<table>
<thead>
<tr>
<th>Title</th>
<th>CRYSTALLINE FLOW STARTING FROM A GENERAL POLYGON</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Giga, Mi-Ho; Giga, Yoshikazu; Kuroda, Ryo; Ochiai, Yusuke</td>
</tr>
<tr>
<td>Citation</td>
<td>Hokkaido University Preprint Series in Mathematics, 1136, 1-24</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2021-03-04</td>
</tr>
<tr>
<td>DOI</td>
<td>10.14943/96862</td>
</tr>
<tr>
<td>Doc URL</td>
<td><a href="http://hdl.handle.net/2115/80552">http://hdl.handle.net/2115/80552</a></td>
</tr>
<tr>
<td>Type</td>
<td>bulletin (article)</td>
</tr>
<tr>
<td>File Information</td>
<td>CryFlowStarting.pdf</td>
</tr>
</tbody>
</table>

Hokkaido University Collection of Scholarly and Academic Papers: HUSCAP
CRYSTALLINE FLOW STARTING FROM A GENERAL POLYGON

MI-HO GIGA
Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan

YOSHIKAZU GIGA
Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan

RYO KURODA
Department of Indian Philosophy and Buddhist Studies, Humanities, Faculty of Letters
The University of Tokyo
7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan

YUSUKE OCHAI
Airitech Co., Ltd.
Masonic 39MT Building, 2-4-5 Azabudai, Minato-ku, Tokyo 106-0041, Japan

Abstract. This paper solves a singular initial value problem for a system of ordinary differential equations describing a polygonal flow called a crystalline flow. Such a problem corresponds to a crystalline flow starting from a general polygon not necessarily admissible in the sense that the corresponding initial value problem is singular. To solve the problem, a self-similar expanding solution constructed by the first two authors with H. Hontani (2006) is effectively used.

1. Introduction

A crystalline mean curvature flow is an example of an anisotropic mean curvature flow whose anisotropy is strong so that its evolution is determined by nonlocal quantities. In mathematical community it was introduced by Taylor [T1] and independently by Angenent and Gurtin [AG] around 1990. In this paper, we restrict ourselves to flows in a plane $\mathbb{R}^2$ so that the mean curvature is just the curvature of a planar curve. To explain a crystalline curvature flow (crystalline flow for short), we consider an example of anisotropic curvature flow for evolving curve $\{\Gamma_t\}$ in a plane of the form

$$V = \kappa_\gamma \quad \text{with} \quad \kappa_\gamma = -\text{div} \xi(n) \quad \text{on} \quad \Gamma_t,$$

Key words and phrases. crystalline flow, non-admissible polygon, self-similar expanding solution, comparison principle, Briot-Bouquet system.

The work of the second author was partly supported by the Japan Society for the Promotion of Science (JSPS) through the grants KAKENHI No. 19H00639, No. 18H05323, No. 17H01091 and by Arithmer Inc. through collaborative grant.

* Corresponding author: Yoshikazu Giga.
where $t$ denotes the time parameter. Here $V$ denotes the normal velocity of $\{\Gamma_t\}$ in the direction of unit normal $\mathbf{n}$ and $\xi = \nabla \gamma$, where $\gamma$ is an interfacial energy density, which is positively one-homogeneous and convex in $\mathbb{R}^2$. The operator $\text{div}$ denotes the surface divergence. The quantity $\kappa_\gamma$ is often called anisotropic curvature in the direction of $\mathbf{n}$. If $\gamma(p) = |p|$, then $\kappa_\gamma$ is the usual curvature and (1) becomes a curve shortening equation. If $\gamma$ is piecewise linear, then (1) is very singular and the speed is not determined by local quantities like curvature. We say that the quantity formally corresponding to $\kappa_\gamma$ is a crystalline curvature if $\gamma$ is piecewise linear. If $V$ is determined by crystalline curvature and the orientation $\mathbf{n}$, i.e.,

\begin{equation}
V = f(\kappa_\gamma, \mathbf{n}) \quad \text{on} \quad \Gamma_t,
\end{equation}

then we say that this equation is a crystalline flow equation. Here we assume that a given function $f$ is nondecreasing with respect to $\kappa_\gamma$ so that the problem is still (degenerate) parabolic.

There are several ways to solve (1) or more generally (2). For example, a level-set method ([CGG], [ES], [G]) can be adjusted for such a problem and gives a global unique (up to fattening) solution starting from any closed curve as studied in [GG1], [GG2], [GG4]. In [T1], [AG] a class of solutions is restricted in a special class of polygons called “admissible”. It turns out that their solution agrees with a level-set solution [GG3] if initial curve is an admissible polygon. One merit of this method is that the solution is given by a system of ordinary differential equations. Moreover, it gives a numerical algorithm to solve a smooth anisotropic curvature flow or even the heat equation by approximating by crystalline flows. Such a topic is studied [FG], [GirK], [GG2] for a graph-like solution and [Gir], [IS], [GG4] for a closed curve.

Let us recall the notion of an admissible polygon introduced in [T1], [AG]. For this purpose, we recall the notion of the Wulff shape

$$W_\gamma = \{ x \in \mathbb{R}^2 \mid x \cdot m \leq \gamma(m) \text{ for all } m \in \mathbb{R}^2 \}.$$ 

If $\gamma$ is piecewise linear, convex one-homogeneous function and $\gamma > 0$ except the origin, then $W_\gamma$ contains zero as an interior point and $W_\gamma$ is a closed, convex polygon. Its weighted curvature $\kappa_\gamma$ formally equals to $-1$ if $\mathbf{n}$ is taken outward so the Wulff shape is considered as a unit disk for isotropic case $\kappa_\gamma = \kappa$. We say that an (oriented) polygon is an admissible crystal if

(i) (direction condition) the orientation of each facet (edge) is one of that in $\partial W_\gamma$ and

(ii) (adjacency condition) the orientations of adjacent facets should be adjacent in $\partial W_\gamma$.

We say that $\{\Gamma_t\}$ is an admissible evolving crystal if $\Gamma_t$ is an admissible crystal at each $t$ and the motion of all vertices of $\Gamma_t$ is $C^1$ in time $t$. This implicitly assumes that each facet moves keeping its orientation. Let $S_j(t)$ denote the $j$-th facet of $\Gamma_t$. Let $\Delta(\mathbf{n}_j)$ be the length of the facet of $\partial W_\gamma$ whose orientation equals the orientation $\mathbf{n}_j$ of $S_j$. For an admissible evolving crystal, the equation (1) turns to be

\begin{equation}
V_j(t) = \Lambda_j(t) \quad \text{on} \quad S_j(t)
\end{equation}

with $\kappa_\gamma$ on $S_j$ equals

$$\Lambda_j(t) = \chi_j \Delta(\mathbf{n}_j)/L_j(t).$$

Here $V_j(t)$ denotes the normal velocity of $S_j(t)$ in the direction of $\mathbf{n}_j$ and $L_j(t)$ denotes the length of $S_j(t)$; $\chi_j$ is the transition number taking only three values
+1, 0, −1 depending on whether $\Gamma_t$ is convex, inflective, concave near $S_j(t)$ in the direction of $n_j$; see Figure 1. Together with transport equations we have a finite system of ordinary differential equations (ODEs), for example, for $L_j$’s, provided that the initial polygon $\Gamma_0$ is a closed polygon. This system is at least locally solvable if $L_j(0) > 0$ for all $j$, i.e., in the case that the initial polygon is admissible. The resulting flow $\{\Gamma_t\}$ is often called a crystalline flow for (1) or (2). It is an admissible evolving crystal satisfying (1) or (2).

Our goal in this paper is to show that a level-set flow for (1) or more general (2) starting from a polygon $\Gamma_0$ satisfying (i) but violating the adjacency condition (ii) immediately becomes admissible (Figure 2): in particular, it satisfies the adjacency condition (ii) as well as (i) for $t > 0$. In particular, the solution becomes an admissible evolving crystal instantaneously. This solution turns to be constructed explicitly by solving the system of ODEs with some $L_j(0)$’s are zero. A typical example is a self-similar expanding solution for (1) in a sector. This corresponds to the case that $L_0(0) = \ell_0 > 0$, $L_{n+1}(0) = \ell_{n+1} > 0$ while $L_1(0) = L_2(0) = \ldots = L_n(0) = 0$ and $\chi_0 = \chi_{n+1} = 0$ so that $S_0$, $S_{n+1}$ are standing. Unique existence of such a solution is shown in [Ca] when $W_\gamma$ is a regular polygon and established for general (1) in [GGH] (Figure 3). In the case that either $\chi_0$ or $\chi_{n+1}$ is not zero, the solution is not exactly a self-similar expanding solution but it can be controlled by these solutions. Based on such observation, we are able to construct a solution when $\chi_0$ or $\chi_{n+1}$ is non-zero. Note that in this case the motion also depends on the facet $S_{-1}$ adjacent to $S_0$ and $S_{n+2}$ adjacent to $S_{n+1}$, but for simplicity, we assume here that transition numbers of these facets are zero. Such a way of construction based on self-similar expanding solutions has been suggested by Taylor [T3, Section
It was carried out in the thesis of Ochiai [O] for (1) under the condition that $\chi_{-1} = 0$ (if $\chi_0 \neq 0$) and $\chi_{n+2} = 0$ (if $\chi_{n+1} \neq 0$). Applying such an idea for each non-admissible corner, one is able to solve the system of ODEs for a polygon violating the adjacency condition (ii).

In this paper, we complete this procedure and also handle more general equation (2) whose typical example is

$$V = M(n)(\kappa_\gamma + C),$$

where $C$ is a constant and $M$ is a positive constant depending on the orientation $n$. The constant $C$ is often called a driving force term and $M$ is called the mobility, which often appears in theory of crystal growth as well as materials science [Gu], [T2]. Since the equation of crystalline flow for (1) is (3), the equation corresponding to (4) is

$$V_j(t) = \chi_j \hat{\Delta}(n_j)/L_j + M(n_j)C, \quad \hat{\Delta}(n) = M(n)\Delta(n).$$

Although there may not exist a Wulff shape having $\hat{\Delta}$ as an edge length, we are still able to derive ODE’s for $L_j$’s and handle them in a similar way. Relation between such a polygonal flow and a level-set flow by now standard [GG3]. We remark that our results can be extended to the problem starting from a general polygon by recalling the notion of weakly admissible crystal [GG96]; see Remark 1.

To construct a solution from non-admissible polygon, we have to be careful since (4) may not have a self-similar expanding solution starting from corners. As for (3), we approximate initial data with admissible one and derive an equicontinuity estimate for distance function of each line containing newly created facet from the corner. In [O] this is done by comparing with self-similar expanding solutions for (3). In our case, we shall use self-similar solutions for (3) which approximates (2) (more precisely (6)). To derive necessary equicontinuity estimates for the distance function, which is called the support function from the corner, we fully compare with self-similar solutions by comparison principle for the distance function.

The system of ODEs for $L_j$’s is formally regarded as a kind of a Briot-Bouquet system, but it does not satisfy a proper condition even for (3). Thus, the existence of a solution does not follow from [GeT] as observed in [O].

Behavior of a crystalline flow for convex initial data has been well-studied especially the case when the equation is (3) or (4) with $C = 0$, i.e., there is no driving force term. It is easy to see that convexity is preserved. For (4) with $M = \gamma$
with $C = 0$, there always exists a self-similar shrinking solution to a point whose profile is the Wulff shape. This is a substitute of a shrinking circle for the curve shortening equation. The uniqueness of self-similar solution is proved when the Wulff shape $W_\gamma$ is symmetric with respect to the origin and the number of its vertices is more than four as shown in [S1]; in the case $W_\gamma$ is a parallelogram all parallelogram shrinks self-similarly. Moreover, it is shown in [S1] all convex solution shrinks asymptotically similar to the self-similar solution. However, if one considers (4) with $C = 0$ with $M$ unrelated to $\gamma$, the solution is more complicated as discussed in [S2], [A]. In [A] rather complete picture is given. Even the flow is orientation free, i.e., $\Delta(n) = \Delta(-n)$ in (5) with $C = 0$, there is a chance that there is no self-similar shrinking and isoperimetric ratio tends to infinity. Moreover, in [A] it is shown that the minimal length of facets at time $t$ behaves like $\{(T - t)/\log(T - t)\}^{1/2}$ or $(T - t)^{\beta}$, $1/2 < \beta < 1$ as $t$ tends to $T$, where $T$ is the extinction time. For a self-similar solution, the length should behave like $(T - t)^{1/2}$ so it is shorter than that of a self-similar solution. If one considers a crystalline flow corresponding to $V = \kappa_\gamma$ for $\alpha > 0$, the situation depends whether $\alpha < 1$, $\alpha > 1$ or $\alpha = 1$; we have already discussed the case $\alpha = 1$. In the case $\alpha \geq 1$, it is shown in [GG3] that there is no self-similar shrinking and isoperimetric ratio tends to infinity. Moreover, in [A] it is shown that the minimal length of facets at time $t$ behaves like $\{(T - t)/\log(T - t)\}^{1/2}$ or $(T - t)^{\beta}$, $1/2 < \beta < 1$ as $t$ tends to $T$, where $T$ is the extinction time. For a self-similar solution, the length should behave like $(T - t)^{1/2}$ so it is shorter than that of a self-similar solution. If one considers a crystalline flow corresponding to $V = \kappa_\gamma$ for $\alpha > 0$, the situation depends whether $\alpha < 1$, $\alpha > 1$ or $\alpha = 1$; we have already discussed the case $\alpha = 1$. In the case $\alpha \geq 1$, it is shown in [GG3] that there is no self-similar shrinking and isoperimetric ratio tends to infinity. Moreover, in [A] it is shown that the minimal length of facets at time $t$ behaves like $\{(T - t)/\log(T - t)\}^{1/2}$ or $(T - t)^{\beta}$, $1/2 < \beta < 1$ as $t$ tends to $T$, where $T$ is the extinction time. For a self-similar solution, the length should behave like $(T - t)^{1/2}$ so it is shorter than that of a self-similar solution. If one considers a crystalline flow corresponding to $V = \kappa_\gamma$ for $\alpha > 0$, the situation depends whether $\alpha < 1$, $\alpha > 1$ or $\alpha = 1$; we have already discussed the case $\alpha = 1$. In the case $\alpha \geq 1$, it is shown in [GG3] that there is no self-similar shrinking and isoperimetric ratio tends to infinity. Moreover, in [A] it is shown that the minimal length of facets at time $t$ behaves like $\{(T - t)/\log(T - t)\}^{1/2}$ or $(T - t)^{\beta}$, $1/2 < \beta < 1$ as $t$ tends to $T$, where $T$ is the extinction time. For a self-similar solution, the length should behave like $(T - t)^{1/2}$ so it is shorter than that of a self-similar solution. If one considers a crystalline flow corresponding to $V = \kappa_\gamma$ for $\alpha > 0$, the situation depends whether $\alpha < 1$, $\alpha > 1$ or $\alpha = 1$; we have already discussed the case $\alpha = 1$. In the case $\alpha \geq 1$, it is shown in [GG3] that there is no self-similar shrinking and isoperimetric ratio tends to infinity. Moreover, in [A] it is shown that the minimal length of facets at time $t$ behaves like $\{(T - t)/\log(T - t)\}^{1/2}$ or $(T - t)^{\beta}$, $1/2 < \beta < 1$ as $t$ tends to $T$, where $T$ is the extinction time. For a self-similar solution, the length should behave like $(T - t)^{1/2}$ so it is shorter than that of a self-similar solution. If one considers a crystalline flow corresponding to $V = \kappa_\gamma$ for $\alpha > 0$, the situation depends whether $\alpha < 1$, $\alpha > 1$ or $\alpha = 1$; we have already discussed the case $\alpha = 1$. In the case $\alpha \geq 1$, it is shown in [GG3] that there is no self-similar shrinking and isoperimetric ratio tends to infinity. Moreover, in [A] it is shown that the minimal length of facets at time $t$ behaves like $\{(T - t)/\log(T - t)\}^{1/2}$ or $(T - t)^{\beta}$, $1/2 < \beta < 1$ as $t$ tends to $T$, where $T$ is the extinction time. For a self-similar solution, the length should behave like $(T - t)^{1/2}$ so it is shorter than that of a self-similar solution. If one considers a crystalline flow corresponding to $V = \kappa_\gamma$ for $\alpha > 0$, the situation depends whether $\alpha < 1$, $\alpha > 1$ or $\alpha = 1$; we have already discussed the case $\alpha = 1$. In the case $\alpha \geq 1$, it is shown in [GG3] that there is no self-similar shrinking and isoperimetric ratio tends to infinity. Moreover, in [A] it is shown that the minimal length of facets at time $t$ behaves like $\{(T - t)/\log(T - t)\}^{1/2}$ or $(T - t)^{\beta}$, $1/2 < \beta < 1$ as $t$ tends to $T$, where $T$ is the extinction time. For a self-similar solution, the length should behave like $(T - t)^{1/2}$ so it is shorter than that of a self-similar solution. If one considers a crystalline flow corresponding to $V = \kappa_\gamma$ for $\alpha > 0$, the situation depends whether $\alpha < 1$, $\alpha > 1$ or $\alpha = 1$; we have already discussed the case $\alpha = 1$. In the case $\alpha \geq 1$, it is shown in [GG3] that there is no self-similar shrinking and isoperimetric ratio tends to infinity. Moreover, in [A] it is shown that the minimal length of facets at time $t$ behaves like $\{(T - t)/\log(T - t)\}^{1/2}$ or $(T - t)^{\beta}$, $1/2 < \beta < 1$ as $t$ tends to $T$, where $T$ is the extinction time. For a self-similar solution, the length should behave like $(T - t)^{1/2}$ so it is shorter than that of a self-similar solution. If one considers a crystalline flow corresponding to $V = \kappa_\gamma$ for $\alpha > 0$, the situation depends whether $\alpha < 1$, $\alpha > 1$ or $\alpha = 1$; we have already discussed the case $\alpha = 1$. In the case $\alpha \geq 1$, it is shown in [GG3] that there is no self-similar shrinking and isoperimetric ratio tends to infinity. Moreover, in [A] it is shown that the minimal length of facets at time $t$ behaves like $\{(T - t)/\log(T - t)\}^{1/2}$ or $(T - t)^{\beta}$, $1/2 < \beta < 1$ as $t$ tends to $T$, where $T$ is the extinction time. For a self-similar solution, the length should behave like $(T - t)^{1/2}$ so it is shorter than that of a self-similar solution. If one considers a crystalline flow corresponding to $V = \kappa_\gamma$ for $\alpha > 0$, the situation depends whether $\alpha < 1$, $\alpha > 1$ or $\alpha = 1$; we have already discussed the case $\alpha = 1$. In the case $\alpha \geq 1$, it is shown in [GG3] that there is no self-similar shrinking and isoperimetric ratio tends to infinity. Moreover, in [A] it is shown that the minimal length of facets at time $t$ behaves like $\{(T - t)/\log(T - t)\}^{1/2}$ or $(T - t)^{\beta}$, $1/2 < \beta < 1$ as $t$ tends to $T$, where $T$ is the extinction time. For a self-similar solution, the length should behave like $(T - t)^{1/2}$ so it is shorter than that of a self-similar solution.
in the initial polygon. It allows a facet whose orientation may not be included in those of the Wulff shape. A convex polygon studied in [Ya] is weakly admissible in the sense of [GG96].

A crystalline flow for the eikonal-curvature flow equation (4) or (5) has been also well-studied. For V-shaped initial data, its evolution was studied in [I11a], [I11b]. For a growth of convex polygon, the large time behavior for a crystalline flow for (4) or (5) is studied by [GG13] with special emphasis on the anisotropic effects of mobility $M$ and $\gamma$. A crystalline flow is also applied to study growth of spirals since the work of [I14], which is further developed in [IO1]. Various methods to compute crystalline flow without solving ODEs are compared with the classical method [IO2]. Although we do not intend to give numerics much, we point out that there is a numerical approach based on a variational inequality, when $\Gamma_t$ is a graph [EGS]. There is a direct numerical approach based on a level-set method for crystalline flow as studied in [OOTT].

Although there is a large number of articles studying crystalline flows, the situation is usually in the case that new facets are not created. A facet creation problem has been observed in [GG1] and further developed in [Mu], [MuR1], [MuR2] mostly for graph-like solutions. However, the number of newly created facets in one point is just one. In [Ca], [GGH], several facets are created just from one corner. The paper [O] is the first paper to handle the situation where several facets are created between convex facets. Since the paper [O] is not well circulated, we write a whole necessary argument to achieve our goal.

It is natural to ask what happens for crystalline flow in a higher dimensional space, for example, evolution of a polytope in three-dimensional space. If one expects that the flow still has a comparison principle, the original curvature flow for (1) with crystalline interfacial energy $\gamma$, i.e., $\gamma$ is piecewise linear is not reduced to (3). In fact, a facet may break or bend as shown in [BNP1], [BNP2]. This indicates that polytope may not stay as polytope during evolution. Thus, the problem is not reduced to a system of ODEs. Nevertheless, a level-set flow has been constructed independently by [CMP], [CMMP] for (4) with “convex” and [GP1], [GP2] for (4) for arbitrary $M$; for non-uniform driving force $C$, see [CMMP], [GP3]. The methods in [GP1], [GP2], [GP3] are quite different from those of [CMP], [CMMP].

This paper is organized as follows. In Section 2, we formulate problems and state our main results. In Section 3, we derive a priori estimate for approximate solutions by comparing with self-similar expanding solutions constructed by [GGH]. We construct a solution for singular initial value problem near one corner. This procedure is essentially done for (3) in [O]; in this particular case one is able to take $\varepsilon = 0$ in the argument. In Section 4, we shall prove our main results stated in Section 2 by extending the argument in Section 3. A weaker form of results is presented in [K] but it is not well circulated so we reproduce several contents of [K]. In Section 5, we give some numerical tests. In Section 6, we explain why the general theory for a Briot-Bouquet system does not apply at least directly.

2. Crystalline flow

To formulate the problem, we recall several notion. Let $\mathcal{N}$ be a finite subset of the unit circle, i.e.,

$$\mathcal{N} = \{ n_k \}_{k=1}^{n} \quad \text{with} \quad n_k = (\cos \theta_k, \sin \theta_k).$$
If the Wulff shape $W_\gamma$ is given, $\mathcal{N}$ is taken so that it is the set of all orientations of $\partial W_\gamma$. We call $\mathcal{N}$ the set of admissible directions. The set $\Theta = \{\theta_k \}_{k=1}^n$ is called the set of admissible angles, which is considered a subset in $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. A polygon whose orientation belonging to $\mathcal{N}$ is called admissible if the angles of adjacent facet are adjacent in $\Theta$ (adjacency condition). An evolving polygon $\{\Gamma_t\}_{t \in I}$ is an admissible evolving crystal if $\Gamma_t$ is an admissible polygon for $t \in I$ and the motion of all vertices is $C^1$ in time $t \in I$, where $I$ is a time interval. We consider the equation (2), i.e.,

$$V = f(\kappa, n),$$

where $f$ is a given locally Lipschitz (continuous) function and is nondecreasing in the first variable. We say that an admissible evolving crystal $\{\Gamma_t\}_{t \in I}$ is a crystalline flow for (2) if

$$V_j(t) = f(\Lambda_j(t), n_j),$$

$$\Lambda_j(t) = \chi_j \Delta(n_j)/L_j(t)$$

holds for each facet $S_j$ of $\Gamma_t$ for $t \in I$. The quantity $\Lambda_j$ is often called a crystalline curvature introduced in Section 1. Setting $f_j(x) = f(\Delta(n_j)x, n_j)$, this equation of the form

$$(6) \quad V_j(t) = f_j(\chi_j/L_j(t))$$

with a given locally Lipschitz function $f_j$. We may call that an admissible evolving crystal is a crystalline flow of (6) if it satisfies (6) for all $t \in I$. We are able to construct a crystalline flow for non-admissible polygon.

**Theorem 2.1.** Assume that $f_j$ is locally Lipschitz and non-decreasing. Assume that

$$(7) \lim_{|x| \to \infty} f_j(x)/x = \lambda_j$$

with $\lambda_j$ depending on $j$ only through $n_j$. Let $\Gamma_0$ be a polygon whose orientations belong to $\mathcal{N}$. Then there is a unique crystalline flow of (6) in some time interval $(0, t_0)$ which converges to $\Gamma_0$ as $t \downarrow 0$ in the Hausdorff distance sense.

We apply Theorem 2.1 to get a main result.

**Theorem 2.2.** Assume that $f$ is locally Lipschitz and non-decreasing in the first variable. Assume that

$$(8) \lim_{|y| \to \infty} f(y, n_j)/y = \lambda_j$$

with $\lambda_j$ depending on $j$ only through $n_j$. Let $\Gamma_0$ be in Theorem 2.1. Then there is a unique local-in-time crystalline flow $\{\Gamma_t\}$ for (2) which converges to $\Gamma_0$ as $t \downarrow 0$ in the Hausdorff distance. Moreover, if $f(0, n)$ satisfies “corner preserving conditions”, this solution is a level-set flow solution of (2).

Let us explain the corner preserving condition [GG13]. We say that $f$ satisfies the corner preserving condition if for each $n_k \in \mathcal{N}$

$$f^0(m) \geq \frac{1}{\sin \phi_{k+1}} (f^0(n_k) \sin \theta_{k+1} + f^0(n_{k+1}) \sin \theta_k), \quad f^0(p) := f(0, p)$$

for all $m = (\cos \theta, \sin \theta)$ with $\theta_k < \theta < \theta_{k+1}$; we consider indices modulo $n$ so that $n+1$ is interpreted as $1$. Geometrically speaking, this is equivalent to say that

$$A_k \subset \{x \in \mathbb{R}^2 \mid x \cdot m \leq f^0(m)\} \subset B_k$$
with
\[ A_k = H_k \cap H_{k+1}, \quad B_k = H_k \cup H_{k+1}, \quad H_{k+j} = \{ x \in \mathbb{R}^2 \mid x \cdot n_{k+j} \leq f^0(n_{k+j}) \} \]
for all \( m = (\cos \theta, \sin \theta) \) with \( \theta_k < \theta < \theta_{k+1} \). This condition says that in the corner all facets whose orientations are between that of facets forming the corner move faster than corner facets for \( V = f^0(n) \). This condition is first pointed out explicitly by [GHK] and independently by [GSS]. It is stated in a different form in [GG96], which proves a graph-like crystalline flow is consistent with viscosity solution defined in [GG1]; see also [GG]. The geometric version is found in [GG3]; however, unfortunately the definition of \( B_k \) was mistyped.

**Remark 1.** In both Theorem 2.1 and Theorem 2.2, one may allow \( \Gamma_0 \) for a general polygon whose orientation \( n_\alpha \) of a facet \( E_\alpha \) not belong to \( N \) by interpreting \( f_\alpha \) is independent of \( L_\alpha \) or \( \Delta(n_\alpha) = 0 \). The resulting crystalline flow is still an evolving crystal if \( \Delta(n_\alpha) = 0 \) is allowed.

If one considers an non-admissible polygon \( \Gamma_0 \) whose facets \( \{E_\alpha\}_{\alpha=1}^K \) has an orientation \( n_\alpha \in N \), there is at least one vertex \( P \) connecting two facets \( S_A \) and \( S_B \) such that there is at least one direction \( n \in N \) between \( n_A \) and \( n_B \). We may assume \( \theta_A < \theta_B \) since the case \( \theta_B < \theta_A \) can be treated similarly.

Let \( \varphi_i = \theta_i - \theta_{i-1} \) for \( i = 0, \ldots, n+2 \), where \( \theta_{-1}, \theta_n, \theta_{n+1} \in N \) and \( (\theta_{-1}, \theta_0) \cap N = \emptyset \), \( (\theta_{n+1}, \theta_{n+2}) \cap N = \emptyset \).

We expect that at the corner by \( S_A \) and \( S_B \), \( n \) facets should be created; see Figure 4. If \( f(x, n_j) = x \) and transition number of both \( S_A \) and \( S_B \) are zero, there is a

![Wulff shape and newly created facets](image)

**Figure 4.** Wulff shape and newly created facets

self-similar expanding solution for (3) as proved in [GGH]. Thus, we are interested in the case that one of \( \chi_A \) and \( \chi_B \) is not zero. Let \( \{S_i(t)\}_{i=1}^n \) be newly created facets with orientation \( n_i = (\cos \theta_i, \sin \theta_i) \). We set \( S_0(t) = S_A(t), S_{n+1}(t) = S_B(t) \).
We shall derive equations for the length $L_i(t)$ of $S_i(t)$. By transport equation, we have

$$\frac{dL_i(t)}{dt} = -\frac{1}{\sin \phi_i} V_{i-1}(t) + (\cot \phi_i + \cot \phi_{i+1}) V_i(t) - \frac{1}{\sin \phi_{i+1}} V_{i+1}(t).$$

See Figure 5. This identity involving 3 adjacent facets holds for any admissible evolving crystal. If we consider the equation (3) for $S_j$'s, then the resulting system of equations are

$$\begin{align*}
\frac{dL_0(t)}{dt} &= \frac{p_0}{L_0(t)} - \frac{r_1}{L_1(t)}, \quad L_0(0) = \ell_0 > 0, \\
\frac{dL_i(t)}{dt} &= -\frac{q_{i-1}}{L_{i-1}(t)} + \frac{p_i}{L_i(t)} - \frac{r_{i+1}}{L_{i+1}(t)}, \quad L_i(0) = 0 \ (i = 1, 2, \ldots, n), \\
\frac{dL_{n+1}(t)}{dt} &= -\frac{q_n}{L_n(t)} + \frac{p_{n+1}}{L_{n+1}(t)}, \quad L_{n+1}(0) = \ell_{n+1} > 0.
\end{align*}$$

Here positive numbers $p_i, q_i, r_i$'s are

$$\begin{align*}
p_i &= (\cot \phi_i + \cot \phi_{i+1}) \Delta, \quad (i = 0, 1, \ldots, n+1), \\
q_i &= \Delta_i / \sin \phi_{i+1}, \quad (i = 0, 1, \ldots, n), \\
r_i &= \Delta_i / \sin \phi_i, \quad (i = 1, 2, \ldots, n+1).
\end{align*}$$

Here we invoke the assumption that the facet touching to $S_A$ (resp. $S_B$) but not to $S_B$ (resp. $S_A$) has zero transition number. This assumption implies that the first and last equation does not contain $q_{-1}$ and $r_{n+2}$.

In [O] the existence of a unique local solution of (9)–(11) been established. In this paper, we extended this result for more general equation corresponding to (6).
than (3). The resulting system of equations is

\begin{align}
(15) \quad \frac{dL_0(t)}{dt} &= -\frac{1}{\sin \varphi_0} f_{-1}(0) + \left(\cot \varphi_0 + \cot \varphi_1\right) f_0 \left(\frac{1}{L_0(t)}\right) - \frac{1}{\sin \varphi_1} f_1 \left(\frac{1}{L_1(t)}\right) \\
(16) \quad \frac{dL_i(t)}{dt} &= -\frac{1}{\sin \varphi_i} f_{i-1} \left(\frac{1}{L_{i-1}(t)}\right) + \left(\cot \varphi_i + \cot \varphi_{i+1}\right) f_i \left(\frac{1}{L_i(t)}\right) - \frac{1}{\sin \varphi_{i+1}} f_{i+1} \left(\frac{1}{L_{i+1}(t)}\right), \quad (i = 1, \ldots, n) \\
(17) \quad \frac{dL_{n+1}(t)}{dt} &= -\frac{1}{\sin \varphi_{n+1}} f_n \left(\frac{1}{L_n(t)}\right) + \left(\cot \varphi_{n+1} + \cot \varphi_{n+2}\right) f_{n+1} \left(\frac{1}{L_{n+1}(t)}\right).
\end{align}

The initial condition is

\begin{align}
(18) \quad L_0(0) &= \ell_0 > 0, \quad L_{n+1}(0) = \ell_{n+1} > 0 \quad L_i(0) = 0 \quad \text{for} \quad i = 1, \ldots, n.
\end{align}

**Theorem 2.3.** Assume that \(f_i : \mathbb{R} \to \mathbb{R}\) is locally Lipschitz and non-decreasing for \(i = 0, \ldots, n + 1\). Assume that \(f_{-1}(0), f_{n+2}(0) \in \mathbb{R}\). Assume furthermore that

\[
\lim_{|x| \to \infty} f_i(x)/x = \lambda_i > 0.
\]

Then the system (15)–(17) with (18) admits a unique solution \(\{L_i(t)\}_{i=0}^{n+1}\) for some time interval \((0, t_0)\) which is \(C^1\) in \((0, t_0)\) and continuous on \([0, t_0]\).

**Remark 2.** This result is proved in [K] under stronger assumption on \(f_i\)’s, namely,

\[
\lim_{|x| \to \infty} |f_i(x) - \lambda_i x| < \infty, \quad f_{-1}(0) = f_{n+2}(0) = 0.
\]

However, the proof works in our setting.

3. **ESTIMATES FOR APPROXIMATE SOLUTIONS BY SELF-SIMILAR SOLUTIONS**

The goal of this section is to prove Theorem 2.3. We first construct approximate solutions. We consider the system (15)–(17) with initial condition

\begin{align}
(19) \quad L_0(0) &= \ell_0 > 0, \quad L_{n+1}(0) = \ell_{n+1} > 0, \quad L_i(0) = a_i s^{1/2} \quad (i = 1, \ldots, n)
\end{align}

with \(s > 0\). Here \(a_i\) is taken so that

\[
\hat{L}_i(t) = a_i t^{1/2}
\]

is the expanding self-similar solution of (10) by setting \(q_0 = r_{n+1} = 0\) and \(\Delta_i = \lambda_i (i = 1, \ldots, n)\) in (12) (13) (14). In other words, \(a_i\)’s is a unique positive solution of the \(n\) system of algebraic equations of the form

\begin{align}
(20) \quad \frac{a_i}{2} &= \frac{q_{i-1}}{a_{i-1}} + \frac{q_i}{a_i} - \frac{r_{i+1}}{a_{i+1}} \quad (i = 1, \ldots, n)
\end{align}

with \(q_0 = r_{n+1} = 0\). The unique existence of such \(a_i\)’s is guaranteed in [GGH]. The initial length \(\ell_0 \leq \ell_0\) and \(\ell_{n+1} \leq \ell_{n+1}\) is taken so that it is the length of intersection of a facet \(S_0\) (resp. \(S_{n+1}\)) and the 0-th (resp. \((n+1)\)-th facet) of the self-similar solution at \(s\). If \(s\) is taken sufficiently small, say \(s \leq s_0\), \(\ell_0^s\) and \(\ell_{n+1}^s\) can be taken positive.
**Proposition 1.** There exists \( s_0 > 0 \) and \( t_0 > 0 \) such that the system (15)\textendash{}(17) admits a unique \( C^1 \) solution \( L^s_i \) \((i = 0, 1, \ldots, n + 1)\) in \([0, t_0]\) and neighboring facets \( S_{-1}, S_{n+2} \) have positive length in \([0, t_0]\) for \( s \in (0, s_0]\).

**Proof.** Since \( f_i \) is locally Lipschitz and initial value is positive, there is a unique solution \( L^s_i \). Moreover, the neighboring facet \( S_{-1}, S_{n+2} \) with transition number zero have still positive length in \([0, t_0]\) by taking \( s \) small. Let \( t_s \) be the maximal time so that \( L^s_i \) exists and this non-vanishing property holds.

We shall prove that there is \( t_0 > 0 \) such that \( t_s \geq t_0 \) for \( s \leq s_0 \), where \( t_{s_0} > 0 \) are given in Proposition 1; see Figure 6. Here is a precise definition.

**Definition 3.1.** Let \( d^s_i(t) \) denote the (signed) distance from the corner vertex \((\text{origin})\) to \( i\)-th facet \( S^s_i(t) \) (in the direction of \( n_i \)) whose length equals \( L^s_i(t) \) for \( s \in (0, s_0]\) and \( t \in [0, t_0] \), where \( s_0, t_0 > 0 \) are given in Proposition 1; see Figure 6. Here \( i = 0, 1, \ldots, n + 1 \). In other words, \( d^s_i \) is the support function of \( S^s_i \) with respect to the corner vertex.

Let \( V^s_i \) denote the speed of \( S^s_i \). By definition

\[
d^s_i(t) = \int_0^t V^s_i(\sigma)d\sigma + d^s_i(0) \quad (i = 0, 1, \ldots, n + 1).
\]

Since \( \hat{L}_1 \) is the self-similar solution,

\[
d^s_i(0) = \int_0^s \frac{\lambda_i}{\hat{L}_1(\sigma)}d\sigma = \frac{2\lambda_i s^{1/2}}{a_i} \quad (i = 1, \ldots, n)
\]
which is the integral of speed of the facet corresponding to $\hat{L}_1$ over $\Omega$. Of course, $d_s^0(0) = d_{n+1}^n(0) = 0$. By (6) we have
\[
\begin{align*}
  d_i^s(t) &= \int_0^t f_i \left( \frac{1}{L_i^s(\tau)} \right) d\tau + \frac{\lambda_i s^{1/2}}{a_i}, \quad (i = 1, \ldots, n) \\
  d_0^0(t) &= \int_0^t f_0 \left( \frac{1}{L_0^0(\tau)} \right) d\tau, \\
  d_{n+1}^s(t) &= \int_0^t f_{n+1} \left( \frac{1}{L_{n+1}^s(\tau)} \right) d\tau.
\end{align*}
\]

By (strong) comparison principle for crystalline curvature (see e.g. [GGu]), we have
monotonicity of $d_i^s$ with respect to $s$.

**Proposition 2.** Assume that $0 < s_1 < s_2 \leq s_0$. Then
\[
d_i^{s_1}(t) < d_i^{s_2}(t) \quad \text{for} \quad t \in [0, t_0], \quad i = 0, 1, \ldots, n + 1.
\]
Here $t_0$, $s_0$ are as in Proposition 1.

**Definition 3.2.** Let $d_i$ for $i = 0, 1, \ldots, n + 1$ denote
\[
d_i(t) = \inf_{0 < s < s_0} d_i^s(t) = \lim_{s \to 0} d_i^s(t) \quad \text{for} \quad t \in [0, t_0].
\]
The last equality follows from the monotonicity (Proposition 2).

We would like to prove the continuity of this $d_i$'s up to $t = 0$. We shall estimate the modulus of continuity of $d_i^s$ uniformly in $s$ for $i = 1, \ldots, n$. For this purpose, we use a self-similar solution in a sector consisting of $S_A$ and $S_B$. While the original self-similar solution consists of $n$ moving facets $S_i$ ($i = 1, \ldots, n$), this new self-similar solution has $n + 2$ moving facets $\hat{S}_i^n$ ($i = 0, \ldots, n + 2$) with small angle $\psi_0$ (resp. $\psi_{n+2}$) between $S_0^n$ and $S_0^n$ (resp. $S_n^n$ and $S_{n+1}^n$); the angles $\varphi_1 \ldots \varphi_n$ are unchanged; see Figure 7, Figure 8. The set of admissible angles contains $\vartheta_1$ ($\vartheta_1 < \vartheta_2 < \theta_0$) with $\psi_0 = \theta_0 - \vartheta_1$ and $\vartheta_{n+2}$ ($\theta_{n+1} < \vartheta_{n+2} < \theta_{n+2}$) with $\psi_{n+2} = \vartheta_{n+2} - \theta_{n+1}$.

![Figure 7. New self-similar solution](image)

This new self-similar solution is of the form
\[
\hat{L}_i^n(t) = b_i t^{1/2} \quad (i = 0, 1, \ldots, n + 1)
\]
Figure 8. Wulff shape

with $\eta = (\psi_0, \psi_{n+2})$. Here $b_i$ solves

$$b_i = \frac{q_{i-1}}{b_{i-1}} + \frac{p_i}{b_i} - \frac{r_{i+1}}{b_{i+1}} \quad (i = 0, 1, \ldots, n+1)$$

with (12), (13), (14) with $\varphi_0, \varphi_{n+2}$ replaced by $\psi_0, \psi_{n+2}$ by setting $q_1 = 0, r_{n+2} = 0$ and $\Delta_i = \lambda_i$ ($i = 1, \ldots, n$); the quantity $\Delta_0$ and $\Delta_{n+1}$ are fixed independent of $\eta$.

Lemma 3.3. The quantity $b_i$'s approximates $a_i$'s, in the sense that $b_0, b_{n+1} \to \infty$ and $b_i \to a_i$ ($i = 1, \ldots, n$) of $\eta = (\psi_0, \psi_{n+2}) \to 0$.

Proof. By definition

$$p_0 = (\cot \psi_0 + \cot \varphi_1) \Delta_0$$
$$p_{n+1} = (\cot \varphi_{n+1} + \cot \psi_{n+2}) \Delta_{n+1}$$

so $p_0 \to \infty$ if $\psi_0 \to 0$ and $p_{n+1} \to \infty$ if $\psi_{n+2} \to 0$. By comparison of self-similar solutions,

$$b_1(\eta) \leq b_i(\eta) \leq q_i, \quad i = 1, \ldots, n$$

for $\eta \in (0, \eta_1) \times (0, \eta_2)$ with $\eta_* = (\eta_1, \eta_2)$. By (21) we see in particular that

$$b_0 = \frac{p_0}{b_0} - \frac{r_1}{b_1}$$
$$b_{n+1} = \frac{q_n}{b_n} + \frac{p_{n+1}}{b_{n+1}}.$$ 

Since $r_1/b_1$ and $q_n/b_n$ are bounded by (22), this implies $b_0, b_{n+1} \to \infty$ as $\eta \to 0$. By a bound of (22), any accumulation point of $b_i(\eta)$ as $\eta \to 0$ must satisfies (20). By uniqueness [GGH], it must agrees with $a_i$, i.e., $\lim_{\eta \to 0} b_i(\eta) = a_i$ for $i = 1, \ldots, n$. The proof is now complete. $\square$

We shall compare with these self-similar solutions. The original self-similar solution corresponds to solution of (3) with $\Delta(n) = \lambda_i$. Since we consider general problems, we need to use self-similar solutions to

$$V_j = \Lambda_j(1 - \varepsilon)$$

or

$$V_j = \Lambda_j(1 + \varepsilon)$$

(23) $V_j = \Lambda_j(1 - \varepsilon)$

(24) $V_j = \Lambda_j(1 + \varepsilon)$
for $\varepsilon > 0$ to estimate. Instead of considering $a_i$, we consider solutions of (20) with $\Delta_i = \lambda_i (1 - \varepsilon)$ and denote it by $a_i^{\varepsilon-}$; fortunately

$$a_i^{\varepsilon-} = \sqrt{1 - \varepsilon} a_i \quad (i = 1, \ldots, n).$$

This corresponds to the self-similar solution to (23). Similarly, let $b_i^{\varepsilon+}$ the solution of (21) by replacing $\Delta_i = \lambda_i (1 + \varepsilon)$, i.e.,

$$b_i^{\varepsilon+} = \sqrt{1 + \varepsilon} b_i \quad (i = 0, \ldots, n + 1).$$

We have an equi-continuity estimate for $d_s^i$.

**Lemma 3.4.** For any $\varepsilon \in (0, 1)$ and $\eta = (\psi_0, \psi_{n+2})$, there is $t_2$ such that

$$\int_{t_1}^{t} \frac{\lambda_i (1 - \varepsilon)}{a_i^{\varepsilon-} \tau^{1/2}} \leq d_s^i(t) - d_s^i(t_1) \leq \int_{t_1}^{t} \frac{\lambda_i (1 + \varepsilon)}{b_i^{\varepsilon+} \tau^{1/2}} d\tau \quad (i = 1, \ldots, n)$$

for all $t_1 \leq t \leq t_2$ and $s \in (0, s_0]$. In particular, $d_s^i(t)$ is increasing in $t \in [0, t_2]$.

**Proof.** We observe that

$$d_s^i(t) - d_s^i(t_1) = \int_{t_1}^{t} V_s^i(\tau) d\tau$$

where $V_s^i$ is the velocity of $i$-th facet $S_s^i$. By (7), for $\varepsilon > 0$, we take $\sigma_0$ small so that

$$\frac{\lambda_i}{\sigma} (1 - \varepsilon) \leq f_i \left( \frac{1}{\sigma} \right) \leq \frac{\lambda_i}{\sigma} (1 + \varepsilon), \quad i = 1, \ldots, n$$

for all $\sigma < \sigma_0$. By a simple comparison, there is small $t_2 < t_0$ such that

$$L_i^s(t) \leq \sigma \quad \text{for} \quad t \in (0, t_2]$$

uniformly in $s \in (0, s_0]$ for $i = 1, \ldots, n$.

We next compare with self-similar solutions (Figure 9). For a given facet $S_s^i(t_1)$,
The lower estimate for $d_i^s(t) - d_i^s(t_1)$ can be established a similar way by comparing with a self-similar solution $\{S_i\}$ from outside of $S_i$ with $a_i$ replaced by $a_i^{-}$. The proof is now complete.

Lemma 3.5. Under the same hypotheses of Lemma 3.4

$$\frac{\lambda_i\sqrt{1-\varepsilon}}{a_i} (t - t_1)^{1/2} \leq d_i^s(t) - d_i^s(t_1) \leq \frac{\lambda_i\sqrt{1+\varepsilon}}{b_i} (t - t_1)^{1/2}, \quad i = 1, \ldots, n,$$

for $0 \leq t \leq t_1 \leq t_2$, $s \in (0, s_0]$. In particular, the limit $d_i$ is $1/2$-Hölder continuous. More precisely,

$$\frac{\lambda_i\sqrt{1-\varepsilon}}{a_i} (t - t_1)^{1/2} \leq d_i(t) - d_i(t_1) \leq \frac{\lambda_i\sqrt{1+\varepsilon}}{b_i} (t - t_1)^{1/2}, \quad i = 1, \ldots, n$$

for $0 \leq t \leq t_1 \leq t_2$. The convergence $d_i^s \to d_i$ is uniform in $[0, t_2]$.

Proof. The estimate follows from (25). This equi-Hölder estimate yields $1/2$-Hölder continuity of the limit. The convergence is uniform by Dini’s theorem since the convergence is monotone (Proposition 2). (One may apply the Ascoli-Arzela theorem to conclude the uniform convergence without using monotonicity with respect to $s$).

Remark 3. For facets $d_0^s$ and $d_{n+1}^s$ since the length of facet $S_0^s$, $S_{n+1}^s$ is bounded away from zero uniformly in $s \in (0, s_0]$ we have a uniform $C^1$ estimates which is inherited to $d_0$ and $d_{n+1}$. We may conclude the estimate from above in Lemma 3.4 and 3.5 still holds for $i = 0$ and $i = n+1$ by taking $t_2$ smaller.

Proof of Theorem 2.3 (Existence). We shall construct a solution $\{L_i\}$ as a limit of $L_i^s$ as $s$ tends to zero.

We integrate (16) on $(0, t)$ to get

$$L_i^s(t) = -\int_0^t \frac{1}{\sin \varphi_i} f_i(t) \left( \frac{1}{L_i^{s-1}(\tau)} \right) d\tau + \int_0^t \left( \cot \varphi_i + \cot \varphi_{i+1} \right) f_i \left( \frac{1}{L_i^s(\tau)} \right) d\tau$$

$$- \int_0^t \frac{1}{\sin \varphi_{i+1}} f_{i+1} \left( \frac{1}{L_i^{s+1}(\tau)} \right) d\tau + a_is^{1/2},$$

where $i = 1, \ldots, n$. Since

$$d_i^s(t) = \int_0^t f_i \left( \frac{1}{L_i^s(t)} \right) d\tau + 2\lambda_i s^{1/2}/a_i,$$

the right-hand side equals

$$- \frac{1}{\sin \varphi_i} d_{i-1}^s(t) + \left( \cot \varphi_i + \cot \varphi_{i+1} \right) d_i^s(t) - \frac{1}{\sin \varphi_{i+1}} d_{i+1}^s(t) + a_is^{1/2}$$

$$+ \left( \frac{2q_i}{a_i} - \frac{2p_i}{a_i} + \frac{2r_{i+1}}{a_{i+1}} \right) s^{1/2}.$$

By (20) this equals

$$L_i^s(t) = -\frac{q_i}{\lambda_i} d_{i-1}^s(t) + \frac{p_i}{\lambda_i} d_i^s(t) - \frac{r_{i+1}}{\lambda_{i+1}} d_{i+1}^s(t).$$

Since $d_i^s$ converges to $d_i$ uniformly in $[0, t_2]$ by Lemma 3.5, the limit $L_i(t) = \lim_{s \to 0} L_i^s(t)$ exists for $i = 1, \ldots, n$ as a $1/2$-Hölder continuous function. It satisfies

$$L_i(t) = -\frac{q_i}{\lambda_i} d_{i-1}(t) + \frac{p_i}{\lambda_i} d_i(t) - \frac{r_{i+1}}{\lambda_{i+1}} d_{i+1}(t).$$
Similarly, integrating (15) over \( (0, t) \), we obtain
\[
L_0^s(t) = -\frac{1}{\sin \varphi_0} f_{-1}(t) + \frac{p_0}{\lambda_0} d_0^s(t) - \frac{r_1}{\lambda_1} d_1^s(t) + \frac{2r_1}{a_1} s^{1/2}(t) + \ell_0^s
\]
and
\[
L_0(t) = -\frac{1}{\sin \varphi_0} f_{-1}(t) + \frac{p_0}{\lambda_0} d_0(t) - \frac{r_1}{\lambda_1} d_1(t) + \ell_0
\]
(27)

since \( \ell_0^s \to \ell_0 \) as \( s \to 0 \). Similarly, from (16) we obtain
\[
L_{n+1}^s(t) = -\frac{q_n}{\lambda_n} d_n^s(t) + \frac{p_{n+1}}{\lambda_{n+1}} d_{n+1}^s(t) - \frac{1}{\sin \varphi_{n+2}} f_{n+2}(0)t + \frac{2q_n}{a_n} s^{1/2} + \ell_{n+1}^s
\]
(28)
\[
L_{n+1}(t) = -\frac{q_n}{\lambda_n} d_n(t) + \frac{p_{n+1}}{\lambda_{n+1}} d_{n+1}(t) - \frac{1}{\sin \varphi_{n+2}} f_{n+2}(0)t + \ell_{n+1}.
\]

At this moment it is not clear that \( L_i \) (\( i = 1, \ldots, n \)) is positive for \( t \in (0, t_2) \). We shall prove that \( L_i(t) > 0 \) for \( t \in (0, t_2) \). By Lemma 3.5 and Remark 3, we estimate \( d_i \)'s in (26) from above and below by taking \( t_1 = 0 \) to get
\[
L_i(t) \geq \left( -\frac{2q_i}{b_i-1} \sqrt{1+\varepsilon} + \frac{2p_i}{a_i} \sqrt{1-\varepsilon} - \frac{2r_{i+1}}{b_{i+1}} \sqrt{1+\varepsilon} \right) t^{1/2} \quad (i = 1, \ldots, n)
\]
for \( t \in (0, t_2) \). Since \( a_i \) fulfills (20) and \( b_i \to a_i \) as \( \eta \to 0 \), by taking \( \varepsilon \) and \( \eta \) small we obtain
\[
-\frac{2q_i}{b_i-1} \sqrt{1+\varepsilon} + \frac{2p_i}{a_i} \sqrt{1-\varepsilon} - \frac{2r_{i+1}}{b_{i+1}} \sqrt{1+\varepsilon} \geq \frac{1}{2} \left( -\frac{2q_i}{b_i-1} + \frac{2p_i}{a_i} - \frac{2r_{i+1}}{b_{i+1}} \right) = \frac{a_i}{2}.
\]
We thus conclude that
(29)
\[
L_i(t) \geq \frac{1}{2} a_i t^{1/2} \quad \text{for} \quad t \in [0, t_2].
\]

Note that \( t_2 \) may tend to zero as \( \varepsilon \to 0, \eta \to 0 \) but we fixed these parameters as above so that \( t_2 > 0 \). The positivity of \( L_0(t) \) and \( L_{n+1}(t) \) in a short time interval \( (0, t_*) \) with \( t < t_* \) is rather clear by formulas (27), (28) for \( L_0 \) and \( L_{n+1} \) by taking \( t_* \) smaller.

It remains to prove that \( L_i \)'s satisfies the equation (15), (16), (17). We recall
\[
d_i^s(t) = \int_{\delta}^{t} f_i \left( \frac{1}{L_i^s(\tau)} \right) d\tau + d_i^s(\delta), \quad \delta \in (0, t_*).
\]

Sending \( s \) to zero yields
\[
d_i(t) = \int_{\delta}^{t} f_i \left( \frac{1}{L_i(\tau)} \right) d\tau + d_i(\delta)
\]
for \( i = 0, 1, \ldots, n + 1 \). By (29) the function \( L_i(\tau) \) is integrable in \( (0, t_*) \). Since \( f(1/L) \leq \lambda_1(1+\varepsilon)/L \) for small \( L \), by the dominated convergence theorem, we conclude that
\[
d_i(t) = \int_{0}^{t} f_i \left( \frac{1}{L_i(\tau)} \right) d\tau.
\]
By (26) we obtain for \( i = 1, \ldots, n \)
\[
L_i(t) = - \frac{q_{i-1}}{\lambda_{i-1}} \int_0^t f_{i-1} \left( \frac{1}{L_{i-1}(\tau)} \right) d\tau + \frac{p_i}{\lambda_i} \int_0^t f_i \left( \frac{1}{L_i(\tau)} \right) d\tau
- \frac{r_{i+1}}{\lambda_{i+1}} \int_0^t f_{i+1} \left( \frac{1}{L_{i+1}(\tau)} \right) d\tau
- \frac{1}{\sin \varphi_i} \int_0^t f_{i-1} \left( \frac{1}{L_{i-1}(\tau)} \right) d\tau + (\cot \varphi_i + \cot \varphi_{i+1}) \int_0^t f_i \left( \frac{1}{L_i(\tau)} \right) d\tau
- \frac{1}{\sin \varphi_{i+1}} \int_0^t f_{i+1} \left( \frac{1}{L_{i+1}(\tau)} \right) d\tau.
\]
This is an integral form of (16). The formula (27) and (28) give the integral form of (15) and (17), respectively. Thus our \( l_i \)'s are \( C^1 \) in \((0, t_\ast)\) for \( i = 0, \ldots, n + 1 \) satisfying (15), (16), (17) in \((0, t_\ast)\). Moreover, it is 1/2-H"older up to \( t = 0 \) and the initial condition (15) is guaranteed. Note that \( L_0 \) and \( L_1 \) is \( C^1 \) up to \( t = 0 \).

(Unique). We appeal a geometric argument involving \( d_i \). Suppose that there exists another solution \( L_i(t) \in C[0, t_\ast) \cap C^1(0, t_\ast) \). It suffices to prove that \( d_i(t) = \overline{d}_i(t) \), where \( \overline{d}_i(t) \) is the distance function corresponding to \( L_i(t) \).

By comparison principle, it is clear that
\[
\overline{d}_i(t) \leq d^*_i(t) \quad (i = 0, \ldots, n + 1).
\]
Sending \( s \) to zero to get
\[
\overline{d}_i(t) \leq d_i(t) \quad (i = 0, \ldots, n + 1).
\]
Since
\[
\overline{d}_i(t) = \int_0^t \overline{V}_i(\tau) d\tau = \int_0^t f_i \left( \frac{1}{L_i(\tau)} \right) d\tau, \quad (i = 0, \ldots, n + 1)
\]
the bound \( \overline{d}_i \leq d_i \) implies that \( d_i \) is finite so that the right-hand side is integrable. This implies \( d_i \) is a continuous function in \([0, t_\ast)\). By comparison principle, for small \( \varepsilon > 0 \)
\[
d_i(t) \leq \overline{d}_i(\varepsilon + t) \quad (i = 0, \ldots, n + 1), \quad t \in [0, t_\ast - \varepsilon).
\]
Sending \( \varepsilon \) to zero, by the continuity we obtain
\[
d_i(t) \leq \overline{d}_i(t) \quad (i = 0, \ldots, n + 1), \quad t \in [0, t_\ast]
\]
which now implies \( d_i \equiv \overline{d}_i \). The proof is now complete. \( \square \)

**Remark 4.** For the proof of Theorem 2.3, it is possible to prove that
\[
\lim_{t \downarrow 0} \frac{L_i(t)}{t^{1/2}} = a_i \quad (i = 1, \ldots, n)
\]
by taking \( \varepsilon, \eta \) smaller. This says that the shape is asymptotically self-similar as \( t \to 0 \).

4. Problems starting from general polygons

We shall prove Theorem 2.1 and Theorem 2.2. We consider a polygon \( \Gamma_0 \) whose facets (edges) \( \{E_\alpha \}_{\alpha = 1}^K \) has an orientation \( n_\alpha \in \mathcal{N} \). Let \( \{P_\beta\} \) be the set of all vertices connecting two facets \( S_{\beta-} \) and \( S_{\beta+} \) such that there is at least one direction \( n \in \mathcal{N} \) between \( n_{\beta-} \) and \( n_{\beta+} \). We may assume that the orientation of \( \Gamma_0 \) is taken inward since other case can be handled similarly. We number \( \{P_\beta\}_{\beta = 1}^h \) so that it moves counterclockwise on \( \Gamma_0 \).
Proof of Theorem 2.1. We first consider the case when there is no facet $E_\alpha$ whose transition number is equal to zero. In other words, $\Gamma_0$ is convex.

Let $S^\beta_i(t)$ denote a newly created facet from the vertex $P_\beta$ for $i = 1, \ldots, n_\beta$. Let $L^\beta_i$ denote its length. Then we derive the equation for $L^\beta_i$ of the form

$$
\frac{dL^\beta_i}{dt} = -\frac{1}{\sin \varphi^\beta_i} f^\beta_i \left( \frac{1}{L^\beta_i(t)} \right) + \left( \cot \varphi^\beta_i + \cot \varphi^\beta_{i+1} \right) f^\beta_i \left( \frac{1}{L^\beta_i(t)} \right)
$$

(30)

where the speed $V_i^\beta$ of $S^\beta_i$ is given by $f^\beta_i \left( 1/L^\beta_i(t) \right)$; the function $L^\beta_0(t)$ denotes the length of evolving facet $S^\beta_-(t)$ starting from $S^\beta_-$ while $L^\beta_n(t)$ denotes the length of evolving facet $S^\beta_+(t)$ starting from $S^\beta_+$. This is just reindexing which also applies to other quantities $\varphi^\beta_i, a^\beta_i, \lambda^\beta_i$ etc. for $\beta = 1, \ldots, h$. We also have an evolution equation for the length $\ell^\beta_\alpha$ of facet $E^\beta_\alpha(t)$ starting from $E^\alpha$. We have a system of ODEs including (30) for $\{\ell^\alpha_i\}_{i=1}^K$ and

$$
\{L^\beta_i \mid i = 1, \ldots, n_\beta, \beta = 1, \ldots, h\}
$$

with initial data $\ell^\beta_\alpha(0) = \ell^\alpha_{0\alpha}, L^\beta_\alpha(0) = 0$. Here $\ell^\alpha_{0\alpha}$ is the length of $E^\alpha$. The total system denotes (S).

To construct a solution we consider an approximate solution by setting $\ell^\alpha_{0\alpha}$, $\ell^\beta_\alpha(0) = a^\beta_i s^{1/2}$ by using a self-similar expanding solution starting from $P_\beta$ in a sector consisting of $S_\beta^-$ and $S_\beta^+$. This is the same as in the previous section.

We use a distance function $d^\beta_i$ from $P_\beta$ for the line containing $S^\beta_i$ and derive a uniform estimate; for $\varepsilon \in (0, 1)$ and $\eta$, there is $t_2 > 0$ such that

$$
\frac{\lambda^\beta_i}{a^\beta_i} \sqrt{1-\varepsilon} (t-t_1)^{1/2} \leq d^\beta_i(t) - d^\beta_i(t_1) \leq \frac{\lambda^\beta_i}{b^\beta_i} \sqrt{1+\varepsilon} (t-t_1)^{1/2}, \quad i = 1, 2, \ldots, n_\beta
$$

for $0 \leq t_1 \leq t \leq t_2$ as in Lemma 3.5. As in Remark 3, the estimate from above also holds for $i = 0$ and $i = n_\beta + 1$, the distance corresponding to $S_\beta^-$ and $S_\beta^+$, respectively. This estimate guarantees the convergence of solution $\left(\ell^\alpha_{0\alpha}, L^\beta_\alpha\right)$ of (S) with initial data $\ell^\alpha_{0\alpha} = \ell^\alpha_{0\alpha}, L^\beta_\alpha(0) = a^\beta_i s^{1/2}$ to a solution of (S) with $\ell^\alpha_{0\alpha} = \ell^\alpha_{0\alpha}, \ell^\beta_\alpha(0) = 0$ for $i = 1, \ldots, n_\beta, \beta = 1, \ldots, h$ as in the proof of Theorem 2.3. Once the solution of (S) is given, then it is easy to see that it satisfies (6) since

$$
\frac{dd^\beta_i}{dt} = V_i^\beta.
$$

We have thus constructed a crystalline flow. Since $d^\beta_i$ is continuous in time up to $t = 0$, the resulting solution $\{\Gamma_t\}$ converges to $\Gamma_0$ as $t \downarrow 0$ in the sense of Hausdorff distance.

The other case is that there is a facet $E^\alpha$, whose transition number equals zero in $\Gamma_0$. In this case we are able to split the problem. After renumbering there is a connected sequence of facets and vertices

$$
E_0, E_1, \ldots, E_k, P_1, \ldots, P_{h_0}, E_{k+1}, \ldots, E_{k+r}
$$
such that transition number of \( E_0 \) and \( E_{k+r} \) equals zero while other facet including those between \( P_\delta \)'s has the positive transition number. The facet \( E_0 \) and \( E_{k+r} \) can be the same facet. We are able to argue the same way to construct a desired crystalline flow in this case too.

The uniqueness follows the same argument as before. The proof is now complete. □

**Proof of Theorem 2.2.** By Theorem 2.1 it remains to prove that the crystalline flow \( \{ \Gamma_t \} \) for (2) is the only level-set flow of (2).

As proved in [GG3], it is known that a crystalline flow is a level-set flow provided that the corner preserving condition is fulfilled. So the only thing one has to check is that there is no fattening phenomenon in our setting. However, this can be easily seen by approximating the polygon from inside and outside and by applying stability of solution. Both solution converges to the same crystalline flow so there is no fattening. We omit the details. □

5. **Numerical test**

We gave here an example of our numerical calculation of (9)–(11). Of course, this initial value problem is singular so we replace the initial condition by (19).

We take a regular dodecagon with side length 1 as the Wulff shape so that \( \Delta_i = 1 \) and \( \varphi_i = \pi/6 \). We set \( n = 3 \) so that three facets are created. The system (20) of algebraic equations is reduced to

\[
\begin{align*}
\frac{a_1}{2} &= \frac{2\sqrt{3}}{a_1} - \frac{2}{a_2}, \\
\frac{a_2}{2} &= -\frac{2}{a_1} + \frac{2\sqrt{3}}{a_2} - \frac{2}{a_3}, \\
\frac{a_3}{2} &= -\frac{2}{a_2} + \frac{2\sqrt{3}}{a_3}.
\end{align*}
\]

This system is explicitly solvable and the unique positive solution is the form

\[
\begin{align*}
a_1 &= \sqrt{\frac{2}{\sqrt{3}(5 - \sqrt{13})}} = 1.2689 \ldots, \\
a_2 &= \left(\frac{\sqrt{3}}{a_1} - \frac{a_1}{4}\right)^{-1} = 0.9544 \ldots, \\
a_3 &= a_1.
\end{align*}
\]

In this particular example, we take

\[
\begin{align*}
\ell_0' &= \ell_0 - \left(a_2 + \sqrt{3}a_1\right)s^{1/2}, \\
\ell_4' &= \ell_4 - \left(\sqrt{3}a_1 + a_2\right)s^{1/2}.
\end{align*}
\]

In the first example, we set \( \ell_0 = \ell_4 = 5, s = 10^{-4} \). We take \( \varphi_0 = \varphi_4 = \pi/2, \varphi_1 = \varphi_2 = \varphi_3 = \pi/6 \). This means our Wulff shape is modified outside \( \{\theta_0, \theta_1, \theta_2, \theta_3\} \subset N \). We draw evolution of facets starting from the corner with angle \( \pi/3 \). The motion is monotone and each line is drawn with time step \( \Delta t = 0.00902 \). The result of calculation is given in Figure 10. We see that newly created facets move quickly while the facets forming corners move slowly. The second example is given in Figure
11, where $\varphi$'s and $\ell_0$, $\ell_4$ are the same but $s = 10^{-6}$, $\Delta t = 5 \times 10^{-4}$ and the picture is magnified.

Figure 10. An example of the numerical calculation

6. UNUSUAL BRIOT-BOUQUET SYSTEM

We consider (9)–(11) and change independent variable $t$ by $\tau = t^{1/2}$ and depending variable $g_i(\tau) = L_i(\tau^2)/\tau$, $i = 1, \ldots, n$. We postulate $g_i(0) = a_i$. For $i = 2, 3, \ldots, n-1$, the equation for $L_i$ becomes

$$
\frac{1}{2\tau} \frac{d}{d\tau} (\tau g_i(\tau)) = -\frac{q_{i-1}}{\tau g_{i-1}(\tau)} + \frac{p_i}{\tau g_i(\tau)} - \frac{r_{i+1}}{\tau g_{i+1}(\tau)}
$$
which yields
\[ \tau \frac{dg_i(\tau)}{d\tau} = -g_i(\tau) - \frac{2q_{i-1}}{g_{i-1}(\tau)} + \frac{2p_i}{g_i(\tau)} - \frac{2r_{i+1}}{g_{i+1}(\tau)}. \]

If one sets \( \mathcal{L}_0(\tau) = L_0(\tau^2), \mathcal{L}_{n+1}(\tau) = L_{n+1}(\tau^2) \), we now observe that the system (9)–(11) will be transformed into
\[
\begin{align*}
\frac{d\mathcal{L}_0(\tau)}{d\tau} &= \frac{2p_0\tau}{\mathcal{L}_0(\tau)} - \frac{2r_1}{g_1(\tau)}, \quad \mathcal{L}_0(0) = \ell_0 \\
\tau \frac{dg_1(\tau)}{d\tau} &= -g_1(\tau) - \frac{2q_1}{\mathcal{L}_0(\tau)} + \frac{2p_0}{g_1(\tau)} - \frac{2r_1}{g_2(\tau)}, \quad g_1(0) = a_1 \\
\tau \frac{dg_i(\tau)}{d\tau} &= -g_i(\tau) - \frac{2q_{i-1}}{g_{i-1}(\tau)} + \frac{2p_i}{g_i(\tau)} - \frac{2r_i}{g_{i+1}(\tau)}, \quad g_i(0) = a_i \ (i = 2, \ldots, n - 1) \\
\tau \frac{dg_n(\tau)}{d\tau} &= -g_n(\tau) - \frac{2q_{n-1}}{g_{n-1}(\tau)} + \frac{2p_n}{g_n(\tau)} - \frac{2r_{n+1}}{\mathcal{L}_{n+1}(\tau)}, \quad g_n(0) = a_n \\
\frac{d\mathcal{L}_{n+1}(\tau)}{d\tau} &= -\frac{2q_n}{g_n(\tau)} + \frac{2p_{n+1}}{\mathcal{L}_{n+1}(\tau)}, \quad \mathcal{L}_{n+1}(0) = \ell_{n+1}.
\end{align*}
\]

To see the structure of this system, we assume that \( \mathcal{L}_0 \) and \( \mathcal{L}_{n+1} \) are already given. Define functions \( F_i : (\mathbb{R}_+) \to \mathbb{R} \ (i = 1, \ldots, n) \), \( \mathbb{R}_+ = (0, \infty) \) as follows
\[
\begin{align*}
F_1(\tau, x_1, \ldots, x_n) &= -\frac{2q_0\tau}{\mathcal{L}_0(\tau)} + \frac{2p_1}{x_1} - \frac{2r_2}{x_2}, \\
F_i(\tau, x_1, \ldots, x_n) &= -\frac{2q_{i-1}}{x_{i-1}} + \frac{2p_i}{x_i} - \frac{2r_i}{x_{i+1}}, \quad (i = 2, 3, \ldots, n - 1) \\
F_n(\tau, x_1, \ldots, x_n) &= -\frac{2q_{n-1}}{x_{n-1}} + \frac{2p_n}{x_n} - \frac{2r_{n+1}}{\mathcal{L}_{n+1}(\tau)}.
\end{align*}
\]

We set \( F = \langle F_1, \ldots, F_n \rangle, \ x = \langle x_1, \ldots, x_n \rangle, \ g = \langle g_1, \ldots, g_n \rangle, \ a = \langle a_1, \ldots, a_n \rangle \). The ODE system for \( g \) is of the form
\[ \tau \frac{dg(\tau)}{d\tau} = -g(\tau) + F(\tau, g(\tau)), \quad g(0) = a \]
and its integral form is
\[ g(\tau) = \frac{1}{\tau} \int_0^{\tau} F(\sigma, g(\sigma)) \, d\sigma, \quad g(0) = a. \]

If \( F \) would satisfy
\[ F(0, a) = 0 \quad \text{and the Jacobi matrix} \quad D_x F(0, a) = 0, \]
then one would apply [GG98, Lemma 8] to conclude the desired existence result. In this case, however, the Jacobi matrix \( D_x F(0, a) \) is not zero so [GG98, Lemma 8] cannot be applied. The existence of the solution for the Briot-Bouquet system is studied by calculating power series for \( \tau \). Although the existence of formal power series is easy, its convergence is not easy to check. A typical criterion is given by eigenvalues of Jacobi matrix \( D_x F \) [GeT]. However, in our case these eigenvalues are difficult to compute. In our case, although there is a formal power series solution but it is not clear whether or not this series actually converges.

If \( n = 1 \), as shown in [O], it is possible to construct a solution by applying a contraction mapping principle in \( C[0, T] \) for
\[ g \mapsto \frac{1}{\tau} \int_0^{\tau} F(\sigma, g(\sigma)) \, d\sigma. \]
However, for $n > 1$ it seems that the mapping is not contraction as indicated in [O].

ACKNOWLEDGMENTS

The work of the third author was done when he was a bachelor student at the Department of Integrated Science, the University of Tokyo. The work of the last author was done when he was a graduate student at the School of Mathematical Sciences, the University of Tokyo.

REFERENCES


T. Ishiwata, *Crystalline undou ni tsuite: heimenjou no takakukei no undou no kaiseki* (Japanese) [On crystalline motion: analysis on motion of a polygon in the plane], Lecture Series in Mathematics GP-TML06, Graduate School of Science, Tohoku University, 2008.


Email address: mihogiga@ms.u-tokyo.ac.jp
Email address: labgiga@ms.u-tokyo.ac.jp
Email address: njtr52627siro.jtmd522@gmail.com
Email address: taro50514@gmail.com