamma K_{\lambda,\epsilon} E_0 f - \gamma \Pi_\epsilon K_{\lambda,\epsilon} E_0 f\right]||_{W^{2-1/q+\delta,q}(\mathbb{T}^2)} \\
&\leq C||f||_{L^q(\Omega)},
\end{aligned}$$

where $\delta > 0$ is small and C is independent of ϵ. □

Lemma 2.19. *Let $1 < q < \infty$, $0 < \epsilon \leq 1$, $0 < \theta < \pi/2$ and $\lambda \in \Sigma_\theta$ satisfying $|\lambda| > R$ for sufficiently large $R > 0$. Then there exists a constant $C = C(q, \theta)$ such that*

$$\left|\left|\nabla^2 (\lambda + A_\epsilon)^{-1} f\right|\right|_{L^q(\Omega)} \leq C||f||_{L^q(\Omega)}$$

for all $f \in L^q(\Omega)$.

Proof. This is a direct consequence of Propositions 2.8, 2.13 and 2.18. □

Lemma 2.20. *Let $1 < q < \infty$, $0 < \epsilon \leq 1$. Then there exists a constant $C = C(q)$ such that*

$$\|\nabla^2 u\|_{L^q(\Omega)} \leq C\|A_\epsilon u\|_{L^q(\Omega)}$$

for all $u \in D(A_\epsilon)$.

Proof of Lemma 1.3. Let u be a solution of (1.4). Our uniform BIP yields

$$\|\partial_t u\|_{\mathbb{E}_0(T)} + \|A_\epsilon u\|_{\mathbb{E}_0(T)} \le C\left(\|f\|_{\mathbb{E}_0(T)} + \|u_0\|_{X_\gamma}\right)$$

by the Dore-Venni theory, where $C > 0$ is independent of ϵ and T. Applying an a priori estimate Lemma 2.20, we can replace $\|A_\epsilon u\|_{\mathbb{E}_0(T)}$ by $\|\nabla^2 u\|_{\mathbb{E}_0(T)}$. Since (u, π) solves (1.4) and $\partial_t u$ and $\nabla^2 u$ are controlled, we are able to estimate $\|\nabla_\epsilon \pi\|_{\mathbb{E}_0(T)}$. This completes the proof of Lemma 1.3. □

It remains to prove Lemma 2.20. We first observe an a priori estimate slightly weaker than Lemma 2.20, which can be proved by using the resolvent estimate Lemma 2.19.

Proposition 2.21. *Let $1 < q < \infty$ and $0 < \epsilon \le 1$. There exists a unique solution $(u, \pi) \in D(A_\epsilon) \times L^q(\Omega)/\mathbb{R}$ to*

$$\begin{aligned} -\Delta u + \nabla_\epsilon \pi &= f && \text{in } \Omega, \\ \operatorname{div}_\epsilon u &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{2.62}$$

for $f \in L^q(\Omega)$, such that

$$\|\nabla^2 u\|_{L^q(\Omega)} + \|\nabla_\epsilon \pi\|_{L^q(\Omega)} \le C\|f\|_{L^q(\Omega)} + C\|u\|_{L^q(\Omega)},$$

where $C > 0$ is independent of ϵ and f.

Proof. The equations are equivalent to

$$\begin{aligned} \lambda_0 u - \Delta u + \nabla_\epsilon \pi &= f + \lambda_0 u && \text{in } \Omega, \\ \operatorname{div}_\epsilon u &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

for sufficiently large $\lambda_0 > 0$. We find from Lemma 2.19 that

$$\begin{aligned} \|\nabla^2 u\|_{L^q(\Omega)} &\le C\|\nabla^2(\lambda_0 + A_\epsilon)^{-1}\|_{L^q(\Omega)\to L^q(\Omega)}\|f + \lambda_0 u\|_{L^q(\Omega)} \\ &\le C\left(\|f\|_{L^q(\Omega)} + \lambda_0\|u\|_{L^q(\Omega)}\right) \end{aligned}$$

for some constant $C > 0$, which is independent of ϵ. The first equation in (2.62) implies

$$\begin{aligned} \|\nabla_\epsilon \pi\|_{L^q(\Omega)} &\le \|\nabla^2 u\|_{L^q(\Omega)} + \|f\|_{L^q(\Omega)} \\ &\le C\left(\|f\|_{L^q(\Omega)} + \lambda_0\|u\|_{L^q(\Omega)}\right). \end{aligned}$$

For uniqueness we multiply u with the first equation and integrating by parts yields $\nabla u = 0$. By the Poincaré inequality it implies $u = 0$. This argument works for $q \ge 2$ since Ω is bounded. Since $(\lambda_0 + A_\epsilon)^{-1}$ is compact in $L^q(\Omega)$, the Riesz-Schauder theorem implies that 0 is in resolvent since $\ker A_\epsilon = \{0\}$. In particular, (2.62) is uniquely solvable for any $f \in L^q(\Omega)$ for $q \ge 2$. By duality argument the solvability of $q \ge 2$ implies the uniqueness of (2.62) for $1 < q < 2$. Again by compactness of $(\lambda_0 + A_\epsilon)^{-1}$ the solvability for (2.62) follows. □

Proof of Lemma 2.20. Assume that the statement were false then there would exist a sequence $\{\epsilon_k\}_{k\in\mathbb{Z}_{\ge 1}}$, $(0 < \epsilon_k \le 1)$ and $u_k \in D(A_\epsilon)$ such that

$$\|\nabla^2 u_k\|_{L^q(\Omega)} > k\|f_k\|_{L^q(\Omega)}, \quad f_k = A_{\epsilon_k} u_k.$$

Since the problem is linear we may assume that

$$\|\nabla^2 u_k\|_{L^q(\Omega)} \equiv 1, \quad \|f_k\|_{L^q(\Omega)} \le \frac{1}{k} \to 0, \ (k \to 0).$$

By $A_{\epsilon_k} u_k = f_k$ and Proposition 2.21, we have

$$1 \le \alpha\left(\|f_k\|_{L^q(\Omega)} + \|u_k\|_{L^q(\Omega)}\right)$$

for some constant $\alpha > 0$, which is independent of ϵ_k. Letting $k \to \infty$ implies

$$\frac{1}{\alpha} \le \liminf_{k\to\infty} \|u_k\|_{L^q(\Omega)}. \tag{2.63}$$

By the Poincaré inequality for u_k our bound $\|\nabla u_k\|_{L^q(\Omega)}$ implies that u_k and ∇u_k are bounded in $L^q(\Omega)$. By Rellich's compactness theorem, we observe that $u_k \to u$ for some $u \in L^q(\Omega)$ strongly in $L^q(\Omega)$ by taking a subsequence. The estimate (2.63) implies that

$$\|u\|_{L^q(\Omega)} \ge \frac{1}{\alpha}.$$

We may assume $\epsilon_k \to \epsilon_* \in [0,1]$ and $u_k \to u$ as $k \to \infty$ by taking a subsequence. The situation is divided into two cases, i.e. $\epsilon_* = 0$ or $\epsilon_* > 0$. By definition,

$$\begin{array}{rcll} -\Delta u_k + \nabla_{\epsilon_k}\pi_k &=& f_k & \text{in } \Omega, \\ \operatorname{div}_\epsilon u_k &=& 0 & \text{in } \Omega, \\ u_k &=& 0 & \text{on } \partial\Omega, \end{array}$$

with some function π_k satisfying $\int_\Omega \pi_k dx = 0$. Since $\|\nabla^2 u_k\|_{L^q(\Omega)} \le 1$, we see that

$$\|\nabla_{\epsilon_k}\pi_k\|_{L^q(\Omega)} \le \|f\|_{L^q(\Omega)} + 1.$$

By the Poincaré inequality $\{\pi_k\}$ is bounded in $L^q(\Omega)$. By Rellich's compactness theorem we may assume $\pi_k \to \pi$ in $L^q(\Omega)$ for some $\pi \in L^q(\Omega)$ strongly by taking a subsequence. If $\epsilon_* = 0$, this implies π is independent of z. Since $\operatorname{div}_{\epsilon_k} u_k = 0$ and the vertical component $w_k = 0$ on $x_3 = \pm 1$, integration vertically on $(-1,1)$ yields that the horizontal limit v satisfies

$$\operatorname{div}_H \overline{v} = 0,$$

where $\operatorname{div}_H = \nabla_H \cdot$. Thus the horizontal component v satisfies the hydrostatic Stokes equations

$$\begin{array}{rcll} -\Delta u + \nabla_H \pi &=& 0 & \text{in } \Omega, \\ \operatorname{div}_H \overline{v} &=& 0 & \text{in } \Omega, \\ v &=& 0 & \text{on } \partial\Omega. \end{array}$$

Since we know the only possible $W^{2,q}$-solution is zero, so we conclude that $v = 0$. Since $\|\nabla v_k\|_{L^q(\Omega)}$ is bounded, $\operatorname{div}_{\epsilon_k}$-free condition implies that the horizontal limit w is independent of the vertical variable. By the boundary condition $w = 0$ at $x_3 = \pm 1$, this implies w must be zero. We thus observe that $u_k \to 0$ strongly in $L^q(\Omega)$, this contradicts $\|u\|_{L^q(\Omega)} \ge 1/\alpha > 0$. The case ϵ_* is easier since the limit satisfies the anisotropic Stokes equations

$$\begin{array}{rcll} -\Delta u + \nabla_{\epsilon_*} \pi &=& 0 & \text{in } \Omega, \\ \operatorname{div}_{\epsilon_*} u &=& 0 & \text{in } \Omega, \\ u &=& 0 & \text{on } \partial\Omega. \end{array}$$

By the uniqueness $u \equiv 0$ in Ω. This again contradicts $\|u\|_{L^q(\Omega)} \ge 1/\alpha > 0$. The proof of Lemma 2.20 is now complete. □

As a direct application of Lemma 1.3 we obtain

Corollary 2.22. *Let $p, q \in (1,\infty)$, $T > 0$, $F = (f_H, f_z) \in \mathbb{E}_0(T)$, $U_0 \in X_\gamma$ and $0 < \epsilon \le 1$. Then there is a unique solution $(U_\epsilon, P_\epsilon) \in \mathbb{E}_1(T) \times \mathbb{E}_0(T)$ to the equations*

$$\left\{ \begin{array}{rcll} \partial_t V - \Delta V + \nabla_H P &=& f_H & \text{in } \Omega \times (0,T), \\ \partial_t(\epsilon W) - \Delta(\epsilon W) + \frac{\partial_3 P}{\epsilon} &=& f_z & \text{in } \Omega \times (0,T), \\ \operatorname{div}_H V + \frac{\partial_3}{\epsilon}(\epsilon W) &=& 0 & \text{in } \Omega \times (0,T), \\ U &=& 0 & \text{on } \partial\Omega \times (0,T), \\ U(0) &=& U_0 & \text{in } \Omega, \end{array} \right. \tag{2.64}$$

where P is unique up to a constant. Moreover, there exist constants $C > 0$ and $C_T > 0$, which is independent of ϵ, such that

$$\|(V, \epsilon W)\|_{\mathbb{E}_1(T)} + \|\nabla_\epsilon P\|_{\mathbb{E}_0(T)} \le C\|F\|_{\mathbb{E}_0(T)} + C_T \|(V_0, \epsilon W_0)\|_{X_\gamma}. \tag{2.65}$$

Proof. Lemma 1.3 implies there exists a solution $(\tilde{U}, \tilde{P})$ to (1.4) with initial data U_0 such that

$$\|\widetilde{U}\|_{\mathbb{E}_1(T)} + \|\nabla_\epsilon \widetilde{P}\|_{\mathbb{E}_0(T)} \le C\|F\|_{\mathbb{E}_0(T)} + C_T\|U_0\|_{X_\gamma}.$$

Set

$$V = \tilde{V}, \quad W = \epsilon \tilde{W}, \quad P = \tilde{P}.$$

Then (U, P) is the desired solution satisfying (2.65). Note that $\lim_{T\to\infty} C_T < \infty$. □

3. Non-linear Estimates and Regularity of w

In this section, we introduce some Propositions on non-linear estimates to estimate F_H, F_z and F and on the regularity of w, which is the vertical component of the solution to the primitive equations. Although the following Propositions have already proved in [10], we introduce them to explain our restriction for p and q and for the reader's convenience.

Proposition 3.1 ([10])**.** *Let $T > 0$, $p, q \in (1,\infty)$ such that $2/3p + 1/q \le 1$. Then there exist a constant $C = C(p,q) > 0$ such that*

$$||v_1 \partial_x v_2||_{\mathbb{E}_0(T)} \le C||v_1||_{\mathbb{E}_1(T)}||v_2||_{\mathbb{E}_1(T)}$$

for all $v_1, v_2 \in \mathbb{E}_1(T)$.

Proposition 3.2 ([10]). *Let $T > 0$ and $z \in (-1, 1)$. Let $p, q \in (1, \infty)$ such that $1/p + 1/q \le 1$. Then there exist a constant $C = C(p, q) > 0$ such that*

$$||w_1 \partial_3 v_2||_{\mathbb{E}_0(T)} \le C ||v_1||_{\mathbb{E}_1(T)} ||v_2||_{\mathbb{E}_1(T)}$$

for all $v_1, v_2 \in \mathbb{E}_2(T)$ and $w_1 := -\int_z^{-1} \mathrm{div}_H v_1 \, d\zeta$.

The restriction for p and q in our theorem is due to Propositions 3.1 and 3.2.

Let us show $w \in \mathbb{E}_1(T)$. In our previous paper [10], we first derive the equation which w satisfies by applying $\int_{-1}^{x_3} \mathrm{div}_H \, \cdot \, d\zeta$ to the equations v satisfies. Then, estimating the corresponding nonlinear terms and applying the maximal regularity principle, we obtain $w \in \mathbb{E}_1(T)$. Note that, in the present paper, we invoke additional regularity for v to deal with the trace of the second derivative.

Although, in [13], the authors treat higher order regularity of the solution to the primitive equations, they do not explicitly write the maximal regularity in fractional Sobolev spaces. However, it is easy to modify their proof to get the maximal regularity in the fractional Sobolev spaces. In [14], the argument to get H^∞-calculus of hydrostatic Stokes operator is based on H^∞-calculus for the Laplace operator and perturbations arguments. Since the Laplace operator admits H^∞-calculus in fractional Sobolev spaces, it is not difficult to establish H^∞-calculus of the hydrostatic Stokes operator in fractional Sobolev spaces. We also find local well-posedness of the primitive equations in fractional maximal regularity space $W^{1,p}(0, T; W^{s,q}(\Omega)) \cap L^p(0, T; W^{2+s,q}(\Omega))$ for $s > 1/q$ in the same way [13] to get local well-posedness, namely, use Lemma 6.1, Corollary 6.2 and Theorem 5.1 in [13].

Remark 3.3. It is already known that $v \in \mathbb{E}_1(T)$ with initial data $v_0 \in X_\gamma$ by Giga, et al. [14].

Proof of Lemma 1.4. Integrating (PE) both sides over $(-1, 1)$, we find $(\overline{v}, \overline{\pi})$ satisfy

$$\begin{array}{rcll} \partial_t \overline{v} - \Delta \overline{v} + \nabla_H \overline{\pi} & = & -\int_{-1}^1 v \cdot \nabla_H v + w \partial_3 v \, d\zeta & \\ & & +(\partial_3 v)|_{x_3=-1}^{x_3=1} & \text{in} \quad \Omega \times (0, T), \\ \mathrm{div}_H \overline{v} & = & 0 & \text{in} \quad \Omega \times (0, T), \\ \overline{v} & = & 0 & \text{on} \quad \partial\Omega \times (0, T), \\ \overline{v}(0) & = & \overline{v_0} & \text{in} \quad \Omega. \end{array} \tag{3.1}$$

Put $\tilde{v} = v - \overline{v}$. Then, $\tilde{u} = (\tilde{v}, w)$ satisfies

$$\begin{array}{rcll} \partial_t \tilde{v} - \Delta \tilde{v} & = & -\tilde{v} \cdot \nabla_H \tilde{v} - w \partial_z \tilde{v} - \overline{v} \cdot \nabla_H \tilde{v} - \tilde{v} \cdot \nabla_H \overline{v} & \\ & & -\frac{1}{2} \int_{-1}^1 \tilde{v} \cdot \nabla_H \tilde{v} - (\mathrm{div}_H \, \tilde{v}) \, d\zeta & \\ & & +\frac{1}{2} (\partial_3 v)|_{x_3=-1}^{x_3=1} & \text{in} \quad \Omega \times (0, T), \\ \mathrm{div}_H \, \tilde{v} + \partial_z w & = & 0 & \text{in} \quad \Omega \times (0, T), \\ \tilde{v} & = & 0 & \text{on} \quad \partial\Omega \times (0, T), \\ \tilde{v}(0) & = & v(0) - \overline{v_0} & \text{in} \quad \Omega. \end{array} \tag{3.2}$$

Note that the pressure term no longer appears in the above equations and

$$\mathrm{div}_H \, \tilde{v} + \partial_3 w = 0.$$

Applying $-\mathrm{div}_H$ to (3.2) and integrating over $(-1, x_3)$ with respect to vertical variable, we find

$$\begin{aligned} &\partial_t w - \Delta w \\ &= \partial_z \mathrm{div}_H \tilde{v}|_{x_3=-1} - \int_{-1}^{x_3} \frac{1}{2} \mathrm{div}_H \left[(\partial_3 v)|_{x_3=-1}^{x_3=1} \right] d\zeta \\ &+ \int_{-1}^{x_3} \mathrm{div}_H (-\tilde{v} \cdot \nabla_H \tilde{v} - w \partial_\zeta \tilde{v} - \overline{v} \cdot \nabla_H \tilde{v} - \tilde{v} \cdot \nabla_H \overline{v}) d\zeta \\ &- \frac{1}{2} \int_{-1}^{x_3} \mathrm{div}_H \int_{-1}^1 \tilde{v} \cdot \nabla_H \tilde{v} - (\mathrm{div}_H \, \tilde{v}) \tilde{v} \, d\zeta \, d\eta \\ &=: I_1 + I_2 + I_3, \end{aligned}$$

with initial data w_0. Since $v_0 \in X_\gamma \cap B^s_{q,p}(\Omega)$ for $s > 2 - 2/p + 1/q$, we have $v \in \mathbb{E}_1(T) \cap L^p(0, T; W^{2+1/q+\delta,q}(\Omega))$ for some $\delta > 0$ by [13] and [14] , and thus $\|I_1\|_{\mathbb{E}_0(T)} \le C$ for some $C > 0$. We use integration by parts to get

$$\begin{aligned} I_2 &= \tilde{v} \cdot \nabla_H w - w \mathrm{div}_H v + \overline{v} \cdot \nabla_H w \\ &+ \int_{-1}^{x_3} \partial_j \tilde{v} \cdot \nabla_H \tilde{v}_j - (\partial_\zeta \tilde{v} \cdot \nabla_H w) + \nabla_H w \cdot \partial_\zeta \tilde{v} - \partial_\zeta w \, \mathrm{div}_H \tilde{v} \, d\zeta \\ &+ \int_{-1}^{x_3} \partial_j \overline{v} \cdot \nabla_H \tilde{v}_j + \partial_j \tilde{v} \cdot \nabla_H \overline{v}_j \, d\zeta. \end{aligned}$$

for $j=1,2$. We can apply Propositions 3.1 and 3.2 to I_2 to get

$$||I_2||_{\mathbb{E}_0(T)} \le C\left(\|w\|_{\mathbb{E}_1(T)}||\tilde{v}||_{\mathbb{E}_1(T)} + ||\tilde{v}||^2_{\mathbb{E}_1(T)}\right). \tag{3.3}$$

Similarly, we have

$$||I_3||_{\mathbb{E}_0(T)} \le C\left(\|w\|_{\mathbb{E}_1(T)}||\tilde{v}||_{\mathbb{E}_1(T)} + ||\tilde{v}||^2_{\mathbb{E}_1(T)}\right). \tag{3.4}$$

Note that constants in (3.3) and (3.4) are independent of T since constants in Propositions 3.1 and 3.2 are independent of T. Thus we find from the maximal regularity of the heat equation, implicit function theorem and Neumann series argument, which is the same way as in Proposition 4.8 in [10], that

$$\|w\|_{\mathbb{E}_1(T)} \le C$$

for some $C>0$. □

4. Justification of the Hydrostatic approximation and Global-well-posedness of the anisotropic Navier-Stokes Equations

Let us prove our main theorem.

Proof of Theorem 1.1. Let $\mathcal{T}>0$. Let C_1 be the maximum of constants C in Propositions 3.1 and 3.2, (2.65) and the constant in the trace theorem. Let us construct a solution $(V_\epsilon, \epsilon W_\epsilon)$ to (1.1) with zero initial data on $[0,\mathcal{T}]$. Set $(u_\epsilon, p_\epsilon) := (v+V_\epsilon, w+W_\epsilon, p+P_\epsilon)$, then this is the desired solution to (SNS). We denote by $\|\cdot\|_{\mathbb{E}_1(mT,(m+1)T)}$ and $\|\cdot\|_{\mathbb{E}_0(mT,(m+1)T)}$ the $\mathbb{E}_1$-norm and $\mathbb{E}_0$-norm on the time interval $[mT,(m+1)T]$, respectively. Let us take $0<T\le 1$ satisfying $\mathcal{T}=NT$ for sufficiently large integer N and

$$\|u\|_{\mathbb{E}_1(mT,(m+1)T)} \le \frac{1}{10C_1}, \tag{4.1}$$

for all integer $m\in(1,N)$. This choice of T is clearly independent of ϵ. We divide the time interval $[0,\mathcal{T}]$ into $\cup_{m=0}^N[mT,(m+1)T]$. Put $F=F(V_\epsilon,W_\epsilon,u) := (F_H(V_\epsilon,W_\epsilon,u)), F_z(V_\epsilon,W_\epsilon,u))$ be the left hand side of (1.1). We denote by $\mathcal{R}(F,U_0)=(\mathcal{R}^u(F,U_0),\mathcal{R}^p(F,U_0))=(U,P)$ the solution to (2.64) with initial data U_0. Set inductively

$$\begin{aligned}
&U_{\epsilon,1}=\mathcal{R}^u(F(0,u),0), \quad P_{\epsilon,1}=\mathcal{R}^p(F(0,u),0),\\
&U_{\epsilon,j+1}=\mathcal{R}^u(F(U_j,u),0), \quad P_{\epsilon,j+1}=\mathcal{R}^p(F(U_j,u),0).
\end{aligned}$$

Propositions 3.1, 3.2 and Corollary 2.22 lead to

$$\begin{aligned}
&\|\left(V_{\epsilon,j+1},\epsilon W_{\epsilon,j+1}\right)\|_{\mathbb{E}_1(T)} + \|\nabla_\epsilon P_{\epsilon,j+1}\|_{\mathbb{E}_0(T)}\\
&\le C_1T^\eta\left(\|u\|_{\mathbb{E}_1(T)}\|\left(V_{\epsilon,j},\epsilon W_{\epsilon,j}\right)\|_{\mathbb{E}_1(T)} + \|\left(V_{\epsilon,j},\epsilon W_{\epsilon,j}\right)\|^2_{\mathbb{E}_1(T)}\right)\\
&+\epsilon C_1T^\eta\left(\|u\|_{\mathbb{E}_1(T)}+\|u\|^2_{\mathbb{E}_1(T)}\right).
\end{aligned} \tag{4.2}$$

This quadratic inequality and (4.1) imply

$$\|\left(V_{\epsilon,j},\epsilon W_{\epsilon,j}\right)\|_{\mathbb{E}_1(T)} + \|\nabla_\epsilon P_{\epsilon,j}\|_{\mathbb{E}_0(T)} \le 2\epsilon C^* \tag{4.3}$$

for $C^*=\left(1/4C_1+1/16C_1^2\right)$ and small $\epsilon>0$. Put

$$\begin{aligned}
&\widetilde{U}_{\epsilon,j}=U_{\epsilon,j+1}-U_{\epsilon,j}\,(j\ge 1), \quad \widetilde{U}_{\epsilon,0}=U_{\epsilon,1},\\
&\widetilde{P}_{\epsilon,j}=P_{\epsilon,j+1}-P_{\epsilon,j}\,(j\ge 1), \quad \widetilde{P}_{\epsilon,0}=P_{\epsilon,1}.
\end{aligned}$$

Then seeking the equation which $(\widetilde{U}_{\epsilon,j},\widetilde{P}_{\epsilon,j})$ satisfies and applying Propositions 3.1, 3.2 and Corollary 2.22, we have

$$\begin{aligned}
&\|(\widetilde{V}_{\epsilon,j+1},\epsilon\widetilde{W}_{\epsilon,j+1})\|_{\mathbb{E}_1(T)} + \|\nabla_\epsilon\tilde{P}_{j+1}\|_{\mathbb{E}_0(T)}\\
&\le C_1T^\eta\left(\|\left(V_{\epsilon,j},\epsilon W_{\epsilon,j}\right)\|_{\mathbb{E}_1(T)} + \|\left(V_{\epsilon,j+1},\epsilon W_{\epsilon,j+1}\right)\|_{\mathbb{E}_1(T)}\right.\\
&\qquad\left.+2\|u\|_{\mathbb{E}_1(T)}\right)\|(\widetilde{V}_{\epsilon,j},\epsilon\widetilde{W}_{\epsilon,j})\|_{\mathbb{E}_1(T)}\\
&\le\frac{3}{4}\left(\|(\widetilde{V}_{\epsilon,j},\epsilon\widetilde{W}_{\epsilon,j})\|_{\mathbb{E}_1(T)} + \|\nabla_\epsilon\tilde{P}_j\|_{\mathbb{E}_0(T)}\right).
\end{aligned}$$

Thus $(U_\epsilon,P_\epsilon) := (\lim_{j\to\infty}U_j, \lim_{j\to\infty}P_j) = (\sum_{j=0}\tilde{U}_{\epsilon,j}, \sum_{j=0}\tilde{P}_{\epsilon,j})$ exists on $[0,T]$ and satisfies

$$\|\left(V_\epsilon,\epsilon W_\epsilon\right)\|_{\mathbb{E}_1(T)} + \|\nabla_\epsilon P_\epsilon\|_{\mathbb{E}_0(T)} \le 2\epsilon C^*. \tag{4.4}$$

By construction (U_ϵ, P_ϵ) satisfies (1.1) on $[0,T]$. Moreover, by trace theorem there exists a constant $C_{tr} > 0$ such that

$$\| (V_\epsilon(T), \epsilon W_\epsilon(T)) \|_{X_\gamma} \le C_{tr} \| (V_\epsilon, \epsilon W_\epsilon) \|_{E_1(0,T)} \le 2\epsilon C^* C_{tr}. \tag{4.5}$$

Next let us construct the solution to (1.1) on $[T, 2T]$ with initial data $U_\epsilon(T)$. By (4.5), we have $\|U_\epsilon(T)\|_{X_\gamma} \le 2\epsilon C^* C_{tr}$. Put $a_{\epsilon,1} = (b_{\epsilon,1}, c_{\epsilon,1}) = \mathcal{R}^u(0, U_\epsilon(T))$ and $\pi_{\epsilon,1} = \mathcal{R}^p(0, U_\epsilon(T))$. Corollary 2.22 implies

$$\| (b_{\epsilon,1}, \epsilon c_{\epsilon,1}) \|_{\mathbb{E}_1(T,2T)} + \|\nabla_\epsilon \pi_{\epsilon,1}\|_{\mathbb{E}_0(T,2T)} \le 2\epsilon C^* C_{tr} C_T. \tag{4.6}$$

Let the vector field $a_\epsilon = (b_\epsilon, c_\epsilon)$ be the solution to

$$\begin{cases} \partial_t b_\epsilon - \Delta b_\epsilon + \nabla_H \pi_\epsilon &= F_H(b_{1,\epsilon} + b_\epsilon, c_{\epsilon,1} + c_\epsilon, u), \\ \partial_t(\epsilon c_\epsilon) - \Delta(\epsilon c_\epsilon) + \frac{\partial_3}{\epsilon}\pi_\epsilon &= \epsilon F_z(b_{1,\epsilon} + b_\epsilon, c_{\epsilon,1} + c_\epsilon, u), \\ \operatorname{div} a_\epsilon &= 0, \\ a_\epsilon(T) &= 0. \end{cases} \tag{4.7}$$

Then $U_\epsilon = a_{\epsilon,1} + a_\epsilon$ and $P_\epsilon = \pi_{\epsilon,1} + \pi_\epsilon$ is a solution to ((1.1) with initial data $U_\epsilon(T)$. Let us construct the solution to (2.64). Let $F(b_{1,\epsilon}+b_\epsilon, c_{\epsilon,1}+c_\epsilon, u) = (F_H(b_{1,\epsilon} + b_\epsilon, c_{\epsilon,1} + c_\epsilon, u), \epsilon F_z(b_{1,\epsilon} + b_\epsilon, c_{\epsilon,1} + c_\epsilon, u))$ Set inductively

$$\begin{aligned} a_{\epsilon,j+1} &= a_{\epsilon,1} + \mathcal{R}^u(F(b_{1,\epsilon} + b_{\epsilon+j}, c_{\epsilon,1} + c_{\epsilon+j}, u), 0), \\ \pi_{\epsilon,j+1} &= \mathcal{R}^p(F(b_{1,\epsilon} + b_{\epsilon+j}, c_{\epsilon,1} + c_{\epsilon+j}, u), 0), \end{aligned}$$

for $j \ge 1$. Applying Propositions 3.1, 3.2 and Corollary 2.22 to (4.7), we find

$$\begin{aligned} &\| (b_{\epsilon,j+1}, \epsilon c_{\epsilon,j+1}) \|_{\mathbb{E}_1(T,2T)} + \|\nabla_\epsilon \pi_{\epsilon,j+1}\|_{\mathbb{E}_0(T,2T)} \\ &\le C_1 T^\eta \|u\|_{\mathbb{E}_1(T,2T)} \| (b_{\epsilon,1} + b_{\epsilon,j}, \epsilon(c_{\epsilon,1} + c_{\epsilon,j})) \|_{\mathbb{E}_1(T,2T)} \\ &+ C_1 T^\eta \| (b_{\epsilon,1} + b_{\epsilon,j}, \epsilon(c_{\epsilon,1} + c_{\epsilon,j})) \|^2_{\mathbb{E}_1(T,2T)} \\ &+ \epsilon C_1 T^\eta \left[\|u\|_{\mathbb{E}_1(T,2T)} + \|u\|^2_{\mathbb{E}_1(T,2T)} \right] \\ &\le C_1 T^\eta \| (b_{\epsilon,j}, \epsilon c_{\epsilon,j}) \|^2_{\mathbb{E}_1(T,2T)} \\ &+ C_1 T^\eta \left(\|u\|_{\mathbb{E}_1(T,2T)} + 2\| (b_{\epsilon,1}, \epsilon c_{\epsilon,1}) \|_{\mathbb{E}_1(T,2T)} \right) \| (b_{\epsilon,j}, \epsilon c_{\epsilon,j}) \|_{\mathbb{E}_1(T,2T)} \\ &+ \epsilon C_1 T^\eta \left[\|u\|_{\mathbb{E}_1(T,2T)} \| (b_{\epsilon,1}, \epsilon c_{\epsilon,1}) \|_{\mathbb{E}_1(T,2T)} + \| (b_{\epsilon,1}, \epsilon c_{\epsilon,1}) \|^2_{\mathbb{E}_1(T,2T)} \right] \\ &+ \epsilon C_1 T^\eta \left[\|u\|_{\mathbb{E}_1(T,2T)} + \|u\|^2_{\mathbb{E}_1(T,2T)} \right]. \end{aligned}$$

If we take ϵ so small that

$$\|a_{\epsilon,1}\|_{\mathbb{E}_1(T,2T)} \le 2\epsilon C^* C_{tr} C_T \le \frac{1}{8C_1},$$

we have

$$\begin{aligned} &\| (b_{\epsilon,j+1}, \epsilon c_{\epsilon,j+1}) \|_{\mathbb{E}_1(T,2T)} + \|\nabla_\epsilon \pi_{\epsilon,j+1}\|_{\mathbb{E}_0(T,2T)} \\ &\le C_1 \| (b_{\epsilon,j}, \epsilon c_{\epsilon,j}) \|^2_{\mathbb{E}_1(T,2T)} + \frac{1}{2} \| (b_{\epsilon,j}, \epsilon c_{\epsilon,j}) \|_{\mathbb{E}_1(T,2T)} + \epsilon C^* C_T C_{tr} + \epsilon C^* \\ &\le C_1 \| (b_{\epsilon,j}, \epsilon c_{\epsilon,j}) \|^2_{\mathbb{E}_1(T,2T)} + \frac{1}{2} \| (b_{\epsilon,j}, \epsilon c_{\epsilon,j}) \|_{\mathbb{E}_1(T,2T)} + \epsilon C^*(1 + C_T C_{tr}). \end{aligned}$$

Thus, we have by induction

$$\| (b_{\epsilon,j}, \epsilon c_{\epsilon,j}) \|_{\mathbb{E}_1(T,2T)} + \|\nabla_\epsilon \pi_{\epsilon,j}\|_{\mathbb{E}_0(T,2T)} \le 2\epsilon C^*(1 + C_T C_{tr})$$

for all $j \ge 1$. Set

$$\begin{aligned} \tilde{a}_{\epsilon,j} &= a_{\epsilon,j+1} - a_{\epsilon,j} \quad (j \ge 1), \quad \tilde{a}_{\epsilon,0} = a_{\epsilon,0}, \\ \tilde{\pi}_{\epsilon,j} &= \pi_{\epsilon,j+1} - \pi_{\epsilon,j} \quad (j \ge 1), \quad \tilde{\pi}_{\epsilon,0} = \pi_{\epsilon,0}. \end{aligned}$$

Applying Propositions 3.1, 3.2 and Corollary 2.22 to the equations that

$$(\tilde{a}_{\epsilon,j+1}, \tilde{\pi}_{\epsilon,j+1})$$

satisfies, we find

$$\begin{aligned}
&\|(\tilde{b}_{\epsilon,j+1}, \epsilon\tilde{c}_{\epsilon,j+1})\|_{\mathbb{E}_1(T,2T)} + \|\nabla_\epsilon \tilde{\pi}_{\epsilon,j+1}\|_{\mathbb{E}_0(T,2T)} \\
&\le C_1 T^\eta \left(\| (b_{\epsilon,j}, \epsilon c_{\epsilon,j}) \|_{\mathbb{E}_1(T,2T)} + \| (b_{\epsilon,j+1}, \epsilon c_{\epsilon,j+1}) \|_{\mathbb{E}_1(T,2T)} + 2\|u\|_{\mathbb{E}_1(T,2T)}\right) \\
&\qquad \times \|(\tilde{b}_{\epsilon,j}, \epsilon\tilde{c}_{\epsilon,j},)\|_{\mathbb{E}_1(T,2T)} \\
&\le \left[C_1 \left(\| (b_{\epsilon,j}, \epsilon c_{\epsilon,j}) \|_{\mathbb{E}_1(T,2T)} + \| (b_{\epsilon,j+1}, \epsilon c_{\epsilon,j+1}) \|_{\mathbb{E}_1(T,2T)}\right) + \frac{1}{2}\right] \\
&\qquad \times \|(\tilde{b}_{\epsilon,j}, \epsilon\tilde{c}_{\epsilon,j},)\|_{\mathbb{E}_1(T,2T)} \\
&\le \frac{3}{4}\|(\tilde{b}_{\epsilon,j}, \epsilon\tilde{c}_{\epsilon,j},)\|_{\mathbb{E}_1(T,2T)}.
\end{aligned}$$

The last inequality holds if ϵ is sufficiently small. Thus,

$$(a_\epsilon, \pi_\epsilon) := (\lim_{j\to\infty} a_{\epsilon,j}, \lim_{j\to\infty} \pi_{\epsilon,j}) = (\sum_{j=0} \tilde{a}_{\epsilon,j}, \sum_{j=0} \tilde{\pi}_{\epsilon,j})$$

exists and satisfies (4.7) such that

$$\| (b_\epsilon, \epsilon c_\epsilon) \|_{\mathbb{E}_1(T,2T)} + \|\nabla_\epsilon \pi_\epsilon\|_{\mathbb{E}_0(T,2T)} \le 2\epsilon C^*(1 + C_T C_{tr}).$$

The functions (U_ϵ, P_ϵ) solves (1.1) on the time interval $[T, 2T]$ with initial data $U_\epsilon(T)$ such that

$$\begin{aligned}
&\| (V_\epsilon, \epsilon W_\epsilon) \|_{\mathbb{E}_1(T)} + \|\nabla_\epsilon P_\epsilon\|_{\mathbb{E}_0(T)} \\
&\le \| (b_{\epsilon,1}, \epsilon c_{\epsilon,1}) \|_{\mathbb{E}_1(T,2T)} + \| (b_\epsilon, \epsilon c_\epsilon) \|_{\mathbb{E}_1(T,2T)} \\
&+ \|\nabla_\epsilon \pi_{\epsilon,1}\|_{\mathbb{E}_0(T,2T)} + \|\nabla_\epsilon \pi_\epsilon\|_{\mathbb{E}_0(T,2T)} \\
&\le C_T\|U_\epsilon(T)\|_{X_\gamma} + 2\epsilon C^*(1 + C_{tr}C_T) \le 2\epsilon C^*(1 + 2C_{tr}CT).
\end{aligned}$$

By induction, the solution (U_ϵ, P_ϵ) constructed by the same way on the time interval $[mT, (m+1)T]$ satisfies

$$\begin{aligned}
&\| (V_\epsilon, \epsilon W_\epsilon) \|_{\mathbb{E}_1(mT,(m+1)T)} + \|\nabla_\epsilon P_\epsilon\|_{\mathbb{E}_0(mT,(m+1)T)} \\
&\le 2\epsilon C^* \left[1 + 3C_T C_{tr} \left(1 + 3C_T C_{tr}(\cdots)\right)\right] =: 2\epsilon\alpha_j.
\end{aligned}$$

Since $\mathcal{T}$ is finite, this induction ends in finite steps. Thus we conclude

$$\| (V_\epsilon, \epsilon W_\epsilon) \|_{\mathbb{E}_1(\mathcal{T})} + \|\nabla_\epsilon P_\epsilon\|_{\mathbb{E}_1(\mathcal{T})} \le 2\epsilon \sum_{1\le j\le N} \alpha_j.$$

□

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