Title	Simple matrix representations of the orthogonal polynomials for a rational spectral density on the unit circle
Author(s)	Inoue, Akihiko; Kasahara, Yukio
Citation	Journal of mathematical analysis and applications, 464(2), 1366-1374 https://doi.org/10.1016/j.jmaa.2018.04.062
Issue Date	2018-08-15
Doc URL	http://hdl.handle.net/2115/79065
Rights	©2018. This manuscript version is made available under the CC-BY-NC-ND 4.0 license http://creativecommons.org/licenses/by-nc-nd/4.0/
Rights(URL)	http://creativecommons.org/licenses/by-nc-nd/4.0/
Type	article (author version)
File Information	IK-JMAA-rev.pdf

[Instructions for use](#)

Simple matrix representations of the orthogonal polynomials for a rational spectral density on the unit circle

Akihiko Inoue^a, Yukio Kasahara^{b,*}

^a*Department of Mathematics, Hiroshima University, Higashi-Hiroshima, 739-8526, Japan*

^b*Department of Mathematics, Hokkaido University, Sapporo, 060-0810, Japan*

Abstract

In this note, by using a discrete analog of a projection formula introduced by A. Seghier in 1978, we calculate the orthogonal polynomials on the unit circle for a rational spectral density having no zeros there, and derive simple matrix representations of themselves, their squared norms, and the Verblunsky coefficients.

Keywords: Orthogonal polynomials on the unit circle, Verblunsky coefficients, rational spectral density, Seghier's formula

MSC: 42C05, 60G10, 60G25.

1. Introduction

Let μ be a probability measure on the unit circle $\mathbb{T}$ that is supported on an infinite set. By applying the Gram–Schmidt method to the sequence of monomials $1, z, z^2, \dots$ in the Hilbert space $L^2(\mu)$ with

$$(\varphi, \psi)_\mu = \int_0^{2\pi} \varphi(e^{i\theta}) \overline{\psi(e^{i\theta})} d\mu(\theta), \quad \|\varphi\|_\mu = \left(\int_0^{2\pi} |\varphi(e^{i\theta})|^2 d\mu(\theta) \right)^{1/2},$$

one obtains a sequence of the *monic orthogonal polynomials* $\Phi_0, \Phi_1, \Phi_2, \dots$ with respect to μ , i.e.,

$$\Phi_n(z) = z^n + \text{lower order}, \quad (\Phi_n, \Phi_{n'})_\mu = 0 \quad (n \neq n').$$

Their constant terms $\alpha_n = \Phi_n(0)$ lie in the open unit disc $\mathbb{D}$ for all $n = 1, 2, \dots$. Therefore, μ yields a sequence $\{\alpha_1, \alpha_2, \dots\}$ in $\mathbb{D}$. Verblunsky's theorem states that every sequence in $\mathbb{D}$ arises in this way from a unique probability measure on $\mathbb{T}$ that is supported on an infinite set, whence α_n are called the *Verblunsky coefficients*. (There are several other names such as the Schur parameters.) As an application, the monic orthogonal polynomials are used for studying the prediction problem for a centered, complex-valued stationary time series $\{X_k\}$ with spectral measure μ , i.e.,

$$\text{Cov}(X_j, X_k) = \int_0^{2\pi} e^{i(j-k)\theta} d\mu(\theta) \quad (j, k = 0, \pm 1, \pm 2, \dots).$$

More precisely, Φ_n is a spectral counterpart of the one-step prediction error in the sense that

$$X_n - \hat{X}_n = X_n - \sum_{k=1}^n \phi_{n,k} X_{n-k} \quad \leftrightarrow \quad \Phi_n(z) = z^n - \sum_{k=1}^n \phi_{n,k} z^{n-k}, \quad \text{Var}(X_n - \hat{X}_n) = \|\Phi_n\|_\mu^2,$$

*Corresponding author

Email addresses: inoue100@hiroshima-u.ac.jp (Akihiko Inoue), y-kasa@math.sci.hokudai.ac.jp (Yukio Kasahara)

where $\hat{X}_n$ is the linear predictor of X_n based on $\{X_0, X_1, \dots, X_{n-1}\}$. Since $\phi_{n,n} = -\alpha_n$, the Verblunsky coefficients are essentially the same as the *partial autocorrelation function* $\{\phi_{n,n}\}$. See Simon [11] for the theory of orthogonal polynomials on the unit circle, and Brockwell–Davis [2] for the prediction problem of stationary time series.

Inoue [3, 4, 5], Inoue–Kasahara [6, 7] and Bingham *et al.* [1] have developed explicit representations of the predictor coefficients and prediction error variances, and studied their applications. Taking a different approach, this note presents simple matrix representations of the monic orthogonal polynomials and related quantities for a probability measure of the form $d\mu(\theta) = w(e^{i\theta}) \frac{d\theta}{2\pi}$, where w is a rational function of $e^{i\theta}$ satisfying $w(e^{i\theta}) \neq 0$ on $\mathbb{T}$. As is well known, the latter can be expressed as

$$w(e^{i\theta}) = \frac{|P(e^{i\theta})|^2}{|Q(e^{i\theta})|^2} \quad (1)$$

with some polynomials P and Q which have no common zeros and satisfy

$$P(0) > 0, \quad Q(0) > 0, \quad P(z) \neq 0, \quad Q(z) \neq 0 \quad (z \in \mathbb{D} \cup \mathbb{T}). \quad (2)$$

Such a rational function is important in time series analysis since it is the spectral density of a causal and invertible ARMA (autoregressive-moving average) process, characterized by the difference equation

$$Q(B)X_k = P(B)Z_k \quad (k = 0, \pm 1, \pm 2, \dots),$$

where B is the backward shift operator ($BX_k = X_{k-1}$) and $\{Z_k\}$ is a white noise ($\text{Cov}(Z_j, Z_k) = \delta_{jk}$). The matrix representations are derived by calculating Φ_n via an analog of a projection formula introduced by Seghier [10], which is explained in Section 2. The representations themselves, including those of the squared norms $\|\Phi_n\|_\mu^2$ and the Verblunsky coefficients α_n , are given in Section 3. Among other results, the representation of α_n is particularly simple.

We dedicate this note to Yusuke Nakamura (1992–2016) who discovered a similar matrix representation of the finite predictor for some continuous-time process, in joint work with the first author. It was this discovery that made the latter realize, after his death, the existence of explicit formulas in a simple case of this paper, and thus led us to the present work.

2. Seghier’s formula

In his work on the prediction problem for a continuous-time stationary process, Seghier [10] introduced a projection formula which is useful when the process has a rational spectral density. This section is devoted to a brief discussion on its discrete analog and related matters. For details, see Kasahara–Bingham [8].

Let $\mathbb{D}$, $\mathbb{T}$ and $\mathbb{E}$ denote the open unit disc, the unit circle, and the region outside the circle in the extended complex plane, respectively, so that $\mathbb{C} \cup \{\infty\} = \mathbb{D} \cup \mathbb{T} \cup \mathbb{E}$ (disjoint union). Recall the Hilbert space $L^2(\mu)$ from Section 1. Let $H^2(\mu)$ and $K^2(\mu)$ be the closed subspaces of $L^2(\mu)$ spanned by $\{1, e^{i\theta}, e^{2i\theta}, \dots\}$ and $\{e^{-i\theta}, e^{-2i\theta}, \dots\}$, respectively. Write L^2 , H^2 and K^2 for these spaces with the normalized Lebesgue measure $d\mu(\theta) = \frac{d\theta}{2\pi}$, which gives $L^2 = H^2 \oplus K^2$ (orthogonal decomposition). The functions in H^2 and K^2 have natural analytic extensions into $\mathbb{D}$ and $\mathbb{E}$, respectively. Denote by $\mathbb{C}[e^{i\theta}]_{<n}$ the space of polynomials of $e^{i\theta}$ with degree less than n . The monic orthogonal polynomial Φ_n is the projection of $e^{in\theta}$ onto the orthogonal complement of $\mathbb{C}[e^{i\theta}]_{<n}$ in $L^2(\mu)$.

Let $d\mu(\theta) = w(e^{i\theta}) \frac{d\theta}{2\pi} + d\mu_s(\theta)$, where μ_s is the singular part. If $\log w$ is integrable, there exists a unique outer function h , called the *Szegő function*, in H^2 such that $w = |h|^2$ and $h(0) > 0$. By [8, Theorem 3.1], the equation

$$H^2(\mu) \cap e^{in\theta} K^2(\mu) = \mathbb{C}[e^{i\theta}]_{<n} \quad (3)$$

holds for every $n = 0, 1, 2, \dots$ if and only if $\mu_s = 0$, $\log w$ is integrable, and h^2 is *rigid*, which means that, except for its positive constant multiples, no functions in the classical Hardy space H^1 share with it the same argument. Set

$$\phi^\dagger(z) = \overline{\phi(1/\bar{z})}.$$

For $n = 0, 1, 2, \dots$, let H_n be the Hankel operator with symbol $e^{in\theta}(h/h^\dagger)$ and H_n^* its adjoint operator, i.e.,

$$H_n(\varphi) = \mathcal{P}_{K^2}\{e^{in\theta}(h/h^\dagger)\varphi\} \quad (\varphi \in H^2), \quad H_n^*(\psi) = \mathcal{P}_{H^2}\{e^{-in\theta}(h^\dagger/h)\psi\} \quad (\psi \in K^2),$$

where $\mathcal{P}_S$ is the orthogonal projection operator of L^2 onto S . By [8, Remark 3.4 and Proposition 3.5], if (3) holds and also if $I - H_n^*H_n$ is invertible, the monic orthogonal polynomial Φ_n can be expressed as

$$\Phi_n = h(0)\{e^{in\theta}(h^\dagger)^{-1}(I - H_n^*H_n)^{-1}1 - h^{-1}H_n(I - H_n^*H_n)^{-1}1\}, \quad (4)$$

which is a discrete analog of Segher's projection formula in [10, Proposition 3].

3. Results

We calculate the monic orthogonal polynomials for a rational spectral density having no zeros on the unit circle, and derive matrix representations of themselves and the related quantities.

Throughout this section, assume that $\mu_s = 0$ and w is a rational function of $e^{i\theta}$ having no zeros on $\mathbb{T}$. In this case, h^2 is rigid since its reciprocal lies in H^1 ; see Yabuta [12]. Also, H_n is strictly contractive for all $n = 0, 1, 2, \dots$ since the Helson–Szego theorem implies $\|H_0\| < 1$, and also $\|H_n\|$ is non-increasing in n ; see Peller [9, Chapter 9]. (The value of $\|H_n\|$ is equal to the cosine of the angle between “past” $K^2(\nu)$ and “future” $e^{in\theta}H^2(\nu)$, where $d\nu = w^{-1}\frac{d\theta}{2\pi}$.) Consequently, (3) holds true, $I - H_n^*H_n$ is invertible, and thus, Segher's formula (4) is available for all $n = 0, 1, 2, \dots$.

Let P and Q be the polynomials satisfying (1) and (2). Without loss of generality, one may assume that $P(0) = 1$, and also that P is non-constant. (For, if $P(z) = 1$, then $\Phi_n(z) = z^n Q^\dagger(z)$ for $n \geq \deg Q$, and $\alpha_n = 0$ for $n > \deg Q$, hence there is nothing to do; see [2, Example 3.4.3] and [11, Corollary 1.7.6].) This assumption permits us to write

$$P(z) = \prod_{i=1}^d (1 - \bar{p}_i z)^{m_i},$$

where $p_1, p_2, \dots, p_d$ are distinct points in $\mathbb{D} \setminus \{0\}$ and $m_1, m_2, \dots, m_d$ are positive integers. Define

$$m = \deg P (= m_1 + m_2 + \dots + m_d), \quad n_0 = \max\{\deg Q - \deg P, 0\}.$$

By the conditions (1) and (2), the Szego function h is given by

$$h(z) = \frac{P(z)}{Q(z)}.$$

Since

$$\frac{1}{h^\dagger(z)} = \frac{z^m Q^\dagger(z)}{\prod_{i=1}^d (z - p_i)^{m_i}},$$

one may write

$$\frac{h(z)}{h^\dagger(z)} = \sum_{i=1}^d \sum_{j=1}^{m_i} \frac{a_{ij}}{(z - p_i)^j} + R(z), \quad (5)$$

where a_{ij} is the j -th coefficient in the principal part of the Laurent series expansion at a pole $z = p_i$, i.e.,

$$a_{ij} = \lim_{z \rightarrow p_i} \frac{1}{(m_i - j)!} \frac{d^{m_i-j}}{dz^{m_i-j}} \frac{(z - p_i)^{m_i} h(z)}{h^\dagger(z)},$$

and R is a rational function which is holomorphic in $\mathbb{D} \cup \mathbb{T}$ except for a pole at $z = 0$ of multiplicity n_0 in the case $n_0 > 0$. Set

$$\phi^\sharp(z) = (1/z)\overline{\phi(1/\bar{z})},$$

which provides us with

$$(\phi^\sharp)^\sharp = \phi, \quad H^2 \ni \varphi \mapsto \varphi^\sharp \in K^2, \quad K^2 \ni \psi \mapsto \psi^\sharp \in H^2, \quad H_n^* \psi = (H_n \psi^\sharp)^\sharp.$$

To check the last equation, use $(\mathcal{P}_{H^2} \phi)^\sharp = \mathcal{P}_{K^2}(\phi^\sharp)$ and $(\chi \psi)^\sharp = \chi^\dagger \psi^\sharp$ in the following way:

$$[\mathcal{P}_{H^2}\{e^{-in\theta}(h^\dagger/h)\psi\}]^\sharp = \mathcal{P}_{K^2}[\{e^{-in\theta}(h^\dagger/h)\psi\}^\sharp] = \mathcal{P}_{K^2}[\{e^{-in\theta}(h^\dagger/h)\}^\dagger \psi^\sharp] = \mathcal{P}_{K^2}\{e^{in\theta}(h/h^\dagger)\psi^\sharp\}.$$

As will be seen presently, the Hankel operators H_n are closely related with the rational functions

$$f_{ij}(z) = \frac{z^{j-1}}{(1 - \bar{p}_i z)^j}, \quad f_{ij}^\sharp(z) = \frac{1}{(z - p_i)^j}.$$

Define a vector-valued function $\mathbf{f}$ by the $(m_1, m_2, \dots, m_d)$ -block representation

$$\mathbf{f}(z) = {}^T(f_{11}(z), \dots, f_{1m_1}(z) \mid f_{21}(z), \dots, f_{2m_2}(z) \mid \dots \mid f_{d1}(z), \dots, f_{dm_d}(z)).$$

Also, for a holomorphic function φ in $\mathbb{D}$, define an m -dimensional column vector $\mathbf{v}[\varphi]$ by

$$\mathbf{v}[\varphi] = {}^T([\varphi]_{10}, \dots, [\varphi]_{1, m_1-1} \mid [\varphi]_{20}, \dots, [\varphi]_{2, m_2-1} \mid \dots \mid [\varphi]_{d0}, \dots, [\varphi]_{d, m_d-1}),$$

where $[\varphi]_{ij}$ is the j -th coefficient in the Taylor series expansion at $z = p_i$, i.e.,

$$[\varphi]_{ij} = \frac{1}{j!} \frac{d^j \varphi(z)}{dz^j} \Big|_{z=p_i}.$$

In addition, let $\mathbf{1}$ denote a constant vector $\mathbf{f}(0) = \mathbf{v}[1] = {}^T(1, 0, \dots, 0 \mid 1, 0, \dots, 0 \mid \dots \mid 1, 0, \dots, 0)$.

To obtain the main results of this note, let us prepare three lemmas.

Lemma 1. *Let $n \geq n_0$. If φ is a holomorphic function in $\mathbb{D} \cup \mathbb{T}$, then $(H_n \varphi)(z)$ is a linear combination of $f_{ij}^\sharp(z)$ expressed in the form*

$$(H_n \varphi)(z) = {}^T \mathbf{f}^\sharp(z) A B_n \mathbf{v}[\varphi],$$

where A and B_n respectively are $d \times d$ block diagonal matrices with $m_i \times m_i$ submatrices

$$A_i = \begin{pmatrix} a_{i,j+k-1} \end{pmatrix} \quad (\text{upper anti-triangular Hankel matrices})$$

and

$$B_{n,i} = \begin{pmatrix} \binom{n}{j-k} p_i^{n-(j-k)} \end{pmatrix} \quad (\text{lower triangular Toeplitz matrices}),$$

the displayed terms being the (j, k) -entries.

Proof. The value of the function $H_n \varphi$ can be evaluated by the formula

$$(H_n \varphi)(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\zeta^n h(\zeta) \varphi(\zeta)}{h^\dagger(\zeta)(z - \zeta)} d\zeta \quad (z \in \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{T})).$$

In view of the partial fraction expansion (5), it is seen that the integrand is holomorphic in $\mathbb{D} \cup \mathbb{T}$ except for poles at $z = p_1, p_2, \dots, p_d$ since the singularity at $z = 0$ is removable for $n \geq n_0$. By Leibniz's rule,

$$\begin{aligned} \frac{1}{(r-1)!} \frac{d^{r-1}}{d\zeta^{r-1}} \frac{\zeta^n \varphi(\zeta)}{z - \zeta} &= \sum_{j=1}^r \frac{1}{(z - \zeta)^j} \cdot \frac{1}{(r-j)!} \frac{d^{r-j}}{d\zeta^{r-j}} \{\zeta^n \varphi(\zeta)\} \\ &= \sum_{j=1}^r \frac{1}{(z - \zeta)^j} \sum_{k=1}^{r-j+1} \binom{n}{r-j+1-k} \zeta^{n-(r-j+1-k)} \frac{1}{(k-1)!} \frac{d^{k-1}}{d\zeta^{k-1}} \varphi(\zeta). \end{aligned}$$

Thus, by the formula (5) and the residue theorem,

$$\begin{aligned}
(H_n \varphi)(z) &= \sum_{i=1}^d \sum_{r=1}^{m_i} a_{ir} \cdot \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\zeta^n \varphi(\zeta)}{(\zeta - p_i)^r (z - \zeta)} d\zeta \\
&= \sum_{i=1}^d \sum_{r=1}^{m_i} a_{ir} \sum_{j=1}^r \frac{1}{(z - p_i)^j} \sum_{k=1}^{r-j+1} \binom{n}{r-j+1-k} p_i^{n-(r-j+1-k)} [\varphi]_{i,k-1} \\
&= \sum_{i=1}^d \sum_{j=1}^{m_i} \frac{1}{(z - p_i)^j} \sum_{k=1}^{m_i-j+1} \left\{ \sum_{l=k}^{m_i-j+1} a_{i,j+l-1} \binom{n}{l-k} p_i^{n-(l-k)} \right\} [\varphi]_{i,k-1},
\end{aligned}$$

in which $l = r - j + 1$, and this may be rewritten into the matrix form of the assertion. $\square$

Lemma 2. Let $\mathbf{u}$ be an m -dimensional column vector. If $\varphi(z) = {}^T \mathbf{f}(z) \mathbf{u}$, then

$$\mathbf{v}[\varphi] = C \mathbf{u},$$

where C is a $d \times d$ block matrix with $m_i \times m_j$ submatrices

$$C_{ij} = \begin{pmatrix} [f_{jl}]_{i,k-1} \end{pmatrix},$$

the displayed term being the (k,l) -entry. The value of $[f_{jl}]_{ik}$ can be evaluated as

$$[f_{jl}]_{ik} = \sum_{r=0}^k \binom{k}{r} \binom{l+k-r-1}{k} \frac{p_i^{l-1-r} \bar{p}_j^{k-r}}{(1 - p_i \bar{p}_j)^{l+k-r}}.$$

Proof. This assertion is merely a matrix representation of the linear equations

$$[\varphi]_{ik} = \sum_{j=1}^d \sum_{l=1}^{m_j} [f_{jl}]_{ik} u_{jl},$$

subject to $\mathbf{u} = {}^T(u_{11}, \dots, u_{1m_1} \mid u_{21}, \dots, u_{2m_2} \mid \dots \mid u_{d1}, \dots, u_{dm_d})$. $\square$

Lemma 3. If $n \geq n_0$, then

$$\{(I - H_n^* H_n)^{-1} \mathbf{1}\}(z) = 1 + {}^T \mathbf{f}(z) \overline{AB}_n G_n (I - \overline{G}_n G_n)^{-1} \mathbf{1},$$

where

$$G_n = \overline{C} A B_n.$$

In particular, this gives

$$\mathbf{v}[(I - H_n^* H_n)^{-1} \mathbf{1}] = (I - \overline{G}_n G_n)^{-1} \mathbf{1}.$$

Proof. Assume $n \geq n_0$. Let $\mathbf{u}$ be an m -dimensional column vector. It follows from Lemmas 1 and 2 that, for $\varphi(z) = {}^T \mathbf{f}(z) \mathbf{u}$,

$$(H_n \varphi)(z) = {}^T \mathbf{f}^\sharp(z) A B_n C \mathbf{u}.$$

Since $(H_n^* \psi)(z) = (H_n \psi^\sharp)^\sharp(z)$, this implies that, for $\psi(z) = {}^T \mathbf{f}^\sharp(z) \mathbf{u}$,

$$(H_n^* \psi)(z) = {}^T \mathbf{f}(z) \overline{AB}_n \overline{C} \mathbf{u}.$$

Clearly, $I - \overline{G}_n G_n$ is invertible if and only if $I - G_n \overline{G}_n$ is invertible. To check the latter, let $(I - G_n \overline{G}_n) \mathbf{u} = \mathbf{0}$. By applying the above results to $\chi(z) = {}^T \mathbf{f}(z) \overline{AB}_n \mathbf{u}$,

$$\{(I - H_n^* H_n) \chi\}(z) = {}^T \mathbf{f}(z) \overline{AB}_n (I - G_n \overline{G}_n) \mathbf{u} = 0.$$

Since $I - H_n^* H_n$ is invertible, one has $\chi(z) = 0$, so that $\overline{AB}_n \mathbf{u} = \mathbf{0}$. Also, $\overline{AB}_n$ is invertible, as is seen from

$$\det(A_i) = \operatorname{sgn}(\sigma)(a_{i,m_i})^{m_i} \neq 0, \quad \det(B_{i,n}) = (p_i^n)^{m_i} \neq 0,$$

where σ is a permutation defined by $\sigma(k) = m_i - k + 1$ ($k = 1, 2, \dots, m_i$). Hence, $\mathbf{u} = \mathbf{0}$, and $I - G_n \overline{G}_n$ is invertible. Now, by using Lemma 1 and the above results,

$$\begin{aligned} \{H_n^* H_n (1 + {}^T \mathbf{f} \overline{AB}_n G_n (I - \overline{G}_n G_n)^{-1} \mathbf{1})\}(z) &= \{{}^T \mathbf{f} \overline{AB}_n G_n (I + \overline{G}_n G_n (I - \overline{G}_n G_n)^{-1}) \mathbf{1}\}(z) \\ &= \{{}^T \mathbf{f} \overline{AB}_n G_n (I - \overline{G}_n G_n)^{-1} \mathbf{1}\}(z), \end{aligned}$$

which gives

$$\{(I - H_n^* H_n)(1 + {}^T \mathbf{f} \overline{AB}_n G_n (I - \overline{G}_n G_n)^{-1} \mathbf{1})\}(z) = 1.$$

Again, since $I - H_n^* H_n$ is invertible, the first half of the assertion holds. The other half can be shown by applying Lemma 2 to $\{(I - H_n^* H_n)^{-1} \mathbf{1}\}(z) = 1 + {}^T \mathbf{f}(z) \mathbf{u}$ with $\mathbf{u} = \overline{AB}_n G_n (I - \overline{G}_n G_n)^{-1} \mathbf{1}$, and then doing a calculation similar to the above one. $\square$

The following result presents an explicit matrix representation of a discrete analog of Seghier's prediction formula in [10, Corollary 2]. (The original result implicitly gives the "form" of the formula under some general assumption.)

Theorem 4. *For $n \geq n_0$, the monic orthogonal polynomials are expressed as*

$$\Phi_n(z) = z^n \frac{h(0)}{h^\dagger(z)} (1 + {}^T \mathbf{f}(z) \overline{AB}_n G_n (I - \overline{G}_n G_n)^{-1} \mathbf{1}) - \frac{h(0)}{h(z)} {}^T \mathbf{f}^\sharp(z) AB_n (I - \overline{G}_n G_n)^{-1} \mathbf{1},$$

and their squared norms admit the representation

$$\|\Phi_n\|_\mu^2 = h(0)^2 (1 + {}^T \mathbf{g}_n (I - \overline{G}_n G_n)^{-1} \overline{G}_n \mathbf{1}),$$

where

$$\mathbf{g}_n = AB_n \mathbf{1}.$$

Proof. The first formula is readily obtained by applying Lemmas 1 and 3 to Seghier's formula (4). To prove the other one, note that $\|\Phi_n\|_\mu^2 = (\Phi_n, z^n)_\mu$, for which Seghier's formula (4) gives

$$\frac{\Phi_n(z)w(z)}{z^n} = h(0)h(z)\{(I - H_n^* H_n)^{-1} \mathbf{1}\}(z) - \frac{h(0)h^\dagger(z)}{z^n} \{H_n (I - H_n^* H_n)^{-1} \mathbf{1}\}(z).$$

Since the second term of the RHS is holomorphic in $\mathbb{T} \cup \mathbb{E}$ and vanishes at $z = \infty$,

$$\begin{aligned} \|\Phi_n\|_\mu^2 &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\Phi_n(\zeta)w(\zeta)}{\zeta^n} \frac{d\zeta}{\zeta} = h(0) \cdot \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{h(\zeta)\{(I - H_n^* H_n)^{-1} \mathbf{1}\}(\zeta)}{\zeta} d\zeta \\ &= h(0)^2 \{(I - H_n^* H_n)^{-1} \mathbf{1}\}(0). \end{aligned}$$

Observed that ${}^T(AB_n) = AB_n$. By Lemma 3 and $\overline{G}_n (I - G_n \overline{G}_n)^{-1} = (I - \overline{G}_n G_n)^{-1} \overline{G}_n$,

$$\begin{aligned} \{(I - H_n^* H_n)^{-1} \mathbf{1}\}(0) &= \overline{\{(I - H_n^* H_n)^{-1} \mathbf{1}\}(0)} = 1 + {}^T \mathbf{1} AB_n \overline{G}_n (I - G_n \overline{G}_n)^{-1} \mathbf{1} \\ &= 1 + {}^T \mathbf{g}_n (I - \overline{G}_n G_n)^{-1} \overline{G}_n \mathbf{1}, \end{aligned}$$

which completes the proof. $\square$

The following matrix representation is particularly simple.

Theorem 5. *For $n > n_0$, the Verblunsky coefficients admit the representation*

$$\alpha_n = {}^T \mathbf{g}_{n-1} (I - \overline{G}_n G_n)^{-1} \mathbf{1}.$$

Proof. Seghier's formula (4) shows that

$$\frac{h(z)\Phi_n(z)}{h(0)} = \frac{z^n h(z)}{h^\dagger(z)} \{(I - H_n^* H_n)^{-1} 1\}(z) - \{H_n(I - H_n^* H_n)^{-1} 1\}(z),$$

in which the last term, i.e., $\{H_n(I - H_n^* H_n)^{-1} 1\}(z)$, is holomorphic in $\mathbb{T} \cup \mathbb{E}$ and vanishes at $z = \infty$. Hence, by definition of the Verblunsky coefficients,

$$\alpha_n = \Phi_n(0) = \frac{h(0)\Phi_n(0)}{h(0)} = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{h(\zeta)\Phi_n(\zeta)}{h(0)} \frac{d\zeta}{\zeta} = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\zeta^{n-1} h(\zeta)}{h^\dagger(\zeta)} \{(I - H_n^* H_n)^{-1} 1\}(\zeta) d\zeta.$$

Now, let φ be a holomorphic function in $\mathbb{D} \cup \mathbb{T}$. By Leibniz's rule,

$$\frac{1}{(r-1)!} \frac{d^{r-1}}{d\zeta^{r-1}} \{\zeta^{n-1} \varphi(\zeta)\} = \sum_{j=1}^r \binom{n-1}{r-j} \zeta^{(n-1)-(r-j)} \cdot \frac{1}{(j-1)!} \frac{d^{j-1}}{d\zeta^{j-1}} \varphi(\zeta).$$

So, as in the proof of Lemma 1, the formula (5) and the residue theorem imply that, for $n > n_0$,

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\zeta^{n-1} h(\zeta) \varphi(\zeta)}{h^\dagger(\zeta)} d\zeta &= \sum_{i=1}^d \sum_{r=1}^{m_i} a_{ir} \cdot \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\zeta^{n-1} \varphi(\zeta)}{(\zeta - p_i)^r} d\zeta \\ &= \sum_{i=1}^d \sum_{r=1}^{m_i} a_{ir} \sum_{j=1}^r \binom{n-1}{r-j} p_i^{(n-1)-(r-j)} [\varphi]_{i,j-1} \\ &= \sum_{i=1}^d \sum_{j=1}^{m_i} \left\{ \sum_{k=1}^{m_i-j+1} a_{i,j+k-1} \binom{n-1}{k-1} p_i^{(n-1)-(k-1)} \right\} [\varphi]_{i,j-1}, \end{aligned}$$

in which $k = r - j + 1$, and this may be expressed as ${}^T \mathbf{g}_{n-1} \mathbf{v}[\varphi]$. So, Lemma 3 gives the representation. $\square$

Remark 6. If $m = d$, i.e., if P has only simple zeros, the above results take simple forms without block structure. Indeed, the basic three matrices are written as

$$A = \text{diag}(a_1, \dots, a_d), \quad B_n = \text{diag}(p_1^n, \dots, p_d^n), \quad C = \begin{pmatrix} \frac{1}{1-p_1 \bar{p}_1} & \cdots & \frac{1}{1-p_1 \bar{p}_d} \\ \vdots & & \vdots \\ \frac{1}{1-p_d \bar{p}_1} & \cdots & \frac{1}{1-p_d \bar{p}_d} \end{pmatrix},$$

where $a_i = a_{i1}$, the residue of $h/h^\dagger$ at its pole $z = p_i$.

Acknowledgement

The authors would like to thank the referee for helpful comments on the presentation of this note.

References

- [1] N. H. Bingham, A. Inoue and Y. Kasahara, An explicit representation of Verblunsky coefficients, *Statist. Probab. Lett.* 82 (2012) 403–410.
- [2] P. J. Brockwell and R. A. Davis, *Time series: theory and methods*, Second edition, Springer-Verlag, New York, 1991.
- [3] A. Inoue, Asymptotics for the partial autocorrelation function of a stationary process, *J. Anal. Math.* 81 (2000) 65–109.
- [4] A. Inoue, Asymptotic behavior for partial autocorrelation functions of fractional ARIMA processes, *Ann. Appl. Probab.* 12 (2002) 1471–1491.
- [5] A. Inoue, AR and MA representation of partial autocorrelation functions, with applications, *Probab. Theory Related Fields* 140 (2008) 523–551.
- [6] A. Inoue and Y. Kasahara, Partial autocorrelation functions of the fractional ARIMA processes with negative degree of differencing, *J. Multivariate Anal.* 89 (2004) 135–147.

- [7] A. Inoue and Y. Kasahara, Explicit representation of finite predictor coefficients and its applications, *Ann. Statist.* 34 (2006) 973–993.
- [8] Y. Kasahara and N.H. Bingham, Verblunsky coefficients and Nehari sequences, *Trans. Amer. Math. Soc.* 366 (2014) 1363–1378.
- [9] V. V. Peller, *Hankel operators and their applications*, Springer-Verlag, New York, 2003.
- [10] A. Seghier, Prédiction d'un processus stationnaire du second ordre de covariance connue sur un intervalle fini, *Illinois J. Math.* 22 (1978) 389–401.
- [11] B. Simon, *Orthogonal polynomials on the unit circle, Part 1. Classical theory*, American Mathematical Society, Providence, RI, 2005.
- [12] K. Yabuta, Some uniqueness theorems for $H^p(U^n)$ functions, *Tôhoku Math. J.* 24 (1972), 353–357.