THE ADJOINT GROUP OF A COXETER QUANDLE

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ABSTRACT. We give explicit descriptions of the adjoint group $\mathrm{Ad}(Q_W)$ of the Coxeter quandle Q_W associated with an arbitrary Coxeter group W. The adjoint group $\mathrm{Ad}(Q_W)$ turns out to be an intermediate group between W and the corresponding Artin group A_W, and fits into a central extension of W by a finitely generated free abelian group. We construct 2-cocycles of W corresponding to the central extension. In addition, we prove that the commutator subgroup of the adjoint group $\mathrm{Ad}(Q_W)$ is isomorphic to the commutator subgroup of W. Finally, the root system Φ_W associated with a Coxeter group W turns out to be a rack. We prove that the adjoint group $\mathrm{Ad}(\Phi_W)$ of Φ_W is isomorphic to the adjoint group of Q_W.

1. INTRODUCTION

A nonempty set Q equipped with a binary operation $Q\times Q\to Q$, $(x,y)\mapsto x*y$ is called a *quandle* if it satisfies the following three conditions:

(1) $x*x=x$ $(x\in Q)$,
(2) $(x*y)*z=(x*z)*(y*z)$ $(x,y,z\in Q)$,
(3) For all $y\in Q$, the map $Q\to Q$ defined by $x\mapsto x*y$ is bijective.

If X satisfies (2) and (3) but not necessarily (1), then X is called a *rack*. Quandles and racks have been studied in low dimensional topology as well as in Hopf algebras (see Nosaka [19] and Andruskiewitsch-Graña [3] for instance). To any quandle or rack Q one can associate a group $\mathrm{Ad}(Q)$ called the *adjoint group* of Q (also called the associated group or the enveloping group in the literature). It is defined by the presentation

$$\mathrm{Ad}(Q):=\langle e_x\ (x\in Q)\mid e_y^{-1}e_xe_y=e_{x*y}\ (x,y\in Q)\rangle.$$

The assignment $Q\mapsto \mathrm{Ad}(Q)$ is a functor from the category of quandles or racks to the category of groups. Although adjoint groups play important roles in the study of quandles and racks, not much is known about the structure of them, partly because the definition of $\mathrm{Ad}(Q)$ by a possibly infinite presentation is difficult to work with in explicit calculations. We refer Eisermann [12] and Nosaka [18, 19] for generality of adjoint groups, and [9, 12, 18, 19] for descriptions of adjoint groups of certain classes of quandles.

In this paper, we will study the adjoint group of a Coxeter quandle. Let (W,S) be a Coxeter system, a pair of a Coxeter group W and the set S of Coxeter generators

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of W. Following Nosaka [18], we define the *Coxeter quandle* Q_W associated with (W,S) to be the set of all reflections of W:

$$Q_W := \bigcup_{w\in W} w^{-1}Sw.$$

The quandle operation is given by the conjugation $x*y := y^{-1}xy = yxy$. The symmetric group $\mathfrak{S}_n$ of n letters is a Coxeter group (of type A_{n-1}), and the associated Coxeter quandle $Q_{\mathfrak{S}_n}$ is nothing but the set of all transpositions. In their paper [2], Andruskiewitsch-Fantino-García-Vendramin obtained remarkable results concerning with $\mathrm{Ad}(Q_{\mathfrak{S}_n})$. Namely, they proved that $\mathrm{Ad}(Q_{\mathfrak{S}_n})$ is an intermediate group between $\mathfrak{S}_n$ and the braid group B_n of n strands, in the sense that the canonical projection $B_n \twoheadrightarrow \mathfrak{S}_n$ splits into a sequence of surjections $B_n \twoheadrightarrow \mathrm{Ad}(Q_{\mathfrak{S}_n}) \twoheadrightarrow \mathfrak{S}_n$, and that $\mathrm{Ad}(Q_{\mathfrak{S}_n})$ fits into a central extension of the form

$$0 \to \mathbb{Z} \to \mathrm{Ad}(Q_{\mathfrak{S}_n}) \to \mathfrak{S}_n \to 1. \tag{1.1}$$

See [2, Proposition 3.2] and the proofs therein. Furthermore, the central extension (1.1) turns out to be the unique nontrivial central extension of $\mathfrak{S}_n$ by $\mathbb{Z}$ (see Corollary 4.6).

The primary purpose of this paper is to generalize those results to arbitrary Coxeter quandles. We will show that $\mathrm{Ad}(Q_W)$ is an intermediate group between W and the Artin group A_W associated with W (Proposition 3.3), and that $\mathrm{Ad}(Q_W)$ fits into a central extension of the form

$$0 \to \mathbb{Z}^{c(W)} \to \mathrm{Ad}(Q_W) \xrightarrow{\phi} W \to 1, \tag{1.2}$$

where $c(W)$ is the number of conjugacy classes of elements in Q_W (Theorem 3.1). As a byproduct of Theorem 3.1, we will determine the rational cohomology ring of $\mathrm{Ad}(Q_W)$ (Corollary 3.6).

In case $c(W) = 1$, the central extension (1.2) turns out to be the unique nontrivial central extension of W by $\mathbb{Z}$ (Corollary 4.6). As is known, the central extension (1.2) corresponds to the cohomology class $u_\phi \in H^2(W,\mathbb{Z}^{c(W)})$. We will construct 2-cocycles of W representing u_ϕ (Proposition 4.3 and Theorem 4.8).

Alternatively, Eisermann [12] claimed that $\mathrm{Ad}(Q_{\mathfrak{S}_n})$ is isomorphic to the semidirect product $A_n \rtimes \mathbb{Z}$ where A_n is the alternating group on n letters, but he did not write down the proof. We will generalize his result to Coxeter quandles (Theorem 5.1 and Corollary 5.2).

In the final section, we deal with root systems. To each Coxeter system (W,S), one can associate the root system $\Phi_W \subset V$ by using the geometric representation $W \to GL(V)$, where V is a real vector space with the basis $\{\alpha_s \mid s \in S\}$. The root system Φ_W turns out to be a rack with the rack operation $\alpha * \beta := t_\beta(\alpha)$ where $t_\beta \in GL(V)$ is the reflection along β. We close this paper by proving $\mathrm{Ad}(\Phi_W) \cong \mathrm{Ad}(Q_W)$ (Theorem 6.3).

Notation. Let G be a group, $g,h \in G$ elements of G and $m \geq 2$ a natural number. Define

$$(gh)_m := \underbrace{ghghg\cdots}_{m} \in G.$$

For example, $(gh)_2 = gh, (gh)_3 = ghg, (gh)_4 = ghgh$. Let $G_{\mathrm{Ab}} := G/[G,G]$ be the abelianization of G, $\mathrm{Ab}_G : G \to G_{\mathrm{Ab}}$ the natural projection, and write $[g] := g[G,G] \in G_{\mathrm{Ab}}$ for $g \in G$. For a quandle Q and elements $x_k \in Q$ $(k = 1,2,3,\dots)$, we denote

$$x_1 * x_2 * x_3 * \cdots * x_\ell := (\cdots((x_1 * x_2) * x_3) * \cdots) * x_\ell,$$

for simplicity.

2. Coxeter groups and Coxeter quandles

2.1. **Coxeter groups.** Let S be a finite set and $m : S \times S \to \mathbb{N} \cup \{\infty\}$ a map satisfying the following conditions:

(1) $m(s,s) = 1$ for all $s \in S$;
(2) $2 \le m(s,t) = m(t,s) \le \infty$ for all distinct $s,t \in S$.

The map m is represented by the *Coxeter graph* Γ whose vertex set is S and whose edges are the unordered pairs $\{s,t\} \subset S$ such that $m(s,t) \ge 3$. The edges with $m(s,t) \ge 4$ are labeled by the number $m(s,t)$. The *Coxeter system* associated with Γ is the pair (W,S), where W is the group generated by $s \in S$ and the fundamental relations $(st)^{m(s,t)} = 1$ $(s,t \in S, m(s,t) < \infty)$:

$$W := \langle s \in S \mid (st)^{m(s,t)} = 1 (s,t \in S, m(s,t) < \infty) \rangle. \tag{2.1}$$

The group W is called the *Coxeter group* (of type Γ), and elements of S are called *Coxeter generators* of W (also called simple reflections in the literature). Note that the order of the product st is precisely $m(s,t)$. In particular, every Coxeter generator $s \in S$ has order 2. It is easy to check that the defining relations in (2.1) are equivalent to the following relations

$$s^2 = 1 \ (s \in S), \ (st)_{m(s,t)} = (ts)_{m(s,t)} \ (s,t \in S, s \neq t, m(s,t) < \infty). \tag{2.2}$$

Finally, the *odd subgraph* Γ_{odd} is a subgraph of Γ whose vertex set is S and whose edges are the unordered pairs $\{s,t\} \subset S$ such that $m(s,t)$ is an odd integer. We refer Björner-Brenti [4], Bourbaki [5], Davis [10], and Humphreys [15] for further details of Coxeter groups.

2.2. **Conjugacy classes of reflections.** Let

$$Q_W := \bigcup_{w \in W} w^{-1} S w$$

be the set of reflections in W as in the introduction (the underlying set of the Coxeter quandle). Let $\mathscr{O}_W$ be the set of conjugacy classes of elements of Q_W under W, and $\mathscr{R}_W$ a complete set of representatives of conjugacy classes. We may choose such that $\mathscr{R}_W \subseteq S$. Let $c(W)$ be the cardinality of $\mathscr{O}_W$. The following two propositions are well-known and easy to prove (see Björner-Brenti [4, Chapter 1, Exercises 16–17] for instance).

Proposition 2.1. *The elements of $\mathscr{O}_W$ are in one-to-one correspondence with the connected components of Γ_{odd}. Consequently, $c(W)$ equals to the number of connected components of Γ_{odd}.*

To be precise, the conjugacy class of $s\in S$ corresponds to the connected component of Γ_{odd} containing $s\in S$.

Proposition 2.2. *W_{Ab} is the elementary abelian 2-group with a basis $\{[s]\in W_{\mathrm{Ab}}\mid s\in\mathscr{R}_W\}$. In particular, $W_{\mathrm{Ab}}\cong(\mathbb{Z}/2)^{c(W)}$.*

2.3. **Coxeter quandles.** Now we turn our attention to Coxeter quandles. Let Q_W be the Coxeter quandle associated with (W,S) and

$$\mathrm{Ad}(Q_W):=\langle e_x\ (x\in Q_W)\mid e_y^{-1}e_xe_y=e_{x*y}\ (x,y\in Q_W)\rangle$$

the adjoint group of Q_W. Observe that

$$e_y^{-1}e_xe_y=e_{x*y}=e_{y^{-1}xy} \tag{2.3}$$

and

$$e_ye_xe_y^{-1}=e_{x\bar{*}y}, \tag{2.4}$$

where (2.4) follows from $e_y^{-1}e_{x\bar{*}y}e_y=e_{x\bar{*}y*y}=e_x$.

Proposition 2.3. *$\mathrm{Ad}(Q_W)$ is generated by e_s $(s\in S)$.*

Proof. Given $x\in Q_W$, we can express x as $x=(s_1s_2\cdots s_\ell)^{-1}s_0(s_1s_2\cdots s_\ell)$ for some $s_i\in S$ $(0\le i\le\ell)$ by the definition of Q_W. Applying (2.3), we have

$$e_x=e_{(s_1s_2\cdots s_\ell)^{-1}s_0(s_1s_2\cdots s_\ell)}=e_{s_0*s_1*s_2*\cdots*s_\ell}=(e_{s_1}e_{s_2}\cdots e_{s_\ell})^{-1}e_{s_0}(e_{s_1}e_{s_2}\cdots e_{s_\ell}),$$

proving the proposition. □

Proposition 2.4. *$\mathrm{Ad}(Q_W)_{\mathrm{Ab}}$ is the free abelian group with a basis $\{[e_s]\mid s\in\mathscr{R}_W\}$. In particular, $\mathrm{Ad}(Q_W)_{\mathrm{Ab}}\cong\mathbb{Z}^{c(W)}$.*

Proof. By the definition of $\mathrm{Ad}(Q_W)$, the abelianization $\mathrm{Ad}(Q_W)_{\mathrm{Ab}}$ is generated by $[e_x]$ $(x\in Q_W)$ subject to the relations $[e_x]=[e_{y^{-1}xy}]$, $[e_x][e_y]=[e_y][e_x]$ $(x,y\in Q_W)$. Consequently, $[e_x]=[e_y]\in\mathrm{Ad}(Q_W)_{\mathrm{Ab}}$ if and only if $x,y\in Q_W$ are conjugate in W. We conclude that $\mathrm{Ad}(Q_W)_{\mathrm{Ab}}$ is the free abelian group with a basis $\{[e_s]\mid s\in\mathscr{R}_W\}$. □

Let $\phi:\mathrm{Ad}(Q_W)\to W$ be a surjective homomorphism defined by $e_x\mapsto x$, which is well-defined by virtue of (2.3), and let $C_W:=\ker\phi$ be its kernel.

Lemma 2.5. *C_W is a central subgroup of $\mathrm{Ad}(Q_W)$.*

Proof. Given $g\in C_W$, it suffices to prove $g^{-1}e_xg=e_x$ for all $x\in Q_W$. To do so, set $g=e_{y_1}^{\varepsilon_1}e_{y_2}^{\varepsilon_2}\cdots e_{y_\ell}^{\varepsilon_\ell}$ $(\varepsilon_i\in\{\pm1\})$. Applying (2.3) and (2.4), we have

$$g^{-1}e_xg=(e_{y_1}^{\varepsilon_1}e_{y_2}^{\varepsilon_2}\cdots e_{y_\ell}^{\varepsilon_\ell})^{-1}e_x(e_{y_1}^{\varepsilon_1}e_{y_2}^{\varepsilon_2}\cdots e_{y_\ell}^{\varepsilon_\ell})=e_{x*y_1*y_2*\cdots*y_\ell}.$$

Now $x*y_1*y_2*\cdots*y_\ell=(y_1y_2\cdots y_\ell)^{-1}x(y_1y_2\cdots y_\ell)$, and the proposition follows from $\phi(g)=y_1^{\varepsilon_1}y_2^{\varepsilon_2}\cdots y_\ell^{\varepsilon_\ell}=y_1y_2\cdots y_\ell=1$. □

Lemma 2.6. *If $x,y\in Q_W$ are conjugate in W, then $e_x^2=e_y^2\in C_W$.*

Proof. It is obvious that $e_x^2\in C_W$ for all $x\in Q_W$. Since C_W is a central subgroup, $e_x^2=e_y^{-1}e_x^2e_y=(e_y^{-1}e_xe_y)^2=e_{y^{-1}xy}^2$ holds for all $x,y\in Q_W$, which implies the lemma. □

3. Artin groups and the proof of the main result

Now we state the main result of this paper:

Theorem 3.1. *The central subgroup C_W is the free abelian group with a basis $\{e_s^2 \mid s \in \mathscr{R}_W\}$. In particular, $C_W \cong \mathbb{Z}^{c(W)}$.*

As a consequence of Theorem 3.1, $\mathrm{Ad}(Q_W)$ fits into a central extension of the form $0 \to \mathbb{Z}^{c(W)} \to \mathrm{Ad}(Q_W) \to W \to 1$ as stated in the introduction. We begin with the determination of the rank of C_W.

Proposition 3.2. $\mathrm{rank}(C_W) = c(W)$.

Proof. The central extension $1 \to C_W \to \mathrm{Ad}(Q_W) \xrightarrow{\phi} W \to 1$ yields the following exact sequence for the rational homology of groups:

$$H_2(W,\mathbb{Q}) \to H_1(C_W,\mathbb{Q})_W \to H_1(\mathrm{Ad}(Q_W),\mathbb{Q}) \to H_1(W,\mathbb{Q})$$

(see Brown [8, Corollary VII.6.4]). Here the co-invariants $H_1(C_W,\mathbb{Q})_W$ coincides with $H_1(C_W,\mathbb{Q})$ because C_W is a central subgroup of $\mathrm{Ad}(Q_W)$. It is known that the rational (co)homology of a Coxeter group is trivial (see Akita [1, Proposition 5.2] or Davis [10, Theorem 15.1.1]). As a result, we have an isomorphism $H_1(C_W,\mathbb{Q}) \cong H_1(\mathrm{Ad}(Q_W),\mathbb{Q})$ and hence we have

$$\begin{aligned}\mathrm{rank}(C_W) &= \dim_{\mathbb{Q}}(C_W \otimes \mathbb{Q}) = \dim_{\mathbb{Q}} H_1(C_W,\mathbb{Q}) \\ &= \dim_{\mathbb{Q}} H_1(\mathrm{Ad}(Q_W),\mathbb{Q}) = \dim_{\mathbb{Q}}(\mathrm{Ad}(Q_W)_{\mathrm{Ab}} \otimes \mathbb{Q}) = c(W)\end{aligned}$$

as desired. □

To proceed further, we need the notions of Artin groups and pure Artin groups. Given a Coxeter system (W,S), the *Artin group* A_W associated with (W,S) is the group defined by the presentation

$$A_W := \langle a_s\ (s \in S) \mid (a_s a_t)_{m(s,t)} = (a_t a_s)_{m(s,t)}\ (s,t \in S, s \neq t, m(s,t) < \infty)\rangle.$$

In view of (2.2), there is an obvious surjective homomorphism $\pi : A_W \to W$ defined by $a_s \mapsto s$ $(s \in S)$. The *pure Artin group* P_W associated with (W,S) is defined to be the kernel of π so that there is an extension

$$1 \to P_W \hookrightarrow A_W \xrightarrow{\pi} W \to 1.$$

In case W is the symmetric group on n letters, A_W is the braid group of n strands, and P_W is the pure braid group of n strands. Artin groups were introduced by Brieskorn-Saito [7]. Little is known about the structure of general Artin groups. Among others, the following questions are still open.

(1) Are Artin groups torsion free?
(2) What is the center of Artin groups?
(3) Do Artin groups have solvable word problem?
(4) Are there finite $K(\pi,1)$-complexes for Artin groups?

See survey articles by Paris [20–22] for further details of Artin groups.

Proposition 3.3. *The assignment $a_s \mapsto e_s$ $(s \in S)$ yields a well-defined surjective homomorphism $\psi : A_W \to \mathrm{Ad}(Q_W)$.*

Proof. As for the well-definedness, it suffices to show $(e_s e_t)_{m(s,t)} = (e_t e_s)_{m(s,t)}$ for all distinct $s,t \in S$ with $m(s,t) < \infty$. Applying the relation $e_x e_y = e_y e_{x*y}$ $(x,y \in Q_W)$ repeatedly as in

$$(e_t e_s)_{m(s,t)} = e_t e_s e_t e_s \cdots = e_s e_{t*s} e_t e_s \cdots = e_s e_t e_{t*s*t} e_s \cdots = \cdots,$$

we obtain

$$(e_t e_s)_{m(s,t)} = (e_s e_t)_{m(s,t)-1} e_x,$$

where

$$\begin{aligned} x = \overbrace{t*s*t*s*\cdots}^{m(s,t)} &= (st)^{-1}_{m(s,t)-1} t (st)_{m(s,t)-1} \\ &= (st)^{-1}_{m(s,t)-1} (ts)_{m(s,t)} \\ &= (st)^{-1}_{m(s,t)-1} (st)_{m(s,t)}. \end{aligned}$$

In the last equality we used the relation $(st)_{m(s,t)} = (ts)_{m(s,t)}$. It follows that x is the last letter in $(st)_{m(s,t)}$, i.e. $x = s$ or $x = t$ according as $m(s,t)$ is odd or $m(s,t)$ is even. We conclude that $(e_s e_t)_{m(s,t)-1} e_x = (e_s e_t)_{m(s,t)}$ as desired. Finally, the surjectivity follows from Proposition 2.3. □

As a result, the adjoint group $\mathrm{Ad}(Q_W)$ is an intermediate group between a Coxeter group W and the corresponding Artin group A_W, in the sense that the canonical surjection $\pi : A_W \twoheadrightarrow W$, $a_s \mapsto s$ $(s \in S)$ splits into a sequence of surjections

$$A_W \overset{\psi}{\twoheadrightarrow} \mathrm{Ad}(Q_W) \overset{\phi}{\twoheadrightarrow} W, \; a_s \mapsto e_s \mapsto s \; (s \in S).$$

Proposition 3.4. *P_W is the normal closure of $\{a_s^2 \mid s \in S\}$ in A_W. In other words, P_W is generated by $g^{-1} a_s^2 g$ $(s \in S, g \in A_W)$.*

Proof. Given a Coxeter system (W,S), let $F(S)$ be the free group on S and put

$$R := \{(st)_{m(s,t)} (ts)^{-1}_{m(s,t)} \mid s,t \in S, s \neq t, m(s,t) < \infty\}, \; Q := \{s^2 \mid s \in S\}.$$

Let $N(R), N(Q), N(R \cup S)$ be the normal closure of $R, Q, R \cup Q$ in $F(S)$, respectively. The third isomorphism theorem yields a short exact sequence of groups

$$1 \to \frac{N(R)N(Q)}{N(R)} \to \frac{F(S)}{N(R)} \overset{p}{\to} \frac{F(S)}{N(R)N(Q)} \to 1.$$

Observe that the left term $N(R)N(Q)/N(R)$ is nothing but the normal closure of Q in $F(S)/N(R)$. Now $F(S)/N(R)N(Q) = F(S)/N(R \cup Q) = W$ by the definition of W and $F(S)/N(R)$ is identified with A_W via $s \mapsto a_s$ $(s \in S)$. Under this identification, the map p is the canonical surjection $\pi : A_W \twoheadrightarrow W$ and hence the left term $N(R)N(Q)/N(R)$ coincides with P_W. The proposition follows. □

Remark 3.5. Digne-Gomi [11, Corollary 6] obtained a presentation of P_W by using Reidemeister-Schreier method. Their presentation is infinite whenever W is infinite. Although Proposition 3.4 may be read off from their presentation, we wrote down the proof because our proof is much simpler than the arguments in [11].

Proof of Theorem 3.1. Consider the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P_W & \hookrightarrow & A_W & \xrightarrow{\pi} & W & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \psi|_{P_W}} & & \downarrow{\scriptstyle \psi} & & \| & & \\ 1 & \longrightarrow & C_W & \hookrightarrow & \mathrm{Ad}(Q_W) & \xrightarrow{\phi} & W & \longrightarrow & 1 \end{array}$$

whose rows are exact. Since $\psi : A_W \to \mathrm{Ad}(Q_W)$ is surjective, one can check that its restriction $\psi|_{P_W} : P_W \to C_W$ is also surjective. As P_W is generated by $g^{-1}a_s^2 g$ $(s \in S, g \in A_W)$ by Proposition 3.4, C_W is generated by

$$\psi(g^{-1}a_s^2 g) = \psi(g)^{-1}e_s^2\psi(g) = e_s^2 \ (s \in S, g \in A_W),$$

where the last equality follows from the fact that C_W is central by Lemma 2.5. Combining with Lemma 2.6, we see that C_W is generated by $\{e_s^2 \mid s \in \mathscr{R}_W\}$. Now $\mathrm{rank}(C_W) = c(W)$ by Proposition 3.2 and $|\mathscr{R}_W| = c(W)$ by the defintion of $\mathscr{R}_W$, C_W must be a free abelian group of rank $c(W)$ and $\{e_s^2 \mid s \in \mathscr{R}_W\}$ must be a basis of C_W. □

As an immediate cosequence of Theorem 3.1, we can determine the rational cohomology ring of $\mathrm{Ad}(Q_W)$:

Corollary 3.6. *For any Coxeter system (W,S), the inclusion $C_W \hookrightarrow \mathrm{Ad}(Q_W)$ induces an isomorphism*

$$H^*(\mathrm{Ad}(Q_W),\mathbb{Q}) \xrightarrow{\cong} H^*(C_W,\mathbb{Q}) \cong \wedge_{\mathbb{Q}}(u_1,u_2,\ldots,u_{c(W)})$$

with $\deg u_k = 1$ $(1 \le k \le c(W))$.

Proof. In the Lyndon-Hochschild-Serre spectral sequence

$$E_2^{pq} = H^p(W, H^q(C_W,\mathbb{Q})) \Rightarrow H^{p+q}(\mathrm{Ad}(Q_W),\mathbb{Q})$$

associated with the central extension $1 \to C_W \to \mathrm{Ad}(Q_W) \to W \to 1$, one has

$$E_2^{pq} \cong H^p(W,\mathbb{Q}) \otimes_{\mathbb{Q}} H^q(C_W,\mathbb{Q}) \cong \begin{cases} H^q(C_W,\mathbb{Q}) & p = 0 \\ 0 & p \neq 0 \end{cases}$$

because $H^p(W,\mathbb{Q}) = 0$ for $p \neq 0$ (see the proof of Proposition 3.2). The first isomorphism follows immediately. The second isomorphism follows from the fact $C_W \cong \mathbb{Z}^{c(W)}$. □

Remark 3.7. For any right-angled Coxeter group W, Kishimoto [16, Theorem 5.3] determined the mod 2 cohomology ring of $\mathrm{Ad}(Q_W)$.

4. Construction of 2-cocycles

Throughout this section, we assume that the reader is familiar with group cohomology and Coxeter groups. The central extension

$$1 \to C_W \to \mathrm{Ad}(Q_W) \xrightarrow{\phi} W \to 1 \tag{4.1}$$

corresponds to the cohomology class $u_\phi \in H^2(W, C_W)$ (see Brown [8, §IV.3] for precise). In this section, we will construct 2-cocycles representing u_ϕ. Before doing so, we claim $u_\phi \neq 0$. The claim is equivalent to the following lemma.

Lemma 4.1. *The central extension* $1 \to C_W \to \mathrm{Ad}(Q_W) \xrightarrow{\phi} W \to 1$ *is nontrivial.*

Proof. If the central extension is trivial, then $\mathrm{Ad}(Q_W) \cong C_W \times W$. But this is not the case because $\mathrm{Ad}(Q_W)_{\mathrm{Ab}} \cong \mathbb{Z}^{c(W)}$ by Proposition 2.4 while $(C_W \times W)_{\mathrm{Ab}} \cong C_W \times W_{\mathrm{Ab}} \cong \mathbb{Z}^{c(W)} \times (\mathbb{Z}/2)^{c(W)}$ by Proposition 2.2. □

Now we invoke the celebrated Matsumoto's theorem:

Theorem 4.2 (Matsumoto [17])**.** *Let* (W, S) *be a Coxeter system,* M *a monoid and* $f : S \to M$ *a map such that* $(f(s)f(t))_{m(s,t)} = (f(t)f(s))_{m(s,t)}$ *for all* $s, t \in S$ *with* $s \neq t$, $m(s,t) < \infty$. *Then there exists a unique map* $F : W \to M$ *such that* $F(w) = f(s_1) \cdots f(s_k)$ *whenever* $w = s_1 \cdots s_k$ $(s_i \in S)$ *is a reduced expression.*

The proof also can be found in [5, Chapitre IV, §1, Proposition 5] and [14, Theorem 1.2.2]. Define a map $f : S \to \mathrm{Ad}(Q_W)$ by $s \mapsto e_s$, then f satisfies the assumption of Theorem 4.2 as in the proof of Proposition 3.3, and hence there exists a unique map $F : W \to \mathrm{Ad}(Q_W)$ such that $F(w) = f(s_1) \cdots f(s_k) = e_{s_1} \cdots e_{s_k}$ whenever $w = s_1 \cdots s_k$ $(s_i \in S)$ is a reduced expression. It is clear that $F : W \to \mathrm{Ad}(Q_W)$ is a set-theoretical section of $\phi : \mathrm{Ad}(Q_W) \to W$. Define $c : W \times W \to C_W$ by

$$c(w_1, w_2) = F(w_1)F(w_2)F(w_1 w_2)^{-1}.$$

The standard argument in group cohomology (see Brown [8, §IV.3]) implies the following result:

Proposition 4.3. c *is a normalized* 2*-cocycle and* $[c] = u_\phi \in H^2(W, C_W)$.

Remark 4.4. In case W is the symmetric group of n letters $\mathfrak{S}_n$, Proposition 4.3 was stated in [2, Remark 3.3].

Now we deal with the case $c(W) = 1$ more precisely. If $c(W) = 1$ then the odd subgraph Γ_{odd} of W is connected and hence W must be irreducible. All finite irreducible Coxeter groups of type other than B_n $(n \geq 2)$, F_4, $I_2(m)$ (m even) satisfy $c(W) = 1$. Among affine irreducible Coxeter groups, those of type $\widetilde{A}_n$ $(n \geq 2)$, $\widetilde{D}_n$ $(n \geq 4)$, $\widetilde{E}_6$, $\widetilde{E}_7$ and $\widetilde{E}_8$ fulfill $c(W) = 1$. For simplifying the notation, we will identify C_W with $\mathbb{Z}$ by $e_s^2 \mapsto 1$ and denote our central extension by

$$0 \to \mathbb{Z} \to \mathrm{Ad}(Q_W) \xrightarrow{\phi} W \to 1.$$

Proposition 4.5. *If* $c(W) = 1$ *then* $H^2(W, \mathbb{Z}) \cong \mathbb{Z}/2$.

Proof. A short exact sequence $0 \to \mathbb{Z} \hookrightarrow \mathbb{C} \to \mathbb{C}^\times \to 1$ of abelian groups, where $\mathbb{C} \to \mathbb{C}^\times$ is defined by $z \mapsto \exp(2\pi\sqrt{-1}z)$, induces the exact sequence

$$H^1(W, \mathbb{C}) \to H^1(W, \mathbb{C}^\times) \xrightarrow{\delta} H^2(W, \mathbb{Z}) \to H^2(W, \mathbb{C})$$

(see Brown [8, Proposition III.6.1]). It is known that $H^k(W,\mathbb{C}) = 0$ for $k > 0$ (see the proof of Proposition 3.2), which implies that the connecting homomorphism $\delta : H^1(W,\mathbb{C}^\times) \to H^2(W,\mathbb{Z})$ is an isomorphism. We claim that $H^1(W,\mathbb{C}^\times) = \mathrm{Hom}(W,\mathbb{C}^\times) \cong \mathbb{Z}/2$. Indeed, W is generated by S consisting of elements of order 2, and all elements of S are mutually conjugate by the assumption $c(W) = 1$. Thus $\mathrm{Hom}(W,\mathbb{C}^\times)$ consists of the trivial homomorphism and the homomorphism ρ defined by $\rho(s) = -1$ $(s \in S)$. □

Combining Lemma 4.1 and Proposition 4.5, we obtain the following corollary.

Corollary 4.6. *If $c(W) = 1$ then $0 \to \mathbb{Z} \to \mathrm{Ad}(Q_W) \xrightarrow{\phi} W \to 1$ is the unique non-trivial central extension of W by $\mathbb{Z}$.*

In general, given a homomorphism of groups $f : G \to \mathbb{C}^\times$, the cohomology class $\delta f \in H^2(G,\mathbb{Z})$, where $\delta : \mathrm{Hom}(G,\mathbb{C}^\times) \to H^2(G,\mathbb{Z})$ is the connecting homomorphism as above, can be described as follows. For each $g \in G$, choose a branch of $\log(f(g))$. We argue such that $\log(f(1)) = 0$. Define $\tau_f : G \times G \to \mathbb{Z}$ by

$$\tau_f(g,h) = \frac{1}{2\pi\sqrt{-1}}\{\log(f(g)) + \log(f(h)) - \log(f(gh))\} \in \mathbb{Z}. \tag{4.2}$$

By a diagram chase, one can prove the following:

Proposition 4.7. *τ_f is a normalized 2-cocycle and $[\tau_f] = \delta f \in H^2(G,\mathbb{Z})$.*

Assuming $c(W) = 1$, let $\rho : W \to \mathbb{C}^\times$ be the homomorphism defined by $\rho(s) = -1$ $(s \in S)$ as in the proof of Proposition 4.5. Note that $\rho(w) = (-1)^{\ell(w)}$ $(w \in W)$ where $\ell(w)$ is the *length* of w (see Humphreys [15, §5.2]). For each $w \in W$, choose a branch of $\log(\rho(w))$ as

$$\log(\rho(w)) = \begin{cases} 0 & \text{if } \ell(w) \text{ is even,} \\ \pi\sqrt{-1} & \text{if } \ell(w) \text{ is odd.} \end{cases}$$

Applying (4.2), the corresponding map $\tau_\rho : W \times W \to \mathbb{Z}$ is given by

$$\tau_\rho(w_1,w_2) = \begin{cases} 1 & \text{if } \ell(w_1) \text{ and } \ell(w_2) \text{ are odd,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

Combining Lemma 4.1, Corollary 4.6, and Proposition 4.7, we obtain the following theorem:

Theorem 4.8. *If $c(W) = 1$ then $\tau_\rho : W \times W \to \mathbb{Z}$ defined by (4.3) is a normalized 2-cocycle and $[\tau_\rho] = u_\phi \in H^2(W,\mathbb{Z})$.*

5. Commutator subgroups of adjoint groups

As was stated in the introduction, Eisermann [12, Example 1.18] claimed that $\mathrm{Ad}(Q_{\mathfrak{S}_n})$ is isomorphic to the semidirect product $A_n \rtimes \mathbb{Z}$ where A_n is the alternating group on n letters. We will generalize his result to Coxeter quandles by showing the following theorem:

Theorem 5.1. *$\phi : \mathrm{Ad}(Q_W) \to W$ induces an isomorphism*

$$\phi : [\mathrm{Ad}(Q_W), \mathrm{Ad}(Q_W)] \xrightarrow{\cong} [W, W].$$

Proof. Consider the following commutative diagram with exact rows and columns.

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C_W & \xrightarrow[\cong]{\mathrm{Ab}_{\mathrm{Ad}(Q_W)}} & \ker \phi_{\mathrm{Ab}} & & \\
 & & & & \big\downarrow & & \big\downarrow & & \\
1 & \longrightarrow & [\mathrm{Ad}(Q_W), \mathrm{Ad}(Q_W)] & \hookrightarrow & \mathrm{Ad}(Q_W) & \xrightarrow{\mathrm{Ab}_{\mathrm{Ad}(Q_W)}} & \mathrm{Ad}(Q_W)_{\mathrm{Ab}} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi_{\mathrm{Ab}}} & & \\
1 & \longrightarrow & [W, W] & \hookrightarrow & W & \xrightarrow{\mathrm{Ab}_W} & W_{\mathrm{Ab}} & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

From Proposition 2.2 and Proposition 2.4, we see that $\ker \phi_{\mathrm{Ab}}$ is the free abelian group with a basis $\{[e_s^2] \mid s \in \mathscr{R}_W\}$, which implies that $\mathrm{Ab}_{\mathrm{Ad}(Q_W)} : C_W \to \ker \phi_{\mathrm{Ab}}$ is an isomorphism because it assigns $[e_s^2]$ to e_s^2 $(s \in \mathscr{R}_W)$. Since ϕ is surjective, it is obvious that $\phi([\mathrm{Ad}(Q_W), \mathrm{Ad}(Q_W)]) = [W, W]$. We will show that $\phi : [\mathrm{Ad}(Q_W), \mathrm{Ad}(Q_W)] \to [W, W]$ is injective. To do so, let $g \in [\mathrm{Ad}(Q_W), \mathrm{Ad}(Q_W)]$ be an element with $g \in \ker \phi = C_W$. Then $[g] := \mathrm{Ab}_{\mathrm{Ad}(Q_W)}(g) = 1$ by the exactness of the middle row. But it implies $g = 1$ because $\mathrm{Ab}_{\mathrm{Ad}(Q_W)} : C_W \to \ker \phi_{\mathrm{Ab}}$ is an isomorphism. ◻

Corollary 5.2. *There is a group extension of the form*

$$1 \to [W, W] \to \mathrm{Ad}(Q_W) \xrightarrow{\mathrm{Ab}_{\mathrm{Ad}(Q_W)}} \mathbb{Z}^{c(W)} \to 1.$$

If $c(W) = 1$ then the extension splits and $\mathrm{Ad}(Q_W) \cong [W, W] \rtimes \mathbb{Z}$.

Since $c(\mathfrak{S}_n) = 1$ and $[\mathfrak{S}_n, \mathfrak{S}_n] = A_n$, we recover the claim by Eisermann mentioned above.

Remark 5.3. By using Theorem 5.1, Kishimoto [16, Theorem 2.5] proved that the following commutative square which appeared in the proof of Theorem 5.1 is actually a pull-back.

$$\begin{array}{ccc}
\mathrm{Ad}(Q_W) & \xrightarrow{\mathrm{Ab}_{\mathrm{Ad}(Q_W)}} & \mathrm{Ad}(Q_W)_{\mathrm{Ab}} \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi_{\mathrm{Ab}}} \\
W & \xrightarrow{\mathrm{Ab}_W} & W_{\mathrm{Ab}}
\end{array}$$

6. Root systems

As was pointed out by Andruskiewitsch-Graña [3, §1.3.5], Brieskorn [6, p. 58, Example 9], and Fenn-Rourke [13, Example 10], a root system turns out to be a rack. In this final section, we turn our attention to root systems. Let us recall the definition and properties of the root system associated with a Coxeter system (W,S). See Björner-Brenti [4, Chapter 4] or Humphreys [15, Part II, Chapter 5] for precise. Let V be the real vector space with a basis $\{\alpha_s \mid s \in S\}$ in one to one correspondence with S. Define a symmetric bilinear form $B(-,-)$ on V by

$$B(\alpha_s, \alpha_t) := -\cos\frac{\pi}{m(s,t)}.$$

Here we understand $B(\alpha_s, \alpha_t) = -1$ in case $m(s,t) = \infty$. For each $s \in S$, define a reflection $\sigma_s \in GL(V)$ by

$$\sigma_s(\lambda) = \lambda - 2B(\alpha_s, \lambda)\alpha_s.$$

The *geometric representation* of W (or the canonical representation in the literature) is the unique homomorphism $\sigma : W \to GL(V)$ satisfying $s \mapsto \sigma_s$ $(s \in S)$. The geometric representaion is faithful and preserves the form $B(-,-)$. For simplicity, we may write $w(\lambda)$ in place of $\sigma(w)(\lambda)$ $(w \in W, \lambda \in V)$. The *root system* Φ_W associated with (W,S) is defined by

$$\Phi_W := \{w(\alpha_s) \mid s \in S, w \in W\}.$$

For each root $\alpha \in \Phi_W$, define a reflection $t_\alpha \in GL(V)$ by $t_\alpha(\lambda) := \lambda - 2B(\alpha,\lambda)\alpha$. Note that if $\alpha = \alpha_s$ $(s \in S)$ then $t_\alpha = \sigma_s$.

Lemma 6.1. *The root system Φ_W turns out to be a rack with the rack operation $\alpha * \beta = t_\beta(\alpha)$ $(\alpha, \beta \in \Phi_W)$.*

Proof. It suffices to prove $(\alpha*\beta)*\gamma = (\alpha*\gamma)*(\beta*\gamma)$ for $\alpha,\beta,\gamma \in \Phi_W$. One has

$$(\alpha*\beta)*\gamma = t_\gamma(\alpha - 2B(\beta,\alpha)\beta) = t_\gamma(\alpha) - 2B(\beta,\alpha)t_\gamma(\beta)$$

while

$$(\alpha*\gamma)*(\beta*\gamma) = t_\gamma(\alpha)*t_\gamma(\beta) = t_\gamma(\alpha) - 2B(t_\gamma(\beta), t_\gamma(\alpha))t_\gamma(\beta),$$

and the assertion follows from $B(t_\gamma(\beta), t_\gamma(\alpha)) = B(\beta,\alpha)$. Note that Φ_W is not a quandle because $\alpha*\alpha = t_\alpha(\alpha) = -\alpha$. □

Now let Q_W be the Coxeter quandle associated with a Coxeter system (W,S) as before, and define a map $p : \Phi_W \to Q_W$ by $\alpha \mapsto t_\alpha$. Here we identify elements of $\sigma(W)$ with those of W, which is possible since σ is injective. Note that t_α belongs to Q_W because $t_\alpha = wsw^{-1}$ holds for $\alpha = w(\alpha_s)$, and that the map p is surjective and two-to-one (see Humphreys [15, p. 116] or Björner-Brenti [4, p. 104]). Indeed, p maps $\pm\alpha \in \Phi_W$ to the same element $t_\alpha = t_{-\alpha}$.

Lemma 6.2. *The map $p\colon \Phi_W \to Q_W$ is a morphism of racks.*

Proof. In general, $wt_\alpha w^{-1} = t_{w(\alpha)}$ holds for $\alpha \in \Phi_W$ and $w \in W$ (see Björner-Brenti [4, p. 104]). Now for $\alpha, \beta \in \Phi_W$, set $\gamma := t_\beta(\alpha)$ and we have

$$p(\alpha*\beta) = p(\gamma) = t_\gamma = t_\beta t_\alpha t_\beta^{-1} = t_\beta^{-1} t_\alpha t_\beta = t_\alpha * t_\beta = p(\alpha)*p(\beta)$$

as desired. □

We close this paper by proving the following result:

Theorem 6.3. *The morphism* $p\colon \Phi_W \to Q_W$ *of racks induces an isomorphism* $\mathrm{Ad}(\Phi_W) \cong \mathrm{Ad}(Q_W)$.

Proof. Recall that $\mathrm{Ad}(\Phi_W)$ is defined by the presentation

$$\mathrm{Ad}(\Phi_W) = \langle e_\alpha\ (\alpha \in \Phi_W) \mid e_{\alpha*\beta} = e_\beta^{-1} e_\alpha e_\beta \rangle.$$

For each $\alpha \in \Phi_W$, we have $e_{\alpha*\alpha} = e_{-\alpha}$ while $e_\alpha^{-1} e_\alpha e_\alpha = e_\alpha$, which proves that $e_{-\alpha} = e_\alpha$. On the other hand, as $p : \Phi_W \to Q_W$ is surjective, $\mathrm{Ad}(Q_W)$ may be defined as

$$\mathrm{Ad}(Q_W) = \langle e_{p(\alpha)}\ (\alpha \in \Phi_W) \mid e_{p(\alpha)*p(\beta)} = e_{p(\beta)}^{-1} e_{p(\alpha)} e_{p(\beta)} \rangle.$$

Since $p^{-1}(\alpha) = \{\pm\alpha\}$ and $p(\alpha * \beta) = p(\alpha) * p(\beta)$ by Lemma 6.2, the assignment $e_\alpha \mapsto e_{p(\alpha)}$ $(\alpha \in \Phi_W)$ induces an isomorphism $\mathrm{Ad}(\Phi_W) \to \mathrm{Ad}(Q_W)$. □

Remark 6.4. For a (crystallographic) root system Φ and its Weyl group $W(\Phi)$, one can prove that $\mathrm{Ad}(\Phi) \cong \mathrm{Ad}(Q_{W(\Phi)})$ in a similar way.

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