General-affine invariants of plane curves and space curves

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Abstract

We present a fundamental theory of curves in the affine plane and the affine space, equipped with the general-affine groups $\mathrm{GA}(2) = \mathrm{GL}(2, \mathbf{R}) \ltimes \mathbf{R}^2$ and $\mathrm{GA}(3) = \mathrm{GL}(3, \mathbf{R}) \ltimes \mathbf{R}^3$, respectively. We define general-affine length parameter and curvatures and show how such invariants determine the curve up to general-affine motions. We then study the extremal problem of the general-affine length functional and derive a variational formula. We give several examples of curves and also discuss some relations with equiaffine treatment and projective treatment of curves.

1 Introduction

Let $\mathbf{A}^n$ be the affine n-space with coordinates $x = (x^1, \dots, x^n)$. It is called a unimodular affine space if it is equipped with a parallel volume element, namely a determinant function. The unimodular affine group $\mathrm{SA}(n) = \mathrm{SL}(n, \mathbf{R}) \ltimes \mathbf{R}^n$ acts as

$$x = (x^i) \longrightarrow gx + a = \Big(\sum_j g^i_j x^j + a^i \Big), \quad g = (g^i_j) \in \mathrm{SL}(n, \mathbf{R}),\ a = (a^i) \in \mathbf{R}^n,$$

which preserves the volume element. Study of geometric properties of submanifolds in $\mathbf{A}^n$ invariant under this group is called equiaffine differential geometry, while the study of properties invariant under the general affine group $\mathrm{GA}(n) = \mathrm{GL}(n, \mathbf{R}) \ltimes \mathbf{R}^n$ is called

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general-affine differential geometry. Furthermore, the study of the geometric properties of submanifolds in the projective space $\mathbf{P}^n$ invariant under the projective linear group $\mathrm{PGL}(n) = \mathrm{GL}(n+1, \mathbf{R})/\text{center}$ is called projective differential geometry. Equiaffine differential geometry, as well as projective differential geometry, has long been studied and has yielded a plentiful amount of results, especially for curves and hypersurfaces; we refer to [Bl, Sc, NS] and [Wi, La, Bol] to name a few references. However, the study of general-affine differential geometry is little known even for curves. The purpose of this paper is to present a basic study of plane curves and space curves in general-affine differential geometry by recalling old results and by adding some new results. In addition, we relate them with the curve theory in equiaffine and projective differential geometry. Although study of invariants of curves of higher-codimension could possibly be given by a similar formulation used in this paper, it probably requires a more complicated presentation and is not attempted here. For the studies of submanifolds in the affine space which correspond to other types of subgroups of $\mathrm{PGL}(n)$, we refer to, *e.g.*, [Sc].

Let us begin with plane curves relative to $\mathrm{SA}(2)$. Let $\mathbf{A}^2$ be the unimodular affine plane with the determinant function $|x\ y| = x^1y^2 - x^2y^1$ for two vectors $x = (x^1, x^2)$ and $y = (y^1, y^2)$, and let $x(t)$ be a curve with parameter t into $\mathbf{A}^2$, which is nondegenerate in the sense that $|x'\ x''| \neq 0$. When $|x'\ x''| = 1$, furthermore, the parameter t is called an equiaffine length parameter. In this case, it holds that $|x'\ x'''| = 0$, namely x''' is linearly dependent on x' and we can write this dependence as

$$x''' = -k_a x',$$

where k_a is a scalar-valued function called the equiaffine curvature. Conversely, given a differential equation of this form, the map defined by two linearly independent non-constant solutions defines a curve whose equiaffine curvature is k_a.

With reference to this presentation of equiaffine notions, we first define the general-affine length parameter and the general-affine curvature relative to the full affine group $\mathrm{GA}(2)$ for a nondegenerate curve in Section 2. In contrast to the differential equation above, we have

$$x''' = -\frac{3}{2}kx'' - \left(\epsilon + \frac{1}{2}k' + \frac{1}{2}k^2\right)x',$$

where k is the general-affine curvature and $\epsilon = \pm 1$ denotes additional information of the curve; see Section 2.3. Conversely, for a given function k and $\epsilon = \pm 1$, there exists a nondegenerate curve x for which x satisfies the above equation and the curvature of x is k, uniquely up to a general-affine motion (Theorem 2.5). We then give remarks on the total curvature of closed curves and on the sextactic points (Corollary 2.7), and we study how to compute the curvature. In particular, we give an expression of the curvature of graph immersions and classify plane curves with constant general-affine curvature (Proposition 2.15), and discuss some relations with equiaffine treatment and projective treatment of plane curves.

We next consider an extremal problem of the general-affine length functional and derive a variational formula, a nonlinear ordinary differential equation that characterizes an extremal plane curve, as

$$k''' + \frac{3}{2}kk'' + \frac{1}{2}k'^2 + \frac{1}{2}k^2k' + \epsilon k' = 0$$

(Proposition 3.1). Here we give remarks on the preceding studies: the formulation used to define general-affine curvature of plane curves in this paper is very similar to that in [Mi1]. For example, the ordinary differential equation above for x and Theorem 2.5 were already given in [Mi1]. The formula of the general-affine plane curvature was given also in [Sc, OST] in a different context. The variational formula above for k was first given in [Mi2, (33)], though some modifications are necessary. The same formula for $\epsilon = 1$ was then rediscovered by S. Verpoort [Ve, p.432] in the equiaffine setting. Furthermore, the author of [Ve] gave a relation of solutions of this nonlinear equation with the coordinate functions of the immersion; we reprove this relation in Corollary 3.2.

We then remark that the differential equation above is very similar to some of the nonlinear differential equations of Chazy type. Furthermore we derive the variational formula for a certain generalized curvature functional (Theorem 3.6).

When the curve is given as a graph immersion, the general-affine curvature is written as a nonlinear form of a certain intermediate function. It is interesting to obtain a graph immersion from a given curvature function, which is treated in Section 4: We see that its integration can be reduced to solving the Abel equations of the first kind and the second kind (Theorem 4.1).

The second aim of this paper is to study general-affine invariants of space curves. The procedure is similar to that for the plane curves. We give the definition of general-affine space curvatures of two kinds k and M, and the ordinary differential equation of rank four

$$x'''' = -3kx''' - \left(2k' + \frac{11}{4}k^2 + \epsilon\right)x'' - \left(M + \frac{1}{2}\epsilon k + \frac{1}{2}k'' + \frac{7}{4}kk' + \frac{3}{4}k^3\right)x'$$

(Lemma 5.3), which defines the immersion. Then we show how to obtain curvatures, and discuss relations with the equiaffine and projective treatment of space curves. In particular, we give a list of nondegenerate space curves of constant general-affine curvature and a new proof of the theorem that a nondegenerate curve in the affine 3-space is extremal relative to the equiaffine length functional if and only if the two kinds of equiaffine curvature vanish, due to [Bl] (Theorem 5.13).

Finally, we solve the extremal problem of the general-affine length functional: a nondegenerate space curve without affine inflection point is general-affine extremal if and only if the pair of ordinary differential equations

$$\begin{aligned} &k''' + \frac{3}{2}kk'' + \frac{1}{2}k^2k' + \frac{1}{2}k'^2 - \frac{1}{5}\epsilon k' + \frac{6}{5}M' = 0, \\ &k'' + \frac{2}{3}kk' + \frac{5}{6}\epsilon k'M - \frac{3}{2}\epsilon kM' - \epsilon M'' = 0 \end{aligned}$$

are satisfied (Theorem 6.1). Then, we discuss a similarity amongst the nonlinear differential equations for the curvature functions, one for plane curves, and the other for space curves belonging to a linear complex, *i.e.*, $M = \epsilon k$ (Corollary 6.3, 6.4).

In Appendix, we discuss the projective treatment of plane curves and space curves and present the variational formula of the projective length functional by use of the method in this paper. Theorem A.2 reproduces the variational formula for projective plane curves due to E. Cartan, and Theorem B.1 gives the variational formula for projective space curves, which is essentially due to [Ki]. Furthermore, we treat nondegenerate projective

homogeneous space curves, called "W-Kurve". The list of such curves may be found elsewhere, but nonetheless, we give here a list in Appendix C for later reference.

In this paper, we use the classical moving frame method; refer to [Ca2, ST]. For the equiaffine differential geometry and its terminologies, we refer to the books [NS, Bl, Sc], and, for the projective treatment of curves, to E. J. Wilczynski [Wi], E. P. Lane [La].

2 General-affine curvature of plane curves

Let $x : M \longrightarrow \mathbf{A}^2$ be a curve into the 2-dimensional affine space, where M is a 1-dimensional parameter space. Let $e = \{e_1, e_2\}$ be a frame along x; at each point of $x(M)$ it is a set of independent vectors of $\mathbf{A}^2$ that depends smoothly on the parameter. The vector-valued 1-form dx is written as

$$dx = \omega^1 e_1 + \omega^2 e_2, \tag{2.1}$$

and the dependence of e_i on the parameter is described by the equation

$$de_i = \sum_j \omega_i^j e_j, \tag{2.2}$$

where ω^j and ω_i^j are 1-forms, and the matrix of 1-forms

$$\Omega = \begin{pmatrix} \omega^1 & \omega^2 \\ \omega_1^1 & \omega_1^2 \\ \omega_2^1 & \omega_2^2 \end{pmatrix}$$

is called the coframe.

2.1 Choice of frames for plane curves and general-affine curvature

We now reduce the choice of frames in order to define certain invariants. First, we assume $\omega^2 = 0$, which means that e_1 is tangent to the curve, and we set $\omega^1 = \omega$ for simplicity. The vector e_2 is arbitrary at present, as long as it is independent of e_1. Let $\tilde{e} = \{\tilde{e}_1, \tilde{e}_2\}$ be another choice of such a frame. Then, it is written as

$$\tilde{e}_1 = \lambda e_1, \qquad \tilde{e}_2 = \mu e_1 + \nu e_2,$$

where $\lambda\nu \neq 0$. The coframe is written as $\tilde{\omega}$ and $\tilde{\omega}_i^j$, which satisfy

$$dx = \tilde{\omega}\tilde{e}_1, \qquad d\tilde{e}_i = \sum_j \tilde{\omega}_i^j \tilde{e}_j.$$

Then we certainly have

$$\tilde{\omega} = \lambda^{-1}\omega. \tag{2.3}$$

Since $d\tilde{e}_1$ is represented in two way, one being

$$d\tilde{e}_1 = (d\lambda)e_1 + \lambda(\omega_1^1 e_1 + \omega_1^2 e_2)$$

and the other being

$$d\tilde{e}_1 = \tilde{\omega}_1^1(\lambda e_1) + \tilde{\omega}_1^2(\mu e_1 + \nu e_2),$$

by comparing the coefficients of e_1 and e_2 in these expressions, we get

$$\begin{aligned} \lambda\omega_1^2 &= \nu\tilde{\omega}_1^2, \\ d\lambda + \lambda\omega_1^1 &= \lambda\tilde{\omega}_1^1 + \mu\tilde{\omega}_1^2. \end{aligned} \tag{2.4}$$

Similarly, by considering $d\tilde{e}_2$, we have

$$\begin{aligned} \mu\omega_1^2 + d\nu + \nu\omega_2^2 &= \nu\tilde{\omega}_2^2, \\ d\mu + \mu\omega_1^1 + \nu\omega_2^1 &= \lambda\tilde{\omega}_2^1 + \mu\tilde{\omega}_2^2. \end{aligned} \tag{2.5}$$

Since the immersion is 1-dimensional, we can set $\omega_1^2 = h\omega$ and $\tilde{\omega}_1^2 = \tilde{h}\tilde{\omega}$, and then the first identity of (2.4) implies that

$$\tilde{h} = \nu^{-1}\lambda^2 h.$$

Hence the property that h is nonvanishing is independent of the frame and we assume in the following that it is nonvanishing. Such a curve is said to be *nondegenerate*. Geometrically, this property means that the curve is locally strictly convex at each point. By the identity above, provided that h is nonzero, we can choose a frame $\tilde{e}$ so that $\tilde{h} = 1$ and we treat such frames with $h = 1$ in the following. Then $\nu = \lambda^2$ immediately follows. Next, we see from (2.4) and (2.5) that

$$\begin{aligned} \tilde{\omega}_1^1 &= \omega_1^1 + \lambda^{-1}d\lambda - \mu\lambda^{-2}\omega, \\ \tilde{\omega}_2^2 &= \omega_2^2 + \nu^{-1}d\nu + \mu\nu^{-1}\omega. \end{aligned}$$

Hence,

$$2\tilde{\omega}_1^1 - \tilde{\omega}_2^2 = 2\omega_1^1 - \omega_2^2 - 3\mu\lambda^{-2}\omega,$$

which means that we can choose μ so that $2\tilde{\omega}_1^1 - \tilde{\omega}_2^2 = 0$, and, by considering only such frames in the following, we must have $\mu = 0$. Thus, we have determined the frame e up to a change of the form

$$\tilde{e}_1 = \lambda e_1, \qquad \tilde{e}_2 = \lambda^2 e_2.$$

We call the direction determined by e_2 the *general-affine normal direction*. Furthermore, we have

$$\tilde{\omega}_1^1 = d\log\lambda + \omega_1^1, \qquad \tilde{\omega}_2^1 = \lambda\omega_2^1.$$

From the first identity, we can choose λ so that $\tilde{\omega}_1^1 = 0$. Hence, we consider the frame with $\omega_1^1 = 0$ and λ is assumed to be constant. From the second identity, by setting

$$\omega_2^1 = -\ell\omega,$$

and, similarly, $\tilde{\omega}_2^1 = -\tilde{\ell}\tilde{\omega}$, we get

$$\tilde{\ell} = \lambda^2\ell. \tag{2.6}$$

We call this scalar function ℓ the *equiaffine curvature*, see Section 2.5, or *affine mean curvature*, in analogy with equiaffine theory of hypersurfaces, though it still depends on the frame chosen. A point where $\ell = 0$ is called an *affine inflection point*. For its

geometrical meaning, we refer to Section 2.4 and [IS]. Thus, we have seen that, given a nondegenerate curve x, there exists a frame e with coframe of the form

$$\begin{pmatrix} \omega & 0 \\ 0 & \omega \\ -\ell\omega & 0 \end{pmatrix} \tag{2.7}$$

and that such frames are related by $\tilde{e}_1 = \lambda e_1$ and $\tilde{e}_2 = \lambda^2 e_2$ for a nonzero constant λ. In the following, given a curve $x = x(t)$ with parameter t, we assume that the vector e_1 is a positive multiple of the tangent vector dx/dt. Then, the choice of λ is limited to be positive and the form ω is a positive multiple of dt.

We now assume $\ell \neq 0$ and let ϵ denote the sign of ℓ:

$$\epsilon = \mathrm{sign}(\ell).$$

It is a locally defined invariant of the curve called the *sign* of the curve. Then we define a form

$$\omega_s = \sqrt{\epsilon\ell}\,\omega, \tag{2.8}$$

which is uniquely defined, independent of the frame, in view of (2.3) and (2.6). We call this form the *general-affine length element* and call the parameter s such that $ds = \omega_s$ the *general-affine length parameter*, determined up to an additional constant.

Definition 2.1 *We call the scalar function k defined as*

$$\frac{d\ell}{\ell} = k\omega_s,$$

the general-affine curvature. *In other words,*

$$k = \frac{d\log\ell}{ds}. \tag{2.9}$$

We define a new frame $\{E_1, E_2\}$ by setting

$$E_1 = \frac{1}{\sqrt{\epsilon\ell}}e_1, \qquad E_2 = \frac{1}{\epsilon\ell}e_2. \tag{2.10}$$

Then,

$$dx = \omega_s E_1.$$

For another frame $\{\tilde{e}_1, \tilde{e}_2\}$ where $\tilde{e}_1 = \lambda e_1$ and $\tilde{e}_2 = \lambda^2 e_2$, we similarly define $\tilde{E}_1$ and $\tilde{E}_2$. Then we can see that

$$\tilde{E}_1 = \frac{1}{\sqrt{\epsilon\tilde{\ell}}}\tilde{e}_1 = \frac{1}{\sqrt{\epsilon\lambda^2\ell}}\lambda e_1 = E_1$$

and

$$\tilde{E}_2 = \frac{1}{\epsilon\tilde{\ell}}\tilde{e}_2 = \frac{1}{\epsilon\lambda^2\ell}\lambda^2 e_2 = E_2.$$

Thus, we have proved the following.

Proposition 2.2 *Assume $\ell \neq 0$. Then, the frame $\{E_1, E_2\}$ is uniquely defined from the immersion and it satisfies a Pfaffian equation*

$$d\begin{pmatrix} x \\ E_1 \\ E_2 \end{pmatrix} = \Omega \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}; \qquad \Omega = \begin{pmatrix} \omega_s & 0 \\ -\frac{1}{2}k\omega_s & \omega_s \\ -\epsilon\omega_s & -k\omega_s \end{pmatrix}, \tag{2.11}$$

where ω_s is the general-affine length form, k is the general-affine curvature and ϵ is $\mathrm{sign}(\ell)$.

By use of this choice of frame, we have the following lemma.

Lemma 2.3 *The immersion x satisfies the ordinary differential equation*

$$x''' + \frac{3}{2}kx'' + \left(\epsilon + \frac{1}{2}k' + \frac{1}{2}k^2\right)x' = 0, \tag{2.12}$$

relative to a general-affine length parameter.

Proof. The equation (2.11) shows that $x' = E_1$, $E_1' = -\frac{1}{2}kE_1 + E_2$, and $E_2' = -\epsilon E_1 - kE_2$, where the derivation $\{'\}$ is taken relative to the length parameter. Then, combining these derivations, we easily obtain the differential equation above.

Remark 2.4 The definition of the curvature depends on the orientation of the parameter t. If we let the parameter be $u = -t$ and denote by an overhead dot the derivation relative to u, then we have

$$\dddot{x} - \frac{3}{2}k\ddot{x} + \left(\epsilon - \frac{1}{2}\dot{k} + \frac{1}{2}k^2\right)\dot{x} = 0.$$

Namely, the curvature changes sign and its absolute value is a true invariant independent of the orientation of the parameter.

With this remark in mind, we have the following theorem.

Theorem 2.5 ([Mi1]) *Given a function $k(t)$ of a parameter t and $\epsilon = \pm 1$, there exists a nondegenerate curve $x(t)$ for which t is an length-parameter, k the curvature function and ϵ the sign of ℓ, uniquely up to a general-affine transformation.*

Proof. Given k and ϵ, we solve the ordinary differential equation in (2.12) to get the vector $x'(t)$, which is determined up to a general linear transformation. Then, we get $x(t)$ up to an additional translation by a constant vector; that is, the curve $x(t)$ is determined up to a transformation in GA(2).

Theorem 2.5 and the ordinary differential equation (2.12) were first given by T. Mihăilescu in [Mi1], to the authors' knowledge; refer also to [Mi2] and [CG].

Example 2.6 *Ellipse and Hyperbola.* Let x denote an ellipse $(a\cos\theta, b\sin\theta)$ or a hyperbola $(a\cosh\theta, b\sinh\theta)$. Then, $x''' = -\epsilon x'$, where $\epsilon = 1$ for the ellipse and $\epsilon = -1$ for the hyperbola. It is easy to see that θ is a general affine length, see (2.18). Hence, $k = 0$.

According to this example, we may call a nondegenerate curve is of *elliptic* (resp. *hyperbolic*) *type* if $\epsilon = 1$ (resp. $\epsilon = -1$).

We call the vector E_2, uniquely defined when $\ell \neq 0$, the *general-affine normal* and the map $t \longmapsto E_2$ the general-affine Gauss map. Then, by an analogy with affine spheres in

equiaffine differential geometry, it is natural to call a curve such that the map E_2 passes through one fixed point a *general-affine circle*. For the ellipse or the hyperbola, $E_2 = x''$ and it holds that

$$x + \epsilon E_2 = 0,$$

and thus it is a general-affine circle. Conversely, for a curve x to be such a circle, there exists a scalar function $r(t)$ and a fixed vector v such that

$$x + rE_2 = v.$$

However, this implies that $dx + drE_2 + rdE_2 = 0$, which induces, by the identity (2.11), $(1 - \epsilon r)\omega E_1 + (dr - kr\omega)E_2 = 0$. Hence, $r = \epsilon$ is constant and $k = 0$. Then, by integrating the differential equation (2.12) when $k = 0$, we see that any general-affine circle is general-affinely congruent to (a part of) an ellipse or a hyperbola. We also refer to Example 2.12.

2.2 Total curvature and sextactic points

The formula (2.11) implies the identity

$$d\log\left(\left|\det\begin{pmatrix} E_1 \\ E_2 \end{pmatrix}\right|\right) = -\frac{3}{2}k\omega_s, \tag{2.13}$$

where det is taken relative to a (any) unimodular structure of the space $\mathbf{R}^2$. This formula shows the following corollary immediately.

Corollary 2.7 *Assume that the curve C is nondegenerate and closed, and has no affine inflection point. Then, the total curvature $\int_C k\omega_s$ vanishes. In particular, such a curve has at least two general-affine flat points.*

As we will see in Section 2.6, any general-affine flat point, where $k = 0$ by definition, is nothing but a sextactic point. We know a classical theorem due to Mukhopadhayaya, also due to G. Herglotz and J. Rado, we refer, *e.g.*, to [ST, TU], that the number of sextactic points of a strictly convex simply closed smooth curve is at least six. In other words, on such a curve there are at least six general-affine flat points.

Furthermore, as an analogue of the Euclidean plane curve, it is natural to introduce a notion of a *general-affine vertex* where k is extremal. The corollary above says that any nondegenerate closed curve without affine inflection point has at least two general-affine vertices. In fact, the example 2.10 shows that there exists a plane curve which has two general-affine vertices.

2.3 Computation of general-affine curvature of plane curves

In this subsection, we will see how to obtain the curvature of a curve given relative to a parameter not necessarily a length parameter.

Let $t \longrightarrow x = x(t) \in \mathbf{A}^2$ be a nondegenerate curve so that the vectors x' and x'' are linearly independent. Then the derivative x''' is written as a linear combination of x' and x'': there are scalar functions $a = a(t)$ and $b = b(t)$ such that

$$x''' = ax'' + bx'. \tag{2.14}$$

Since $dx = x'\,dt$, the frame vector e_1 is a scalar multiple of x':

$$dx = \omega\, e_1; \qquad e_1 = \lambda x', \quad \omega = \lambda^{-1} dt. \tag{2.15}$$

Then, the derivation

$$de_1 = \lambda(\lambda x'' + \lambda' x')\,\omega$$

implies that the second frame vector is

$$e_2 = \lambda(\lambda x'' + \lambda' x').$$

In order for the frame $\{e_1, e_2\}$ to be chosen as in Section 2.1, the vector de_2 must be a multiple of e_1. Since

$$\begin{aligned} de_2 &= \lambda(\lambda^2 x''' + 3\lambda\lambda' x'' + (\lambda\lambda'' + \lambda'^2)x')\,\omega \\ &= \{\lambda^2(\lambda a + 3\lambda')x'' + \lambda(\lambda^2 b + \lambda\lambda'' + \lambda'^2)x'\}\,\omega, \end{aligned}$$

we have

$$\lambda a + 3\lambda' = 0, \quad i.e. \quad \lambda = e^{-\frac{1}{3}\int a(t)dt} \tag{2.16}$$

up to a positive constant multiple, and by definition,

$$\ell = -(\lambda^2 b + \lambda\lambda'' + \lambda'^2). \tag{2.17}$$

We now assume that $\ell \neq 0$ and recall that $\epsilon = \mathrm{sign}(\ell)$. Then, we have

$$ds^2 = -\epsilon\left(b + \frac{\lambda\lambda'' + \lambda'^2}{\lambda^2}\right) dt^2.$$

In terms of a and b,

$$ds^2 = -\epsilon\left(b + \frac{2}{9}a^2 - \frac{1}{3}a'\right) dt^2. \tag{2.18}$$

Hence, a length parameter s which is a function of t is obtained by solving the equation

$$\left(\frac{ds}{dt}\right)^2 = -\epsilon\left(b + \frac{2}{9}a^2 - \frac{1}{3}a'\right).$$

If in particular t itself is a length parameter, then we must have

$$\ell = \lambda^2\epsilon, \qquad -b - \frac{2}{9}a^2 + \frac{1}{3}a' = \epsilon. \tag{2.19}$$

Assume now that the curve $x(t)$ is given relative to a length parameter t. Then the differential equation (2.14) is written as

$$x''' = ax'' + \left(\frac{1}{3}a' - \frac{2}{9}a^2 - \epsilon\right) x', \tag{2.20}$$

and the curvature is

$$k = \frac{d\log(\epsilon\lambda^2)}{dt} = -\frac{2}{3}a, \tag{2.21}$$

which agrees with the expression of the coefficient in the equation (2.12).

For another parameter $\sigma = \sigma(t)$, we write

$$y(\sigma) = x(t).$$

Then, a calculation shows that

$$\dddot{y}\ (\sigma) = A(\sigma)\ddot{y}(\sigma) + B(\sigma)\dot{y}(\sigma),$$

where

$$A(\sigma) = \left(a - \frac{3\sigma''}{\sigma'}\right)\frac{1}{\sigma'}, \tag{2.22}$$

$$B(\sigma) = \left(b + a\frac{\sigma''}{\sigma'} - \frac{\sigma'''}{\sigma'}\right)\frac{1}{\sigma'^2}. \tag{2.23}$$

We note that there holds a covariance relation:

$$B + \frac{2}{9}A^2 - \frac{1}{3}\dot{A} = \left(b + \frac{2}{9}a^2 - \frac{1}{3}a'\right)\frac{1}{\sigma'^2}.$$

Thus, we have the following procedure to obtain curvature:

Procedure for computing curvature

1. Given a curve $x(t)$, derive the differential equation (2.14).
2. Compute $L = \left(-b - \frac{2}{9}a^2 + \frac{1}{3}a'\right)$ and define ϵ by $\epsilon = \mathrm{sign}(L)$.
3. Compute the length parameter σ by solving $d\sigma = \sqrt{\epsilon L}dt$.
4. Compute A by (2.22); then, $-\frac{2}{3}A$ is the curvature.

Example 2.8 A logarithmic spiral is the curve $x(t) = e^{\gamma t}(\cos\alpha t, \sin\alpha t)$. It is easy to see that $x''' = 2\gamma x'' - (\gamma^2 + \alpha^2)x'$. Hence, $a = 2\gamma$, $b = -\gamma^2 - \alpha^2$, and $L = \gamma^2/9 + \alpha^2$; hence, $\epsilon = 1$. The length parameter is $s = \sqrt{\gamma^2/9 + \alpha^2}t$ and, by rewriting the equation with this s, the coefficient a is multiplied by $1/\sqrt{\gamma^2/9 + \alpha^2}$. Hence, the curvature is the constant $-4\gamma/\sqrt{\gamma^2 + 9\alpha^2}$. We require here that $\gamma \neq 0$, since the curve when $\gamma = 0$ is a circle, which we will consider in Example 2.12, and also $\alpha \neq 0$, because the curve is nondegenerate. Then, it is easy to see that the possible values of k range in $0 < |k| < 4$.

Example 2.9 The catenary curve is defined as $x(t) = (t, \cosh(t))$. The equation is

$$x''' = \tanh(t)x''.$$

Then $L = -(2\cosh(t)^2 - 5)/(9\cosh(t)^2)$, which vanishes at $2\cosh(t)^2 - 5 = 0$ ($t = \pm 1.031\ldots$) , where the curvature is undefined. For the value t where $L < 0$, on the dotted curved of the left-hand side picture in Figure 1, $\epsilon = 1$ and, elsewhere $\epsilon = -1$. As the picture shows, it is difficult to "see" where L vanishes and how ϵ changes the value.

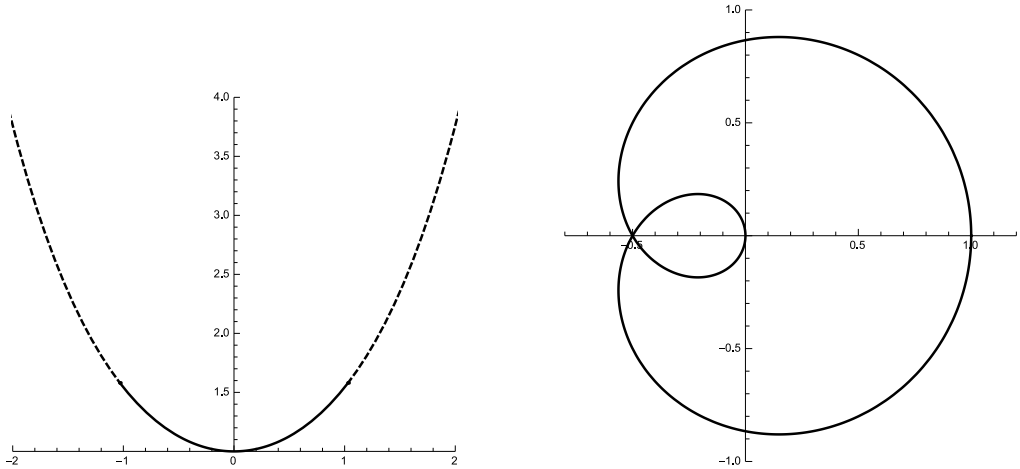


Figure 1. Catenary (left) and rose curve (right)

Example 2.10 The curve $x(t) = (\cos(nt)\cos(t), \cos(nt)\sin(t))$ is called a rose curve. Now choose $n = 1/3$ with the range $t \in [0, 3\pi]$; shown in Figure 1 (right). It satisfies the equation

$$x''' = -\frac{8\sin(t/3)T}{3(1+4T^2)}x'' - \frac{4(7+8T^2)}{9(1+4T^2)}x',$$

where $T = \cos(t/3)$. A computation shows that $L > 0$, $\epsilon = 1$ and the length parameter σ is defined by $d\sigma = \frac{2\sqrt{256T^2+320T^4+69}}{9(1+4T^2)}dt$, and the curvature $k(t)$ is defined for all value t. It vanishes at $t = 0, 3\pi/2$. Moreover, it is easy to see that the number of general-affine vertices is two. Thus the rose curve with $n = 1/3$ attains the minimum number of general-affine vertices amongst general-affine plane closed curves.

2.4 General-affine curvature of a graph immersion

Let us consider the nondegenerate curve given by a graph immersion $x(t) = (t, f(t))$. We will find the formula of the curvature given by the function f and show fundamental examples of graph immersions.

Note that since x is nondegenerate, $x'' = (0, f'') \neq 0$, we can assume $f'' > 0$ in the following. Since $x''' = (0, f''')$, the coefficients of the differential equation (2.14) are

$$a = \frac{f'''}{f''}, \qquad b = 0.$$

Hence,

$$\lambda = e^{-\frac{1}{3}\int a(t)dt} = (f'')^{-1/3}$$

up to a constant multiple. With this λ, we have

$$\ell = -(\lambda\lambda'' + (\lambda')^2)$$

and the length element is

$$ds^2 = -\epsilon\lambda^{-2}(\lambda\lambda'' + (\lambda')^2)dt^2.$$

If we set $\mu = \lambda^2 = (f'')^{-2/3}$, then

$$\ell = -\frac{\mu''}{2}, \qquad ds^2 = -\frac{\epsilon\mu''}{2\mu}dt^2. \tag{2.24}$$

Hence, we have the formula

$$k = \frac{d\log\ell}{ds} = \sqrt{\frac{-2\epsilon\mu}{\mu''}}\frac{\mu'''}{\mu''}. \tag{2.25}$$

Lemma 2.11 *The quantities ds^2 and k^2 are expressed by use of the function f as follows*:

$$\begin{aligned} ds^2 &= \frac{|3f''f'''' - 5(f''')^2|}{9(f'')^2}dt^2, \\ k^2 &= \frac{|9(f'')^2 f''''' - 45f''f'''f'''' + 40(f''')^3|^2}{|3f''f'''' - 5(f''')^2|^3}. \end{aligned} \tag{2.26}$$

This is seen by expressing μ'' and μ''' explicitly in terms of derivations of f. As we will see in Section 2.5, $-\mu''(= 2\ell)$ equals the equiaffine curvature up to a multiplicative constant. The factor in the right-hand side of the first expression is known, see [Bl, p. 14]. The second expression of k^2 was already presented in [Sc, p. 54], which was proved from a different point of view. We refer also to [OST, p. 343] and [CQ]. The differential polynomials in the numerator and denominator in the right-hand side of k^2 are known classically; Berzolari [Be] stated that those go back to G. Monge in 1810: Sur les Équations différentielles des courbes du second degré, Corresp. sur l'École imp. Polytechn., Paris, N° II, 1810, pp. 51–54.

We consider a nondegenerate curve around the point $t = 0$. For appropriate affine coordinates, we have an expression

$$f = \frac{1}{2}t^2 + \frac{p}{4!}t^4 + \frac{q}{5!}t^5 + \cdots.$$

Then, we see

$$\mu = 1 - \frac{p}{3}t^2 - \frac{q}{9}t^3 + \cdots$$

and

$$\mu'' = -\frac{2p}{3} - \frac{2q}{3}t + \cdots. \tag{2.27}$$

Hence, the property $\ell = 0$ at $t = 0$ means that $p = 0$ and, hence, that the osculating parabola touches the curve to at least 5-th order. In particular, $\ell \equiv 0$ for any parabola. Conversely, we have the following example.

Example 2.12 *Curves with constant ℓ.* We first let $\ell \equiv 0$. Then, $\mu = at + b$ or $\mu = a$, and it follows that $f = c_1(at+b)^{1/2} + c_2t + c_3$ or $f = c_1t^2 + c_2t + c_3$ for certain constants a, b, c_1, c_2, c_3. Namely, the curve $x(t) = (t, f(t))$ is (general-affinely equivalent to) a parabola.

If ℓ is a nonzero constant: $\ell = -c \neq 0$, then $\mu = ct^2 + at + b$, and $f(t) = c_1(ct^2 + at + b)^{1/2} + c_2t + c_3$ for constants a, b, c, c_1, c_2, c_3. This implies that the curve is an ellipse or a hyperbola. Thus, we have seen that the curve with constant ℓ is a quadric and hence the curvature satisfies $k = 0$.

For the ellipse $f = -(\alpha^2 - t^2)^{1/2}$ and the hyperbola $f = (\alpha^2 + t^2)^{1/2}$, we can see that $\mu = \alpha^{-4/3}(\alpha^2 \pm t^2)$ and $ds^2 = \dfrac{1}{\alpha^2 \pm t^2} dt^2$. Relative to the reparametrization by use of angular variable $t = \alpha \sin\theta$ or $\alpha \sinh\theta$, we have $ds^2 = d\theta^2$; namely, ds is independent of α, the "size" of the curve in euclidean sense.

Example 2.13 For the curve $f = e^t$, we see that $\epsilon = -1$ and $k = -\sqrt{2}$. For the curve $f = t \log t$ $(t > 0)$, we have $\epsilon = 1$ and $k = -4$.

Example 2.14 For the curve $f = t^\alpha$, we see that the curvature is

$$k(\alpha) = \frac{-2(\alpha + 1)}{\sqrt{|(2\alpha - 1)(\alpha - 2)|}}.$$

Here we assume $\alpha \neq 0,\ \pm 1,\ 1/2,\ 2$ so that the curve is neither trivial nor quadratic. Since $\mu'' = (1/9)(2\alpha - 4)(2\alpha - 1)t^{-2(\alpha+1)/3}$, we have $\epsilon = 1$ when $1/2 < \alpha < 2$, and $\epsilon = -1$ when $\alpha > 2$ or $\alpha < 1/2$. Note that $k(1/\alpha) = k(\alpha)$ when $\alpha > 0$, and $k(1/\alpha) = -k(\alpha)$ when $\alpha < 0$. By this symmetry, in order to know the possible values of $k = k(\alpha)$, it is sufficient to consider the case $-1 < \alpha < 1$. Then, $\epsilon = -1$ and $k \in (-\infty, -\sqrt{2})$ for $\alpha \in (0, 1/2)$; $\epsilon = -1$ and $k \in (-\sqrt{2}, 0)$ for $\alpha \in (-1, 0)$; $\epsilon = 1$ and $k \in (-\infty, -4)$ for $\alpha \in (1/2, 1)$.

Due to Theorem 2.5, the curves with constant curvature are obtained by solving the equation (2.20) for constant a:

$$x''' = ax'' + \left(-\frac{2}{9}a^2 - \epsilon\right) x'.$$

The case where $a = 0$ is the case $\ell = 0$: Example 2.12. When $a \neq 0$ we have the following:

Proposition 2.15 *When the curvature k is a nonzero constant, the curve is general-affinely congruent to one of the curves in Examples* 2.8, 2.13 *and* 2.14*: $e^{\gamma t}(\cos t, \sin t)$, (t, e^t), $(t, t \log t)$, (t, t^α) $(\alpha \neq 0,\ \pm 1,\ 2,\ 1/2)$.*

Proof. We set $u = e^{-at/2}x'$. Then u satisfies the equation $u'' + pu = 0$, $p = \epsilon - a^2/36$. According to $p = 0$, $p < 0$, $p > 0$, a set of independent solutions gives a map x which is congruent to the curves $(t, t \log t)$, (t, t^α) and (t, e^t), and $e^{\gamma t}(\cos t, \sin t)$ with parameter renewed appropriately. More precisely, when $\epsilon = 1$, we have the curve (t, t^α) $(1/2 < \alpha < 1)$, $(t, t \log t)$, and $e^{\gamma t}(\cos t, \sin t)$ according to $k \in (-\infty, -4)$, $k = -4$, and $k \in (-4, 0)$, respectively. When $\epsilon = -1$, we have (t, t^α) $(0 < \alpha < 1/2)$, (t, e^t), and (t, t^α) $(-1 < \alpha < 0)$ according to $k \in (-\infty, -\sqrt{2})$, $k = -\sqrt{2}$, and $k \in (-\sqrt{2}, 0)$, respectively.

We remark that any curve with constant curvature is an orbit of a 1-parameter subgroup of GA(2), because of the unique and existence in Theorem 2.5.

Table 1 is the classification of general-affine curves with constant curvature k. We let $k \leq 0$, see Remark 2.4, and note that the case $k = -\infty$ corresponds to the parabola defined in Example 2.12.

curvatures	curves with $\epsilon = +1$	Examples
$k = 0$	$(t, -(\alpha^2 - t^2)^{1/2})$	2.12
$-4 < k < 0$	$e^{\gamma t}(\cos \alpha t, \sin \alpha t)$ $(\gamma \neq 0, \alpha \neq 0)$	2.8
$k = -4$	$(t, t \log t)$ $(t > 0)$	2.13
$-\infty < k < -4$	(t, t^α) $(\alpha \in (1/2, 1))$	2.14

curvatures	curves with $\epsilon = -1$	Examples
$k = 0$	$(t, (\alpha^2 + t^2)^{1/2})$	2.12
$k = -\sqrt{2}$	(t, e^t)	2.13
$-\infty < k < 0,\ k \neq -\sqrt{2}$	(t, t^α) $(\alpha \in (0, 1/2)$ or $(\alpha \in (-1, 0))$	2.14

Table 1: Plane curves with constant general-affine curvature

2.5 From equiaffine to general-affine

Since the group of equiaffine motions $\mathrm{SA}(2) = \mathrm{SL}(2, \mathbf{R}) \ltimes \mathbf{R}^2$ is a subgroup of the general-affine group $\mathrm{GA}(2) = \mathrm{GL}(2, \mathbf{R}) \ltimes \mathbf{R}^2$, any general-affine invariant is obviously an equiaffine invariant. In this subsection, we give the expression of the general-affine length parameter and the general-affine curvature by use of the equiaffine length parameter and the equiaffine curvature.

Let us consider the coframe (2.7):

$$de = \begin{pmatrix} \omega & 0 \\ 0 & \omega \\ -\ell\omega & 0 \end{pmatrix} e \tag{2.28}$$

for $e = \{e_1, e_2\}$. In the equiaffine treatment, it is enough to consider only the unimodular change of frame, *i.e.* $\lambda\nu = 1$. Because $\nu = \lambda^2$, we have $\lambda^3 = 1$. Thus ℓ is an absolute invariant. The scalar ℓ is usually denoted by k_a and called the *equiaffine curvature* of a plane curve. The parameter t for which $\omega = dt$ holds is called the *equiaffine length parameter*. Let $x(t)$ be a curve with equiaffine length parameter t. Then, it is easy to see by (2.28) that x satisfies

$$x''' = -k_a x'. \tag{2.29}$$

On the other hand, when the curve is given by a graph immersion $x(t) = (t, f(t))$, where t is a parameter that is not necessarily equiaffine, the equiaffine length element is $(f'')^{1/3}dt$ and the equiaffine curvature is $k_a = -\frac{1}{2}\mu''$ for $\mu = (f'')^{-2/3}$. As we have seen in (2.24), the equiaffine curvature is nothing but ℓ up to a constant multiple. We refer to [Bl, p. 13–14].

Now we consider the curve in view of the group GA(2). Then, the differential equation (2.29) above shows that the general-affine length element is

$$ds = |k_a|^{1/2}dt.$$

We rewrite the differential equation using the parameter $s = s(t)$: we set $y(s) = x(t)$ and let $\{\dot{}\}$ denote the derivation relative to s. Then, we get the equation:

$$\dddot{y} = A(s)\ddot{y} + B(s)\dot{y},$$

where $A(s)$ and $B(s)$ are given by use of (2.22) and (2.23). For simplicity, we set $K = |k_a|$. Since $s' = ds/dt = K^{1/2}$ and $s'' = \frac{1}{2}K'K^{-1/2}$, we see that $A(s) = -\frac{3}{2}K'K^{-3/2}$. Therefore, the general-affine curvature of $x(t)$ is

$$k = K'K^{-3/2}.$$

The quantity of this form was already treated in [Bl, p. 24] by dimension considerations to get an invariant relative to similarity transformation. There a remark was given that the curves with constant $K'K^{-3/2}$ consist of parts of curves called "W-Kurve", discussed by Klein and Lie [KL].

2.6 From general-affine to projective

It was G. H. Halphen [Ha] who began a systematic study of projective curves in view of ordinary differential equations. Later, E. J. Wilczynski gave a classical treatment of projective curves in the book [Wi]. Also, the books by E. P. Lane [La] are standard references for this subject. In this subsection, we recall the definition of the projective length element and the projective curvature, and give the expressions of such invariants in terms of general-affine invariants.

A nondegenerate curve in $\mathbf{P}^2$ with parameter t is described by an ordinary differential equation of the form

$$y''' + P_2 y' + P_3 y = 0, \tag{2.30}$$

such that a set of three independent solutions, say, x^1, x^2, x^3 defines a map $t \to [x^1, x^2, x^3] \in \mathbf{P}^2$, where $[\quad]$ denotes homogeneous coordinates. For this equation, the form

$$P^{1/3}dt, \quad \text{where} \quad P = P_3 - \frac{1}{2}P_2', \tag{2.31}$$

is called the projective length element. Furthermore, when t itself is a projective length parameter, the equation can be transformed by a certain change of variables from y to $z = \lambda y$ into the equation of the form

$$z''' + 2k_p z' + (1 + k_p')z = 0, \tag{2.32}$$

which is called the Halphen canonical form. Then, the coefficient k_p is called the *projective curvature* and is given by the formula

$$k_p = P^{-2/3}\left(\frac{1}{2}P_2 - \frac{1}{3}\frac{P''}{P} + \frac{7}{18}\left(\frac{P'}{P}\right)^2\right); \tag{2.33}$$

we refer to [Ca2, p. 71]. In particular, when k_p is constant, the curve is called an anharmonic curve and is obtained by integrating the differential equation $z''' + 2k_p z' + z = 0$; we refer to [Wi, p. 86–91]. The list of anharmonic curves is the same as the list of plane curves with constant general-affine curvature in Section 2.4, up to projective equivalence; we note that the sign ϵ plays no role in the projective classification.

In the general-affine setting we had a differential equation (2.20), which can be transformed into the equation of the form (2.30) by changing x into $y = e^{-\frac{1}{3}\int a dt}x$. The result is

$$y''' = \left(\frac{1}{9}a^2 - \frac{2}{3}a' - \epsilon\right)y' + \left(\frac{1}{9}aa' - \frac{1}{3}a'' - \frac{1}{3}a\epsilon\right)y.$$

Hence, we can see

$$P = -\frac{1}{3}a\epsilon;$$

this implies that the projective length element is $a^{1/3}dt$ up to a scalar multiple, while $-\frac{2}{3}a$ is the general-affine curvature. In particular, the point where the general-affine curvature vanishes is the point where the invariant P vanishes, which is classically called a *sextactic point*.

We remark that, in [SS], S. Sasaki showed how to obtain the projective length parameter and the projective curvature directly from the equiaffine curvature.

3 General-affine extremal plane curves and the associated differential equation

In Section 2.1, we have defined the general-affine length element $\omega_s = \sqrt{\epsilon\ell}\omega$ with $\epsilon = \mathrm{sign}(\ell)$. It defines the length functional for general-affine curves, and in this section we consider the curves that are extremal with respect to this functional. First, we prove the variational formula, which is a nonlinear differential equation relative to the general-affine curvature, then discuss some special solutions with reference to Chazy equations. Second, we compute the variational formula for a more generally defined curvature functional.

3.1 Extremal plane curves relative to the length functional

We have shown that there exists a unique frame $e = \{x, E_1, E_2\}$ such that (2.11) holds. Recall that Ω denotes the 3×2 matrix in (2.11). Let $x_\eta(t)$ be a family of curves parametrized by η around $\eta = 0$ and $x_0 = x$. We assume that $x_\eta(t) = x(t)$ outside a compact set C and $x_0(t)$ is parametrized by general-affine arc length. For simplicity, we further assume that x_η does not have an affine inflection point, *i.e.*, the invariant ω_s does not vanish anywhere for all η. The length functional L is given by

$$L(\eta) = \int_C \omega_s(\eta).$$

For the sake of brevity, we use the notation δ to denote the derivative with respect to η evaluated at $\eta = 0$:

$$\delta a = \frac{da(\eta)}{d\eta}\Big|_{\eta=0}.$$

Then the curve x is called *general-affine extremal* if

$$\delta L = 0$$

for any compactly supported deformation of x.

We want to derive a differential equation for affine extremality. Since $\{E_1, E_2\}$ are linearly independent, there exists a 3×2-matrix τ such that

$$\delta \begin{pmatrix} x \\ E_1 \\ E_2 \end{pmatrix} = \tau \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}, \qquad \tau = \begin{pmatrix} \tau_0^1 & \tau_0^2 \\ \tau_1^1 & \tau_1^2 \\ \tau_2^1 & \tau_2^2 \end{pmatrix}$$

holds. Components of Ω and τ are denoted by ω_α^β and τ_α^β, where $\alpha = 0, 1, 2$ and $\beta = 1, 2$. Since $\delta de = d\delta e$ with $e = \{x, E_1, E_2\}$, we have

$$\delta\omega_\alpha^\beta - d\tau_\alpha^\beta = \sum_{\gamma=1,2} \tau_\alpha^\gamma \omega_\gamma^\beta - \omega_\alpha^\gamma \tau_\gamma^\beta.$$

In terms of entries of Ω and τ, we have

$$\delta\omega_s - d\tau_0^1 = -\left(\frac{1}{2}k\tau_0^1 + \epsilon\tau_0^2 + \tau_1^1\right)\omega_s, \tag{3.1}$$

$$-d\tau_0^2 = \left(\tau_0^1 - \tau_1^2 - k\tau_0^2\right)\omega_s, \tag{3.2}$$

$$-\frac{1}{2}\delta(k\omega_s) - d\tau_1^1 = -\left(\epsilon\tau_1^2 + \tau_2^1\right)\omega_s, \tag{3.3}$$

$$\delta\omega_s - d\tau_1^2 = \left(\tau_1^1 - \frac{1}{2}k\tau_1^2 - \tau_2^2\right)\omega_s, \tag{3.4}$$

$$-\epsilon\delta\omega_s - d\tau_2^1 = \epsilon\left(\tau_1^1 + \frac{\epsilon}{2}k\tau_2^1 - \tau_2^2\right)\omega_s, \tag{3.5}$$

$$-\delta(k\omega_s) - d\tau_2^2 = \left(\tau_2^1 + \epsilon\tau_1^2\right)\omega_s. \tag{3.6}$$

Here we use $\omega_0^1 = \omega_1^2 = -\epsilon\omega_2^1 = \omega_s$, $\omega_0^2 = 0$, $\omega_1^1 = -1/2k\omega_s$, and $\omega_2^2 = -k\omega_s$. Then adding (3.4) and $-\epsilon$(3.5),

$$2\delta\omega_s - d\tau_1^2 + \epsilon d\tau_2^1 = -\frac{1}{2}k(\tau_1^2 + \epsilon\tau_2^1)\omega_s$$

holds. Then adding (3.6) and -2(3.3), we have

$$2d\tau_1^1 - d\tau_2^2 = 3\epsilon(\tau_1^2 + \epsilon\tau_2^1)\omega_s. \tag{3.7}$$

Combining these equations, we have

$$2\delta\omega_s = d\tau_1^2 - \epsilon d\tau_2^1 + \frac{\epsilon}{6}k(-2d\tau_1^1 + d\tau_2^2). \tag{3.8}$$

Recall that the deformation is compactly supported. Then by using Stokes' theorem and integration by parts, we have

$$\delta L = -\frac{\epsilon}{12}\int_C \left(-2\tau_1^1 + \tau_2^2\right) dk.$$

We now compute $-2\tau_1^1 + \tau_2^2$ as follows. From (3.1) and (3.4), we have

$$d\tau_0^1 - d\tau_1^2 = \left(2\tau_1^1 - \tau_2^2 + \epsilon\tau_0^2\right)\omega_s + \frac{1}{2}k\left(\tau_0^1 - \tau_1^2\right)\omega_s.$$

Inserting (3.2) into the above equation, we have

$$-2\tau_1^1 + \tau_2^2 = \tau_0^{2\prime\prime} - \frac{3}{2}k\tau_0^{2\prime} + \left(\epsilon + \frac{1}{2}k^2 - k'\right)\tau_0^2. \tag{3.9}$$

Here $\{'\}$ denotes the derivative with respect to the general-affine arc length ω_s, *i.e.* $a' = da/\omega_s$ for a function a. Finally, using integration by parts again,

$$\delta L = -\frac{\epsilon}{12}\int_C \left(k''' + \frac{3}{2}kk'' + \frac{1}{2}k'^2 + \frac{1}{2}k^2k' + \epsilon k'\right)\tau_0^2\omega_s \tag{3.10}$$

holds. If we now take $x(t)+\eta\{v^1(t,\eta)E_1(t)+v^2(t,\eta)E_2(t)\}$ for the family of curves $x_\eta(t)$, where v^1 and v^2 are arbitrary smooth functions with compact support relative to t, then $\delta x = v^1(t,0)E_1+v^2(t,0)E_2$. This means that $\tau_0^2 = v^2(t,0)$ is arbitrary for this family and the vanishing of the integral implies the following proposition.

Proposition 3.1 ([Mi2]) *A nondegenerate plane curve without affine inflection point is general-affine extremal relative to the length functional if and only if*

$$k''' + \frac{3}{2}kk'' + \frac{1}{2}k'^2 + \frac{1}{2}k^2k' + \epsilon k' = 0 \tag{3.11}$$

holds. In particular, any curves of constant general-affine curvature are extremal.

We remark here that the differential equation (3.11) was first given in [Mi2, (33)], though some modifications are necessary. The formula for $\epsilon = 1$ was then rediscovered by S. Verpoort in [Ve, p.432] by making use of his general variational formula of equiaffine invariants: The differential equation is written in terms of both of the equiaffine curvature and the general-affine curvature, and looks much simpler than (3.11). As a result, he proved the following corollary, which we now state in our setting.

Corollary 3.2 ([Ve]) *Let $x(t) = (x_1(t), x_2(t))$ be a curve parametrized by general-affine parameter t. Assume that it is general-affine extremal. Then, there exist constants c_1, c_2 and c_3 such that the general-affine curvature k can be written as*

$$k = c_1x_1 + c_2x_2 + c_3.$$

Proof. Let us consider the ordinary linear differential equation

$$z''' + \frac{3}{2}kz'' + \left(\epsilon + \frac{1}{2}k' + \frac{1}{2}k^2\right)z' = 0,$$

with unknown function z. We note that this is nothing but of the same form as the equation (2.12), therefore, x_1, x_2 are solutions. Also any constant is obviously a solution. On the other hand, if x is extremal, then the equation (3.11) shows that $k(t)$ itself is a solution. Therefore, k can be expressed as claimed.

We also have the following property on the curvature integral.

Proposition 3.3 *The variation of total curvature on any compact interval always vanishes, i.e.,*

$$\delta\int_C k\omega_s = 0$$

holds.

Proof. Adding (3.3) and (3.6),

$$-\frac{3}{2}\delta(k\omega_s) - d\tau_1^1 - d\tau_2^2 = 0$$

holds. Then Stokes' theorem implies the proposition.

Remark 3.4 The differential equation (3.11) associated to general-affine extremal plane curves is the third order nonlinear differential equation of Chazy type. It is known that J. Chazy [Ch] classified third order nonlinear ordinary differential equations of Painlevé type, *i.e.* the solutions only admit poles as movable singularities. Then Chazy equations are classified into I to XIII classes of equations and the full list of equations can be found in [Ba]. We here cite Chazy equations for IV, V and VI, which are respectively given by

$$k''' + 3kk'' + 3k'^2 + 3k^2k' - Sk' - S'k - T = 0, \tag{3.12}$$

$$k''' + 2kk'' + 4k'^2 + 2k^2k' - 2Rk' - R'k = 0, \tag{3.13}$$

$$k''' + kk'' + 5k'^2 + k^2k' - 3Qk' - Q'k + Q'' = 0, \tag{3.14}$$

where S, T, R, Q are certain analytic functions of t, [Ba]. Then (3.11) is clearly of the form of the above Chazy equations with the coefficients of kk'', k'^2 and k^2k' replaced by the half integers 3/2, 1/2 and 1/2, respectively, and with S, T, R or Q chosen properly.

Example 3.5 We can see that the following $k(t)$ are solutions of (3.11):

$$k(t) = 3\sqrt{2}\tanh(\sqrt{2}(t-c)) \quad \text{and} \quad k(t) = 3\sqrt{2}\coth(\sqrt{2}(t-c))$$

for $\epsilon = 1$ ([Ve, Example 14]), and

$$k(t) = -3\sqrt{2}\tan(\sqrt{2}(t-c)), \quad k(t) = 3\sqrt{2}\cot(\sqrt{2}(t-c)) \quad \text{and} \quad k(t) = \pm\sqrt{2} + \frac{3}{t-c}$$

for $\epsilon = -1$. For each of these solutions, we can compute the associated plane curve by integrating the differential equation, using computer software. Since the expression is not simple, we give here one example for the case $\epsilon = -1$ and $k(t) = \sqrt{2} + 3/t$, and the curve is written as (x_1, x_2) for $t > 0$:

$$x_1 = 3\sqrt{2} - \frac{1}{t},$$

$$x_2 = \frac{\sqrt{3\pi}(\sqrt{2} - 6t)\ \mathrm{erfi}\left(\frac{\sqrt{3t}}{2^{1/4}}\right)}{t} + \frac{62^{1/4}\exp\left(\frac{3t}{\sqrt{2}}\right)}{\sqrt{t}},$$

where erfi is the error function defined by

$$\mathrm{erfi}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{t^2}\ dt.$$

3.2 Extremal problem for a generalized curvature functional

More generally, one can consider a variational problem for the following curvature functional:

$$F(\eta) = \int_C f(k)\omega_s, \tag{3.15}$$

where f is a smooth function of one variable and η is the parameter for the variation of curves. Then we have

$$\delta F = \int_C f'(k)(\delta k)\omega_s + f(k)\delta\omega_s. \tag{3.16}$$

The computation of δk is done as follows: Adding (3.4) and ϵ(3.5), we have

$$-d\tau_1^2 - \epsilon d\tau_2^1 = 2(\tau_1^1 - \tau_2^2)\omega_s - \frac{1}{2}k(\tau_1^2 - \epsilon\tau_2^1)\omega_s.$$

By taking a derivative of this equation and by using (3.7), we have

$$\frac{\epsilon}{3}\left(-2\tau_1^1 + \tau_2^2\right)''' = 2(\tau_1^1 - \tau_2^2)' - \frac{1}{2}k'(\tau_1^2 - \epsilon\tau_2^1) - \frac{1}{2}k(\tau_1^2 - \epsilon\tau_2^1)'. \tag{3.17}$$

Subtracting (3.6) from (3.3), we have

$$\frac{1}{2}(\delta k)\omega_s + \frac{1}{2}k\delta\omega_s - d\tau_1^1 + d\tau_2^2 = -2(\tau_2^1 + \epsilon\tau_1^2)\omega_s.$$

Then, using (3.8) we see that

$$\delta k = -k\left(\frac{1}{2}\left(\tau_1^2 - \epsilon\tau_2^1\right)' + \frac{\epsilon}{12}k(-2\tau_1^1 + \tau_2^2)'\right) + 2(\tau_1^1 - \tau_2^2)' - 4(\tau_2^1 + \epsilon\tau_1^2).$$

Again, by using (3.7),

$$\delta k = -\frac{1}{2}k\left(\tau_1^2 - \epsilon\tau_2^1\right)' + \left(\frac{4}{3} - \frac{\epsilon}{12}k^2\right)\left(-2\tau_1^1 + \tau_2^2\right)' + 2(\tau_1^1 - \tau_2^2)',$$

and finally, by using (3.17), we have

$$\delta k = \frac{1}{2}k'\left(\tau_1^2 - \epsilon\tau_2^1\right) + \frac{\epsilon}{3}\left(-2\tau_1^1 + \tau_2^2\right)''' + \left(\frac{4}{3} - \frac{\epsilon}{12}k^2\right)\left(-2\tau_1^1 + \tau_2^2\right)'. \tag{3.18}$$

Then, by inserting the expressions $\delta\omega_s$ (3.8) and δk (3.18) into (3.16), and by using integration by parts, we get

$$\delta F = \int_C \dot{f}(k)\left\{\frac{\epsilon}{3}\left(-2\tau_1^1 + \tau_2^2\right)''' + \left(-\frac{\epsilon k^2}{12} + \frac{4}{3}\right)\left(-2\tau_1^1 + \tau_2^2\right)'\right\}\omega_s$$
$$+ f(k)\left\{\frac{\epsilon}{12}k\left(-2\tau_1^1 + \tau_2^2\right)'\right\}\omega_s.$$

Again applying integration by parts, we have

$$\delta F = -\frac{\epsilon}{12}\int_C G(-2\tau_1^1 + \tau_2^2)\omega_s,$$

where

$$G = 4\dddot{f}(k)k'^3 + 12\ddot{f}(k)k'k'' + \ddot{f}(k)(4k''' - k'k^2 + 16\epsilon k') - \dot{f}(k)kk' + f(k)k'. \tag{3.19}$$

Thus, by use of (3.9), we have the following theorem.

Theorem 3.6 *A plane curve without affine inflection points is general-affine extremal with respect to the curvature functional* (3.15) *if and only if*

$$G'' + \frac{3}{2}G'k + \frac{1}{2}Gk' + \frac{1}{2}Gk^2 + \epsilon G = 0 \tag{3.20}$$

holds, where G is the function defined in (3.19).

Remark 3.7 *Variation of energy integral.* When $f = \frac{1}{2}k^2$, the integral F may be called the energy integral. For this f, we see that

$$G = 4k''' - \frac{3}{2}k^2k' + 16\epsilon k'$$

and the equation (3.20) give an extremal curve relative to the energy functional.

4 How to find plane curves with given general-affine curvature

In Section 2.4, we have derived the expression (2.25) of the general-affine curvature for a graph immersion $x(t) = (t, f(t))$ with $\mu = (f'')^{-2/3} > 0$. Making use of this expression, we study how to find a graph immersion of plane curves with given general-affine curvature, by considering the following nonlinear differential equation directly,

$$\mu(\mu''')^2 = -\epsilon\frac{k^2}{2}(\mu'')^3. \tag{4.1}$$

We regard the function μ' of t as a function of μ and set

$$w(\mu) = \mu'(t) = \frac{d\mu}{dt}.$$

Then, by the chain rule, we have

$$\mu'' = w\dot{w}, \quad \mu''' = w\dot{w}^2 + w^2\ddot{w}.$$

Hence, the equation (4.1) is written as

$$\mu w^2(\dot{w}^2 + w\ddot{w})^2 + \epsilon\frac{k^2}{2}w^3\dot{w}^3 = 0, \tag{4.2}$$

which can be reduced to the Abel equation as follows:
(i) First reduction: We introduce s by setting

$$w(x) = \pm\exp\left(-\epsilon\int s^2 dx\right).$$

Here we choose the sign properly, depending on the function w. Then, we get the equation

$$8x(-\epsilon\dot{s} + s^3)^2 - k^2 s^4 = 0, \qquad (x > 0). \tag{4.3}$$

Therefore, the original differential equation (4.1) is equivalent to

$$\epsilon\dot{s} = \frac{k}{2\sqrt{2x}}s^2 + s^3, \tag{4.4}$$

which is an *Abel equation of the first kind*.

It is easy to see that for constant $k < 0$ with $\epsilon = -1$ or $k \leq -4$ with $\epsilon = 1$, the solution s of (4.4) can be explicitly obtained as

$$s(x) = \frac{a}{\sqrt{2x}} \quad \text{with} \quad a = \frac{-k \pm \sqrt{-16\epsilon + k^2}}{4}.$$

The corresponding curves are given in Examples 2.13 and 2.14. Moreover, in the case of $k = 0$ (both $\epsilon = \pm 1$), the solution s can be obtained as

$$s(x) = \frac{1}{\sqrt{\epsilon(a - 2x)}},$$

where a is some constant. The corresponding curves are given in Example 2.12. On the contrary, in the case of $-4 < k < 0$ for $\epsilon = 1$, the solution s of (4.4) is not easy to write down explicitly. The corresponding curves are given in Example 2.8.

(ii) Second reduction: We define s by

$$w(x) = \pm \exp\left(-\epsilon \int s^{-2} dx\right),$$

by choosing the sign properly. Then a straightforward computation shows that the equation (4.2) is transformed to

$$-k^2 s^2 + 8x(\epsilon + s\dot{s})^2 = 0,$$

which is equivalent to

$$s\dot{s} = \frac{k}{2\sqrt{2x}} s - \epsilon. \tag{4.5}$$

This is a particular case of the *Abel equation of the second kind*. We refer to [PZ, Section 1.3.2] for integrable Abel equations.

Theorem 4.1 *For any general-affine plane curve with graph immersion $(t, f(t))$, there exists a function s given as above such that s satisfies the Abel equation of the first kind or second kind,* (4.4) *or* (4.5)*, respectively. Conversely, for given any function k, a solution s of* (4.4) *or* (4.5) *gives rise to a plane curve of graph immersion $(t, f(t))$ with general-affine curvature k.*

5 General-affine curvature of space curves

The equiaffine treatment of space curves as well as the projective treatment of space curves are classically known. However, it seems that a general-affine treatment of space curves is not fully developed. In this section, we will introduce several notions such as curvature, length parameter and ordinary differential equation associated with space curves from a general-affine point of view.

5.1 Choice of frames for space curves and general-affine curvatures

Let $x : t \longrightarrow x(t) \in \mathbf{A}^3$ be a curve in a 3-dimensional affine space with parameter t and let $e = \{e_1, e_2, e_3\}$ be a frame along x; it is a set of independent vectors of $\mathbf{A}^3$. The vector-valued 1-form dx is written as

$$dx = \omega^1 e_1 + \omega^2 e_2 + \omega^3 e_3, \tag{5.1}$$

and the dependence of e_i on the parameter is described by the equation

$$de_i = \sum_j \omega_i^j e_j, \tag{5.2}$$

where ω^j and ω_i^j are 1-forms as before in the 2-dimensional case and $1 \leq i, j \leq 3$. We call $\{\omega^i, \omega_i^j\}$ the coframe.

We assume in the following that the curve is nondegenerate in the sense that the vectors x', x'' and x''' are linearly independent and that $\omega^2 = \omega^3 = 0$ and $\omega_1^3 = 0$, so that e_1 is tangent to the curve and that $\{e_1, e_2\}$ is the first osculating space of the curve. We write $\omega^1 = \omega$ for simplicity.

Let $\tilde{e} = \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ be another choice of such a frame. Then, it can be written as

$$\tilde{e}_1 = \lambda e_1, \qquad \tilde{e}_2 = \mu e_1 + \nu e_2, \qquad \tilde{e}_3 = \alpha e_1 + \beta e_2 + \gamma e_3,$$

where $\lambda\nu\gamma \neq 0$. The associated coframe is written as $\tilde{\omega}$ and $\tilde{\omega}_i^j$, which satisfies

$$dx = \tilde{\omega}\tilde{e}_1, \qquad d\tilde{e}_i = \sum_j \tilde{\omega}_i^j \tilde{e}_j.$$

Then we have

$$\tilde{\omega} = \lambda^{-1}\omega. \tag{5.3}$$

Since $d\tilde{e}_1$ is represented in two ways, one being

$$d\tilde{e}_1 = (d\lambda)e_1 + \lambda(\omega_1^1 e_1 + \omega_1^2 e_2)$$

and the other being

$$d\tilde{e}_1 = \tilde{\omega}_1^1(\lambda e_1) + \tilde{\omega}_1^2(\mu e_1 + \nu e_2),$$

by comparing the coefficients of e_1 and e_2 in these expressions, we get

$$\nu\tilde{\omega}_1^2 = \lambda\omega_1^2, \tag{5.4}$$

$$\lambda\tilde{\omega}_1^1 + \mu\tilde{\omega}_1^2 = d\lambda + \lambda\omega_1^1. \tag{5.5}$$

Similarly, by considering $d\tilde{e}_2$, we have

$$\gamma\tilde{\omega}_2^3 = \nu\omega_2^3, \tag{5.6}$$

$$\nu\tilde{\omega}_2^2 + \beta\tilde{\omega}_2^3 = d\nu + \mu\omega_1^2 + \nu\omega_2^2, \tag{5.7}$$

$$\lambda\tilde{\omega}_2^1 + \mu\tilde{\omega}_2^2 + \alpha\tilde{\omega}_2^3 = d\mu + \mu\omega_1^1 + \nu\omega_2^1, \tag{5.8}$$

and by $d\tilde{e}_3$ we have

$$\gamma\tilde{\omega}_3^3 = d\gamma + \beta\omega_2^3 + \gamma\omega_3^3, \tag{5.9}$$

$$\nu\tilde{\omega}_3^2 + \beta\tilde{\omega}_3^3 = d\beta + \alpha\omega_1^2 + \beta\omega_2^2 + \gamma\omega_3^2, \tag{5.10}$$

$$\lambda\tilde{\omega}_3^1 + \mu\tilde{\omega}_3^2 + \alpha\tilde{\omega}_3^3 = d\alpha + \alpha\omega_1^1 + \beta\omega_2^1 + \gamma\omega_3^1. \tag{5.11}$$

First note that, from the generality assumption, we have $\omega_1^2 \neq 0$ and $\omega_2^3 \neq 0$. Then, by an appropriate choice of ν and γ, in view of (5.4) and (5.6), we can assume that $\tilde{\omega}_1^2 = \tilde{\omega}$ and $\tilde{\omega}_2^3 = \tilde{\omega}$. Hence, we can restrict our consideration to the case

$$\omega_1^2 = \omega \quad \text{and} \quad \omega_2^3 = \omega$$

in the following. In particular,

$$\nu = \lambda^2 \quad \text{and} \quad \gamma = \lambda^3 \tag{5.12}$$

are necessary. We next see that, from (5.5), (5.7) and (5.9), we have

$$\begin{aligned} 2\tilde{\omega}_1^1 - \tilde{\omega}_2^2 &= 2\omega_1^1 - \omega_2^2 - 3\lambda^{-2}\mu\omega + \lambda^{-3}\beta\omega, \\ 3\tilde{\omega}_1^1 - \tilde{\omega}_3^3 &= 3\omega_1^1 - \omega_3^3 - 3\lambda^{-2}\mu\omega - \lambda^{-3}\beta\omega. \end{aligned}$$

Thus an appropriate choice of the parameters μ and β makes the identities $\tilde{\omega}_2^2 = 3\tilde{\omega}_1^1$ and $\tilde{\omega}_3^3 = 2\tilde{\omega}_1^1$ hold. To keep this condition it is necessary to have $\mu = \beta = 0$. Now (5.8) can be rephrased as

$$\lambda\tilde{\omega}_2^1 + \alpha\tilde{\omega}_2^3 = \lambda^2\omega_2^1,$$

and we choose α so that $\tilde{\omega}_2^1 = 0$. Thus, we can assume that $\omega_2^1 = 0$ and $\alpha = 0$ in the following. Moreover (5.5) is

$$\tilde{\omega}_1^1 = \lambda^{-1}d\lambda + \omega_1^1,$$

and we can choose λ so that $\tilde{\omega}_1^1 = 0$. Therefore $\omega_1^1 = 0$, and to keep this condition, λ is a non-zero constant. With these considerations, the last identities (5.10) and (5.11) turn out to be

$$\tilde{\omega}_3^2 = \lambda\omega_3^2 \quad \text{and} \quad \tilde{\omega}_3^1 = \lambda^2\omega_3^1,$$

respectively. We set

$$\omega_3^2 = -\ell\omega, \quad \omega_3^1 = -m\omega, \tag{5.13}$$

and similarly for $\tilde{\omega}_3^2$ and $\tilde{\omega}_3^1$. Then, we have the covariance

$$\tilde{\ell} = \lambda^2\ell, \quad \text{and} \quad \tilde{m} = \lambda^3 m. \tag{5.14}$$

Thus, we have seen that, given a nondegenerate curve x, there exists a frame e with the coframe of the form

$$\begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \\ -m\omega & -\ell\omega & 0 \end{pmatrix}. \tag{5.15}$$

We remark here that, in the equiaffine treatment of space curves, the scalars ℓ and m above are known to be absolute invariants, called the *equiaffine curvature* and the *equiaffine torsion*, respectively; we refer to Section 5.4. In this paper, we call the point where $\ell = 0$ an *affine inflection point*.

In the following we assume $\ell \neq 0$ and let ϵ denote the sign of ℓ:

$$\epsilon = \mathrm{sign}(\ell).$$

It is an invariant of the curve. Then we define the *general-affine length element* by

$$\omega_s = \sqrt{\epsilon\ell}\,\omega, \tag{5.16}$$

which is well-defined independent of the frame in view of (5.14), and a parameter s for which $ds = \omega_s$ holds is the *general-affine length parameter* determined up to an additive constant.

Definition 5.1 *We call the scalar function k defined by*

$$\frac{d\ell}{\ell} = k\omega_s$$

the first general-affine curvature. In other words,

$$k = \frac{d\log\ell}{ds}. \tag{5.17}$$

We call the scalar function M defined by

$$M = \frac{m}{(\epsilon\ell)^{3/2}} \tag{5.18}$$

the second general-affine curvature of the space curve.

Both curvatures defined above are absolute invariants.

We next define a new frame $\{E_1, E_2, E_3\}$ by setting

$$E_1 = \frac{1}{(\epsilon\ell)^{1/2}}e_1, \qquad E_2 = \frac{1}{\epsilon\ell}e_2, \qquad E_3 = \frac{1}{(\epsilon\ell)^{3/2}}e_3.$$

It is easy to see that this frame does not depend on the choice of λ; hence, it is determined uniquely.

Thus we have proved the following:

Proposition 5.2 *Assume $\ell \neq 0$. Then, the frame $\{E_1, E_2, E_3\}$ is uniquely defined from the immersion and it satisfies the Pfaffian equation*

$$d\begin{pmatrix} x \\ E_1 \\ E_2 \\ E_3 \end{pmatrix} = \Omega \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}, \qquad \Omega = \begin{pmatrix} \omega_s & 0 & 0 \\ -\frac{1}{2}k\omega_s & \omega_s & 0 \\ 0 & -k\omega_s & \omega_s \\ -M\omega_s & -\epsilon\omega_s & -\frac{3}{2}k\omega_s \end{pmatrix}, \tag{5.19}$$

where ω_s is the general-affine length form, k and M are the first and second general-affine curvatures, respectively, and $\epsilon = \mathrm{sign}(\ell)$.

By use of this choice of frame, we can see the following lemma, by a similar reasoning to that for Lemma 2.3.

Lemma 5.3 *The immersion x satisfies the ordinary differential equation*

$$x'''' + 3kx''' + \left(2k' + \frac{11}{4}k^2 + \epsilon\right)x'' + \left(M + \frac{1}{2}\epsilon k + \frac{1}{2}k'' + \frac{7}{4}kk' + \frac{3}{4}k^3\right)x' = 0, \tag{5.20}$$

relative to a general-affine length parameter.

In the definition of the curvature, we had an ambiguity of orientation of the chosen parameter: by the change of the parameter from t to $-t$, the equation transforms to

$$x'''' - 3kx''' + \left(-2k' + \frac{11}{4}k^2 + \epsilon\right)x'' + \left(-M - \frac{1}{2}\epsilon k - \frac{1}{2}k'' + \frac{7}{4}kk' - \frac{3}{4}k^3\right)x' = 0.$$

Namely, the transform $(k, M) \to (-k, -M)$ keeps the form of the equation.

Thus, up to this ambiguity, we have the following theorem.

Theorem 5.4 *Given functions $k(t)$ and $M(t)$ of a parameter t, and $\epsilon = \pm 1$, there exists a nondegenerate space curve $x(t)$ for which t is a general-affine length parameter, k is the first general-affine curvature, M is the second general-affine curvature, and ϵ is the sign of ℓ, uniquely up to a general-affine transformation.*

Analogously to the case of plane curves, we have the following property on the total general-affine curvature:

Corollary 5.5 *Assume that the curve C is nondegenerate and closed, and has no affine inflection point. Then, the total curvature $\int_C k\omega_s$ vanishes. In particular, such a curve has at least two general-affine flat points.*

5.2 Computation of general-affine curvatures of space curves

Let $t \longrightarrow x = x(t) \in \mathbf{A}^3$ be a nondegenerate curve such that the vectors x', x'' and x''' are linearly independent. Since x'''' is written as a linear combination of x', x'' and x''', there are scalar functions $a = a(t)$, $b = b(t)$ and $c = c(t)$ such that

$$x'''' = ax''' + bx'' + cx'. \tag{5.21}$$

We give a formula to compute these coefficients by use of the general-affine curvatures of such a curve. The method is similar to that used for plane curves.

Since $dx = x'\,dt$, the frame vector e_1 is a scalar multiple of x':

$$dx = \omega\, e_1; \qquad e_1 = \lambda x', \quad \omega = \lambda^{-1} dt. \tag{5.22}$$

Then, the differential

$$de_1 = (\lambda^2 x'' + \lambda\lambda' x')\,\omega$$

implies that the second frame vector is

$$e_2 = \lambda^2 x'' + \lambda\lambda' x'.$$

The derivation of e_2 is

$$de_2 = (\lambda^3 x''' + 3\lambda^2\lambda' x'' + (\lambda^2\lambda'' + \lambda\lambda'^2)x')\omega,$$

which is equal to ωe_3:

$$e_3 = (\lambda^3 x''' + 3\lambda^2\lambda' x'' + (\lambda^2\lambda'' + \lambda\lambda'^2)x').$$

Its derivation is

$$de_3 = \Big((\lambda^3 a + 6\lambda^2\lambda')x''' + (\lambda^3 b + 7\lambda\lambda'^2 + 4\lambda^2\lambda'')x'' + (\lambda^3 c + 4\lambda\lambda'\lambda'' + \lambda^2\lambda''' + \lambda'^3)x'\Big)\,dt$$

by use of (5.21). Since de_3 has no e_3-component, we have

$$\lambda a + 6\lambda' = 0, \quad i.e. \quad \lambda = e^{-\frac{1}{6}\int a(t)dt} \tag{5.23}$$

up to a multiplicative constant. Then, de_3 is written as

$$de_3 = (\lambda^2 b + 7\lambda'^2 + 4\lambda\lambda'')\omega e_2 + (\lambda^3 c - \lambda^2\lambda' b - 6\lambda'^3 + \lambda^2\lambda''')\omega e_1.$$

By the definition in (5.13), we have

$$\ell = -(\lambda^2 b + 7\lambda'^2 + 4\lambda\lambda''). \tag{5.24}$$

Also, by the definition of m, we have

$$m = -\lambda^3 c + \lambda^2\lambda' b + 6\lambda'^3 - \lambda^2\lambda'''. \tag{5.25}$$

We now assume that $\ell \neq 0$ and recall that $\epsilon = \mathrm{sign}(\ell)$. Then, we have

$$ds^2 = \epsilon\ell\omega\omega = -\epsilon\left(b + 7\frac{\lambda'^2}{\lambda^2} + 4\frac{\lambda''}{\lambda}\right)dt^2.$$

In terms of a and b,

$$ds^2 = -\epsilon\left(b + \frac{11}{36}a^2 - \frac{2}{3}a'\right)dt^2. \tag{5.26}$$

Hence, a length parameter s which is a function of t is obtained by solving the equation

$$\left(\frac{ds}{dt}\right)^2 = -\epsilon\left(b + \frac{11}{36}a^2 - \frac{2}{3}a'\right).$$

If, in particular, t itself is a length parameter, then we must have

$$\ell = \epsilon\lambda^2, \qquad b = -\epsilon - \frac{11}{36}a^2 + \frac{2}{3}a'. \tag{5.27}$$

By definition, the first curvature k is

$$k = -\frac{1}{3}a. \tag{5.28}$$

We next treat the second curvature M defined in (5.18): From the formula (5.25) above,

$$M = -c + \frac{\lambda'}{\lambda}b + 6\left(\frac{\lambda'}{\lambda}\right)^3 - \frac{\lambda'''}{\lambda}. \tag{5.29}$$

Hence, by (5.23), we can see that

$$c = -M + \frac{1}{6}a\epsilon + \frac{1}{6}a'' - \frac{7}{36}aa' + \frac{1}{36}a^3. \tag{5.30}$$

Thus, we have seen that the differential equation (5.21) agrees with the equation (5.20).

For another parameter $\sigma = \sigma(t)$, we write

$$y(\sigma) = x(t).$$

Then, using the notation $\{\dot{}\}$ for the derivation by σ and $\{'\}$ for the derivation by t, we see that

$$\begin{aligned}
x' &= \dot{y}\sigma', \\
x'' &= \ddot{y}\sigma'^2 + \dot{y}\sigma'', \\
x''' &= \dddot{y}\,\sigma'^3 + 3\ddot{y}\sigma'\sigma'' + \dot{y}\sigma''', \\
x'''' &= \ddddot{y}\,\sigma'^4 + 6\,\dddot{y}\,\sigma'^2\sigma'' + \ddot{y}(3\sigma''^2 + 4\sigma'\sigma''') + \dot{y}\sigma''''.
\end{aligned}$$

Making use of these formulas, we can show that

$$\dddot{\dddot{y}} = A(\sigma)\,\dddot{y} + B(\sigma)\ddot{y} + C(\sigma)\dot{y}, \tag{5.31}$$

where

$$A(\sigma) = \left(a - 6\frac{\sigma''}{\sigma'}\right)\frac{1}{\sigma'}, \tag{5.32}$$

$$B(\sigma) = \left(b + 3a\frac{\sigma''}{\sigma'} - 3\left(\frac{\sigma''}{\sigma'}\right)^2 - 4\frac{\sigma'''}{\sigma'}\right)\frac{1}{\sigma'^2}, \tag{5.33}$$

$$C(\sigma) = \left(c + b\frac{\sigma''}{\sigma'} + a\frac{\sigma'''}{\sigma'} - \frac{\sigma''''}{\sigma'}\right)\frac{1}{\sigma'^3}. \tag{5.34}$$

The differential polynomials that appeared in the representation of b and c in (5.27) and (5.30) have a covariant property with respect to this change of parameters:

Lemma 5.6 *By the change of parameter, the following covariant relations hold.*

$$\begin{aligned} B - \frac{2}{3}\dot{A} + \frac{11}{36}A^2 &= \left(b - \frac{2}{3}a' + \frac{11}{36}a^2\right)\frac{1}{\sigma'^2}, \\ C - \frac{1}{6}\ddot{A} + \frac{7}{36}A\dot{A} - \frac{1}{36}A^3 &= \left(c - \frac{1}{6}a'' + \frac{7}{36}aa' - \frac{1}{36}a^3\right)\frac{1}{\sigma'^3} \\ &\quad + \left(b - \frac{2}{3}a' + \frac{11}{36}a^2\right)\frac{\sigma''}{\sigma'^4}. \end{aligned} \tag{5.35}$$

Thanks to the formulas above, we can compute curvatures according to a procedure similar to that in Section 2.3.

Example 5.7 *Viviani's curve.* This curve is given by the mapping

$$(1 + \cos(2t), \sin(2t), 2\sin(t)).$$

The associated differential equation is

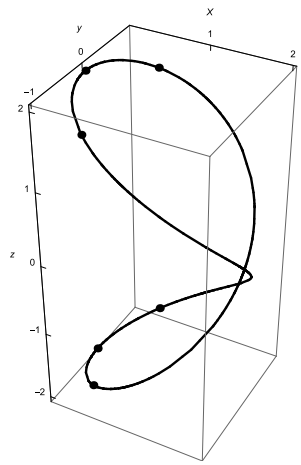
$$x'''' = -\tan(t)x''' - 4x'' - 4\tan(t)x',$$

which is singular at t with $\cos(t) = 0$; in the left figure, this corresponds to $z = \pm 2$. A simple calculation shows the identity

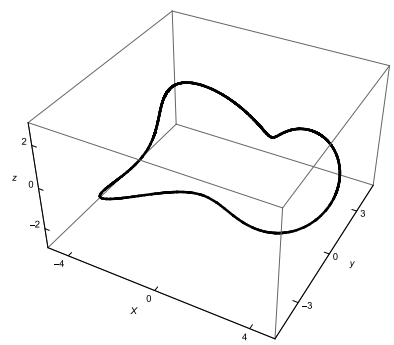
$$-b - \frac{11}{36}a^2 + \frac{2}{3}a' = \frac{5(31\cos(t)^2 - 7)}{36\cos(t)^2};$$

hence, at the values t with $\cos(t)^2 = 7/31$, the general-affine length parameter cannot be defined, namely, $\ell = 0$ at these values; in the left figure, there correspond to the four points with $z = \pm 1.75....$ Except for these six values of t (we marked these points as dots in the figure), ϵ is determined and the curvatures are computable. The first curvature k has the absolute value

$$\frac{2|\sin t|(49 - 31\cos^2 t)}{\sqrt{5}|31\cos^2 t - 7|^{3/2}}.$$



Viviani's curve



Torus knot

Example 5.8 *Torus knot.* The mapping

$$x = ((4 + \cos(3t))\cos(t), (4 + \cos(3t))\sin(t), \sin(3t))$$

defines one of the torus knots. The equation is computed as

$$\begin{aligned} x'''' &= \frac{-3\sin(3t)(12T^2 - 152T - 35)}{P} x''' + \frac{2(52T^3 - 178T^2 + 562T - 891)}{P} x'' \\ &\quad + \frac{12\sin(3t)(8T^2 + 82T + 281)}{P} x', \end{aligned}$$

where

$$T = \cos(3t) \quad \text{and} \quad P = 4T^3 - 76T^2 - 35T + 198.$$

Since $P > 0$ for all value t, the equation is non-singular. With computer assistance, we can see that the length parameter is well-defined and $\epsilon = 1$, and that k has period $2\pi/3$ and symmetry $k(t) = -k(2\pi/3 - t) = -k(-t)$, with values $-4 < k < 4$.

5.3 Space curves with constant curvatures

The curves with constant k and M have a special interest, because such a curve is an orbit of a 1-parameter subgroup of general-affine motions. In [Sc, p. 36–39] a classification of such groups is given. We here give a list of such curves by use of the differential equation treated in the previous subsection.

If the curvatures are constant, then the differential equation

$$x'''' = ax''' + bx'' + cx',$$

has constant coefficients a, b and c. Conversely, assume that a curve satisfies a differential equation with constant coefficients. Then, the length parameter s is obtained by the identity

$$ds^2 = -\epsilon \left(b + \frac{11}{36} a^2 \right) dt^2,$$

where

$$\epsilon = \operatorname{sign}\left(-b - \frac{11}{36}a^2\right).$$

We set

$$q = \sqrt{-\epsilon\left(b + \frac{11}{36}a^2\right)}.$$

Then, the first curvature is

$$k = -\frac{1}{3}\left(\frac{a}{q}\right),$$

and the second curvature is

$$M = -\frac{c}{q^3} + \frac{\epsilon}{6}\left(\frac{a}{q}\right) + \frac{1}{36}\left(\frac{a}{q}\right)^3.$$

Therefore, any differential equation of the form above with constant coefficients defines a curve with constant general-affine curvatures. It is sufficient to solve

$$y''' = ay'' + by' + cy$$

and integrate it to get x. If the function $y = e^{\lambda t}$ is a solution of the equation, then λ is a root of the algebraic equation

$$\lambda^3 - a\lambda^2 - b\lambda - c = 0.$$

Depending on whether λ is a single or multiple root or a real or imaginary root, the form of the solution varies. Without showing detailed computations, we list the result of the classification as follows, which agrees with the classification of the 1-parameter subgroup of general-affine motions. First, we start with some examples.

Example 5.9 Curves with $k = M = 0$. The equation is

$$x'''' = -\epsilon x''.$$

When $\epsilon = 1$, the curve is $x = (t, \cos t, \sin t)$ called a *circular helix* and, when $\epsilon = -1$, the curve is $x = (t, \cosh t, \sinh t)$, called a *hyperbolic helix*.

Example 5.10 *Logarithmic spiral.* This curve is given by $x = (e^{-2\lambda t}, e^{\lambda t}\cos pt, e^{\lambda t}\sin pt)$. The equation is

$$x'''' = (3\lambda^2 - p^2)x'' - 2\lambda(p^2 + \lambda^2)x'.$$

We see that $\epsilon = 1$ (resp. $\epsilon = -1$) when $3\lambda^2 < p^2$ (resp. $3\lambda^2 > p^2$), $k = 0$, and $M = 2\lambda(p^2 + \lambda^2)|3\lambda^2 - p^2|^{-3/2}$.

Example 5.11 Curves given by $x = (e^{\lambda t}, e^{\mu t}, e^{\nu t})$, where the values of λ, μ, ν are distinct, satisfy

$$x'''' = (\lambda + \mu + \nu)x''' - (\lambda\mu + \lambda\nu + \mu\nu)x'' + \lambda\mu\nu x',$$

where ϵ takes both ± 1. When, in addition, $\lambda + \mu + \nu = 0$, we have $k = 0$, $\epsilon = -1$, $q = \sqrt{\lambda^2 + \lambda\mu + \mu^2}$, and $M = \lambda\mu(\lambda + \mu)/q^3$.

Example 5.12 For curves given by $x = (t, e^{\lambda t}, te^{\lambda t})$, the equation is

$$x'''' = 2\lambda x''' - \lambda^2 x'',$$

and $\epsilon = -1$, $k = -\sqrt{2}\mathrm{sign}(\lambda)$, $M = \sqrt{2}\mathrm{sign}(\lambda)$. In particular, the identity $M - k\epsilon = 0$ holds.

Together with these examples, the curves in the next table exhaust the list of nondegenerate curves with constant general-affine curvatures. Note that the hyperbolic helix in Example 5.9 is listed also in the class of curves numbered 1, and Example 5.10 is in the class numbered 7 below. We list also the associated differential equations.

	curves	differential equations
1	$(t, e^{\lambda t}, e^{\mu t})$	$x'''' = (\lambda + \mu)x''' - \lambda\mu x''$
2	$(e^t, te^t, e^{\lambda t})$	$x'''' = (\lambda + 2)x''' - (2\lambda + 1)x'' + \lambda x'$
3	$(t, \frac{1}{2}t^2, e^{\lambda t})$	$x'''' = \lambda x'''$
4	(e^t, te^t, t^2e^t)	$x'''' = 3x''' - 3x'' + x'$
5	$(t, e^t\cos(pt), e^t\sin(pt))$	$x'''' = 2x''' - (p^2 + 1)x''$
6	$(e^{\lambda t}, \cosh(pt), \sinh(pt))$	$x'''' = \lambda x''' + p^2x'' - \lambda p^2 x'$
7	$(e^{\lambda t}, e^{\mu t}\cos(pt), e^{\mu t}\sin(pt))$	$x'''' = (\lambda + 2\mu)x''' - (p^2 + \mu(2\lambda + \mu))x'' + \lambda(p^2 + \mu^2)x'$
8	$(t, \frac{1}{2}t^2, \frac{1}{6}t^3)$	$x'''' = 0$

5.4 From equiaffine to general-affine for space curves

Let us pay some attention to the equiaffine theory of space curves in comparison with the general-affine treatment. Recall the choice of the frame $e = \{e_1, e_2, e_3\}$ and the scalars ℓ and m in Section 5.1, (5.14).

$$d\begin{pmatrix} x \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \\ -m\omega & -\ell\omega & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{5.36}$$

In the equiaffine treatment, it is enough to consider only the unimodular change of frames: *i.e.* $\lambda\nu\gamma = 1$ and e takes values in $\mathrm{SL}(3, \mathbf{R})$. By (5.12), we have $\lambda^6 = 1$. This means that the scalar ℓ is an absolute invariant and the scalar m is an invariant determined up to ± 1 by (5.14). As was remarked in Section 5.1, the scalar ℓ is usually called the *equiaffine curvature* of the space curve and the scalar m is called the *equiaffine torsion*; we refer to the books [Bl, Sc]. The invariant ℓ measures how the space curve differs from the osculating cubic parabola, which is defined to be the curve $(t, t^2/2, t^3/6)$ relative to certain affine coordinates.

The parameter t for which $\omega = dt$ holds is called an *equiaffine length parameter*. Then the equation (5.36) implies that the immersion $x(t)$ satisfies the differential equation

$$x'''' + \ell x'' + mx' = 0, \tag{5.37}$$

which is written in the form of equation (5.21) where $a = 0$, $b = -\ell$, $c = -m$. By (5.16), the general-affine length parameter σ is determined by use of the equiaffine curvature ℓ as

$$d\sigma^2 = Ldt^2, \qquad \text{where} \quad L = \epsilon\ell, \quad \epsilon = \text{sign}(\ell)$$

and, by (5.17) and (5.18), the first and second general-affine curvatures are given as

$$k = L'L^{-3/2}, \qquad M = mL^{-3/2}$$

in terms of equiaffine curvature and equiaffine torsion. Relative to the parameter σ, the map $y(\sigma) = x(t)$ is seen to satisfy the equation (5.31) whose coefficients are given by

$$\begin{aligned} A(\sigma) &= -\frac{3L'}{L^{3/2}}, \\ B(\sigma) &= -\epsilon + \frac{L'}{4L^3} - \frac{2L''}{L^2}, \\ C(\sigma) &= L^{-3/2}\left(-m - \frac{\epsilon L'}{2} - \frac{L'''}{2L} + \frac{3L'L''}{4L^2} - \frac{3L'^3}{8L^3}\right), \end{aligned}$$

by use of (5.32) – (5.34).

The curves for which ℓ and m are constant can be listed, by solving the equation (5.37), as follows; see [Sc, p.75].

1. $(e^{\lambda t}, e^{\mu t}, e^{-(\lambda+\mu)t})$, 2. $(te^{\lambda t}, e^{\lambda t}, e^{-2\lambda t})$, 3. $(e^{-2\alpha t}, e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t))$,
4. $(t, \cosh t, \sinh t)$, 5. $(t, \cos t, \sin t)$, 6. $(t, \frac{1}{2}t^2, \frac{1}{6}t^3)$,

where λ, μ, α, β are nonzero constants. They are homogeneous under equiaffine transformations. The value m is nonzero for the first three and is zero for the last three. The value ℓ is -1, 1 and 0 for the last three, in this order. Except for the last example, the general-affine curvature k is defined, and it is vanishing because ℓ is constant. The listed curves are general-affinely equivalent to some of the examples in the previous section.

5.4.1 Extremal equiaffine space curves

W. Blaschke [Bl] gave a variational formula of the equiaffine length and showed that extremal curves of this variation are the curves with $\ell = m = 0$; hence, the cubic parabola. This will be seen as follows in the present setting.

Theorem 5.13 ([Bl]) *A nondegenerate curve in the affine 3-space is extremal relative to the equiaffine length functional if and only if the equiaffine curvatures ℓ and m are vanishing.*

Proof. Let $x_\eta(t)$ denote a family of curves parametrized by η around $\eta = 0$ and $x_0 = x$ as before. We assume that $x_\eta(t) = x(t)$ outside of a compact set C and $x_0(t)$ is parametrized by equiaffine arc length, and that ω does not vanish anywhere for all η. The equiaffine length functional is given by

$$L(\eta) = \int_C \omega(\eta).$$

Then the curve x is *equiaffine extremal* if $\delta L = 0$. Let $e = \{x, e_1, e_2, e_3\}$ be the frame defined as in (5.36), and set Ω to be 4×3 coefficient matrix. Since $\{e_1, e_2, e_3\}$ are linearly

independent, there exists a 4×3-matrix τ such that

$$\delta \begin{pmatrix} x \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} = \tau \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

We denote the components of Ω and τ by ω_α^β and τ_α^β, respectively, where $0 \leq \alpha \leq 3$ and $1 \leq \beta \leq 3$. Since $\delta de = d\delta e$, we have $\delta\omega_\alpha^\beta - d\tau_\alpha^\beta = \tau_\alpha^\gamma \omega_\gamma^\beta - \omega_\alpha^\gamma \tau_\gamma^\beta$; in terms of entries of Ω and τ, we have

$$\delta\omega - d\tau_0^1 = -(m\tau_0^3 + \tau_1^1)\omega, \tag{5.38}$$
$$-d\tau_0^3 = (\tau_0^2 - \tau_1^3)\omega, \tag{5.39}$$
$$\delta\omega - d\tau_1^2 = (\tau_1^1 - \ell\tau_1^3 - \tau_2^2)\omega, \tag{5.40}$$
$$\delta\omega - d\tau_2^3 = (\tau_2^2 - \tau_3^3)\omega. \tag{5.41}$$

First, note that since $\{e_1, e_2, e_3\}$ takes values in $\mathrm{SL}(3, \mathbf{R})$, we have

$$\tau_1^1 + \tau_2^2 + \tau_3^3 = 0. \tag{5.42}$$

Adding (5.38), (5.40) and (5.41), we have

$$3\delta\omega - d\tau_0^1 - d\tau_1^2 - d\tau_2^3 = -(m\tau_0^3 + \ell\tau_1^3 + \tau_3^3)\omega.$$

On the one hand, subtracting (5.38) from (5.41), we have

$$-d\tau_2^3 + d\tau_0^1 = (m\tau_0^3 - 2\tau_3^3)\omega,$$

where we use the relation (5.42). Thus we have

$$3\delta\omega = d\tau_0^1 + d\tau_1^2 + d\tau_2^3 - (m\tau_0^3 + \ell\tau_1^3)\omega - \frac{1}{2}(d\tau_2^3 - d\tau_0^1 + m\tau_0^3\omega),$$

and therefore

$$3\delta L = \int_C \left(-\frac{3}{2}m\tau_0^3 - \ell\tau_1^3\right)\omega$$

holds. Finally, using (5.39) and integration by parts, we obtain

$$\begin{aligned} 3\delta L &= \int_C \left\{-\frac{3}{2}m\tau_0^3 - \ell\left(\tau_0^{3\prime} + \tau_0^2\right)\right\}\omega \\ &= \int_C \left\{\left(-\frac{3}{2}m + \ell'\right)\tau_0^3 - \ell\tau_0^2\right\}\omega, \end{aligned}$$

where the $\{'\}$ denotes the derivative with respect to the arc length. Since τ_0^3 and τ_0^2 are independent variation vector fields, we have completed the proof.

5.5 From general-affine to projective for space curves

A space curve in $\mathbf{P}^3$ is given by the immersion $t \longmapsto x(t) \in \mathbf{A}^4$ in homogeneous coordinates satisfying an ordinary differential equation of the form

$$x'''' + 4p_1x''' + 6p_2x'' + 4p_3x' + p_4x = 0.$$

By multiplying a nonzero factor to the indeterminate x, the equation is transformed to the equation

$$x''''' + 6P_2x'' + 4P_3x' + P_4x = 0,$$

where

$$\begin{aligned} P_2 &= p_2 - p_1^2 - p_1', \\ P_3 &= p_3 - 3p_1p_2 + 2p_1^3 - p_1'', \\ P_4 &= p_4 - 4p_1p_3 + 6p_1^2p_2 - 6p_1'p_2 - 3p_1^4 + 6p_1^2p_1' + 3(p_1')^2 - p_1'''. \end{aligned}$$

Then, the two forms $\theta_3 dt^3$ and $\theta_4 dt^4$, where

$$\begin{aligned} \theta_3 &= P_3 - \tfrac{3}{2}P_2', \\ \theta_4 &= P_4 - \tfrac{9}{5}P_2'' - \tfrac{81}{25}P_2^2 - 2\theta_3', \end{aligned} \tag{5.43}$$

are fundamental invariant forms: [La]. Provided that $\theta_3 \neq 0$, the parameter s defined as

$$ds = \theta_3^{1/3} dt$$

is called the *projective length parameter*. Relative to this parameter, we can define projective curvatures; we refer to Appendix B.1. When $\theta_3 \equiv 0$, the curve x has a special property that the curve formed by the tangent vectors to the curve x, which lies in the 5-dimensional projective space consisting of lines in $\mathbf{P}^3$, is degenerate in the sense that it belongs to a 4-dimensional hyperplane. Such a curve was said to belong to a *linear complex* and is named Gewindekurve in [Bl].

Given a nondegenerate curve $x(t)$ in the affine space $\mathbf{A}^3$, which is described by the differential equation (5.20), we associate a curve in $\mathbf{P}^3$ by a mapping $t \longmapsto (1, x(t)) \in \mathbf{A}^4$, where 1 is a constant function. Then, the projective invariants are computed by the definition above. In fact, a straightforward computation shows that

$$\theta_3 = \frac{1}{4}(M - \epsilon k), \tag{5.44}$$

$$\theta_4 = -\frac{3}{4}kM - \frac{1}{2}M' + \frac{1}{5}\epsilon k' + \frac{3}{10}\epsilon k^2 - \frac{9}{100}. \tag{5.45}$$

In particular, when $\theta_3 \neq 0$, the projective length parameter s is given as above by use of the general-affine curvatures k and M. When $M = \epsilon k$, the curve belongs to a linear complex. Example 5.12 in the previous subsection is such an example.

6 General-affine extremal space curves and the associated differential equations

In Section 5.1, we have defined a frame for a general-affine space curve under the condition that the curve has no affine inflection point. In this section, we obtain the condition under

which a space curve is extremal relative to the length functional and, in particular, show that any curve with constant general-affine curvatures is extremal.

Let $x_\eta(t)$ be a family of curves parametrized by η around $\eta = 0$ and $x_0 = x$. We assume that $x_\eta(t) = x(t)$ outside a compact set C, and that the invariant ω_s does not vanish anywhere for all η. Then x_η and the corresponding frame $\{E_1, E_2, E_3\}$ satisfy the equation in (5.19). Then the length functional L is given by

$$L(\eta) = \int_C \omega_s(\eta),$$

and the curve $x = x_0$ is said to be *general-affine extremal* if

$$\delta L = \frac{dL}{d\eta}\Big|_{\eta=0} = 0$$

holds for any compactly supported deformation of x.

We now consider the variation

$$\delta \begin{pmatrix} x_\eta \\ E_1 \\ E_2 \\ E_3 \end{pmatrix} = \tau \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}, \qquad \tau = (\tau_\alpha^\beta)_{0\le\alpha\le3, 1\le\beta\le3}.$$

Then the compatibility condition $d\delta = \delta d$ implies that

$$\delta\omega_\alpha^\beta - d\tau_\alpha^\beta = \tau_\alpha^\gamma\omega_\gamma^\beta - \omega_\alpha^\gamma\tau_\gamma^\beta,$$

where we set the entries of Ω in (5.19) as $(\omega_\alpha^\beta)_{0\le i\le3, 1\le\beta\le3}$. Then they are explicitly given by

$$\delta\omega_s - d\tau_0^1 = \left(-\frac{1}{2}k\tau_0^1 - M\tau_0^3 - \tau_1^1\right)\omega_s, \tag{6.1}$$

$$-d\tau_0^2 = (\tau_0^1 - k\tau_0^2 - \epsilon\tau_0^3 - \tau_1^2)\omega_s, \tag{6.2}$$

$$-d\tau_0^3 = \left(\tau_0^2 - \frac{3}{2}k\tau_0^3 - \tau_1^3\right)\omega_s, \tag{6.3}$$

$$\delta\omega_s - d\tau_1^2 = \left(\tau_1^1 - \frac{1}{2}k\tau_1^2 - \epsilon\tau_1^3 - \tau_2^2\right)\omega_s, \tag{6.4}$$

$$-d\tau_1^3 = \left(\tau_1^2 - k\tau_1^3 - \tau_2^3\right)\omega_s, \tag{6.5}$$

$$-d\tau_2^1 = \left(\frac{1}{2}k\tau_2^1 - M\tau_2^3 - \tau_3^1\right)\omega_s, \tag{6.6}$$

$$\delta\omega_s - d\tau_2^3 = \left(\tau_2^2 - \frac{1}{2}k\tau_2^3 - \tau_3^3\right)\omega_s, \tag{6.7}$$

$$-\epsilon\delta\omega_s - d\tau_3^2 = \left(-\epsilon(\tau_3^3 - \tau_2^2) + \tau_3^1 + \frac{1}{2}k\tau_3^2 + M\tau_1^2\right)\omega_s, \tag{6.8}$$

$$-\frac{1}{2}\delta(k\omega_s) - d\tau_1^1 = (-M\tau_1^3 - \tau_2^1)\omega_s, \tag{6.9}$$

$$-\delta(k\omega_s) - d\tau_2^2 = (\tau_2^1 - \epsilon\tau_2^3 - \tau_3^2)\omega_s, \tag{6.10}$$

$$-\frac{3}{2}\delta(k\omega_s) - d\tau_3^3 = (\tau_3^2 + M\tau_1^3 + \epsilon\tau_2^3)\omega_s, \tag{6.11}$$

$$-\delta(M\omega_s) - d\tau_3^1 = (k\tau_3^1 + M(\tau_1^1 - \tau_3^3) + \epsilon\tau_2^1)\omega_s. \tag{6.12}$$

Adding (6.7) and $-\epsilon$(6.8), we have

$$2\delta\omega_s - d\tau_2^3 + \epsilon d\tau_3^2 = \left(-\frac{1}{2}k(\tau_2^3 + \epsilon\tau_3^2) - \epsilon\tau_3^1 - \epsilon M\tau_1^2\right)\omega_s.$$

Then, by Stokes' theorem, we have

$$2\delta\int_C \omega_s = \int_C \left(-\frac{1}{2}k(\tau_2^3 + \epsilon\tau_3^2) - \epsilon\tau_3^1 - \epsilon M\tau_1^2\right)\omega_s. \tag{6.13}$$

Next, from (6.9) + (6.10) $-$(6.11), we get

$$(-\tau_2^2 - \tau_1^1 + \tau_3^3)' = -2(\epsilon\tau_2^3 + \tau_3^2) - 2M\tau_1^3,$$

which is written as

$$-\frac{1}{2}k(\tau_3^2 + \epsilon\tau_2^3) = -\frac{1}{4}\epsilon k(\tau_1^1 + \tau_2^2 - \tau_3^3)' + \frac{1}{2}\epsilon kM\tau_1^3. \tag{6.14}$$

Here the $\{'\}$ denotes $\frac{d}{\omega_s}$. On the one hand, by (6.6),

$$\tau_3^1 = \tau_2^{1\prime} + \frac{1}{2}k\tau_2^1 - M\tau_2^3 \tag{6.15}$$

and (6.10)+(6.11)$-$5(6.9) implies that

$$(-\tau_2^2 - \tau_3^3 + 5\tau_1^1)' = 6(\tau_2^1 + M\tau_1^3). \tag{6.16}$$

Then by use of (6.5) and (6.16), (6.15) can be rephrased as

$$\tau_3^1 = \tau_2^{1\prime} + \frac{1}{12}k(5\tau_1^1 - \tau_2^2 - \tau_3^3)' - M\left(\tau_1^{3\prime} + \tau_1^2 - \frac{1}{2}k\tau_1^3\right). \tag{6.17}$$

Finally (6.14) and (6.17) implies that

$$\begin{aligned} 2\delta\int_C \omega_s &= \frac{\epsilon}{12}\int_C \left(-k\left(8\tau_1^1 + 2\tau_2^2 - 4\tau_3^3\right)' + 12M\tau_1^{3\prime}\right)\omega_s \\ &= \frac{\epsilon}{12}\int_C \left(k'\left(8\tau_1^1 + 2\tau_2^2 - 4\tau_3^3\right) + 12M\tau_1^{3\prime}\right)\omega_s. \end{aligned} \tag{6.18}$$

Here we use integration by parts for the second equality.

We now compute -6(6.1)+2(6.4)+4(6.7). A straightforward computation shows that

$$\begin{aligned} 8\tau_1^1 + 2\tau_2^2 - 4\tau_3^3 &= (6\tau_0^1 - 2\tau_1^2 - 4\tau_2^3)' - 3k\tau_0^1 - 6M\tau_0^3 + k\tau_1^2 + 2\epsilon\tau_1^3 + 2k\tau_2^3 \\ &= 6X' - 4Y' - 3kX + 2kY - 6M\tau_0^3 + 2\epsilon\tau_1^3. \end{aligned}$$

Here $X = \tau_0^1 - \tau_1^2$ and $Y = \tau_2^3 - \tau_1^2$. Thus (6.18) can be again rephrased, by using integration by parts, as

$$\begin{aligned} 24\epsilon\delta\int_C \omega_s = \int_C \{&(-6k'' - 3k'k)X + (4k'' + 2k'k)Y - 6k'M\tau_0^3 \\ &+(2\epsilon k' - 12M')\tau_1^3\}\,\omega_s. \end{aligned} \tag{6.19}$$

Then by (6.5) and (6.2) we have

$$X = \tau_0^1 - \tau_1^2 = -\tau_0^{2\prime} + k\tau_0^2 + \epsilon\tau_0^3, \quad Y = \tau_2^3 - \tau_1^2 = \tau_1^{3\prime} - k\tau_1^3.$$

Finally, making use of (6.3) to erase the τ_1^3-term, we can see that the τ_0^2 part of the integrand of (6.19) is computed as

$$-10k''' - 15k''k - 5k'k^2 - 5k'^2 + 2\epsilon k' - 12M'. \tag{6.20}$$

Similarly the τ_0^3 part of the integrand of (6.19) can be computed as

$$4k'''' + 12k'''k + (11k^2 + 10k' - 8\epsilon)k'' + 7k'^2k - 6\epsilon k'k + 3k'k^3 - 6k'M + 12M'' + 18M'k. \tag{6.21}$$

Theorem 6.1 *A nondegenerate space curve without affine inflection point is general-affine extremal if and only if the following pair of ordinary differential equations is satisfied*:

$$k''' + \frac{3}{2}kk'' + \frac{1}{2}k'^2 + \frac{1}{2}k^2k' - \frac{1}{5}\epsilon k' + \frac{6}{5}M' = 0 \tag{6.22}$$

and

$$k'' + \frac{2}{3}k'k + \frac{5}{6}\epsilon k'M - \frac{3}{2}\epsilon kM' - \epsilon M'' = 0. \tag{6.23}$$

In particular, all space curves which have constant general-affine curvatures are general-affine extremal.

Proof. Inserting (6.20)= 0 into (6.21)= 0, we have the differential equation (6.23).

Example 6.2 *Extremal curves with constant M.* First, assume $M = 0$. Then (6.23) can be easily integrated as

$$k(t) = -3a\tan(at) \quad \text{and} \quad 3a\tanh(at),$$

where a is constant. Inserting this expression into (6.22), we get solutions

$$k(t) = -3a\tan(at), \quad a = \sqrt{2/5} \quad \text{when} \quad \epsilon = 1$$

and

$$k(t) = 3a\tanh(at), \quad a = \sqrt{2/5} \quad \text{when} \quad \epsilon = -1.$$

Second, assume M is a nonzero constant. Then

$$k(t) = -\frac{5}{4}\epsilon M + 3a\tanh(at)$$

is a solution of (6.23) and it satisfies (6.22) if and only if

$$a^2(80a^2 - 125\epsilon^2M^2 + 32\epsilon) = 0.$$

Thus, except for a constant solution, we have the above $k(t)$, where $a = \sqrt{-32 + 125M^2}/(4\sqrt{5})$ when $\epsilon = 1$ and $a = \sqrt{32 + 125M^2}/(4\sqrt{5})$ when $\epsilon = -1$. If we started with $-5/4\epsilon M - 3a\tan(at)$, an another solution of (6.23), then a turns out to be pure imaginary and we get the same curvature function.

We here recall the invariant θ_3 given by the equation (5.44):

$$\theta_3 = \frac{1}{4}(M - \epsilon k).$$

Then, the differential equations (6.22) and (6.23) are written as

$$k''' + \frac{3}{2}kk'' + \frac{1}{2}k'^2 + \frac{1}{2}k^2k' + \epsilon k' + \frac{24}{5}\theta_3' = 0, \tag{6.24}$$

$$\theta_3'' + \frac{3}{2}k\theta_3' - \frac{5}{6}k'\theta_3 = 0. \tag{6.25}$$

Since $\theta_3 = 0$ characterizes a curve belonging to a linear complex, see Section 5.5, in view of the equation (3.11), we have the following corollary.

Corollary 6.3 *The general-affine extremal space curve belongs to a linear complex if and only if $M = \epsilon k$ and k satisfies* (3.11).

Since the differential equation (3.11) is the equation for the plane extremality, we have the following method of constructing an extremal space curve belonging to a linear complex:

Corollary 6.4 *Let k be the general-affine curvature of an extremal plane curve without affine inflection point. Let ϵ denote the sign of this curve. Then, the set $\{k, M, \epsilon\}$, where $M = \epsilon k$, defines a space curve that is general-affine extremal and belonging to a linear complex.*

Thanks to Example 3.5, we can give concrete examples of such curves in Corollary 6.4. The explicit integration of the associated differential equation can be carried out with computer assistance. For example, when $\epsilon = -1$ and $k(t) = \sqrt{2} + 3/t$, we get the curve (x_1, x_2, x_3) for $t > 0$, where

$$\begin{aligned}
x_1 &= \frac{1}{t}, \\
x_2 &= 2^{1/4}\sqrt{\pi}\ \mathrm{erf}\left(\frac{\sqrt{3t}}{2^{1/4}}\right) - \frac{1-\sqrt{2}\ 3t}{(3t)^{3/2}}\exp\left(-\frac{3}{\sqrt{2}}t\right), \\
x_3 &= \int\left\{\frac{6}{t^2}\int H(t)\ dt + \frac{1}{t^{5/2}(\sqrt{2}+6t)}\exp\left(-\frac{3}{\sqrt{2}}t\right)\right\}\ dt,
\end{aligned}$$

with $\mathrm{erf}(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-t^2}dt$ and $H(t) = \dfrac{1}{\sqrt{t}(\sqrt{2}+6t)^2}\exp\left(-\dfrac{3}{\sqrt{2}}t\right)$.

Appendix

A Projective treatment of plane curves

A study of basic notions such as projective length parameter and projective curvature of projective plane curves by use of moving frames was originally done by E. Cartan in [Ca1], [Ca2, Chapitre 2]; nonetheless, in this appendix, we recall how to define such notions in a fairly detailed way because to the authors' knowledge it may not be so familiar in the present day and it is useful for comparison with the general-affine treatment.

By a projective plane curve we mean a nondegenerate immersion into $\mathbf{P}^2$: $t \longrightarrow \underline{x}(t) \in \mathbf{P}^2$. We denote its lift to the affine space $\mathbf{A}^3$ by $t \longrightarrow x(t) \in \mathbf{A}^3 - \{0\}$.

Let $e = \{e_0, e_1, e_2\}$ be a frame along x; at each point of x it is a set of independent vectors of $\mathbf{A}^3$ which depends smoothly on the parameter. We choose $e_0 = x$ for simplicity. Then, the vector-valued 1-form de_0 is written as

$$de_0 = \omega_0^0 e_0 + \omega_0^1 e_1 + \omega_0^2 e_2,$$

and the dependence of e_i on the parameter is similarly written as

$$de_i = \sum_{j=0}^{2} \omega_i^j e_j, \qquad i = 1, 2.$$

In the following, we consider frames such that the space generated by e_0 and e_1 is the space generated by the vector $x(t)$ and the tangent vector $x'(t)$. Then, $\omega := \omega_0^1$ is nontrivial and $\omega_0^2 = 0$. We consider furthermore frames such that the condition $\omega_0^0 + \omega_1^1 + \omega_2^2 = 0$ always holds. Thus, we have the equation

$$\begin{pmatrix} de_0 \\ de_1 \\ de_2 \end{pmatrix} = \begin{pmatrix} \omega_0^0 & \omega & 0 \\ \omega_1^0 & \omega_1^1 & \omega_1^2 \\ \omega_2^0 & \omega_2^1 & \omega_2^2 \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \end{pmatrix}, \tag{A.1}$$

where the matrix (ω_i^j) is called the coframe.

Once we choose a frame e with the required property, then another frame $\tilde{e}$ is given as

$$\begin{pmatrix} \tilde{e}_0 \\ \tilde{e}_1 \\ \tilde{e}_2 \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 \\ \mu & \alpha & 0 \\ \nu & \beta & \gamma \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \end{pmatrix}, \tag{A.2}$$

where $\lambda\alpha\gamma = 1$.

We write the coframe for the frame $\tilde{e}$ by $\tilde{\omega}_i^j$. Since the derivation $d\tilde{e}_0$ has two expressions $d\tilde{e}_0 = \tilde{\omega}_0^0 \tilde{e}_0 + \tilde{\omega}\tilde{e}_1$ and $d\tilde{e}_0 = d(\lambda e_0) = d\lambda e_0 + \lambda(\omega_0^0 e_0 + \omega e_1)$, we get the identities

$$\alpha\tilde{\omega} = \lambda\omega, \tag{A.3}$$

$$\lambda\tilde{\omega}_0^0 + \mu\tilde{\omega} = \lambda\omega_0^0 + d\lambda. \tag{A.4}$$

Similarly considering the frames $\tilde{e}_1$ and $\tilde{e}_2$, we get

$$\gamma\tilde{\omega}_1^2 = \alpha\omega_1^2, \tag{A.5}$$
$$\alpha\tilde{\omega}_1^1 + \beta\tilde{\omega}_1^2 = \alpha\omega_1^1 + \mu\omega + d\alpha, \tag{A.6}$$
$$\lambda\tilde{\omega}_1^0 + \mu\tilde{\omega}_1^1 + \nu\tilde{\omega}_1^2 = \alpha\omega_1^0 + \mu\omega_0^0 + d\mu, \tag{A.7}$$
$$\gamma\tilde{\omega}_2^2 = \gamma\omega_2^2 + \beta\omega_1^2 + d\gamma, \tag{A.8}$$
$$\alpha\tilde{\omega}_2^1 + \beta\tilde{\omega}_2^2 = \gamma\omega_2^1 + \beta\omega_1^1 + \nu\omega + d\beta, \tag{A.9}$$
$$\lambda\tilde{\omega}_2^0 + \mu\tilde{\omega}_2^1 + \nu\tilde{\omega}_2^2 = \gamma\omega_2^0 + \beta\omega_1^0 + \nu\omega_0^0 + d\nu. \tag{A.10}$$

A.1 Projective curvature of plane curves

Making use of these identities, we reduce the freedom of choice of frames by a stepwise procedure. First, by (A.3),

$$\tilde{\omega} = \frac{\lambda}{\alpha}\omega.$$

If we set $\omega_1^2 = h\omega$ and $\tilde{\omega}_1^2 = \tilde{h}\tilde{\omega}$, then by (A.5), we have

$$\tilde{h} = \alpha^3 h.$$

We assume that the curve is nondegenerate: $h \neq 0$. Then, by a choice of α, we see that there is a frame with $\tilde{h} = 1$. Then, we can restrict our consideration to frames with $h = 1$ and, therefore we necessarily have that $\alpha = 1$ and $\lambda\gamma = 1$. This means that

$$\omega_1^2 = \omega, \qquad \tilde{\omega} = \lambda\omega, \qquad \tilde{\omega}_1^2 = \lambda\omega.$$

By (A.4), we can choose μ so that $\tilde{\omega}_0^0 = 0$ and to keep this condition for all frames, $\omega_0^0 = 0$, we have

$$d\lambda = \lambda\mu\omega.$$

Also by (A.8), we can assume $\omega_2^2 = 0$ and

$$d\gamma = -\beta\omega.$$

Since $\omega_0^0 + \omega_1^1 + \omega_2^2 = 0$ by assumption, we conclude that $\omega_1^1 = 0$, and by (A.6) we see that

$$\mu = \lambda\beta.$$

Now (A.7) reduces to $\tilde{\omega}_1^0 = \gamma\omega_1^0 - \nu\omega + \gamma d\mu$ and (A.9) reduces to $\tilde{\omega}_2^1 = \gamma\omega_2^1 + \nu\omega + d\beta$; therefore

$$\tilde{\omega}_1^0 - \tilde{\omega}_2^1 = \gamma(\omega_1^0 - \omega_2^1) - 2\nu\omega + \gamma d\mu - d\beta.$$

Then, by a choice of ν, there exists a frame with $\tilde{\omega}_1^0 = \tilde{\omega}_2^1$. Assuming this identity for all frames, we necessarily have

$$2\nu\omega = \gamma d\mu - d\beta.$$

Using the formulas of $d\lambda$ and $d\gamma$ above, and the identity $\lambda\mu = 1$, we see that

$$2\nu = \mu\beta.$$

Finally, by simplifying (A.10) using (A.7), we see that

$$\lambda\tilde{\omega}_2^0 = \gamma\omega_2^0.$$

We set

$$\omega_2^0 = \rho\omega$$

and $\tilde{\omega}_2^0 = \tilde{\rho}\tilde{\omega}$, and then we have $\tilde{\rho} = \gamma^3\rho$. Here we have two cases $\rho = 0$ and $\rho \neq 0$. In the latter case, we can find a frame with $\rho = -1$ and, therefore, we can restrict the change of frame to the case $\gamma = 1$. In this case, by the formulas already obtained above, we see that $\lambda = 1$, $\mu = \nu = \beta = 0$; namely, we have uniquely determined the form ω and the identity (A.7) implies $\tilde{\omega}_1^0 = \omega_1^0$. If we set

$$\omega_1^0 = -k_p\omega, \tag{A.11}$$

and $\tilde{\omega}_1^0 = -\tilde{k}_p\tilde{\omega}$, then it turns out that $\tilde{k}_p = k_p$.

Now, we call ω the *projective length element* and k_p the *projective curvature*. Thus we have the following:

Lemma A.1 *Assume that the curve is nondegenerate and the scalar ρ is nonvanishing. Then, the frame is uniquely defined such that the coframe has the form*

$$\Omega = \begin{pmatrix} 0 & \omega & 0 \\ -k_p\omega & 0 & \omega \\ -\omega & -k_p\omega & 0 \end{pmatrix}, \tag{A.12}$$

where k_p is the projective curvature and the 1-form ω is the projective length element.

E. Cartan [Ca2] called this coframe the formula of Frenet.

When $\rho = 0$, the form ω_1^0 is not uniquely determined and we need a separate consideration. The condition $\rho = 0$ is equivalent to $P = 0$, where P is defined in (2.31) in Section 2.6.

A.2 Extremal projective plane curves

In this subsection, we derive the differential equation of an extremal projective plane curve according to [Ca1]. Let $x_\eta(t)$ be a family of projective plane curves such that $x_0 = x$, and let ω be the projective length element of x_t. Assume that $x = x_0$ is parametrized by projective arc length, $x_\eta(t) = x(t)$ outside a compact set C and ω does not vanish everywhere for all η. Consider the length functional

$$L = \int_C \omega(\eta).$$

It is natural to call the curve $x = x_0$ *projective extremal* with respect to the length functional if $\delta L = 0$ holds. In the following we compute the associated differential equation for an extremal projective plane curve. Let k denote the curvature k_p in this subsection. Let $e = \{e_0, e_1, e_2\}$ be the frame for a family of projective plane curves defined as in (A.12). Then the variation of the frame e can be defined as

$$\delta \begin{pmatrix} e_0 \\ e_1 \\ e_2 \end{pmatrix} = \tau \begin{pmatrix} e_0 \\ e_1 \\ e_2 \end{pmatrix},$$

where τ is a 3×3 matrix. Denote the entries of Ω and τ by ω_α^β and τ_α^β. Here the Greek letters run from 0 through 2. Then the compatibility $d\delta e = \delta de$ implies that

$$\delta\omega - d\tau_0^1 = (\tau_0^0 - k\tau_0^2 - \tau_1^1)\omega, \tag{A.13}$$
$$-d\tau_0^2 = (\tau_0^1 - \tau_1^2)\omega, \tag{A.14}$$
$$-d\tau_1^1 = (\tau_1^0 - k\tau_1^2 + k\tau_0^1 - \tau_2^1)\omega, \tag{A.15}$$
$$\delta\omega - d\tau_1^2 = (\tau_1^1 + k\tau_0^2 - \tau_2^2)\omega, \tag{A.16}$$
$$-\delta\omega - d\tau_2^0 = (-k\tau_2^1 - \tau_2^2 + \tau_0^0 + k\tau_1^0)\omega. \tag{A.17}$$

We first note that since the frame $e = \{e_0, e_1, e_2\}$ takes values in $\mathrm{SL}(3, \mathbf{R})$,

$$\tau_0^0 + \tau_1^1 + \tau_2^2 = 0 \tag{A.18}$$

holds. Now adding (A.13), (A.16) and -1(A.17), we have

$$3\delta\omega - d\tau_0^1 - d\tau_1^2 + d\tau_2^0 = k(\tau_2^1 - \tau_1^0)\omega. \tag{A.19}$$

Then by (A.15) and (A.14)

$$\tau_2^1 - \tau_1^0 = \tau_1^{1\prime} - k\tau_0^{2\prime} \tag{A.20}$$

holds. Here the $\{'\}$ denotes the derivation with respect to the projective length element, *i.e.* $a' = \frac{da}{\omega}$ for a function a. Moreover, subtracting (A.13) from (A.16) and using (A.18), we have

$$d(\tau_0^1 - \tau_1^2) = (3\tau_1^1 + 2k\tau_0^2)\omega.$$

Rephrasing the above equation by using (A.14), we have

$$3\tau_1^1 = -2k\tau_0^2 + (\tau_0^1 - \tau_1^2)' = -2k\tau_0^2 - \tau_0^{2\prime\prime}. \tag{A.21}$$

Inserting (A.21) and (A.20) into (A.19), we have

$$3\delta\omega = d\tau_0^1 + d\tau_1^2 - d\tau_2^0 + \left(-\frac{2}{3}kk'\tau_0^2 - \frac{5}{3}k^2\tau_0^{2\prime} - \frac{1}{3}k\tau_0^{2\prime\prime\prime}\right)\omega. \tag{A.22}$$

Finally, by integration by parts and Stokes' theorem,

$$9\delta L = \int_C (k''' + 8kk')\,\tau_0^2\omega.$$

Therefore we have the following:

Theorem A.2 [Ca1] *A plane curve without inflection points is projective extremal relative to the length functional if and only if*

$$k''' + 8kk' = 0$$

holds.

B Projective treatment of space curves

We give a summary on how to define two kinds of projective curvatures of space curves in projective 3-space. We recall the normalization of frame by use of the Halphen canonical form of the differential equation, [La], and another normalization by G. Bol [Bol]. Then, following M. Kimpara [Ki], we compute the variational formula of the projective length functional for the curve with $\theta_3 \neq 0$, according to which we can see that such a curve is extremal if and only if both projective curvatures are constant; such curves are classified in Appendix C.

B.1 Projective curvatures for space curves

In Section 5.5, we introduced the differential equation

$$x'''' + 6P_2x'' + 4P_3x' + P_4x = 0, \tag{B.1}$$

which describes any nondegenerate space curve in projective space : $t \longrightarrow x(t) \in \mathbf{P}^3$, and defined two invariant forms $\theta_3 dt^3$ and $\theta_4 dt^4$ in (5.43). We now assume $\theta_3 \neq 0$ and choose the parameter t so that $\theta_3 = 1$; namely, let t be a projective length parameter. Then, the equation is written as

$$x'''' + 6P_2x'' + 2(2 + 3P_2')x' + P_4x = 0, \tag{B.2}$$

which is called the *Halphen canonical form*, and the two scalars P_2 and P_4 (or θ_4 instead of P_4) are called the *projective curvatures*; we refer to [FC, p.26]. For the sake of later reference, we set

$$k = \frac{3}{5}P_2, \tag{B.3}$$

$$\theta = \theta_4, \tag{B.4}$$

and call them the *first projective curvature* and the *second projective curvature*, respectively. The curve with constant curvatures is called an *anharmonic* curve and we give a complement to the study in [Wi, Section 3 of Chapter 14] by giving a classification of such curves in Appendix C.

With this preparation, let us choose a frame $(e_1, e_2, e_3, e_4) = (x, x', x'', x''')$; then the coframe is given as

$$d\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -P_4 & -4-6P_2' & -6P_2 & 0 \end{pmatrix} dt \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}. \tag{B.5}$$

On the other hand, in the book [Bol, Section 39], another choice of frame was given by directly using the invariants θ_3 and θ_4. It is done, for the differential equation (B.1), by choosing $\{x, u, y, z\}$, where

$$u = x', \quad y = u' + \frac{9}{5}P_2x, \quad z = y' + \frac{12}{5}P_2u + 2\theta_3x.$$

In terms of the frame $\{e_1, e_2, e_3, e_4\}$, it is given as

$$x = e_1,\ u = e_2,\ y = e_3 + \frac{9}{5}P_2 e_1, z = e_4 + \frac{21}{5}P_2 e_2 + \left(\frac{9}{5}P_2' + 2\theta_3\right) e_1. \tag{B.6}$$

Then, we can see that

$$d\begin{pmatrix} x \\ u \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -3k & 0 & 1 & 0 \\ -2\theta_3 & -4k & 0 & 1 \\ -\theta_4 & -2\theta_3 & -3k & 0 \end{pmatrix} dt \begin{pmatrix} x \\ u \\ y \\ z \end{pmatrix}.$$

When this choice of the frame is applied to the equation (B.2), the frame equation is

$$d\begin{pmatrix} x \\ u \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -3k & 0 & 1 & 0 \\ -2 & -4k & 0 & 1 \\ -\theta & -2 & -3k & 0 \end{pmatrix} dt \begin{pmatrix} x \\ u \\ y \\ z \end{pmatrix}. \tag{B.7}$$

B.2 Extremal projective space curves

We now derive the differential equation of an extremal projective space curve according to [Ki]. Let $x_\eta(t)$ be a family of projective space curves such that $x_0 = x$, and let ω be the projective length element of x_t. Assume that $x = x_0$ is parametrized by projective arc length, $x_\eta(t) = x(t)$ outside a compact set C, and that ω does not vanish anywhere for all η. Let k_1 and k_2, respectively, denote the curvatures k and θ in this subsection. Let $e = \{e_1, e_2, e_3, e_4\}$ be the frame for a family of projective space curves defined in (B.6). Then, according to (B.7), we have

$$de = \begin{pmatrix} 0 & \omega & 0 & 0 \\ -3k_1\omega & 0 & \omega & 0 \\ -2\omega & -4k_1\omega & 0 & \omega \\ -k_2\omega & -2\omega & -3k_1\omega & 0 \end{pmatrix} e.$$

The variation of e can be computed as $\delta e = (\tau_i^j)e$. Then the compatibility condition $d\delta e = \delta de$ is equivalent to

$$-d\tau_0^0 = (-3k_1\tau_0^1 - 2\tau_0^2 - k_2\tau_0^3 - \tau_1^0)\omega \tag{B.8}$$
$$\delta\omega - d\tau_0^1 = (\tau_0^0 - 4k_1\tau_0^2 - 2\tau_0^3 - \tau_1^1)\omega \tag{B.9}$$
$$-d\tau_0^2 = (\tau_0^1 - 3k_1\tau_0^3 - \tau_1^2)\omega \tag{B.10}$$
$$-d\tau_0^3 = (\tau_0^2 - \tau_1^3)\omega \tag{B.11}$$
$$-3\delta(k_1\omega) - d\tau_1^0 = (3k_1\tau_0^0 - 3k_1\tau_1^1 - 2\tau_1^2 - k_2\tau_1^3 - \tau_2^0)\omega \tag{B.12}$$
$$-d\tau_1^1 = (3k_1\tau_0^1 + \tau_1^0 - 4k_1\tau_1^2 - 2\tau_1^3 - \tau_2^1)\omega \tag{B.13}$$
$$\delta\omega - d\tau_1^2 = (3k_1\tau_0^2 + \tau_1^1 - 3k_1\tau_1^3 - \tau_2^2)\omega \tag{B.14}$$
$$-d\tau_1^3 = (3k_1\tau_0^3 + \tau_1^2 - \tau_2^3)\omega \tag{B.15}$$

$$-2\delta\omega - d\tau_2^0 = (2\tau_0^0 + 4k_1\tau_1^0 - 3k_1\tau_2^1 - 2\tau_2^2 - k_2\tau_2^3 - \tau_3^0)\omega \tag{B.16}$$
$$-4\delta(k_1\omega) - d\tau_2^1 = (2\tau_0^1 + 4k_1\tau_1^1 + \tau_2^0 - 4k_1\tau_2^2 - 2\tau_2^3 - \tau_3^1)\omega \tag{B.17}$$
$$-d\tau_2^2 = (2\tau_0^2 + 4k_1\tau_1^2 + \tau_2^1 - 3k_1\tau_2^3 - \tau_3^2)\omega \tag{B.18}$$
$$\delta\omega - d\tau_2^3 = (\tau_0^0 + 2\tau_0^3 + \tau_1^1 + 4k_1\tau_1^3 + 2\tau_2^2)\omega \tag{B.19}$$
$$-\delta(k_2\omega) - d\tau_3^0 = (2k_2\tau_0^0 + 2\tau_1^0 + k_2\tau_1^1 + 3k_1\tau_2^0 + k_2\tau_2^2 - 3k_1\tau_3^1 - 2\tau_3^2)\omega \tag{B.20}$$
$$-2\delta\omega - d\tau_3^1 = (2\tau_0^0 + k_2\tau_0^1 + 4\tau_1^1 + 3k_1\tau_2^1 + 2\tau_2^2 + \tau_3^0 - 4k_1\tau_3^2)\omega \tag{B.21}$$
$$-3\delta(k_1\omega) - d\tau_3^2 = (3k_1\tau_0^0 + k_2\tau_0^2 + 3k_1\tau_1^1 + 2\tau_1^2 + 6k_1\tau_2^2 + \tau_3^1)\omega \tag{B.22}$$
$$-d\tau_3^3 = (k_2\tau_0^3 + 2\tau_1^3 + 3k_1\tau_2^3 + \tau_3^2)\omega. \tag{B.23}$$

Here we use the relation $\tau_3^3 = -\tau_0^0 - \tau_1^1 - \tau_2^2$. First adding (B.10) and (B.15),

$$\tau_0^1 - \tau_2^3 = -\tau_0^{2\prime} - \tau_1^{3\prime},$$

and by using (B.11), the above equation can be rephrased as

$$\tau_0^1 - \tau_2^3 = -2\tau_0^{2\prime} - \tau_0^{3\prime\prime}. \tag{B.24}$$

Here the $\{'\}$ denotes the derivation with respect to the arc length. Next by (B.13)

$$\begin{aligned}\tau_1^0 - \tau_2^1 &= -\tau_1^{1\prime} + 2\tau_1^3 - k_1(3\tau_0^1 - 4\tau_1^2)\\ &= -\tau_1^{1\prime} + 2(\tau_0^2 - \tau_0^{3\prime}) + k_1^2\tau_0^1 + 4k_1(\tau_0^{2\prime} - 3k_1\tau_0^3),\end{aligned}$$

where we use (B.11) and (B.10). Subtracting (B.9) from (B.14),

$$\tau_2^2 = 2\tau_1^1 - \tau_0^0 + 7k_1\tau_0^2 - 3k_1\tau_1^3 + 2\tau_0^3 + (\tau_1^2 - \tau_0^1)'.$$

Then, by (B.10) and (B.11), we can rephrase the above equation as

$$\tau_2^2 = 2\tau_1^1 - \tau_0^0 + 4k_1\tau_0^2 + 2\tau_0^3 - 3k_1{}'\tau_0^3 - 6k_1\tau_0^{3\prime} + \tau_0^{2\prime\prime}. \tag{B.25}$$

Subtracting (B.9) from (B.19), we have

$$\tau_1^1 + \tau_2^2 = \frac{1}{2}(\tau_0^1 - \tau_2^3)' - 2\tau_0^3 - 2k_1(\tau_1^3 + \tau_0^2).$$

Then, by (B.24) and (B.11), we can rephrase the above equation as

$$\tau_1^1 + \tau_2^2 = \frac{1}{2}\left(-2\tau_0^{2\prime\prime} - \tau_0^{3\prime\prime\prime}\right) - 2\tau_0^3 - 2k_1(2\tau_0^2 + \tau_0^{3\prime}). \tag{B.26}$$

Then, subtracting (B.26) from (B.25), we have

$$3\tau_1^1 - \tau_0^0 = -8k_1\tau_0^2 - 2\tau_0^{2\prime\prime} + (3k_1{}' - 4)\tau_0^3 + 4k_1\tau_0^{3\prime} - \frac{1}{2}\tau_0^{3\prime\prime\prime}. \tag{B.27}$$

Now, adding (B.13) and (B.18),

$$\tau_3^2 - \tau_1^0 = (\tau_1^{1\prime} + \tau_2^{2\prime}) + 3k_1(\tau_0^1 - \tau_2^3) - 2\tau_1^3 + 2\tau_0^2.$$

By using (B.26) and (B.24), we can rephrase the above equation as

$$\tau_3^2 - \tau_1^0 = -4k_1{}'\tau_0^2 - 10k_1\tau_0^{2\prime} - \tau_0^{2\prime\prime\prime} - 2\tau_0^{3\prime} - 2k_1{}'\tau_0^{3\prime} - 5k_1\tau_0^{3\prime\prime} - \frac{1}{2}\tau_0^{3\prime\prime\prime\prime}. \tag{B.28}$$

Adding 4(B.9), (B.16) and (B.21), we have

$$8\tau_0^0 = -(4\tau_0^1 + \tau_2^0 + \tau_3^1)' + 16k_1\tau_0^2 + 8\tau_0^3 + 4k_1(\tau_3^2 - \tau_1^0) - k_2(\tau_0^1 - \tau_2^3).$$

Then, by using (B.28) and (B.24), we can rephrase the above equation as

$$\begin{aligned} -2\tau_0^0 =& \frac{1}{4}(4\tau_0^1 + \tau_2^0 + \tau_3^1)' + 4k_1({k_1}' - 1)\tau_0^2 + \left(10k_1^2 - \frac{1}{2}k_2\right)\tau_0^{2\prime} + k_1\tau_0^{2\prime\prime\prime} \\ & - 2\tau_0^3 + 2k_1\left(1 + {k_1}'\right)\tau_0^{3\prime} + \left(5k_1^2 - \frac{1}{4}k_2\right)\tau_0^{3\prime\prime} + \frac{1}{2}k_1\tau_0^{3\prime\prime\prime\prime}. \end{aligned} \tag{B.29}$$

Adding (B.29) and (B.27), we have

$$\begin{aligned} 3(\tau_1^1 - \tau_0^0) =& \frac{1}{4}(4\tau_0^1 + \tau_2^0 + \tau_3^1)' + 4k_1({k_1}' - 3)\tau_0^2 + \left(10k_1^2 - \frac{1}{2}k_2\right)\tau_0^{2\prime} - 2\tau_0^{2\prime\prime} + k_1\tau_0^{2\prime\prime\prime} \\ & + (3{k_1}' - 6)\tau_0^3 + 2k_1(3 + {k_1}')\tau_0^{3\prime} + \left(5k_1^2 - \frac{1}{4}\right)\tau_0^{3\prime\prime} - \frac{1}{2}\tau_0^{3\prime\prime\prime} + \frac{1}{2}{k_1}''''\tau_0^3. \end{aligned} \tag{B.30}$$

Therefore we have

$$\begin{aligned} & \tau_0^0 - \tau_1^1 - 2\tau_0^3 - 4k_1\tau_0^2 \\ =& -\frac{1}{12}(4\tau_0^1 + \tau_2^0 + \tau_3^1)' - \frac{4}{3}k_1{k_1}'\tau_0^2 - \frac{1}{6}(20k_1^2 - k_2)\tau_0^{2\prime} + \frac{2}{3}\tau_0^{2\prime\prime} - \frac{1}{3}k_1\tau_0^{2\prime\prime\prime} \\ & - {k_1}'\tau_0^3 - \frac{2}{3}k_1({k_1}' + 3)\tau_0^{3\prime} - \frac{1}{12}\left(20k_1^2 - k_2\right)\tau_0^{3\prime\prime} + \frac{1}{6}\tau_0^{3\prime\prime\prime} - \frac{1}{6}k_1\tau_0^{3\prime\prime\prime\prime}. \end{aligned} \tag{B.31}$$

Finally, by using Stokes' theorem, we obtain

$$\begin{aligned} \delta\int_C \omega = \int_C \Big\{ & -\frac{4}{3}k_1{k_1}'\tau_0^2 - \frac{1}{6}(20k_1^2 - k_2)\tau_0^{2\prime} - \frac{1}{3}k_1\tau_0^{2\prime\prime\prime} - {k_1}'\tau_0^3 - \frac{2}{3}k_1({k_1}' + 3)\tau_0^{3\prime} \\ & - \frac{1}{12}\left(20k_1^2 - k_2\right)\tau_0^{3\prime\prime} - \frac{1}{6}k_1\tau_0^{3\prime\prime\prime\prime}\Big\}\,\omega. \end{aligned}$$

By using integration by parts, we finally obtain differential equations for an extremal space curve as

$$\begin{cases} {k_1}''' + 16k_1{k_1}' - \frac{1}{2}{k_2}' = 0, \\ {k_1}'''' + 16k_1{k_1}'' + 16({k_1}')^2 + 6{k_1}' - \frac{1}{2}{k_2}'' = 0. \end{cases} \tag{B.32}$$

From these equations, the following theorem holds.

Theorem B.1 ([Ki], pp. 233-234 in [Bl]) *A space curve without inflection points is projective extremal relative to the length functional if and only if the both curvatures are constant.*

Proof. Subtracting the derivative of the first equation from the second equation (B.32), we have ${k_1}' = 0$. Then ${k_2}' = 0$ follows immediately.

C Classification of space curves with constant projective curvatures

In Section B.1, we have seen that any nondegenerate curve with constant projective curvatures is defined by a differential equation

$$x'''' = ax'' + bx' + cx, \tag{C.1}$$

where the coefficients a, b and c are constants. Such a curve is described by linearly independent solutions and we get a classification of such curves by listing such solutions.

Let $e^{\lambda t}$ be a solution of the differential equation, then λ is a solution of the algebraic equation

$$\lambda^4 - a\lambda^2 - b\lambda - c = 0.$$

Since the sum of the four roots is zero, we have the following cases to consider separately.

1. four distinct real roots: $(\lambda, \mu, \nu, -(\lambda+\mu+\nu))$
2. one set of double real roots: $(2\lambda, 2\mu, -\lambda-\mu, -\lambda-\mu)$
3. one set of complex conjugate roots: $(2\lambda, 2\mu, -(\lambda+\mu)+ip, -(\lambda+\mu)-ip)$
4. two distinct sets of complex roots: $(\lambda+ip, \lambda-ip, -\lambda+iq, -\lambda-iq)$
5. one set of complex roots and double real roots: $(-\lambda, -\lambda, \lambda+ip, \lambda-ip)$
6. two sets of double roots: $(\lambda, \lambda, -\lambda, -\lambda)$
7. pure imaginary roots: $(ip, ip, -ip, -ip)$
8. triple roots: $(\lambda, \lambda, \lambda, -3\lambda)$
9. trivial case: $(0, 0, 0, 0)$,

where λ, μ, ν, p, q are real constants. For each case above, we compute the set of solutions to define the immersion: we denote by CVi the curve in the case i above. The associated differential equation defining the immersion is also listed, where a, b, c denote the coefficients of the differential equation.

Curves	mapping
CV1	$[e^{-(\lambda+\mu+\nu)t}, e^{\lambda t}, e^{\mu t}, e^{\nu t}]$
CV2	$[e^{-(\lambda+\mu)t}, te^{-(\lambda+\mu)t}, e^{2\lambda t}, e^{2\mu t}]$
CV3	$[e^{2\lambda t}, e^{2\mu t}, e^{-(\lambda+\mu)t}\cos(pt), e^{-(\lambda+\mu)t}\sin(pt)]$
CV4	$[e^{\lambda t}\cos(pt), e^{\lambda t}\sin(pt), e^{-\lambda t}\cos(qt), e^{-\lambda t}\sin(qt)]$
CV5	$[e^{\lambda t}\cos(pt), e^{\lambda t}\sin(pt), e^{-\lambda t}, te^{-\lambda t}]$
CV6	$[e^{\lambda t}, te^{\lambda t}, e^{-\lambda t}, te^{-\lambda t}]$
CV7	$[\cos(pt), \sin(pt), t\cos(pt), t\sin(pt)]$
CV8	$[e^{\lambda t}, te^{\lambda t}, t^2e^{\lambda t}, e^{-3\lambda t}]$
CV9	$[1, t, t^2, t^3]$

Curves	a	b	c
CV1	$\lambda^2+\lambda\mu+\lambda\nu+\mu^2+\mu\nu+\nu^2$	$-(\mu+\nu)(\nu+\lambda)(\lambda+\mu)$	$\lambda\mu\nu(\lambda+\mu+\nu)$
CV2	$3\lambda^2+2\lambda\mu+3\mu^2$	$2(\lambda+\mu)(-\mu+\lambda)^2$	$-4\lambda\mu(\lambda+\mu)^2$
CV3	$3\lambda^2+2\lambda\mu+3\mu^2-p^2$	$2(\lambda+\mu)(\lambda^2-2\lambda\mu+\mu^2+p^2)$	$-4\lambda\mu(\lambda^2+2\lambda\mu+\mu^2+p^2)$
CV4	$2\lambda^2-q^2-p^2$	$-2\lambda(p-q)(p+q)$	$-(\lambda^2+q^2)(\lambda^2+p^2)$
CV5	$2\lambda^2-p^2$	$-2p^2\lambda$	$-\lambda^2(\lambda^2+p^2)$
CV6	$2\lambda^2$	0	$-\lambda^4$
CV7	$-2p^2$	0	$-p^4$
CV8	$6\lambda^2$	$-8\lambda^3$	$3\lambda^4$
CV9	0	0	0

Table A1: Space curves of constant projective curves

All curves, except CV4 and CV7, are general-affine homogeneous and already were listed in Section 5.3.

C.1 1-parameter subgroups defining anharmonic curves

Thanks to Theorem 5.4, each curve CVi is an orbit of a point p under a 1-parameter subgroup G. We list them as follows:

Curves	1-parameter subgroup G	Point p
CV1	$\begin{pmatrix} e^{-(\lambda+\mu+\nu)t} & & & \\ & e^{\lambda t} & & \\ & & e^{\mu t} & \\ & & & e^{\nu t} \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$
CV2	$\begin{pmatrix} e^{-(\lambda+\mu)t} & 0 & 0 & 0 \\ te^{-(\lambda+\mu)t} & e^{-(\lambda+\mu)t} & 0 & 0 \\ & & e^{2\lambda t} & 0 \\ & & 0 & e^{2\mu t} \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$
CV3	$\begin{pmatrix} e^{2\lambda t} & & & \\ & e^{2\mu t} & & \\ & & e^{-(\lambda+\mu)t}\cos(pt) & e^{-(\lambda+\mu)t}\sin(pt) \\ & & e^{-(\lambda+\mu)t}\sin(pt) & e^{-(\lambda+\mu)t}\cos(pt) \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$
CV4	$\begin{pmatrix} e^{\lambda t}\cos(pt) & -e^{\lambda t}\sin(pt) & & \\ e^{\lambda t}\sin(pt) & e^{\lambda t}\cos(pt) & & \\ & & e^{-\lambda t}\cos(qt) & e^{-\lambda t}\sin(qt) \\ & & e^{-\lambda t}\sin(qt) & e^{-\lambda t}\cos(qt) \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$
CV5	$\begin{pmatrix} e^{\lambda t}\cos(pt) & -e^{\lambda t}\sin(pt) & & \\ e^{\lambda t}\sin(pt) & e^{\lambda t}\cos(pt) & & \\ & & e^{-\lambda t} & 0 \\ & & te^{-\lambda t} & e^{-\lambda t} \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$
CV6	$\begin{pmatrix} e^{\lambda t} & 0 & & \\ te^{\lambda t} & e^{\lambda t} & & \\ & & e^{-\lambda t} & 0 \\ & & te^{-\lambda t} & e^{-\lambda t} \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$
CV7	$\begin{pmatrix} \cos(pt) & -\sin(pt) & & \\ \sin(pt) & \cos(pt) & & \\ t\cos(pt) & -t\sin(pt) & \cos(pt) & \sin(pt) \\ t\sin(pt) & t\cos(pt) & \sin(pt) & \cos(pt) \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$
CV8	$\begin{pmatrix} e^{\lambda t} & 0 & 0 & 0 \\ te^{\lambda t} & e^{\lambda t} & 0 & 0 \\ \frac{1}{2}t^2e^{\lambda t} & te^{\lambda t} & e^{\lambda t} & 0 \\ 0 & 0 & 0 & e^{-3\lambda t} \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$
CV9	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ \frac{1}{2}t^2 & t & 1 & 0 \\ \frac{1}{6}t^3 & \frac{1}{2}t^2 & t & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

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