Doctoral Dissertation

Semigroups and Geometry, and Link invariants constructed by semigroups

(半群と幾何学及び半群から構成される絡み目の不変量に関して)

Toshinori Miyatani

Depertment of Mathematics
Graduate School of Science, Hokkaido University

2020.06

Preface

We shall consider two theories related to semigroups. In part I, we consider the theory of $\mathbb{P}M$-monoids. The braid groups and the symmetric groups have deep relations and have rich theories. The braid groups are generalized to the Artin groups and the symmetric groups to the Coxeter groups [3]. We will consider two monoids analogous to the symmetric group S_n and the braid group B_n, respectively. We first define a monoid $\mathscr{R}_n$, which we call a $\mathbb{P}M$-monoid. A $\mathbb{P}M$-monoid can be seen as an analogue of the rook monoid defined by L. Solomon [27]. A $\mathbb{P}M$-monoid is obtained in the context of a compactification of the projective linear group defined by Mutsumi Saito [25]. The structure of a $\mathbb{P}M$-monoid is described by a matched pair of S_n and a collection of the ordered partition (Proposition 2.1.2). We show that a $\mathbb{P}M$-monoid has a presentation with generators and relations (Proposition 2.2.2). This is an analogue of the fact that the rook monoid has a presentation with generators and relations [27]. In the context of this presentation, we define a braid $\mathbb{P}M$-monoid denoted by $\mathscr{R}\mathscr{B}_n$ (Definition 2.3.1). The braid $\mathbb{P}M$-monoid is an analogue of the inverse braid monoid defined by D. Easdown and T. G. Lavers [5]. As a main result of the part I, we will show that the braid $\mathbb{P}M$-monoid has a presentation by geometric braids and contains the braid group (Theorem 2.3.7). This is an analogue of the fact that the braid groups and the inverse braid monoids have the presentation by the geometric braids [14],[5]. Moreover, we shall find a solution to the word problem of the braid $\mathbb{P}M$-monoid (Theorem 2.3.12). This statement is an analogue of the fact that the braid groups and the inverse braid monoids have a solution to the word problem [8],[33].

In part II, we develop the theory of knot semigroups, which were defined by A. Vernitski in [33]. A knot semigroup is a cancellative semigroup whose defining relations come in pairs of the form $xy = yx$ and $yx = zy$ arizing from crossing points of a given knot diagram. This construction is similar to the Wirtinger presentation of a knot group. Vernitski proved in [33] that the knot semigroup of torus knots and twist knots are isomorphic to what he calles alternating sum semigroups and conjectured that the knot semigroup of

2-bridge knot is isomorphic to an alternating sum semigroup [33]. To support this conjecture, we shall prove that the knot semigroup of the double twist knot is isomorphic to an alternating sum semigroup (Theorem 4.1.2). Next, we consider the growth of knot semigroups. To investigate the growth of knot semigroups, we use the Gelfand-Kirillov dimension of semigroup algebra. As a main result of this part, we construct a link invariant arizing from the Gelfand-Kirillov dimension of algebra (Theorem 5.2.1). This research is a first step connecting knot theory and semigroup theory.

Contents

Part I

$\mathbb{P}$M-monoids and braid $\mathbb{P}$M-monoids

Chapter 1

Preliminaries

1.1 Semigroups

We shall explain basic concept related to semigroups.

1.1.1 Definitions

Definition 1.1.1. A semigroup is a set S equipped with a binary operation $S \times S \to S$ that is associative. Moreover, if S has an identity element 1, then S is called a monoid.

Next we define the cancellative semigroups.

Definition 1.1.2. A semigroup S is called cancellative if it satisfies two conditions : if $xz = yz$ then $x = y$, and if $xy = xz$ then $y = z$ for $x, y, z \in S$.

We shall define an inverse semigroup.

Definition 1.1.3. An inverse semigroup is a semigroup S such that, for each $x \in S$, there exists a unique $y \in S$ such that

$$xyx = x, \text{ and } yxy = y.$$

1.1.2 Presentations of a semigroup

Let X be a set, and X^* the set of words of X. Let R be the subset of $X^* \times X^*$. We define $\sim$ the smallest equivalence relation on X^* containing all pairs $(w_1 r w_2, w_1 r' w_2)$, where $(r, r') \in R$ and $w_1, w_2 \in X^*$. We define

$$\langle X \mid R \rangle := X^* / \sim .$$

$\langle X \mid R \rangle$ is called a semigroup presentation. The elements of X are called generators and the elements of R are called relations.

The word problem for a presentation $\langle X \mid R \rangle$ is the following: given two words $w, w' \in X^*$ representing certain $a, a' \in \langle X \mid R \rangle$, determine whether $a = a'$.

1.2 Braid groups

We will review braid groups.

Definition 1.2.1. The braid group B_n is the group generated by $n - 1$ elements $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ with the braid relations

$$\begin{aligned} \sigma_i\sigma_j &= \sigma_j\sigma_i && (i = 1, 2, \ldots, n-1, |i-j| \geq 2), \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} && (i = 1, 2, \ldots, n-2). \end{aligned}$$

An element of the braid group can be represented by a braid diagram. We denote by I the closed interval $[0, 1]$ in $\mathbb{R}$. By a topological interval, we mean a topological space homeomorphic to $I = [0, 1]$.

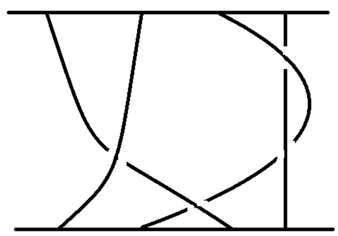
Definition 1.2.2. A braid diagram on n strands is a set $D \subset \mathbb{R} \times I$ split as a union of n topological intervals called the strands of D such that the following three conditions are met:

(1) The projection $\mathbb{R} \times I \to I$ maps each strand homeomorphically onto I.

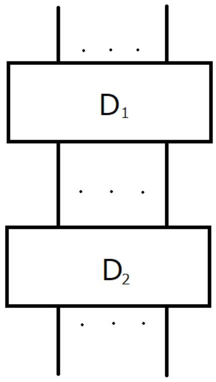
(2) Every point of $\{1, 2, \ldots, n\} \times \{0, 1\}$ is the endpoint of a unique strand.

(3) Every point of $\mathbb{R} \times I$ belongs to at most two strands. At each intersection point of two strands, these strands meet transversely, and one of them is distinguished and said to be undergoing, the other strands being overgoing.

An example of a braid diagram is given below.



Given two braid diagrams D_1, D_2 on n strands, their product D_1D_2 is obtained by placing D_1 on the top of D_2 and squeezing the resulting diagram into $\mathbb{R} \times I$.

$D_1D_2 =$ 

Denote by $\mathcal{B}_n$ the set of braid diagrams on n strands with multiplication defined above. In fact, $\mathcal{B}_n$ becomes a group and the following theorem holds.

Theorem 1.2.3. *The groups B_n and $\mathcal{B}_n$ are isomorphic.*

1.3 Algebraic monoids

The definition of $\mathbb{P}M$-monoid is motivated by the definition of the Renner monoids. The Renner monoid appears in the algebraic monoid theory. So we shall explain the algebraic monoid theory. Let K be an algebraically closed field. Let $M_n = M_n(K)$ denote the set of all $n \times n$ matrices over K.

Definition 1.3.1. A linear algebraic monoid is a submonoid of M_n which is a Zariski closed subset.

Let M be a reductive monoid, i.e., M is a linear algebraic monoid which is irreducible as an algebraic set and has a connected reductive group G of units. Let T be a maximal torus of G. Then

$$R = \overline{N_G(T)}/T$$

is called a Renner monoid ([24] Definition 11.2), where the closure is taken in Zariski topology. This contains the Weyl group $W = N_G(T)/T$ of G. Renner monoids play the central role in the linear algebraic monoid theory like Weyl groups do in the linear algebraic group theory, and have the following properties. Let $E(M)$ be the set of idempotents of M, and $P(e) = \{g \in G \mid ge = ege\}$ for $e \in E(M)$. Let B be a Borel subgroup containing T, and $\Lambda(B) = \{e \in E(\overline{T}) \mid P(e) \supseteq B\}$.

Theorem 1.3.2 ([28] Theorem 5.10.)**.** *Let M be a reductive monoid, and $e \in \Lambda(B)$. Then*

(1) *R is a finite inverse monoid.*

(2) *The group of units of R coincide with W, and $R = WE(R)$.*

(3) $E(R) \simeq E(\overline{T})$.

(4) $M = \sum_{\rho \in R} B\rho B$, *and* $B\rho B = B\rho' B$ implies $\rho = \rho'$.

(5) *If s is a Coxeter generator, then $BsB \cdot B\rho B \subseteq Bs\rho B \cup B\rho B$.*

(6) $GeG = \sum_{\rho \in WeW} B\rho B.$

(7) *If $w_0 \in W$ is the longest element, then Bw_0eB is open and dense in GeG.*

1.4 Rook monoids

Let R_n be the set of $n \times n$ zero-one matrices which have at most one entry equal to 1 in each row and in each column. The monoid R_n is called the rook monoid, which is the Renner monod of type A. The rook monoid R_n has the following presentation using a generating set and relations:

Theorem 1.4.1 ([10] Prop 1.6.)**.** *The rook monoid has a monoid presentation with generating set $\{s_1, \ldots, s_{n-1}, e_0, \ldots, e_{n-1}\}$ and defining relations:*

$$
\begin{aligned}
s_i^2 &= 1 && (1 \le i \le n-1),\\
s_i s_j &= s_j s_i && (1 \le i, j \le n-1,\ |i-j| \ge 2),\\
s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && (1 \le i \le n-1),\\
e_i e_j &= e_j e_i = e_{\min(i,j)} && (0 \le i, j \le n-1),\\
e_j s_i &= s_i e_j && (1 \le i < j \le n-1),\\
e_j s_i &= s_i e_j = e_j && (0 \le j < i \le n-1),\\
e_i s_i e_i &= s_i e_{i-1} && (1 \le i \le n-1).
\end{aligned}
$$

1.5 Inverse braid monoids

We explain inverse braid monoids defined by D. Easdown and T.G. Lavers [5].

Definition 1.5.1. The inverse braid group IB_n is the group generated by $\sigma_1^{\pm}, \sigma_2^{\pm}, \ldots, \sigma_{n-1}^{\pm}, \epsilon$ with relations

$$
\begin{aligned}
\sigma_i\sigma_i^{-1} &= \sigma_i^{-1}\sigma_i = 1 && (i = 1, 2, \ldots, n-1),\\
\sigma_i\sigma_j &= \sigma_j\sigma_i && (i = 1, 2, \ldots, n-1, |i-j| \geq 2),\\
\sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} && (i = 1, 2, \ldots, n-2),\\
\epsilon^2 &= \epsilon = \epsilon\sigma_{n-1}^2 = \sigma_{n-1}^2\epsilon, &&\\
\epsilon\sigma_i &= \sigma_i\epsilon && (i = 1, 2, \ldots, n-2),\\
\epsilon\sigma_{n-1}\epsilon &= \sigma_{n-1}\epsilon\sigma_{n-1}\epsilon = \epsilon\sigma_{n-1}\epsilon\sigma_{n-1}. &&
\end{aligned}
$$

The relation of the symmetric group and the braid group is generalized to the relation of the rook monoid and the inverse braid monoid.

We next define a partial braid diagram. Take the usual coordinate system for $\mathbb{R}^3$ (in which we think of the z-axis as pointing downwards). Choose $z_0 < z_1$ and call the planes $z = z_0$ and z_1 upper and lower, respectively. Mark $n \geq 1$ distinct points $P_1, \ldots, P_n$ on a line in the upper plane and project them orthogonally onto the lower plane yielding points $P'_1, \ldots, P'_n$. An arc is the image of an embedding from interval $[0, 1]$ into $\mathbb{R}^3$. A partial braid on n strings is a system

$$\beta = \{\beta_1, \ldots, \beta_m\}$$

of m arcs for some $m \leq n$ such that

(1) there is a rank m partial one-one mapping $\Phi^\beta : \{1, \ldots, n\} \to \{1, \ldots, n\}$ with domain $\{i_1, \ldots, i_m\}$ such that β_i connects P_{i_j} to $P_{\phi^\beta(i_j)}$ for $j = 1, \ldots, m$,

(2) each arc intersects each intermediate parallel plane between (and including) the upper and lower planes exactly once,

(3) the union $\beta_1 \cup \cdots \cup \beta_m$ of the arcs intersect each intermediate parallel plane between the upper and lower planes in exactly m distinct points.

If $m = 0$, we get the empty braid.

Two partial braids $\beta = \{\beta_1, \ldots, \beta_m\}$ and $\gamma = \{\gamma_1, \ldots, \gamma_{m'}\}$ are called equivalent if

(1) $\Phi^\beta = \Phi^\gamma$. In particular $m = m'$ and we may write the domain of Φ^β as $\{i_1, \ldots, i_m\}$ for some $i_1 < \cdots < i_m$,

(2) β and γ are homotopy equivalent, which we mean that there exist continuous maps
$$F_j : [0,1] \times [0,1] \to \mathbb{R}^3$$
for $j = 1, \ldots, m$, such that, for all $t \in [0,1]$,
$$F_j(t,0) = \beta_j(t) \text{ and } F_j(t,1) = \gamma_j(t),$$
for all $s \in [0,1]$,
$$F_j(0,s) = P_{i_j} \text{ and } F_j(1,s) = P'_{\Phi^\beta(i_j)},$$
and, for each $s \in [0,1]$, if we define
$$\beta^s = \{\beta_1^s, \ldots, \beta_m^s\}$$
where
$$\beta_j^s(t) = F_j(s,t) \text{ for } j = 1, \ldots, m,$$
then β^s is itself a partial braid. Write $\beta \equiv \gamma$ if β and γ are equivalent and $[\beta]$ for the equivalence class containing β.

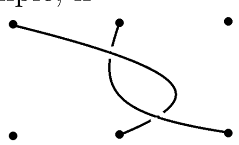
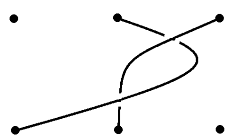
Define the product $\beta_1\beta_2$ of two partial braids β_1 and β_2 as follows:

(1) translate β_2 parallel to itself in space so that the upper plane of β_2 and the lower plane of β_1 coincide,

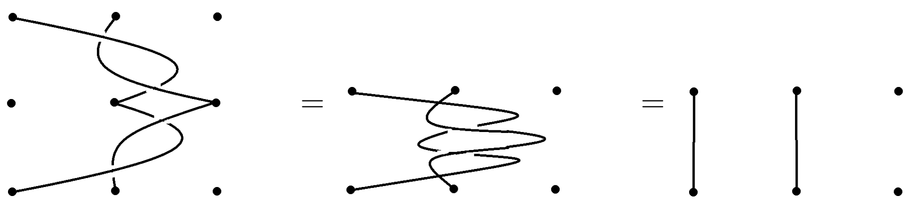
(2) keeping the plane $z = z_0$ fixed, contract the resulting system of arcs so that the translated lower plane of β_2 moves into the position of the plane $z = z_1$,

(3) remove any arcs that do not join the upper and lower planes.

For example, if

$\alpha =$  $\beta =$ 

Then

$\alpha\beta =$ 

Put

$$M_n = \{[\beta] \mid \beta \text{ is a partial braid }\}.$$

Then we have the following theorem.

Theorem 1.5.2 ([5] Theorem 3.1.). *The monoids M_n and IB_n are isomorphic.*

1.6 Compactifications of projective linear groups

We explain the compactification of the projective linear group constructed by M. Saito [25].

1.6.1 Motivation

One strategy of compactification is constructing a "limit". Then we consider the set of all limit points and introduce a topology compatible with the limit. For instance Y. A. Neretin constructed a compactification of the projective linear group by this strategy called hinge [22].

Let V be an n-dimensional vector space over $\mathbb{C}$ and $A_i \in \mathrm{End}(V)$, $(i = 1, 2, \dots)$. Suppose that the linear map

$$A_\epsilon := \sum_{i=0}^{m} A_i \epsilon^i$$

is in $\mathrm{GL}(V)$ for $\epsilon \in \mathbb{R}\backslash\{0\}$. Dividing by nonzero scalar matrices we consider the projective linear map

$$\overline{A_\epsilon} \in \mathrm{PGL}(V). \tag{1.1}$$

We want to define a "limit" $\lim_{\epsilon\to 0} \overline{A_\epsilon}$. To define a limit, we observe the action of $\overline{A_\epsilon}$ on $\mathbb{P}(V)$. For $\overline{x} \in \mathbb{P}(V)$ we have

$$\lim_{\epsilon\to 0} \overline{A_\epsilon(x)} = \begin{cases} \overline{A_0 x} & (x \notin \mathrm{Ker}A_0) \\ \overline{A_1 x} & (x \notin \mathrm{Ker}A_0 \backslash \mathrm{Ker}A_1) \\ \overline{A_2 x} & (x \notin \mathrm{Ker}A_0 \cap \mathrm{Ker}A_1 \backslash \mathrm{Ker}A_2) \\ \vdots & \end{cases}.$$

Thus we define the limit of (1.1) as

$$\lim_{\epsilon\to 0} \overline{A_\epsilon} := (\overline{A_0}, \overline{A_1}|_{\mathbb{P}(\mathrm{Ker}A_0)}, \overline{A_2}|_{\mathbb{P}(\mathrm{Ker}A_0 \cap \mathrm{Ker}A_1)}, \dots). \tag{1.2}$$

1.6.2 Definition of $\mathbb{P}M$

In order to construct a compactification of the projective linear group, we consider the set of forms of the right hand side of (1.2). We define the following sets. Let V be an n-dimensional vector space over $\mathbb{C}$. Set

$$M := M(V) = \left\{ (A_0, A_1, \ldots, A_m) \mid \begin{array}{c} m = 0, 1, 2, \ldots \\ 0 \neq A_i \in \mathrm{Hom}(V_i, V) \ (0 \leq i \leq m) \\ V_0 = V, V_{m+1} = 0 \\ V_{i+1} = \mathrm{Ker}(A_i) \ (0 \leq i \leq m) \end{array} \right\}$$

and

$$\widetilde{M} := \widetilde{M}(V) = \left\{ (A_0, A_1, \ldots, A_m) \mid \begin{array}{c} m = 0, 1, 2, \ldots \\ 0 \neq A_i \in \mathrm{End}(V) \ (0 \leq i \leq m) \\ \cap_{k=0}^{i-1} \mathrm{Ker} A_k \not\subseteq \mathrm{Ker} A_i \\ \cap_{k=0}^{m} \mathrm{Ker} A_k = 0 \end{array} \right\}.$$

Let $\mathbb{A} := (A_0, A_1, \ldots, A_m) \in M$. Since $A_i \in \mathrm{Hom}(V_i, V) \backslash \{0\}$, we can consider the element $\overline{A_i} \in \mathbb{P}\mathrm{Hom}(V_i, V)$ represented by A_i, and we can define

$$\mathbb{P}\mathbb{A} := (\overline{A_0}, \overline{A_1}, \ldots, \overline{A_m}).$$

Let $\mathbb{P}M = \mathbb{P}M(V)$ denote the image of M under $\mathbb{P}$. $\mathbb{P}\widetilde{M}$ can be defined similarly.

1.6.3 Topology of $\mathbb{P}M$

We introduce a topology in $\mathbb{P}M$ which we can deal with the limit (1.2). We fix a Hermitian inner product on V. Let W be a subspace of V. By considering $V = W \oplus W^{\perp}$ via this inner product, we regard $\mathrm{Hom}(W, V)$ as a subspace of $\mathrm{End}(V)$. We consider the classical topology in $\mathbb{P}\mathrm{Hom}(W, V)$ for any subspace W of V.

Let $\mathbb{A} = (A_0, A_1, \ldots, A_m) \in M$. Then $A_i \in \mathrm{Hom}(V_i, V)$, where $V_i = V(\mathbb{A})_i = \mathrm{Ker}(A_{i-1})$. Let U_i be a neighborhood of $\overline{A_i}$ in $\mathbb{P}\mathrm{Hom}(V(\mathbb{A})_i, V)$. Then set

$$U_{\mathbb{P}\mathbb{A}}(U_0, \ldots, U_m) = \left\{ \mathbb{P}\mathbb{B} = (\overline{B_0}, \overline{B_1}, \ldots, \overline{B_n}) \mid \begin{array}{c} {}^{\forall} i = 1, \ldots, m, {}^{\exists} j \in \{1, \ldots, n\} \text{ s.t.} \\ V(\mathbb{B})_j \supseteq V(\mathbb{A})_i \text{ and} \\ \overline{B_j|_{V(\mathbb{A})_i}} \in U_i \end{array} \right\}. \tag{1.3}$$

We will explain the fact that the sets (1.3) can define a topology that deal with the limit (1.2) by using the following example.

Example 1.6.1. Let V be a 4-dimensional vector space over $\mathbb{C}$. Taking the standard basis, we identify $V \cong \mathbb{C}^4$. Let

$$\mathbb{A} = (A_0, A_1, A_2, A_3) = \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right),$$

$$\mathbb{B}(t) = (B_0(t), B_1(t)) = \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & t \end{pmatrix} \right),$$

and let U_i be a neighborhood of A_i, $(i = 0, 1, 2, 3)$. In the rule of (1.2), $\mathbb{B}(t)$ converges to $\mathbb{A}$ when $t \to 0$. In terms of (1.3), we want to have

$$\mathbb{B}(t) \in U_{\mathbb{PA}}(U_0, U_1, U_2, U_3) \tag{1.4}$$

when $t << 0$. In fact, (1.4) holds by the following :

$$V(\mathbb{B})_0 \supseteq V(\mathbb{A})_0,\ \overline{B_0(t)|_{V(\mathbb{A})_0}} \in U_0 \text{ since } \lim_{t\to 0} \overline{B_0(t)|_{V(\mathbb{A})_0}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$V(\mathbb{B})_0 \supseteq V(\mathbb{A})_1,\ \overline{B_0(t)|_{V(\mathbb{A})_1}} \in U_1 \text{ since } \lim_{t\to 0} \overline{B_0(t)|_{V(\mathbb{A})_1}} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$V(\mathbb{B})_1 \supseteq V(\mathbb{A})_2,\ \overline{B_1(t)|_{V(\mathbb{A})_2}} \in U_2 \text{ since } \lim_{t\to 0} \overline{B_1(t)|_{V(\mathbb{A})_2}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$V(\mathbb{B})_1 \supseteq V(\mathbb{A})_3,\ \overline{B_1(t)|_{V(\mathbb{A})_3}} \in U_3 \text{ since } \lim_{t\to 0} \overline{B_1(t)|_{V(\mathbb{A})_3}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

In fact (1.3) induces a topology on $\mathbb{P}M$ by the following lemma.

Lemma 1.6.2 ([25] Lemma 3.2.)**.** *The sets*

$$\{U_{\mathbb{PA}}(U_0, \ldots, U_m) | U_i \text{ is a neighborhood of } \overline{A_i}\ (0 \le i \le m)\}$$

satisfy the axiom of a base of neighborhoods of $\mathbb{PA}$*, and hence define a topology in* $\mathbb{P}M$*.*

Moreover the following theorem holds.

Theorem 1.6.3 ([25] Theorem 5.1., Proposition 3.9, 3.10.)**.** *The set* $\mathbb{P}M$ *is compact, and* $\mathrm{PGL}(V)$ *is dense open in* $\mathbb{P}M$*.*

Here we regard an element of $\mathrm{PGL}(V)$ as a one-term element of $\mathbb{P}M$, and $\mathbb{P}M$ is a compactification of $\mathrm{PGL}(V)$.

1.6.4 Monoid structure of $\mathbb{P}\widetilde{M}$

For $\mathbb{A} = (A_0, A_1, \ldots, A_m)$, $\mathbb{B} = (B_0, B_1, \ldots, B_n) \in \mathbb{P}\widetilde{M}$, define $\mathbb{A}\mathbb{B}$ by removing the redundant matrices from

$$\begin{aligned} \mathbb{A}\mathbb{B} = (A_0B_0, A_1B_0, \ldots, &A_mB_0, A_0B_1, \\ &\ldots, A_mB_1, \ldots, A_0B_n, \ldots, A_mB_n). \end{aligned} \tag{1.5}$$

This defines a monoid structure on $\mathbb{P}\widetilde{M}$ ([25] Proposition 6.6.).

Chapter 2

$\mathbb{P}$M-monoids

We shall define a $\mathbb{P}M$-monoid denoted by $\mathscr{R}_n$.

2.1 $\mathbb{P}$M-monoids and their structures

We shall define a $\mathbb{P}M$-monoid denoted by $\mathscr{R}_n$, and reveal its structue. Let T be a maximal torus of PGL_n. Then we consider the following monoid

$$\mathscr{R}_n = \overline{N_{\mathrm{PGL}_n}(T)}/T,$$

where the closure is taken in the topology of $\mathbb{P}M$. We call this monoid $\mathscr{R}_n$ a $\mathbb{P}M$-monoid. We next consider the structure of a $\mathbb{P}M$-monoid. The structure of a $\mathbb{P}M$-monoid can be described in terms of a matched pairs. We first explain a matched pairs (cf. [18],[29]). Let S be a monoid. We denote the unit element of S by 1_S.

A matched pair of monoids means a triple (S, B, σ) such that S acts on B and B acts on S and these actions are compatible with each other.

Definition 2.1.1. Let S, B be monoids which have binary operations $\rightharpoonup$: $S \times B \to B$ and $\leftharpoonup$: $S \times B \to S$. A matched pair of monoids means a triple (S, B, σ), where S, B are monoids and

$$\sigma : S \times B \to B \times S,\ (s, b) \mapsto (s \rightharpoonup b, s \leftharpoonup b)$$

is a map satisfying the following conditions :

(1) $s \rightharpoonup (t \rightharpoonup b) = st \rightharpoonup b$,

(2) $st \leftharpoonup b = (s \leftharpoonup (t \rightharpoonup b))(t \leftharpoonup b)$,

(3) $(s \leftharpoonup b) \leftharpoonup c = s \leftharpoonup bc$,

(4) $s \rightharpoonup bc = (s \rightharpoonup b)((s \leftharpoonup b) \rightharpoonup c)$,

(5) $1_S \rightharpoonup b = b$,

(6) $s \rightharpoonup 1_B = 1_B$,

(7) $s \leftharpoonup 1_B = s$,

(8) $1_S \leftharpoonup b = 1_S$

for $s, t \in S$, $b, c \in B$.

The product $B \times S$ forms a monoid with product

$$(b, s)(c, t) = (b(s \rightharpoonup c), (s \leftharpoonup c)t).$$

This monoid is denoted by $B \bowtie_\sigma S$.

An ordered set partition of a set T is a list of pairwise disjoint nonempty subsets of T such that the union of these subsets is S. Let

$$P_n = \left\{ (\{i_1, \dots, i_{k_1}\}, \{i_{k_1+1}, \dots, i_{k_2}\}, \dots, \{i_{k_m+1}, \dots, i_n\}) \mid \begin{matrix} \{i_1 \dots, i_n\} = \{1, \dots, n\} \\ 1 \le k_1 < k_2 < \dots < k_{m-1} < n \end{matrix} \right\}.$$

An element of P_n is called an ordered set partitions of $[n] := \{1, 2, \dots, n\}$. The set P_n has a monoid structure defined by

$$(p_1, \dots, p_m) * (p'_1, \dots, p'_{m'}) := (p_1 \cap p'_1, \dots, p_m \cap p'_1, \dots, p_1 \cap p'_{m'}, \dots, p_m \cap p'_{m'}).$$

Then the following proposition holds.

Proposition 2.1.2. Let $\mathscr{R}_n$ be the $\mathbb{P}M$-monoid, S_n the symmetric group and P_n the collection of the ordered set partitions of $[n]$. Define a map

$$\varphi : P_n \times S_n \to S_n \times P_n,\ ((p_1, \dots, p_m), w) \mapsto (w, (w^{-1}(p_1), \dots, w^{-1}(p_m))).$$

Then

$$\mathscr{R}_n \simeq S_n \bowtie_\varphi P_n.$$

Proof. Since $N_{\mathrm{PGL}_n}(T) = \{\sum_{j=1}^{n} t_j E_{\pi(j)j} \mid t_j \in \mathbb{C}^*, \pi \in S_n\}$, we have

$$\overline{N_{\mathrm{PGL}_n}(T)} = \left\{ \left(\sum_{j \in p_1} t_j E_{\pi(j)j}, \dots, \sum_{j \in p_m} t_j E_{\pi(j)j} \right) \mid \begin{matrix} t_j \in \mathbb{C}^*, \\ (p_1, \dots, p_m) \in P_n, \pi \in S_n \end{matrix} \right\}.$$

Thus

$$\overline{N_{\mathrm{PGL}_n}(T)}/T = \left\{ \left(\sum_{j\in p_1} E_{\pi(j)j}, \dots, \sum_{j\in p_m} E_{\pi(j)j} \right) \mid (p_1,\dots,p_m) \in P_n, \pi \in S_n \right\}.$$

Then we have the following bijective correspondence as sets.

$$\begin{aligned} &\overline{N_{\mathrm{PGL}_n}(T)}/T \simeq S_n \times P_n : \\ &\left(\sum_{j\in p_1} E_{\pi(j)j}, \dots, \sum_{j\in p_m} E_{\pi(j)j} \right) \mapsto (\pi, (p_1,\dots,p_m)). \end{aligned} \tag{2.1}$$

To introduce a monoid structure on $S_n \times P_n$, we recall a monoid structure of $\overline{N_{\mathrm{PGL}_n}(T)}/T$ (cf. (1.5)).

$$\begin{aligned} &\left(\sum_{j\in p_1} E_{\sigma(j)j}, \dots, \sum_{j\in p_m} E_{\sigma(j)j} \right) \cdot \left(\sum_{k\in p'_1} E_{\sigma'(k)k}, \dots, \sum_{k\in p'_n} E_{\sigma'(k)k} \right) \\ &= \left(\sum_{j\in p_1} E_{\sigma(j)j} \sum_{k\in p'_1} E_{\sigma(k)k}, \sum_{j\in p_2} E_{\sigma(j)j} \sum_{k\in p'_1} E_{\sigma(k)k}, \dots, \sum_{j\in p_m} E_{\sigma'(j)j} \sum_{k\in p'_n} E_{\sigma'(k)k} \right) \\ &= \left(\sum_{l\in {\sigma'}^{-1}(p_1)\cap p'_1} E_{\sigma\sigma'(l)l}, \sum_{l\in {\sigma'}^{-1}(p_2)\cap p'_1} E_{\sigma\sigma'(l)l}, \dots, \sum_{l\in {\sigma'}^{-1}(p_m)\cap p'_n} E_{\sigma\sigma'(l)l} \right). \end{aligned}$$

By the above calculation, we define a product on $S_n \times P_n$

$$\begin{aligned} &(\sigma, (p_1,\dots,p_m)) \cdot (\sigma', (p'_1,\dots,p'_n)) \\ &\qquad := (\sigma\sigma', ({\sigma'}^{-1}(p_1),\dots,{\sigma'}^{-1}(p_m)) * (p'_1,\dots,p'_n)). \end{aligned} \tag{2.2}$$

Then (2.1) becomes an isomorphism of monoids. On the other hand, we define a map

$$\varphi : P_n \times S_n \to S_n \times P_n : ((p_1,\dots,p_m),\sigma) \mapsto (\sigma, (\sigma^{-1}(p_1),\dots,\sigma^{-1}(p_m))).$$

Then (P_n, S_n, φ) satisfies (1)-(8) in Definition 2.1.1, and becomes a matched pair. The monoid structure of $S_n \bowtie_\varphi P_n$ coincides with (2.2). Therefore

$$\mathscr{R}_n \simeq S_n \bowtie_\varphi P_n$$

as monoids. □

2.2 Properties of $\mathbb{P}$M-monoids

A $\mathbb{P}M$-monoid has the following properties analogous to Theorem 1.3.2.

Proposition 2.2.1. Let $\mathscr{R}_n = \overline{N_{\mathrm{PGL}_n}(T)}/T$, and

$$\Lambda_n = \left\{ (\sum_{j\in p_1} E_{jj}, \sum_{j\in p_2} E_{jj}, \ldots, \sum_{j\in p_n} E_{jj}), \right. \\ \left. \mid \begin{array}{c} (p_1,\ldots,p_n) = (\{1,\ldots,k_1\},\ldots,\{k_{m-1}+1,\ldots,n\}) \\ 1 \le k_1 < k_2 < \cdots < k_{m-1} < n \end{array} \right\}.$$

(a) $\mathscr{R}_n$ is a finite inverse monoid. Moreover the number of its elements is

$$\begin{aligned} &|\mathscr{R}_n| \\ &= \sum_{r_1+\cdots+r_m=n} \binom{n}{r_1}^2 r_1! \binom{n-r_1}{r_2}^2 r_2! \\ &\qquad\qquad \cdots \binom{n-r_1-\cdots-r_{n-1}}{r_n}^2 r_n! \\ &= n! \sum_{m=1}^{n} m! S(n,m), \end{aligned} \tag{2.3}$$

where $S(n,m)$ is the Stirling number of the second kind, i.e., $S(n,m)$ is the number of ways of partitioning a set of n elements into m non-empty subsets.

(b) The unit group of $\mathscr{R}_n$ coincide with $W := N_{\mathrm{PGL}}(T)/T$, and $\mathscr{R}_n = WE(\mathscr{R}_n)$.

(c) $E(\mathscr{R}_n) = \bigcup_{w\in W} w\Lambda_n w^{-1}$.

(d) $\mathscr{R}_n = \bigsqcup_{e\in\Lambda_n} WeW$.

Proof. (a) For any $(\sigma,p) \in \mathscr{R}_n$, let $(\sigma,p)^* := (\sigma^{-1},\sigma(p))$. Then

$$(\sigma,p) = (\sigma,p)(\sigma,p)^*(\sigma,p),\ (\sigma,p)^* = (\sigma,p)^*(\sigma,p)(\sigma,p)^*.$$

Thus $\mathscr{R}_n$ is an inverse monoid. We next consider the number $|\mathscr{R}_n|$. We fix a partition $r_1 + r_2 + \cdots + r_m = n$. We first choose r_1 columns and r_1 rows among the n columns and n rows, and choose a placement of 1's in the place of $r_1 \times r_1$ permutation matrices. Next we choose r_2 columns and r_2 rows among the $n - r_1$ columns and $n - r_1$ rows, and choose a placement of 1's in the place of $r_2 \times r_2$ permutation matrices. We repeat this process and sum up over partitions $r_1 + r_2 + \cdots + r_m = n$, and then we obtain the first equality of (2.3).

On the other hand, let $P_{n,m}$ be the collection of ordered set partitions of $[n]$ with m blocks. Then $|P_{n,m}| = m!S(n,m)$ by the definition of the Stirling numbers of the second kind. Thus we obtain the second equality of (2.3) since $|P_n| = \sum_{m=1}^{n} |P_{n,m}|$ and $|\mathscr{R}_n| = |S_n||P_n|$.

(b) First we see

$$E(\mathscr{R}_n) = \left\{ (\sum_{j\in p_1} E_{jj}, \sum_{j\in p_2} E_{jj}, \ldots, \sum_{j\in p_n} E_{jj}) \mid (p_1, \ldots, p_n) \in P_n \right\}. \tag{2.4}$$

In fact, if $(\sigma, p)(\sigma, p) = (\sigma, p)$ for $(\sigma, p) \in \mathscr{R}_n$, then $\sigma^2 = \sigma$, i.e., $\sigma = e$. Then $\mathscr{R}_n = WE(\mathscr{R}_n)$.

(c) follows from (2.4)

(d)

$$\begin{aligned}
&\bigsqcup_{e\in\Lambda_n} WeW \\
&= \left\{ \sigma(\sum_{j\in p_1} E_{jj}, \ldots, \sum_{j\in p_n} E_{jj})\tau \mid \begin{array}{c}(p_1, \ldots, p_n) = (k_1, \ldots, k_{m-1}) \\ \sigma, \tau \in W\end{array} \right\} \\
&= \left\{ (\sum_{j\in p_1} E_{\sigma^{-1}(j)\tau(j)}, \ldots, \sum_{j\in p_n} E_{\sigma^{-1}(j)\tau(j)}) \mid \begin{array}{c}(p_1, \ldots, p_n) = (k_1, \ldots, k_{m-1}) \\ \sigma, \tau \in W\end{array} \right\} \\
&= \left\{ (\sum_{k\in\tau(p_1)} E_{(\tau\sigma)^{-1}(k)k}, \ldots, \sum_{k\in\tau(p_n)} E_{(\sigma\tau)^{-1}(k)k}) \mid \right. \\
&\qquad\qquad \left. \begin{array}{c}(p_1, \ldots, p_n) = (k_1, \ldots, k_{m-1}) \\ \sigma, \tau \in W\end{array} \right\} \\
&= \mathscr{R}_n,
\end{aligned}$$

where we denote

$$(k_1, \ldots, k_{m-1}) = (\{1, \ldots, k_1\}, \ldots, \{k_{m-1} + 1, \ldots, n\}). \tag{2.5}$$

□

We construct a presentation for $\mathscr{R}_n = \overline{N_{\mathrm{PGL}}(T)}/T$, similar to that of the rook monoid. We first define some notations. For $i = 1, \dots, n-1$ and a partition $(k_1, \dots, k_{m-1})$ (cf. (2.5)), if there exists $j \in \{1, \dots, n\}$ such that $\{i, i+1\} \subseteq \{k_{j-1}+1, \dots, k_j\}$, then we set

$$i_* := j.$$

For $\sigma \in S_n$ we define a map $\varphi_\sigma : P_n \to P_n$ by

$$(p_1, \dots, p_m) \mapsto (\sigma^{-1}(p_1), \dots, \sigma^{-1}(p_m)).$$

We define a set

$$\Pi_n = \{(k_1, \dots, k_{m-1}) : 1 \leq k_1 < \cdots < k_{m-1} < n\},$$

where $(k_1, \dots, k_{m-1})$ is (2.5). For $p \in P_n$, take an element $w \in S_n$ such that $wpw^{-1} \in \Pi_n$, and set

$$u^w(p) := wpw^{-1} \in \Pi_n.$$

We also set

$$\mathrm{Ad}(\sigma)(e) := \sigma^{-1} e \sigma.$$

Using these notations we obtain the following monoid presentation of the $\mathbb{P}M$-monoid $\mathscr{R}_n$.

Proposition 2.2.2. The $\mathbb{P}M$-monoid $\mathscr{R}_n$ has a monoid presentation with generating set

$$\{s_1, \dots, s_{n-1}, e_{k_1,\dots,k_{m-1}} \ (1 \leq k_1 < \cdots < k_{m-1} < n)\}$$

and defining relations

$$s_i^2 = 1 \qquad (1 \leq i \leq n-1), \tag{2.6}$$

$$s_i s_j = s_j s_i \qquad (1 \leq i, j \leq n-1,\ |i-j| \geq 2), \tag{2.7}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \qquad (1 \leq i \leq n-1), \tag{2.8}$$

$$e_{k_1,\dots,k_{i_*},\dots,k_{m-1}} s_i = s_i e_{k_1,\dots,k_{i_*},\dots,k_{m-1}} \tag{2.9}$$
$$\begin{pmatrix} 1 \leq i \leq n-1 \\ 1 \leq k_1 < \cdots < k_{i_*} < \cdots < k_{m-1} < n \end{pmatrix},$$

$$e_{k_1,\dots,k_{m-1}} s_{i_1} \dots s_{i_r} e_{l_1,\dots,l_{m'-1}} = \mathrm{Ad}(s_{j_1} \dots s_{j_t})(e_q) s_{i_1} \dots s_{i_r} \tag{2.10}$$
$$\begin{pmatrix} 1 \leq k_1 < \cdots < k_{m-1} < n \\ 1 \leq l_1 < \cdots < l_{m'-1} < n \\ \{i_1, i_1+1\} \not\subseteq \{k_{l-1}+1, \dots, k_l\}, {}^\forall l = 1, \dots, n \\ q = u^{s_{j_1} \cdots s_{j_t}}((k_1, \dots, k_{m-1}) * \varphi_{(s_{i_1} \dots s_{i_r})^{-1}}((l_1, \dots, l_{m'-1}))) \end{pmatrix}.$$

Proof. Let

$$e_{k_1,\dots,k_{m-1}} := \left(\sum_{j=1}^{k_1} E_{jj}, \sum_{j=k_1+1}^{k_2} E_{jj}, \dots, \sum_{j=k_{m-1}+1}^{n} E_{jj} \right),$$

and $s_i = (i, i+1)$. These elements satisfy the above relations. Let $\mathscr{R}'_n$ be the monoid generated by elements $s'_1, \dots, s'_{n-1}, e'_{k_1,\dots,k_{m-1}}$ subject to the defining relations (2.6)-(2.10). Since $\mathscr{R}_n$ satisfies (2.6)-(2.10), there is a surjective monoid homomorphism $\theta : \mathscr{R}'_n \to \mathscr{R}_n$ such that $\theta(s'_i) = s_i$ and $\theta(e'_{k_1,\dots,k_{m-1}}) = e_{k_1,\dots,k_{m-1}}$. Let $S'_n = \langle s'_1, \dots, s'_{n-1} \rangle \subseteq \mathscr{R}'_n$. To show that $\theta : \mathscr{R}'_n \to \mathscr{R}_n$ is an isomorphism of monoids, it suffices to show that $|\mathscr{R}'_n| \leq |\mathscr{R}_n|$, where $|\mathscr{R}_n|$ is given by (2.3).
We consider the following set

$$\bigcup_{1 \leq k_1 < \dots < k_m < n} S'_n e'_{k_1,\dots,k_{m-1}} S'_n. \tag{2.11}$$

Using relations (2.9) and (2.10), we can show that the set (2.11) is stable under the left multiplication by $e'_{k_1,\dots,k_{m-1}}$ and S'_n. The set (2.11) contains $e_n = 1$. Thus the set (2.11) contains $\mathscr{R}_n$. Therefore we have

$$\mathscr{R}'_n = \bigcup_{1 \leq k_1 < \dots < k_m < n} S'_n e'_{k_1,\dots,k_{m-1}} S'_n.$$

We fix $1 \leq k_1 < \dots < k_{m-1} < n$ and let

$$\begin{aligned} S'_{k_1,\dots,k_{m-1}} &:= \langle s'_1, \dots, s'_{k_1-1}, s'_{k_1+1}, \dots, s'_{k_2-1}, s'_{k_2+1}, \dots, s'_{k_{m-1}-1}, s'_{k_{m-1}+1}, \dots, s'_{n-1} \rangle \\ &\simeq S_{k_1} \times S_{k_2-k_1} \times \dots \times S_{n-k_{m-1}}. \end{aligned}$$

Write $S'_n = S'_{k_1,\dots,k_{m-1}} X_{k_1,\dots,k_{m-1}}$, where $X_{k_1,\dots,k_{m-1}}$ is a set of coset representatives. Then by the relation (2.9) of the above relations,

$$\begin{aligned} e'_{k_1,\dots,k_{m-1}} S'_n &= e'_{k_1,\dots,k_{m-1}} S'_{k_1,\dots,k_{m-1}} X_{k_1,\dots,k_{m-1}} \\ &= S'_{k_1,\dots,k_{m-1}} e'_{k_1,\dots,k_{m-1}} X_{k_1,\dots,k_{m-1}} \\ &\subseteq S'_n e'_{k_1,\dots,k_{m-1}} X_{k_1,\dots,k_{m-1}}. \end{aligned}$$

Thus

$$\begin{aligned} |S'_n e'_{k_1,\dots,k_{m-1}} S'_n| &\leq |S'_n e'_{k_1,\dots,k_{m-1}} X_{k_1,\dots,k_{m-1}}| \\ &\leq |S'_n e'_{k_1,\dots,k_{m-1}}||X_{k_1,\dots,k_{m-1}}| \\ &= \frac{n!}{k_1!(k_2-k_1)!\dots(n-k_{m-1})!} |S'_n e_{k_1,\dots,k_{m-1}}| \\ &\leq \frac{(n!)^2}{k_1!(k_2-k_1)!\dots(n-k_{m-1})!}. \end{aligned}$$

Therefore

$$|\bigcup_{1\le k_1<\cdots<k_m<n} S'_n e'_{k_1,\dots,k_{m-1}} S'_n| \le \sum_{1\le k_1<\cdots<k_m<n} \frac{(n!)^2}{k_1!(k_2-k_1)!\dots(n-k_{m-1})!}$$
$$= \sum_{r_1+\cdots+r_m=n} \frac{(n!)^2}{r_1!r_2!\dots r_m!}.$$

On the other hand

$$\sum_{r_1+\cdots+r_m=n} \binom{n}{r_1}^2 r_1! \binom{n-r_1}{r_2}^2 r_2! \dots \binom{n-r_1-\cdots-r_{n-1}}{r_n}^2 r_n!$$
$$= \sum_{r_1+\cdots+r_m=n} \frac{(n!)^2}{r_1!r_2!\dots r_m!}.$$

□

Remark 2.2.3. In the relation (2.10), the adjustment by Ad is necessary since the right hand of (2.10) is not necessarily of the form $e_{k_1,\dots,k_{m-1}} s_{i_1} \dots s_{i_r}$. For example, in $\mathscr{R}_3$

$$e_2(s_2s_1s_2)e_1 = s_1s_2e_1s_2s_1(s_2s_1s_2).$$

2.3 Braid $\mathbb{P}$M-monoids

2.3.1 Definitions of Braid $\mathbb{P}$M-monoids

We define a braid monoid according to Proposition 2.2.2. The notations are the same as those in Proposition 2.2.2, and we add the following notation. We denote by $b|_I$ an element of braid group of $\#I$-strings which has strings in the place of any $i \in I$ for $b \in B_n$ and $I \subset \{1,\dots,n\}$. If $s_{i_1},\dots,s_{i_r} \in B_n$ satisfy $s_{i_1}\dots s_{i_r}|_I = id|_I$, where $I \subset \{1,\dots,n\}$ and id is the identity braid in B_n, then we abbreviate this condition as $\{i_1,\dots,i_r\}|_I = id$.

Definition 2.3.1. The braid $\mathbb{P}M$-monoid is a monoid which is defined by the monoid presentation with generating set

$$\{s_1^{\pm 1},\dots,s_{n-1}^{\pm 1}, e_{k_1,\dots,k_{m-1}}\ (1 \le k_1 < \cdots < k_{m-1} < n)\}$$

and defining relations

$$s_is_i^{-1} = s_i^{-1}s_i = 1 \qquad (1 \le i \le n-1), \tag{2.12}$$
$$s_is_j = s_js_i, \qquad (1 \le i,j \le n-1,\ |i-j| \ge 2), \tag{2.13}$$
$$s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \qquad (1 \le i \le n-1), \tag{2.14}$$

$$s_{i_1}^{\pm 1}\dots s_{i_r}^{\pm 1}e_{k_1,\dots,k_{m-1}}s_{j_1}^{\pm 1}\dots s_{j_t}^{\pm 1}=e_{k_1,\dots,k_{m-1}} \tag{2.15}$$
$$\begin{pmatrix}\{i_1,\dots,i_r,j_1,\dots,j_t\}|_{\{k_{j-1}-1,\dots,k_j\}}=id\\ {}^{\forall}j=1,\dots,m\end{pmatrix},$$
$$e_{k_1,\dots,k_{m-1}}s_{i_1}^{\pm 1}\dots s_{i_r}^{\pm 1}e_{l_1,\dots,l_{m'-1}}=\mathrm{Ad}(s_{j_1}^{\pm 1}\dots s_{j_t}^{\pm 1})(e_q)s_{i_1}^{\pm 1}\dots s_{i_r}^{\pm 1} \tag{2.16}$$
$$\begin{pmatrix}1\le k_1<\dots<k_{m-1}<n\\ 1\le l_1<\dots<l_{m'-1}<n\\ \{i_1,i_1+1\}\nsubseteq\{k_{l-1}+1,\dots,k_l\},{}^{\forall}l=1,\dots,m\\ q=u^{s_{j_1}^{\pm 1}\dots s_{j_t}^{\pm 1}}((k_1,\dots,k_{m-1})*\varphi_{(s_{i_1}\dots s_{i_r})^{-1}}((l_1,\dots,l_{m'-1})))\end{pmatrix}.$$

2.3.2 Braid diagrams of braid $\mathbb{P}M$-monoids

We denote by $\mathscr{M}$ the monoid defined in Definition 2.3.1. To describe the monoid $\mathscr{M}$ geometrically we shall define $\mathbb{P}M$-braid and $\mathbb{P}M$-braid diagram. First, we explain a formal definition of $\mathbb{P}M$-braid and $\mathbb{P}M$-braid diagram. Next, we describe an informal explanation of $\mathbb{P}M$-braid diagrams.

First, we shall define an arc.

Definition 2.3.2. An arc is the image of an embedding from the unit interval $[0,1]$ into $\mathbb{R}^3$.

Let n be a fixed natural number, and m a natural number such that $m\le n$. Take the usual coordinate system (x,y,z) for $\mathbb{R}^3$. Choose planes $z=z_j^{(i)}$ $(i=1,\dots,m, j=0,1)$ where

$$z_j^{(i)}=\begin{cases}2m-2i+1 & (j=1)\\ 2m-2i & (j=0).\end{cases}$$

Mark $n\ge 0$ distinct points $P_1^i,\dots,P_n^i$ on a line in the plane $z=z_1^{(i)}$, and project this orthogonally on the plane $z=z_0^{(i)}$, yielding points $Q_1^i,\dots,Q_n^i$ for each $i=1,\dots,m$.

A $\mathbb{P}M$-braid on n strings is a system

$$\beta=\{\beta_1,\dots,\beta_{k_1},\beta_{k_1+1},\dots,\beta_{k_2},\beta_{k_2+1},\dots,\beta_{k_{m-1}+1},\dots,\beta_n\}$$

of n arcs for some $1\le k_1<k_2<\dots<k_{m-1}<n$ such that

(1) There exists a partial one-one mapping of rank k_1

$$\Phi_1^{\beta}:\{1,\dots,n\}\to\{1,\dots,n\}$$

with domain $\{i_1, \ldots, i_{k_1}\}$ such that β_j connects $P^1_{i_j}$ to $Q^1_{\Phi^\beta_1(i_j)}$ for $j = 1, \ldots, k_1$.
There exists a partial one-one mapping of rank $k_2 - k_1$

$$\Phi^\beta_2 : \{1, \ldots, n\}\backslash\{i_1, \ldots, i_{k_1}\} \to \{1, \ldots, n\}\backslash\{i_1, \ldots, i_{k_1}\}$$

with domain $\{i_{k_1+1}, \ldots, i_{k_2}\}$ such that β_j connects $P^2_{i_j}$ to $Q^2_{\Phi^\beta_1(i_j)}$ for $j = k_1 + 1, \ldots, k_2$.

$$\ldots\ldots$$

There exists a partial one-one mapping of rank $n - k_{m-1}$

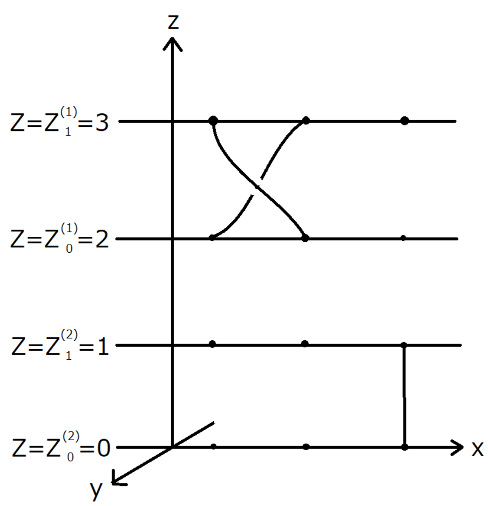
$$\Phi^\beta_m : \{1, \ldots, n\}\backslash\{i_1, \ldots, i_{k_{m-1}}\} \to \{1, \ldots, n\}\backslash\{i_1, \ldots, i_{k_{m-1}}\}$$

with domain $\{i_{k_{m-1}+1}, \ldots, i_n\}$ such that β_j connects $P^m_{i_j}$ to $Q^m_{\Phi^\beta_m(i_j)}$ for $j = k_{m-1} + 1, \ldots, n$.

(2) For $j = 1, \ldots, m$, the arc β_l intersects the plane $z = z^{(j)}_0$ exactly once, and β_l intersects the plane $z = z^{(j)}_1$ exactly once, for $l = k_{j-1}+1, \ldots, k_j$, and β_s does not intersect $z = z^{(t)}_0$, $z = z^{(t)}_1$ for $s \neq t$.

(3) For $j = 1, \ldots, m$ the union $\beta_{k_{j-1}+1} \cup \cdots \cup \beta_{k_j}$ of the arcs intersects each of parallel planes $z = z^{(j)}_0, z = z^{(j)}_1$ at exactly $k_j - k_{j-1}$ distinct points.

Example 2.3.3. Let $n = 3$ and $m = 2$. The following is a $\mathbb{P}M$-braid.



Two $\mathbb{P}M$-braids

$$\beta = \{\beta_1, \dots, \beta_{k_1}, \beta_{k_1+1}, \dots, \beta_{k_2}, \beta_{k_2+1}, \dots, \beta_{k_{m-1}+1}, \dots, \beta_n\},$$
$$\gamma = \{\gamma_1, \dots, \gamma_{k_1}, \gamma_{k_1+1}, \dots, \gamma_{k_2}, \gamma_{k_2+1}, \dots, \gamma_{k_{m'-1}+1}, \dots, \gamma_n\}$$

are defined to be equivalent if

(1) $m = m'$ and $\Phi_i^\beta = \Phi_i^\gamma$ for $i = 1, \dots, m$,

(2) β and γ are homotopy equivalent, i.e., there exist continuous maps

$$F_j : [0,1] \times [0,1] \to \mathbb{R}^3, \qquad (j = 1, \dots, m)$$

such that for all $s, t \in [0,1]$,

$$\begin{array}{rcll} F_j(t,0) &=& \beta_j(t) & \\ F_j(t,1) &=& \gamma_j(t) & (j = 1, \dots, m), \\ F_j(0,s) &=& P^1_{i_j} & \\ F_j(1,s) &=& Q^1_{\Phi_1^\beta(i_j)} & (j = 1, \dots, k_1), \\ F_j(0,s) &=& P^2_{i_j} & \\ F_j(1,s) &=& Q^2_{\Phi_2^\beta(i_j)} & (j = k_1+1, \dots, k_2), \\ & & & \dots \\ F_j(0,s) &=& P^m_{i_j} & \\ F_j(1,s) &=& P^m_{\Phi_m^\beta(i_j)} & (j = k_{m-1}+1, \dots, n), \end{array}$$

and, for each $s \in [0,1]$ if we define

$$\beta^s = \{\beta_1^s, \dots, \beta_m^s\},$$

where

$$\beta_j^s(t) = F_j(s,t) \text{ for } j = 1, \dots, n,$$

then β^s itself is a $\mathbb{P}M$-braid.

Define the product $\beta\gamma$ of two braids

$$\beta = \{\beta_1, \dots, \beta_{k_1}, \beta_{k_1+1}, \dots, \beta_{k_2}, \beta_{k_2+1}, \dots, \beta_{k_{m-1}+1}, \dots, \beta_n\},$$
$$\gamma = \{\gamma_1, \dots, \gamma_{k_1}, \gamma_{k_1+1}, \dots, \gamma_{k_2}, \gamma_{k_2+1}, \dots, \gamma_{k_{m'-1}+1}, \dots, \gamma_n\}$$

as follows.

We first define an operation (k_il_j). Take $z_1^{(11)} > z_0^{(11)} > z_1^{(21)} > z_0^{(21)} > \cdots > z_1^{(m1)} > z_0^{(m1)} > z_1^{(12)} > z_0^{(12)} > \cdots > z_1^{(m2)} > z_0^{(m2)} > \cdots > z_1^{(mm')} > z_0^{(mm')}$.

(k_il_j) :

(1) Translate $\{\gamma_{l_{j-1}+1}, \ldots, \gamma_{l_j}\}$ parallel to itself so that the upper plane of $\{\gamma_{l_{j-1}+1}, \ldots, \gamma_{l_j}\}$ coincides with the lower plane of $\{\beta_{k_{i-1}+1}, \ldots, \beta_{k_j}\}$;

(2) Translate the above system of arcs so that the upper plane of $\{\beta_{k_{i-1}+1}, \ldots, \beta_{k_j}\}$ coincides with $z = z_1^{(ij)}$. Keeping $z = z_1^{(ij)}$ fixed, contract the resulting systems of arcs so that the translated lower plane of $\{\gamma_{l_{j-1}+1}, \ldots, \gamma_{l_j}\}$ lies into the position of $z = z_0^{(ij)}$;

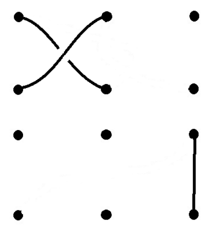
(3) Remove any arc that do not now join the upper plane to the lower plane.

Then take the operations (k_1l_1),...,(k_ml_1),(k_1l_2),...,(k_ml_2),$(k_1l_{m'})$,...,$(k_ml_{m'})$, finally remove empty systems of arcs. The resulting $\mathbb{P}M$-braid is denoted by $\beta\gamma$.

Similar to the braid diagram, we define a $\mathbb{P}M$-braid diagram. A $\mathbb{P}M$-braid diagram is an image of the projection of $\mathbb{P}M$-braid concerning the second coordinate. At each intersection point of two strands, these strands meet transversely, and one of them is distinguished and said to be undergoing, the other strand being overgoing.

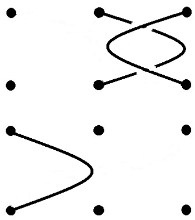
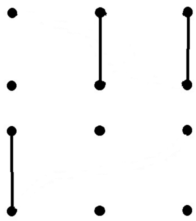
Informally, $\mathbb{P}M$-braid diagrams are considered braid diagrams which have some layers. n corresponds to the number of strings and m corresponds to the number of layers. The $z = z_1^{(i)}$ corresponds to the upper plane of each layers and $z = z_0^{(i)}$ corresponds to the lower plane of each layers. The strings of each layer are disjoint and the union of strings is full braids.

Example 2.3.4. The following is a $\mathbb{P}M$-braid diagram.

$\beta =$ 

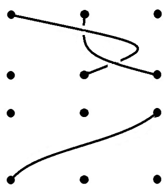
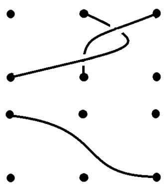
The equivalence of $\mathbb{P}M$-braid diagram is considered at each layer.

Example 2.3.5. The following $\mathbb{P}M$-braids are equivalent.

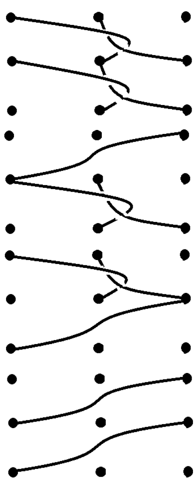
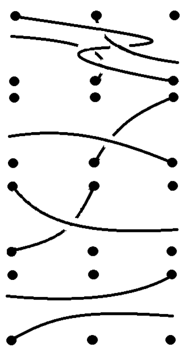
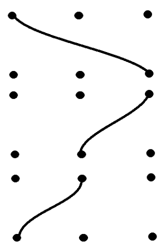
$\beta =$  $\gamma =$ 

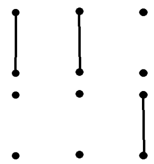
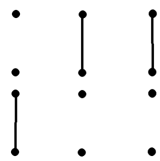
The product of $\mathbb{P}M$-braid diagrams are similar to the product of $\mathbb{P}M$-monoid. The product of each layer is the product of partial braids.

Example 2.3.6. Let β and γ be the following $\mathbb{P}M$-braids:

$\beta = \begin{matrix}\beta_1 \\ \beta_2\end{matrix} =$  $\gamma = \begin{matrix}\gamma_1 \\ \gamma_2\end{matrix} =$ 

then $\beta^2, \beta\gamma, \gamma\beta$ are obtained as follows:

$\beta^2 = \begin{matrix}\beta_1\beta_1 \\ \beta_2\beta_1 \\ \beta_1\beta_2 \\ \beta_2\beta_2\end{matrix} =$  $=$  $=$ 

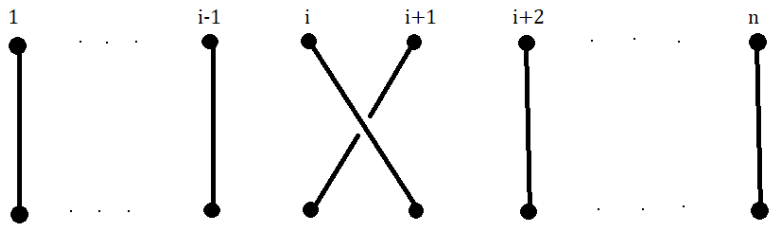
$\beta\gamma =$  $\gamma\beta =$ 

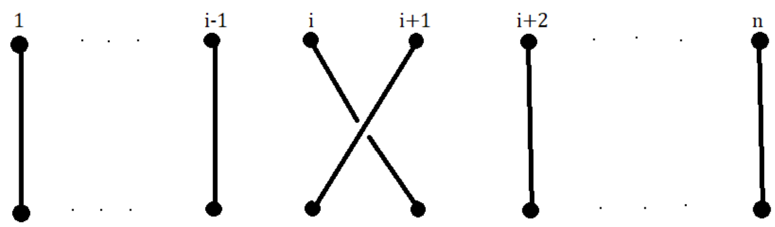
We denote by $[\beta]$ the homotopy equivalence class of β. Put

$$\mathscr{RB}_n = \{[\beta] : \beta \text{ is a } \mathbb{P}M\text{-braid}\}.$$

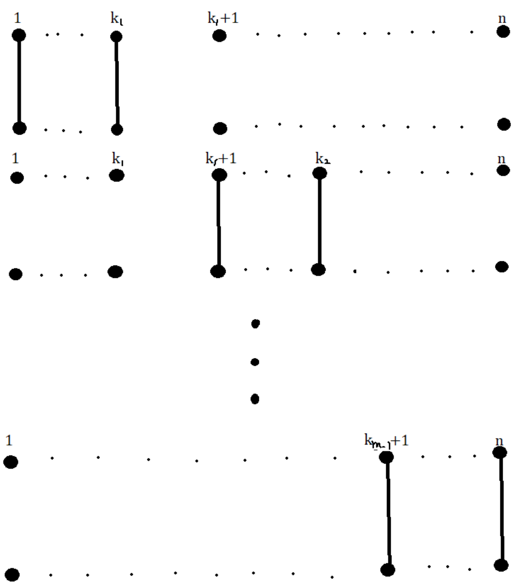
Theorem 2.3.7. *The braid $\mathbb{P}M$-monoid $\mathscr{M}$ is isomorphic to the monoid $\mathscr{RB}_n$.*

Proof. Let Ψ denote a map from the set of generators for $\mathscr{M}$ into $\mathscr{RB}_n$ given by

$s_i \mapsto$ 

$s_i^{-1} \mapsto$ 

$e_{k_1,\dots,k_{m-1}} \mapsto$



The relations in the presentation for $\mathscr{M}$ hold for the images of the generators, so that Ψ induces a well-defined homomorphism, which we also denote by Ψ. To prove the assertion, it suffices to prove that Ψ is bijective. First we prove that Ψ is surjective. Take any $\theta \in \mathscr{RB}_n$, and say θ has $k_1, k_2 - k_1, \dots, n - k_m - 1$-strings. Then there exist $s_{j_1}, \dots, s_{j_t}, s_{i_1}, \dots, s_{i_r}$ and $e_{k_1,\dots,k_{m-1}}$ such that

$$\theta = (\Psi(s_{j_1}) \dots \Psi(s_{j_t}))^{-1} \Psi(e_{k_1,\dots,k_{m-1}})(\Psi(s_{j_1}) \dots \Psi(s_{j_t}))(\Psi(s_{i_1}) \dots \Psi(s_{i_1})).$$

Thus Ψ is surjective. We next show that Ψ is injective. By the discussion in the proof of Proposition 2.2.2,

$$\mathscr{M} = \bigsqcup_{1 \leq k_1 < \cdots < k_{m-1} < n} B_n e_{k_1,\dots,k_{m-1}} B_n,$$

where B_n is the Braid group. Let

$$\Psi(b_1 e_{k_1,\dots,k_{m-1}} b_2) = \Psi(b_1^{'} e_{l_1,\dots,l_{m'-1}} b_2^{'}),$$

where $b_1, b_2, b_1^{'}, b_2{'} \in B_n$. Since Ψ is a homomorphism of monoids and any of $b_1, b_2, b_1^{'}, b_2{'}$ has an inverse element, respectively, we can assume

$$\Psi(b_1 e_{k_1,\dots,k_{m-1}} b_2) = \Psi(e_{l_1,\dots,l_{m'}-1}).$$

First, it must be $m = m'$ and $k_1 = l_1, \ldots, k_{m-1} = l_{m-1}$. Since $\Psi(b_1)\Psi(b_2)|_{\{k_{l-1}+1,\ldots,k_l\}} = id|_{\{k_{l-1}+1,\ldots,k_l\}}$ for all $l = 1, \ldots, n$. Then as elements of the braid group $b_1 b_2|_{\{k_{l-1}+1,\ldots,k_l\}} = id|_{\{k_{l-1}+1,\ldots,k_l\}}$. Therefore by the relation (2.15), $b_1 e_{k_1,\ldots,k_{m-1}} b_2 = e_{k_1,\ldots,k_{m-1}}$. Thus Ψ is injective. $\square$

2.3.3 Automorphisms of free group and word problems

We will give a presentation of a $\mathbb{P}M$-braids by automorphisms of a free group and find a solution to the word problem in $\mathscr{RB}_n$. We recall the cases : the classical braid group and an inverse braid monoid.
Let $F_n = F(x_1, \ldots, x_n)$ be the free group of rank n generated by $\{x_1, \ldots, x_n\}$. For $1 \leq k \leq n-1$, let $\tau_k : F_n \to F_n$ be the automorphism defined by

$$\tau_k : \begin{cases} x_k \mapsto x_k^{-1} x_{k+1} x_k \\ x_{k+1} \mapsto x_k \\ x_l \mapsto x_l & \text{if } l \neq k, k+1. \end{cases}$$

Then the mapping $s_k \mapsto \tau_k$ $(1 \leq k \leq n-1)$ determines a representation $\rho : B_n \to \mathrm{Aut}(F_n)$ called Artin representation. The following theorem was proved by E. Artin.

Theorem 2.3.8 ([1],[2])**.** (1) *The Artin representation $\rho : B_n \to \mathrm{Aut}(F_n)$ is faithful.*
(2) *An automorphism $\alpha \in \mathrm{Aut}(F_n)$ belongs to* Imρ *if and only if $\alpha(x_n \ldots x_2 x_1) = x_n \ldots x_2 x_1$ and there exists a permutation $\sigma \in S_n$ such that $\alpha(x_k)$ is conjugate to $x_{\sigma(k)}$ for all $1 \leq k \leq n$.*

The braid group B_n can be viewed as a subgroup of $\mathrm{Aut}(F_n)$. Moreover this yields a solution to the word problem in B_n.

Next recall the case of the inverse braid monoid studied by V. V. Vershinin [33]. Let EF_n be a monoid of partial isomorphisms of a free group defined as follows. Let a be an element of rook monoid R_n, and J_k the image of a. Let elements $i_1, \ldots, i_k$ belong to the domain of definition of a. The monoid EF_n consists of isomorphisms

$$F(x_{i_1}, \ldots, x_{i_k}) \to F(x_{j_1}, \ldots, x_{j_k})$$

expressed by

$$f_a(x_i) = \begin{cases} w_i^{-1} x_{a(i)} w_i & (i \in \{i_1, \ldots, i_k\}) \\ \text{not defined} & (\text{otherwise}) \end{cases}.$$

We define a map ϕ_n from IB_n to EF_n expanding the Artin representation ρ by the condition that $\phi_n(e_j)$ as a partial isomorphism of F_n is given by the formula

$$\phi_n(e_j)(x_i) = \begin{cases} x_i & (i \leq j) \\ \text{not defined} & (i > j). \end{cases}$$

Theorem 2.3.9 ([33] Theorem 2.2.)**.** *The homomorphism ϕ_n is a monomorphism.*

Theorem 2.3.10 ([33] Theorem 2.3.)**.** *The monomorphism ϕ_n gives a solution to the word problem for the inverse braid monoid.*

Let $\mathscr{E}F_n$ be a monoid of sequence of partial isomorphisms of the free group F_n defined as follows.

Let $\mathbb{A} = (\sigma, (\{i_1, \dots, i_{k_1}\}, \dots, \{i_{k_{m-1}+1}, \dots, n\})) \in \mathscr{R}_n$. The monoid $\mathscr{E}F_n$ consists of sequence of isomorphisms $\boldsymbol{f}_{\mathbb{A}} = (f_{A_1}, \dots, f_{A_m})$, where for $j = 1, \dots, m$

$$f_{A_j} : F(x_{i_{k_{j-1}+1}}, \dots, x_{i_{k_j}}) \to F(x_{\sigma(i_{k_{j-1}+1})}, \dots, x_{\sigma(i_{k_j})})$$

is defined by

$$f_{A_j}(x_l) = \begin{cases} w_l^{-1} x_{A_j(l)} w_l & (l \in \{i_{k_{j-1}+1}, \dots, i_{k_j}\}) \\ \text{not defined} & (\text{otherwise}) \end{cases},$$

where w_i is a word on $x_{\sigma(i_{k_{j-1}+1})}, \dots, x_{\sigma(i_{k_j})}$. We define a map φ_n from $\mathscr{R}\mathscr{B}_n$ to $\mathscr{E}F_n$ extending the Artin representation ρ by the condition that $\varphi_n(e_{k_1,\dots,k_{m-1}})$ as a sequence of partial isomorphisms of F_n is given by the formula

$$\varphi_n(e_{k_1,\dots,k_{m-1}})(x_i) = (f_1(x_i), \dots, f_{m-1}(x_i)),$$

where

$$f_j(x_i) = \begin{cases} x_i & (k_{j-1} \leq i \leq k_j) \\ \text{not defined} & (i < k_{j+1}, i > k_j) \end{cases}$$

for $j = 1, \dots, m-1$.

Proposition 2.3.11. The homomorphism φ_n is a monomorphism.

Proof. Let $\mathscr{R}\mathscr{B}_n^{(m)}$ be the set of $\mathbb{P}M$-braids which have m layers. Then as a set we have the following decomposition :

$$\begin{aligned} \mathscr{R}\mathscr{B}_n &= \bigsqcup_{m\geq 1} \mathscr{R}\mathscr{B}_n^{(m)} \\ &= \bigsqcup_{m\geq 1} \bigsqcup_{(p_1,\dots,p_m)\in P_n, \sigma\in S_n} B(p_1, \sigma(p_1)) \times \cdots \times B(p_m, \sigma(p_m)), \end{aligned}$$

where $B(p_i, \sigma(p_i))$ is the braid group starting at p_i and ending at $\sigma(p_i)$. Let $\#p_i = r_i$. Then consider the following diagram:

$$\begin{array}{ccc} B_{r_1} \times \cdots \times B_{r_m} & \xrightarrow{Id \times \cdots \times Id} & B_{r_1} \times \cdots \times B_{r_m} \\ \Big\downarrow{\scriptstyle \rho_1 \times \cdots \times \rho_m} & & \Big\downarrow{\scriptstyle \psi(p_1, \sigma(p_1)) \times \cdots \times \psi(p_m, \sigma(p_m))} \\ B(p_1, \sigma(p_1)) \times \cdots \times B(p_m, \sigma(p_m)) & \xrightarrow{\varphi_n} & \mathscr{E}F_n. \end{array}$$

The above diagram is commutative since the diagram (2.6) in [33] is commutative. Thus φ_n is a monomorphism. □

Theorem 2.3.12. *The morphism ϕ_n gives a solution to the word problem for the braid $\mathbb{P}M$-monoid.*

Proof. This assertion holds by the following fact : Two words represent the same element of the monoid if and only if they have the same action on the finite set of generators of the free group. □

Part II

Link invariants constructed by semigroups

Chapter 3

Preliminaries

3.1 Knots and Links

We will explain basic concepts related to knots and links.

3.1.1 definitions

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, and S^n the n-dimensional sphere.

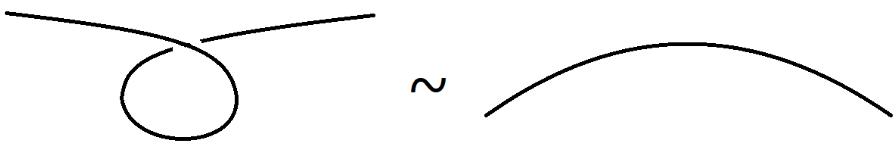
Definition 3.1.1. A link L of m components is a subset of S^3, or of $\mathbb{R}^3$ that consists of m disjoint, simple closed curves. A link of one component is a knot.

Definition 3.1.2. Links L_1 and L_2 in S^3 are equivalent if there is an orientation preserving homeomorphism $h : S^3 \to S^3$ such that $h(L_1) = h(L_2)$.

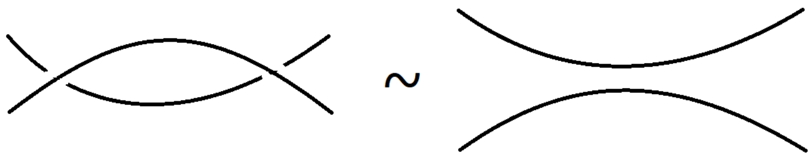
Let $p : \mathbb{R}^3 \to \mathbb{R}^2$ be a projection on plane. Let L be a link. The image of L in $\mathbb{R}^2$ by p together with "over and under" information at the crossing is called the link diagram of L.

Theorem 3.1.3. *Any two links L_1 and L_2 are equivalent if and only if the link diagrams of L_1 and L_2 are deformed by a sequence of Reidemeister moves.*

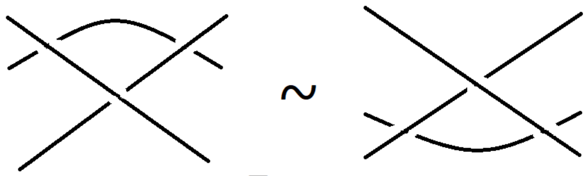
The Reidemister moves are of three types, shown below.



Type I



Type II



Type III

3.2 Knot semigroups

We will explain knot semigroups.

3.2.1 Definitions

We define a knot semigroup defined by A. Vernitski [33]. By an arc we mean a continuous line on a knot diagram from one undercrossing to another undercrossing. For example, consider the knot diagram $T(2,3)$ on Figure 1. It has three arcs, denoted by a, b, and c.

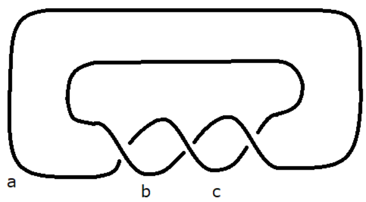


Figure 1 : $T(2,3)$

Let K be a knot diagram. We shall define a semigroup, which we call the knot semigroup of K, and denote by $M(K)$. We assume that each arc is denoted by a letter. Then we define two defining relations $xy = yz$ and $yx = zy$ at crossing, where arcs x and z form the undercrossing and arc y is the overcrossing. We define these relations at every crossing. The cancellative semigroup generated by arc letters with these defining relations is the knot semigroup of the knot diagram. This construction is the analogy of the Wirtinger presentation of knot group [34].

The definition of the knot semigroup naturally generalizes from diagrams of knots to diagrams of links. For example, the diagram of the Hopf link in Figure 2 contains two arcs a, b and two crossings, each defining a single relation $ab = ba$. Hence, its

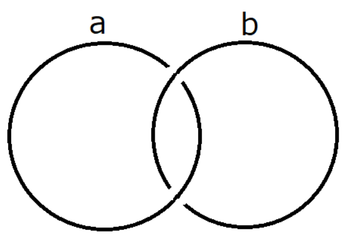


Figure 2 : Hopf link

knot semigroup is the free commutative semigroup with two generators.

3.2.2 Alternating sum semigroups

We shall recall an alternating sum semigroup defined by A. Vernitski [33]. Let G be either $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ or $\mathbb{Z}$. Let B be a subset of G, and B^+ the set of words of B. By the alternating sum of a word $b_1b_2b_3b_4\dots b_k \in B^+$ we shall mean the value of $b_1 - b_2 + b_3 - b_4 + \cdots + (-1)^{k+1}b_k$ calculated in G. The notation $\mathrm{alt}(b_1b_2\dots b_k)$ denote the value, i.e.,

$$\mathrm{alt}(b_1b_2\dots b_k) = b_1 - b_2 + b_3 - b_4 + \cdots + (-1)^{k+1}b_k.$$

We also define the following notation. $|b_1b_2\dots b_k|$ denote the length of the word $b_1b_2\dots b_k$. We shall say that two words $u, v \in B^+$ are in relation $\sim$ if and only if

(1) $|u| = |v|$,

(2) $\mathrm{alt}(u) = \mathrm{alt}(u)$.

The relation $\sim$ is a congruence on B^+. The semigroup $\mathrm{AS}(G, B)$ denote the factor semigroup $B^+/\sim$. $\mathrm{AS}(G, B)$ is called an alternating sum semigroup.

We shall also recall a strong alternating sum semigroup defined by A. Vernitski [33]. The sets G and B are as above. Let us say that $g \in G$ is even (resp. odd) in G if g can be represented in the form $g = 2h$ (resp. $g = 2h + 1$) for some $h \in G$. Let $w \in B^+$. The notation $|w|_e$ denote the number of entries in w which are even in G. We shall say that two words $u, v \in B^+$ are in relation $\approx$ if and only if

(1) $|u| = |v|$,

(2) $\mathrm{alt}(u) = \mathrm{alt}(v)$,

(3) $|u|_e = |v|_e$.

The relation $\approx$ is a congruence on B^+. The semigroup $\mathrm{SAS}(G, B)$ denote the factor semigroup $B^+/\approx$. $\mathrm{SAS}(G, B)$ is called a strong alternating sum semigroup.

3.2.3 Examples of knot semigroups

Trivial knots

Let $\mathbb{N}$ be the semigroup of positive integers. The semigroup $\mathbb{N}$ is a cancellative semigroup. The diagram of the trivial knot contains one arc and no crossings (Figure 3). Therefore, its knot semigroup is isomorphic to the semigroup $\mathbb{N}$.

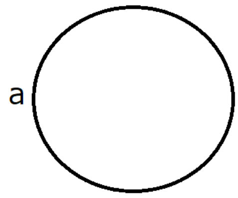


Figure 3 : Trivial knots

Torus knots and torus links

A torus knot $T(2, n)$ consists of n half-twists (Figure 4.). We recall the knot semigroup $M(T(2, n))$ of a knot diagram $T(2, n)$ (with an odd n) and the knot semigroup of a link diagram $T(2, n)$ (with an even n) proved by A. Vernitski [33].

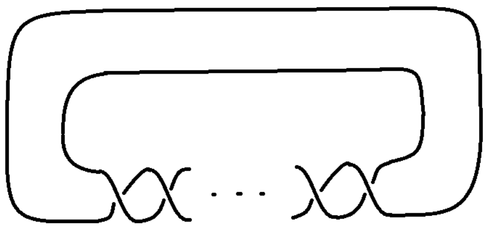


Figure 4 : Torus knots

Theorem 3.2.1 ([33] Theorem 3.)**.** *Let n be an odd integer. The knot semigroup $M(T(2, n))$ of the torus knot diagram $T(2, n)$ is isomorphic to the alternating sum semigroup* $\mathrm{AS}(\mathbb{Z}_n, \mathbb{Z}_n)$.

Theorem 3.2.2 ([33] Theorem 13.)**.** *Let n be an even integer. The knot semigroup $M(T(2, n))$ of the torus link diagram $T(2, n)$ is isomorphic to the strong alternating sum semigroup* $\mathrm{SAS}(\mathbb{Z}_n, \mathbb{Z}_n)$.

Since $\mathrm{SAS}(\mathbb{Z}_n, \mathbb{Z}_n) = \mathrm{AS}(\mathbb{Z}_n, \mathbb{Z}_n)$ for odd values of n, we have the following corollary.

Corollary 3.2.3 ([33] Corollary 14.)**.** *The knot semigroup $M(T(2, n))$ of the diagram $T(2, n)$ for every positive n is isomorphic to the strong alternating sum semigroup* $\mathrm{SAS}(\mathbb{Z}_n, \mathbb{Z}_n)$.

Twist knots

A twist knot, which we shall denote $\mathfrak{tw}_n$ consists of n clockwise half-twists and 2 anticlockwise half-twists (Figure 5). We recall the knot semigroup $M(\mathfrak{tw}_n)$ of a knot diagram $\mathfrak{tw}_n$ proved by A. Vernitski [33]. The notation $[n + 2]$ denote the set $\{0, 1, \dots, n + 2\}$.

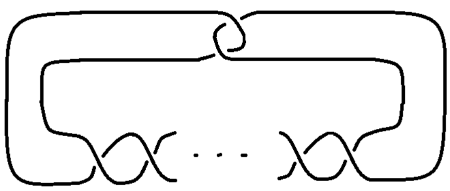


Figure 5 : Twist knots

Theorem 3.2.4 ([33] Theorem 15.)**.** *The knot semigroup $M(\mathfrak{tw}_n)$ of the twist knot diagram $\mathfrak{tw}_n$ is isomorphic to the alternating sum semigroup* $\mathrm{AS}(\mathbb{Z}_{2n+1}, [n + 2])$.

Chapter 4

Knot semigroups of double twist knots

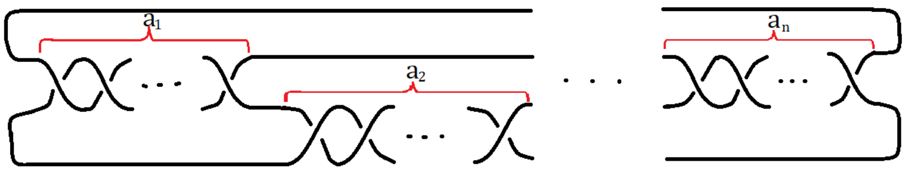
In this chapter we will explain knot semigroups of double twist knots.

4.1 Statements

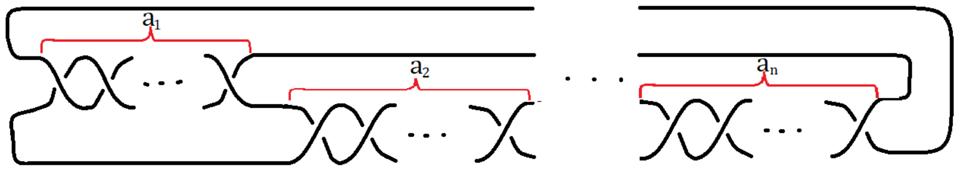
The plan of this section faithfully follows that of Section 7 of [33]. First we explain Conway's normal form of 2-bridge knot.

4.1.1 Conway's normal forms

Any 2-bridge knot has a presentation, which can be deformed as in figure, where a_i indicates $|a_i|$ $(\neq 0)$ crossing points with sign $\epsilon_i = a_i/|a_i| = \pm 1$.



(n is odd)



(n is even)

We denote the 2-bridge knot with this knot diagram by $C(a_1, a_2, \ldots, a_n)$, which is called Conway's normal form.

4.1.2 A conjecture of the knot semigroups of 2-bridge knots

The torus knots and the twist knots are the 2-bridge knots. Then we have the following conjecture by A. Vernitski ([33] Conjecture 23.).

Conjecture 4.1.1. *The knot semigroup of the 2-bridge knot is isomorphic to an alternating sum semigroup.*

4.1.3 Double twist knots

To support Conjecture 4.1.1 we prove that the knot semigroup of the double twist knot is isomorphic to an alternating sum semigroup.

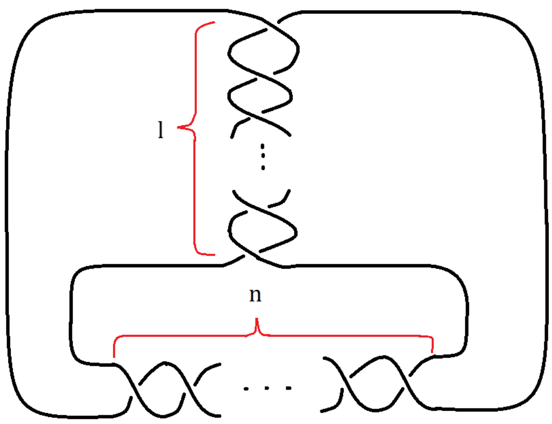


Figure6 : Double twist knots

A double twist knot, which we shall denote by $\mathfrak{dtw}_n^l$ consists of n clockwise half-twists and l anticlockwise half-twists, where l, n indicate the number of crossing points (Figure 5). Then we have the following theorem.

Theorem 4.1.2. *Let $n, l \geq 1$ be integers. Suppose the integer nl is an even integer. Then the knot semigroup $M(\mathfrak{dtw}_n^l)$ of the double twist knot diagram $\mathfrak{dtw}_n^l$ is isomorphic to the alternating sum semigroup*

$$\mathrm{AS}(\mathbb{Z}_{ln+1}, \{0, 1, \ldots, n, 1 \cdot n + 1, \ldots (l-1) \cdot n + 1\}).$$

Remark 4.1.3. The double twist knot is the 2-bridge knot $C(l, n)$.

Remark 4.1.4. Let $l = 2$ in Theorem 4.1.2. Then

$$\{0, 1, \ldots, 1 \cdot n + 1\} = [n + 2].$$

Thus Theorem 4.1.2 implies Theorem 3.2.4.

Remark 4.1.5. Let $l = 1$ in Theorem 4.1.2. Then

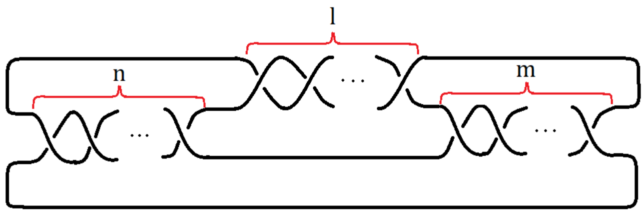
$$\begin{aligned} M(\mathfrak{dtw}_n^1) &\simeq \mathrm{AS}(\mathbb{Z}_{n+1}, \{0, 1, \ldots, n, 0 \cdot n + 1\}) \\ &\simeq \mathrm{AS}(\mathbb{Z}_{n+1}, \{0, 1, \ldots, n\}) \\ &\simeq \mathrm{AS}(\mathbb{Z}_{n+1}, \mathbb{Z}_{n+1}). \end{aligned}$$

On the other hand, $\mathfrak{dtw}_n^1 \simeq T(2, n + 1)$ as knots. By Theorem 3.2.1,

$$M(T(2, n + 1)) \simeq \mathrm{AS}(\mathbb{Z}_{n+1}, \mathbb{Z}_{n+1}).$$

Thus Theorem 4.1.2 holds in the case of $l = 1$.

Remark 4.1.6. We consider the following knot $C(m, l, n)$.



Then we have the following conjecture.

Conjecture 4.1.7. *Let $l, m, n \geq 1$ be integers. Suppose the integer $(ml + 1)n + m$ is odd. Then the knot semigroup $M(C(m, l, n))$ of the knot diagram $C(m, l, n)$ is isomorphic to the alternating sum semigroup*

$$\mathrm{AS}(\mathbb{Z}_{(ml+1)n+m}, \bigcup_{i=0}^{n+1}\{i\} \cup \bigcup_{j=0}^{l+1}\{jn + 1\} \cup \bigcup_{k=0}^{m-1}\{(kl + 1)n + k\}).$$

4.2 Proof of theorem

We shall prove Theorem 4.1.2.

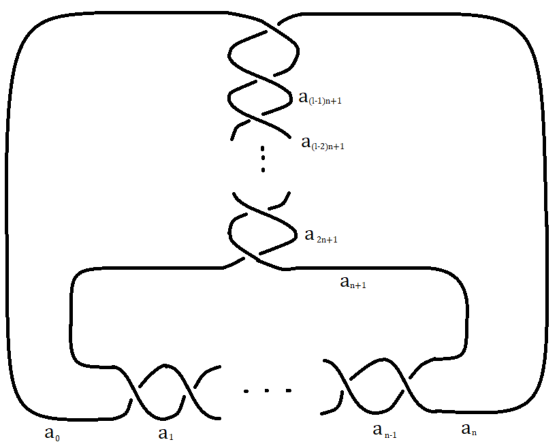
Suppose that A^+/κ is a knot semigroup, where A is the set of arcs and κ is a cancellative congruence on the free semigroup A^+ induced by the defining relations of the knot semigroup. Let $\sim$ be a congruence on B^+, where B is an alphabet of the same size as A. We shall establish an isomorphism between A^+/κ and $B^+/\sim$ by the following lemma.

Lemma 4.2.1 ([33] Lemma 2.)**.** *Suppose A and B are sets. Consider a bijection $\phi : A \to B$. It induces an isomorphism between A^+ and B^+, which we shall denote by ϕ^+. Suppose a congruence κ on A^+ and a congruence $\sim$ on B^+ are such that for each $u, v \in A^+$ if $u \,\kappa\, v$ then $\phi(u) \sim \phi(v)$. Then ϕ induces a mapping from A^+ to B^+, which we shall denote by ψ. Moreover, ψ is a homomorphism. Suppose a subset of B^+ exists, which we shall call the set of canonical words, such that in each class of $\sim$ there is exactly one canonical word and at least one word of each class of κ is mapped by ϕ^+ to a canonical word. Then ψ is an isomorphism between A^+/κ and $B^+/\sim$.*

Let

$$A = \{a_0, \dots, a_n, a_{n+1}, a_{2n+1}, \dots, a_{(l-2)n+1}, a_{(l-1)n+1}\}$$

be the set of arcs as in the following figure.



Denote the set $\{0, 1, \dots, n, 0 \cdot n+1, 1 \cdot n+1, \dots (l-1) \cdot n+1\}$ by $C_{n,l}$. Consider a mapping ϕ from A to $C_{n,l}$ defined as $a_i \mapsto i$. It induces an isomorphism A^+ to $C^+_{n,l}$, which we shall denote by ϕ^+. Then we have the following Lemma.

Lemma 4.2.2. *The equality $a_i a_{i+j} = a_{i+k} a_{i+j+k}$ is true in $M(\mathfrak{dtw}^l_n)$ for all values of i, j, k such that $0 \le i \le i+j \le i+j+k \le n+1$.*

Proof. The relations in $M(\mathfrak{dtw}^l_n)$ are the equalities

$$a_{i-1} a_i = a_i a_{i+1},$$

and

$$a_i a_{i-1} = a_{i+1} a_i$$

for all $i = 1, 2, \dots, n$ (from the crossings at the bottom of the diagram), and the equalities

$$a_{(l-j-1)n+1}a_{(l-j)n+1} = a_{(l-j)n+1}a_{(l-j+1)n+1},$$

and

$$a_{(l-j)n+1}a_{(l-j-1)n+1} = a_{(l-j+1)n+1}a_{(l-j)n+1}$$

for all $j = 0, 1, \dots, l-1$ (from the crossings at the top of the diagram), where $a_{ln+1} = a_0$. Applying relations of the type $a_{i-1}a_i = a_ia_{i+1}$ repeatedly, we obtain $a_ia_{i+1} = a_{i+k}a_{i+1+k}$ for all values of i, k such that $0 \leq i \leq i+1 \leq i+1+k \leq n+1$. Similarly, we can obtain $a_ia_{i-1} = a_{i+k}a_{i-1+k}$ for all values of i, k such that $0 \leq i-1 \leq i \leq i+k \leq n+1$. Consider

$$a_ia_{i+j}a_{i+j+1} = a_ia_{i+1}a_{i+2} = a_{i+2}a_{i+3}a_{i+2} = a_{i+2}a_{i+j+2}a_{i+j+1}.$$

Hence $a_ia_{i+j} = a_{i+2}a_{i+j+2}$ by the cancellative rule. This proves that $a_ia_{i+j} = a_{i+k}a_{i+j+k}$ for all values of i, j, k such that $0 \leq i \leq i+j \leq i+j+k \leq n+1$ and even k.

We shall prove that $a_0a_0 = a_1a_1$.

(1) Suppose the integer l is an even number.
Consider

$$\begin{aligned} a_{n+1}a_{\{l-(l-2)\}n+1}a_{\{l-(l-2)\}n+1} &= a_{\{l-(l-2)\}n+1}a_{\{l-(l-3)\}n+1}a_{\{l-(l-2)\}n+1} \\ &= a_{\{l-(l-2)\}n+1}a_{\{l-(l-2)\}n+1}a_{n+1} \\ &= a_{\{l-(l-2)\}n+1}a_{n+1}a_1 \\ &= a_{n+1}a_1a_1. \end{aligned}$$

Hence we have $a_{\{l-(l-2)\}n+1}a_{\{l-(l-2)\}n+1} = a_1a_1$.
Next consider

$$\begin{aligned} a_{\{l-(l-3)\}n+1}a_{\{l-(l-4)\}n+1}a_{\{l-(l-4)\}n+1} &= a_{\{l-(l-4)\}n+1}a_{\{l-(l-5)\}n+1}a_{\{l-(l-4)\}n+1} \\ &= a_{\{l-(l-4)\}n+1}a_{\{l-(l-4)\}n+1}a_{\{l-(l-3)\}n+1} \\ &= a_{\{l-(l-4)\}n+1}a_{\{l-(l-3)\}n+1}a_{\{l-(l-2)\}n+1} \\ &= a_{\{l-(l-3)\}n+1}a_{\{l-(l-2)\}n+1}a_{\{l-(l-2)\}n+1}. \end{aligned}$$

Hence $a_{\{l-(l-4)\}n+1}a_{\{l-(l-4)\}n+1} = a_{\{l-(l-2)\}n+1}a_{\{l-(l-2)\}n+1}$.
Since l is even, $a_0a_0 = a_1a_1$.

(2) Suppose the integer l is an odd number.
Since nl is even, n is an even number. Consider

$$\begin{aligned} a_{(l-1)n+1}a_0a_0 &= a_0a_na_0 \\ &= a_0a_0a_{(l-1)n+1} \\ &= a_0a_{(l-1)n+1}a_{(l-2)n+1} \\ &= a_{(l-1)n+1}a_{(l-2)n+1}a_{(l-2)n+1}. \end{aligned}$$

Hence $a_0a_0 = a_{(l-2)n+1}a_{(l-2)n+1}$. Since this equation holds and n is an even number,

$$a_0a_0 = a_{(l-2)n+1}a_{(l-2)n+1} = \cdots = a_{n+1}a_{n+1} = a_1a_1.$$

We shall prove that $a_0a_j = a_1a_{j+1}$ for all values of $j = 0, 1, \ldots, n$. Let j be an odd number. Consider

$$a_0a_0a_j = a_{j-1}a_{j-1}a_j = a_{j-1}a_ja_{j+1} = a_0a_1a_{j+1}.$$

Hence $a_0a_j = a_1a_{j+1}$.
Let j be even and positive. Consider

$$\begin{aligned} a_0a_0a_j &= a_1a_1a_j \quad (\text{since}\, a_0a_0 = a_1a_1.) \\ &= a_{j-1}a_{j-1}a_j \\ &= a_{j-1}a_ja_{j+1} \\ &= a_{j-2}a_{j-1}a_{j+1} \\ &= a_0a_1a_{j+1}. \end{aligned}$$

Hence $a_0a_j = a_1a_{j+1}$.
Now suppose k is odd. If i is even, we have

$$\begin{aligned} a_ia_{i+j} &= a_0a_j \\ &= a_1a_{j+1} \\ &= a_{i+1}a_{i+j+1} \\ &= a_{(i+1)+(k-1)}a_{(i+j+1)+(k-1)} \\ &= a_{i+k}a_{i+j+k}. \end{aligned}$$

If i is odd, we have

$$\begin{aligned} a_ia_{i+j} &= a_1a_{j+1} \\ &= a_0a_j \\ &= a_{i+1}a_{i+j+1} \\ &= a_{(i+1)+(k-1)}a_{(i+j+1)+(k-1)} \\ &= a_{i+k}a_{i+j+k}. \end{aligned}$$

$\square$

Lemma 4.2.3. *The equality* $a_{in+1}a_{(i+j)n+1} = a_{(i+k)n+1}a_{(i+j+k)n+1}$ *is true in* $M(\mathfrak{dtw}_n^l)$ *for all values of* i, j, k *such that* $0 \leq i \leq i+j \leq i+j+k \leq l+1$.

Proof. Applying relations of the type

$$a_{(l-j-1)n+1}a_{(l-j)n+1} = a_{(l-j)n+1}a_{(l-j+1)n+1}$$

repeatedly, we obtain

$$a_{in+1}a_{(i+1)n+1} = a_{(i+k)n+1}a_{(i+1+k)n+1}$$

for all values of i, k such that $0 \leq i \leq i+1 \leq i+1+k \leq l+1$. Similarly, we can obtain

$$a_{in+1}a_{(i-1)n+1} = a_{(i+k)n+1}a_{(i-1+k)n+1}$$

for all values of i, k such that $0 \leq i-1 \leq i \leq i+k \leq l+1$. Consider

$$\begin{aligned} a_{in+1}a_{(i+j)n+1}a_{(i+j+1)n+1} &= a_{in+1}a_{(i+1)n+1}a_{(i+2)n+1} \\ &= a_{(i+2)n+1}a_{(i+3)n+1}a_{(i+2)n+1} \\ &= a_{(i+2)n+1}a_{(i+j+2)n+1}a_{(i+j+1)n+1}. \end{aligned}$$

Hence $a_{in+1}a_{(i+j)n+1} = a_{(i+2)n+1}a_{(i+j+2)n+1}$. This proves that

$$a_{in+1}a_{(i+j)n+1} = a_{(i+k)n+1}a_{(i+j+k)n+1}$$

for all values of i, j, k such that $0 \leq i \leq i+j \leq i+j+k \leq l+1$ and even k. We shall prove that $a_1a_1 = a_{n+1}a_{n+1}$.

(1) Suppose l is odd.
Since n is even, $a_1a_1 = a_{n+1}a_{n+1}$.

(2) Suppose l is even, and n is even.
Consider

$$a_2a_1a_1 = a_1a_0a_1 = a_1a_1a_2 = a_1a_2a_3 = a_2a_3a_3.$$

Hence $a_1a_1 = a_3a_3$. This proves that $a_1a_1 = a_{n+1}a_{n+1}$.

(3) Suppose l is even and n is odd.
Consider

$$\begin{aligned} a_na_{n+1}a_{n+1} &= a_{n-1}a_na_{n+1} \\ &= a_{n-1}a_{n-1}a_n \\ &= a_{n-1}a_{n-2}a_{n-1} \\ &= a_na_{n-1}a_{n-1}. \end{aligned}$$

Hence $a_{n+1}a_{n+1} = a_{n-1}a_{n-1}$.Since n is odd, $a_{n+1}a_{n+1} = a_0a_0$. Since l is even, $a_0a_0 = a_1a_1$. Thus $a_{n+1}a_{n+1} = a_1a_1$.

We shall prove that $a_1a_{jn+1} = a_{n+1}a_{(j+1)n+1}$ for all values $j = 0, 1, \dots, n$. Let j be an odd number. Consider

$$\begin{aligned} a_1a_1a_{jn+1} &= a_{(j-1)n+1}a_{(j-1)n+1}a_{jn+1} \\ &= a_{(j-1)n+1}a_{jn+1}a_{(j+1)n+1} \\ &= a_1a_{n+1}a_{(j+1)n+1}. \end{aligned}$$

Hence $a_1a_{jn+1} = a_{n+1}a_{(j+1)n+1}$.
Let j be even and positive. Consider

$$\begin{aligned} a_1a_1a_{jn+1} &= a_{n+1}a_{n+1}a_{jn+1} \\ &= a_{(j-1)n+1}a_{(j-1)n+1}a_{jn+1} \\ &= a_{(j-1)n+1}a_{jn+1}a_{(j+1)n+1} \\ &= a_{(j-2)n+1}a_{(j-1)n+1}a_{(j+1)n+1} \\ &= a_1a_{n+1}a_{(j+1)n+1}. \end{aligned}$$

Hence $a_1a_{jn+1} = a_{n+1}a_{(j+1)n+1}$.
Suppose k is odd. If i is even, we have

$$\begin{aligned} a_{in+1}a_{(i+j)n} &= a_1a_{jn+1} \\ &= a_{n+1}a_{(j+1)n+1} \\ &= a_{(i+1)n+1}a_{(i+j+1)n+1} \\ &= a_{\{(i+1)+(k-1)\}n+1}a_{\{(i+j+1)+(k-1)\}n+1} \\ &= a_{(i+k)n+1}a_{(i+j+k)n+1}. \end{aligned}$$

If i is odd, we have

$$\begin{aligned} a_{in+1}a_{(i+j)n} &= a_{n+1}a_{(j+1)n+1} \\ &= a_1a_{jn+1} \\ &= a_{(i+1)n+1}a_{(i+j+1)n+1} \\ &= a_{\{(i+1)+(k-1)\}n+1}a_{\{(i+j+1)+(k-1)\}n+1} \\ &= a_{(i+k)n+1}a_{(i+j+k)n+1}. \end{aligned}$$

□

Lemma 4.2.4. *The equality* $a_{pn+1}a_qa_{rn+1} = a_{(p+1)n+1}a_qa_{(r-1)n+1}$ *is true in* $M(\mathfrak{dtw}_n^l)$ *for all values* p, q, r *such that* $0 \le p \le n-1$, $1 \le r \le l$, *and* $q \notin \{0, 2n+1, \dots, (l-1)n+1\}$.

Proof. Consider

$$\begin{aligned}
a_{(p+1)n+1}a_{pn+1}a_q a_{rn+1} &= a_n a_0 a_q a_{rn+1}\\
&= a_n a_{n-q+1} a_{n+1} a_{rn+1}\\
&= a_n a_{n-q+1} a_1 a_{(r-1)n+1}\\
&= a_n a_n a_q a_{(r-1)n+1}\\
&= a_{n+1} a_{n+1} a_q a_{(r-1)n+1}\\
&= a_{(p+1)n+1} a_{(p+1)n+1} a_q a_{(r-1)n+1}.
\end{aligned}$$

Hence $a_{pn+1}a_q a_{rn+1} = a_{(p+1)n+1}a_q a_{(r-1)n+1}$. □

Lemma 4.2.5. *The equality $a_p a_q a_r = a_{p+1}a_q a_{r-1}$ is true in $M(\mathfrak{dtw}_n^l)$ for all values p, q, r such that $p \in \{0, 1, \ldots, n\}$, $r \in \{1, 2, \ldots, n+1\}$, $q \in \{0, 2n+1, \ldots, (l-1)n+1\}$.*

Proof. Suppose $q = 0$. Then

$$a_p a_0 a_r = a_{p+1} a_1 a_r = a_{p+1} a_0 a_{r-1}.$$

Suppose $q \in \{2n+1, \ldots, (l-1)n+1\}$. Consider

$$\begin{aligned}
a_{p+1}a_p a_{kn+1} a_r &= a_1 a_0 a_{kn+1} a_r\\
&= a_1 a_{(l-k)n+1} a_1 a_r\\
&= a_1 a_{(l-k)n+1} a_0 a_{r-1}\\
&= a_{kn+1} a_0 a_0 a_{r-1}\\
&= a_{kn+1} a_1 a_1 a_{r-1}\\
&= a_{kn+1} a_{(k-1)n+1} a_{(k-1)n+1} a_{r-1}\\
&= a_{n+1} a_1 a_{(k-1)n+1} a_{r-1}\\
&= a_{2n+1} a_{n+1} a_{(k-1)n+1} a_{r-1}\\
&= a_{n+1} a_{n+1} a_{kn+1} a_{r-1}\\
&= a_{p+1} a_{p+1} a_{kn+1} a_{r-1}.
\end{aligned}$$

Hence $a_p a_q a_r = a_{p+1} a_q a_{r-1}$. □

Canonical words in $C_{n,l}^{+}$ will be defined as words of the form 000^{t-2} or $c00^{t-2}$ or $0c0^{t-2}$ or $dc0^{t-2}$, where $t \geq 2$ is the length of the word and $c \in \{1, 2, \ldots, n\}$, $d \in \{2n+1, 3n+1, \ldots, (l-1)n+1\}$.

Consider a non-negative integer valued parameter $\pi(w)$ of a word w in $C_{n,l}^{+}$, which is 0 if the first two entries in w are 00 or $c0$ or $0c$ or dc for some $c \in \{1, \ldots, n\}$, $d \in \{2n+1, \ldots, (l-1)n+1\}$ and which is 1 otherwise. Define the defect of a word $w = b_1 b_2 \ldots b_t$ in $C_{n,l}^{+}$ as a word $\pi(w) b_3 \ldots b_t$. Defects are assumed to be ordered antilexicographically.

Lemma 4.2.6. *A word in* $C^+_{n,l}$ *is canonical if and only if its defect is a word consisting of* 0*s.*

Proof. The result follows from the form of canonical word. □

Lemma 4.2.7. *Let* u *be a word in* A^+*. Unless the defect of* $\phi(u)$ *is a word consisting of* 0*s, there is a word* v *in* A^+ *such that* $u = v$ *in* $M(\mathfrak{dtw}^l_n)$ *and the defect of* $\phi(v)$ *is less than the defect of* $\phi(u)$*.*

Proof. Suppose the defect of $\phi(u)$ has a non-zero entry at a position which is not the first one. This means that at some position $d \geq 3$ there is a non-zero entry r in $\phi(u)$. Let $u = u'a_pa_qa_ru''$, where $u', u'' \in A^+$ and $p, q, r \in C_{n,l}$ with $r \neq 0$.

(1) Suppose $q \notin \{0, 2n+1, 3n+1, \ldots, (l-1)n+1\}$.

1. If $r \notin \{2n+1, \ldots, (l-1)n+1\}$, then we define

$$v = u'a_pa_{q-1}a_{r-1}u''.$$

2. Suppose $r \in \{2n+1, 3n+1, \ldots, (l-1)n+1\}$.
   - If $p \notin \{0, 2n+1, \ldots, (l-1)n+1\}$, then we define

   $$v = u'a_{p-1}a_{q-1}a_ru''.$$

   - If $p \in \{0, 2n+1, \ldots, (l-1)n+1\}$, then define

   $$v = u'a_{(k'+1)n+1}a_qa_{(k-1)n+1}u'',$$

   where $p = k'n+1, r = kn+1$. The words u and v are equal by the Lemma 4.2.4.

(2) Suppose $q \in \{0, 2n+1, \ldots, (l-1)n+1\}$.

1. If $p \in \{2n+1, \ldots, (l-1)n+1\}$ or $r \in \{2n+1, \ldots, (l-1)n+1\}$, then we define

$$v = u'a_{(k'-1)n+1}a_{(k-1)n+1}a_ru'',$$

where $p = k'n+1, q = kn+1$, or

$$v = u'a_pa_{(k-1)n+1}a_{(k''-1)n+1}u'',$$

where $q = kn+1, r = k''n+1$.

2. If $p, r \notin \{2n+1, 3n+1 \ldots, (l-1)n+1\}$, then we define

$$v = u' a_{p+1} a_q a_{r-1} u''.$$

By the Lemma 4.2.5, $u = v$.

In each case, $u = v$ in $M(\mathfrak{dtw}_n^l)$, and the defect of $\phi(v)$ is less than the defect of $\phi(u)$.

Suppose the defect of $\phi(u)$ has a non-zero entry only at the first position. Let $u = a_p a_q u'$, where $u' \in A^+$ and $p, q \in C_n^l$.

(1) If $p, q \in \{1, 2, \ldots, n+1\}$, let $m = \min(p, q)$, then define

$$v = a_{p-m} a_{q-m} u'.$$

(2) If $p, q \in \{2n+1, \ldots, (l-1)n+1\}$, let $p = kn+1, q = k'n+1$ and $m = \min(k, k')$. If $m = k'$, then we define

$$v = a_{(k-m)n+1} a_{(k'-m)n+1} u'.$$

If $m = k$, then we use the following case (3).

(3) If $p \in \{1, 2, \ldots, n\}, q \in \{2n+1, \ldots, (l-1)n+1\}$, then consider

$$\begin{aligned} a_p a_{kn+1} a_{n-p+1} &= a_0 a_{kn+1} a_{n+1} \\ &= a_{(l-k)n+1} a_1 a_{n+1} \\ &= a_{(l-k+1)n+1} a_{n+1} a_{n+1} \\ &= a_{(l-k+1)n+1} a_{n-p+1} a_{n-p+1}. \end{aligned}$$

Hence $a_p a_{kn+1} = a_{(l-k+1)n+1} a_{n-p+1}$. Then we define

$$v = a_{(l-k+1)n+1} a_{n-p+1} u',$$

where $q = kn+1$.

(4) If $p \in \{2n+1 \ldots, (l-1)n+1\}, q = n+1$, then we define

$$v = a_{(k-1)n+1} a_1 u',$$

where $p = kn+1$.

(5) If $p = n+1, q \in \{2n+1, \ldots, (l-1)n+1\}$, then we consider

$$a_{n+1} a_{kn+1} = a_{(l-k+1)n+1} a_0 = a_{(l-k+2)n+1} a_n.$$

Then we define

$$v = a_{(l-k+2)n+1} a_n u'.$$

(6) If $p \in \{n+1, 2n+1, \ldots, (l-1)n+1\}, q = 0$, then we consider

$$a_{kn+1}a_0 = a_1 a_{(l-k)n+1} = a_{(k+1)n+1}a_n.$$

Then we define

$$v = a_{(k+1)n+1}a_n u',$$

where $p = kn + 1$.

(7) If $p = 0, q \in \{n+1, 2n+1, \ldots, (l-1)n+1\}$, then consider

$$a_0 a_{kn+1} = a_{(l-k)n+1}a_1$$

Then we define

$$v = a_{(l-k)n+1}a_1 u',$$

where $q = kn + 1$.

In each case, $u = v$ in $M(\mathfrak{dtw}_n^l)$, and the defect $\phi(v)$ is a word consisting of 0s. □

Then we have the following corollary.

Corollary 4.2.8. *Every word in A^+ is equal in $M(\mathfrak{dtw}_n^l)$ to a word in A^+ which is mapped by ϕ to a word with a defect consisting of* 0*s.*

Proof of Theorem 4.1.2. The relations in $M(\mathfrak{dtw}_n^l)$ are listed in the proof of Lemma 4.2.2. For each relation $u = v$ the words $\phi(u)$ and $\phi(v)$ have the same length and the same alternating sum calculated in $\mathbb{Z}_{ln+1}$. Thus by Lemma 4.2.1, ϕ^+ induces a homomorphism $\psi : M(\mathfrak{dtw}_n^l) \to C_{n,l}^+/\sim$.
Consider two canonical words u, v which are $\sim$ equivalent. We shall show that each class of $\sim$ contains at most one canonical word.

(1) Suppose their alternating sums are both 0.

1. If u, v are form of $c00^{t-2}$ or $0c0^{t-2}$, where $c \in \{1, 2, \ldots, n\}$, then since the canonical word can have at most one non-zero entry, both words consist only of 0s and, therefore are equal.
2. If u or v is form of $dc0^{t-2}$, where $c \in \{1, 2, \ldots, n\}$, $d \in \{2n+1, \ldots, (l-1)n+1\}$, then the alternating sum of $dc0^{t-2}$ is $d - c$. If $d \neq 0$ or $c \neq 0$, then since $d - c = 0$, this contradicts $c \in \{1, 2, \ldots, n\}, d \in \{2n+1, \ldots, (l-1)n+1\}$. Therefore both words u, v consist only of 0s, and are equal.

(2) Suppose two canonical words share the same non-zero alternating sum.

1. If $u = c_1 0 0^{t-2}$, $v = c_2 0 0^{t-2}$, then since both alternating sums are the same, $c_1 = c_2$. Thus $u = v$.
2. If $u = 0 c_1 0^{t-2}$, $v = 0 c_2 0^{t-2}$, then $u = v$ by the same reason of 1.
3. If $u = c_1 0 0^{t-2}$, $v = 0 c_2 0^{t-2}$, then $c_1 = -c_2$ in $\mathbb{Z}_{nl+1}$. Since $c_1, c_2 \in \{1, 2, \ldots, n\}$, this case is impossible.
4. If $u = c_1 0 0^{t-2}$, $v = 0 c_2 0^{t-2}$, then $c_1 = d_2 - c_2$ in $\mathbb{Z}_{ln+1}$. Since $c_1, c_2 \in \{1, 2, \ldots, n\}$, $d \in \{2n+1, \ldots, (l-1)n+1\}$, this case is impossible ($c_1 + c_2 \leq 2n, d_2 \geq 2n$).
5. If $u = 0 c_1 0^{t-2}$, $v = d_2 c_2 0^{t-2}$, then $-c_1 = d_2 - c_2$ in $\mathbb{Z}_{ln+1}$. If $c_2 - c_1 > 0$, this case is impossible. If $c_2 - c_1 > 0$, this case is also impossible ($ln + 1 + c_2 - c_1 > (l-1)n + 1$, $d_2 \leq (l-1)n + 1$).
6. If $u = d_1 c_1 0^{t-2}$, $v = d_2 c_2 0^{t-2}$, then $d_1 - c_1 = d_2 - c_2$ in $\mathbb{Z}_{ln+1}$. Since $c_1, c_2 \in \{1, 2, \ldots, n\}$, $d_1, d_2 \in \{2n+1, \ldots, (l-1)n+1\}$, this case is impossible.

Thus each class of $\sim$ contains at most one canonical word.
Consider a word $w \in C^+_{n,l}$ which has length t and alternating sum s. We shall show that each class of $\sim$ contains at least one canonical word.

(1) If $s \in \{0, 1, \ldots, n\}$, then canonical word $s 0 0^{t-2}$ is $\sim$ equivalent to w.

(2) If $s \in \{(l-1)n+1, \ldots, ln+1\}$, let $q = -s$. Then $0 q 0^{t-2}$ is $\sim$ equivalent to w.

(3) If $s \in \{n+1, \ldots, (l-1)n\}$, then w is $\sim$ equivalent to $d(-c)0^{t-2}$ for some $c \in \{1, 2, \ldots, n\}$, $d \in \{2n+1, \ldots, (l-1)n+1\}$.

Thus each class of $\sim$ contains at least one canonical word.
By Corollary 4.2 and Lemma 4.2.6, each word in A^+ is equal in $M(\mathfrak{dtw}^l_n)$ to a word mapped by ϕ to a canonical word. Now Theorem 4.1.2 follows from Lemma 4.2.1.

Chapter 5

Link invariants

In this section, we consider the growth of semigroup algebras of knot semigroups. To investigate the growth, we use the Gelfand-Kirillov dimension. As an application we construct a link invariant.

5.1 Gelfand-Kirillov dimensions

First, we explain the definition of Gelfand-Kirillov dimension.

Let k be a field and A a finitely generated algebra over k. If V is a subspace of A, we denote by V^n the subspace spanned by all products of elements of V of length n. By generating subspace, we will mean a finite-dimensional subspace of A which generates A as algebra, and which contains 1. The growth function associated to a generating subspace V is

$$f_V(n) = \dim_k V^n.$$

An algebra A is said to have polynomial growth if there are positive real numbers c, r such that

$$f_V(n) \leq cn^r \tag{5.1}$$

for all n. We will see below that this is independent of the choice of generating spaces.

Lemma 5.1.1. *Let A be a finitely generated k-algebra, and let V, W be generating subspaces. If $f_V(n) \leq cn^r$ for all n, then there is a c' such that $f_W(n) \leq c'n^r$ for all n.*

Proof. Assume that $f_V(n) \leq cn^r$. Since $A = \cup V^n$ and W is finite-dimensional, $W \subset V^s$ for some s. Then $W^n \subset V^{sn}$, hence

$$f_W(n) \leq f_V(sn) \leq cs^r n^r = c'n^r.$$

□

Definition 5.1.2. The Gelfand-Kirillov dimension of an algebra A with polynomial growth is the infimum of the real numbers r such that (5.1) holds for some c:

$$\operatorname{GK}\dim(A) = \inf\{r \mid f_V(n) \leq cn^r\}.$$

If A does not have polynomial growth, then we define $\operatorname{GK}\dim(A) = \infty$.

The Gelfand-Kirillov dimension can be represented as follows:

Lemma 5.1.3. *Let A be an algebra. Then we have*

$$\operatorname{GK}\dim(A) = \inf\{r \mid f(n) \leq p(n)\}$$

for some polynomial p of degree r.

Proof. Since $\{r \mid f(n) \leq cn^r\} \subset \{r \mid f(n) \leq p(n)\}$, we have

$$\inf\{r \mid f_V(n) \leq cn^r\} \geq \inf\{r \mid f(n) \leq p(n)\}.$$

Let $r = \inf\{r \mid f_V(n) \leq cn^r\}$ and, $r' = \inf\{r \mid f(n) \leq p(n)\}$. Assume that $r > r'$. We write $p(n) = c_{r'}n^{r'} + c_{r'-1}n^{r'-1} + \cdots + c_0$. Then we have

$$\begin{aligned} p(n) &= c_{r'}n^{r'} + c_{r'-1}n^{r'-1} + \cdots + c_0 \\ &\leq c_{r'}n^{r'} + c_{r'-1}n^{r'} + \cdots + c_0n^{r'} \\ &= (c_{r'} + c_{r'-1} + \cdots + c_0)n^{r'}. \end{aligned}$$

Since $r > r'$, this contradicts the equality $r = \inf\{r \mid f_V(n) \leq cn^r\}$. Therefore

$$\inf\{r \mid f_V(n) \leq cn^r\} = \inf\{r \mid f(n) \leq p(n)\}.$$

□

Example 5.1.4. (1) Let $A = k[x_1, \ldots, x_d]$. If V is the space spanned by $\{1, x_1, \ldots, x_d\}$, then V^n is the space of polynomials of degree $\leq n$. The dimension of this space is $\binom{n+d}{d}$ since the number of the basis of V^n is the number of combination which we choose n elements from $\{1, x_1, \ldots, x_d\}$ allowing the overwrapping. Since $f(n) = \dim V^n = \binom{n+d}{d}$ is the polynomial of degree d and by Lemma 5.1.3, we have $\operatorname{GK}\dim(A) = d$.
(2) Let $A = k\langle x_1, \ldots, x_d\rangle$ be the noncommutative polynomial algebra. Let V be the space spanned by $\{1, x_1, \ldots, x_d\}$. The dimension of this space is $d^n + d^{n-1} + \cdots + d + 1$. Thus $\operatorname{GK}\dim(A) = \infty$.

5.2 Link invariants

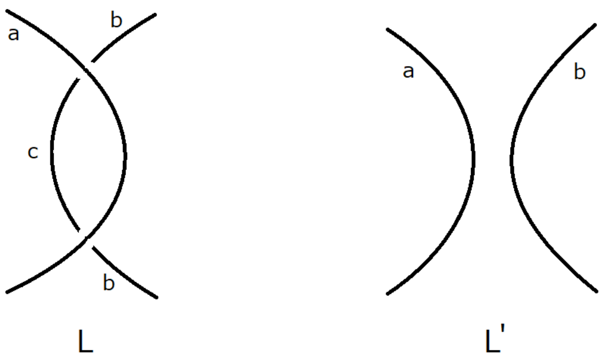
We can prove that the Gelfand-Kirillov dimension of semigroup algebra of knot semigrop is a link invariant. Let L be a link. Then we let $k(M(L))$ be a semigroup algebra of a knot semigroup of L.

Theorem 5.2.1. *Let L_1, L_2 be links. If L_1 and L_2 are equivalent then*

$$\operatorname{GK}\dim(k(M(L_1))) = \operatorname{GK}\dim(k(M(L_2))).$$

Proof. For a link L, by Theorem 3.1.3 it is enough to prove that $\operatorname{GK}\dim(k(M(L)))$ is invariant under Reidemeister move I, II, III.

$\operatorname{GK}\dim(k(M(L)))$ is invariant under Reidemeister move I since $M(L)$ is. Let L,L' be the same links, except in the neighborhood of a point where they are as shown in the following figure.



We can assume $\operatorname{GK}\dim(k(M(L))), \operatorname{GK}\dim(k(M(L'))) < \infty$. Then

$$k(M(L)) \simeq k\langle x_1, \ldots, x_m, a, b, c\rangle / \langle I_{a,b} \cup \begin{Bmatrix} ba - ac \\ ab - ca \end{Bmatrix} \rangle,$$
$$k(M(L')) \simeq k\langle x_1, \ldots, x_m, a, b\rangle / \langle I_{a,b}\rangle,$$

where $x_1, \ldots, x_m$ are labels of arcs except a, b, c, and $I_{a,b}$ is a set of relations except $ba - ac, ab - ca$. Let

$$V = \langle x_1, \ldots, x_m, a, b, c\rangle / \langle I_{a,b} \cup \begin{Bmatrix} ba - ac \\ ab - ca \end{Bmatrix} \rangle,$$
$$V' = \langle x_1, \ldots, x_m, a, b\rangle / \langle I_{a,b}\rangle.$$

We can prove that

$$(V/\langle c-1\rangle)^n \simeq (V'/\langle b-1\rangle)^n$$

for $n \geq 2$ as follows: we have

$$(V/\langle c-1\rangle)^n \simeq (\langle x_1, \ldots, x_m, a, b, c\rangle / \langle I_{a,b} \cup \begin{Bmatrix} ba - ac \\ ab - ca \end{Bmatrix} \cup \{c-1\}\rangle)^n.$$

In this space, $c = 1$ and then $ba = ac = a \cdot 1 = a$. By the cancellativity of knot semigroups, we have $b = 1$. Thus we have,

$$\begin{aligned}(V/\langle c-1\rangle)^n &\simeq (\langle x_1, \dots, x_m, a, b, c\rangle / \langle I_{a,b} \cup \begin{Bmatrix} ba - ac \\ ab - ca \end{Bmatrix} \cup \{c-1\}\rangle)^n \\ &\simeq \langle x_1, \dots, x_m, a\rangle / \langle I_{a,1}\rangle.\end{aligned}$$

On the other hand

$$\begin{aligned}(V'/\langle b-1\rangle)^n &\simeq \langle x_1, \dots, x_m, a, b\rangle / \langle I_{a,b} \cup \{b-1\}\rangle \\ &\simeq \langle x_1, \dots, x_m, a\rangle / \langle I_{a,1}\rangle.\end{aligned}$$

Therefore we have

$$(V/\langle c-1\rangle)^n \simeq (V'/\langle b-1\rangle)^n.$$

Then we have the following exact sequences for $n \geq 2$.

$$\begin{aligned}0 \to \langle c-1\rangle^n \to V^n \to (V^n/\langle c-1\rangle)^n \to 0,\\ 0 \to \langle b-1\rangle^n \to V^n \to (V^n/\langle b-1\rangle)^n \to 0.\end{aligned}$$

Since

$$\begin{aligned}f_V(n) &= \dim V^n = \dim(V/\langle c-1\rangle)^n + \dim\langle c-1\rangle^n,\\ f_{V'}(n) &= \dim V'^n = \dim(V'/\langle b-1\rangle)^n + \dim\langle b-1\rangle^n,\end{aligned}$$

and

$$(V/\langle c-1\rangle)^n \simeq (V'/\langle b-1\rangle)^n,$$

we have

$$f_V(n) = f_{V'}(n) \text{ for all } n \geq 2.$$

Let c be a real number such that

$$f_V(n), f_{V'}(n) \leq c.$$

Let

$$R = \inf\{r \mid f_V(n) \leq cn^r\}, R' = \inf\{r \mid f'_V(n) \leq cn^r\}.$$

Assume that $R < R'$. Since $\operatorname{GK}\dim(k(M(L))) = r$ is finite, there exists a real number c' such that

$$f_{V'}(n) = f_V(n) \leq c'n^r.$$

Thus we have

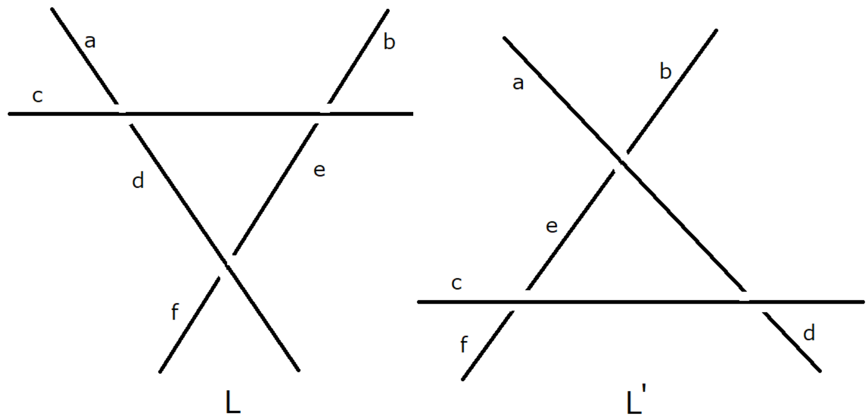
$$f_{V'}(1) \leq \max\{c, c'\}, f_{V'}(n) \leq \max\{c, c'\}n^r.$$

Since $R < R'$, this contradicts $R' = \inf\{r \mid f'_V(n) \leq cn^r\}$. Therefore,

$$\mathrm{GK}\dim(k(M(L))) = \mathrm{GK}\dim(k(M(L'))).$$

Thus $\mathrm{GK}\dim(k(M(L)))$ is invariant under the Reidemeister move II.

Next let L, L' be the same links, except in the neighborhood of a point where they are as shown in Figure.



We can assume $\mathrm{GK}\dim(k(M(L))), \mathrm{GK}\dim(k(M(L'))) < \infty$. Then

$$k(M(L)) \simeq k\langle x_1, \ldots, x_m, a, b, c, d, e, f\rangle / \langle I_{a,b,c,d,f} \cup J\rangle,$$
$$k(M(L')) \simeq k\langle x_1, \ldots, x_m, a, b\rangle / \langle I'_{a,b,c,d,f} \cup J'\rangle,$$

where

$$J = \begin{Bmatrix} ac - cd \\ ca - dc \end{Bmatrix} \cup \begin{Bmatrix} bc - ce \\ cb - ec \end{Bmatrix} \cup \begin{Bmatrix} fd - de \\ df - ed \end{Bmatrix},$$
$$J' = \begin{Bmatrix} ea - ab \\ ae - ba \end{Bmatrix} \cup \begin{Bmatrix} fc - ce \\ cf - ec \end{Bmatrix} \cup \begin{Bmatrix} dc - ca \\ cd - ac \end{Bmatrix},$$

and $x_1, \ldots, x_m$ are labels of arcs except a, b, c, d, e, f, and $I_{a,b,c,d,f}, I'_{a,b,c,d,f}$ is a set of relations except J, J'.

$$V = \langle x_1, \ldots, x_m, a, b, c, d, e, f\rangle / \langle I_{a,b,c,d,f} \cup J\rangle,$$
$$V' = \langle x_1, \ldots, x_m, a, b, c, d, e, f\rangle / \langle I'_{a,b,c,d,f} \cup J'\rangle.$$

We can prove

$$(V/\langle a-1\rangle)^n \simeq (V'/\langle d-1\rangle)^n$$

for $n \geq 2$ as follows: we have

$$(V/\langle a-1\rangle)^n \simeq (\langle x_1, \ldots, x_m, a, b, c, d, e, f\rangle / \langle I_{a,b,c,d,f} \cup J \cup \{a-1\}\rangle)^n.$$

In this space, $a = 1$ and then $cd = ac = 1 \cdot c = c$. By the cancellativity of knot semigroups, we have $d = 1$. Since $fc = ce$, we obtain $f = e$. Thus we have

$$\begin{aligned}(V/\langle a-1\rangle)^n &\simeq (\langle x_1, \ldots, x_m, a, b, c, d, e, f\rangle/\langle I_{a,b,c,d,f} \cup J \cup \{a-1\}\rangle)^n \\ &\simeq (\langle x_1, \ldots, x_m, b, c, f\rangle/\langle I_{1,b,c,1,f} \cup \begin{Bmatrix} bc-cf \\ cb-fc \end{Bmatrix}\rangle)^n.\end{aligned}$$

On the other hand,

$$(V'/\langle d-1\rangle)^n \simeq (\langle x_1, \ldots, x_m, a, b, c, d, e, f\rangle/\langle I'_{a,b,c,d,f} \cup J' \cup \{d-1\}\rangle)^n.$$

In this space, $d = 1$ and then $ca = dc = 1 \cdot c = c$. By the cancellativity of knot semigroups, we have a=1. Since $ea = ab$, we have $e = b$. Thus we have

$$\begin{aligned}(V'/\langle d-1\rangle)^n &\simeq (\langle x_1, \ldots, x_m, a, b, c, d, e, f\rangle/\langle I'_{a,b,c,d,f} \cup J' \cup \{d-1\}\rangle)^n \\ &\simeq (\langle x_1, \ldots, x_m, b, c, f\rangle/\langle I'_{1,b,c,1,f} \cup \begin{Bmatrix} fc-cb \\ cf-bc \end{Bmatrix}\rangle)^n.\end{aligned}$$

Therefore

$$(V/\langle a-1\rangle)^n \simeq (V'/\langle d-1\rangle)^n.$$

Then we have the following exact sequences for $n \geq 2$.

$$\begin{aligned}0 \to \langle a-1\rangle^n \to V^n \to (V^n/\langle a-1\rangle)^n \to 0, \\ 0 \to \langle e-1\rangle^n \to V^n \to (V^n/\langle d-1\rangle)^n \to 0.\end{aligned}$$

Since

$$\begin{aligned}f_V(n) &= \dim V^n = \dim(V/\langle a-1\rangle)^n + \dim\langle a-1\rangle^n, \\ f_{V'}(n) &= \dim V'^n = \dim(V'/\langle d-1\rangle)^n + \dim\langle d-1\rangle^n,\end{aligned}$$

and

$$(V/\langle a-1\rangle)^n \simeq (V'/\langle d-1\rangle)^n,$$

we have

$$f_V(n) = f_{V'}(n) \text{ for all } n \geq 2.$$

Then we can prove that $\operatorname{GK}\dim(k(M(L))) = \operatorname{GK}\dim(k(M(L')))$ by the similar way of the case of the Reidemeister move Ⅱ. Therefore $\operatorname{GK}\dim(k(M(L)))$ is invariant under the Reidemeister move Ⅲ. □

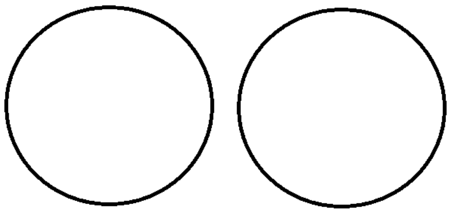
5.3 Examples

We calculate some examples of $\operatorname{GK}\dim(k(M(L)))$ for a link L.

Example 5.3.1. Let L_1 be a Hopf link (Figure 2). Since

$$\begin{aligned} k(M(L_1)) &\simeq k\langle a,b\rangle/\langle ab-ba\rangle \\ &\simeq k[a,b], \end{aligned}$$

$\operatorname{GK}\dim(k(M(L_1)))=2$. On the other hand let L_2 be the following link.



Since

$$k(M(L_2)) \simeq k\langle a,b\rangle, \tag{5.2}$$

$\operatorname{GK}\dim(k(M(L_2)))=\infty$. Therefore we can conclude that $L_1 \not\simeq L_2$.

Example 5.3.2. We shall consider torus knots $T(2,n)$ and double twist knots $\mathfrak{dtw}^l_m$. Let V be a generating subspace of $k(M(T(2,n)))$. Then by Theorem 3.2.1,

$$f_V(d)=\dim V^d = nd+1.$$

Since $f_V(d)$ is a polynomial of degree 1 and by Lemma 5.1.3, $\operatorname{GK}\dim(k(M(T(2,n))))=1$. Let V' be a generating subspace of $k(M(\mathfrak{dtw}^l_m))$. Then by Theorem 4.1.2,

$$f_{V'}(d)=\begin{cases} 1 & (d=0) \\ (lm+1)d+m+l-lm & (d>0) \end{cases}.$$

Since $f_{V'}(d)$ is a polynomial of degree 1 and by Lemma 5.1.3, $\operatorname{GK}\dim(k(M(\mathfrak{dtw}^l_m)))=1$.

Bibliography

[1] E. Artin, Theorie der Zpfe. Abh. Math. Sem. Hamburgischen Univ. 4 (1925), 4772.

[2] E. Artin, Theory of braids. Annals of Math. 48 (1947), 101 126.

[3] E. Brieskorn, K. Saito, Artin-Gruppen und Coxeter-Gruppen. Inventiones math. **17**, 245-271, (1972).

[4] D. Easdown, J. East, D. G. FitzGerald, Presentations of factorizable inverse monoids. Acta Universitatis Szeged., Acta Sci. Math. 71, 509520 (2005) .

[5] D. Easdown, T. G. Lavers, The inverse braid monoid. Advances in Mathematics 186 (2004) 438-455.

[6] E. Fedell, J. V. Buskirk, The braid groups of E^2 and S^2. Duke Math. J. 29 (1962) 243-257.

[7] B. Farb, D. Margalit, A primer on mapping class groups. Princeton university press (2012).

[8] J. Gonzalez-Meneses, Basic results on braid groups. Annales mathematiques BlaisePascal Volume 18, no1 (2011), 15-59.

[9] E. Godelle, A note on Renner monoids. hal-00373601.

[10] E. Godelle, The braid rook monoid. International Journal of Algebra and Computation Vol. 18, No.4 (2008) 779-802.

[11] A. Joyal, R. Street, Braded monoidal categories. Macquaria Math. Reports No. 8500067 (1985).

[12] A. Joyal, R. Street, Braded tensor categories. Adv. Math. 102 no. 1, 20-78 (1993).

[13] R. Karoui, V. V. Vershinin, On the inverse mapping class monoids. Journal of Knot Theory and Its Ramifications Vol. 19, No. 02, 163-179 (2010).

[14] C. Kassel, V. Turaev. Braid Groups. Graduate Texts in Mathematics 247 Springer (2000).

[15] A. Kawauchi, A survey of knot theory. Birkhauser (1996).

[16] G. R. Krause, T. H. Lenagan, Growth of Algebras and Gelfand-Kirillov Dimension. Graduate Studies in Mathematics. American Mathematical Society (1991).

[17] W. B. R. Lickorish, An Introduction to Knot Theory. Graduate Texts in Mathematics 175 (1997), Springer.

[18] D. K. Matsumoto, Y. Shibukawa, Quantum Yang-Baxter Equation, Braded Semigroups, and Dynamical Yang-Baxter Maps. Tokyo Journal of Mathematics, Vol. 38, No. 1, (2015) 227-237.

[19] T. Miyatani, On $\mathbb{P}M$-monoids and braid $\mathbb{P}M$-monoids. Topology and its applications, 274 (2020) 107118.

[20] T. Miyatani, On knot semigroups and Gelfand-Kirillov dimensions. arXiv:1907.00569 (2019).

[21] K. Murasugi, Knot theory and its applications. Springer Science & Business Media (2007).

[22] Y. A. Neretin, Geometry of $\mathrm{GL}_n(\mathbb{C})$ at infinity: Hinges, complete collineations, projective compactifications, and universal boundary, in The orbit method in geometry and physics (Marseille 2000), 297327, Progress in Mathematics, vol. 213, Birkhauser, Boston, 2003.

[23] L. Paris, Braid groups and Artin groups. Handbook on Teichmuller theory , Volume II, EMS Publishing House, 2008.

[24] M. S. Putcha, Linear algebraic Monoids. London Mathematical Society Lecture Note Series 133, 1988.

[25] M. Saito, Projective Linear Monoids and Hinges. arXiv:1711.01397, 2017.

[26] Y. Shibukawa, Dynamical braided monoids and dynamical Yang-Baxter maps. Quantum groups and quantum topology (edited by A. Masuoka), 80-89, RIMS Kokyuroku 1714, 2010.

[27] L. Solomon, Representation of the rook monoid. Journal of Algebra 256 (2002) 309-342.

[28] L. Solomon, An introduction to reductive monoids, Semigroups, formal languages and groups, J. Fountain, Ed., Kluwer Academic Publishers, 1995, 295-352.

[29] M. Takeuchi, Survey on matched pairs of groups-an elementary approach to the ESS-LYZ theory, Non commutative geometry and quantum groups (Warsaw, 2001), Banach Center Publ., **61**, Polish Acad. Sci., Warsaw (2003), 305-331 .

[30] M Takeuchi, Matched pairs of groups and bismash products of Hopf algebras. Comm. Algebra 9 (1981), no. 8, 841-882.

[31] A. Vernitski, Describing semigroups with defining relations of the form $xy = yz$ and $yx = zy$ and connection with knot theory. Semigroup Forum, Volume 95, 1-17 (2016).

[32] A. Vernitski, L. Tunsi, C. Ponchel, A. Lisitsa, Dihedral semigroups, their defining relations and an application to describing knot semigroups of rational links. Semigroup Forum, Volume 97, 75-86 (2018).

[33] V. V. Vershinin, On the inverse braid monoid. Topology and its Applications Volume 156, Issue 6, 1153-1166, 2009.

[34] S. K. Winker, Quandles, knot invariants, and the n-fold branched cover. PhD thesis, University of Illinois at Chicago (1984).

[35] O. Zariski, On the Poincare group of rational plane curves, Am. J. Math. (1936), 58, 607-619.