Normal trace for vector fields of bounded mean oscillation

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Abstract

We introduce various spaces of vector fields of bounded mean oscillation (BMO) defined in a domain so that normal trace on the boundary is bounded when its divergence is well controlled. The behavior of "normal" component and "tangential" component may be different for our BMO vector fields. As a result zero extension of the normal component stays in BMO although such property may not hold for tangential components.

Keywords: BMO, normal trace, duality, Jones' extension, Triebel-Lizorkin space.

1 Introduction

One of basic questions on vector fields defined on a domain Ω in $\mathbf{R}^n$ is whether the normal trace is well controlled without estimating all partial derivatives when the divergence is well controlled. Such a type of estimates is well known when a vector field is L^p $(1 < p < \infty)$ or L^∞. Here are examples. Let Ω be a bounded domain with smooth boundary Γ. Let $\mathbf{n}$ denotes its exterior unit normal vector field on Γ. For simplicity we assume that a vector field v satisfies $\operatorname{div} v = 0$. Then there is a constant C independent of v such that

$$\|v \cdot \mathbf{n}\|_{W^{-1/p,p}(\Gamma)} \leq C\|v\|_{L^p(\Omega)} \tag{1}$$ {N1}

$$\|v \cdot \mathbf{n}\|_{L^\infty(\Gamma)} \leq C\|v\|_{L^\infty(\Omega)}. \tag{2}$$ {N2}

Here $W^{s,p}$ denotes the Sobolev space which is actually a Besov space $B^s_{p,p}$ for non-integer s. The first estimate is a key to establish the Helmholtz decomposition of an L^p vector field; see e.g. [FM]. The second estimate is important to study for example total variation flow; see e.g. [ACM, Appendix C1]. These estimates (1), (2) hold for various domains including the case that Ω is a half space $\mathbf{R}^n_+$ i.e.,

$$\mathbf{R}^n_+ = \big\{(x_1, \ldots, x_n) \bigm| x_n > 0\big\}.$$

Our goal in this paper is to extend (2) by replacing $\|v\|_{L^\infty(\Omega)}$ by some BMO type norm.

However, it turns out that the normal trace of divergence free BMO vector fields may not be bounded. Indeed, consider

$$v = (v^1, v^2), \quad v^1(x) = v^2(x) = \log|x_1 - x_2|. \quad x = (x_1, x_2) \in \mathbf{R}^2.$$

This vector field is in $BMO(\mathbf{R}^2)$ and it is divergence free in distribution sence. Indeed,

$$\int_{\mathbf{R}^2} v \cdot \nabla\varphi \, dx = \frac{1}{2} \int_{\mathbf{R}^2} \log|\zeta| \big((\partial_\zeta - \partial_\zeta)\tilde{\varphi} + (\partial_\eta + \partial_\eta)\tilde{\varphi} \big) \, d\zeta d\eta = 0,$$

$$\zeta = x_1 - x_2, \quad \eta = x_1 + x_2$$

for all compactly supported smooth function φ, i.e., $\varphi \in C_c^\infty(\mathbf{R}^2)$. Here, $\tilde{\varphi}(\zeta, \eta) = \varphi\big((\zeta + \eta)/2, (\eta - \zeta)/2\big)$. However, if we consider $\Omega = \mathbf{R}^2_+$ and $\Gamma = \{x_2 = 0\}$, then $v \cdot \mathbf{n} = -v_2$ on Γ is clearly unbounded. This example indicates that we need some control near the boundary. Such a control is introduced in [BG], [BGS], [BGMST], [BGST]. More precisely, for $f \in L^1_{\mathrm{loc}}(\Omega)$, $\nu \in (0, \infty]$ they introduced a seminorm

$$[f]_{b^\nu} := \sup \left\{ r^{-n} \int_{\Omega \cap B_r(x)} |f(y)| \, dy \;\middle|\; x \in \Gamma, \ 0 < r < \nu \right\},$$

where $B_r(x)$ denotes the closed ball of radius r centered at x. For $\mu \in (0, \infty]$ they define

$$[f]_{BMO^\mu} := \sup \left\{ \frac{1}{|B_r(x)|} \int_{B_r(x)} \left| f - f_{B_r(x)} \right| \, dy \;\middle|\; B_r(x) \subset \Omega, \ r < \mu \right\},$$

where $f_B = \dfrac{1}{|B|} \displaystyle\int_B f(y) \, dy$, the average over B; here $|B|$ denotes the Lebesgue measure of B. The BMO type space $BMO_b^{\mu,\nu}$ introduced in these papers is the space of $f \in L^1_{\mathrm{loc}}(\Omega)$ having finite

$$\|f\|_{BMO_b^{\mu,\nu}} := [f]_{BMO^\mu} + [f]_{b^\nu}.$$

This space is very convenient to study the Stokes semigroup in [BG], [BGS], [BGMST], [BGST] as well as the heat semigroup [BGST]. One of our main results (Theorem 25) yields

$$\|v \cdot \mathbf{n}\|_{L^\infty(\Gamma)} \leq C \|v\|_{BMO_b^{\mu,\nu}} \tag{3}$$ {N3}

for any $\mu, \nu \in (0, \infty]$ for any uniformly $C^{1+\beta}$ domain.

However, for applications, especially to establish the Helmholtz decomposion $BMO_b^{\mu,\nu}$ norm for all components in too strong so we would like to estimate by a weaker norm. We only use b^ν seminorm for normal component of a vector field v. To decompose the vector field let $d_\Omega(x)$

be the distance of $x \in \Omega$ from the boundary Γ, i.e.,

$$d_\Omega(x) := \inf\bigl\{|x-y| \bigm| y \in \Gamma\bigr\}.$$

If Ω is uniformly C^2, then d_Ω is C^2 in a δ-tubular neighborhood Γ_δ of Γ for some $\delta < R_*$, where R_* is the reach of Γ [GT, Chapter 14, Appendix], [KP, §4.4]; here

$$\Gamma_\delta := \bigl\{x \in \Omega \bigm| d_\Omega(x) < \delta\bigr\}.$$

Instead of (3), our main results (Theorem 21, 22) together with Theorem 9 read:

$$\|v \cdot \mathbf{n}\|_{L^\infty(\Gamma)} \leq C\bigl([v]_{BMO^\mu} + [\nabla d_\Omega \cdot v]_{b^\nu}\bigr) \tag{4}$$ {MTB}

for $\nu \leq \delta$, $\mu \in (0,\infty]$ provided that Ω is a bounded $C^{2+\beta}$ domain with $\beta \in (0,1)$. The quantity $\nabla d_\Omega \cdot v$ is a kind of normal component.

Our main strategy is to use the formula

$$\int_\Gamma (v \cdot \mathbf{n})\psi \, d\mathcal{H}^{n-1} = \int_\Omega (\operatorname{div} v)\rho \, dx - \int_\Omega v \cdot \nabla\varphi \, dx$$

for any $\varphi \in C_c^\infty(\overline{\Omega})$ with $\varphi|_\Gamma = \psi$, where $d\mathcal{H}^{n-1}$ denotes the surface element. This formula is obtained by integration by parts. If $\operatorname{div} v = 0$, then it reads

$$\int_\Gamma (v \cdot \mathbf{n})\psi \, d\mathcal{H}^{n-1} = -\int_\Omega v \cdot \nabla\varphi \, dx. \tag{5}$$ {SINT}

Our estimate (4) follows from localization, flattening the boundary and duality argument. To get the flavor we explain the case when Ω is the half space $\mathbf{R}^n_+$. For $\psi \in L^1(\Gamma)$ it is known that there is $\varphi \in F^1_{1,2}(\mathbf{R}^n)$ such that its trace to the boundary equals to ψ; see e.g. [Tr92, Section 4.4.3]. Here $F^1_{1,2}$ denotes the Triebel-Lizorkin space which means that $\nabla\varphi \in h^1$, a localized Hardy space. We may assume that φ is even in x_n. We extend $v = (v', v_n)$ even in x_n for tangential part v' and odd in x_n for the normal part $v_n = \nabla d_\Omega \cdot v$. Although extended v' is still in $BMO^\infty(\mathbf{R}^n)$, the extended v_n may not be in $BMO^\infty(\mathbf{R}^n)$ unless we assume $[v_n]_{b^\nu} < \infty$. Here we invoke $[\nabla d_\Omega \cdot v]_{b^\nu} < \infty$. By these extensions our (5) yields

$$\int_\Gamma (v \cdot \mathbf{n})\psi \, d\mathcal{H}^{n-1} = -\frac{1}{2}\int_{\mathbf{R}^n} v \cdot \nabla\varphi \, dx, \tag{6}$$ {KID}

where v denotes the extended vector field. We apply h^1-*bmo* duality [Sa, Theorem 3.22] for (6) to get

$$\left|\int_\Gamma (v \cdot \mathbf{n})\psi \, d\mathcal{H}^{n-1}\right| \leq C\|v\|_{bmo}\|\varphi\|_{F^1_{1,2}},$$

where $bmo = BMO \cap L^1_{\mathrm{ul}}$ a localized BMO space. Here L^1_{ul} denotes a uniformly local L^1 space; see Section 2 for details. Since $\|\varphi\|_{F^1_{1,2}} \leq C\|\psi\|_{L^1}$, this implies

$$
\begin{aligned}
\|v \cdot \mathbf{n}\|_{L^\infty(\Gamma)} &\leq C\|v\|_{bmo(\mathbf{R}^n)} && (7) \quad \{\mathrm{MTH}\} \\
&\leq C\left([v]_{BMO^\infty(\Omega)} + [v]_{L^1_{\mathrm{ul}}(\Omega)} + [\nabla d_\Omega \cdot v]_{b^\infty}\right). && (8)
\end{aligned}
$$

Here and hereafter C denotes a constant depends only on space dimension and its numerical value may be different line by line.

In the case of a curved domain we need localization and flattening procedure by using normal (principal) coordinates. The localized space $bmo^\mu_\delta = BMO^\mu \cap L^1_{\mathrm{ul}}(\Gamma_\delta)$ is convenient for this purpose. Again we have to handle normal component $\nabla d_\Omega \cdot v$ separately. If the domain has a compact boundary, we are able to remove L^1_{ul} term in (7) and we deduce the estimate (4). Note that in this trace estimate only the behavior of v near Γ is important so one may use finite exponents in BMO^μ and b^ν.

As a byproduct we notice the extension problem of BMO functions. In general, zero extension of $v \in BMO^\mu(\Omega)$ may not belong to $BMO^\mu(\mathbf{R}^n)$ but if v is in $BMO^{\mu,\nu}_b$, as noticed in [BGST], its zero extension belongs to $BMO^\mu(\mathbf{R}^n)$ for $\nu \geq 2\mu$. We also note that it is possible to extend general $bmo^\mu_\delta(\Omega)$ to BMO^μ whose support is near $\overline{\Omega}$. We develop such a theory to explain the role of b^ν.

This paper is organized as follows. In Section 2 we introduce several localized BMO spaces and compared these spaces. Some of them are discussed in [BGST]. We introduce a new space $vbmo^{\mu,\nu}_\delta$ which requires that the b^ν seminorm of the normal component is bounded in $(bmo^\mu_\delta)^n$. A key observation is that if the boundary of the domain is compact, i.e., either a bounded or an exterior domain, the requirement in $L^1_{\mathrm{ul}}(\Gamma_\delta)$ is redudent in the definition of $vbmo^{\mu,\nu}_\delta$. In Section 3 we discuss extension problem as well as localization problem. In Section 4 we shall prove our main results. In Appendix we discuss coordinate change of vector fields by normal coordinates for the reader's convenience.

2 Spaces

{PS}

In this section we fix notation of important function spaces. Let $L^1_{\mathrm{ul}}(\mathbf{R}^n)$ be a uniformly L^1 space, i.e., for a fixed $r_0 > 0$

$$
L^1_{\mathrm{ul}}(\mathbf{R}^n) := \left\{ f \in L^1_{\mathrm{loc}}(\mathbf{R}^n) \;\middle|\; \|f\|_{L^1_{\mathrm{ul}}} := \sup_{x \in \mathbf{R}^n} \int_{B_{r_0}(x)} \left|f(y)\right| dy < \infty \right\}.
$$

The space is independent of the choice of r_0. For a domain Ω the space L^1_{ul} is the space of all L^1_{loc} functions f in Ω whose zero extension belongs to $L^1_{\mathrm{ul}}(\mathbf{R}^n)$. In other words

$$L^1_{\mathrm{ul}}(\Omega) := \left\{ f \in L^1_{\mathrm{loc}}(\Omega) \;\middle|\; \|f\|_{L^1_{\mathrm{ul}}(\Omega)} := \sup_{x\in\mathbf{R}^n} \int_{B_{r_0}(x)\cap\Omega} \bigl|f(y)\bigr|\, dy < \infty \right\}.$$

As in [BG] we set

$$BMO^\mu(\Omega) := \left\{ f \in L^1_{\mathrm{loc}}(\Omega) \;\middle|\; [f]_{BMO^\mu} < \infty \right\}.$$

For $\delta \in (0,\infty]$ we set

$$bmo^\mu_\delta(\Omega) := BMO^\mu(\Omega) \cap L^1_{\mathrm{ul}}(\Gamma_\delta) = \bigl\{ f \in BMO^\mu(\Omega) \bigm| \text{restriction of } f \text{ on } \Gamma_\delta \text{ is in } L^1_{\mathrm{ul}}(\Gamma_\delta) \bigr\}.$$

This is a Banach space equipped with the norm

$$\|f\|_{bmo^\mu_\delta} := [f]_{BMO^\mu(\Omega)} + [f]_{\Gamma_\delta}, \quad [f]_{\Gamma_\delta} := \|f\|_{L^1_{\mathrm{ul}}(\Gamma_\delta)},$$

where the restriction of f on Γ_δ is still denoted by f. If there is no boundary we set

$$bmo(\mathbf{R}^n) := BMO^\infty(\mathbf{R}^n) \cap L^1_{\mathrm{ul}}(\mathbf{R}^n)$$

which is a local BMO space and it agrees with the Triebel-Lizorkin space $F^0_{\infty,2}$; see e.g, [Tr92, Section 1.7.1], [Sa, Theorem 3.26].

For vector-valued function spaces we still write BMO^μ instead of $(BMO^\mu)^n$. For example for vector field v by $v \in bmo^\mu_\delta(\Omega)$ we mean that

$$v = (v_1, \dots, v_n), \quad v_i \in bmo^\mu_\delta(\Omega), \quad 1 \le i \le n.$$

We next introduce the space of vector fields whose normal component has finite b^ν of the form

$$vbmo^{\mu,\nu}_\delta(\Omega) := \bigl\{ v \in bmo^\mu_\delta(\Omega) \bigm| [\nabla d_\Omega \cdot v]_{b^\nu} < \infty \bigr\}$$

for $\nu \in (0,\infty]$. This space is a Banach space equipped with the norm

$$\|v\|_{vbmo^{\mu,\nu}_\delta} := \|v\|_{bmo^\mu_\delta} + [\nabla d_\Omega \cdot v]_{b^\nu}.$$

Similarly, we introduce another space

$$vBMO^{\mu,\nu}(\Omega) := \bigl\{ v \in BMO^\mu(\Omega) \bigm| [\nabla d_\Omega \cdot v]_{b^\nu} < \infty \bigr\}$$

equipped with a seminorm

$$[v]_{vBMO^{\mu,\nu}} := [v]_{BMO^\mu} + [\nabla d_\Omega \cdot v]_{b^\nu}.$$

Of course, this is strictly larger than the Banach space

$$BMO_b^{\mu,\nu}(\Omega) := \left\{ v \in BMO^\mu(\Omega) \;\middle|\; [v]_{b^\nu} < \infty \right\}$$

equipped with the norm

$$\|v\|_{BMO_b^{\mu,\nu}} := [v]_{BMO^\mu} + [v]_{b^\nu}$$

introduced essentially in [BG]. Indeed, in the case when Ω is the half space $\mathbf{R}^n_+$

$$vBMO^{\mu,\nu}(\mathbf{R}^n_+) = \left(BMO^\mu(\mathbf{R}^n_+) \right)^{n-1} \times BMO_b^{\mu,\nu}(\mathbf{R}^n_+), \tag{9}$$ {DC}

where in the right-hand side the each space denotes the space of scalar functions not of vector fields. This shows that $vBMO^{\mu,\nu}(\mathbf{R}^n_+)$ is strictly larger than $BMO_b^{\mu,\nu}(\mathbf{R}^n_+)$ for $n \geq 2$.

Although there are many exponents, the spaces may be the same for different exponents. By definition for $0 < \mu_1 \leq \mu_2 \leq \infty$, $0 < \nu_1 \leq \nu_2 \leq \infty$, $0 < \delta_1 \leq \delta_2 \leq \infty$,

$$[f]_{BMO^{\mu_1}} \leq [f]_{BMO^{\mu_2}}, \quad [f]_{b^{\nu_1}} \leq [f]_{b^{\nu_2}}, \quad [f]_{\Gamma_{\delta_1}} \leq [f]_{\Gamma_{\delta_2}}.$$

{E1}

Proposition 1. *Let Ω be an arbitrary domain in $\mathbf{R}^n$.*

(i) Let $0 < \mu_1 < \mu_2 < \infty$. Then seminorms $[\cdot]_{BMO^{\mu_1}}$ and $[\cdot]_{BMO^{\mu_2}}$ are equivalent. If Ω is bounded, one may take $\mu_2 = \infty$.

(ii) Let $0 < \delta_1 < \delta_2 < \infty$ and $\mu \in (0, \infty]$. Then there exists a constant $C > 0$ depending only on n, μ, δ_1, δ_2 and Ω such that

$$[f]_{\Gamma_{\delta_2}} \leq C \left([f]_{BMO^\mu} + [f]_{\Gamma_{\delta_1}} \right).$$

In particular, the norms $\|\cdot\|_{bmo^\mu_{\delta_1}}$ and $\|\cdot\|_{bmo^\mu_{\delta_2}}$ are equivalent. If Ω is bounded, one may take $\delta_2 = \infty$.

Proof. (i) This is [BGST, Theorem 4] which follows from [BGST, Theorem 3].

(ii) Since the space $L^1_{\mathrm{ul}}(\Gamma_\delta)$ is independent of the radius r_0 in its definition, without loss of generality, we may assume that $r_0 > \delta_1$. Let us firstly consider the case where the dimension

$n > 1$. Let k be the smallest integer such that $2^{-k} < \frac{\delta_1}{\sqrt{n}}$ and $x \in \mathbf{R}^n$. Notice that

$$\int_{B_{r_0}(x)\cap\Gamma_{\delta_2}} |f|\,dy = \int_{B_{r_0}(x)\cap\Gamma_{\delta_1}} |f|\,dy + \int_{B_{r_0}(x)\cap(\Gamma_{\delta_2}\setminus\Gamma_{\delta_1})} |f|\,dy,$$

we can estimate $\|f\|_{L^1(B_{r_0}(x)\cap\Gamma_{\delta_1})}$ directly by $[f]_{\Gamma_{\delta_1}}$. Assume that $\Gamma_{\delta_2}\setminus\Gamma_{\delta_1} \neq \emptyset$. Let $D_k(x)$ be the set of dyadic cubes of side length 2^{-k} that intersect with $B_{r_0}(x)\cap(\Gamma_{\delta_2}\setminus\Gamma_{\delta_1})$. For a dyadic cube $Q_j \in D_k(x)$, we define B_j to be the ball which has radius $\frac{\sqrt{n}}{2}\cdot 2^{-k}$ and shares the same center with Q_j. Let $C_k(x) := \{B_j \,|\, Q_j \in D_k(x)\}$ and $\Sigma := \{x \in \Omega \,|\, d_\Omega(x) = \delta_1\}$.

For $Q_j \in D_k(x)$ that intersects Σ, we seek to estimate $\|f\|_{L^1(B_j)}$. Let c_j be a point on $\Sigma\cap Q_j$, we have that $B_{\delta_1}(c_j) \subset \Omega$. Indeed as otherwise, there exists $z \in B_{\delta_1}(c_j)\cap\Omega^c$. Then the line segment joining c_j and z must intersect Γ at some point, say z^*. Then $|z^*-c_j| \le |z-c_j| < \delta_1$. This contradicts the fact that $d_\Omega(c_j) = \delta_1$. For $y \in B_j$, $|y-c_j| < \sqrt{n}\cdot\ell(Q_j) = \sqrt{n}\cdot 2^{-k} < \delta_1$. So $B_j \subset B_{\delta_1}(c_j)$. Let $d_j \in \Gamma$ be a point such that $|c_j - d_j| = \delta_1$, then on the line segment joining c_j and d_j, we can find a point o_j such that $|o_j - d_j| = \frac{\sqrt{n}}{2}\cdot 2^{-k}$. For $y \in B_{\frac{\sqrt{n}}{2}\cdot 2^{-k}}(o_j)$, we have that $|d_\Omega(y) - d_\Omega(o_j)| \le |y-o_j|$. Hence $d_\Omega(y) \le d_\Omega(o_j) + |y-o_j| < \sqrt{n}\cdot 2^{-k} < \delta_1$. This means that $B_{\frac{\sqrt{n}}{2}\cdot 2^{-k}}(o_j) \subset \Gamma_{\delta_1}$. Moreover,

$$|c_j - y| \le |c_j - o_j| + |o_j - y| \le \delta_1 - \frac{\sqrt{n}}{2}\cdot 2^{-k} + \frac{\sqrt{n}}{2}\cdot 2^{-k} = \delta_1.$$

Thus $B_{\frac{\sqrt{n}}{2}\cdot 2^{-k}}(o_j) \subset B_{\delta_1}(c_j)$. Denote $B_{\frac{\sqrt{n}}{2}\cdot 2^{-k}}(o_j)$ by B_j^*. We have that

$$\int_{B_j} |f|\,dy \le \int_{B_{\delta_1}(c_j)} |f - f_{B_{\delta_1}(c_j)}|\,dy + \int_{B_{\delta_1}(c_j)} |f_{B_{\delta_1}(c_j)} - f_{B_j^*}|\,dy + \int_{B_{\delta_1}(c_j)} |f_{B_j^*}|\,dy.$$

Notice that

$$\begin{aligned}
\int_{B_{\delta_1}(c_j)} |f - f_{B_{\delta_1}(c_j)}|\,dy &\le C_n\cdot\delta_1^n\cdot[f]_{BMO^\mu},\\
\int_{B_{\delta_1}(c_j)} |f_{B_{\delta_1}(c_j)} - f_{B_j^*}|\,dy &\le \frac{|B_{\delta_1}(c_j)|^2}{|B_j^*|}\cdot[f]_{BMO^\mu},\\
\int_{B_{\delta_1}(c_j)} |f_{B_j^*}|\,dy &\le \frac{|B_{\delta_1}(c_j)|}{|B_j^*|}\cdot[f]_{\delta_1}.
\end{aligned}$$

Since $|B_{\delta_1}(c_j)| = C_n\cdot\delta_1^n$ and $\frac{|B_{\delta_1}(c_j)|}{|B_j^*|} = \frac{C_n\cdot\delta_1^n}{(\frac{\sqrt{n}}{2}\cdot 2^{-k})^n} \le \frac{C_n\cdot\delta_1^n}{(\frac{\delta_1}{4})^n} = C_n$, $\|f\|_{L^1(B_j)}$ is therefore controlled by $C_{\delta_1,n}\cdot\Big([f]_{BMO^\mu} + [f]_{\Gamma_{\delta_1}}\Big)$.

Next we consider $Q_j' \in D_k(x)$ that does not intersect Σ. Suppose that $Q_j \in D_k(x)$ has a touching edge with Q_j'. There exists a ball B_i^j of radius $\frac{\sqrt{n}-1}{2}\cdot 2^{-k}$ which is contained in $B_j \cap B_j'$ where B_j, B_j' are the smallest balls that contain Q_j, Q_j' respectively. Similar to above, as $B_i^j \subset B_j$,

$$\begin{aligned}\int_{B_j'} |f|\,dy &\leq \int_{B_j'} |f - f_{B_j'}|\,dy + \int_{B_j'} |f_{B_j'} - f_{B_i^j}|\,dy + \int_{B_j'} |f_{B_i^j}|\,dy \\ &\leq |B_j'| \cdot [f]_{BMO^\mu} + \frac{|B_j'|^2}{|B_i^j|} \cdot [f]_{BMO^\mu} + \frac{|B_j'|}{|B_i^j|} \cdot \int_{B_j} |f|\,dy.\end{aligned}$$

Therefore if $\|f\|_{L^1(B_j)}$ is controlled by $C_{\delta_1,n} \cdot \Big([f]_{BMO^\mu} + [f]_{\Gamma_{\delta_1}}\Big)$, $\|f\|_{L^1(B_j')}$ is also controlled by $C_{\delta_1,n} \cdot \Big([f]_{BMO^\mu} + [f]_{\Gamma_{\delta_1}}\Big)$.

Since $B_{r_0}(x) \cap (\Gamma_{\delta_2} \backslash \Gamma_{\delta_1})$ is connected, we can estimate $\|f\|_{L^1(B_j)}$ for every $Q_j \in D_k(x)$ where B_j is the smallest ball that contains Q_j. For each $Q_j \in D_k(x)$, there exists $y \in Q_j \cap B_{r_0}(x)$, so for any $z \in B_j$, $|z-x| \leq |z-y| + |y-x| < \sqrt{n} \cdot 2^{-k} + r_0 < r_0 + \delta_1$. Thus $\bigcup_{Q_j \in D_k(x)} B_j \subset B_{r_0+\delta_1}(x)$. Let $N(D_k(x))$ be the number of cubes in $D_k(x)$, we have that

$$N(D_k(x)) \leq \frac{|B_{r_0+\delta_1}(x)|}{2^{-kn}} \leq C_n \cdot \left(\frac{r_0+\delta_1}{\delta_1}\right)^n.$$

Therefore,

$$\begin{aligned}\int_{B_{r_0}(x)\cap(\Gamma_{\delta_2}\backslash\Gamma_{\delta_1})} |f|\,dy &\leq \sum_{B_j \in C_{r_0}(x)} \int_{B_j} |f|\,dy \\ &\leq N(D_k(x)) \cdot C_{\delta_1,n} \cdot \Big([f]_{BMO^\mu} + [f]_{\Gamma_{\delta_1}}\Big) \\ &\leq C_{n,\delta_1,r_0} \cdot \Big([f]_{BMO^\mu} + [f]_{\Gamma_{\delta_1}}\Big).\end{aligned}$$

For the case where the dimension $n = 1$, we let k to be the smallest integer such that $2^{-k} < \frac{\delta_1}{2}$ and D_k to be the set of dyadic cubes of side length 2^{-k} that intersects $\Gamma_{\delta_2} \backslash \Gamma_{\delta_1}$. Notice that the region $\Gamma_{\delta_2} \backslash \Gamma_{\delta_1}$ is indeed a union of intervals. Without loss of generality, we can assume Ω to be $(0, \infty)$ and take $\mu = \infty$ by part (i) of this proposition. Thus in this case $\Gamma_{\delta_2} \backslash \Gamma_{\delta_1} = (\delta_1, \delta_2)$. For $Q_0 \in D_k$ such that $\delta_1 \in Q_0$,

$$\begin{aligned}\int_{Q_0} |f|\,dy \leq \int_{2Q_0} |f|\,dy &\leq \int_{2Q_0} |f - f_{2Q_0}|\,dy + \int_{2Q_0} |f_{2Q_0} - f_{Q_0^*}|\,dy + \int_{2Q_0} |f_{Q_0^*}|\,dy \\ &\leq C \cdot \Big([f]_{BMO^\infty} + [f]_{\Gamma_{\delta_1}}\Big),\end{aligned}$$

where $Q_0^* = 2Q_0 \backslash (Q_0 \cup [\delta_1, \infty))$ and $\ell(Q_0^*) = \frac{1}{2}\ell(Q_0) = 2^{-(k+1)}$.

We then put an ordering on the elements of D_k in the following way. For $j \in \mathbf{N}$, suppose that we have ordered intervals $Q_0, Q_1, ..., Q_{j-1}$, we pick $Q_j \in D_k \backslash \{Q_0, Q_1, ..., Q_{j-1}\}$ such that

Q_j has a touching edge with Q_{j-1}. For $Q_j \in D_k$, similarly we have that

$$\int_{Q_j} |f|\, dy \le \int_{2Q_j} |f|\, dy \le \int_{2Q_j} |f - f_{2Q_j}|\, dy + \int_{2Q_j} |f_{2Q_j} - f_{Q_j^*}|\, dy + \int_{2Q_j} |f_{Q_j^*}|\, dy$$
$$\le C \cdot \left([f]_{BMO^\infty} + [f]_{\Gamma_{\delta_1}} \right),$$

where $Q_j^* = 2Q_{j-1} \cap 2Q_j$ and $\ell(Q_j^*) = \ell(Q_j) = 2^{-k}$.

Let $N(D_k)$ be the number of elements of D_k, we have that

$$N(D_k) \le \frac{\delta_2 - \delta_1}{2^{-k}} + 2 \le \frac{4(\delta_2 - \delta_1)}{\delta_1} + 2$$

and therefore

$$\int_{\Gamma_{\delta_2} \setminus \Gamma_{\delta_1}} |f|\, dy \le C_{\delta_2, \delta_1} \cdot \left([f]_{BMO^\mu} + [f]_{\Gamma_{\delta_1}} \right).$$

The proof is now complete. □

By this observation when we discuss the space bmo_δ^μ there are only four types of spaces

$$bmo_\delta^\mu, \quad bmo_\delta^\infty, \quad bmo_\infty^\mu, \quad bmo_\infty^\infty$$

for finite $\mu, \delta > 0$. If Ω is bounded, it is clear that these four spaces agree with each other. However, if Ω is unbounded, there four spaces may be different because they requires different growth at infinity. Indeed, if $\Omega = (0, \infty)$

$$bmo_\infty^\infty \subsetneq bmo_\delta^\infty$$

since $\log(x+1) \in bmo_\delta^\infty$ while it does not belong to bmo_∞^∞. Moreover, since $x \in bmo_\delta^\mu$ but it does not belong to neither bmo_∞^μ nor bmo_δ^∞, we see that

$$bmo_\delta^\infty \subsetneq bmo_\delta^\mu, \quad bmo_\infty^\mu \subsetneq bmo_\delta^\mu.$$

It is possible to prove that $bmo_\infty^\infty = bmo_\infty^\mu$. Indeed as $bmo_\infty^\infty(\Omega) \subset bmo_\infty^\mu(\Omega)$ is simply by the definition of the BMO seminorm. It is sufficient to show the contrary, i.e. $[f]_{BMO^\infty} \le C \cdot ([f]_{BMO^\mu} + [f]_{\Gamma_\infty})$. Without loss of generality, in defining the seminorm $[\cdot]_{L^1_{\mathrm{ul}}(\Gamma_\infty)}$, we set the radius of the ball to be $\frac{\sqrt{n}}{2}$. For $B_r(x) \subset \Omega$ with $r < \mu$,

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |f - f_{B_r(x)}|\, dy \le [f]_{BMO^\mu}.$$

For $B_r(x) \subset \Omega$ with $r \geq \mu$, if $r \leq \frac{\sqrt{n}}{2}$, then

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} |f - f_{B_r(x)}|\,dy \leq \frac{2}{|B_r(x)|}\int_{B_{\frac{\sqrt{n}}{2}}(x)\cap\Omega} |f|\,dy \leq C_{\mu,n}\cdot[f]_{\Gamma_\infty}.$$

If $r > \frac{\sqrt{n}}{2}$, $B_r(x)$ is contained in the cube Q_r with center x and side length $2([r]+1)$, here $[r]$ is the largest integer less than or equal to r. By dividing each side length of Q_r equally into $2([r]+1)$ parts, we can divide the cube Q_r into $(2[r]+2)^n$ subcubes of side length 1. Let S_{Q_r} be the set of these $(2[r]+2)^n$ subcubes of Q_r. For $Q_r^i \in S_{Q_r}$, let B_r^i be the smallest ball that contains Q_r^i. Let $C_{Q_r} := \{B_r^i \mid Q_r^i \in S_{Q_r}\}$. We have that

$$\int_{B_r(x)} |f|\,dy \leq \sum_{i=1}^{(2[r]+2)^n}\int_{B_r^i\cap\Omega} |f|\,dy \leq (2[r]+2)^n\cdot[f]_{\Gamma_\infty}.$$

Since $r \geq \mu$,

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} |f - f_{B_r(x)}|\,dy \leq \frac{2}{|B_r(x)|}\int_{B_r(x)} |f|\,dy \leq C_{\mu,n}\cdot[f]_{\Gamma_\infty}.$$

Therefore $bmo_\infty^\infty = bmo_\infty^\mu$ and thus $bmo_\infty^\mu \subsetneq bmo_\mu^\infty$.

We summarize these equivalences. {B1}

Theorem 2. *Let Ω be an arbitrary domain in $\mathbf{R}^n$. Then*

$$bmo_\infty^\infty(\Omega) = bmo_\infty^\mu(\Omega) \subset bmo_\delta^\infty(\Omega) \subset bmo_\delta^\mu(\Omega)$$

for finite $\delta, \mu > 0$. The inclusions can be strict when Ω is unbounded. If Ω is bounded, all four spaces are the same.

As a simple application of Proposition 1, we conclude that the space $BMO_b^{\mu,\nu}$ is included in bmo_ν^μ since $[f]_\nu \leq c[f]_{b^\nu}$ $(\nu < \infty)$ with $c > 0$ depending on ν. {B2}

Theorem 3. *Let Ω be an arbitrary domain in $\mathbf{R}^n$. For $\mu \in (0,\infty]$ the inclusion*

$$BMO_b^{\mu,\nu}(\Omega) \subset bmo_\nu^\mu(\Omega)$$

holds for $\nu \in (0,\infty)$.

Since b^ν-seminorm controls boundary growth stronger than L^1 sense, this inclusion is in general strict even when Ω is bounded. Here is a simple example when $\Omega = (0,1)$. The b^ν-seminorm of $f(x) = \log x$ is infinite but $\|f\|_{L^1(\Omega)}$ is finite.

We next discuss the space $vbmo_\delta^{\mu,\nu}$.

{MY}

Remark 4. As proved in [BGST, Theorem 9], if Ω is a bounded Lipschitz domain, the space $BMO_b^{\mu,\nu}$ ($\mu,\nu \in (0,\infty]$) agrees with the Miyachi BMO space [Miy] defined by

$$\begin{aligned} BMO^M(\Omega) &= \left\{ f \in L^1_{\mathrm{loc}}(\Omega) \;\middle|\; \|f\|_{BMO^M} < \infty \right\}, \\ \|f\|_{BMO^M} &:= [f]_{BMO^M} + [f]_{b^M}, \\ [f]_{BMO^M} &:= \sup \left\{ \frac{1}{|B_r(x)|} \int_{B_r(x)} |f - f_{B_r(x)}| \, dy \;\middle|\; B_{2r}(x) \subset \Omega \right\}, \\ [f]_{b^M} &:= \sup \left\{ \frac{1}{|B_r(x)|} \int_{B_r(x)} |f| \, dy \;\middle|\; B_{2r}(x) \subset \Omega \text{ and } B_{5r}(x) \cap \Omega^c \neq \emptyset \right\}. \end{aligned}$$

{E2}

Proposition 5. *Let Ω be an arbitrary domain in $\mathbf{R}^n$. Let $0 < \nu_1 \leq \nu_2 \leq \delta \leq \infty$. Then there exists a constant $c > 0$ depending only on n, ν_1, ν_2, δ such that*

$$[\nabla d_\Omega \cdot v]_{b^{\nu_2}} \leq [\nabla d_\Omega \cdot v]_{b^{\nu_1}} + c[v]_{\Gamma_\delta}$$

for all $v \in L^1_{\mathrm{ul}}(\Gamma_\delta)$.

Proof. We may assume that $\nu_1 < \infty$. Let $Q_r(x)$ denote a cube centered at x with side length $2r$, Since $|\nabla d| = 1$ and $B_r(x) \subset Q_r(x)$, we see that

$$\begin{aligned} [\nabla d_\Omega \cdot v]_{b^{\nu_2}} - [\nabla d_\Omega \cdot v]_{b^{\nu_1}} &\leq \sup \left\{ \frac{1}{r^n} \int_{B_r(x)\cap\Omega} |\nabla d_\Omega \cdot v| \, dy \;\middle|\; x \in \partial\Omega,\ \nu_1 \leq r < \nu_2 \right\} \\ &\leq \sup \left\{ \frac{1}{r^n} \int_{Q_r(x)} |\tilde{v}| \, dy \;\middle|\; x \in \partial\Omega,\ \nu_1 \leq r \leq \nu_2 \right\} \end{aligned}$$

where $\tilde{v}$ denotes the zero extension of v to $\mathbf{R}^n$. Since $\nu_2 \leq \delta$ so that $Q_r(x) \cap \Omega \subset \Gamma_\delta$, we see that

$$\sup_{x\in\partial\Omega} \int_{Q_r(x)} |\tilde{v}| \, dy \leq \|v\|_{L^1_{\mathrm{ul}}(\Gamma_\delta)} \quad \text{for} \quad \nu_1 \leq r \leq \nu_2$$

provided that ν_2 is finite by taking an equivalent norm of L^1_{ul}; in fact we take $r_0 = \sqrt{n}\,\nu_2$ This implies that

$$[\nabla d_\Omega \cdot v]_{b^{\nu_2}} - [\nabla d_\Omega \cdot v]_{b^{\nu_1}} \leq \frac{1}{\nu_1^n} [v]_{\Gamma_\delta}.$$

If $\nu_2 = \delta = \infty$, we may assume $r = 2^\ell \nu_1$. We divide $Q_r(x)$ into subcube Q_j, $j = 1, \ldots, 2^{\ell n}$ of side length $2\nu_1$. Then

$$\frac{1}{|Q_r(x)|} \int_{Q_r(x)} |\tilde{v}| \, dy \leq \frac{1}{2^{\ell n}(2\nu_1)^n} \sum_{j=1}^{2^{\ell n}} \int_{Q_j} |\tilde{v}| \, dy \leq \frac{2^{\ell n}}{2^{\ell n}(2\nu_1)^n} \|\tilde{v}\|_{L^1_{\mathrm{ul}}} \leq \frac{1}{(2\nu_1)^n} \|\tilde{v}\|_{L^1_{\mathrm{ul}}}$$

where r_0 in L^1_{ul} norm is taken as $\sqrt{n}\,\nu_1$. We thus observe that

$$[\nabla d \cdot v]_{b^{\nu_2}} - [\nabla d \cdot v]_{b^{\nu_1}} \leq c[v]_{\Gamma_\delta}.$$

$\square$

By Propositions 1, 5 we do not need to care about ν. More precisely, {F3}

Theorem 6. *Let Ω be an arbitrary domain in $\mathbf{R}^n$. Assume that $\mu \in (0,\infty]$ and that $\delta \in (0,\infty]$. Then norms $\|\cdot\|_{vbmo^{\mu,\nu_1}_\delta}$ and $\|\cdot\|_{vbmo^{\mu,\nu_2}_\delta}$ are equivalent provided that $0 < \nu_1 < \nu_2 < \infty$. In the case $\delta = \infty$, we may take $\nu_2 = \infty$.*

In general, different from Theorem 3, the space $vBMO^{\mu,\nu}$ may not be included in bmo^μ_ν even for finite μ by the decomposition (9) and the fact that BMO^μ is not contained in $L^1_{\mathrm{ul}}(\Gamma_\delta)$ for any δ. However, if each connected component of the boundary Γ of Ω has a curved part, we are able to compare these spaces. {FC}

Definition 7. Let Ω be a uniformly C^1 domain in $\mathbf{R}^n$ and Γ^0 be a connected component of the boundary Γ of Ω. We say that Γ^0 has a *fully curved part* if the set of all normals of Γ^0 spans $\mathbf{R}^n$. In other words, the set $\left\{\mathbf{n}(x) \in \mathbf{R}^n \mid x \in \Gamma^0\right\}$ contains n linearly independent vectors, when $\mathbf{n}$ denotes the unit exterior normal of Γ^0.

We introduce $b^\nu(\Gamma^0)$-seminorm for convenience. Let us decompose Γ into its connected component Γ^j so that $\Gamma = \bigcup_{j=1}^m \Gamma^j$. We set

$$[f]_{b^\nu(\Gamma^j)} := \sup\left\{ r^{-n} \int_{\Omega\cap B_r(x)} \left|f(y)\right| dy \;\middle|\; x \in \Gamma^j,\ 0 < r < \nu \right\}.$$

Evidently, $[f]_{b^\nu} = \max_{1\leq j\leq m} [f]_{b^\nu(\Gamma^j)}$ at least for small $\nu > 0$.

The existence of a fully curved part implies "non-degeneracy" of the seminorm $[\nabla d \cdot f]_{b^\nu}$. {ND}

Lemma 8. *Let Ω be a uniformly C^2 domain in $\mathbf{R}^n$. Let Γ^j be a connected component of the boundary Γ of Ω. If $c \in \mathbf{R}^n$ satisfies*

$$[\nabla d_\Omega \cdot c]_{b^\nu(\Gamma^j)} = 0,$$

for some $\nu > 0$, then $c = 0$ provided that Γ^j has a fully curved part.

Proof. If Ω is uniformly C^2, then d_Ω is C^2 in $(\Gamma^j)_\delta$ for sufficiently small $\delta > 0$. Since $-\nabla d_\Omega(x)$ at $x \in \Gamma^j$ equals $\mathbf{n}(x)$, we see that

$$\frac{1}{r^n} \int_{B_r(x)\cap\Omega} \nabla d_\Omega(y)\,dy \to c_0 \mathbf{n}(x) \quad \text{as} \quad r \to 0$$

with scalar constant c_0. Our assumption implies that $c \cdot \mathbf{n}(x) = 0$ for $x \in \Gamma^j$. If Γ^j has a curved part, then by definition this implies that $c = 0$. $\square$

Here is a few comments on examples of such domains. All connected components of the boundary of a bounded domain, exterior domain has a fully curved part. A perturbed half space

$$\mathbf{R}^n_\psi = \left\{ (x', x_n) \in \mathbf{R}^n \;\middle|\; x_n > \psi(x'),\ x' = (x_1, \ldots, x_{n-1}) \in \mathbf{R}^{n-1} \right\}$$

with $\psi \in C^1_c(\mathbf{R}^{n-1})$, $\psi \not\equiv 0$ is another example. However, a half-space $\mathbf{R}^n_+$, cylindrical domain $G \times \mathbf{R}^{n-k}$ with $k \geq 1$, $G \subset \mathbf{R}^k$ does not have a boundary having a fully curved part. Our goal is to show that for a domain with boundary components having a fully curved part the space $vBMO^{\mu,\nu}$ is comparable with $vbmo^{\mu,\nu}_\delta$ space if the boundary is compact.

{V1}

Theorem 9. *Let Ω be a C^2 bounded or exterior domain in $\mathbf{R}^n$ so that each component of the boundary has a fully curved part. For $\mu \in (0, \infty]$ and $\nu \in (0, R_*)$ the identity holds:*

$$vBMO^{\mu,\nu}(\Omega) = vbmo^{\mu,\nu}_\nu.$$

Proof. Let Γ^j be a j-th connected component of the boundary $\Gamma = \partial\Omega$ such that $\Gamma = \bigcup_{j=1}^m \Gamma^j$. Since $(\Gamma^j)_\nu$ is C^2 and compact, there is a number $r_0 \in (0, \nu/2)$ such that

$$(\Gamma^j)_\nu = \bigcup_{x \in \Lambda} \operatorname{int} B_{r_0}(x), \quad \Lambda \subset (\Gamma^j)_\nu,$$

where Γ^j is a connected component of Γ and $(\Gamma^j)_\nu$ denotes its ν-neighborhood. The next lemma shows that

$$vBMO^{\nu,\nu}(\Omega) \subset L^1_{\mathrm{ul}}(\Gamma_\nu)$$

which yields the desired result. Note that we may assume $\nu \leq \mu$ by Proposition 1. $\square$

{V2}

Lemma 10. *Under the same assumption of Theorem 9 with $\mu \leq \nu$ assume that $r_0 < \nu/2 < R_*/2$ is taken so that*

$$(\Gamma^i)_\nu = \bigcup_{x \in \Lambda} \operatorname{int} B_{r_0}(x)$$

with some $\Lambda \subset (\Gamma^j)_\nu$. Then there exists $C > 0$ depending only on r_0, n, Γ^j, ν such that

$$\sup_{x \in \Lambda} \frac{1}{|B_{r_0}(x)|} \int_{B_{r_0}(x)} |f(y)|\, dy \leq C\Big([f]_{BMO^\mu((\Gamma^j)_\nu)} + [\nabla d_\Omega \cdot f]_{b^\nu(\Gamma^j)} \Big).$$

Proof. We shall suppress r_0 dependence since it is fixed. We shall prove the average $f_{B(x)} =$

$\dfrac{1}{|B(x)|}\displaystyle\int_{B(x)} f\,dy$ has an estimate

$$\sup_{x\in\Lambda}\left|f_{B(x)}\right| \leq C\Big([f]_{BMO^\mu((\Gamma^j)_\nu)} + [\nabla d\cdot f]_{b^\nu(\Gamma^j)}\Big). \tag{10}$$ {KI}

If this is proved, applying the triangle inequality

$$(|f|)_{B(x)} \leq \frac{1}{|B(x)|}\int_{B(x)}\left|f-f_{B(x)}\right|\,dy + \left|f_{B(x)}\right|$$

yields the desired result.

We shall prove the key inequality (10) by contradiction argument. Assume the inequality (10) were false. Then, there would exist a sequence $\{f^k\}_{k=1}^\infty$ such that

$$1=\sup_{x\in\Lambda}\left|f^k_{B(x)}\right| \geq k\left(\left[f^k\right]_{BMO^\mu} + \left[\nabla d_\Omega\cdot f^k\right]_{b^\nu}\right).$$

Here we suppress $(\Gamma^i)_\nu$ and Γ^j in the right-hand side. Since

$$\sup_{x\in\Lambda}\left|c^k(x)\right| = 1 \quad \text{with} \quad c^k(x)=f^k_{B(x)}\in\mathbf{R}^n,$$

there is a sequence $\{x_k\}_{k=1}^\infty$ in Λ with the property

$$1\geq\left|c^k(x_k)\right|\geq 1/2.$$

By taking a subsequence we may assume that x_k converges to some $\hat{x}\in(\Gamma^j)_\nu$ since Γ^j is compact and $d\left(x_k,\partial(\Gamma^j)_\nu\right)\geq r_0$, where $d(x_k,A)$ denotes the distance from a point x_k to a set A. Since Γ^j is connected, there is an increasing sequence $\{K_\ell\}_{\ell=1}^\infty$ of connected compact sets in $(\Gamma^j)_\nu$ such that $\operatorname{int} K_\ell \ni \hat{x}$ for $\ell\geq 1$ and $(\Gamma^j)_\nu=\bigcup_{\ell=1}^\infty K_\ell$. By compactness, there is a finite subset Λ_ℓ of Λ with the property that

$$K_\ell\subset\bigcup_{x\in\Lambda_\ell}\operatorname{int} B(x), \quad \Lambda_\ell\subset\Lambda_{\ell+1}$$

and the right-hand side is connected. By taking a further subsequence we may assume that $c^k(x)\to c(x)$ for $x\in\Lambda_\ell$. However, since $\left[f^k\right]_{BMO^\mu}\to 0$ so that

$$\int_{B(x)}\left|f^k-c^k\right|\,dx\to 0$$

as $k\to\infty$, we see that $c(x)=c(y)$ if $\operatorname{int} B(x)\cap\operatorname{int} B(y)\neq\emptyset$. Since

$$\bigcup_{x\in\Lambda_\ell}\operatorname{int} B(x)$$

is connected, $c(x)$ is independent of $x \in \Lambda_\ell$, say $c = c_\ell$. By taking a further subsequence of $\{f^k\}$ we may assume that $c^k(x) \to c_\ell$ in Λ_ℓ. By a diagonal argument there is a subsequence of $\{f^k\}$ such that

$$c^k(x) \to c \quad \text{for} \quad x \in \bigcup_{\ell=1}^{\infty} \Lambda_\ell =: \Lambda_\infty \subset \Lambda.$$

We thus observe that

$$\int_{B(x)} \left| f^k(y) - c \right| dy \to 0 \quad \text{for} \quad x \in \Lambda_\infty \quad \text{as} \quad k \to \infty.$$

If we take $B(x)$ such that $\hat{x} \in \operatorname{int} B(x)$, c should not be equal to zero since $\left| c^k(x_k) \right| \geq 1/2$ and $x_k \to \hat{x}$ as $k \to \infty$. We now invoke the property that

$$\left[\nabla d_\Omega \cdot f^k \right]_{b^\nu} \to 0.$$

Since

$$(\Gamma^j)_\nu = \bigcup_{x \in \Lambda_\infty} B(x),$$

we observe that $f^k \to c$ in $L^1_{\mathrm{loc}}\left((\Gamma^j)_\nu\right)$. By taking a subsequence we may assume that $f^k(x) \to c$ for a.e. $x \in (\Gamma^j)_\nu$ so that $\nabla d_\Omega \cdot f^k \to \nabla d_\Omega \cdot c$, a.e. By lower semicontinuity of integrals (Fatou's lemma) and supremum operation, the seminorm b^ν is lower semicontinuous under this convergence. We thus conclude that

$$[\nabla d_\Omega \cdot c]_{b^\nu} \leq \varliminf_{k \to \infty} \left[\nabla d_\Omega \cdot f^k \right]_{b^\nu} = 0.$$

By Lemma 8 this c must be zero which leads a contradiction. We thus proved the key estimate (10). This completes the proof of Lemma 10. □

3 A variant of Jones' extension theorem

{JE}

Different from L^∞ function it is in general impossible to extend BMO function by setting zero outside the domain. Indeed, the zero-extension of $\log \min(x, 1) \in bmo^\infty_\infty(\mathbf{R}^1_+)$ does not belong to $BMO^\infty(\mathbf{R})$. The goal in this section is to give a linear, extention operator of BMO type function so that the support of extended function is contained in an ε-neighborhood of the original domain, of a function.

For this purpose we recall an extension given by P. W. Jones [PJ]. Since we modify the way of construction, we will give a sketch of this construction. We first recall a dyadic Whitney decomposition of a set A in $\mathbf{R}^n$. Let $\mathcal{A} = \{Q_j\}_{j \in \mathbf{N}}$ be a set of dyadic closed cubes with side length $\ell(Q_j)$ contained in A satisfying following four conditions.

(i) $A = \cup_j Q_j$,

(ii) $\operatorname{int} Q_j \cap \operatorname{int} Q_k = \emptyset$ if $j = k$,

(iii) $\sqrt{n} \le d(Q_j, \mathbf{R}^n \setminus A)/\ell(Q_j) \le 4\sqrt{n}$ for all $j \in \mathbf{N}$,

(iv) $1/4 \le \ell(Q_k)/\ell(Q_j) \le 4$ if $Q_j \cap Q_k \neq \emptyset$.

We say that $\mathcal{A}$ is called a dyadic Whitney decomposition of A. Such a decomposition exists for any open sets; see [Ste, Chapter VI, Theorem 1]. Here $d(B,C)$ for sets B, C in $\mathbf{R}^n$ is defined as

$$d(B,C) = \inf \left\{ |x-y| \;\middle|\; x \in B,\ y \in C \right\}.$$

If B is a point x, we write $d(x,C)$ instead of $d(\{x\}, C)$.

There are at least two important distance functions on $\mathcal{A}$. For $Q_j, Q_k \in \mathcal{A}$, a family $\{Q(\ell)\}_{\ell=0}^m \subset \mathcal{A}$ is called a Whitney chain of length m if $Q(0) = Q_j$ and $Q(m) = Q_k$ such that $Q(\ell) \cap Q(\ell+1) \neq \emptyset$ for ℓ with $0 \le \ell \le m-1$. Then the length of the shortest Whitney chain connecting Q_j and Q_k gives a distance on $\mathcal{A}$, which is denoted by $d_1(Q_j, Q_k)$. The second distance for $Q_j, Q_k \in \mathcal{A}$ is defined as

$$d_2(Q_j, Q_k) := \log \left| \frac{\ell(Q_j)}{\ell(Q_k)} \right| + \log \left| \frac{\ell(Q_j, Q_k)}{\ell(Q_j) + \ell(Q_k)} + 1 \right|.$$

Note that d_1 and d_2 are invariant under dilation as well as translation and rotation. P. W. Jones [PJ] gives a necessary and sufficient conditon for a domain such that there exists a linear extension operator. A domain Ω is called a uniform domain if there exists constants $a, b > 0$ such that for all $x, y \in \Omega$ there exists a rectifiable curve $\gamma \subset \Omega$ of length $s(\gamma) \le a|x-y|$ with $\min \{ s(\gamma(x,z)),\ s(\gamma(y,z)) \} \le b d(z, \partial\Omega)$, where $\gamma(x,z)$ denotes the part of γ between x and z on the curve; see e.g, [GO]. It is equivalent to say that there is a constant $K > 0$ such that

$$d_1(Q_j, Q_k) \le K d_2(Q_j, Q_k) \tag{11}$$ {UD}

for all $Q_j, Q_k \in \mathcal{A}$ and some dyadic Whitney decomposition $\mathcal{A}$ of Ω.

{PJ}

Theorem 11. *Let $A \subset \mathbf{R}^n$ be a uniform domain. Then there is a constant $C(K)$ depending only on K in (11) such that for each $f \in BMO^\infty(A)$ there is an extension $\overline{f} \in BMO^\infty(\mathbf{R}^n)$ satysfying*

$$[\overline{f}]_{BMO^\infty(\mathbf{R}^n)} \le C(K)[f]_{BMO^\infty(A)}.$$

The operator $f \mapsto \overline{f}$ is a bounded linear operator. Conversely, if there exists such an extension, then A is a uniform domain.

A bounded Lipschitz domain is a typical example of a uniform domain. The constant K in (11) depends only on the Lipschitz regularity of the domain. A Lipshitz half space $\mathbf{R}^n_\psi$ is another example of a uniform domain; here ψ is a Lipshitz function on $\mathbf{R}^{n-1}$.

We next note that if we modify the construction by P. W. Jones, the support of the extension $\overline{f}$ is contained in an ε-neighborhood of $\overline{\Omega}$ if f is also in L^1_{ul} type space.

{PJM}

Theorem 12. *Let $\Omega \subset \mathbf{R}^n$ be a uniform domain. For each $\varepsilon > 0$ there is a constant $C = C(K, \varepsilon)$ with K in* (11) *such that for each $f \in bmo^\infty_\infty(\Omega)$ there is an extension $\overline{f} \in bmo^\infty_\infty(\Omega_{2\varepsilon})$ such that*

$$[\overline{f}]_{bmo^\infty_\infty(\Omega_{2\varepsilon})} \leq C[f]_{bmo^\infty_\infty(\Omega)}$$

and $\operatorname{supp} \overline{f} \subset \overline{\Omega_\varepsilon}$, *where*

$$\Omega_\varepsilon := \left\{ x \in \mathbf{R}^n \;\middle|\; d(x, \overline{\Omega}) < \varepsilon \right\}.$$

The operator $f \mapsto \overline{f}$ is a bounded linear operator.

This can be proved almost along the same way as in [PJ]. We shall give an explicit proof.

Proof. Let k_ε be the smallest integer such that $2^{-k_\varepsilon} < \frac{\varepsilon}{5\sqrt{n}}$. So $2^{-k_\varepsilon} \geq \frac{\varepsilon}{10\sqrt{n}}$. Let $E = \{Q_j\}$ be the Whitney decomposition of Ω and $E' = \{Q'_j\}$ be the Whitney decomposition of Ω^c. Let E_* be the set of Whitney cubes in E whose side length is strictly greater than 2^{-k_ε}. For each $Q_m \in E_*$, we define a function g_m on Ω by

$$g_m(x) := \begin{cases} f_{Q_m}, & \text{if } x \in Q_m \\ 0, & \text{else} \end{cases}$$

and we further define a function g on Ω by

$$g := \sum_{Q_m \in E_*} g_m.$$

Here $f_{Q_m} = \frac{1}{|Q_m|} \int_{Q_m} f(y) dy$ for each $Q_m \in E_*$. Let $\widetilde{g}$ be the zero extension of g from Ω to $\mathbf{R}^n$.

Without loss of generality, we assume that the radius r_0 of the ball equals 1 in defining the space $L^1_{\mathrm{ul}}(\Omega)$. Notice that

$$\|g_m\|_{L^\infty(\Omega)} \leq \frac{1}{|Q_m|} \cdot \int_{Q_m} |f|\, dy.$$

Let k_0 be the smallest integer such that $2^{-k_0} < \frac{2}{\sqrt{n}}$. If $\ell(Q_m) \leq 2^{-k_0}$, then $\|f\|_{L^1(Q_m)} \leq [f]_{\Gamma_\infty}$. In this case, as $\ell(Q_m) > 2^{-k_\varepsilon}$,

$$\|g_m\|_{L^\infty(\Omega)} \leq \frac{1}{|Q_m|} \cdot \int_{Q_m} |f|\, dy \leq (\frac{10\sqrt{n}}{\varepsilon})^n \cdot [f]_{\Gamma_\infty}.$$

If $\ell(Q_m) > 2^{-k_0}$, we divide Q_m into $(\frac{\ell(Q_m)}{2^{-k_0}})^n$ small subcubes of side length 2^{-k_0}. Hence,

$$\int_{Q_m} |f|\,dy = \sum_{i=1}^{(\ell(Q_m)/2^{-k_0})^n} \int_{Q_m^i} |f|\,dy \le (\frac{\ell(Q_m)}{2^{-k_0}})^n \cdot [f]_{\Gamma_\infty} \le |Q_m| \cdot n^{\frac{n}{2}} \cdot [f]_{\Gamma_\infty},$$

in this case $\|g_m\|_{L^\infty(\Omega)} \le n^{\frac{n}{2}} \cdot [f]_{\Gamma_\infty}$. Therefore,

$$\|g\|_{L^\infty(\Omega)} \le C_{n,\varepsilon} \cdot [f]_{\Gamma_\infty}$$

and we deduce that $g \in bmo_\infty^\infty(\Omega)$ as $L^\infty(\Omega) \subset bmo_\infty^\infty(\Omega)$.

Let $f^* := f - g \in bmo_\infty^\infty(\Omega)$. We do Jones extension to f^*. If Ω is unbounded, for each $Q_j^{'} \in E^{'}$, we find a nearset $Q_j \in E$ satisfying $\ell(Q_j) \ge \ell(Q_j^{'})$. We define that $\widetilde{f^*} = f^*$ on Ω and $\widetilde{f^*}(x) = f^*_{Q_j}$ for $x \in Q_j^{'}$. If Ω is bounded, we pick $Q_0 \in E$ such that $\ell(Q_0) = \sup_{Q_j \in E} \ell(Q_j)$. We define that $\widetilde{f^*} = f^*$ on Ω, $\widetilde{f^*}(x) = f^*_{Q_j}$ for $x \in Q_j^{'}$ where $\ell(Q_j^{'}) \le \ell(Q_0)$ and $\widetilde{f^*}(x) = f^*_{Q_0}$ for $x \in Q_j^{'}$ where $\ell(Q_j) > \ell(Q_0)$. By Jones [PJ], $\widetilde{f^*} \in BMO$ and $[\widetilde{f^*}]_{BMO} \le C_K \cdot [f^*]_{BMO^\infty(\Omega)}$. By this extension, for $\widetilde{f^*}(x) \ne 0$, either $x \in \Omega$ or $x \in Q_j^{'}$ such that $\ell(Q_j^{'}) \le 2^{-k_\varepsilon}$. Since $d(Q_j^{'}, \Omega) \le 4\sqrt{n} \cdot \ell(Q_j^{'})$, pick $x \in \overline{Q_j^{'}}$ and $z \in \Gamma$ such that $|x - z| = d(Q_j^{'}, \Omega)$. For any $y \in Q_j^{'}$, $|y - z| \le |y - x| + |x - z| \le 5\sqrt{n} \cdot \ell(Q_j^{'})$. So $\operatorname{int} Q_j^{'} \subset B_{5\sqrt{n}\cdot\ell(Q_j^{'})}(z)$ for some $z \in \Gamma$. Since $5\sqrt{n} \cdot \ell(Q_j^{'}) \le 5\sqrt{n} \cdot 2^{-k_\varepsilon} < \varepsilon$, $\operatorname{int} Q_j^{'} \subset \Omega_\varepsilon$. Let $\widetilde{f} := \widetilde{f^*} + \widetilde{g}$ and $\overline{f} = \widetilde{f}|_{\Omega_{2\varepsilon}}$, we have that $\operatorname{supp} \overline{f} \subset \overline{\Omega_\varepsilon}$ and by previous calculation,

$$\begin{aligned}[\overline{f}]_{BMO^\infty(\Omega_{2\varepsilon})} &\le [\widetilde{f}]_{BMO} \le [\widetilde{f^*}]_{BMO} + [\widetilde{g}]_{BMO} \le C_K \cdot [f^*]_{BMO^\infty(\Omega)} + 2\|g\|_\infty \\ &\le C_{K,n,\varepsilon} \cdot ([f]_{BMO^\infty(\Omega)} + [f]_{\Gamma_\infty}).\end{aligned}$$

Let $B(x)$ denotes the ball of radius 1 centered at x and $\Gamma^\varepsilon := \{x \in \Omega^c \,|\, d_\Omega(x) < \varepsilon\}$. For $B(x) \cap \Omega_\varepsilon \ne \emptyset$,

$$\int_{B(x)\cap\Omega_\varepsilon} |\overline{f}|\,dy = \int_{B(x)\cap\Omega} |f|\,dy + \int_{B(x)\cap\Gamma^\varepsilon} |\overline{f}|\,dy.$$

The first integral on the right hand side is directly estimated by $[f]_\infty$, so we only need to consider the second integral. Let $Q_*^{'}$ be a largest Whitney cube in $E^{'}$ that intersects $B(x) \cap \Gamma^\varepsilon$. For $Q_j^{'} \in E^{'}$, [PJ, Lemma 2.10] says that if $Q_j \in E$ is a nearest Whitney cube satisfying $\ell(Q_j) \ge \ell(Q_j^{'})$, then $d(Q_j, Q_j^{'}) \le 65K^2 \cdot \ell(Q_j^{'})$. Consider $Q_j^{'} \in E^{'}$ such that $Q_j^{'} \cap B(x) \cap \Gamma^\varepsilon \ne \emptyset$, let $x_j \in Q_j$ where Q_j is a nearest Whitney cube satisfying $\ell(Q_j) \ge \ell(Q_j^{'})$, let $x_j^{'} \in Q_j^{'} \cap B(x) \cap \Gamma^\varepsilon$ and $x_*^{'} \in Q_*^{'} \cap B(x) \cap \Gamma^\varepsilon$. By choosing K large such that $K^2 \ge 2\sqrt{n}$, we have that

$$|x_j - x_*^{'}| \le |x_*^{'} - x_j^{'}| + |x_j^{'} - x_j| \le 2 + 2\sqrt{n} \cdot \ell(Q_j^{'}) + 65K^2 \cdot \ell(Q_j^{'}) \le 2 + 66K^2 \cdot \ell(Q_j).$$

Since $\ell(Q_j) \le 2\ell(Q_j^{'}) \le 2\ell(Q_*^{'}) \le 2\ell(Q_*)$ where $Q_* \in E$ is a nearest cube satisfying $\ell(Q_*) \ge$

$\ell(Q_*^{'})$, $|x_j - x_*^{'}| \le 2 + 132K^2 \cdot \ell(Q_*)$.

If $B(x) \cap \Gamma \neq \emptyset$, then $\sqrt{n} \cdot \ell(Q_*^{'}) \le d(Q_*^{'}, \Omega) \le 2$. Hence $\ell(Q_*) \le 2\ell(Q_*^{'}) \le \frac{4}{\sqrt{n}}$, for any $x_j \in Q_j$, $|x_j - x_*^{'}| < 2 + 133K^2 \cdot \frac{4}{\sqrt{n}}$. Consider the cube $\widetilde{Q_*^{'}}$ with center $x_*^{'}$ and side length $4 + \frac{1064K^2}{\sqrt{n}}$. For each $Q_j^{'} \in E^{'}$ such that $Q_j^{'} \cap B(x) \cap \Gamma^\varepsilon \neq \emptyset$, the corresponding nearest $Q_j \in E$ such that $\ell(Q_j) \ge \ell(Q_j^{'})$ we choose to define $\widetilde{f}^*$ is contained in $\widetilde{Q_*^{'}}$, i.e. $Q_j \subset \widetilde{Q_*^{'}}$. Hence,

$$\int_{B(x)\cap\Gamma^\varepsilon} |\overline{f}|\, dy = \sum_{\substack{Q_j^{'} \in E^{'}, \\ Q_j^{'} \cap B(x) \cap \Gamma^\varepsilon \neq \emptyset}} \int_{Q_j^{'} \cap B(x) \cap \Gamma^\varepsilon} |f_{Q_j}^*|\, dy \le \int_{\widetilde{Q_*^{'}} \cap \Omega} |f^*|\, dy.$$

Let p be the largest integer such that $2^{-p} > 4 + \frac{1064K^2}{\sqrt{n}}$, so $2^{-p} \le 8 + \frac{2128K^2}{\sqrt{n}}$. Let $\widetilde{Q_*^{'}}$ be contained in a larger cube $\widetilde{Q}$ where $\widetilde{Q}$ has center $x_*^{'}$ and side length 2^{-p}. We can divide $\widetilde{Q}$ into $(\frac{2^{-p}}{2^{-k_0}})^n$ subcubes of side length 2^{-k_0}, thus

$$\int_{\widetilde{Q_*^{'}} \cap \Omega} |f^*|\, dy \le \sum_{i=1}^{(2^{-p}/2^{-k_0})^n} \int_{\widetilde{Q_i} \cap \Omega} |f^*|\, dy \le (\frac{2^{-p}}{2^{-k_0}})^n \cdot [f^*]_{\Gamma_\infty} \le C_{K,n} \cdot [f^*]_{\Gamma_\infty}.$$

If $B(x) \cap \Gamma = \emptyset$, i.e. $B(x) \subset \overline{\Omega}^c$. Let $E_1^{'} := \{Q_j^{'} \in E^{'} \mid Q_j^{'} \cap B(x) \neq \emptyset\}$. Let $\ell_m := \inf_{Q_j^{'} \in E_1^{'}} \ell(Q_j^{'})$ and $Q_*^{'}$ be a largest $Q_j^{'} \in E_1^{'}$. If $\ell_m = 0$, then there exists $z \in \Gamma \cap \partial B(x)$. In this case, $\sqrt{n} \cdot \ell(Q_*^{'}) \le d(Q_*^{'}, \Omega) \le 2$. Therefore same argument as in the case where $B(x) \cap \Gamma \neq \emptyset$ gives that $\|\overline{f}\|_{L^1(B(x)\cap\Gamma^\varepsilon)} \le C_{K,n} \cdot [f^*]_\infty$. If $0 < \ell_m \le 2$, then pick $Q_m^{'} \in E_1^{'}$ such that $\ell(Q_m^{'}) = \ell_m$. Since $\sqrt{n} \cdot \ell(Q_*^{'}) \le d(Q_*^{'}, \Omega) \le 2 + \sqrt{n} \cdot \ell(Q_m^{'}) + d(Q_m^{'}, \Omega) \le 2 + 10\sqrt{n}$, we have that $\ell(Q_*) \le \frac{4}{\sqrt{n}} + 20$. Hence $|x_j - x_*^{'}| \le 2 + 133K^2 \cdot (\frac{4}{\sqrt{n}} + 20)$. Following the argument as in the case where $B(x) \cap \Gamma \neq \emptyset$, we can deduce that $\|\overline{f}\|_{L^1(B(x)\cap\Gamma^\varepsilon)} \le C_{K,n} \cdot [f^*]_{\Gamma_\infty}$. If $\ell_m > 2$, then $B(x)$ intersects at most 2^n Whitney cubes in $E^{'}$. Without loss of generality, assume that $E_1^{'}$ has 2^n elements. Then

$$\int_{B(x)\cap\Gamma^\varepsilon} |\overline{f}|\, dy \le \sum_{Q_i^{'} \in E_1^{'}} \int_{B(x) \cap Q_i^{'}} |f_{Q_i}^*|\, dy \le \sum_{Q_i^{'} \in E_1^{'}} \frac{|B(x) \cap Q_i^{'}|}{|Q_i|} \cdot \int_{Q_i} |f^*|\, dy.$$

Divide Q_i into $\left(\frac{\ell(Q_i)}{2^{-k_0}}\right)^n$ subcubes of side length 2^{-k_0}, we have that

$$\int_{Q_i} |f^*|\, dy \le \left(\frac{\ell(Q_i)}{2^{-k_0}}\right)^n \cdot [f^*]_{\Gamma_\infty} \le |Q_i| \cdot n^{\frac{n}{2}} \cdot [f^*]_{\Gamma_\infty}.$$

Therefore,

$$\int_{B(x)\cap\Gamma^\varepsilon} |\overline{f}|\, dy \le \Big(\sum_{Q_i^{'} \in E_1^{'}} |B(x) \cap Q_i^{'}| \Big) \cdot n^{\frac{n}{2}} \cdot [f^*]_{\Gamma_\infty} \le C_n \cdot [f^*]_{\Gamma_\infty}.$$

Since $[f^*]_{\Gamma_\infty} \le [f]_{\Gamma_\infty} + [g]_{\Gamma_\infty}$ and $[g]_{\Gamma_\infty}$ is estimated by $C_{n,\varepsilon} \cdot [f]_{\Gamma_\infty}$, we are done. □

As an application we give an estimate for the product of a Hölder function and a function in bmo^∞_∞. We first recall properties of point multipliers. It is known that for a local hardy space $h^1 = F^0_{1,2}$ [Sa, Theorem 3.18], there is a constant C such that

$$\|\varphi g\|_{F^0_{1,2}} \le C\|\varphi\|_{C^\gamma}\|g\|_{F^0_{1,2}} \quad g \in F^0_{1,2} \tag{12}$$ {MHA}

for $\varphi \in C^\gamma(\mathbf{R}^n)$, $\gamma \in (0,1)$, where

$$\|\varphi\|_{C^\gamma} = \sup_{x\in\mathbf{R}^n} |\varphi(x)| + \sup_{\substack{x,y\in\mathbf{R}^n\\ x\neq y}} |\varphi(x)-\varphi(y)| \,/\, |x-y|^\gamma;$$

see e.g. [Sa, Remark 4.4]. Since

$$bmo = BMO^\infty(\mathbf{R}^n) \cap L^1_{\mathrm{ul}}(\mathbf{R}^n)$$

equals to $F^0_{\infty,2}$ [Sa, Theorem 3.26], it is a dual space of $h^1 = F^0_{1,2}$ [Sa, Theorem 3.22]. Thus

$$\|\varphi f\|_{bmo} \le C\|\varphi\|_{C^\gamma}\|f\|_{bmo}. \tag{13}$$ {ME}

{PMU}

Theorem 13. *Let $\Omega \subset \mathbf{R}^n$ be a uniform domain. Let $\varphi \in C^\gamma(\Omega)$, $\gamma \in (0,1)$. For each $f \in bmo^\infty_\infty(\Omega)$, the function $\varphi f \in bmo^\infty_\infty(\Omega)$ satisfies*

$$\|\varphi f\|_{bmo^\infty_\infty(\Omega)} \le C\|\varphi\|_{C^\gamma(\Omega)}\|f\|_{bmo^\infty_\infty(\Omega)}$$

with C independent of φ and f.

Proof. By [McS], there exists $\overline{\varphi} \in C^\gamma(\mathbf{R}^n)$ such that $\overline{\varphi}|_\Omega = \varphi$ and

$$\|\overline{\varphi}\|_{C^\gamma(\mathbf{R}^n)} \le \|\varphi\|_{C^\gamma(\Omega)}.$$

For our current purpose it suffices to set $\overline{\varphi} = \max\{\min\{\varphi_*, \|\varphi\|_\infty\}, -\|\varphi\|_\infty\}$ with

$$\varphi_*(x) = \inf_{y\in\Omega} \left\{\varphi(y) + [\varphi]_{C^\gamma}\cdot|x-y|^\gamma\right\},$$

where $\|\varphi\|_{C^\gamma(\Omega)} = \|\varphi\|_{L^\infty(\Omega)} + [\varphi]_{C^\gamma(\Omega)}$, $\|\varphi\|_{L^\infty(\Omega)} = \sup_{x\in\Omega}|\varphi(x)|$ and $[\varphi]_{C^\gamma(\Omega)} = \sup_{x,y\in\Omega}\frac{|\varphi(x)-\varphi(y)|}{|x-y|^\gamma}$; we often suppress Ω. By definition $\varphi_*(x) \le \varphi(x)$. Moreover, since $\varphi(x) \le \varphi(y) + [\varphi]_{C^\gamma}\cdot|x-y|^\gamma$ for $x,y \in \Omega$, we see that $\varphi(x) \le \varphi_*(x)$ which implies $\varphi = \varphi_*$ on Ω. For any $x \in \mathbf{R}^n$ and $\varepsilon > 0$ there is $y_\varepsilon \in \Omega$ such that

$$\varphi(y_\varepsilon) + [\varphi]_{C^\gamma}\cdot|x-y_\varepsilon|^\gamma \le \varphi_*(x) + \varepsilon.$$

For $x_1 \in \mathbf{R}^n$ we observe that

$$\varphi_*(x_1) - \varphi_*(x) \le \varphi(y_\varepsilon) + [\varphi]_{C^\gamma} \cdot |x_1 - y_\varepsilon|^\gamma - \{\varphi(y_\varepsilon) + [\varphi]_{C^\gamma} \cdot |x - y_\varepsilon|^\gamma\} + \varepsilon \le [\varphi]_{C^\gamma} \cdot |x - x_1|^\gamma + \varepsilon.$$

Since ε is arbitrary, we see that $\varphi_*(x_1) - \varphi_*(x) \le [\varphi]_{C^\gamma} \cdot |x - x_1|^\gamma$. Interchanging the role of x_1 and x, we conclude that

$$[\varphi_*]_{C^\gamma(\mathbf{R}^n)} \le [\varphi]_{C^\gamma(\Omega)}.$$

Since $\|\varphi\|_\infty < \infty$, $\overline{\varphi} = \varphi$ on Ω and $\overline{\varphi}$ is still Hölder. More precisely, $[\overline{\varphi}]_{C^\gamma} \le [\varphi_*]_{C^\gamma}$. By definition $\|\overline{\varphi}\|_\infty \le \|\varphi\|_\infty$ so we conclude that $\|\overline{\varphi}\|_{C^\gamma} \le \|\varphi\|_{C^\gamma}$.

Extending $f \in bmo^\infty_\infty(\Omega)$ to $\overline{f} \in bmo$ by Theorem 12, we conclude from multiplication estimate (13) that

$$\begin{aligned}
\|\varphi f\|_{bmo^\infty_\infty(\Omega)} &\le \|\overline{\varphi}\overline{f}\|_{bmo} \\
&\le C \cdot \|\overline{\varphi}\|_{C^\gamma(\mathbf{R}^n)} \cdot \|\overline{f}\|_{bmo} \\
&\le C \cdot \|\varphi\|_{C^\gamma(\Omega)} \cdot \|f\|_{bmo^\infty_\infty(\Omega)}.
\end{aligned}$$

□ {POZ}

Remark 14. If we prove that the extension $f \mapsto \overline{f}$ constructed in Theorem 11 is bounded from bmo^∞_∞ to $bmo = BMO \cap L^1_{\mathrm{ul}}$, then the support condition will follow by taking $\varphi \in C^\gamma(\mathbf{R}^n)$ in Theorem 13 as a cutoff function of Ω, i.e. $\varphi \equiv 1$ on Ω with $\mathrm{supp}\varphi \subset \Omega_\varepsilon$. In other words, we consider $f \mapsto \varphi\overline{f}$. However, the proof that $\overline{f} \in L^1_{\mathrm{ul}}$ needs some argument so we give a direct proof of Theorem 12.

For $BMO^{\mu,\infty}_b$ function in Ω it is easy to see that its zero extension is in BMO space; see e.g. [BGST, Lemma 4].

{ZE}

Theorem 15. *Let Ω be an arbitrary domain in $\mathbf{R}^n$. Assume that $\mu \in (0,\infty]$. For $f \in BMO^{\mu,\nu}_b(\Omega)$ with $\nu \ge 2\mu$, let f_0 be the zero extension to $\mathbf{R}^n$, i.e. $f_0(x) = 0$ for $x \in \Omega^c$ and $f_0(x) = f(x)$ for $x \in \Omega$. Then $f_0 \in BMO^\mu(\mathbf{R}^n)$ and $[f_0]_{BMO^\mu} \le C[f]_{BMO^{\mu,\nu}_b}$ with C independent of f.*

Proof. If the ball B of radius $\le \mu$ is in Ω, then

$$\frac{1}{|B|}\int_B |f_0 - f_{0B}|\, dy \le [f]_{BMO^\mu}.$$

If B is in Ω^c, then $\int_B |f_0 - f_{0B}|\, dy = 0$. It remains to estimate the integral if B has nonempty intersection with the boundary $\Gamma = \partial\Omega$. For each $B_r(x) \cap \Gamma \ne \emptyset$, $r < \mu$, we take $x_0 \in B_r(x) \cap \Gamma$.

Then, $B_r(x) \subset B_{2r}(x_0)$ and thus

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} |f_0 - f_{0B_r(x)}|\,dy \le \frac{2}{|B_r(x)|}\int_{B_{2r}(x_0)} |f_0|\,dy \le \frac{2^{n+1}}{\omega_n}\cdot [f]_{b^{2\mu}},$$

where ω_n is the volume of an n-dimensional ball. $\square$

{BGST}

Remark 16. In [BGST, Lemma 4], it is assumed that $\Omega = \Omega' \times \mathbf{R}^{n-k}$ where Ω' is a bounded Lipschitz domain in $\mathbf{R}^k$. However, from the proof above it is clear that we do not need this requirement. Thus we give a full proof here.

As an application of boundedness of multiplication, we give invariance of function spaces under coordinate changes. We say that Ψ is a global $C^{k+\beta}$ (resp. C^k)-diffeomorphism if $C^{k+\beta}$ (resp. C^k)-norm of Ψ and Ψ^{-1} are bounded in $\mathbf{R}^n$, where $k \in \mathbf{N}$ and $\beta \in (0,1)$.

{CC}

Proposition 17. *The space bmo is invariant under bi-Lipschitz coordinate change and the space h^1 is invariant under global $C^{1+\beta}$-diffeomorphism.*

Proof. For $f \in bmo$, by a simple change of variables on the equivalent definition of the seminorm $[f]_{BMO}$ where

$$[f]_{BMO} = \sup_{B\subset\mathbf{R}^n} \inf_{c\in\mathbf{R}} \int_B |f(y)-c|\,dy,$$

see e.g. [Gra, Proposition 3.1.2], we can easily deduce that *bmo* is invariant under bi-Lipschitz coordinate change.

Let $g \in h^1(\mathbf{R}^n)$ and Ψ be a global $C^{1+\beta}$-diffeomorphism. We have that

$$\|g\circ\Psi\|_{h^1} = \sup_{\|f\|_{bmo}\le 1} \left|\int_{\mathbf{R}^n} f\cdot g\circ\Psi\,dy\right|.$$

By change of variable we have that

$$\left|\int_{\mathbf{R}^n} f(y)\cdot g\circ\Psi(y)\,dy\right| = \left|\int_{\mathbf{R}^n} f\circ\Psi^{-1}(x)\cdot g(x)\cdot J_{\Psi^{-1}}(x)\,dx\right|,$$

where $J_{\Psi^{-1}}$ is the Jacobian which is of regularity C^β. Then by the $bmo-h^1$ duality [Sa, Theorem 3.22] and multiplication estimate (12), we deduce that

$$\left|\int_{\mathbf{R}^n} f\circ\Psi^{-1}\cdot g\cdot J_{\Psi^{-1}}\,dx\right| \le \|f\circ\Psi^{-1}\|_{bmo}\cdot\|gJ_{\Psi^{-1}}\|_{h^1} \le \|f\circ\Psi^{-1}\|_{bmo}\cdot\|J_{\Psi^{-1}}\|_{C^\beta}\cdot\|g\|_{h^1}.$$

Since *bmo* is independent of bi-Lipschitz coordinate change, we have that

$$\|g\circ\Psi\|_{h^1} \le C\cdot\|\nabla\Psi^{-1}\|_{L^\infty}\cdot\|J_{\Psi^{-1}}\|_{C^\beta}\cdot\|g\|_{h^1}$$

for some constant C independent of g and Ψ. $\square$ {CCL}

Proposition 18. *The space $F^1_{1,2}(\mathbf{R}^n)$ is independent of global $C^{1+\beta}$-diffeomorphism.*

Proof. Let $g \in F^1_{1,2}$ and Ψ be a global $C^{1+\beta}$-diffeomorphism. By multiplication estimate (12) and Proposition 17, we have that

$$\|\nabla(g\circ\Psi)\|_{F^0_{1,2}} \leq C\cdot\|\nabla\Psi\|_{C^\beta}\cdot\|(\nabla g)\circ\Psi\|_{F^0_{1,2}} \leq C\cdot\|\nabla\Psi\|_{C^\beta}\cdot\|\nabla\Psi^{-1}\|_{L^\infty}\cdot\|J_{\Psi^{-1}}\|_{C^\beta}\cdot\|g\|_{F^1_{1,2}},$$

where $J_{\Psi^{-1}}$ is the Jacobian for Ψ^{-1} and C is a constant independent of g and Ψ. Hence $\nabla(g\circ\Psi) \in F^0_{1,2}$. Since the differentiation mapping is bounded from $F^s_{p,q}$ to $F^{s-1}_{p,q}$ for $p \in (0,\infty)$, $q \in (0,\infty]$ and $s \in \mathbf{R}$, see e.g. [Sa, Theorem 2.12], we have that $\Delta(g\circ\Psi) \in F^{-1}_{1,2}$. Since $F^1_{1,2} \hookrightarrow F^0_{1,2}$, Proposition 17 tells us that $g\circ\Psi \in F^0_{1,2}$ and thus $g\circ\Psi \in F^{-1}_{1,2}$. Therefore, $(I-\Delta)(g\circ\Psi) \in F^{-1}_{1,2}$. Notice that [Sa, Theorem 2.12] also tells us that for $\sigma \in \mathbf{R}$, $(I-\Delta)^\sigma$ is an isomorphism from $F^s_{p,q}$ to $F^{s-2\sigma}_{p,q}$. Hence by letting $\sigma = -1$, we deduce that

$$\begin{aligned}
\|g\circ\Psi\|_{F^1_{1,2}} &= \|(I-\Delta)^{-1}(I-\Delta)(g\circ\Psi)\|_{F^1_{1,2}} \\
&= C\cdot\|(I-\Delta)(g\circ\Psi)\|_{F^{-1}_{1,2}} \\
&\leq C\cdot\Big(\|g\circ\Psi\|_{F^0_{1,2}} + \|\nabla(g\circ\Psi)\|_{F^0_{1,2}}\Big) \\
&\leq C\cdot(1+\|\nabla\Psi\|_{C^\beta})\cdot\|\nabla\Psi^{-1}\|_{L^\infty}\cdot\|J_{\Psi^{-1}}\|_{C^\beta}\cdot\|g\|_{F^1_{1,2}},
\end{aligned}$$

where C is a constant independent of g and Ψ. $\square$ {CCLR}

Remark 19. The proof of Proposition 18 also says that $F^1_{1,2} = \{f \in F^0_{1,2} \mid \nabla f \in (F^0_{1,2})^n\}$.

4 Trace problems

{TR}

In this section we show that the normal trace of a vector field in $vbmo^{\mu,\nu}_\delta$ is in $L^\infty(\Gamma)$ if its divergence is well controlled. We begin with the case that Ω is the half space $\mathbf{R}^n_+$.

We first recall that the trace operator $(Trf)(x') = f(x',0)$ for $f \in F^1_{1,2}(\mathbf{R}^n)$ gives a surjective bounded linear operator from $F^1_{1,2}(\mathbf{R}^n)$ to $L^1(\mathbf{R}^{n-1})$; see [Tr92, Section 4.4.3].

{HA}

Proposition 20 ([Tr92])**.** *The operator Tr from $F^1_{1,2}$ to $L^1(\mathbf{R}^{n-1})$ is surjective for $n \geq 2$. Actually, surjectivity holds for a smaller space $B^1_{1,1}$. The inverse operator is called the extension and it is a bounded operator.*

For a C^2 domain Ω a normal trace $v\cdot\mathbf{n}$ on $\Gamma = \partial\Omega$ of v is well-defined as an element of $W^{-1/p}_{p,\mathrm{loc}}(\Gamma)$ if v and $\operatorname{div} v$ is in L^p_{loc}; see e.g. [FM] or [Gal]. If $v \in vbmo^{\mu,\nu}_\delta(\Omega)$ so that $v \in L^1_{\mathrm{loc}}$, then by an interpolation inequality (see e.g. [BGST, Theorem 11]) v is in L^p_{loc} for any $p \geq 1$. Thus if $\operatorname{div} v$ is in L^p_{loc}, $v \cdot \mathbf{n}$ is well-defined. We derive L^∞ estimate for $v \cdot \mathbf{n}$ when Ω is the half space.

{NTH}

Theorem 21. *Let μ, ν, δ be in $(0, \infty]$ and $n \geq 2$. Then there is a constant $C = C(\mu, \nu, \delta, n)$ such that*

$$\|v \cdot \mathbf{n}\|_{L^\infty(\mathbf{R}^{n-1})} \leq C \left(\|v\|_{vbmo^{\mu,\nu}_\delta(\mathbf{R}^n_+)} + \|\operatorname{div} v\|_{L^n_{\mathrm{ul}}(\Gamma_\delta)} \right)$$

for all $v \in vbmo^{\mu,\nu}_\delta(\mathbf{R}^n_+)$.

Proof. Let $v \in vbmo^{\mu,\nu}_\delta(\mathbf{R}^n_+)$, by definition the n-th component v_n of $v = (v', v_n)$ belongs to $BMO^{\mu,\nu}_b(\mathbf{R}^n_+)$. For $x_0' \in \mathbf{R}^{n-1}$, we consider the region $U = B_1(x_0') \times (-\delta, \delta)$ where $B_1(x_0')$ denotes the ball in $\mathbf{R}^{n-1}$ centered at x_0' with radius 1. Let v_{re} denotes the restriction of v on $U \cap \mathbf{R}^n_+$, i.e. $v_{\mathrm{re}} = v\,|_{U \cap \mathbf{R}^n_+}$. We have that $v_{\mathrm{re}} \in bmo^\infty_\infty(U \cap \mathbf{R}^n_+)$ and

$$\sup_{\substack{x' \in B_1(x_0') \\ r < \nu}} \frac{1}{|B_r((x', 0))|} \int_{B_r((x',0))} |(v_{\mathrm{re}})_n| \, dy < \infty.$$

Let $\overline{(v_{\mathrm{re}})_n}$ be the zero extension of $(v_{\mathrm{re}})_n$ to U. By Theorem 15, $\overline{(v_{re})_n}$ is in $BMO^\infty(U)$. Let $\overline{v_{\mathrm{re}}'}$ be the even extension of v_{re}' to U of the form

$$\overline{v_{\mathrm{re}}'}(x', x_n) = \begin{cases} v_{\mathrm{re}}'(x', x_n), & x' \in B_1(x_0') \text{ and } x_n > 0 \\ v_{\mathrm{re}}'(x', -x_n), & x' \in B_1(x_0') \text{ and } x_n < 0 \end{cases} \tag{14}$$

and set $\widetilde{v} = (\overline{v_{\mathrm{re}}'}, \overline{(v_{\mathrm{re}})_n})$. We have that $\widetilde{v} \in bmo^\infty_\infty(U)$. By Theorem 12 its Jones' extension v_U belongs to $bmo^\infty_\infty(\mathbf{R}^n)$.

Integration by parts formally yields

$$\int_{\mathbf{R}^{n-1}} v_U \cdot \mathbf{n} \rho \, dx' = \int_{\mathbf{R}^n_+} (\operatorname{div} v_U) \rho \, dx - \int_{\mathbf{R}^n_+} v_U \cdot \nabla \rho \, dx. \tag{15}$$ {INT}

By Proposition 20 there is an extension operator $Ext : L^1(\mathbf{R}^{n-1}) \to F^1_{1,2}(\mathbf{R}^n)$ such that $Tr \circ Ext$ is the identity operator on L^1. For $\varphi \in C^\infty_c \left(B_{\frac{1}{2}}(x_0') \right)$ we set $\sigma = Ext\, \varphi$. By multiplying a cut off function $\theta \in C^\infty_c(U)$ such that $\theta \equiv 1$ in $\frac{1}{2}U$ and consider $\rho = \theta\sigma$ we still find $\rho \in F^1_{1,2}(\mathbf{R}^n)$ by a multiplier theorem [Sa, Theorem 3.18], [Tr92, Section 4.2.2]. We estimate (15) to get

$$\begin{aligned} \left| \int_{\mathbf{R}^{n-1}} v_U \cdot \mathbf{n} \rho \, dx' \right| &\leq \left| \int_U (\operatorname{div} v_U) \rho \, dx \right| + \left| \int_{\mathbf{R}^n_+} v_U' \cdot \nabla' \rho \, dx \right| \\ &\quad + \left| \int_{\mathbf{R}^n} v_{U^n} \frac{\partial \rho}{\partial x_n} dx \right| = I + I\!I + I\!I\!I. \end{aligned}$$

We may assume that ρ is even in x_n by taking $(\rho(x',x_n)+\rho(x',-x_n))/2$ so that the second term is estimated by *bmo*-h^1 duality $(h^1)^*=(F^0_{1,2})^*=F^0_{\infty,2}=bmo$ as follows

$$
\begin{aligned}
I\!I &= \left| \int_{\mathbf{R}^n_+} v'_U \cdot \nabla' \rho \, dx \right| = \frac{1}{2} \left| \int_{\mathbf{R}^n} v'_U \cdot \nabla' \rho \, dx \right| \\
&\leq C \| v'_U \|_{bmo} \| \nabla' \rho \|_{h^1}.
\end{aligned}
$$

The third term is estimated as

$$
I\!I\!I \leq C \| v_{U^n} \|_{bmo} \left\| \frac{\partial \rho}{\partial x_n} \right\|_{h^1}.
$$

The first term is estimated by

$$
\begin{aligned}
I &\leq \| \operatorname{div} v_U \|_{L^n(U)} \| \rho \|_{L^{n/(n-1)}(U)} \\
&\leq C \| \operatorname{div} v \|_{L^n_{\mathrm{ul}}(\Gamma_\delta)} \| \nabla \rho \|_{L^1(U)}
\end{aligned}
$$

by the Sobolev inequality. Since $\|\nabla\rho\|_{L^1} \leq \|\nabla\rho\|_{h^1}$ and $\|\nabla\rho\|_{h^1} \leq \|\rho\|_{F^1_{1,2}} \leq C\|\varphi\|_{L^1\left(B_{\frac{1}{2}}(x'_0)\right)}$, collecting these estimates yields

$$
\left| \int_{B_{\frac{1}{2}}(x'_0)} v \cdot \mathbf{n} \varphi \, dx' \right| \leq C \|\varphi\|_{L^1\left(B_{\frac{1}{2}}(x'_0)\right)} \left(\|v\|_{vbmo^{\mu,\nu}_\delta(\mathbf{R}^n_+)} + \|\operatorname{div} v\|_{L^n_{\mathrm{ul}}(\Gamma_\delta)} \right).
$$

This yields the desired estimate since $C^\infty_c\left(B_{\frac{1}{2}}(x'_0)\right)$ is dense in $L^1\left(B_{\frac{1}{2}}(x'_0)\right)$ and C in the right-hand side is independent of $x'_0 \in \mathbf{R}^{n-1}$. □

We now consider a curved domain. Let Ω be a uniformly C^2 domain in $\mathbf{R}^n$ so that the reach R_* of Γ is positive and $\beta \in (0,1)$.

{NTG}

Theorem 22. *Let Ω be a uniformly $C^{2+\beta}$ domain in $\mathbf{R}^n$ with $n \geq 2$. Let μ,ν,δ be in $(0,\infty]$. Then there is a constant $C=C(\mu,\nu,\delta,\Omega)$ such that*

$$
\|v \cdot \mathbf{n}\|_{L^\infty(\Gamma)} \leq C \left(\|v\|_{vbmo^{\mu,\nu}_\delta(\Omega)} + \|\operatorname{div} v\|_{L^n_{\mathrm{ul}}(\Gamma_\delta)} \right)
$$

for all $v \in vbmo^{\mu,\nu}_\delta(\Omega)$.

We shall prove this result by localizing the problems near the boundary and use normal (principal) coordinates. Let Ω be a uniformly $C^{2+\beta}$ domain. In other words, there exist $r_*, \delta_* > 0$ such that for each $z_0 \in \Gamma$, up to translation and rotation, there exists a function $h_{z_0} \in$

$C^{2+\beta}(B_{r_*}(0'))$ with

$$|\nabla^k h_{z_0}| \leq L \text{ in } B_{r_*}(0') \text{ for } k = 0, 1, 2,$$
$$[\nabla^2 h_{z_0}]_{C^\beta(B_{r_*}(0'))} < \infty, \, \nabla' h_{z_0}(0') = 0', \, h_{z_0}(0') = 0$$

such that the neighborhood

$$U_{r_*,\delta_*,h_{z_0}}(z_0) := \{(x', x_n) \in \mathbf{R}^n \,|\, h_{z_0}(x') - \delta_* < x_n < h_{z_0}(x') + \delta_*, \, |x'| < r_*\}$$

satisfies

$$\Omega \cap U_{r_*,\delta_*,h_{z_0}}(z_0) = \{(x', x_n) \in \mathbf{R}^n \,|\, h_{z_0}(x') < x_n < h_{z_0}(x') + \delta_*, \, |x'| < r_*\}$$

and

$$\partial\Omega \cap U_{r_*,\delta_*,h_{z_0}}(z_0) = \{(x', x_n) \in \mathbf{R}^n \,|\, x_n = h_{z_0}(x'), \, |x'| < r_*\}.$$

For $x \in \Omega$, let πx be a point on Γ such that $|x - \pi x| = d_\Omega(x)$. If x is within the reach of Γ, then this πx is unique. There exist $r < r_*$ and $\delta < \delta_*$ such that

$$U(z_0) = \{x \in U_{r_*,\delta_*,h_{z_0}}(z_0) \,|\, (\pi x)' \in B_r(0'), \, d_\Gamma(x) < \delta\}$$

is contained in $U_{r_*,\delta_*,h_{z_0}}(z_0)$. Since d_Ω is $C^{2+\beta}$ in $\overline{\Gamma}_\sigma$ for $\sigma < R_*$ [GT, Chap. 14, Appendix] [KP, §4.4], we may take δ smaller (independent of z_0) so that d_Ω is $C^{2+\beta}$ in $\overline{U(z_0) \cap \Omega}$.

We next consider the normal coordinate in $U(z_0)$

$$\begin{cases} x' = y' + y_n \nabla' d_\Omega\,(y', \psi(y')) \\ x_n = \psi(y') + y_n \partial_{x_n} d_\Omega\,(y', \psi(y')) \end{cases} \tag{16}$$ {NC}

or shortly

$$x = \pi x - d_\Omega(x)\mathbf{n}(\pi x).$$

Let this coordinate change be denoted by $x = \psi(y)$, $\psi \in C^{1+\beta}(B_r(0'))$. Notice that $\nabla\psi(0) = I$. If we consider r and δ small, this coordinate change is indeed a local C^1-diffeomorphism which maps $U(z_0)$ to V where $V := B_r(0') \times (-\delta, \delta)$. Moreover, by [Lew], we extend ψ to a global C^1-diffeomorphism $\widetilde{\psi}$ such that $\widetilde{\psi}|_V = \psi$ and $\|\nabla\widetilde{\psi}\|_{L^\infty(\mathbf{R}^n)} < 2$. Let the inverse of ψ in V be denoted by ϕ, i.e. $\phi = \psi^{-1}$.

{VFG}

Lemma 23. *Let W be a vector field with measureble coefficient in Γ_σ, $\sigma < R_*$ of the form*

$$W = \sum_{i=1}^{n} w_i \frac{\partial}{\partial x_i}.$$

Let y be the normal coordinate such that $y_n = d_\Omega(x)$. Let $\tilde{W}$ be W in y coordinate of the form $\overline{W} = \sum_{j=1}^n \tilde{w}_j(y)\partial/\partial y_j$. Then

$$\tilde{w}_n(y) = \nabla d_\Omega\left(x(y)\right) \cdot w\left(x(y)\right).$$

We shall prove this lemma in Appendix which follows from a simple linear algebra.

Proof of Theorem 22. We first observe that the restriction v on $U(z_0)\cap\Omega$ is in $bmo_\infty^\infty\left(U(z_0)\cap\Omega\right)$. By considering the following equivalent definition of the seminorm $[f]_{BMO^\infty(D)}$ where

$$[f]_{BMO^\infty(D)} = \sup_{B_r(x)\subset D} \inf_{c\in\mathbf{R}} \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y)-c|\,dy,$$

see [Gra, Proposition 3.1.2], we can deduce that the space bmo_∞^∞ on a bounded domain is independent of bi-Lipschitz coordinate change. We introduce normal coordinate for a vector field $v = \sum_{i=1}^n v_i\partial/\partial x_i$ with $v_i \in bmo_\infty^\infty\left(U(z_0)\cap\Omega\right)$. Let w be the transformed vector field of normal coordinate y. By Lemma 23, w_n of $w = \sum_{i=1}^n w_i\partial/\partial y_i$ fulfills $w_n = \nabla d_\Omega\left(x(y)\right)\cdot v\left(x(y)\right)$. Since $v \in vbmo_\delta^{\mu,\nu}(\Omega)$, this implies that $w \in bmo_\infty^\infty(V\cap\mathbf{R}_+^n)$ and moreover,

$$\sup_{\ell<\delta,\, B_\ell(x)\subset V} \ell^{-n}\int_{B_\ell(x)\cap\mathbf{R}_+^n} |w_n|\,dy < \infty.$$

Thus, as in the proof of Theorem 21 the zero extension of w_n for $y_n<0$ is in $bmo_\infty^\infty(V)$, we still denote this extension by w_n. Let $J = J(y)$ denote the Jacobian of the mapping $y\mapsto x$ in V. For tangential part $w^{'}$ of $w = (w^{'}, w_n)$, we take an even extension with weight J of the form

$$\hat{w}'(y', y_n) = \begin{cases} w'(y', y_n), & y_n > 0 \\ w'(y', -y_n)J(y', -y_n)/J(y', y_n), & y_n < 0 \end{cases} \tag{17}$$

and set $\tilde{w}(y', y_n) = (\hat{w}', w_n)$. Let $\overline{w}^{'}$ denote the normal even extension of $w^{'}$ to V, thus $w \in bmo_\infty^\infty(V\cap\mathbf{R}_+^n)$ implies that $\overline{w}^{'} \in bmo_\infty^\infty(V)$. Let f be the function defined on V such that $f\equiv 1$ for $y_n \geq 0$ and $f = J(y', -y_n)/J(y', y_n)$ for $y_n < 0$. Since $J(y)^{-1} = |\det D\psi(y)|^{-1} = |\det D\phi(\psi(y))|$ for $y\in V$, we have that $f \in C^\beta(V)$. Notice that $\hat{w}'(y) = \overline{w}'(y)f(y)$, therefore by Theorem 13, we can deduce that $\tilde{w}$ belongs to $bmo_\infty^\infty(V)$. By Theorem 12, the Jones' extension w_U of $\tilde{w}$ belongs to $bmo_\infty^\infty(\mathbf{R}^n)$. Its expression in x coodinate is v_U which is only defined near Γ.

If the support of ρ is in $U(z_0)$, then integration by parts implies that

$$\int_\Gamma v_U\cdot\mathbf{n}\rho\,d\mathcal{H}^{n-1} = \int_\Omega (\operatorname{div} v_U)\rho\,dx - \int_\Omega v_U\cdot\nabla\rho\,dx. \tag{18}$$ {INT2}

We shall estimate the left-hand side as in the case of $\mathbf{R}_+^n$. The first integral in the right-hand side can be estimated similarly as in the proof of Theorem 21. It is sufficient to only consider the second integral. Let $\Psi : B_r(0') \to \Gamma \cap U(z_0)$ by $(y', 0) \mapsto (y', h_{z_0}(y'))$. Extend $h_{z_0} \in C^2(B_r(0'))$ to $\tilde{h} \in C_c^2(\mathbf{R}^{n-1})$ such that $\tilde{h}\,|_{B_r(0')} = h_{z_0}$. Define $\widetilde{\Psi} : \mathbf{R}^{n-1} \to \tilde{h}(\mathbf{R}^{n-1})$ by $(y', 0) \mapsto (y', \tilde{h}(y'))$. Hence $\widetilde{\Psi}\,|_{B_r(0')} = \Psi$. Extend further $\widetilde{\Psi}$ to $\widetilde{\Psi}^* : \mathbf{R}^n \to \mathbf{R}^n$ by $(y', d) \mapsto \widetilde{\Psi}(y', 0) + (0', d)$. Notice that this $\widetilde{\Psi}^*$ is a global C^2-diffeomorphism whose derivatives are bounded in $\mathbf{R}^n$ up to second-order. Let $\zeta > 0$ be a constant, for $\varphi \in C_c^1(\Gamma \cap \zeta U(z_0))$, we have that $\varphi \circ \Psi \in C_c^1(B_\zeta(0'))$. Let $\widetilde{\sigma} = \mathrm{Ext}\,(\varphi \circ \Psi)$ as in the proof of Theorem 21 and let $\sigma = \widetilde{\sigma} \circ (\widetilde{\Psi}^*)^{-1}$. With this choice of σ, we have that for $(y', h_{z_0}(y')) \in \Gamma \cap \zeta U(z_0)$,

$$\sigma(y', h_{z_0}(y')) = \widetilde{\sigma} \circ (\widetilde{\Psi}^*)^{-1}(y', h_{z_0}(y')) = \widetilde{\sigma}(y', 0) = \varphi \circ \Psi(y', 0) = \varphi(y', h_{z_0}(y')).$$

Thus φ is an extension of σ. Since $(\widetilde{\Psi}^*)^{-1}$ is a global C^2-diffeomorphism and $\widetilde{\sigma} \in F^1_{1,2}(\mathbf{R}^n)$, we observe that $\sigma \in F^1_{1,2}(\mathbf{R}^n)$, see e.g. see Proposition 18 or [Tr92, Section 4.3.1].

For each $z_0 \in \Gamma$, there exists $\epsilon_{z_0} > 0$ such that we can find a cutoff function $\theta_{z_0} \in C_c^\infty(U(z_0))$ for which $\theta_{z_0} \equiv 1$ within $\epsilon_{z_0} U(z_0)$ and

$$\sum_{|\alpha| \leq 2} \|D^\alpha \theta_{z_0}\|_{L^\infty(\mathbf{R}^n)} \leq M$$

for some fixed universal constant $M > 1$ independent of z_0. By multiplying this cutoff function θ_{z_0}, we have that $\rho = \theta_{z_0}\sigma \in F^1_{1,2}(\mathbf{R}^n)$ and $\|\rho\|_{F^1_{1,2}(\mathbf{R}^n)} \leq M \cdot \|\sigma\|_{F^1_{1,2}(\mathbf{R}^n)}$. Hence we take the constant ζ above to be ϵ_{z_0}.

By coodinate change, we observe that

$$\int_\Omega v_U \cdot \nabla \rho \, dx = \int_{U(z_0) \cap \Omega} \sum_{i=1}^n v_i \frac{\partial}{\partial x_i} \rho \, dx = \int_{V \cap \mathbf{R}^n_+} \sum_{j=1}^n w_{U_j}(y) J(y) \frac{\partial}{\partial y_j} \big(\rho \circ \psi(y)\big) dy.$$

The n-th component equals

$$\int_{V \cap \mathbf{R}^n_+} w_{U_n}(y) J(y) \frac{\partial}{\partial y_n} \big(\rho \circ \psi(y)\big) dy = \int_V w_{U_n}(y) J(y) \frac{\partial}{\partial y_n} \big(\rho \circ \psi(y)\big) dy$$

since w_{U_n} equals zero for $y_n < 0$. Consider extensions of Hölder functions [McS] and local diffeomorphism [Lew], by the $F^0_{1,2} - F^0_{\infty,2}$ duality [Sa, Theorem 3.22] and Proposition 17, we conclude that

$$\begin{aligned} \left| \int_V w_{U_n}(y) J(y) \frac{\partial}{\partial y_n} \big(\rho \circ \psi(y)\big) dy \right| &\leq C \cdot \sum_{i=1}^n \|w_{U_n}\|_{bmo} \cdot \|J\|_{C^\beta(V)} \cdot \|\partial_{y_n} \psi\|_{C^\beta(V)} \cdot \|\nabla \rho \circ \widetilde{\psi}\|_{h^1} \\ &\leq C \cdot \|w_{U_n}\|_{bmo} \cdot \|\nabla \rho\|_{h^1}. \end{aligned}$$

For tangential part we may assume that

$$(\rho \circ \psi)(y', y_n) = (\rho \circ \psi)(y', -y_n) \quad \text{for} \;\; y_n < 0. \tag{19}$$ {EEV}

In fact, for a given ρ we take

$$g(y', y_n) = \big(\rho \circ \psi(y', y_n) + \rho \circ \psi(y', -y_n)\big)/2$$

which satisfies evenness $g(y', y_n) = g(y', -y_n)$ and

$$g(y', 0) = \theta \circ \psi(y', 0) \cdot \sigma \circ \psi(y', 0) = \theta(y', h_{z_0}(y')) \cdot \varphi(y', h_{z_0}(y')).$$

It suffices to take ρ such that $\rho \circ \psi(y) = g(y)$. Thus, we may assume that $\rho \circ \psi$ is even in y_n so that $\partial_{y_j}(\rho \circ \psi)$ is also even in y_n for $j = 1, 2, \ldots, n-1$. Since $w_{U_j} J$ is even in y_n for y in V, we observe that

$$\int_{V \cap \mathbf{R}^n_+} w_{U_j}(y) J(y) \frac{\partial}{\partial y_j} \big(\rho \circ \psi\big)\, dy = \frac{1}{2} \int_V w_{U_j}(y) J(y) \frac{\partial}{\partial y_j} \big(\rho \circ \psi\big)\, dy$$

for $1 \leq j \leq n-1$. Similar to the case for the n-th component, we thus conclude that

$$\left| \int_V w_{U_j}(y) J(y) \frac{\partial}{\partial y_j} (\rho \circ \psi)\, dy \right| \leq C \cdot \|w_{U_j}\|_{bmo} \cdot \|\nabla \rho\|_{h^1}.$$

Collecting these estimates, we conclude that

$$\begin{aligned} \left| \int_{\Gamma \cap \epsilon_{z_0} U(z_0)} v \cdot \mathbf{n}\, \varphi \, dx^{n-1} \right| &\leq C \|w_U\|_{bmo} \|\nabla \rho\|_{h^1} \\ &\leq C \|v\|_{vbmo^{\mu,\nu}_\delta(\Omega)} \|\varphi\|_{L^1(\Gamma \cap \epsilon_{z_0} U(z_0))}. \end{aligned}$$

Thus $\|v \cdot \mathbf{n}\|_{L^\infty} \leq C \|v\|_{vbmo^{\mu,\nu}_\delta(\Omega)}$. □

{NTGB}

Remark 24. (i) Since $BMO^{\mu,\nu}_b \subset vbmo^{\mu,\nu}_\delta$ for $\delta < \infty$, the estimate in Theorem 22 holds if we replace $vbmo^{\mu,\nu}_\delta$ by $BMO^{\mu,\nu}_b$. Moreover, since we are able to use zero extension in this case. We can follow the proof of Theorem 22 directly without the necessity to invoke normal coordinates. We shall state a version of Theorem 22 for $BMO^{\mu,\nu}_b$ in the end of this section.

(ii) By Theorem 9 we may replace $vbmo^{\mu,\nu}_\delta$ by $vBMO^{\mu,\nu}(\Omega)$ in the estimate in Theorem 22 since we may always take $\delta \leq \nu < R_*$ provided that Ω is a bounded or an exterior domain.

{TRBB}

Theorem 25. *Let Ω be a uniformly $C^{1+\beta}$ domain in $\mathbf{R}^n$ with $n \geq 2$. Let μ, ν, δ be in $(0, \infty]$.*

Then there is a constant $C = C(\mu, \nu, \delta, \Omega)$ such that

$$\|v \cdot \mathbf{n}\|_{L^\infty(\Gamma)} \leq C \cdot (\|v\|_{BMO_b^{\mu,\nu}(\Omega)} + \|\mathrm{div}\, v\|_{L^n_{\mathrm{ul}}(\Gamma_\delta)})$$

for all $v \in BMO_b^{\mu,\nu}(\Omega)$.

Proof. For $z_0 \in \Gamma$, let $U(z_0) = U_{r_*,\delta_*,h_{z_0}}(z_0)$ with $\delta_* \leq R_*$. We then follow the proof of Theorem 22 without invoking the normal coordinates. For $v \in BMO_b^{\mu,\nu}(\Omega)$, let v_0 be the zero extension of v. We have that $v_0 \in bmo_\infty^\infty(U(z_0))$. Let v_U be the Jones' extension of $r_{U(z_0)}v_0$ by Theorem 12 where $r_{U(z_0)}v_0$ denotes the restriction of v_0 on $U(z_0)$. For $\varphi \in C_c^1(\Gamma \cap \frac{1}{2}U(z_0))$, we construct the function σ in the same way as in the proof of Theorem 22. Since the boundary Γ is uniformly $C^{1+\beta}$, $\widetilde{\Psi}^*$ is a global $C^{1+\beta}$-diffeomorphism. By Proposition 18, we have that $\sigma = \widetilde{\sigma} \circ (\widetilde{\Psi}^*)^{-1} \in F^1_{1,2}(\mathbf{R}^n)$. Pick θ in $C_c^\infty(U(z_0))$ such that $\theta \equiv 1$ within $\frac{1}{2}U(z_0)$ and let $\rho = \theta\sigma$, we deduce that $\rho \in F^1_{1,2}(\mathbf{R}^n)$ and

$$\left| \int_\Omega v_U \cdot \nabla\rho \, dx \right| \leq C \cdot \|v_U\|_{bmo} \cdot \|\nabla\rho\|_{h^1} \leq C \cdot \|v\|_{BMO_b^{\mu,\nu}(\Omega)} \cdot \|\nabla\rho\|_{h^1}.$$

Therefore,

$$\left| \int_{\Gamma\cap\frac{1}{2}U(z_0)} v \cdot \mathbf{n}\, \varphi \, dx^{n-1} \right| \leq C \cdot \|v\|_{BMO_b^{\mu,\nu}(\Omega)} \cdot \|\varphi\|_{L^1(\Gamma\cap\frac{1}{2}U(z_0))}.$$

The proof is therefore complete. □

5 Appendix

{APP}

We shall prove Lemma 23. We first recall a simple property of a matrix.

{LA}

Proposition 26. *Let A be an invertible matrix*

$$A = (\vec{a}_1, \ldots, \vec{a}_n)$$

when $\vec{a}_j =^t (a_{ij})_{1\leq i\leq n}$ is an column vector. Assume that $\vec{a}_n$ is a unit vector and orthogonal to $\vec{a}_j$ with $1 \leq j \leq n-1$. Then n-row vector of A^{-1} equals ${}^t\vec{a}_n$. In other words, if one writes $A^{-1} = (b_{ij})_{1\leq i,j\leq n}$, then $b_{nj} = a_{jn}$ for $1 \leq j \leq n$.

Proof. By definition the row vector $\vec{b} = (b_{nj})_{1\leq j\leq n}$ must satisfies $\vec{b} \cdot \vec{a}_j = 0$ $(j = 1, \ldots, n-1)$, $\vec{b} \cdot \vec{a}_n = 1$. Since $\{\vec{a}_j\}_{j=1}^{n-1}$ spans $\mathbf{R}^{n-1}$ orthogonal to $\vec{a}_n$, first identities imply that $\vec{b}$ is parallel to $\vec{a}_n$. We thus conclude that $\vec{b} = \vec{a}_n$ since $\vec{b} \cdot \vec{a}_n = 1$ and $|\vec{a}_n| = 1$. □

Proof of Lemma 23. We recall the explicit representation (16) of the normal coordinate. The Jacobi matrix from $y \longmapsto x$ is of the form

$$A = (\vec{a}_1, \dots, \vec{a}_n)$$

with $\vec{a}_j =^t \left(\delta_{ij} - y_n \partial_j \mathbf{n}_i \left(y', \psi(y')\right),\ \partial_j \psi(y') - y_n \partial_j \mathbf{n}_n \left(y', \psi(y')\right)\right)_{1 \le i \le n-1}$, $1 \le j \le n-1$,

$$\vec{a}_n = -^t\mathbf{n}\left(y', \psi(y')\right) \quad \text{where} \quad \mathbf{n} = -\nabla d_\Omega.$$

Note that the vector $(\delta_{ij}, \partial_j \psi(y'))_{1 \le i \le n-1}$ is a tangential vector to Γ. Moreover, $(\partial_j \mathbf{n}_1, \dots, \partial_j \mathbf{n}_n)$ is also tangential since $\partial_j \mathbf{n} \cdot \mathbf{n} = \partial_j |\mathbf{n}|^2/2 = 0$. Thus $\vec{a}_j$ is orthogonal to $\vec{a}_n$ for $1 \le j \le n-1$. The invertibility of A is guaranteed if $y_n < R_*$.

By a chain rule we have

$$\begin{aligned}\overline{w} &= \sum_{j=1}^{n} \tilde{w_j}(y)(\partial/\partial y_j) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{w_j} \frac{\partial x_i}{\partial y_j} \frac{\partial}{\partial x_i}\end{aligned}$$

so that

$$w_i\left(x(y)\right) = \sum_{j=1}^{n} \tilde{w_j}(y) \frac{\partial x_i}{\partial y_j} \quad \text{i.e.,} \quad w = A\tilde{w},$$

where $A = (\partial x_i/\partial y_j)_{1 \le i,j \le n}$, $\tilde{w} =^t (\tilde{w_i}, \dots, \tilde{w_n})$, $w =^t (w_1, \dots, w_n)$. Thus

$$\tilde{w} = A^{-1} w.$$

By Proposition 26, the last row of A^{-1} equals ∇d_Ω.

We thus conclude that $\tilde{w_n} = \nabla d_\Omega \cdot w$. This is what we would like to prove. □

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