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CRYSTALLINE FLOW STARTING FROM A GENERAL POLYGON

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Abstract. This paper solves a singular initial value problem for a system of ordinary differential equations describing a polygonal flow called a crystalline flow. Such a problem corresponds to a crystalline flow starting from a general polygon not necessarily admissible in the sense that the corresponding initial value problem is singular. To solve the problem, a self-similar expanding solution constructed by the first two authors with H. Hontani (2006) is effectively used.

1. Introduction

A crystalline mean curvature flow is an example of an anisotropic mean curvature flow whose anisotropy is strong so that its evolution is determined by nonlocal quantities. In mathematical community it was introduced by Taylor [T1] and independently by Angenent and Gurtin [AG] around 1990. In this paper, we restrict ourselves to flows in a plane $\mathbf{R}^2$ so that the mean curvature is just the curvature of a planar curve. To explain a crystalline curvature flow (crystalline flow for short), we consider an example of anisotropic curvature flow for evolving curve $\{\Gamma_t\}$ in a plane of the form

$$V = \kappa_\gamma \quad \text{with} \quad \kappa_\gamma = -\operatorname{div} \xi(\mathbf{n}) \quad \text{on} \quad \Gamma_t, \tag{1}$$

Key words and phrases. crystalline flow, non-admissible polygon, self-similar expanding solution, comparison principle, Briot-Bouquet system.

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where t denotes the time parameter. Here V denotes the normal velocity of $\{\Gamma_t\}$ in the direction of unit normal $\mathbf{n}$ and $\xi = \nabla\gamma$, where γ is an interfacial energy density, which is positively one-homogeneous and convex in $\mathbf{R}^2$. The operator div denotes the surface divergence. The quantity κ_γ is often called anisotropic curvature in the direction of $\mathbf{n}$. If $\gamma(p) = |p|$, then κ_γ is the usual curvature and (1) becomes a curve shortening equation. If γ is piecewise linear, then (1) is very singular and the speed is not determined by local quantities like curvature. We say that the quantity formally corresponding to κ_γ is a crystalline curvature if γ is piecewise linear. If V is determined by crystalline curvature and the orientation $\mathbf{n}$, i.e.,

$$V = f(\kappa_\gamma, \mathbf{n}) \quad \text{on} \quad \Gamma_t, \tag{2}$$

then we say that this equation is a crystalline flow equation. Here we assume that a given function f is nondecreasing with respect to κ_γ so that the problem is still (degenerate) parabolic.

There are several ways to solve (1) or more generally (2). For example, a level-set method ([CGG], [ES], [G]) can be adjusted for such a problem and gives a global unique (up to fattening) solution starting from any closed curve as studied in [GG1], [GG2], [GG4]. In [T1], [AG] a class of solutions is restricted in a special class of polygons called "admissible". It turns out that their solution agrees with a level-set solution [GG3] if initial curve is an admissible polygon. One merit of this method is that the solution is given by a system of ordinary differential equations. Moreover, it gives a numerical algorithm to solve a smooth anisotropic curvature flow or even the heat equation by approximating by crystalline flows. Such a topic is studied [FG], [GirK], [GG2] for a graph-like solution and [Gir], [IS], [GG4] for a closed curve.

Let us recall the notion of an admissible polygon introduced in [T1], [AG]. For this purpose, we recall the notion of the Wulff shape

$$W_\gamma = \left\{ x \in \mathbf{R}^2 \;\middle|\; x \cdot m \le \gamma(m) \text{ for all } m \in \mathbf{R}^2 \right\}.$$

If γ is piecewise linear, convex one-homogeneous function and $\gamma > 0$ except the origin, then W_γ contains zero as an interior point and W_γ is a closed, convex polygon. Its weighted curvature κ_γ formally equals to -1 if $\mathbf{n}$ is taken outward so the Wulff shape is considered as a unit disk for isotropic case $\kappa_\gamma = \kappa$. We say that an (oriented) polygon is an *admissible* crystal if

(i) (direction condition) the orientation of each facet (edge) is one of that in ∂W_γ and
(ii) (adjacence condition) the orientations of adjacent facets should be adjacent in ∂W_γ.

We say that $\{\Gamma_t\}$ is an *admissible evolving crystal* if Γ_t is an admissible crystal at each t and the motion of all vertices of Γ_t is C^1 in time t. This implicitly assumes that each facet moves keeping its orientation. Let $S_j(t)$ denote the j-th facet of Γ_t. Let $\Delta(\mathbf{n}_j)$ be the length of the facet of ∂W_γ whose orientation equals the orientation $\mathbf{n}_j$ of S_j. For an admissible evolving crystal, the equation (1) turns to be

$$V_j(t) = \Lambda_j(t) \quad \text{on} \quad S_j(t) \tag{3}$$

with κ_γ on S_j equals

$$\Lambda_j(t) = \chi_j \Delta(\mathbf{n}_j)/L_j(t).$$

Here $V_j(t)$ denotes the normal velocity of $S_j(t)$ in the direction of $\mathbf{n}_j$ and $L_j(t)$ denotes the length of $S_j(t)$; χ_j is the transition number taking only three values

$+1, 0, -1$ depending on whether Γ_t is convex, inflective, concave near $S_j(t)$ in the direction of $\mathbf{n}_j$; see Figure 1. Together with transport equations we have a finite

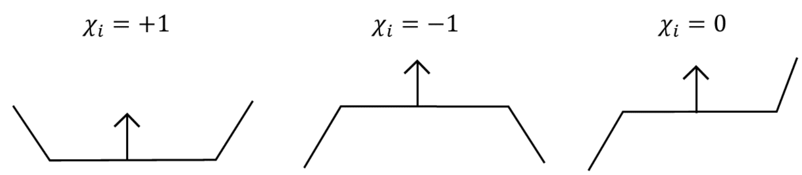

FIGURE 1. Each arrow indicates the positive direction

system of ordinary differential equations (ODEs), for example, for L_j's, provided that the initial polygon Γ_0 is a closed polygon. This system is at least locally solvable if $L_j(0) > 0$ for all j, i.e., in the case that the initial polygon is admissible. The resulting flow $\{\Gamma_t\}$ is often called a *crystalline flow* for (1) or (2). It is an admissible evolving crystal satisfying (1) or (2).

Our goal in this paper is to show that a level-set flow for (1) or more general (2) starting from a polygon Γ_0 satisfying (i) but violating the adjacence condition (ii) immediately becomes admissible (Figure 2); in particular, it satisfies the adjacence condition (ii) as well as (i) for $t > 0$. In particular, the solution becomes an

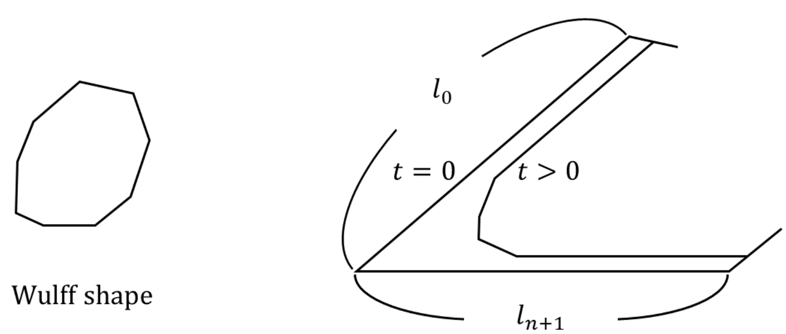

FIGURE 2. A given Wulff shape and initial condition

admissible evolving crystal instanteneously. This solution turns to be constructed explicitly by solving the system of ODEs with some $L_j(0)$'s are zero. A typical example is a self-similar expanding solution for (1) in a sector. This corresponds to the case that $L_0(0) = \ell_0 > 0$, $L_{n+1}(0) = \ell_{n+1} > 0$ while $L_1(0) = L_2(0) = \ldots = L_n(0) = 0$ and $\chi_0 = \chi_{n+1} = 0$ so that S_0, S_{n+1} are standing. Unique existence of such a solution is shown in [Ca] when W_γ is a regular polygon and established for general (1) in [GGH] (Figure 3). In the case that either χ_0 or χ_{n+1} is not zero, the solution is not exactly a self-similar expanding solution but it can be controlled by these solutions. Based on such observation, we are able to construct a solution when χ_0 or χ_{n+1} is non-zero. Note that in this case the motion also depends on the facet S_{-1} adjacent to S_0 and S_{n+2} adjacent to S_{n+1}, but for simplicity, we assume here that transition numbers of these facets are zero. Such a way of construction based on self-similar expanding solutions has been suggested by Taylor [T3, Section

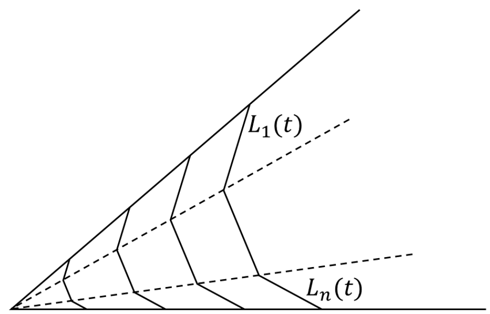

FIGURE 3. Self-similar solution

2.2]. It was carried out in the thesis of Ochiai [O] for (1) under the condition that $\chi_{-1} = 0$ (if $\chi_0 \neq 0$) and $\chi_{n+2} = 0$ (if $\chi_{n+1} \neq 0$). Applying such an idea for each non-admissible corner, one is able to solve the system of ODEs for a polygon violating the adjacence condition (ii).

In this paper, we complete this procedure and also handle more general equation (2) whose typical example is

$$V = M(\mathbf{n})(\kappa_\gamma + C), \tag{4}$$

where C is a constant and M is a positive constant depending on the orientation $\mathbf{n}$. The constant C is often called a driving force term and M is called the mobility, which often appears in theory of crystal growth as well as materials science [Gu], [T2]. Since the equation of crystalline flow for (1) is (3), the equation corresponding to (4) is

$$V_j(t) = \chi_j \tilde{\Delta}(n_j)/L_j + M(\mathbf{n}_j)C, \quad \tilde{\Delta}(\mathbf{n}) = M(\mathbf{n})\Delta(\mathbf{n}). \tag{5}$$

Although there may not exist a Wulff shape having $\tilde{\Delta}$ as an edge length, we are still able to derive ODE's for L_j's and handle them in a similar way. Relation between such a polygonal flow and a level-set flow by now standard [GG3]. We remark that our results can be extended to the problem starting from a general polygon by recalling the notion of weakly admissible crystal [GG96]; see Remark 1.

To construct a solution from non-admissible polygon, we have to be careful since (4) may not have a self-similar expanding solution starting from corners. As for (3), we approximate initial data with admissible one and derive an equicontinuity estimate for distance function of each line containing newly created facet from the corner. In [O] this is done by comparing with self-similar expanding solutions for (3). In our case, we shall use self-similar solutions for (3) which approximates (2) (more precisely (6)). To derive necessary equicontinuity estimates for the distance function, which is called the support function from the corner, we fully compare with self-similar solutions by comparison principle for the distance function.

The system of ODEs for L_j's is formally regarded as a kind of a Briot-Bouquet system, but it does not satisfy a proper condition even for (3). Thus, the existence of a solution does not follow from [GeT] as observed in [O].

Behavior of a crystalline flow for convex initial data has been well-studied especially the case when the equation is (3) or (4) with $C = 0$, i.e., there is no driving force term. It is easy to see that convexity is preserved. For (4) with $M = \gamma$

with $C = 0$, there always exists a self-similar shrinking solution to a point whose profile is the Wulff shape. This is a substitute of a shrinking circle for the curve shortening equation. The uniqueness of self-similar solution is proved when the Wulff shape W_γ is symmetric with respect to the origin and the number of its verteces is more than four as shown in [S1]; in the case W_γ is a parallelogram all pararellogram shrinks self-similarly. Moreover, it is shown in [S1] all convex solution shrinks asymptotically similar to the self-similar solution. However, if one considers (4) with $C = 0$ with M unrelated to γ, the solution is more complicated as discussed in [S2], [A]. In [A] rather complete picture is given. Even the flow is orientation free, i.e., $\tilde{\Delta}(\mathbf{n}) = \tilde{\Delta}(-\mathbf{n})$ in (5) with $C = 0$ unless M is parallel to γ, there is a chance that there is no self-similar shrinking and isoperimetric ratio tends to infinity. Moreover, in [A] it is shown that the minimal length of facets at time t behaves like $\{(T-t)/\log(T-t)\}^{1/2}$ or $(T-t)^\beta$, $1/2 < \beta < 1$ as t tends to T, where T is the extinction time. For a self-similar solution, the length should behave like $(T-t)^{1/2}$ so it is shorter than that of a self-similar solution. If one considers a crystalline flow corresponding to $V = \kappa_\gamma^\alpha$ for $\alpha > 0$, the situation depends whether $\alpha < 1$, $\alpha > 1$ or $\alpha = 1$; we have already discussed the case $\alpha = 1$. In the case $\alpha \geq 1$, it is shown in [GG3] that there is no degenerate pinching at the extinction time. By degenerate pinching we mean that two parallel facets touches with positive length at the extinction time. Moreover, as shown in [A] for $\alpha > 1$ all (convex) solution shrinks to a point in a self-similar way.

Results for $V = \gamma\kappa_\gamma$ in [S1] are parallel to those for conventional anisotropic curve shortening flow as established in [Ga93], [GaL94], [DGM], [DG]. However, for $V = M\kappa_\gamma$ with M not parallel to γ, as discussed above, there may exist non self-similar shrinking for a crystalline flow [A] even if the flow is orientation free. This has a strong contrast compared with the conventional orientation-free anisotropic curvature flow, where all flows shrinks in a self-similar way. This indicates that qualitative property of a solution may be different from conventional curve shortening equation depending upon the Wulff shape. If the motion is not orientation free, it is shown in [IUYY] that crystalline flow may not become convex. In fact, in [IUYY] the existence of non-convex self-similar shrinking solution is proved when the Wulff shape is a square or a regular triangle for $V = M\kappa_\gamma$ with M not parallel to γ. Moreover, the existence of such a non-convex self-similar solution for $V = M\kappa_\gamma^\alpha$ is proved for $\alpha \in (0,1)$ even if it is orientation free [IUYY].

For the curve shortening flow, it is proved that flow becomes convex in finite time [Gr]. It seems that the corresponding result is not established when W_γ is symmetric with respect to the origin and the flow is $V = \gamma\kappa_\gamma$. To our best knowledge flow becomes almost convex in the sense that all facet have positive crystalline curvature possibly except two adjacent inflection facets, i.e., facets with $\chi = 0$. This is proved in [I08]. Note that it is already proved in [T3, Theorem 3.2] that for the equation $V = \gamma\kappa_\gamma$, admissible polygon stays admissible and possibly miss inflection facets at finitely many times until it shrinks to a point. However, the proof there is sketchy. In [IS] a full proof is given when W_γ is a regular polygon with even faces. In all above discussion, the initial polygon is an admissible crystal.

If there is a facet whose orientation is not contained in the set of orientations of the Wulff shape, one should be careful. For example, if initial polygon is convex, one can track whole evolution until it shrinks to a point provided that all orientation of the Wulff shape exists in the initial polygon as studied in [Ya]; note that non-admissibility in [Ya] does not allow to miss any orientation of that of Wulff shape in the intial polygon. It allows a facet whose orientation may not be included in those of the Wulff shape. A convex polygon studied in [Ya] is weakly admissible in the sense of [GG96].

A crystalline flow for the eikonal-curvature flow equation (4) or (5) has been also well-studied. For V-shaped initial data, its evolution was studied in [I11a], [I11b]. For a growth of convex polygon, the large time behavior for a crystalline flow for (4) or (5) is studied by [GG13] with special emphasis on the anisotropic effects of mobility M and γ. A crystalline flow is also applied to study growth of spirals since the work of [I14], which is further developed in [IO1]. Various methods to compute crystalline flow without solving ODEs are compared with the classical method [IO2]. Although we do not intend to give numerics much, we point out that there is a numical approach based on a variational inequality, when Γ_t is a graph [EGS]. There is a direct numerical approach based on a level-set method for crystalline flow as studied in [OOTT].

Although there is a large number of articles studying crystalline flows, the situation is usually in the case that new facets are not created. A facet creation problem has been observed in [GG1] and further developed in [Mu], [MuR1], [MuR2] mostly for graph-like solutions. However, the number of newly created facets in one point is just one. In [Ca], [GGH], several facets are created just from one corner. The paper [O] is the first paper to handle the situation where several facets are created between convex facets. Since the paper [O] is not well circulated, we write a whole necessary argument to achieve our goal.

It is natural to ask what happens for crystalline flow in a higher dimensional space, for example, evolution of a polytope in three-dimensional space. If one expects that the flow still has a comparison principle, the original curvature flow for (1) with crystalline interfacial energy γ, i.e., γ is piecewise linear is not reduced to (3). In fact, a facet may break or bend as shown in [BNP1], [BNP2]. This indicates that polytope may not stay as polytope during evolution. Thus, the problem is not reduced to a system of ODEs. Nevertheless, a level-set flow has been constructed independently by [CMP], [CMMP] for (4) with "convex" and [GP1], [GP2] for (4) for arbitrary M; for non-uniform driving force C, see [CMMP], [GP3]. The methods in [GP1], [GP2], [GP3] are quite different from those of [CMP], [CMMP].

This paper is organized as follows. In Section 2, we formulate problems and state our main results. In Section 3, we derive a priori estimate for approximate solutions by comparing with self-similar expanding solutions constructed by [GGH]. We construct a solution for singular initial value problem near one corner. This procedure is essentially done for (3) in [O]; in this particular case one is able to take $\varepsilon = 0$ in the argument. In Section 4, we shall prove our main results stated in Section 2 by extending the argument in Section 3. A weaker form of results is presented in [K] but it is not well circulated so we reproduce several contents of [K]. In Section 5, we give some numerical tests. In Section 6, we explain why the general theory for a Briot-Bouquet system does not apply at least directly.

2. Crystalline flow

To formulate the problem, we recall several notion. Let $\mathcal{N}$ be a finite subset of the unit circle, i.e.,

$$\mathcal{N} = \{\mathbf{n}_k\}_{k=1}^{n} \quad \text{with} \quad \mathbf{n}_k = (\cos\theta_k, \sin\theta_k).$$

If the Wulff shape W_γ is given, $\mathcal{N}$ is taken so that it is the set of all orientations of ∂W_γ. We call $\mathcal{N}$ the set of *admissible directions*. The set $\Theta = \{\theta_k\}_{k=1}^n$ is called the set of *admissible angles*, which is considered a subset in $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$. A polygon whose orientation belonging to $\mathcal{N}$ is called *admissible* if the angles of adjacent facet are adjacent in Θ (adjacence condition). An evolving polygon $\{\Gamma_t\}_{t\in I}$ is an *admissible evolving crystal* if Γ_t is an admissible polygon for $t \in I$ and the motion of all vertices is C^1 in time $t \in I$, where I is a time interval. We consider the equation (2), i.e.,

$$V = f(\kappa_\gamma, \mathbf{n}),$$

where f is a given locally Lipschitz (continuous) function and is nondecreasing in the first variable. We say that an admissible evolving crystal $\{\Gamma_t\}_{t\in I}$ is a *crystalline flow* for (2) if

$$V_j(t) = f\left(\Lambda_j(t), \mathbf{n}_j\right),$$
$$\Lambda_j(t) = \chi_j \Delta(\mathbf{n}_j)/L_j(t)$$

holds for each facet S_j of Γ_t for $t \in I$. The quantity Λ_j is often called a crystalline curvature introduced in Section 1. Setting $f_j(x) = f\left(\Delta(\mathbf{n}_j)x, \mathbf{n}_j\right)$, this equation of the form

$$V_j(t) = f_j\left(\chi_j/L_j(t)\right) \tag{6}$$

with a given locally Lipschitz function f_j. We may call that an admissible evolving crystal is a *crystalline flow* of (6) if it satisfies (6) for all $t \in I$. We are able to construct a crystalline flow for non-admissible polygon.

Theorem 2.1. *Assume that f_j is locally Lipschitz and non-decreasing. Assume that*

$$\lim_{|x|\to\infty} f_j(x)/x = \lambda_j \tag{7}$$

with λ_j depending on j only through $\mathbf{n}_j$. Let Γ_0 be a polygon whose orientations belong to $\mathcal{N}$. Then there is a unique crystalline flow of (6) in some time interval $(0, t_0)$ which converges to Γ_0 as $t \downarrow 0$ in the Hausdorff distance sense.

We apply Theorem 2.1 to get a main result.

Theorem 2.2. *Assume that f is locally Lipschitz and non-decreasing in the first variable. Assume that*

$$\lim_{|y|\to\infty} f(y, \mathbf{n}_j)/y = \lambda_j \tag{8}$$

with λ_j depending on j only through $\mathbf{n}_j$. Let Γ_0 be in Theorem 2.1. Then there is a unique local-in-time crystalline flow $\{\Gamma_t\}$ for (2) which converges to Γ_0 as $t \downarrow 0$ in the Hausdorff distance. Moreover, if $f(0, \mathbf{n})$ satisfies "corner preserving conditions", this solution is a level-set flow solution of (2).

Let us explain the corner preserving condition [GG13]. We say that f satisfies the corner preserving condition if for each $\mathbf{n}_k \in \mathcal{N}$

$$f^0(\mathbf{m}) \geq \frac{1}{\sin\varphi_{k+1}}\left(f^0(\mathbf{n}_k)\sin\theta_{k+1} + f^0(\mathbf{n}_{k+1})\sin\theta_k\right), \quad f^0(p) := f(0, p)$$

for all $\mathbf{m} = (\cos\theta, \sin\theta)$ with $\theta_k < \theta < \theta_{k+1}$; we consider indices modulo n so that $n+1$ is interpreted as 1. Geometrically speaking, this is equivalent to say that

$$A_k \subset \left\{x \in \mathbf{R}^2 \;\middle|\; x \cdot \mathbf{m} \leq f^0(\mathbf{m})\right\} \subset B_k$$

with

$$A_k = H_k \cap H_{k+1}, \quad B_k = H_k \cup H_{k+1}. \quad H_{k+j} = \left\{ x \in \mathbf{R}^2 \;\middle|\; x \cdot \mathbf{n}_{k+j} \leq f^0(\mathbf{n}_{k+j}) \right\}$$

for all $\mathbf{m} = (\cos\theta, \sin\theta)$ with $\theta_k < \theta < \theta_{k+1}$. This condition says that in the corner all facets whose orientations are between that of facets forming the corner move faster than corner facets for $V = f^0(\mathbf{n})$. This condition is first pointed out explicitly by [GHK] and independently by [GSS]. It is stated in a different form in [GG96], which proves a graph-like crystalline flow is consistent with viscosity solution defined in [GG1]; see also [GG]. The geometric version is found in [GG3]; however, unfortunately the definition of B_k was mistyped.

Remark 1. In both Theorem 2.1 and Theorem 2.2, one may allow Γ_0 for a general polygon whose orientation $\mathbf{n}_\alpha$ of a facet E_α not belong to $\mathcal{N}$ by interpreting f_α is independent of L_α or $\Delta(\mathbf{n}_\alpha) = 0$. The resulting crystalline flow is still an evolving crystal if $\Delta(\mathbf{n}_\alpha) = 0$ is allowed.

If one considers an non-admissible polygon Γ_0 whose facets $\{E_\alpha\}_{\alpha=1}^K$ has an orientation $\mathbf{n}_\alpha \in \mathcal{N}$, there is at least one vertex P connecting two facets S_A and S_B such that there is at least one direction $\mathbf{n} \in \mathcal{N}$ between $\mathbf{n}_A$ and $\mathbf{n}_B$. Below we consider a special case that the transition number χ_j of facet touching to S_A (resp. S_B) but not to S_B (resp. S_A) is zero. Let $\mathbf{n}_A = (\cos\theta_A, \sin\theta_A)$, $\mathbf{n}_B = (\cos\theta_B, \sin\theta_B)$ be orientations of S_A and S_B. We may assume $|\theta_A - \theta_B| < \pi$. To fix the idea, we assume that $\theta_A < \theta_B$ since the case $\theta_B < \theta_A$ can be treated similarly. Let $\{\theta_j\}_{j=0}^{n+1}$ be the intersection of $[\theta_A, \theta_B]$ and Θ and satisfies

$$\theta_{-1} < \theta_A = \theta_0 < \theta_1 < \ldots < \theta_n < \theta_{n+1} = \theta_B < \theta_{n+2}.$$

We set $\varphi_i = \theta_i - \theta_{i-1}$ for $i = 0, \ldots, n+2$, where $\theta_{-1}, \theta_{n+2} \in \mathcal{N}$ and $(\theta_{-1}, \theta_0) \cap \mathcal{N} = \emptyset$, $(\theta_{n+1}, \theta_{n+2}) \cap \mathcal{N} = \emptyset$.

We expect that at the corner by S_A and S_B, n facets should be created; see Figure 4. If $f(x, \mathbf{n}_j) = x$ and transition number of both S_A and S_B are zero, there is a

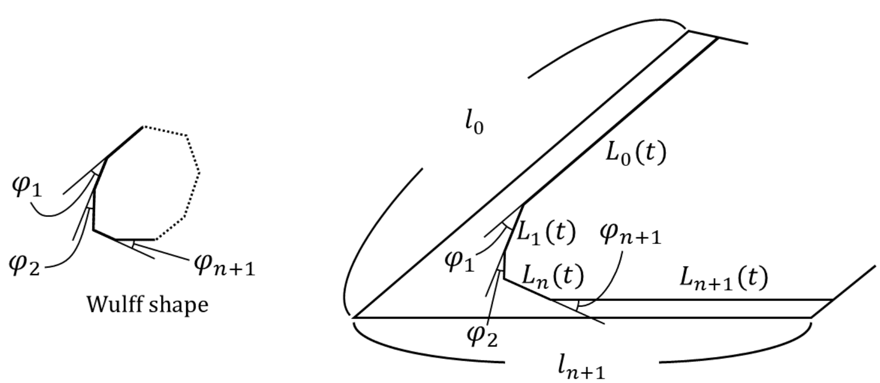

FIGURE 4. Wulff shape and newly created facets

self-similar expanding solution for (3) as proved in [GGH]. Thus, we are interested in the case that one of χ_A and χ_B is not zero. Let $\{S_i(t)\}_{i=1}^n$ be newly created facets with orientation $\mathbf{n}_i = (\cos\theta_i, \sin\theta_i)$. We set $S_0(t) = S_A(t)$, $S_{n+1}(t) = S_B(t)$.

We shall derive equations for the length $L_i(t)$ of $S_i(t)$. By transport equation, we have

$$\begin{aligned}\frac{dL_i(t)}{dt} &= -\frac{1}{\sin\varphi_i}V_{i-1}(t) + (\cot\varphi_i + \cot\varphi_{i+1})\,V_i(t)\\ &\quad -\frac{1}{\sin\varphi_{i+1}}V_{i+1}(t).\end{aligned}$$

See Figure 5. This indentity involving 3 adjacent facets holds for any admissible evolving crystal. If we consider the equation (3) for S_j's, then the resulting system

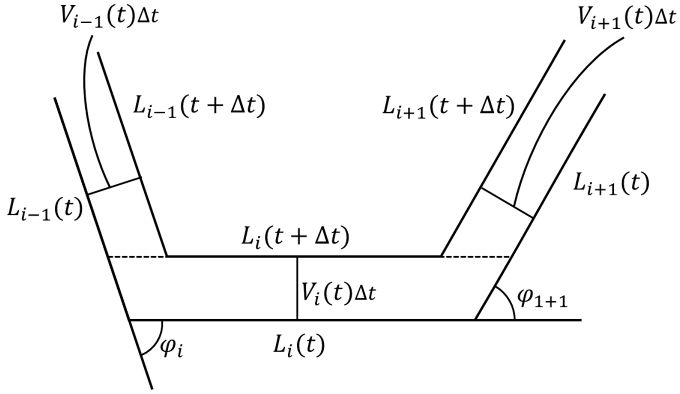

FIGURE 5. i-th, $(i-1)$-th and $(i+1)$-th facet

of equations are

$$\begin{cases}\dfrac{dL_0(t)}{dt} = \dfrac{p_0}{L_0(t)} - \dfrac{r_1}{L_1(t)}, & L_0(0) = \ell_0 > 0 \qquad (9)\\[2mm] \dfrac{dL_i(t)}{dt} = -\dfrac{q_{i-1}}{L_{i-1}(t)} + \dfrac{p_i}{L_i(t)} - \dfrac{r_{i+1}}{L_{i+1}(t)}, & L_i(0) = 0 \ (i = 1, 2, \dots, n), \qquad (10)\\[2mm] \dfrac{dL_{n+1}(t)}{dt} = -\dfrac{q_n}{L_n(t)} + \dfrac{p_{n+1}}{L_{n+1}(t)}, & L_{n+1}(0) = \ell_{n+1} > 0. \qquad (11)\end{cases}$$

Here positive numbers p_i, q_i, r_i's are

$$p_i = (\cot\varphi_i + \cot\varphi_{i+1})\,\Delta_i \quad (i = 0, 1, \dots, n+1), \tag{12}$$

$$q_i = \Delta_i / \sin\varphi_{i+1} \quad (i = 0, 1, \ \ , n), \tag{13}$$

$$r_i = \Delta_i / \sin\varphi_i \quad (i = 1, 2, \dots, n+1). \tag{14}$$

Here we invoke the assumption that the facet touching to S_A (resp. S_B) but not to S_B (resp. S_A) has zero transition number. This assumption implies that the first and last equation does not contain q_{-1} and r_{n+2}.

In [O] the existence of a unique local solution of (9)–(11) been established. In this paper, we extended this result for more general equation corresponding to (6) than (3). The resulting system of equations is

(15)
$$\frac{dL_0(t)}{dt} = -\frac{1}{\sin\varphi_0} f_{-1}(0) + (\cot\varphi_0 + \cot\varphi_1) f_0\left(\frac{1}{L_0(t)}\right) - \frac{1}{\sin\varphi_1} f_1\left(\frac{1}{L_1(t)}\right)$$

$$\frac{dL_i(t)}{dt} = -\frac{1}{\sin\varphi_i} f_{i-1}\left(\frac{1}{L_{i-1}(t)}\right) + (\cot\varphi_i + \cot\varphi_{i+1}) f_i\left(\frac{1}{L_i(t)}\right)$$

$$- \frac{1}{\sin\varphi_{i+1}} f_{i+1}\left(\frac{1}{L_{i+1}(t)}\right), \ (i = 1, \ldots, n) \tag{16}$$

$$\frac{dL_{n+1}(t)}{dt} = -\frac{1}{\sin\varphi_{n+1}} f_n\left(\frac{1}{L_n(t)}\right) + (\cot\varphi_{n+1} + \cot\varphi_{n+2}) f_{n+1}\left(\frac{1}{L_i(t)}\right)$$

$$- \frac{1}{\sin\varphi_{n+2}} f_{n+2}(0). \tag{17}$$

The initial condition is

$$L_0(0) = \ell_0 > 0, \quad L_{n+1}(0) = \ell_{n+1} > 0 \quad L_i(0) = 0 \ \text{ for } \ i = 1, \ldots, n. \tag{18}$$

Theorem 2.3. *Assume that $f_i : \mathbf{R} \to \mathbf{R}$ is locally Lipschitz and non-decreasing for $i = 0, \ldots, n+1$. Assume that $f_{-1}(0), f_{n+2}(0) \in \mathbf{R}$. Assume furthermore that*

$$\lim_{|x|\to\infty} f_i(x)/x = \lambda_i > 0.$$

Then the system (15)–(17) *with* (18) *admits a unique solution $\{L_i(t)\}_{i=0}^{n+1}$ for some time interval $(0, t_0)$ which is C^1 in $(0, t_0)$ and continuous on $[0, t_0)$.*

Remark 2. This result is proved in [K] under stronger assumption on f's, namely,

$$\overline{\lim_{|x|\to\infty}} |f_i(x) - \lambda_i x| < \infty, \quad f_{-1}(0) = f_{n+2}(0) = 0.$$

However, the proof works in our setting.

3. Estimates for approximate solutions by self-similar solutions

The goal of this section is to prove Theorem 2.3. We first construct approximate solutions. We consider the system (15)–(17) with initial condition

$$L_0(0) = \ell_0^s > 0, \quad L_{n+1}(0) = \ell_{n+1}^s > 0, \quad L_i(0) = a_i s^{1/2} \quad (i = 1, \ldots, n) \tag{19}$$

with $s > 0$. Here a_i is taken so that

$$\hat{L}_i(t) = a_i t^{1/2}$$

is the expanding self-similar solution of (10) by setting $q_0 = r_{n+1} = 0$ and $\Delta_i = \lambda_i$ $(i = 1, \ldots, n)$ in (12) (13) (14). In other words, a_i's is a unique positive solution of the n system of algebraic equations of the form

$$\frac{a_i}{2} = \frac{q_{i-1}}{a_{i-1}} + \frac{q_i}{a_i} - \frac{r_{i+1}}{a_{i+1}} \quad (i = 1, \ldots, n) \tag{20}$$

with $q_0 = r_{n+1} = 0$. The unique existence of such a_i's is guaranteed in [GGH]. The initial length $\ell_0^s \le \ell_0$ and $\ell_{n+1}^s \le \ell_{n+1}$ is taken so that it is the length of intersection of a facet S_0 (resp. S_{n+1}) and the 0-th (resp. $(n+1)$-th facet) of the self-similar solution at s. If s is taken sufficiently small, say $s \le s_0$, ℓ_0^s and ℓ_{n+1}^s can be taken positive.

Proposition 1. *There exists* $s_0 > 0$ *and* $t_0 > 0$ *such that the system* (15)–(17) *admits a unique* C^1 *solution* L_i^s $(i = 0, 1, \dots, n+1)$ *in* $[0, t_0]$ *and neighboring facets* S_{-1}, S_{n+2} *have positive length in* $[0, t_0]$ *for* $s \in (0, s_0]$.

Proof. Since f_i is locally Lipschitz and initial value is positive, there is a unique solution L_i^s. Moreover, the neighboring facet S_{-1}, S_{n+2} with transition number zero have still positive length in $[0, t_s)$ by taking s small. Let t_s be the maximal time so that L_i^s exists and this non-vanishing property holds.

We shall prove that there is $t_0 > 0$ such that $t_s \geq t_0$ for $s \leq s_0$, where $\ell_0^{s_0}, \ell_{n+1}^{s_0} > 0$. There is a nontrivial existence time t_{s_0} such that the non-vanishing property holds in $(0, t_{s_0})$. We first note that there is no possibility that some facet becomes of length zero since otherwise all facets must shrink to a point which is absurd in our setting since S_{-1} and S_{n+2} remains; see e.g. [IJ]. By comparison principle (see e.g. [GGu]), non-vanishing property holds for $s \leq s_0$ if $t_s \geq t_{s_0}$; we invoke the property that f_i is non-decreasing. The proof is now complete by taking $t_0 = t_{s_0}$. □

We shall construct a solution of (15)–(17) with (18) by taking a limit of L_i^s as s tends to zero. We set the corner vertex $S_A \cap S_B$ as the origin O. To estimate L_i^s, it is convenient to introduce the distance from the origin to the line containing facet S_i. Here is a precise definition.

Definition 3.1. Let $d_i^s(t)$ denote the (signed) distance from the corner vertex (origin) to i-th facet $S_i^s(t)$ (in the direction of $\mathbf{n}_i$) whose length equals $L_i^s(t)$ for $s \in (0, s_0]$ and $t \in [0, t_0]$, where $s_0, t_0 > 0$ are given in Proposition 1; see Figure 6. Here $i = 0, 1, \dots, n+1$. In other words, d_i^s is the support function of S_i^s with

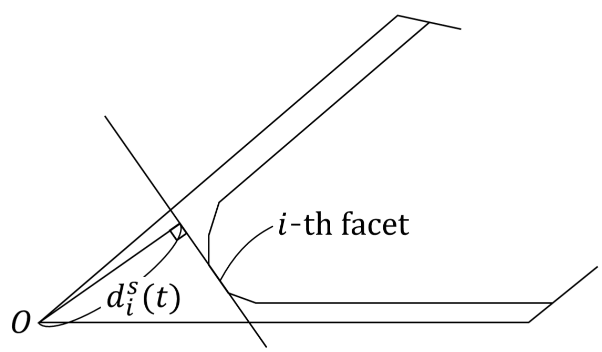

FIGURE 6. Distance function $d_i^s(t)$

respect to the corner vertex.

Let V_i^s denote the speed of S_i^s. By definition

$$d_i^s(t) = \int_0^t V_i^s(t)d\tau + d_i^s(0) \quad (i = 0, 1, \dots, n+1).$$

Since $\hat{L}_1$ is the self-similar solution,

$$d_i^s(0) = \int_0^s \frac{\lambda_i}{\hat{L}_1(\sigma)} d\sigma = \frac{2\lambda_i s^{1/2}}{a_i} \quad (i = 1, \dots, n)$$

which is the integral of speed of the facet corresponding to $\hat{L}_1$ over $(0, s)$. Of course, $d_0^s(0) = d_{n+1}^s(0) = 0$. By (6) we have

$$d_i^s(t) = \int_0^t f_i\left(\frac{1}{L_i^s(\tau)}\right) d\tau + \frac{\lambda_i s^{1/2}}{a_i}, \quad (i = 1, \ldots, n)$$
$$d_0^s(t) = \int_0^t f_0\left(\frac{1}{L_0^s(\tau)}\right) d\tau,$$
$$d_{n+1}^s(t) = \int_0^t f_{n+1}\left(\frac{1}{L_{n+1}^s(\tau)}\right) d\tau.$$

By (strong) comparison principle for crystalline curvature (see e.g. [GGu]), we have monotonicity of d_i^s with respect to s.

Proposition 2. *Assume that* $0 < s_1 < s_2 \leq s_0$. *Then*

$$d_i^{s_1}(t) < d_i^{s_2}(t) \quad \textit{for} \quad t \in [0, t_0], \quad i = 0, 1, \ldots, n+1.$$

Here t_0, s_0 *are as in Proposition 1.*

Definition 3.2. Let d_i for $i = 0, 1, \ldots, n+1$ denote

$$d_i(t) = \inf_{0<s<s_0} d_i^s(t) \ \left(= \lim_{s\to 0} d_i^s(t)\right) \quad \text{for} \quad t \in [0, t_0].$$

The last equality follows from the monotonicity (Proposition 2).

We would like to prove the continuity of this d_i's up to $t = 0$. We shall estimate the modulus of continuity of d_i^s uniformly in s for $i = 1, \ldots, n$. For this purpose, we use a self-similar solution in a sector consisting of S_A and S_B. While the original self-similar solution consists of n moving facets $\hat{S}_i$ $(i = 1, \ldots, n)$, this new self-similar solution has $n + 2$ moving facets $\hat{S}_i^\eta$ $(i = 0, \ldots, n+2)$ with small angle ψ_0 (resp. ψ_{n+2}) between S_0^η and S_1^η (resp. S_n^η and S_{n+1}^η); the angles $\varphi_1 \ldots \varphi_n$ are unchanged; see Figure 7, Figure 8. The set of admissible angles contains ϑ_{-1} $(\theta_{-1} < \vartheta_{-1} < \theta_0)$

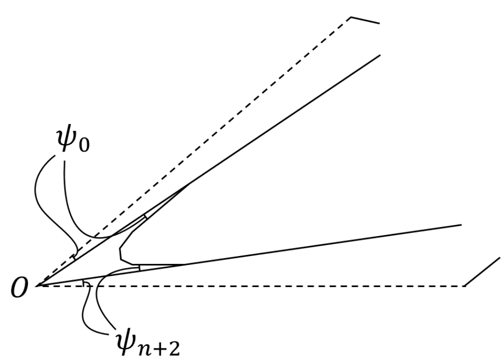

FIGURE 7. New self-similar solution

with $\psi_0 = \theta_0 - \vartheta_{-1}$ and ϑ_{n+2} $(\theta_{n+1} < \vartheta_{n+2} < \theta_{n+2})$ with $\psi_{n+2} = \vartheta_{n+2} - \theta_{n+1}$. This new self-similar solution is of the form

$$\hat{L}_i^\eta(t) = b_i t^{1/2} \quad (i = 0, 1, \ldots, n+1)$$

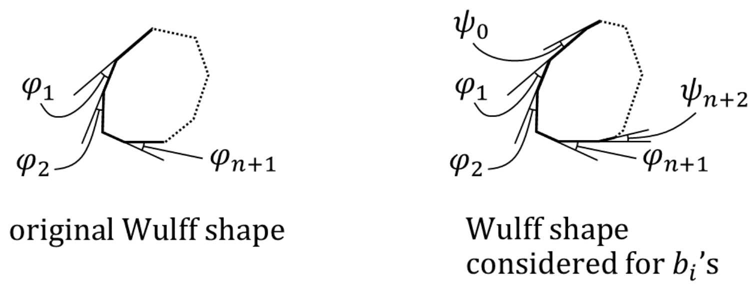

FIGURE 8. Wulff shape

with $\eta = (\psi_0, \psi_{n+2})$. Here b_i solves

$$\frac{b_i}{2} = \frac{q_{i-1}}{b_{i-1}} + \frac{p_i}{b_i} - \frac{r_{i+1}}{b_{i+1}} \quad (i = 0, 1, \ldots, n+1) \tag{21}$$

with (12), (13), (14) with φ_0, φ_{n+2} replaced by ψ_0, ψ_{n+2} by setting $q_{-1} = 0$, $r_{n+2} = 0$ and $\Delta_i = \lambda_i$ $(i = 1, \ldots, n)$; the quantity Δ_0 and Δ_{n+1} are fixed independent of η.

Lemma 3.3. *The quantity b_i's approximates a_i's, in the sense that $b_0, b_{n+1} \to \infty$ and $b_i \to a_i$ $(i = 1, \ldots, n)$ of $\eta = (\psi_0, \psi_{n+2}) \to 0$.*

Proof. By definition

$$p_0 = (\cot \psi_0 + \cot \varphi_1)\,\Delta_0$$
$$p_{n+1} = (\cot \varphi_{n+1} + \cot \psi_{n+2})\,\Delta_{n+1}$$

so $p_0 \to \infty$ if $\psi_0 \to 0$ and $p_{n+1} \to \infty$ if $\psi_{n+2} \to 0$. By comparison of self-similar solutions,

$$b_1(\eta_*) \leq b_i(\eta) \leq q_i, \quad i = 1, \ldots, n \tag{22}$$

for $\eta \in (0, \eta_{*1}) \times (0, \eta_{*2})$ with $\eta_* = (\eta_{*1}, \eta_{*2})$. By (21) we see in particular that

$$\frac{b_0}{2} = \frac{p_0}{b_0} - \frac{r_1}{b_1}$$
$$\frac{b_{n+1}}{2} = -\frac{q_n}{b_n} + \frac{p_{n+1}}{b_{n+1}}.$$

Since r_1/b_1 and q_n/b_n are bounded by (22), this implies $b_0, b_{n+1} \to \infty$ as $\eta \to 0$. By a bound of (22), any accumulation point of $b_i(\eta)$ as $\eta \to 0$ must satisfies (20). By uniqueness [GGH], it must agrees with a_i, i.e., $\lim_{\eta \to 0} b_i(\eta) = a_i$ for $i = 1, \ldots, n$. The proof is now complete. □

We shall compare with these self-similar solutions. The original self-similar solution corresponds to solution of (3) with $\Delta(\mathbf{n}_i) = \lambda_i$. Since we consider general problems, we need to use self-similar solutions to

$$V_j = \Lambda_j(1 - \varepsilon) \tag{23}$$

or

$$V_j = \Lambda_j(1 + \varepsilon) \tag{24}$$

for $\varepsilon > 0$ to estimate. Instead of considering a_i, we consider solutions of (20) with $\Delta_i = \lambda_i(1-\varepsilon)$ and denote it by $a_i^{\varepsilon-}$; fortunately

$$a_i^{\varepsilon-} = \sqrt{1-\varepsilon}\,a_i \quad (i = 1, \ldots, n).$$

This corresponds the self-similar solution to (23). Similarly, let be $b_i^{\varepsilon+}$ the solution of (21) by replacing $\Delta_i = \lambda_i(1+\varepsilon)$, i.e.,

$$b_i^{\varepsilon+} = \sqrt{1+\varepsilon}\,b_i \quad (i = 0, \ldots, n+1).$$

We have an equi-continuity estimate for d_i^s.

Lemma 3.4. *For any $\varepsilon \in (0,1)$ and $\eta = (\psi_0, \psi_{n+2})$, there is t_2 such that*

$$\int_0^{t-t_1} \frac{\lambda_i(1-\varepsilon)}{a_i^{\varepsilon-}\tau^{1/2}} \le d_i^s(t) - d_i^s(t_1) \le \int_0^{t-t_1} \frac{\lambda_i(1+\varepsilon)}{b_i^{\varepsilon+}\tau^{1/2}} d\tau \quad (i = 1, \ldots, n) \tag{25}$$

for all $t_1 \le t \le t_2$ and $s \in (0, s_0]$. In particular, $d_i^s(t)$ is increasing in $t \in [0, t_2]$.

Proof. We observe that

$$d_i^s(t) - d_i^s(t_1) = \int_{t_1}^t V_i^s(\tau) d\tau$$

where V_i^s is the velocity of i-th facet S_i^s. By (7), for $\varepsilon > 0$, we take σ_0 small so that

$$\frac{\lambda_i}{\sigma}(1-\varepsilon) \le f_i\left(\frac{1}{\sigma}\right) \le \frac{\lambda_i}{\sigma}(1+\varepsilon), \quad i = 1, \ldots, n$$

for all $\sigma < \sigma_0$. By a simple comparison, there is small $t_2 < t_0$ such that

$$L_i^s(t) \le \sigma \quad \text{for} \quad t \in (0, t_2]$$

uniformly in $s \in (0, s_0]$ for $i = 1, \ldots, n$.

We next compare with self-similar solutions (Figure 9). For a given facet $S_i^s(t_1)$,

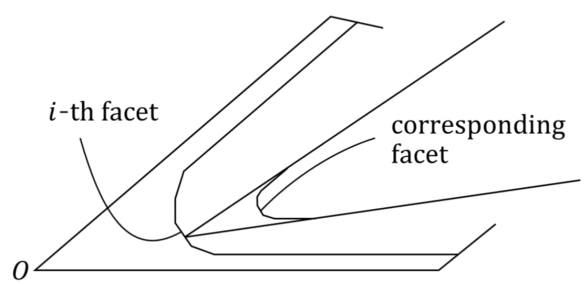

FIGURE 9. Compare with a self-similar solution

we touch over self-similar solution $\{\hat{S}_i^\eta\}$ at size zero with b_i replaced by $b_i^{\varepsilon+}$. The speed of S_i^s is dominated by that of $\hat{S}_i^\eta$ in a short time so by taking t_2 smaller we have

$$V_i^s(t) = f_i\left(\frac{1}{L_i^s(t)}\right) \le \frac{\lambda_i(1+\varepsilon)}{b_i^{\varepsilon+}t^{1/2}} \quad \text{for} \quad t \in (0, t_2), \quad i = 1, \ldots, n.$$

(We have to take t_2 smaller as $\eta \to 0$ since the self-similar solution Γ^η should be inside Γ^s, the S_A^η, S_B^η should be a segment with translation number zero.)

The lower estimate for $d_i^s(t) - d_i^s(t_1)$ can be established a similar way by comparing with a self-similar solution $\{\hat{S}_i\}$ from outside of S_i with a_i replaced by $a_i^{\varepsilon-}$. The proof is now complete. □

Lemma 3.5. *Under the same hypotheses of Lemma 3.4*

$$\frac{\lambda_i\sqrt{1-\varepsilon}}{a_i}(t-t_1)^{1/2} \le d_i^s(t) - d_i^s(t_1) \le \frac{\lambda_i\sqrt{1+\varepsilon}}{b_i}(t-t_1)^{1/2},\ i=1,\dots,n,$$

for $0 \le t \le t_1 \le t_2$, $s \in (0, s_0]$. *In particular, the limit* d_i *is* 1/2*-Hölder continuous. More precisely,*

$$\frac{\lambda_i\sqrt{1-\varepsilon}}{a_i}(t-t_1)^{1/2} \le d_i(t) - d_i(t_1) \le \frac{\lambda_i\sqrt{1+\varepsilon}}{b_i}(t-t_1)^{1/2},\ i=1,\dots,n$$

for $0 \le t \le t_1 \le t_2$. *The convergence* $d_i^s \to d_i$ *is uniform in* $[0, t_2]$.

Proof. The estimate follows from (25). This equi-Hölder estimate yields 1/2-Hölder continuity of the limit. The convergence is uniform by Dini's theorem since the convergence is monotone (Proposition 2). (One may apply the Ascoli-Arzela theorem to conclude the uniform convergence without using monotonicity with respect to s.) □

Remark 3. For facets d_0^s and d_{n+1}^s since the length of facet S_0^s, S_{n+1}^s is bounded away from zero uniformly in $s \in (0, s_0]$ we have a uniform C^1 estimates which is inherited to d_0 and d_{n+1}. We may conclude the estimate from above in Lemma 3.4 and 3.5 still holds for $i = 0$ and $i = n+1$ by taking t_2 smaller.

Proof of Theorem 2.3 (Existence). We shall construct a solution $\{L_i\}$ as a limit of L_i^s as s tends to zero.

We integrate (16) on $(0, t)$ to get

$$L_i^s(t) = -\int_0^t \frac{1}{\sin\varphi_i} f_{i-1}\left(\frac{1}{L_{i-1}^s(\tau)}\right)d\tau + \int_0^t (\cot\varphi_i + \cot\varphi_{i+1})\, f_i\left(\frac{1}{L_i^s(\tau)}\right)d\tau$$
$$- \int_0^t \frac{1}{\sin\varphi_{i+1}} f_{i+1}\left(\frac{1}{L_{i+1}^s(\tau)}\right)d\tau + a_i s^{1/2},$$

where $i = 1, \dots, n$. Since

$$d_i^s(t) = \int_0^t f_i\left(\frac{1}{L_i^s(t)}\right)d\tau + 2\lambda_i s^{1/2}/a_i,$$

the right-hand side equals

$$-\frac{1}{\sin\varphi_1} d_{i-1}^s(t) + (\cot\varphi_i + \cot\varphi_{i+1})\, d_i^s(t) - \frac{1}{\sin\varphi_{i+1}} d_{i+1}^s(t) + a_i s^{1/2}$$
$$+ \left(\frac{2q_{i-1}}{a_{i-1}} - \frac{2p_i}{a_i} + \frac{2r_{i+1}}{a_{i+1}}\right)s^{1/2}.$$

By (20) this equals

$$L_i^s(t) = -\frac{q_{i-1}}{\lambda_{i-1}} d_{i-1}^s(t) + \frac{p_i}{\lambda_i} d_i^s(t) - \frac{r_{i+1}}{\lambda_{i+1}} d_{i+1}^s(t).$$

Since d_i^s converges to d_i uniformly in $[0, t_2]$ by Lemma 3.5, the limit $L_i(t) = \lim_{s\to 0} L_i^s(t)$ exists for $i = 1, \dots, n$ as a 1/2-Hölder continuous function. It satisfies

$$L_i(t) = -\frac{q_{i-1}}{\lambda_{i-1}} d_{i-1}(t) + \frac{p_i}{\lambda_1} d_i(t) - \frac{r_{i-1}}{\lambda_{i+1}} d_{i+1}(t). \tag{26}$$

Similarly, integrating (15) over $(0,t)$, we obtain

$$L_0^s(t) = -\frac{1}{\sin\varphi_0} f_{-1}(t) + \frac{p_0}{\lambda_0} d_0^s(t) - \frac{r_1}{\lambda_1} d_1^2(t) + \frac{2r_1}{a_1} s^{1/2}(t) + \ell_0^s$$

and

$$L_0(t) = -\frac{1}{\sin\varphi_0} f_{-1}(t) + \frac{p_0}{\lambda_0} d_0(t) - \frac{r_1}{\lambda_1} d_1(t) + \ell_0 \tag{27}$$

since $\ell_0^s \to \ell_0$ as $s \to 0$. Similarly, from (16) we obtain

$$\begin{aligned} L_{n+1}^s(t) &= -\frac{q_n}{\lambda_n} d_n^s(t) + \frac{p_{n+1}}{\lambda_{n+1}} d_{n+1}^s(t) - \frac{1}{\sin\varphi_{n+2}} f_{n+2}(0)t + \frac{2q_n}{a_n} s^{1/2} + \ell_{n+1}^s \\ L_{n+1}(t) &= -\frac{q_n}{\lambda_n} d_n(t) + \frac{p_{n+1}}{\lambda_{n+1}} d_{n+1}(t) - \frac{1}{\sin\varphi_{n+2}} f_{n+2}(0)t + \ell_{n+1}. \end{aligned} \tag{28}$$

At this moment it is not clear that L_i $(i = 1,\dots,n)$ is positive for $t \in (0,t_2]$. We shall prove that $L_i(t) > 0$ for $t \in (0,t_2]$. By Lemma 3.5 and Remark 3, we estimate d_i's in (26) from above and below by taking $t_1 = 0$ to get

$$L_i(t) \geq \left(-\frac{2q_{i-1}}{b_{i-1}}\sqrt{1+\varepsilon} + \frac{2p_i}{a_i}\sqrt{1-\varepsilon} - \frac{2r_{i+1}}{b_{i+1}}\sqrt{1+\varepsilon} \right) t^{1/2} \quad (i = 1,\dots,n)$$

for $t \in (0,t_2]$. Since a_i fulfills (20) and $b_i \to a_i$ as $\eta \to 0$, by taking ε and η small we obtain

$$-\frac{2q_{i-1}}{b_{i-1}}\sqrt{1+\varepsilon} + \frac{2p_i}{a_i}\sqrt{1-\varepsilon} - \frac{2r_{i+1}}{b_{i+1}}\sqrt{1+\varepsilon} \geq \frac{1}{2}\left(-\frac{2q_{i-1}}{b_{i-1}} + \frac{2p_i}{a_i} - \frac{2r_{i+1}}{b_{i+1}} \right) = \frac{a_i}{2}.$$

We thus conclude that

$$L_i(t) \geq \frac{1}{2} a_i t^{1/2} \quad \text{for} \quad t \in [0,t_2]. \tag{29}$$

Note that t_2 may tend to zero as $\varepsilon \to 0$, $\eta \to 0$ but we fixed these parameters as above so that $t_2 > 0$. The positivity of $L_0(t)$ and $L_{n+1}(t)$ in a short time interval $(0,t_*)$ with $t < t_*$ is rather clear by formulas (27), (28) for L_0 and L_{n+1} by taking $t_* \in (0,t_2)$ smaller.

It remains to prove that L_i's satisfies the equation (15), (16), (17). We recall

$$d_i^s(t) = \int_\delta^t f_i\left(\frac{1}{L_i^s(\tau)}\right) d\tau + d_i^s(\delta), \quad \delta \in (0,t_*).$$

Sending s to zero yields

$$d_i(t) = \int_\delta^t f_i\left(\frac{1}{L_i^s(\tau)}\right) d\tau + d_i(\delta)$$

for $i = 0,1,\dots,n+1$. By (29) the function $L_i(\tau)$ is integrable in $(0,t_*)$. Since $f(1/L) \leq \lambda_i(1+\varepsilon)/L$ for small L, by the dominated convergence theorem, we conclude that

$$d_i(t) = \int_0^t f_i\left(\frac{1}{L_i(\tau)}\right) d\tau.$$

By (26) we obtain for $i = 1, \dots, n$

$$\begin{aligned} L_i(t) = & -\frac{q_{i-1}}{\lambda_{i-1}} \int_0^t f_{i-1}\left(\frac{1}{L_{i-1}(\tau)}\right) d\tau + \frac{p_i}{\lambda_i} \int_0^t f_i\left(\frac{1}{L_i(\tau)}\right) d\tau \\ & - \frac{r_{i+1}}{\lambda_{i+1}} \int_0^t f_{i+1}\left(\frac{1}{L_{i+1}(\tau)}\right) d\tau \\ = & -\frac{1}{\sin\varphi_i} \int_0^t f_{i-1}\left(\frac{1}{L_{i-1}(\tau)}\right) d\tau + (\cot\varphi_i + \cot\varphi_{i+1}) \int_0^t f_i\left(\frac{1}{L_i(\tau)}\right) d\tau \\ & - \frac{1}{\sin\varphi_{i+1}} \int_0^t f_{i+1}\left(\frac{1}{L_{i+1}(\tau)}\right) d\tau. \end{aligned}$$

This is an integral form of (16). The formula (27) and (28) give the integral form of (15) and (17), respectively. Thus our L_i's are C^1 in $(0, t_*)$ for $i = 0, \dots, n+1$ satisfying (15), (16), (17) in $(0, t_*)$. Moreover, it is 1/2-Hölder up to $t = 0$ and the initial condition (15) is guaranteed. Note that L_0 and L_1 is C^1 up to $t = 0$.

(Uniqueness). We appeal a geometric argument involving d_i. Suppose that there exists another solution $\overline{L}_i(t) \in C[0, t_*) \cap C^1(0, t_*)$. It suffices to prove that $d_i(t) = \overline{d}_i(t)$, where $\overline{d}_i(t)$ is the distance function corresponding to $\overline{L}_i(t)$.

By comparison principle, it is clear that

$$\overline{d}_i(t) \le d_i^s(t) \quad (i = 0, \dots, n+1).$$

Sending s to zero to get

$$\overline{d}_i(t) \le d_i(t) \quad (i = 0, \dots, n+1).$$

Since

$$\overline{d}_i(t) = \int_0^t \overline{V}_i(\tau) d\tau = \int_0^t f_i\left(\frac{1}{\overline{L}_i(\tau)}\right) d\tau, \quad (i = 0, \dots, n+1)$$

the bound $\overline{d}_i \le d_i$ implies that d_i is finite so that the right-hand side is integrable. This implies $\overline{d}_i$ is a continuous function in $[0, t_*)$. By comparison principle, for small $\varepsilon > 0$

$$d_i(t) \le \overline{d}_i(\varepsilon + t) \quad (i = 0, \dots, n+1), \quad t \in [0, t_* - \varepsilon).$$

Sending ε to zero, by the continuity we obtain

$$d_i(t) \le \overline{d}_i(t) \quad (i = 0, \dots, n+1), \quad t \in [0, t_* - \varepsilon]$$

which now implies $d_1 \equiv \overline{d}_i$. The proof is now complete. □

Remark 4. For the proof of Theorem 2.3, it is possible to prove that

$$\lim_{t \downarrow 0} L_i(t)/t^{1/2} = a_i \quad (i = 1, \dots, n)$$

by taking ε, η smaller. This says that the shape is asymptotically self-similar as $t \to 0$.

4. Problems starting from general polygons

We shall prove Theorem 2.1 and Theorem 2.2. We consider a polygon Γ_0 whose facets (edges) $\{E_\alpha\}_{\alpha=1}^K$ has an orientation $\mathbf{n}_\alpha \in \mathcal{N}$. Let $\{P_\beta\}$ be the set of all vertices connecting two facets $S_{\beta-}$ and $S_{\beta+}$ such that there is at least one direction $\mathbf{n} \in \mathcal{N}$ between $\mathbf{n}_{\beta-}$ and $\mathbf{n}_{\beta+}$. We may assume that the orientation of Γ_0 is taken inward since other case can be handled similarly. We number $\{P_\beta\}_{\beta=1}^h$ so that it moves counterclockwise on Γ_0.

Proof of Theorem 2.1. We first consider the case when there is no facet E_α whose transition number is equal to zero. In other words, Γ_0 is convex.

Let $S_i^\beta(t)$ denote a newly created facet from the vertex P_β for $i = 1, \dots, n_\beta$. Let L_i^β denote its length. Then we derive the equation for L_i^β of the form

$$
\begin{aligned}
\frac{dL_i^\beta}{dt} = &- \frac{1}{\sin\varphi_i^\beta} f_{i-1}^\beta\left(\frac{1}{L_{i-1}^\beta(t)}\right) + \left(\cot\varphi_i^\beta + \cot\varphi_{i+1}^\beta\right) f_i^\beta\left(\frac{1}{L_i^\beta(t)}\right) \\
&- \frac{1}{\sin\varphi_{i+1}^\beta} f_{i+1}^\beta\left(\frac{1}{L_{i+1}^\beta(t)}\right),
\end{aligned} \tag{30}
$$

where the speed V_i^β of S_i^β is given by $f_i^\beta\left(1/L_{\beta_i}(t)\right)$; the function $L_0^\beta(t)$ denotes the length of evolving facet $S_{\beta-}(t)$ starting from $S_{\beta-}$ while $L_{n_\beta+1}^\beta(t)$ denotes the length of evolving facet $S_{\beta+}(t)$ starting from $S_{\beta+}$. This is just reindexing which also applies to other quantities φ_i^β, a_i^β, λ_i^β etc. for $\beta = 1, \dots, h$. We also have an evolution equation for the length ℓ_α of facet $E_\alpha(t)$ starting from E_α. We have a system of ODEs including (30) for $\{\ell_\alpha\}_{\alpha=1}^K$ and

$$\left\{ L_i^\beta \;\middle|\; i = 1, \dots, n_\beta, \beta = 1, \dots, h \right\}$$

with initial data $\ell_\alpha(0) = \ell_{0\alpha}$, $L_i^\beta(0) = 0$. Here $\ell_{0\alpha}$ is the length of E_α. The total system denotes (S).

To construct a solution we consider an approximate solution by setting

$$\ell_\alpha(0) = \ell_{0\alpha}^s, \ell_i^{\beta s}(0) = a_i^\beta s^{1/2}$$

by using a self-similar expanding solution starting from P_β in a sector consisting of $S_{\beta-}$ and $S_{\beta+}$. This is the same as in the previous section.

We use a distance function d_i^β from P_β for the line containing S_i^β and derive a uniform estimate; for $\varepsilon \in (0,1)$ and η, there is $t_2 > 0$ such that

$$\frac{\lambda_i^\beta\sqrt{1-\varepsilon}}{a_i^\beta}(t-t_1)^{1/2} \le d_i^\beta(t) - d_i^\beta(t_1) \le \frac{\lambda_i^\beta\sqrt{1+\varepsilon}}{b_i^\beta}(t-t_1)^{1/2}, \quad i = 1, 2, \dots, n_\beta$$

for $0 \le t_1 \le t \le t_2$ as in Lemma 3.5. As in Remark 3, the estimate from above also holds for $i = 0$ and $i = n_\beta + 1$, the distance corresponding to $S_{\beta-}$ and $S_{\beta+}$, respectively. This estimate guarantees the convergence of solution $\left(\ell_\alpha^s, L_i^{\beta s}\right)$ of (S) with initial data $\ell_\alpha(0) = \ell_{0\alpha}^s$, $L_i^{\beta s}(0) = a_i^\beta s^{1/2}$ to a solution of (S) with $\ell_\alpha(0) = \ell_{0\alpha}$, $\ell_i^\beta(0) = 0$ for $i = 1, \dots, n_\beta, \beta = 1, \dots, h$ as in the proof of Theorem 2.3. Once the solution of (S) is given, then it is easy to see that it satisfies (6) since

$$\frac{dd_i^\beta}{dt} = V_i^\beta.$$

We have thus consctructed a crystalline flow. Since d_i^β is continuous in time up to $t = 0$, the resulting solution $\{\Gamma_t\}$ converges to Γ_0 as $t \downarrow 0$ in the sense of Hausdorff distance.

The other case is that there is a facet E_{α_i} whose transition number equals zero in Γ_0. In this case we are able to split the problem. After renumbering there is a connected sequence of facets and vertices

$$E_0, E_1, \dots, E_k, P_1, \dots, P_{h_0}, E_{k+1}, \dots, E_{k+r}$$

such that transition number of E_0 and E_{k+r} equals zero while other facet including those between P_β's has the positive transition number. The facet E_0 and E_{k+r} can be the same facet. We are able to argue the same way to construct a desired crystalline flow in this case too.

The uniqueness follows the same argument as before. The proof is now complete. □

Proof of Theorem 2.2. By Theorem 2.1 it remains to prove that the crystalline flow $\{\Gamma_t\}$ for (2) is the only level-set flow of (2).

As proved in [GG3], it is known that a crystalline flow is a level-set flow provided that the corner preserving condition is fulfilled. So the only thing one has to check is that there is no fattening phenomenon in our setting. However, this can be easily seen by approximating the polygon from inside and outside and by applying stability of solution. Both solution converges to the same crystalline flow so there is no fattening. We omit the details. □

5. Numerical test

We gave here an example of our numerical calculation of (9)–(11). Of course, this initial value problem is singular so we replace the initial condition by (19).

We take a regular dodecagon with side length 1 as the Wulff shape so that $\Delta_i = 1$ and $\varphi_i = \pi/6$. We set $n = 3$ so that three facets are created. The system (20) of algebraic equations is reduced to

$$\begin{cases} \dfrac{a_1}{2} = \dfrac{2\sqrt{3}}{a_1} - \dfrac{2}{a_2}, \\ \dfrac{a_2}{2} = -\dfrac{2}{a_1} + \dfrac{2\sqrt{3}}{a_2} - \dfrac{2}{a_3}, \\ \dfrac{a_3}{2} = -\dfrac{2}{a_2} + \dfrac{2\sqrt{3}}{a_3}. \end{cases} \tag{31}$$

This system is explicitly solvable and the unique positive solution is the form

$$\begin{cases} a_1 = \sqrt{\dfrac{2}{\sqrt{3}}(5 - \sqrt{13})} = 1.2689\ldots, \\ a_2 = \left(\dfrac{\sqrt{3}}{a_1} - \dfrac{a_1}{4}\right)^{-1} = 0.9544\ldots, \\ a_3 = a_1. \end{cases} \tag{32}$$

In this particular example, we take

$$\ell_0^s = \ell_0 - \left(a_2 + \sqrt{3}\, a_1\right) s^{1/2},$$
$$\ell_4^s = \ell_4 - \left(\sqrt{3}\, a_1 + a_2\right) s^{1/2}.$$

In the first example, we set $\ell_0 = \ell_4 = 5$, $s = 10^{-4}$. We take $\varphi_0 = \varphi_4 = \pi/2$, $\varphi_1 = \varphi_2 = \varphi_3 = \pi/6$. This means our Wulff shape is modified outside $\{\theta_0, \theta_1, \theta_2, \theta_3\} \subset \mathcal{N}$. We draw evolution of facets starting from the corner with angle $\pi/3$. The motion is monotone and each line is drawn with time step $\Delta t = 0.0902$. The result of calculation is given in Figure 10. We see that newly created facets move quickly while the facets forming corners move slowly. The second example is given in Figure

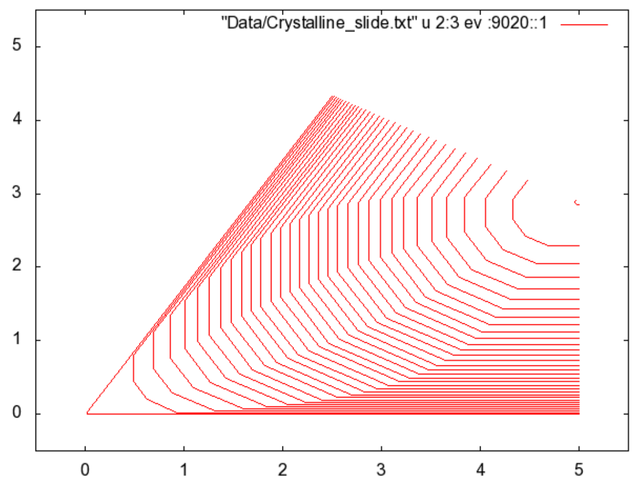

FIGURE 10. An example of the numerical calculation

11, where φ's and ℓ_0, ℓ_4 are the same but $s = 10^{-6}$, $\Delta t = 5 \times 10^{-4}$ and the picture is magnified.

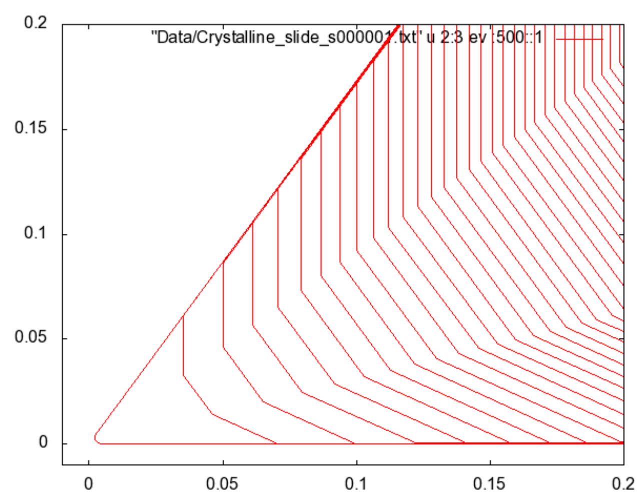

FIGURE 11. Another example of the numerical calculation (enlarged)

6. Unusual Briot-Bouquet system

We consider (9)–(11) and change independent variable t by $\tau = t^{1/2}$ and depending variable $g_i(\tau) = L_i(\tau^2)/\tau$, $i = 1, \ldots, n$. We postulate $g_i(0) = a_i$. For $i = 2, 3, \ldots, n-1$, the equation for L_i becomes

$$\frac{1}{2\tau}\frac{d}{d\tau}\left(\tau g_i(\tau)\right) = -\frac{q_{i-1}}{\tau g_{i-1}(\tau)} + \frac{p_i}{\tau g_i(\tau)} - \frac{r_{i+1}}{\tau g_{i+1}(\tau)}$$

which yields

$$\tau \frac{dg_i(\tau)}{d\tau} = -g_i(\tau) - \frac{2q_{i-1}}{g_{i-1}(\tau)} + \frac{2p_i}{g_i(\tau)} - \frac{2r_{i+1}}{g_{i+1}(\tau)}.$$

If one sets $\mathcal{L}_0(\tau) = L_0(\tau^2)$, $\mathcal{L}_{n+1}(\tau) = L_{n+1}(\tau^2)$, we now observe that the system (9)–(11) will be transformed into

$$\begin{aligned}
\frac{d\mathcal{L}_0(\tau)}{d\tau} &= \frac{2p_0\tau}{\mathcal{L}_0(\tau)} - \frac{2r_1}{g_1(\tau)}, \quad \mathcal{L}_0(0) = \ell_0 \\
\tau \frac{dg_1(\tau)}{d\tau} &= -g_1(\tau) - \frac{2q_1\tau}{\mathcal{L}_0(\tau)} + \frac{2p_0}{g_1(\tau)} - \frac{2r_1}{g_2(\tau)}, \quad g_1(0) = a_1 \\
\tau \frac{dg_i(\tau)}{d\tau} &= -g_i(\tau) - \frac{2q_{i-1}}{g_{i-1}(\tau)} + \frac{2p_i}{g_i(\tau)} - \frac{2r_{i+1}}{g_{i+1}(\tau)}, \quad g_i(0) = a_i \ (i = 2, \dots, n-1) \\
\tau \frac{dg_n(\tau)}{d\tau} &= -g_n(\tau) - \frac{2q_{n-1}}{g_{n-1}(\tau)} + \frac{2p_n}{g_n(\tau)} - \frac{2r_{n+1}\tau}{\mathcal{L}_{n+1}(\tau)}, \quad g_n(0) = a_n \\
\frac{d\mathcal{L}_{n+1}}{d\tau} &= -\frac{2q_n}{g_n(\tau)} + \frac{2p_{n+1}\tau}{\mathcal{L}_{n+1}(\tau)}, \quad \mathcal{L}_{n+1}(0) = \ell_{n+1}.
\end{aligned}$$

To see the structure of this system, we assume that $\mathcal{L}_0$ and $\mathcal{L}_{n+1}$ are already given. Define functions $F_i : (\mathbf{R}_+) \to \mathbf{R}$ $(i = 1, \dots, n)$, $\mathbf{R}_+ = (0, \infty)$ as follows

$$\begin{aligned}
F_1(\tau, x_1, \dots, x_n) &= -\frac{2q_0\tau}{\mathcal{L}_0(\tau)} + \frac{2p_1}{x_1} - \frac{2r_2}{x_2}, \\
F_i(\tau, x_1, \dots, x_n) &= -\frac{2q_{i-1}}{x_{i-1}} + \frac{2p_i}{x_i} - \frac{2r_{i+1}}{x_{i+1}}, \ (i = 2, 3, \dots, n-1) \\
F_n(\tau, x_1, \dots, x_n) &= \frac{-2q_{n-1}}{x_{n-1}} + \frac{2p_n}{x_n} - \frac{2r_{n+1}\tau}{\mathcal{L}_{n+1}(\tau)}.
\end{aligned}$$

We set $F = {}^t(F_1, \dots, F_n)$, $x = {}^t(x_1, \dots, x_n)$, $g = {}^t(g_1, \dots, g_n)$, $a = {}^t(a_1, \dots, a_n)$. The ODE system for g is of the form

$$\tau \frac{dg(\tau)}{d\tau} = -g(\tau) + F(\tau, g(\tau)), \quad g(0) = a$$

and its integral form is

$$g(\tau) = \frac{1}{\tau} \int_0^\tau F(\sigma, g(\sigma))\, d\sigma, \quad g(0) = a.$$

If F would satisfy

$$F(0, a) = 0 \quad \text{and the Jacobi matrix} \quad D_x F(0, a) = 0,$$

then one would apply [GG98, Lemma 8] to conclude the desired existance result. In this case, however, the Jacobi matrix $D_x F(0, a)$ is not zero so [GG98, Lemma 8] cannot be applied. The existence of the solution for the Briot-Bouquet system is studied by calculating power series for τ. Although the existence of formal power series is easy, its convergence is not easy to check. A typical criterion is given by eigenvalues of Jacobi matrix $D_x F$ [GeT]. However, in our case these eigenvalues are difficult to compute. In our case, although there is a formal power series solution but it is not clear whether or not this series actually converges.

If $n = 1$, as shown in [O], it is possible to construct a solution by applying a contraction mapping principle in $C[0, T]$ for

$$g \longmapsto \frac{1}{\tau} \int_0^\tau F(\sigma, g(\sigma))\, d\sigma.$$

However, for $n > 1$ it seems that the mapping is not contraction as indicated in [O].

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