Three-Term Relations for ${}_3F_2(1)$*

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Abstract

For the hypergeometric function of unit argument ${}_3F_2(1)$ we prove the existence and uniqueness of three-term contiguous relations with arbitrary integer shifts. We show that not only the original ${}_3F_2(1)$ function but also other five functions related to it satisfy one and the same three-term relation. This fact together with the uniqueness mentioned above provides three-term relations with a group symmetry of order 72.

1 Introduction

If $p \le q+1$ then the hypergeometric function ${}_pF_q$ admits $(q+2)$-term contiguous relations (see Rainville [8]). Under certain conditions, they may reduce to ones with only a smaller number of terms. In the case of ${}_3F_2$ the general contiguous relations are four-term ones, while Kummer observed that it was possible to obtain three-term contiguous relations for ${}_3F_2$ when the argument was 1, that is, for ${}_3F_2(1)$; see Andrews *et al.* [1, §3.7]. Bailey [2] gave a procedure to produce those relations using differential equations and Wilson [11] gave a simpler method.

A contiguous function in the narrow sense is a function obtained from the original ${}_pF_q$ by altering one of the parameters by ± 1. A three-term contiguous relation in the narrow sense is then a linear relation between the original ${}_pF_q$ and two other functions contiguous to it in the above sense. In the case of ${}_3F_2(1)$ there are a total of twelve such relations, excluding the ones derived by permuting numerator or denominator parameters, a complete list of which can be found in Wilson [11, formulas (13)–(24)]. We refer to them as *basic* three-term relations.

We can also speak of contiguous relations in the wider sense where the parameters may differ by arbitrary integers. A linear relation among three contiguous functions in the wider sense is referred to as a *general* three-term relation. We may safely say that the study of basic three-term relations for ${}_3F_2(1)$ is finished by the works of Bailey [2] and Wilson [11]. This is not the case with general three-term relations. It is often said that a general three-term relation can be obtained by an iterative application of the basic ones, but this procedure involves

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some nontrivial issues that should be taken seriously and there remains something new for the subject. We shall establish such results (Theorem 1.1) regarding the existence, uniqueness and simultaneousness of general three-term relations for ${}_3F_2(1)$, where the last property means that one three-term relation is commonly satisfied by certain six functions associated with the original ${}_3F_2(1)$ function. As a corollary to the uniqueness and simultaneousness we are also able to obtain a group symmetry of order 72 on three-term relations. The results of this article are fundamental in developing a general theory of ${}_3F_2(1)$ continued fractions in the article [6].

For the Gauss hypergeometric function ${}_2F_1(\boldsymbol{a}; z) := {}_2F_1(a, b; c; z)$ with parameters $\boldsymbol{a} := (a, b; c) \in \mathbb{C}^3$, Vidunas [10] considered three-term relations representing ${}_2F_1(\boldsymbol{a}+\boldsymbol{k}; z)$ for $\boldsymbol{k} \in \mathbb{Z}^3$ in terms of ${}_2F_1(\boldsymbol{a}; z)$ and ${}_2F_1(\boldsymbol{a} + \boldsymbol{e}_1; z)$, where $\boldsymbol{e}_1 := (1, 0; 0)$. He showed the existence and uniqueness of three-term relations of this form [10, Theorem 1.1] and obtained simultaneousness for them [10, formulas (19)–(23)]. There is a similar approach that carries over when the particular shift vector $\boldsymbol{e}_1$ is replaced by an arbitrary nonzero integer vector. To discuss similar issues for ${}_3F_2(1)$, however, we shall develop a quite different method that works for every integer shift. As for ${}_2F_1$, Ebisu [4, 5] gave a useful formula for three-term relations expressing ${}_2F_1(\boldsymbol{a} + \boldsymbol{k}; z)$ in terms of ${}_2F_1(\boldsymbol{a}; z)$ and ${}_2F_1(\boldsymbol{a} + \mathbf{1}; z)$ with $\mathbf{1} := (1, 1; 1)$, derived symmetry on them from simultaneousness and moreover applied these results to special values.

Recall that the hypergeometric series ${}_3F_2(\boldsymbol{a}; z)$ is a power series of z defined by

$$ {}_3F_2(\boldsymbol{a}; z) := \sum_{k=0}^{\infty} \frac{(a_0, k)\,(a_1, k)\,(a_2, k)}{(1, k)\,(a_3, k)\,(a_4, k)}\, z^k, \qquad (a, k) := \frac{\Gamma(a + k)}{\Gamma(a)}, \tag{1} $$

with complex parameters $\boldsymbol{a} = (a_0, a_1, a_2; a_3, a_4) \in \mathbb{C}^5$, which are often denoted by

$$ \boldsymbol{a} := \begin{pmatrix} a_0, & a_1, & a_2 \\ & a_3, & a_4 \end{pmatrix} = \begin{pmatrix} a_0, & a_1, & a_2 \\ & b_1, & b_2 \end{pmatrix}. $$

Throughout we will freely switch between (a_3, a_4) and (b_1, b_2) according to the situation.

In this article it is more convenient to work with a renormalized version of the series (1):

$$ {}_3f_2(\boldsymbol{a}; z) := \sum_{k=0}^{\infty} \frac{\Gamma(a_0 + k)\,\Gamma(a_1 + k)\,\Gamma(a_2 + k)}{\Gamma(1 + k)\,\Gamma(a_3 + k)\,\Gamma(a_4 + k)}\, z^k = \frac{\Gamma(a_0)\,\Gamma(a_1)\,\Gamma(a_2)}{\Gamma(a_3)\,\Gamma(a_4)}\, {}_3F_2(\boldsymbol{a}; z). \tag{2} $$

If $\boldsymbol{a}$ satisfies $a_i \notin \mathbb{Z}_{\le 0}$, $i = 0, 1, 2, 3, 4$, then (2) is a well-defined power series in z with non-vanishing leading coefficient. To ensure this condition for all $\boldsymbol{a} + \boldsymbol{p}$ with $\boldsymbol{p} \in \mathbb{Z}^5$ we assume

$$ a_0,\ a_1,\ a_2,\ a_3,\ a_4 \notin \mathbb{Z} \qquad \text{for} \quad \boldsymbol{a} = (a_0, a_1, a_2; a_3, a_4) \in \mathbb{C}^5. \tag{3} $$

It is well known that ${}_3f_2(\boldsymbol{a}; z)$ converges in $|z| < 1$ and solves a Fuchsian differential equation

$$ L(\boldsymbol{a})\, y := \{\, \theta(\theta + b_1 - 1)(\theta + b_2 - 1) - z\,(\theta + a_0)(\theta + a_1)(\theta + a_2)\,\}\, y = 0, \quad \theta := z\tfrac{d}{dz}, \tag{4} $$

the third-order hypergeometric equation, whose Riemann scheme is given by

$$ \left\{ \begin{matrix} z = 0 & z = 1 & z = \infty \\ 1 - b_0 & 0 & a_0 \\ 1 - b_1 & 1 & a_1 \\ 1 - b_2 & s(\boldsymbol{a}) & a_2 \end{matrix} \right\}, \qquad s(\boldsymbol{a}) := b_1 + b_2 - a_0 - a_1 - a_2, \tag{5} $$

$$\sigma_0^{(0)}(\boldsymbol{a}) := \boldsymbol{a} = \begin{pmatrix} a_0, & a_1, & a_2 \\ & b_1, & b_2 \end{pmatrix}$$

$$\sigma_1^{(0)}(\boldsymbol{a}) = \tau_1(\boldsymbol{a}) := \begin{pmatrix} a_0+1-b_1, & a_1+1-b_1, & a_2+1-b_1 \\ & 2-b_1, & b_2+1-b_1 \end{pmatrix}$$

$$\sigma_2^{(0)}(\boldsymbol{a}) = \tau_2(\boldsymbol{a}) := \begin{pmatrix} a_0+1-b_2, & a_1+1-b_2, & a_2+1-b_2 \\ & b_1+1-b_2, & 2-b_2 \end{pmatrix}$$

$$\sigma_0^{(\infty)}(\boldsymbol{a}) = \sigma_0(\boldsymbol{a}) := \begin{pmatrix} a_0, & a_0+1-b_1, & a_0+1-b_2 \\ & a_0+1-a_1, & a_0+1-a_2 \end{pmatrix}$$

$$\sigma_1^{(\infty)}(\boldsymbol{a}) = \sigma_1(\boldsymbol{a}) := \begin{pmatrix} a_1+1-b_1, & a_1, & a_1+1-b_2 \\ & a_1+1-a_0, & a_1+1-a_2 \end{pmatrix}$$

$$\sigma_2^{(\infty)}(\boldsymbol{a}) = \sigma_2(\boldsymbol{a}) := \begin{pmatrix} a_2+1-b_2, & a_2+1-b_1, & a_2 \\ & a_2+1-a_1, & a_2+1-a_0 \end{pmatrix}$$

Table 1: Six parameter involutions (including identity).

where $b_0 := 1$ by convention and $s(\boldsymbol{a})$ is referred to as the *parametric excess* of $\boldsymbol{a}$.

To describe other solutions to equation (4) we introduce six parameter involutions $\sigma_i^{(\nu)}$, $i = 0, 1, 2$, $\nu = 0, \infty$, as in Table 1 and impose condition (3) for all of them, which becomes

$$a_i, \ a_i - a_j \not\in \mathbb{Z}, \qquad i, j = 0, 1, 2, 3, 4, \ i \neq j. \tag{6}$$

Let $\mathcal{S}^{(\nu)}(\boldsymbol{a})$ be the space of local solutions to equation (4) around $z = \nu \in \{0, \infty\}$. Under condition (6) none of the local exponent differences at $z = 0$ is an integer, so the local monodromy operator $\mathrm{M}^{(0)} : \mathcal{S}^{(0)}(\boldsymbol{a}) \circlearrowleft$ has three distinct eigenvalues $\lambda_i^{(0)}(\boldsymbol{a}) := \exp(-2\mathrm{i}\,\pi\, b_i)$, $i = 0, 1, 2$, where $\mathrm{i} := \sqrt{-1}$, and the corresponding eigen-solutions to (4) are given by

$$y_i^{(0)}(\boldsymbol{a}; z) := z^{1-b_i}\, {}_3f_2(\sigma_i^{(0)}(\boldsymbol{a}); z), \qquad i = 0, 1, 2. \tag{7}$$

Similarly, under condition (6) none of the local exponent differences at $z = \infty$ is an integer, so the local monodromy operator $\mathrm{M}^{(\infty)} : \mathcal{S}^{(\infty)}(\boldsymbol{a}) \circlearrowleft$ has three distinct eigenvalues $\lambda_i^{(\infty)}(\boldsymbol{a}) := \exp(2\mathrm{i}\,\pi\, a_i)$, $i = 0, 1, 2$, and the corresponding eigen-solutions to (4) are given by

$$y_i^{(\infty)}(\boldsymbol{a}; z) := e^{\mathrm{i}\pi s(\boldsymbol{a})}\, z^{-a_i}\, {}_3f_2(\sigma_i^{(\infty)}(\boldsymbol{a}); 1/z), \qquad i = 0, 1, 2, \tag{8}$$

where $s(\boldsymbol{a})$ is defined in (5); the constant factor $\exp(\mathrm{i}\,\pi\, s(\boldsymbol{a}))$ plays no role in differential equation (4), but it will be important later when we discuss contiguous operators (see Lemma 2.1). As a summary the six functions in (7)-(8) are characterized by the eigen-relations

$$\mathrm{M}^{(\nu)}\, y_i^{(\nu)}(\boldsymbol{a}; z) = \lambda_i^{(\nu)}(\boldsymbol{a})\, y_i^{(\nu)}(\boldsymbol{a}; z) \qquad i = 0, 1, 2, \ \nu = 0, \infty, \tag{9}$$

uniquely up to nonzero constant multiples, where $\lambda_i^{(\nu)}(\boldsymbol{a})$ is invariant under any $\mathbb{Z}^5$-shift of $\boldsymbol{a}$.

Our main concern in this article is the hypergeometric series ${}_3f_2(\boldsymbol{a}) := {}_3f_2(\boldsymbol{a}; 1)$ of unit argument $z = 1$ as well as its six companions (including itself):

$$y_i^{(\nu)}(\boldsymbol{a}) := y_i^{(\nu)}(\boldsymbol{a}; 1), \qquad i = 0, 1, 2, \quad \nu = 0, \infty, \tag{10}$$

where ${}_3f_2(\boldsymbol{a}) = y_0^{(0)}(\boldsymbol{a})$. It is well known that ${}_3f_2(\boldsymbol{a})$ is convergent if and only if

$$\operatorname{Re} s(\boldsymbol{a}) > 0, \tag{11}$$

in which case the convergence is absolute and uniform in any compact subset so that the series ${}_3f_2(\boldsymbol{a})$ defines a holomorphic function in domain (11) with genericity condition (3). Observe that the parametric excess is invariant under the six parameter changes in Table 1,

$$s(\boldsymbol{a}) = s(\sigma_i^{(\nu)}(\boldsymbol{a})), \qquad i = 0, 1, 2, \ \nu = 0, \infty,$$

so all the series in (10) are simultaneously convergent under the single condition (11) together with genericity condition (6). We shall denote by ${}_3h_2(\boldsymbol{a}; z)$ any function in (7)-(8) and by ${}_3h_2(\boldsymbol{a})$ any function in (10) respectively. The aim of this article is to establish the following.

Theorem 1.1 *Let ${}_3h_2(\boldsymbol{a})$ be any member of the six functions in* (10). *If $\boldsymbol{p}$, $\boldsymbol{q} \in \mathbb{Z}^5$ are distinct shift vectors then the following assertions hold true:*

(1) *${}_3h_2(\boldsymbol{a}+\boldsymbol{p})$ and ${}_3h_2(\boldsymbol{a}+\boldsymbol{q})$ are linearly independent over the field $\mathbb{C}(\boldsymbol{a})$.*

(2) *There exist unique rational functions $u(\boldsymbol{a})$, $v(\boldsymbol{a}) \in \mathbb{Q}(\boldsymbol{a})$ such that*

$$ {}_3h_2(\boldsymbol{a}) = u(\boldsymbol{a})\, {}_3h_2(\boldsymbol{a}+\boldsymbol{p}) + v(\boldsymbol{a})\, {}_3h_2(\boldsymbol{a}+\boldsymbol{q}). \tag{12}$$

(3) *The rational functions $u(\boldsymbol{a})$ and $v(\boldsymbol{a})$ are common for all choices of ${}_3h_2(\boldsymbol{a})$, that is, equation* (12) *is a simultaneous three-term relation for all functions in* (10).

(4) *There is a systematic recipe to determine $u(\boldsymbol{a})$ and $v(\boldsymbol{a})$ in finite steps* (Recipe 5.4).

(5) *Three-term relation* (12) *admits an $S_2 \ltimes (S_3 \times S_3)$-symmetry of order seventy-two.*

Three-term relation (12) is interesting only when $\boldsymbol{p}$ and $\boldsymbol{q}$ are both different from zero, but the theorem itself is of course true even when one of them is zero. Obviously, for any triple of mutually distinct vectors $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r} \in \mathbb{Z}^5$, we can construct a general three-term relation

$$u(\boldsymbol{a})\, {}_3h_2(\boldsymbol{a}+\boldsymbol{p}) + v(\boldsymbol{a})\, {}_3h_2(\boldsymbol{a}+\boldsymbol{q}) + w(\boldsymbol{a})\, {}_3h_2(\boldsymbol{a}+\boldsymbol{r}) = 0, \tag{13}$$

having polynomial coefficients $u(\boldsymbol{a})$, $v(\boldsymbol{a})$, $w(\boldsymbol{a}) \in \mathbb{Q}[\boldsymbol{a}]$ without common factors, from a three-term relation of the form (12) by multiplying it by the common denominator of its coefficients and translating $\boldsymbol{a}$ by an integer vector; (13) is unique up to a constant multiple.

Remark 1.2 If $K \supset \mathbb{Q}$ is a field of functions in $\boldsymbol{a} = (a_0, a_1, a_2; a_3, a_4)$ such that

$$a_0,\ a_1,\ a_2,\ a_3,\ a_4 \text{ are algebraically independent over } K, \tag{14}$$

then assertion (1) of Theorem 1.1 remains true over the field $K(\boldsymbol{a})$. For example this condition is fulfilled by the field $K = \mathcal{M}_\Lambda$ of Λ-periodic meromorphic functions on $\mathbb{C}^5$ for a lattice Λ, where a *lattice* in $\mathbb{C}^5$ is the $\mathbb{Z}$-linear span of five vectors that form a $\mathbb{C}$-linear basis of $\mathbb{C}^5$.

This remark is of particular interest because the six functions in (10) satisfy some linear relations over the periodic function field $\mathcal{M}_\Lambda$ with period lattice $\Lambda = (2\mathbb{Z})^5$ but not over the field $\mathbb{C}$. For example the three functions associated with $z = 0$ satisfy a linear relation

$$c_0(\boldsymbol{a})\, y_0^{(0)}(\boldsymbol{a}) + c_1(\boldsymbol{a})\, y_1^{(0)}(\boldsymbol{a}) + c_2(\boldsymbol{a})\, y_2^{(0)}(\boldsymbol{a}) = 0, \tag{15}$$

where the coefficients $c_i(\boldsymbol{a}) \in \mathcal{M}_\Lambda$ are given in terms of trigonometric functions by

$$c_i(\boldsymbol{a}) := c(\sigma_i^{(0)}(\boldsymbol{a})) \qquad \text{with} \quad c(\boldsymbol{a}) := \frac{\sin \pi a_0 \cdot \sin \pi a_1 \cdot \sin \pi a_2}{\sin \pi b_1 \cdot \sin \pi b_2}.$$

One way to prove relation (15) is to take the difference of two connection formulas

$$e^{-\mathrm{i}\pi s(\boldsymbol{a})} y_0^{(\infty)}(\boldsymbol{a}) = \frac{\sin \pi a_1 \cdot \sin \pi a_2}{\sin \pi b_j \cdot \sin \pi (b_k - a_0)}\, y_0^{(0)}(\boldsymbol{a}) - \frac{\sin \pi (a_1 - b_j) \cdot \sin \pi (a_2 - b_j)}{\sin \pi b_j \cdot \sin \pi (b_k - a_0)}\, y_j^{(0)}(\boldsymbol{a}), \tag{16}$$

with $(j, k) = (1, 2)$, $(2, 1)$. Either formula in (16) is a renormalized version of Thomae's second fundamental relation in [9, page 72], whereas another representation of it with a different rescaling can be found in Beyer, Louck and Stein [3, formulas (2.7) and (2.8ab)].

In Theorem 1.1 the symmetry on three-term relations in assertion (5) is just a corollary to the uniqueness and simultaneousness in assertions (2) and (3), but this issue is interesting in its own light and hence discussed in §6, which contains a detailed account of assertion (5).

2 Differential Operators

We take the differential operator approach to the contiguous relations as in Bailey [2], but we are more systematic with this method and keep it minimum for the sake of conciseness. Let

$$\begin{aligned} &\boldsymbol{e}_0 := (1,0,0;0,0), \qquad \boldsymbol{e}_1 := (0,1,0;0,0), \qquad \boldsymbol{e}_2 := (0,0,1;0,0), \\ &\boldsymbol{e}_3 := (0,0,0;1,0), \qquad \boldsymbol{e}_4 := (0,0,0;0,1), \qquad \boldsymbol{1} := (1,1,1;1,1). \end{aligned}$$

Our approach to the ${}_3F_2$ contiguous relations is based on the following.

Lemma 2.1 *For any member ${}_3h_2(\boldsymbol{a}; z)$ of the six functions in* (7)-(8)*, we have*

$$(\theta + a_i)\, {}_3h_2(\boldsymbol{a}; z) = {}_3h_2(\boldsymbol{a} + \boldsymbol{e}_i; z) \qquad i = 0, 1, 2, \tag{17a}$$

$$(\theta + a_i - 1)\, {}_3h_2(\boldsymbol{a}; z) = {}_3h_2(\boldsymbol{a} - \boldsymbol{e}_i; z) \qquad i = 3, 4, \tag{17b}$$

$$\partial\, {}_3h_2(\boldsymbol{a}; z) = {}_3h_2(\boldsymbol{a} + \boldsymbol{1}; z) \qquad \partial := \tfrac{\partial}{\partial z}. \tag{17c}$$

Proof. Using the basic commutation relations $\theta \cdot z = z \cdot (\theta + 1)$ and $\theta \cdot \partial = \partial \cdot (\theta - 1)$, we can easily observe that the differential operator $L(\boldsymbol{a})$ in (4) admits commutation relations

$$\begin{aligned} L(\boldsymbol{a} + \boldsymbol{e}_i)\,(\theta + a_i) &= (\theta + a_i)\, L(\boldsymbol{a}) \qquad && i = 0, 1, 2, \\ L(\boldsymbol{a} - \boldsymbol{e}_i)\,(\theta + a_i - 1) &= (\theta + a_i - 2)\, L(\boldsymbol{a}) \qquad && i = 3, 4, \\ z\, L(\boldsymbol{a} + \boldsymbol{1})\, \partial &= (\theta - 1)\, L(\boldsymbol{a}). \end{aligned}$$

This implies that for any $j = 0,1,2$, $\nu = 0,\infty$ we have on the one hand,

$$\begin{aligned} (\theta + a_i)\, y_j^{(\nu)}(\boldsymbol{a}; z) &\in \mathcal{S}^{(\nu)}(\boldsymbol{a} + \boldsymbol{e}_i; z) \qquad i = 0,1,2, \\ (\theta + a_i - 1)\, y_j^{(\nu)}(\boldsymbol{a}; z) &\in \mathcal{S}^{(\nu)}(\boldsymbol{a} - \boldsymbol{e}_i; z) \qquad i = 3,4, \\ \partial\, y_j^{(\nu)}(\boldsymbol{a}; z) &\in \mathcal{S}^{(\nu)}(\boldsymbol{a} + \boldsymbol{1}; z). \end{aligned}$$

On the other hand, since the local monodromy operator $\mathrm{M}^{(\nu)}$ is commutative with differential operators with rational coefficients in z and the eigenvalues $\lambda_j^{(\nu)}(\boldsymbol{a})$ are $\mathbb{Z}^5$-periodic in $\boldsymbol{a}$,

$$\begin{aligned} \mathrm{M}^{(\nu)} \cdot (\theta + a_i)\, y_j^{(\nu)}(\boldsymbol{a}; z) &= \lambda_j^{(\nu)}(\boldsymbol{a} + \boldsymbol{e}_i) \cdot (\theta + a_i)\, y_j^{(\nu)}(\boldsymbol{a}; z) \qquad i = 0,1,2, \\ \mathrm{M}^{(\nu)} \cdot (\theta + a_i - 1)\, y_j^{(\nu)}(\boldsymbol{a}; z) &= \lambda_j^{(\nu)}(\boldsymbol{a} - \boldsymbol{e}_i) \cdot (\theta + a_i - 1)\, y_j^{(\nu)}(\boldsymbol{a}; z) \qquad i = 3,4, \\ \mathrm{M}^{(\nu)} \cdot \partial\, y_j^{(\nu)}(\boldsymbol{a}; z) &= \lambda_j^{(\nu)}(\boldsymbol{a} + \boldsymbol{1}) \cdot \partial\, y_j^{(\nu)}(\boldsymbol{a}; z). \end{aligned}$$

Since eigen-solutions (7)-(8) are uniquely determined by eigen-relations (9) up to constant multiples, there must be constants $\alpha_{ij}^{(\nu)}(\boldsymbol{a})$ and $\beta_j^{(\nu)}(\boldsymbol{a})$ with respect to z such that

$$\begin{aligned} (\theta + a_i)\, y_j^{(\nu)}(\boldsymbol{a}; z) &= \alpha_{ij}^{(\nu)}(\boldsymbol{a}) \cdot y_j^{(\nu)}(\boldsymbol{a} + \boldsymbol{e}_i; z) \qquad i = 0,1,2, \\ (\theta + a_i - 1)\, y_j^{(\nu)}(\boldsymbol{a}; z) &= \alpha_{ij}^{(\nu)}(\boldsymbol{a}) \cdot y_j^{(\nu)}(\boldsymbol{a} - \boldsymbol{e}_i; z) \qquad i = 3,4, \\ \partial\, y_j^{(\nu)}(\boldsymbol{a}; z) &= \beta_j^{(\nu)}(\boldsymbol{a}) \cdot y_j^{(\nu)}(\boldsymbol{a} + \boldsymbol{1}; z). \end{aligned} \tag{18}$$

These constants can be determined by equating the leading coefficients of both sides in (18). Thanks to the normalization in (2) and the constant factor $\exp(\mathrm{i}\,\pi\, s(\boldsymbol{a}))$ in (8) we have

$$\alpha_{ij}^{(\nu)}(\boldsymbol{a}) = \beta_j^{(\nu)}(\boldsymbol{a}) = 1 \qquad i = 0,1,2,3,4,\ j = 0,1,2,\ \nu = 0,\infty \quad \text{(simultaneously one !)},$$

which together with (18) yields the desired formulas (17). □

The differential equation (4) is written in terms of θ. Expressed in ∂, it is equivalent to

$$\begin{aligned} M(\boldsymbol{a})\, y := \big[(1-z)\, z^2\, \partial^3 + z\, \{\, (a_3 + a_4 + 1) - (\varphi_1(\boldsymbol{a}) + 3)\, z\,\}\, \partial^2 \\ + \{\, a_3 a_4 - (\, \varphi_2(\boldsymbol{a}) + \varphi_1(\boldsymbol{a}) + 1\,)\, z\,\}\, \partial - \varphi_3(\boldsymbol{a})\big]\, y = 0, \end{aligned} \tag{19}$$

where $\varphi_i(\boldsymbol{a})$ is the i-th elementary symmetric polynomial in the numerator parameters a_0, a_1, a_2,

$$\varphi_1(\boldsymbol{a}) := a_0 + a_1 + a_2, \qquad \varphi_2(\boldsymbol{a}) := a_0 a_1 + a_1 a_2 + a_2 a_0, \qquad \varphi_3(\boldsymbol{a}) := a_0 a_1 a_2.$$

The differential equation (19) leads to an important three-term relation

$$\{s(\boldsymbol{a}) - 2\}\, {}_3h_2(\boldsymbol{a} + \boldsymbol{2}) + \psi(\boldsymbol{a})\, {}_3h_2(\boldsymbol{a} + \boldsymbol{1}) - \varphi_3(\boldsymbol{a})\, {}_3h_2(\boldsymbol{a}) = 0, \tag{20}$$

with $\boldsymbol{2} := (2,2,2;2,2)$ and $\psi(\boldsymbol{a}) := a_3 a_4 - \varphi_2(\boldsymbol{a}) - \varphi_1(\boldsymbol{a}) - 1$. Indeed, substitute $y = {}_3h_2(\boldsymbol{a}; z)$ into equation (19), use formula (17c) and put $z = 1$ to get equation (20). It indicates explicitly how ${}_3h_2(\boldsymbol{a} + \boldsymbol{2})$ can be written as a linear combination of ${}_3h_2(\boldsymbol{a})$ and ${}_3h_2(\boldsymbol{a} + \boldsymbol{1})$.

For $i = 0,1,2$ with $\{i,j,k\} = \{0,1,2\}$, one has

$${}_3h_2(\boldsymbol{a} + \boldsymbol{e}_i) = a_i\, {}_3h_2(\boldsymbol{a}) + {}_3h_2(\boldsymbol{a} + \boldsymbol{1}), \tag{21a}$$

$${}_3h_2(\boldsymbol{a} + \boldsymbol{e}_i + \boldsymbol{1}) = \frac{\varphi_3(\boldsymbol{a})}{s(\boldsymbol{a}) - 2}\, {}_3h_2(\boldsymbol{a}) + \frac{a_j a_k - (a_i - a_3 + 1)(a_i - a_4 + 1)}{s(\boldsymbol{a}) - 2}\, {}_3h_2(\boldsymbol{a} + \boldsymbol{1}), \tag{21b}$$

while for $i = 3, 4$, one has

$$ {}_3h_2(\boldsymbol{a} - \boldsymbol{e}_i) = (a_i - 1)\, {}_3h_2(\boldsymbol{a}) + {}_3h_2(\boldsymbol{a} + \boldsymbol{1}), \tag{22a} $$

$$ {}_3h_2(\boldsymbol{a} - \boldsymbol{e}_i + \boldsymbol{1}) = \frac{\varphi_3(\boldsymbol{a})}{s(\boldsymbol{a}) - 2}\, {}_3h_2(\boldsymbol{a}) + \frac{(a_i - 1)^2 - \varphi_1(\boldsymbol{a})\,(a_i - 1) + \varphi_2(\boldsymbol{a})}{s(\boldsymbol{a}) - 2}\, {}_3h_2(\boldsymbol{a} + \boldsymbol{1}). \tag{22b} $$

Indeed, formula (21a) resp. (22a) is obtained by using (17c) in (17a) resp. (17b) and putting $z = 1$. Formulas (21b) resp. (22b) is then obtained by replacing $\boldsymbol{a}$ with $\boldsymbol{a} + \boldsymbol{1}$ in (21a) resp. (22a) and eliminating ${}_3h_2(\boldsymbol{a} + \boldsymbol{2})$ through three-term relation (20).

3 Contiguous and Connection Matrices

To establish our main theorem it is crucial to put contiguous relations into matrix forms. If

$$ {}_3\boldsymbol{h}_2(\boldsymbol{a}) := \begin{pmatrix} {}_3h_2(\boldsymbol{a}) \\ {}_3h_2(\boldsymbol{a} + \boldsymbol{1}) \end{pmatrix}, \tag{23} $$

then the matrix version of contiguous relations is represented as

$$ {}_3\boldsymbol{h}_2(\boldsymbol{a} + \varepsilon\, \boldsymbol{e}_i) = A_i^{\varepsilon}(\boldsymbol{a})\, {}_3\boldsymbol{h}_2(\boldsymbol{a}) \qquad i = 0, 1, 2, 3, 4, \;\; \varepsilon = \pm, \tag{24} $$

where the matrices $A_i^{\varepsilon}(\boldsymbol{a})$ are given in (25) of Table 2; they are invertible over the field $\mathbb{Q}(\boldsymbol{a})$, having non-vanishing determinants as in (26). Formulas (25a) and (25d) come directly from (21) and (22), while the remaining formulas (25b) and (25c) are derived from them via the relation $A_i^{\varepsilon}(\boldsymbol{a}) = A_i^{-\varepsilon}(\boldsymbol{a} + \varepsilon\, \boldsymbol{e}_i)^{-1}$. The contiguous matrices satisfy compatibility conditions:

$$ A_i^{\varepsilon_i}(\boldsymbol{a} + \varepsilon_j\, \boldsymbol{e}_j)\, A_j^{\varepsilon_j}(\boldsymbol{a}) = A_j^{\varepsilon_j}(\boldsymbol{a} + \varepsilon_i\, \boldsymbol{e}_i)\, A_i^{\varepsilon_i}(\boldsymbol{a}), \qquad i, j = 0, 1, 2, 3, 4, \quad \varepsilon_i, \varepsilon_j = \pm. \tag{28} $$

Given an integer vector $\boldsymbol{p} \in \mathbb{Z}^5$, a finite sequence $\boldsymbol{p}_0, \boldsymbol{p}_1, \dots, \boldsymbol{p}_m$ of integer vectors in $\mathbb{Z}^5$ is said to be a *lattice path* from the origin $\boldsymbol{0}$ to $\boldsymbol{p}$ if there exist an m-tuple of indices $(i_1, \dots, i_m) \in \{0, 1, 2, 3, 4\}^m$ and an m-tuple of signs $(\varepsilon_1, \dots, \varepsilon_m) \in \{\pm\}^m$ such that

$$ \boldsymbol{p}_0 = \boldsymbol{0}, \qquad \boldsymbol{p}_m = \boldsymbol{p}, \qquad \boldsymbol{p}_k - \boldsymbol{p}_{k-1} = \varepsilon_k\, \boldsymbol{e}_{i_k}, \qquad k = 1, \dots, m. \tag{29} $$

To such a lattice path one can associate the products of contiguous matrices

$$ A(\boldsymbol{a}; \boldsymbol{p}) := A_{i_m}^{\varepsilon_m}(\boldsymbol{a} + \boldsymbol{p}_{m-1})\, A_{i_{m-1}}^{\varepsilon_{m-1}}(\boldsymbol{a} + \boldsymbol{p}_{m-2}) \cdots A_{i_2}^{\varepsilon_2}(\boldsymbol{a} + \boldsymbol{p}_1)\, A_{i_1}^{\varepsilon_1}(\boldsymbol{a} + \boldsymbol{p}_0). \tag{30} $$

Thanks to the compatibility condition (28) the ensuing matrix $A(\boldsymbol{a}; \boldsymbol{p})$ depends only on the terminal vector $\boldsymbol{p}$, being independent of which lattice path is chosen. In view of (24) and (30) the effect of translation $\boldsymbol{a} \mapsto \boldsymbol{a} + \boldsymbol{p}$ on the function ${}_3\boldsymbol{h}_2(\boldsymbol{a})$ is represented by

$$ {}_3\boldsymbol{h}_2(\boldsymbol{a} + \boldsymbol{p}) = A(\boldsymbol{a}; \boldsymbol{p})\, {}_3\boldsymbol{h}_2(\boldsymbol{a}). \tag{31} $$

We refer to $A(\boldsymbol{a}; \boldsymbol{p})$ as the *connection matrix* for a given shift vector $\boldsymbol{p} \in \mathbb{Z}^5$. Induction on the length m in (30) based on the determinant formulas (26) yields

$$ \det A(\boldsymbol{a}; \boldsymbol{p}) = \delta(\boldsymbol{a}; \boldsymbol{p}) := \frac{(-1)^{p_0 + p_1 + p_2} \cdot (s(\boldsymbol{a}) - 1,\, s(\boldsymbol{p})) \cdot \prod_{i=0}^{2} (a_i,\, p_i)}{\prod_{i=0}^{2} \prod_{j=3}^{4} (a_j - a_i,\, p_j - p_i)} \neq 0 \quad \text{in} \quad \mathbb{Q}(\boldsymbol{a}), \tag{32} $$

$$A_i^+(\boldsymbol{a}) = \begin{pmatrix} a_i & 1 \\ \dfrac{\varphi_3(\boldsymbol{a})}{s(\boldsymbol{a})-2} & \dfrac{a_j a_k - (a_i - a_3 + 1)(a_i - a_4 + 1)}{s(\boldsymbol{a})-2} \end{pmatrix} \qquad i = 0,1,2, \tag{25a}$$

$$A_i^+(\boldsymbol{a}) = \begin{pmatrix} \dfrac{a_i^2 - \varphi_1(\boldsymbol{a})\, a_i + \varphi_2(\boldsymbol{a})}{(a_i - a_0)(a_i - a_1)(a_i - a_2)} & -\dfrac{s(\boldsymbol{a})-1}{(a_i - a_0)(a_i - a_1)(a_i - a_2)} \\ -\dfrac{\varphi_3(\boldsymbol{a})}{(a_i - a_0)(a_i - a_1)(a_i - a_2)} & \dfrac{a_i\,\{s(\boldsymbol{a})-1\}}{(a_i - a_0)(a_i - a_1)(a_i - a_2)} \end{pmatrix} \qquad i = 3,4, \tag{25b}$$

$$A_i^-(\boldsymbol{a}) = \begin{pmatrix} \dfrac{(a_i - a_3)(a_i - a_4) - a_j a_k}{(a_i - 1)(a_i - a_3)(a_i - a_4)} & \dfrac{s(\boldsymbol{a})-1}{(a_i - 1)(a_i - a_3)(a_i - a_4)} \\ \dfrac{a_j a_k}{(a_i - a_3)(a_i - a_4)} & -\dfrac{s(\boldsymbol{a})-1}{(a_i - a_3)(a_i - a_4)} \end{pmatrix} \qquad i = 0,1,2, \tag{25c}$$

$$A_i^-(\boldsymbol{a}) = \begin{pmatrix} a_i - 1 & 1 \\ \dfrac{\varphi_3(\boldsymbol{a})}{s(\boldsymbol{a})-2} & \dfrac{(a_i - 1)^2 - \varphi_1(\boldsymbol{a})\,(a_i - 1) + \varphi_2(\boldsymbol{a})}{s(\boldsymbol{a})-2} \end{pmatrix} \qquad i = 3,4. \tag{25d}$$

$$\det A_i^+(\boldsymbol{a}) = -\frac{a_i(a_i - a_3 + 1)(a_i - a_4 + 1)}{s(\boldsymbol{a})-2} \qquad i = 0,1,2, \tag{26a}$$

$$\det A_i^+(\boldsymbol{a}) = \frac{s(\boldsymbol{a})-1}{(a_i - a_0)(a_i - a_1)(a_i - a_2)} \qquad i = 3,4, \tag{26b}$$

$$\det A_i^-(\boldsymbol{a}) = -\frac{s(\boldsymbol{a})-1}{(a_i - 1)(a_i - a_3)(a_i - a_4)} \qquad i = 0,1,2, \tag{26c}$$

$$\det A_i^-(\boldsymbol{a}) = \frac{(a_i - a_0 - 1)(a_i - a_1 - 1)(a_i - a_2 - 1)}{s(\boldsymbol{a})-2} \qquad i = 3,4, \tag{26d}$$

Table 2: Contiguous matrices and their determinants; $\{i,j,k\} = \{0,1,2\}$ for $i = 0,1,2$.

$$B_i(\boldsymbol{a}) = \begin{pmatrix} a_i & 1 \\ \dfrac{\varphi_3(\boldsymbol{a})}{s(\boldsymbol{a})} & \dfrac{a_j a_k - (a_i - a_3)(a_i - a_4)}{s(\boldsymbol{a})} \end{pmatrix} \qquad i = 0,1,2, \tag{27a}$$

$$B_i(\boldsymbol{a}) = \begin{pmatrix} \dfrac{a_i^2 - \varphi_1(\boldsymbol{a})\, a_i + \varphi_2(\boldsymbol{a})}{(a_i - a_0)(a_i - a_1)(a_i - a_2)} & -\dfrac{s(\boldsymbol{a})}{(a_i - a_0)(a_i - a_1)(a_i - a_2)} \\ -\dfrac{\varphi_3(\boldsymbol{a})}{(a_i - a_0)(a_i - a_1)(a_i - a_2)} & \dfrac{a_i\, s(\boldsymbol{a})}{(a_i - a_0)(a_i - a_1)(a_i - a_2)} \end{pmatrix} \qquad i = 3,4, \tag{27b}$$

Table 3: Principal parts of contiguous matrices; $\{i,j,k\} = \{0,1,2\}$ for $i = 0,1,2$.

where $\boldsymbol{p} = (p_0, p_1, p_2; p_3, p_4)$ and $s(\boldsymbol{p})$ is its parametric excess.

Focusing on the upper row of connection matrix $A(\boldsymbol{a}; \boldsymbol{p})$, if we write

$$A(\boldsymbol{a}; \boldsymbol{p}) = \begin{pmatrix} r_1(\boldsymbol{a}; \boldsymbol{p}) & r(\boldsymbol{a}; \boldsymbol{p}) \\ * & * \end{pmatrix}, \tag{33}$$

then the upper entry of matrix equation (31) leads to a three-term relation

$${}_3h_2(\boldsymbol{a} + \boldsymbol{p}) = r_1(\boldsymbol{a}; \boldsymbol{p})\, {}_3h_2(\boldsymbol{a}) + r(\boldsymbol{a}; \boldsymbol{p})\, {}_3h_2(\boldsymbol{a} + \mathbf{1}). \tag{34}$$

Similarly, for another vector $\boldsymbol{q} \in \mathbb{Z}^5$ different from $\boldsymbol{p}$ one has a second three-term relation

$${}_3h_2(\boldsymbol{a} + \boldsymbol{q}) = r_1(\boldsymbol{a}; \boldsymbol{q})\, {}_3h_2(\boldsymbol{a}) + r(\boldsymbol{a}; \boldsymbol{q})\, {}_3h_2(\boldsymbol{a} + \mathbf{1}). \tag{35}$$

The linear system (34)-(35) can be written in the matrix form

$$\begin{pmatrix} {}_3h_2(\boldsymbol{a} + \boldsymbol{p}) \\ {}_3h_2(\boldsymbol{a} + \boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} r_1(\boldsymbol{a}; \boldsymbol{p}) & r(\boldsymbol{a}; \boldsymbol{p}) \\ r_1(\boldsymbol{a}; \boldsymbol{q}) & r(\boldsymbol{a}; \boldsymbol{q}) \end{pmatrix} \begin{pmatrix} {}_3h_2(\boldsymbol{a}) \\ {}_3h_2(\boldsymbol{a} + \mathbf{1}) \end{pmatrix}, \tag{36}$$

and we wonder whether the system is invertible, that is, if the determinant

$$\varDelta(\boldsymbol{a}; \boldsymbol{p}, \boldsymbol{q}) := \begin{vmatrix} r_1(\boldsymbol{a}; \boldsymbol{p}) & r(\boldsymbol{a}; \boldsymbol{p}) \\ r_1(\boldsymbol{a}; \boldsymbol{q}) & r(\boldsymbol{a}; \boldsymbol{q}) \end{vmatrix} \tag{37}$$

is non-vanishing in $\mathbb{Q}(\boldsymbol{a})$. To answer this question we first prove the following lemma.

Lemma 3.1 *With formula* (32) *and definition* (37) *one has*

$$\varDelta(\boldsymbol{a}; \boldsymbol{p}, \boldsymbol{q}) = \det A(\boldsymbol{a}; \boldsymbol{p}) \cdot r(\boldsymbol{a} + \boldsymbol{p}; \boldsymbol{q} - \boldsymbol{p}) = \delta(\boldsymbol{a}; \boldsymbol{p}) \cdot r(\boldsymbol{a} + \boldsymbol{p}; \boldsymbol{q} - \boldsymbol{p}). \tag{38}$$

Proof. By the chain rule $A(\boldsymbol{a}; \boldsymbol{q}) = A(\boldsymbol{a} + \boldsymbol{p}; \boldsymbol{q} - \boldsymbol{p}) A(\boldsymbol{a}; \boldsymbol{p})$ one has

$$\begin{aligned} A(\boldsymbol{a} + \boldsymbol{p}; \boldsymbol{q} - \boldsymbol{p}) = A(\boldsymbol{a}; \boldsymbol{q}) A(\boldsymbol{a}; \boldsymbol{p})^{-1} &= \begin{pmatrix} r_1(\boldsymbol{a}; \boldsymbol{q}) & r(\boldsymbol{a}; \boldsymbol{q}) \\ * & * \end{pmatrix} \frac{1}{\det A(\boldsymbol{a}; \boldsymbol{p})} \begin{pmatrix} * & -r(\boldsymbol{a}; \boldsymbol{p}) \\ * & r_1(\boldsymbol{a}; \boldsymbol{p}) \end{pmatrix} \\ &= \frac{1}{\det A(\boldsymbol{a}; \boldsymbol{p})} \begin{pmatrix} * & \varDelta(\boldsymbol{a}; \boldsymbol{p}, \boldsymbol{q}) \\ * & * \end{pmatrix}, \end{aligned}$$

the $(1, 2)$ entry of which leads to $r(\boldsymbol{a} + \boldsymbol{p}; \boldsymbol{q} - \boldsymbol{p}) = \varDelta(\boldsymbol{a}; \boldsymbol{p}, \boldsymbol{q}) / \det A(\boldsymbol{a}; \boldsymbol{p})$. □

Thus the vanishing of $\varDelta(\boldsymbol{a}; \boldsymbol{p}, \boldsymbol{q})$ in $\mathbb{Q}(\boldsymbol{a})$ is equivalent to the condition that the matrix $A(\boldsymbol{a} + \boldsymbol{p}; \boldsymbol{q} - \boldsymbol{p})$ or more simply $A(\boldsymbol{a}; \boldsymbol{q} - \boldsymbol{p})$ be lower triangular over $\mathbb{Q}(\boldsymbol{a})$. In the next section we show that this never happens unless $\boldsymbol{p} = \boldsymbol{q}$ (see Proposition 4.3).

4 Principal Parts and Triangular Matrices

The non-commutative nature of contiguous matrices makes the structure of $A(\boldsymbol{a}; \boldsymbol{p})$ so intricate that a direct tackle to our problem may be difficult. In such a situation, taking the "principal parts" of non-commutative objects often turns things commutative, although much information is usually lost in this process, but the ensuing commutative objects, perhaps much easier to handle, may still retain something important. Such an idea is employed in Iwasaki [7, §9] to analyze contiguous matrices of Gauss's hypergeometric functions ${}_2F_1$.

We now apply a similar thought to ${}_3F_2(1)$, but with a necessary twist involving

$$D(t) := \mathrm{diag}\{1,\, t\}, \qquad \delta_i := \begin{cases} -1 & (i = 0, 1, 2), \\ +1 & (i = 3, 4), \end{cases}$$

where t is a formal parameter. An inspection shows that there is a formal Laurent expansion

$$t^{\delta_i \varepsilon} \cdot D(t)^{-1} A_i^{\varepsilon}(t\,\boldsymbol{a})\, D(t) = B_i(\boldsymbol{a})^{\varepsilon} + O(1/t) \qquad \text{around} \quad t = \infty, \tag{39}$$

where $B_i(\boldsymbol{a})^{\varepsilon} := B_i(\boldsymbol{a})^{\pm 1}$ for $\varepsilon = \pm$ and $B_i(\boldsymbol{a})$ is given as in (27) of Table 3. The matrix $B_i(\boldsymbol{a})$ is obtained from $A_i^{+}(\boldsymbol{a})$ by taking the top homogeneous components of its entries, where the top homogeneous component of a rational function is the ratio of the top homogeneous polynomials of its numerator and denominator. Observe that

$$\det B_i(\boldsymbol{a}) = -\frac{a_i(a_i - a_3)(a_i - a_4)}{s(\boldsymbol{a})} \qquad i = 0, 1, 2, \tag{40a}$$

$$\det B_i(\boldsymbol{a}) = \frac{s(\boldsymbol{a})}{(a_i - a_0)(a_i - a_1)(a_i - a_2)} \qquad i = 3, 4, \tag{40b}$$

so the matrices $B_i(\boldsymbol{a})$ are invertible over $\mathbb{Q}(\boldsymbol{a})$. Compatibility condition (28) for contiguous matrices naturally leads to the commutativity of their principal parts:

$$B_i(\boldsymbol{a}) B_j(\boldsymbol{a}) = B_j(\boldsymbol{a}) B_i(\boldsymbol{a}) \qquad i, j = 0, 1, 2, 3, 4. \tag{41}$$

Lemma 4.1 *For any $\boldsymbol{p} = (p_0, p_1, p_2; p_3, p_4) \in \mathbb{Z}^5$ there is a formal Laurent expansion*

$$t^{s(\boldsymbol{p})} \cdot D(t)^{-1} A(t\,\boldsymbol{a}; \boldsymbol{p})\, D(t) = B(\boldsymbol{a}; \boldsymbol{p}) + O(1/t) \qquad \textit{around} \quad t = \infty, \tag{42}$$

where the matrix $B(\boldsymbol{a}; \boldsymbol{p})$, called the principal part of $A(\boldsymbol{a}; \boldsymbol{p})$, is given by

$$B(\boldsymbol{a}; \boldsymbol{p}) = B_0(\boldsymbol{a})^{p_0}\, B_1(\boldsymbol{a})^{p_1}\, B_2(\boldsymbol{a})^{p_2}\, B_3(\boldsymbol{a})^{p_3}\, B_4(\boldsymbol{a})^{p_4}. \tag{43}$$

Proof. Replacing $\boldsymbol{a}$ with $t\,\boldsymbol{a}$ in formula (30), substituting the result into the left-hand side of (42) and using formula (39), we observe that expansion (42) holds true with

$$B(\boldsymbol{a}; \boldsymbol{p}) = B_{i_m}(\boldsymbol{a})^{\varepsilon_m}\, B_{i_{m-1}}(\boldsymbol{a})^{\varepsilon_{m-1}} \cdots B_{i_2}(\boldsymbol{a})^{\varepsilon_2}\, B_{i_1}(\boldsymbol{a})^{\varepsilon_1}.$$

The commutativity (41) allows us to rearrange the order of the matrix products so that

$$B(\boldsymbol{a}; \boldsymbol{p}) = \prod_{k \in \Lambda_0} B_0(\boldsymbol{a})^{\varepsilon_k} \prod_{k \in \Lambda_1} B_1(\boldsymbol{a})^{\varepsilon_k} \prod_{k \in \Lambda_2} B_2(\boldsymbol{a})^{\varepsilon_k} \prod_{k \in \Lambda_3} B_3(\boldsymbol{a})^{\varepsilon_k} \prod_{k \in \Lambda_4} B_4(\boldsymbol{a})^{\varepsilon_k},$$

where $\Lambda_j := \{ k = 1, \dots, m \,:\, i_k = j \}$ for $j = 0, 1, 2, 3, 4$. Since $\sum_{k \in \Lambda_j} \varepsilon_k = p_j$ we have (43). □

Remark 4.2 Since $D(t)$ is a diagonal matrix, expansion formula (42) implies that if $A(\boldsymbol{a}; \boldsymbol{p})$ is a lower triangular matrix over the field $\mathbb{Q}(\boldsymbol{a})$, then so must be $B(\boldsymbol{a}; \boldsymbol{p})$.

The aim of this section is to prove the following proposition.

Proposition 4.3 *For any shift vector $\boldsymbol{p} = (p_0, p_1, p_2; p_3, p_4) \in \mathbb{Z}^5$, if $A(\boldsymbol{a}; \boldsymbol{p})$ is lower triangular over $\mathbb{Q}(\boldsymbol{a})$, that is, if $r(\boldsymbol{a}; \boldsymbol{p})$ vanishes in $\mathbb{Q}(\boldsymbol{a})$ then $\boldsymbol{p}$ must be $\mathbf{0}$.*

By Remark 4.2, to prove the proposition it is sufficient to show the following lemma.

Lemma 4.4 *If $B(\boldsymbol{a}; \boldsymbol{p})$ is lower triangular over $\mathbb{Q}(\boldsymbol{a})$ then $\boldsymbol{p}$ must be $\mathbf{0}$.*

Thanks to the commutative nature of principal part matrices, Lemma 4.4 is much more tractable because simultaneous diagonalization technique is available. Our strategy to prove the lemma is *specializations*, that is, to work with some special but convenient values of $\boldsymbol{a}$. For this purpose we may take any complex value of $\boldsymbol{a}$ as far as the matrices $B_i(\boldsymbol{a})$, $i = 0, 1, 2, 3, 4$, have no zero denominators or no zero determinants; such a value is called *good*. In view of the denominators in Table 3 and the determinants in (40) the goodness condition is given by

$$a_0 a_1 a_2 \cdot s(\boldsymbol{a}) \cdot \prod_{i=1}^{2} \prod_{j=0}^{2} (b_i - a_j) \neq 0. \tag{44}$$

Here we do not have to mind the genericity condition (6), which has nothing to do with our current concern. If Lemma 4.4 is proved for some good values of $\boldsymbol{a}$, then we are done.

An invertible matrix $P = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is said to be *admissible* if $\alpha\,\beta \neq 0$.

Our discussion is then based on the following simple lemma for lower triangular matrices.

Lemma 4.5 *If an admissible matrix P diagonalizes a lower triangular matrix B as $P^{-1}BP = \mathrm{diag}\{\lambda_1,\, \lambda_2\}$, then one must have $\lambda_1 = \lambda_2$ and so B must be a scalar matrix.*

Proof. A straightforward calculation shows

$$B = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1} = \frac{1}{\Delta} \begin{pmatrix} \lambda_1\,\alpha\,\delta - \lambda_2\,\beta\,\gamma & (\lambda_2 - \lambda_1)\,\alpha\,\beta \\ (\lambda_1 - \lambda_2)\,\gamma\,\delta & \lambda_2\,\alpha\,\delta - \lambda_1\,\beta\,\gamma \end{pmatrix}, \qquad \Delta := \det P \neq 0.$$

Since B is lower triangular, we have $(\lambda_2 - \lambda_1)\alpha\,\beta = 0$ and hence $\lambda_2 - \lambda_1 = 0$ by $\alpha\,\beta \neq 0$. □

When working with a special value of $\boldsymbol{a}$, we abbreviate $B(\boldsymbol{a}; \boldsymbol{p})$ to B and $B_i(\boldsymbol{a})$ to B_i.

Lemma 4.6 *If $B(\boldsymbol{a}; \boldsymbol{p})$ is lower triangular over $\mathbb{Q}(\boldsymbol{a})$ then*

$$p_0 + p_1 + p_2 = p_3 + p_4 = 0. \tag{45}$$

Proof. Taking a specialization $\boldsymbol{a} = (2, 2, 2; -7, -7)$, which is good from (44), we have

$$B_0 = B_1 = B_2 = \begin{pmatrix} 2 & 1 \\ -2/5 & 77/20 \end{pmatrix}, \qquad B_3 = B_4 = \begin{pmatrix} -103/729 & -20/729 \\ 8/729 & -140/729 \end{pmatrix}.$$

These matrices are simultaneously diagonalized by an admissible matrix

$$P = \begin{pmatrix} 5/8 & 4 \\ 1 & 1 \end{pmatrix} \qquad \text{as} \qquad \begin{aligned} P^{-1}\,B_0\,P &= \mathrm{diag}\{\,2 \cdot 3^2 \cdot 5^{-1}, \quad 2^{-2} \cdot 3^2\,\}, \\ P^{-1}\,B_3\,P &= \mathrm{diag}\{\,-3^{-3} \cdot 5, \quad -2^2 \cdot 3^{-3}\,\}. \end{aligned}$$

Formula (43) then implies $B = B_0^m\, B_3^n$ with $m := p_0 + p_1 + p_2$ and $n := p_3 + p_4$. So the lower triangular matrix B is diagonalized as $P^{-1}\, B\, P = \mathrm{diag}\{\lambda_1,\, \lambda_2\}$, where

$$\lambda_1 = (2 \cdot 3^2 \cdot 5^{-1})^m (-3^{-3} \cdot 5)^n = (-1)^n \cdot 2^m \cdot 3^{2m-3n} \cdot 5^{n-m},$$
$$\lambda_2 = (2^{-2} \cdot 3^2)^m (-2^2 \cdot 3^{-3})^n = (-1)^n \cdot 2^{2n-2m} \cdot 3^{2m-3n}.$$

By Lemma 4.5 one must have $\lambda_1 = \lambda_2$, that is, $\lambda_1 \lambda_2^{-1} = 2^{3m-2n} \cdot 5^{n-m} = 1$. The unique factorization property of rational numbers then forces $3m - 2n = n - m = 0$, which yields $m = n = 0$, that is condition (45). $\square$

Lemma 4.7 *Suppose* (45)*. If* $B(\boldsymbol{a}; \boldsymbol{p})$ *is lower triangular over* $\mathbb{Q}(\boldsymbol{a})$ *then* $p_3 = p_4 = 0$*.*

Proof. Taking a specialization $\boldsymbol{a} = (4, 4, 4; 0, 7)$, which is good from (44), we have

$$B_0 = B_1 = B_2 = \begin{pmatrix} 4 & 1 \\ -64/5 & -28/5 \end{pmatrix}, \quad B_3 = \begin{pmatrix} -3/4 & -5/64 \\ 1 & 0 \end{pmatrix} \quad B_4 = \begin{pmatrix} 13/27 & 5/27 \\ -64/27 & -35/27 \end{pmatrix}.$$

By formula (43) we have $B = B_0^{p_0+p_1+p_2}\, B_3^{p_3}\, B_4^{p_4} = (B_3 B_4^{-1})^{p_3}$ where condition (45) is also used. Observe that the matrix $B_3 B_4^{-1}$ is diagonalized by an admissible matrix

$$P = \begin{pmatrix} -1/8 & -5/8 \\ 1 & 1 \end{pmatrix} \qquad \text{as} \qquad P^{-1}\, (B_3\, B_4^{-1})\, P = \mathrm{diag}\{\, 2^{-3},\; -2^{-3} \cdot 3^3 \,\},$$

and so the lower triangular matrix B is diagonalized as $P^{-1}\, B\, P = \mathrm{diag}\{\lambda_1,\, \lambda_2\}$ with $\lambda_1 = (2^{-3})^{p_3}$ and $\lambda_2 = (-2^{-3} \cdot 3^3)^{p_3}$. Lemma 4.5 then forces $\lambda_1 = \lambda_2$, that is, $\lambda_1 \lambda_2^{-1} = (-1)^{p_3} \cdot 3^{-3p_3} = 1$. This immediately shows that $p_3 = -p_4 = 0$. $\square$

Lemma 4.8 *Suppose* $p_0 + p_1 + p_2 = p_3 = p_4 = 0$*. If* $B(\boldsymbol{a}; \boldsymbol{p})$ *is lower triangular over* $\mathbb{Q}(\boldsymbol{a})$ *then* $p_0 = p_1 = p_2 = p_3 = p_4 = 0$*, namely,* $\boldsymbol{p} = \boldsymbol{0}$*.*

Proof. Taking a specialization $\boldsymbol{a} = (-6, 1, 1; 0, 0)$, which is good from (44), we have

$$B_0 = \begin{pmatrix} -6 & 1 \\ -3/2 & -35/4 \end{pmatrix}, \quad B_1 = B_2 = \begin{pmatrix} 1 & 1 \\ -3/2 & -7/4 \end{pmatrix} \quad B_3 = B_4 = \begin{pmatrix} -11/6 & -2/3 \\ 1 & 0 \end{pmatrix}.$$

By formula (43) we have $B = B_0^{p_0}\, B_1^{p_1+p_2}\, (B_3\, B_4)^0 = (B_0 B_1^{-1})^{p_0}$ where condition $p_0 + p_1 + p_2 = p_3 = p_4 = 0$ is also used. Observe that $B_0 B_1^{-1}$ is diagonalized by an admissible matrix

$$P = \begin{pmatrix} -1/2 & -4/3 \\ 1 & 1 \end{pmatrix} \qquad \text{as} \qquad P^{-1}\, (B_0\, B_1^{-1})\, P = \mathrm{diag}\{\, 2^3,\; -3^3 \,\},$$

and so the lower triangular matrix B is diagonalized as $P^{-1}\, B\, P = \mathrm{diag}\{\lambda_1,\, \lambda_2\}$ with $\lambda_1 = 2^{3p_0}$ and $\lambda_2 = (-3^3)^{p_0}$. Lemma 4.5 then forces $\lambda_1 = \lambda_2$, that is, $\lambda_1 \lambda_2^{-1} = (-1)^{p_0} \cdot 2^{3p_0} \cdot 3^{-3p_0} = 1$. This immediately yields $p_0 = -p_1 - p_2 = 0$. In a similar manner $p_1 = -p_2 - p_0 = 0$ follows from the good specialization $\boldsymbol{a} = (1, -6, 1; 0, 0)$. Thus we have $p_0 = p_1 = p_2 = p_3 = p_4 = 0$. $\square$

Lemma 4.4 and hence Proposition 4.3 follows immediately from Lemmas 4.6, 4.7 and 4.8. Proposition 4.3 can be strengthened if the shift vectors $\boldsymbol{p}$ are restricted somewhat.

Proposition 4.9 *Let $\boldsymbol{p} = (p_0, p_1, p_2; p_3, p_4) \in \mathbb{Z}^3_{\ge 0} \times \mathbb{Z}^2$ and $\boldsymbol{c} \in \mathbb{Q}^5$. Then the specialization*

$$\hat{r}(a_3, a_4; \boldsymbol{p}) := r(\boldsymbol{a} + \boldsymbol{c}; \boldsymbol{p})\big|_{a_0=a_1=a_2=0} \in \mathbb{Q}(a_3, a_4), \qquad \boldsymbol{a} = (a_0, a_1, a_2; a_3, a_4),$$

is well defined. If $\hat{r}(a_3, a_4; \boldsymbol{p})$ vanishes in $\mathbb{Q}(a_3, a_4)$ then $\boldsymbol{p}$ must be $\boldsymbol{0}$.

Proof. Since p_0, p_1, $p_2 \ge 0$, one can take a lattice path from $\boldsymbol{0}$ to $\boldsymbol{p}$ in such a manner that if $i_k \in \{0, 1, 2\}$ then $\varepsilon_k = +$ in (29), so that the product (30) contains no contiguous matrices of type (25c) in Table 2. In view of the denominators of contiguous matrices of types (25a), (25b), (25d), any irreducible factor of the denominator of $A(\boldsymbol{a}+\boldsymbol{c}; \boldsymbol{p})$ must be either $s(\boldsymbol{a})+$a rational number, or $a_j - a_i +$ a rational number, for some $i = 0, 1, 2$, $j = 3, 4$. Thus one can safely take the specialization $\hat{A}(a_3, a_4; \boldsymbol{p}) := A(\boldsymbol{a} + \boldsymbol{c}; \boldsymbol{p})\big|_{a_0=a_1=a_2=0}$ as a matrix over $\mathbb{Q}(a_3, a_4)$, the principal part of which is independent of $\boldsymbol{c}$ and given by $\hat{B}(a_3, a_4; \boldsymbol{p}) = \hat{B}_0^{p_0} \hat{B}_1^{p_1} \hat{B}_2^{p_2} \hat{B}_3^{p_3} \hat{B}_4^{p_4}$, where

$$\hat{B}_i := \begin{pmatrix} 0 & 1 \\ 0 & -a_3 a_4 (a_3 + a_4)^{-1} \end{pmatrix}, \quad i = 0, 1, 2; \qquad \hat{B}_i := \begin{pmatrix} a_i^{-1} & -a_i^{-3}(a_3 + a_4) \\ 0 & a_i^{-2}(a_3 + a_4) \end{pmatrix}, \quad i = 3, 4,$$

are derived from formulas (27) in Table 3 by putting $a_0 = a_1 = a_2 = 0$. Since $\hat{B}_0 = \hat{B}_1 = \hat{B}_2$, one has $\hat{B}(a_3, a_4; \boldsymbol{p}) = \hat{B}_0^{p} \hat{B}_3^{p_3} \hat{B}_4^{p_4}$ with $p := p_0 + p_1 + p_2$. In place of (44) the goodness condition is now $a_3 a_4 (a_3 + a_4) \neq 0$. Take a good specialization $a_3 = -2$, $a_4 = 3$. After some manipulations we have

$$\hat{B}(-2, 3; \boldsymbol{p}) = \begin{pmatrix} (-2)^{-p_3} \cdot 3^{-p_4} & 2^{-2p_3-1} \cdot 3^{-2p_4-1}\{1 - (-2)^{p_3} \cdot 3^{p_4}\} \\ 0 & 2^{-2p_3} \cdot 3^{-2p_4} \end{pmatrix}, \qquad p = 0,$$

$$\hat{B}(-2, 3; \boldsymbol{p}) = \begin{pmatrix} 0 & 2^{p-2p_3-1} \cdot 3^{p-2p_4-1} \\ 0 & 2^{p-2p_3} \cdot 3^{p-2p_4} \end{pmatrix}, \qquad p \ge 1.$$

If $\hat{r}(a_3, a_4; \boldsymbol{p}) = 0$ in $\mathbb{Q}(a_3, a_4)$ then $\hat{A}(a_3, a_4; \boldsymbol{p})$ and so $\hat{B}(-2, 3; \boldsymbol{p})$ must be lower triangular. This forces $p = p_0 + p_1 + p_2 = 0$ and $(-2)^{p_3} \cdot 3^{p_4} = 1$. Since p_0, p_1, $p_2 \ge 0$, we must have $p_0 = p_1 = p_2 = p_3 = p_4 = 0$ and hence the proposition is proved. □

Proposition 4.9 is used to define the concept of *specializations* of ${}_3F_2(1)$ continued fractions in [6, §8]. Although the present article focuses mostly on the case where the parameters $\boldsymbol{a}$ are unrestricted, the idea itself in this section would work more widely when they are restricted or specialized to a larger extent.

5 Linear Independence

The aim of this section is to establish the linear independence of ${}_3h_2(\boldsymbol{a}+\boldsymbol{p})$ and ${}_3h_2(\boldsymbol{a}+\boldsymbol{q})$ for any distinct $\boldsymbol{p}$, $\boldsymbol{q} \in \mathbb{Z}^5$ (Proposition 5.3); this is indispensable for the existence and uniqueness of solutions to the linear system (34)-(35). Once this is done then the proof of our main theorem (Theorem 1.1) is an easy task and hence completed in this and next sections.

We begin by dealing with an important special case where $\boldsymbol{p} = \boldsymbol{0}$ and $\boldsymbol{q} = \boldsymbol{1}$. For this case three-term relation (20) can be used to show the following fundamental lemma.

Lemma 5.1 *${}_3h_2(\boldsymbol{a})$ and ${}_3h_2(\boldsymbol{a} + \boldsymbol{1})$ are linearly independent over $\mathbb{C}(\boldsymbol{a})$.*

Proof. To get a contradiction, suppose that ${}_3h_2(\boldsymbol{a})$ and ${}_3h_2(\boldsymbol{a}+\boldsymbol{1})$ are linearly dependent over $\mathbb{C}(\boldsymbol{a})$. Then there exists a rational function $R(\boldsymbol{a}) \in \mathbb{C}(\boldsymbol{a})$ such that ${}_3h_2(\boldsymbol{a}+\boldsymbol{1}) = R(\boldsymbol{a})\, {}_3h_2(\boldsymbol{a})$. This implies ${}_3h_2(\boldsymbol{a}+\boldsymbol{2}) = R(\boldsymbol{a}+\boldsymbol{1})\, R(\boldsymbol{a})\, {}_3h_2(\boldsymbol{a})$. Substituting these into (20) yields

$$\{s(\boldsymbol{a})-2\}\, R(\boldsymbol{a}+\boldsymbol{1})\, R(\boldsymbol{a}) + \psi(\boldsymbol{a})\, R(\boldsymbol{a}) - \varphi_3(\boldsymbol{a}) = 0, \tag{46}$$

where $\psi(\boldsymbol{a}) = b_1 b_2 - \varphi_2(\boldsymbol{a}) - \varphi_1(\boldsymbol{a}) - 1$. Putting $R(\boldsymbol{a}) = P(\boldsymbol{a})/Q(\boldsymbol{a})$ with $P(\boldsymbol{a})$, $Q(\boldsymbol{a}) \in \mathbb{C}[\boldsymbol{a}]$ and multiplying (46) by $Q(\boldsymbol{a}+\boldsymbol{1})\, Q(\boldsymbol{a})$, we have a polynomial equation

$$\{s(\boldsymbol{a})-2\}\, P(\boldsymbol{a}+\boldsymbol{1})\, P(\boldsymbol{a}) + \psi(\boldsymbol{a})\, P(\boldsymbol{a})\, Q(\boldsymbol{a}+\boldsymbol{1}) - \varphi_3(\boldsymbol{a})\, Q(\boldsymbol{a}+\boldsymbol{1})\, Q(\boldsymbol{a}) = 0. \tag{47}$$

If $d_p := \deg P(\boldsymbol{a})$ and $d_q := \deg Q(\boldsymbol{a})$, the three terms in the left side of (47) have degrees $2d_p+1$, d_p+d_q+2, $2d_q+3$, at least two of which must coincide. This implies $d_p = d_q+1$ and hence the three degrees must be equal. Thus the top homogeneous component of (47) gives

$$s(\boldsymbol{a})\, \bar{P}^2(\boldsymbol{a}) + \{b_1 b_2 - \varphi_2(\boldsymbol{a})\}\, \bar{P}(\boldsymbol{a})\, \bar{Q}(\boldsymbol{a}) - \varphi_3(\boldsymbol{a})\, \bar{Q}^2(\boldsymbol{a}) = 0, \tag{48}$$

where $\bar{P}(\boldsymbol{a})$ and $\bar{Q}(\boldsymbol{a})$ are the top homogeneous components of $P(\boldsymbol{a})$ and $Q(\boldsymbol{a})$ respectively. We write $\bar{P}(\boldsymbol{a}) = p(\boldsymbol{a})\, r(\boldsymbol{a})$ and $\bar{Q}(\boldsymbol{a}) = q(\boldsymbol{a})\, r(\boldsymbol{a})$ with $p(\boldsymbol{a})$, $q(\boldsymbol{a})$, $r(\boldsymbol{a}) \in \mathbb{C}[\boldsymbol{a}]$, where $p(\boldsymbol{a})$ and $q(\boldsymbol{a})$ have no factors in common. Then (48) and $d_p = d_q + 1$ yield

$$s(\boldsymbol{a})\, p^2(\boldsymbol{a}) + \{b_1 b_2 - \varphi_2(\boldsymbol{a})\}\, p(\boldsymbol{a})\, q(\boldsymbol{a}) - \varphi_3(\boldsymbol{a})\, q^2(\boldsymbol{a}) = 0, \quad \deg p(\boldsymbol{a}) = \deg q(\boldsymbol{a}) + 1. \tag{49}$$

The former equation in (49) implies $q(\boldsymbol{a}) \mid s(\boldsymbol{a})\, p^2(\boldsymbol{a})$ and $p(\boldsymbol{a}) \mid \varphi_3(\boldsymbol{a})\, q^2(\boldsymbol{a})$. Since $p(\boldsymbol{a})$ and $q(\boldsymbol{a})$ have no factors in common, we must have $q(\boldsymbol{a}) \mid s(\boldsymbol{a})$ and $p(\boldsymbol{a}) \mid \varphi_3(\boldsymbol{a}) = a_1 a_2 a_3$. These division relations and the latter equation in (49) lead to a dichotomy

$$\begin{array}{lll} \text{(i)} & q(\boldsymbol{a}) = c_q, & p(\boldsymbol{a}) = c_p\, a_i, \\ \text{(ii)} & q(\boldsymbol{a}) = c_q\, s(\boldsymbol{a}), & p(\boldsymbol{a}) = c_p\, a_i\, a_j, \end{array}$$

where c_p, $c_q \in \mathbb{C}^{\times}$ and $\{i, j, k\} = \{1, 2, 3\}$. The former equation in (49) then gives

$$\begin{array}{ll} \text{(i)} & c_p^2\, a_i\, s(\boldsymbol{a}) + c_p c_q\, \{b_1 b_2 - \varphi_2(\boldsymbol{a})\} - c_q^2\, a_j\, a_k = 0, \\ \text{(ii)} & c_p^2\, a_i\, a_j + c_p c_q\, \{b_1 b_2 - \varphi_2(\boldsymbol{a})\} - c_q^2\, a_k\, s(\boldsymbol{a}) = 0, \end{array}$$

neither of which is feasible as an equation in $\mathbb{C}[\boldsymbol{a}]$. This contradiction proves the lemma. □

Remark 5.2 We make two remarks about Lemma 5.1.

(1) Vidunas [10] showed the linear independence over $\mathbb{C}(\boldsymbol{a}; z)$ of ${}_2F_1(\boldsymbol{a}; z)$ and ${}_2F_1(\boldsymbol{a}+\boldsymbol{e}_1; z)$ with $\boldsymbol{e}_1 := (1, 0; 0)$, using Kummer's ${}_2F_1(-1)$ formula as in [10, (14)]. Lemma 5.1 might be proved in a similar spirit, while our proof here is purely algebraic and totally independent of special-value formulas, which we believe is of its own merit and interest; e.g., availability of field extensions as in item (2) below and potential applicability to restricted ${}_3F_2(\boldsymbol{a})$'s.

(2) From the way in which Lemma 5.1 is proved it is evident that the lemma remains true over $K(\boldsymbol{a})$ for any function field $K \supset \mathbb{Q}$ in $\boldsymbol{a}$ satisfying condition (14).

Proposition 4.3, Lemmas 3.1 and 5.1 are put together to establish the following.

Proposition 5.3 *If* $\boldsymbol{p},\ \boldsymbol{q} \in \mathbb{Z}^5$ *are distinct integer vectors, then*

(1) *the determinant* $\varDelta(\boldsymbol{a};\boldsymbol{p},\boldsymbol{q})$ *in* (37) *is non-vanishing in* $\mathbb{Q}(\boldsymbol{a})$*;*

(2) ${}_3h_2(\boldsymbol{a}+\boldsymbol{p})$ *and* ${}_3h_2(\boldsymbol{a}+\boldsymbol{q})$ *are linearly independent over* $K(\boldsymbol{a})$ *for any function field* $K \supset \mathbb{Q}$ *in* $\boldsymbol{a}$ *satisfying condition* (14)*;*

(3) *there exist unique rational functions* $u(\boldsymbol{a}),\ v(\boldsymbol{a}) \in \mathbb{Q}(\boldsymbol{a})$ *such that*

$$ {}_3h_2(\boldsymbol{a}) = u(\boldsymbol{a})\, {}_3h_2(\boldsymbol{a}+\boldsymbol{p}) + v(\boldsymbol{a})\, {}_3h_2(\boldsymbol{a}+\boldsymbol{q}), \tag{50} $$

where $u(\boldsymbol{a})$ *and* $v(\boldsymbol{a})$ *are common for all choices of* ${}_3h_2(\boldsymbol{a})$ *and explicitly given by*

$$ u(\boldsymbol{a}) = \frac{r(\boldsymbol{a};\boldsymbol{q})}{\varDelta(\boldsymbol{a};\boldsymbol{p},\boldsymbol{q})} = \frac{r(\boldsymbol{a};\boldsymbol{q})}{\delta(\boldsymbol{a};\boldsymbol{p}) \cdot r(\boldsymbol{a}+\boldsymbol{p};\boldsymbol{q}-\boldsymbol{p})}, \tag{51a} $$

$$ v(\boldsymbol{a}) = -\frac{r(\boldsymbol{a};\boldsymbol{p})}{\varDelta(\boldsymbol{a};\boldsymbol{p},\boldsymbol{q})} = -\frac{r(\boldsymbol{a};\boldsymbol{p})}{\delta(\boldsymbol{a};\boldsymbol{p}) \cdot r(\boldsymbol{a}+\boldsymbol{p};\boldsymbol{q}-\boldsymbol{p})}, \tag{51b} $$

where $\delta(\boldsymbol{a};\boldsymbol{p})$ *is defined in* (32).

Proof. Proposition 4.3 implies that $r(\boldsymbol{a}+\boldsymbol{p};\boldsymbol{q}-\boldsymbol{p})$ is nonzero in $\mathbb{Q}(\boldsymbol{a})$, then assertion (1) follows from equation (38) in Lemma 3.1. By assertion (1) the matrix in (36) is invertible over $\mathbb{Q}(\boldsymbol{a})$ and hence over $K(\boldsymbol{a})$, so assertion (2) follows from Lemma 5.1 together with item (2) of Remark 5.2. The second equations in (51a) and (51b) come from Lemma 3.1. Converting equation (36) one has three-term relation (50) with $u(\boldsymbol{a})$ and $v(\boldsymbol{a})$ given by (51). The uniqueness of $u(\boldsymbol{a})$ and $v(\boldsymbol{a})$ in (50) is an easy consequence of assertion (2). Assertion (3) is thus established. □

Summarizing the discussions in §3 and in this section we have the following recipe to determine the coefficients of three-term relation (12). It works effectively on computers.

Recipe 5.4 Given any distinct $\boldsymbol{p},\ \boldsymbol{q} \in \mathbb{Z}^5$, take a lattice path from $\boldsymbol{0}$ to $\boldsymbol{p}$ and calculate the successive product (30) of contiguous matrices along it to produce the connection matrix $A(\boldsymbol{a};\boldsymbol{p})$. From it extract its upper components $r_1(\boldsymbol{a};\boldsymbol{p})$ and $r(\boldsymbol{a};\boldsymbol{p})$ as in (33). With $\boldsymbol{q}$ in place of $\boldsymbol{p}$, proceed exactly in the same manner to get $r_1(\boldsymbol{a};\boldsymbol{q})$ and $r(\boldsymbol{a};\boldsymbol{q})$. Then the coefficients $u(\boldsymbol{a})$ and $v(\boldsymbol{a})$ of three-term relation (12) are given by the first equations in (51a) and (51b). Alternatively one may use the second equations in (51a) and (51b), in which case one should also take the product along a lattice path from $\boldsymbol{0}$ to $\boldsymbol{q}-\boldsymbol{p}$ to deduce $r(\boldsymbol{a}+\boldsymbol{p};\boldsymbol{q}-\boldsymbol{p})$.

With Proposition 5.3 and Recipe 5.4 all assertions of Theorem 1.1 have been established except for assertion (5), which is treated in the next section (see Proposition 6.3).

6 Symmetry

The uniqueness and simultaneousness for the three-term relations provide themselves with a group symmetry of order seventy-two. To see this let

$$ G := \langle \sigma_0,\, \sigma_1,\, \sigma_2,\, \sigma_3,\, \sigma_4 \rangle \qquad \text{with} \quad \sigma_3 := \tau_1, \quad \sigma_4 := \tau_2 $$

be the group of affine transformations generated by the involutions in Table 1. The parametric excess $s(\boldsymbol{a})$ is preserved by the generators and hence by the group G. Note also that

$$\boldsymbol{c} := (2/3, 2/3, 2/3; 1, 1)$$

is the unique fixed point of the generators and hence of G with null parametric excess.

Taking the linear parts of affine transformations induces a surjective group homomorphism

$$G \to \bar{G} := \langle \bar{\sigma}_0,\, \bar{\sigma}_1,\, \bar{\sigma}_2,\, \bar{\sigma}_3,\, \bar{\sigma}_4 \rangle, \qquad \sigma \mapsto \bar{\sigma},$$

where $\bar{\sigma}$ stands for the linear part of an affine $\sigma \in G$. This is just an isomorphism because

$$\bar{\sigma} = t^{-1} \cdot \sigma \cdot t \qquad \text{with} \quad t : \boldsymbol{a} \mapsto \boldsymbol{a} + \boldsymbol{c} \quad \text{being the parallel translation by } \boldsymbol{c}.$$

Each $\bar{\sigma}_i$ is a $\mathbb{Z}$-linear transformation of determinant $\det \bar{\sigma}_i = \pm 1$, actually $+1$ for $i = 0, 1, 2$ and -1 for $i = 3, 4$, so that every $\bar{\sigma} \in \bar{G}$ maps the lattice $\mathbb{Z}^5$ isomorphically onto itself.

Put $u(\boldsymbol{a}; \boldsymbol{p}, \boldsymbol{q}) := r(\boldsymbol{a}; \boldsymbol{q})/\Delta(\boldsymbol{a}; \boldsymbol{p}, \boldsymbol{q})$. Note that $u(\boldsymbol{a}) = u(\boldsymbol{a}; \boldsymbol{p}, \boldsymbol{q})$ and $v(\boldsymbol{a}) = u(\boldsymbol{a}; \boldsymbol{q}, \boldsymbol{p})$ in formulas (51). We show that $u(\boldsymbol{a}; \boldsymbol{p}, \boldsymbol{q})$ enjoys the following G-covariance.

Lemma 6.1 *For any distinct* $\boldsymbol{p}$, $\boldsymbol{q} \in \mathbb{Z}^5$ *and any* $\sigma \in G$ *we have*

$$u(\boldsymbol{a}; \bar{\sigma}(\boldsymbol{p}), \bar{\sigma}(\boldsymbol{q})) = u(\sigma^{-1}(\boldsymbol{a}); \boldsymbol{p}, \boldsymbol{q}). \tag{52}$$

Proof. In three-term relation (50) we take ${}_3h_2(\boldsymbol{a}) = {}_3f_2(\sigma_i(\boldsymbol{a}))$ for each $i = 0, 1, 2, 3, 4$. Formula (50) with ${}_3h_2$, $\boldsymbol{a}$, $\boldsymbol{p}$ and $\boldsymbol{q}$ replaced by ${}_3f_2$, $\sigma_i(\boldsymbol{a})$, $\bar{\sigma}_i(\boldsymbol{p})$ and $\bar{\sigma}_i(\boldsymbol{q})$ yields

$$\begin{aligned}
{}_3h_2(\boldsymbol{a}) &= {}_3f_2(\sigma_i(\boldsymbol{a})) \\
&= u(\sigma_i(\boldsymbol{a}); \bar{\sigma}_i(\boldsymbol{p}), \bar{\sigma}_i(\boldsymbol{q}))\, {}_3f_2(\sigma_i(\boldsymbol{a}) + \bar{\sigma}_i(\boldsymbol{p})) + u(\sigma_i(\boldsymbol{a}); \bar{\sigma}_i(\boldsymbol{q}), \bar{\sigma}_i(\boldsymbol{p}))\, {}_3f_2(\sigma_i(\boldsymbol{a}) + \bar{\sigma}_i(\boldsymbol{q})) \\
&= u(\sigma_i(\boldsymbol{a}); \bar{\sigma}_i(\boldsymbol{p}), \bar{\sigma}_i(\boldsymbol{q}))\, {}_3f_2(\sigma_i(\boldsymbol{a} + \boldsymbol{p})) + u(\sigma_i(\boldsymbol{a}); \bar{\sigma}_i(\boldsymbol{q}), \bar{\sigma}_i(\boldsymbol{p}))\, {}_3f_2(\sigma_i(\boldsymbol{a} + \boldsymbol{q})) \\
&= u(\sigma_i(\boldsymbol{a}); \bar{\sigma}_i(\boldsymbol{p}), \bar{\sigma}_i(\boldsymbol{q}))\, {}_3h_2(\boldsymbol{a} + \boldsymbol{p}) + u(\sigma_i(\boldsymbol{a}); \bar{\sigma}_i(\boldsymbol{q}), \bar{\sigma}_i(\boldsymbol{p}))\, {}_3h_2(\boldsymbol{a} + \boldsymbol{q}),
\end{aligned}$$

which must coincide with (50) by the uniqueness of three-term relation. Thus we have

$$u(\sigma_i(\boldsymbol{a}); \bar{\sigma}_i(\boldsymbol{p}), \bar{\sigma}_i(\boldsymbol{q})) = u(\boldsymbol{a}; \boldsymbol{p}, \boldsymbol{q}), \quad u(\sigma_i(\boldsymbol{a}); \bar{\sigma}_i(\boldsymbol{q}), \bar{\sigma}_i(\boldsymbol{p})) = u(\boldsymbol{a}; \boldsymbol{q}, \boldsymbol{p}), \quad i = 0, 1, 2, 3, 4,$$

and hence $u(\sigma(\boldsymbol{a}); \bar{\sigma}(\boldsymbol{p}), \bar{\sigma}(\boldsymbol{q})) = u(\boldsymbol{a}; \boldsymbol{p}, \boldsymbol{q})$ for every $\sigma \in G$. Formula (52) then follows from the last equation by replacing $\boldsymbol{a}$ with $\sigma^{-1}(\boldsymbol{a})$. □

We are interested in the group structure of G or $\bar{G}$. It is easy to see that

$$\rho_i := \tau_i\, \sigma_0\, \tau_i\, \sigma_0\, \tau_i = \tau_i\, \sigma_i\, \tau_i\, \sigma_i\, \tau_i \in G, \qquad i = 1, 2,$$

are involutions such that $\sigma_i = \rho_i\, \sigma_0\, \rho_i$ and $\tau_i = \sigma_0\, \rho_i\, \sigma_0$ for $i = 1, 2$, so that we have

$$G = \langle \sigma_0,\, \rho_1,\, \rho_2,\, \tau_1,\, \tau_2 \rangle = \langle \sigma_0,\, \rho_1,\, \rho_2 \rangle.$$

Lemma 6.2 *The group structure of* $\bar{G}$ *is given by*

$$\bar{G} = \langle \bar{\sigma}_0 \rangle \ltimes (\langle \bar{\rho}_1,\, \bar{\rho}_2 \rangle \times \langle \bar{\tau}_1,\, \bar{\tau}_2 \rangle) \cong S_2 \ltimes (S_3 \times S_3), \tag{53}$$

in particular $\bar{G}$ *is a group of order* $2! \times (3! \times 3!) = 72$. *Viewed as acting on the real vector space* $\mathbb{R}^5$ *with coordinates* $\boldsymbol{p} = (p_0, p_1, p_2; q_1, q_2)$, *the group* $\bar{G}$ *has a fundamental domain*

$$p_0 \geq p_1 \geq p_2, \quad q_2 \geq q_1 \geq p_0 - p_1. \tag{54}$$

Proof. As a linear basis of $\mathbb{R}^5$ we take five vectors

$$\boldsymbol{u}_1 := (1/3, -2/3, 1/3; 0, 0), \quad \boldsymbol{v}_1 := (1/3, 1/3, 1/3; 1, 0), \quad \boldsymbol{w} := (-1/3, -1/3, -1/3; 0, 0),$$
$$\boldsymbol{u}_2 := (1/3, 1/3, -2/3; 0, 0), \quad \boldsymbol{v}_2 := (1/3, 1/3, 1/3; 0, 1),$$

along with two auxiliary vectors such that $\boldsymbol{u}_0 + \boldsymbol{u}_1 + \boldsymbol{u}_2 = \boldsymbol{0}$ and $\boldsymbol{v}_0 + \boldsymbol{v}_1 + \boldsymbol{v}_2 = \boldsymbol{0}$, namely,

$$\boldsymbol{u}_0 := (-2/3, 1/3, 1/3; 0, 0), \qquad \boldsymbol{v}_0 := (-2/3, -2/3, -2/3; -1, -1).$$

The linear action of $\bar{G}$ on $\mathbb{R}^5$ is faithfully represented by permutations on these vectors:

$$\bar{\rho}_i : \boldsymbol{u}_0 \leftrightarrow \boldsymbol{u}_i, \quad \bar{\tau}_i : \boldsymbol{v}_0 \leftrightarrow \boldsymbol{v}_i, \quad i = 1, 2; \qquad \bar{\sigma}_0 : \boldsymbol{u}_0 \leftrightarrow \boldsymbol{v}_0, \ \boldsymbol{u}_1 \leftrightarrow \boldsymbol{v}_1, \ \boldsymbol{u}_2 \leftrightarrow \boldsymbol{v}_2,$$

where the vectors left invariant are not indicated. Thus we have two actions

$$S_3 \cong \langle \bar{\rho}_1, \bar{\rho}_2 \rangle \curvearrowright U := \mathbb{R}\,\boldsymbol{u}_1 \oplus \mathbb{R}\,\boldsymbol{u}_2, \qquad S_3 \cong \langle \bar{\tau}_1, \bar{\tau}_2 \rangle \curvearrowright V := \mathbb{R}\,\boldsymbol{v}_1 \oplus \mathbb{R}\,\boldsymbol{v}_2,$$

which are commutative and permuted by $\bar{\sigma}_0$, together with a line $\mathbb{R}\,\boldsymbol{w}$ fixed pointwise by $\bar{G}$. These observations clearly show that $\bar{G}$ has the group structure (53). If $\boldsymbol{p}$ is represented as

$$\boldsymbol{p} = x_1\,\boldsymbol{u}_1 + x_2\,\boldsymbol{u}_2 + y_1\,\boldsymbol{v}_1 + y_2\,\boldsymbol{v}_2 + z\boldsymbol{w} \in U \oplus V \oplus \mathbb{R}\,\boldsymbol{w},$$

then $x_1 = p_0 - p_1$, $x_2 = p_0 - p_2$, $y_1 = q_1$, $y_2 = q_2$ and $z = s(\boldsymbol{p})$. As a fundamental domain of $\langle \bar{\rho}_1, \bar{\rho}_2 \rangle \times \langle \bar{\tau}_1, \bar{\tau}_2 \rangle$ we can take $D_0 := \{ x_2 \ge x_1 \ge 0, \ y_2 \ge y_1 \ge 0 \}$. Note that $\bar{\sigma}_0$ maps D_0 onto itself while swapping two conditions $x_2 \ge x_1 \ge 0$ and $y_2 \ge y_1 \ge 0$. Thus we can impose condition $y_1 \ge x_1$ to get a fundamental domain $D := \{ x_2 \ge x_1 \ge 0, \ y_2 \ge y_1 \ge 0, \ y_1 \ge x_1 \}$ of the whole group $\bar{G}$. In terms of the original coordinates of $\boldsymbol{p}$ the domain D is given by (54). □

Putting Lemmas 6.1 and 6.2 together we have the following.

Proposition 6.3 *Three-term relation* (12) *admits a* $S_2 \ltimes (S_3 \times S_3)$*-symmetry*

$$ {}_3h_2(\boldsymbol{a}) = {}^{\sigma}u(\boldsymbol{a})\, {}_3h_2(\boldsymbol{a} + \bar{\sigma}(\boldsymbol{p})) + {}^{\sigma}v(\boldsymbol{a})\, {}_3h_2(\boldsymbol{a} + \bar{\sigma}(\boldsymbol{q})), \qquad \sigma \in G, \tag{55}$$

where ${}^{\sigma}w(\boldsymbol{a}) := w(\sigma^{-1}(\boldsymbol{a}))$ *is the induced action of* σ *on a function* $w(\boldsymbol{a})$*.*

Remark 6.4 A couple of remarks are in order at this stage.

(1) Formula (55) is of particular interest when $\boldsymbol{q}$ is $\boldsymbol{e} := (1, 1, 1; 0, 0)$ or a nonzero integer multiple of it, because in that case $\boldsymbol{q}$ is $\bar{G}$-invariant and hence (55) becomes

$$ {}_3h_2(\boldsymbol{a}) = {}^{\sigma}u(\boldsymbol{a})\, {}_3h_2(\boldsymbol{a} + \bar{\sigma}(\boldsymbol{p})) + {}^{\sigma}v(\boldsymbol{a})\, {}_3h_2(\boldsymbol{a} + \boldsymbol{q}), \qquad \sigma \in G,$$

so without loss of generality $\boldsymbol{p}$ may lie within a fundamental domain of $\bar{G}$ as in (54).

(2) In §3 the contiguous and connection matrices $A_i^{\varepsilon}(\boldsymbol{a})$ and $A(\boldsymbol{a}; \boldsymbol{p})$ were formulated in terms of the basis ${}_3\boldsymbol{h}_2(\boldsymbol{a}) = {}^t({}_3h_2(\boldsymbol{a}), {}_3h_2(\boldsymbol{a}+\boldsymbol{1}))$ as in (23), where $\boldsymbol{1}$ was not $\bar{G}$-invariant. From the viewpoint of symmetry it is better to reformulate them in term of an alternative basis ${}_3\tilde{\boldsymbol{h}}_2(\boldsymbol{a}) := {}^t({}_3h_2(\boldsymbol{a}), {}_3h_2(\boldsymbol{a} + \boldsymbol{e}))$. The revised contiguous and connection matrices are

$$\tilde{A}_i^{\varepsilon}(\boldsymbol{a}) = P(\boldsymbol{a} + \varepsilon\, \boldsymbol{e}_i)\, A_i^{\varepsilon}(\boldsymbol{a})\, P(\boldsymbol{a})^{-1}, \qquad \tilde{A}(\boldsymbol{a}; \boldsymbol{p}) = P(\boldsymbol{a} + \boldsymbol{p})\, A(\boldsymbol{a}; \boldsymbol{p})\, P(\boldsymbol{a})^{-1},$$

where $P(\boldsymbol{a})$ is the invertible matrix over $\mathbb{Q}(\boldsymbol{a})$ such that ${}_3\tilde{\boldsymbol{h}}_2(\boldsymbol{a}) = P(\boldsymbol{a})\, {}_3\boldsymbol{h}_2(\boldsymbol{a})$. An advantage of this base change is that $\tilde{A}(\boldsymbol{a}; \boldsymbol{p})$ gains the nice G-covariance property

$$\tilde{A}(\boldsymbol{a}; \bar{\sigma}(\boldsymbol{p})) = \tilde{A}(\sigma^{-1}(\boldsymbol{a}); \boldsymbol{p}), \qquad \sigma \in G,$$

but unfortunately explicit formula for $\tilde{A}_i^{\varepsilon}(\boldsymbol{a})$ is too complicated to be presented here.

(3) Let $\rho_3 := \tau_1\tau_2\tau_1 = \tau_2\tau_1\tau_2 \in G$. Then G contains a subgroup $G_0 := \langle \rho_1, \rho_2\rangle \times \langle \rho_3 \rangle \cong S_3 \times S_2$ that acts on ${}_3f_2(\boldsymbol{a})$ trivially permuting its numerator or denominator parameters. The left quotient set $G_0\backslash G$ is then represented by σ_i, $i = 0, 1, 2, 3, 4$, and the unit element, so the G-orbit of ${}_3f_2(\boldsymbol{a})$ is exactly the six functions in (10) up to the trivial symmetry G_0.

7 Applications

We conclude this article by touching on two applications, one major and one minor.

A major application is regarding continued fractions. Three-term contiguous relations give rise to three-term recurrence relations, which in turn generate continued fractions, so the results and methods of this article are used in a fundamental manner to develop a general theory of ${}_3F_2(1)$ continued fractions, in which an infinite number of continued fraction expansions with exact error term estimates are established for the ratios of two ${}_3F_2(1)$ hypergeometric series. For the details we refer to the article [6].

A minor application concerns the impossibility of representing unrestricted ${}_3F_2(1)$ function as products and quotients of gamma functions like

$$ {}_3F_2(\boldsymbol{a}; 1) = C \cdot d^{k(\boldsymbol{a})} \frac{\prod_{i=1}^{I} \Gamma(l_i(\boldsymbol{a}))}{\prod_{j=1}^{J} \Gamma(m_j(\boldsymbol{a}))}, $$

where C, $d \in \mathbb{C}^{\times}$ and $k(\boldsymbol{a})$, $l_i(\boldsymbol{a})$, $m_j(\boldsymbol{a})$ are complex affine polynomials in $\boldsymbol{a}$ such that the homogeneous linear parts of $l_i(\boldsymbol{a})$ and $m_j(\boldsymbol{a})$ have coefficients in $\mathbb{Q}$. Indeed, if such an identity existed then Proposition 5.3 and the recursion formula $\Gamma(z+1) = z\,\Gamma(z)$ would readily lead to a contradiction. This gives an alternative approach to Wimp's results [12, Theorems 3 and 4].

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