Absolutely continuous σ-finite invariant measures for Markov operators and dissipative behavior of random dynamical systems

(マルコフ作用素に対する絶対連続σ-有限不変測度とランダム力学系の散逸挙動について)

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1 Introduction

1.1 Background

Classical ergodic theory deals with statistical properties of dynamical systems given by a point transformation T on a phase space X. The iterates of T are defined by $T^0 := \mathrm{id}_X$ and $T^n := T \circ T^{n-1}$ and the trajectory of an initial point $x \in X$ is $\{T^n x\}_n$. The asymptotic behavior of trajectories of typical chaotic transformations is hard to predict. Hence it is natural to consider the time evolution of an initial distribution, introducing measurable structure in X, instead of looking only at single initial point. Then an *invariant probability measure* or a *stationary measure* (a measure μ such that $\mu \circ T^{-1} = \mu$) for a given system is very important from the measure-theoretic viewpoint. If one finds an *ergodic* invariant probability measure μ for T ($T^{-1}E = E$ implies $\mu(E) = 0$ or $\mu(X \setminus E) = 0$), then the Birkhoff ergodic theorem is one of the most powerful tools to predict the system in probabilistic manner: for any $L^1(X, \mu)$ function f, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^n(x) = \int_X f d\mu$$

for μ-almost every $x \in X$. This theorem is a version of the law of large number for deterministic dynamical systems. In order to relate observation through the reference measure (e.g. Lebesgue measure) and statistical properties given by an invariant probability measure such as the above Birkhoff ergodic theorem, an invariant probability measure may be required to be absolutely continuous with respect to the reference measure. Therefore, necessary and sufficient conditions for the existence of an absolutely continuous invariant probability measure for a given transformation have been asked for a long time.

The conventional tool to construct an absolutely continuous invariant probability measure is the *Perron–Frobenius operator* introduced as follows. Non-singularity of a given transformation T with respect to m (absolute continuity of $m \circ T^{-1}$ with respect to m) naturally raises the well-defined bounded linear operator on $L^1(X, m)$, say P, called the Perron–Frobenius operator or transfer operator by

$$\int_X Pf \cdot g dm = \int_X f \cdot g \circ T dm$$

for $f \in L^1(X, m)$ and $g \in L^\infty(X, m)$. Then Perron–Frobenius operators belong to the class of *Markov operators* and the sequence $\{P^n f\}_n$ represents the time evolution of an initial distribution $f \in L^1(X, m)$. If the weak limit of $\{P^n f\}_n$ exists for some $f \in L^1_+(X, m)$, then the limit point is the very density of an absolutely continuous invariant probability measure. Moreover, lots of statistical properties of dynamical systems can be analyzed through the asymptotic behavior of their Perron–Frobenius operators.

Let us now turn our attention to random dynamical systems. Random dynamical systems are Markov processes arising from, for example, a single transformation + "noises" or a randomly selected transformation from some family of transformations. Then under some weak conditions random dynamical systems possess well-defined Markov operators. For random dynamical systems having the corresponding Markov operators, we can naturally define their absolutely continuous invariant probability measures as non-negative fixed points of the operators. Thanks to considering Markov operators, we can deal with not only deterministic dynamical systems but also random dynamical systems.

From a historical point of view, only for invertible transformations, Hajian and Kakutani [26] in 1964 established the necessary and sufficient conditions for the existence of an *equivalent* invariant probability measure as follows, by using Banach limits. The following conditions for an invertible and non-singular transformation T on a probability space $(X, \mathscr{B}, m)$ are mutually equivalent to each other:

(1) There exists an equivalent invariant probability measure for T;

(2) $m(A) > 0$ implies $\liminf_{n\to\infty} m(T^{-n}A) > 0$;

(3) $m(A) > 0$ implies $\liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} m(T^{-i}A) > 0$;

(4) $m(A) > 0$ implies $\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} m(T^{-i}A) > 0$;

(5) There is no set W of positive measure such that $\{T^{-n_i}W\}_i$ are mutually disjoint for some $\{n_i\}_i \subset \mathbb{N}$.

Then Sucheston [66] in 1964 gave a proof for the non-invertible case. Furthermore, Ito [33] in 1964 and Dean and Sucheston [15] in 1966 generalized the above equivalent conditions to the Markov operators case. A set W in the condition (5) is called a weakly wandering set and this concept is unexpected and surprising. This is because any transformations preserving not probability but σ-finite infinite measures always admit weakly wandering sets, even though almost all initial points will return to near neighbors of them infinitely many times by the processes of systems. The appearance of weakly wandering sets make researchers of ergodic theory turn their attention to infinite measure preserving systems.

By virtue of the celebrated results by Thaler [67, 68] in 1980s, lots of simple transformations admitting indifferent fixed points on the unit interval were shown to naturally have σ-finite infinite invariant measures which are absolutely continuous with respect to Lebesgue measure. Furthermore, certain sufficient conditions for the existence of an absolutely continuous σ-finite (infinite) invariant measure for a given transformation and a Markov operator were shown in [23, 45, 46, 58, 59, 77, 78, 79]. The main idea of construction of σ-finite invariant measures is that one observes systems by changing coordinate of time, the method called *induced transformations*, *jump transformations* or *Young towers*. For an ergodic σ-finite infinite measure preserving transformation, it is known that the Birkhoff ergodic theorem no longer holds. Then several statistical properties and limit theorems instead of the Birkhoff ergodic theorem were studied, for example see [1, 2, 3, 6, 69, 70] and references therein.

We also mention previous researches of application to concrete random dynamical systems that one chooses a transformation from some family of transformations (in an independent and identically distributed way) and the selected transformation applies to the system. The existence of absolutely continuous invariant probability measures and several statistical properties for random dynamical systems of which transformations are all uniformly expanding or uniformly expanding on average were studied for example in [7, 17, 25, 30, 48, 53]. In those researches, the scheme of the Lasota–Yorke inequality or quasi-compactness is relied on to analyze their statistical properties. For random dynamical systems which admits a common indifferent fixed point (such systems are sometime called weakly or non-uniformly expanding random maps), the existence of absolutely continuous finite or σ-finite infinite invariant measures and their limit theorems were studied in [8, 9, 10, 11, 31, 51, 52]. In those recent progress of non-uniformly expanding random dynamical systems, they frequently use spectral gap or quasi-compactness of the induced system (or Young tower) instead of the original system.

1.2 Main results

As we mentioned in the previous subsection, there are a lot of results on the existence of an absolutely continuous finite or σ-finite infinite invariant measure for a given transformation or Markov operator. However, most of these results concern only with sufficient conditions or only with equivalent measure (i.e., the reference measure and an invariant measure are required to be mutually absolutely continuous). Moreover, the relation between the induced transformations and the jump transformations were not well-understood. Therefore, the purpose of this thesis is the following:

(A) We give equivalent conditions for the existence of an absolutely continuous finite invariant measure for a Markov operator;

(B) We give equivalent conditions for the existence of an absolutely continuous σ-finite invariant measure for a Markov operator through generalizing the induced transformation and the jump transformation and clarifying the relations of them;

(C) We construct new examples of random dynamical systems which are essentially different from those dealt with in previous researches, and show the existence of absolutely continuous σ-finite invariant measures for our examples.

Our investigation into (A), (B) and (C) are mainly based on our paper [71, 72]

As written in (A) and (B) above, we will consider not only sufficient but also necessary conditions for the existence of an absolutely continuous finite or σ-finite invariant measure for a given Markov operator. For this reason, we introduce a new requirement for an invariant measure called the *maximal support condition*, that is, almost all initial points will concentrate on the support of the invariant measure sooner or later. In other words, the support of an invariant measure is required to asymptotically spread the whole state space. We attribute it to the maximal support condition that we can show the equivalence between the existence of an absolutely continuous finite invariant measure and weak almost periodicity of a given Markov operator (see Definition 3.1 and Theorem 3.5).

In order to construct an absolutely continuous σ-finite invariant measure for a given Markov operator related to (B), we generalize and develop the method of the induced transformation and the jump transformation to the case of Markov operators, called the *induced operator* and the *the jump operator* (Definition 4.2 and Definition 4.4). Here the induced operator was already introduced in [22, 23] but the jump operator was introduced by the author [71]. Because of introducing the jump operator, we can show the equivalence of the existence of an absolutely continuous σ-finite invariant measure with the maximal support and weak almost periodicity of the jump operator. Through the existence of an absolutely continuous σ-finite invariant measure with the maximal support, we can relate the induced operator and the jump operator. See Theorem 4.10 for precise statements.

Through the investigation into (A) and (B), we can derive the definition of *eventual conservativeness* of a system (Definition 4.9), weaker property of conservativeness, from considering the maximal support condition of an invariant measure. Furthermore, our results related to (A) and (B) are interpreted as the equivalent conditions for the existence of absolutely continuous finite or σ-finite invariant measures for eventually conservative Markov operators.

For our purpose (C), we will consider random dynamical systems which have not only a common indifferent fixed point but also uniformly contractive branches, called *random iterations of intermittent Markov maps with uniformly contractive part*, defined in [72]. Our model is, as long as we know, not dealt with in the previous researches mentioned in the last paragraph in subsection 1.1. Applying our result associated with (A) and (B) to the model, we can show that our random dynamical systems always admit absolutely continuous σ-finite invariant measures (Theorem 5.1). Moreover, we can calculate the exact formula of invariant densities (Corollary 5.2), so that we succeed in pointing the critical value where the invariant measure varies finite to infinite. The critical value of our systems are different from that of deterministic systems as in [67]. See Example 9 and Example 10 in subsection 5.2.

Our results presented in this thesis do neither assume systems to be conservative nor weakly expanding. Therefore, the contents of the thesis would be interpreted as a first step toward understanding dissipative behavior of random dynamical systems.

1.3 Organization

The organization of this thesis is as follows. In section 2, we prepare necessary definitions and notations which are frequently used through this thesis.

In section 3, we will give our first main result related to (A), equivalent conditions for the existence of absolutely continuous finite invariant measures with the maximal supports for Markov operators relying on weak almost periodicity of the operators (Theorem 3.5). We also consider asymptotically periodic Markov operators and relate them to weakly almost periodic operators (Proposition 3.22). Furthermore, we relate asymptotic periodicity and exactness (Proposition 3.21) as well as showing the Lasota–Mackey type equivalence (Proposition 3.23).

In section 4, we will give our second main result related to (B), equivalent conditions for the existence of absolutely continuous σ-finite invariant measures with the maximal supports for Markov operators (Theorem 4.10). In order to show the theorem, we recall the induced operators (the generalization of the induced transformations) and define the jump operators (the generalization of the jump transformations). We also investigate some ergodic property of random iterations of weakly expanding transformations (Corollary 4.17).

In section 5, motivated by the purpose (C), we introduce random iterations of piecewise linear intermittent Markov maps with uniformly contractive parts and show the existence of absolutely continuous σ-finite invariant measures for them (Theorem 5.1 and Corollary 5.2). We also give several examples of them which exhibit different behavior from deterministic transformations or random iterations of weakly expanding transformations.

2 Preliminaries

In this section, we prepare precise definitions and concepts in ergodic theory in order to discuss statistical properties of dynamical systems or random dynamical systems.

Throughout this thesis, $(X, \mathscr{B}, m)$ denotes a probability space as a phase space and $L^1(X, m)$ and $L^\infty(X, m)$ denote (the quotient by equality m-almost everywhere of) the set of all integrable functions and the set of all essentially bounded functions, respectively. A characteristic function of $E \in \mathscr{B}$ is denoted by 1_E. We remark that if m is σ-finite infinite measure then we can always construct a probability measure m' which is equivalent to m by setting

$$m'(E) := \sum_{i \geq 1} \frac{m(E \cap X_i)}{2^i m(X_i)}$$

for $E \in B$ where $\{X_i\}_i$ is a partition of X into finite m-measure sets: $X = \bigcup_i X_i$ with $m(X_i \cap X_j) = 0$ $(i \neq j)$ and $m(X_i) < \infty$. That is, without loss of generality, we can always assume a reference measure m is a probability measure. We frequently ignore a set of measure zero and we consider the quotient spaces $L^1(X, m)$ and $L^\infty(X, m)$ so that difference of functions on sets of measure zero is negligible. If two measurable sets A and B are equivalent up to an m-measure zero set, then we will write $A = B \pmod{m}$. For a non-negative measurable function f, we define the support of f by $\operatorname{supp} f = \{x \in X : f(x) > 0\}$. For an m-absolutely continuous measure ν (i.e., for each $N \in \mathscr{B}$, $m(N) = 0$ implies $\nu(N) = 0$), we set $\operatorname{supp} \nu = \operatorname{supp} d\nu/dm$ where $d\nu/dm$ is the Radon–Nikodým derivative of ν with respect to m. Since we do not introduce any topology in X, the support of function is not necessarily the closure of the set. Further, as convention, $\mathscr{B}_+$ denotes the collection of all measurable sets of positive m-measure, $L^1_+(X, m)$ denotes the set of non-negative functions in $L^1(X, m)$ and $D(X, m)$ denotes the set of all density functions, namely, functions $f \in L^1_+(X, m)$ such that $\|f\|_{L^1(m)} = 1$.

2.1 Non-singular transformations, Markov operators, and invariant measures

In this subsection, we recall the definition of non-singularity of transformations with respect to the reference measure m and then define the Perron–Frobenius operators via non-singularity. This definition naturally leads to the definition of Markov operators over the probability space. Finally, for non-singular transformations and Markov operators, we define finite and σ-finite infinite, respectively, invariant measures which are required to be absolutely continuous with respect to the reference measure m.

Let $(X, \mathscr{B}, m)$ be a probability space. Then a transformation $T : X \to X$ is called `measurable` if $T^{-1}\mathscr{B} \subset \mathscr{B}$ and is called `non-singular` if T is measurable and $m(T^{-1}N) = 0$ holds for any measurable set N with $m(N) = 0$, that is, the pushforward of m, $m \circ T^{-1}$ is absolutely continuous with respect to m. For a given non-singular transformation, we can define associated operator called the Perron–Frobenius operator on $L^1(X, m)$.

Definition 2.1. For a non-singular transformation T on a probability space $(X, \mathscr{B}, m)$ into itself, the `Perron–Frobenius operator` corresponding to T is the bounded linear operator P over $L^1(X, m)$ defined by

$$\int_X Pf \cdot g dm = \int_X f \cdot g \circ T dm \quad \text{for } f \in L^1(X, m) \text{ and } g \in L^\infty(X, m)$$

which is well-defined by the Radon–Nikodým theorem. An alternative form of the Perron–Frobenius operator is the Radon–Nikodým derivative

$$Pf = \frac{d\left(m_f \circ T^{-1}\right)}{dm}$$

for $f \in L^1(X, m)$ where m_f is the signed measure defined by $m_f(A) = \int_A f dm$. The adjoint operator of P which is the composition operator of T on $L^\infty(X, m)$ is called the `Koopman operator` and is denoted by P^*.

The following proposition is basic properties of the Perron–Frobenius operators (see [43, 58] for example).

Proposition 2.2. *Let $P : L^1(X, m) \to L^1(X, m)$ be the Perron–Frobenius operator associated with some non-singular transformation T. Then P satisfies the following properties:*

(1) $Pf \geq 0$ *for any* $f \in L^1_+(X, m)$*;*

(2) $\int_X Pf dm = \int_X f dm$ *for any* $f \in L^1(X, m)$*.*

Consequently, we have $P(D(X, m)) \subset D(X, m)$*.*

Example 1 (The Perron–Frobenius operators for fibered systems)**.** Let (X, T) be a fibered system in the Schweiger's sense [58]: there is a partition of X into measurable and countable number of subsets $\{X_j\}_{j \in J}$ with $J \subset \mathbb{N}$ such that

$$T|_{X_j} : X_j \to TX_j$$

is a bijective, bi-measurable (i.e., $T|_{X_j}$ is measurable and $T|_{X_j}^{-1}$ is measurable) and forward non-singular (i.e., $m \circ (T|_{X_j})$ is absolutely continuous with respect to m) map for each $j \in J$. Then, by using the local inverse of T,

$$V_j := T|_{X_j}^{-1} : TX_j \to X_j$$

for $j \in J$, the Perron–Frobenius operator P of T can be expressed as

$$Pf(x) = \sum_{\substack{j \in J \\ x \in TX_j}} \frac{d(m \circ V_j)}{dm}(x) f(V_j x)$$

for any integrable or non-negative measurable function f.

A Markov operator, defined below, is a generalization of the Perron–Frobenius operator corresponding to a non-singular transformation in terms of the properties (1) and (2) in Proposition 2.2. Note that the definition of Markov operators does not necessarily require transformations.

Definition 2.3. A linear operator over $L^1(X, m)$ is called a `Markov operator` if it satisfies

(1) $Pf \geq 0$ for any $f \in L^1_+(X, m)$;

(2) $\int_X Pf dm = \int_X f dm$ for any $f \in L^1(X, m)$.

The adjoint operator of P defined on $L^\infty(X, m)$ is denoted by P^* which satisfies

$$\int_X Pf \cdot g dm = \int_X f \cdot P^* g dm \quad \text{for } f \in L^1(X, m) \text{ and } g \in L^\infty(X, m).$$

Remark 1. The domain of Markov operators can be naturally extended to $\mathcal{M}_+(X)$ the set of all non-negative measurable functions or $\mathcal{M}_+^\sigma(X, m)$ the set of all non-negative locally integrable functions (see [22, 37] for more detail).

We give a few examples of Markov operators which correspond some random dynamical systems to which we can apply our results in this thesis. We will investigate several statistical properties for a spacial type of the following Example 2, in section 4 and 5.

Example 2. Let $I \subset \mathbb{N}$ and T_i be non-singular transformations over a probability space $(X, \mathscr{B}, m)$ for $i \in I$. A special type of random dynamical system, called random iteration of transformations T_i $(i \in I)$, is defined as follows. For an arbitrary probability vector $\{p_i\}_{i \in I}$ (i.e., $p_i \geq 0$ for any $i \in I$ and $\sum_{i \in I} p_i = 1$), we define the transition probability from m-almost all point $x \in X$ into a set $E \in \mathscr{B}$, say $\mathbb{P}(x, E)$, by

$$\mathbb{P}(x, E) := \sum_{i \in I} p_i 1_E(T_i x).$$

Non-singularity of T_i for each $i \in I$ implies null-preserving property of the transition probability: if a set $N \in \mathscr{B}$ satisfies $m(N) = 0$ then $\mathbb{P}(x, N) = 0$ for m-almost every $x \in X$. Then we can define the Markov operator P corresponding to this random iteration by

$$\int_E Pf dm = \int_X f \cdot \mathbb{P}(x, E) dm(x)$$

for $f \in L^1(X, m)$.

Remark 2. A parameter space of transformations I in Example 2 can be indeed replaced by uncountable set such as a closed interval $I \subset \mathbb{R}$ with Lebesgue-absolutely continuous probability measure p on I. Then the transition probability of the random dynamical system is of the form

$$\mathbb{P}(x, E) = \int_I 1_E(T_i x) dp(i). \tag{1}$$

Since each transformation T_i is non-singular for $i \in I$, the transition probability is null-preserving and define the corresponding Markov operator.

Definition 2.4. A random dynamical system of which transition probability is given by (1) is called the `random iteration` $\{T_i, dp(i) : i \in I\}$.

Example 3. We consider additive noisy type random dynamical systems in the following sense. Let T be a non-singular transformation on $[0, 1]$ with the Borel σ-algebra and Lebesgue measure λ. We further put a sequence of mutually independent random variables $\{\xi_n\}_{n\geq 0}$ where ξ_n is a $[0, 1]$-valued measurable function over some probability space $(\Omega, \mathscr{B}, \nu)$ having the same density $g \in D([0, 1], \lambda)$ (i.e., $\nu(\{\omega \in \Omega : \xi_n(\omega) \in A\}) = \int_A g d\lambda$). Then we consider the stochastic process defined by

$$x_{n+1}(\omega) = T x_n(\omega) + \xi_n(\omega) \pmod 1$$

for $\omega \in \Omega$. If x_0 has the distribution $f_0 \in D([0, 1], \lambda)$ as the initial density, then it was shown in Proposition 2.2 in [34] that the Markov operator P representing the time evolution of the initial density, namely P satisfies

$$\nu(\{\omega \in \Omega : x_n(\omega) \in A\}) = \int_A P^n f_0 d\lambda$$

for $A \subset [0, 1]$ measurable and $n \in \mathbb{N}$, is given by

$$Pf_0(x) = \int_{[0,1]} f_0(y) \left(\sum_{i=0}^{1} g(x - Ty + i) \right) d\lambda(y). \tag{2}$$

Remark 3. In general, if we have a doubly measurable stochastic kernel $K : X \times X \to \mathbb{R}_+$ then we can define the Markov operator associated with K. More precisely, assume that K satisfies $K(x, y) \geq 0$, $m \times m$-almost everywhere and $\int_X K(x, y) dm(y) = 1$ m-almost every $x \in X$. Then the Markov operator is defined by

$$Pf(y) = \int_X f(x) K(x, y) dm(x) \quad \text{for } f \in L^1(X, m).$$

Now we give the definition of absolutely continuous invariant measures for Markov operators, the main concept of this thesis.

Definition 2.5. Let $(X, \mathscr{B}, m)$ be a probability space and P be a Markov operator over $L^1(X, m)$. A σ-finite measure μ is called an `absolutely continuous` σ`-finite invariant measure` for P if μ is m-absolutely continuous (i.e., $m(N) = 0$ implies $\mu(N) = 0$) and the Radon–Nikodým derivative of μ with respect to m is a non-negative (non-trivial) fixed point of P, namely,

$$P\frac{d\mu}{dm} = \frac{d\mu}{dm}.$$

In particular, if μ is a finite measure, then μ is called an absolutely continuous `finite` invariant measure for P. The function $d\mu/dm$ is called a `(`σ`-)finite invariant density` of P.

Remark 4. (i) Without loss of generality, an absolutely continuous finite invariant measure is assumed to be a probability measure by replacing normalized one. By definition of absolutely continuous σ-finite infinite invariant measures for Markov operators, the domain of Markov operators should contain $\mathcal{M}_+^\sigma(X, m)$. From Remark 1, the definition of absolutely continuous σ-finite infinite invariant measures makes sense.

(ii) When one considers a non-singular transformation T then the condition of invariance i.e.,

$$P\frac{d\mu}{dm} = \frac{d\mu}{dm},$$

is equivalent to $\mu \circ T^{-1} = \mu$.

(iii) For a random iteration $\{T_i, dp(i) : i \in I\}$ defined in Definition 2.4, μ is an absolutely continuous σ-finite invariant measure if and only if μ is absolutely continuous with respect to m and

$$\mu = \sum_{i \in I} p_i \mu \circ T_i^{-1}.$$

2.2 Conservativeness, dissipation and ergodicity

Considering dynamical systems or random dynamical systems, characterization and properties of sets of recurrent points (and non-recurrent points) are important. If one considers a deterministic system, that is a non-singular transformation T, then non-recurrent points are characterized by wandering sets: A set W is called a `wandering set` for T if $\{T^{-n}W\}_{n\geq 0}$ is mutually disjoint modulo m. This means that any point in W will never return to W in the future and W is exactly the set of non-recurrent points. In this subsection, we recall the concept of recurrent property, conservativeness and dissipation for the general case, namely Markov operators case.

For a given Markov operator, we can always decompose the whole space into two spaces: the *conservative part* where the recurrent property holds and the *dissipative part* where almost all points are never to return. This dichotomy is guaranteed by the Hopf decomposition theorem (see §3.1 in [37]):

Theorem 2.6 (The Hopf decomposition theorem)**.** *Let P be a Markov operator over $L^1(X, m)$. Then the set X is uniquely decomposed into two sets $\mathfrak{C}$ and $\mathfrak{D}$ up to m-measure zero sets:*

$$\mathfrak{C} = \left\{ x \in X : \sum_{n\geq 0} P^n 1_X(x) = \infty \right\}, \quad \mathfrak{D} = \left\{ x \in X : \sum_{n\geq 0} P^n 1_X(x) < \infty \right\}.$$

Furthermore, the characteristic function 1_X in $\mathfrak{C}$ and $\mathfrak{D}$ above can be replaced by any strictly positive L^1-function and the sets $\mathfrak{C}$ and $\mathfrak{D}$ do not change.

By the Hopf decomposition theorem, we can define conservativeness and dissipation for Markov operators. In section 4, we will give eventual conservativeness for Markov operators (see also Definition 4.9).

Definition 2.7. For a Markov operator P, $\mathfrak{C}$ and $\mathfrak{D}$ from the Hopf decomposition theorem are called the `conservative part` and the `dissipative part` of P, respectively. In particular, if $X = \mathfrak{C} \pmod m$ (resp. $X = \mathfrak{D} \pmod m$) then the system (P, m) is called `conservative` (resp. `totally dissipative`).

Remark 5. If P is the Perron–Frobenius operator of a non-singular transformation T, we have

$$\mathfrak{D} = \bigcup_{W \in \mathcal{W}} W$$

the measurable union of wandering sets ($\mathcal{W}$ is the collection of all measurable wandering sets). Thus $\mathfrak{C}$ is the complement of all wandering sets and hence for any set $C \subset \mathfrak{C} \cap \mathscr{B}$, we have $C \cap T^{-n}C \neq \emptyset \pmod m$ for some $n \in \mathbb{N}$.

For systems admitting finite invariant densities, they are always conservative with respect to the invariant measure known as the Poincaré recurrence theorem (see for example [3]). Note that the converse is not always true.

Now we recall the definition of invariant sets and ergodicity for Markov operators (see also [22, 38]).

Definition 2.8. For a Markov operator P over $L^1(X, m)$, the sub-algebra of `invariant sets` is defined by $\Sigma_i(P^*) = \{A \in \mathscr{B} : P^* 1_A = 1_A\}$. $\Sigma_i(P^*)$ is obviously sub-σ-algebra of $\mathscr{B}$. If $\Sigma_i(P^*)$ is purely atomic, then each atom of $\Sigma_i(P^*)$ is called an `ergodic component`. Further a system (P, m) is called `ergodic` if $\Sigma_i(P^*) = \{\emptyset, X\} \pmod m$.

Remark 6. If P is the Perron–Frobenius operator corresponding to T, then invariant sets are nothing but a set $E \in \mathscr{B}$ such that $T^{-1}E = E \pmod m$ and ergodicity of (T, m) is equivalent to that for each E with $T^{-1}E = E \pmod m$ we have $m(E) = 0$ or $m(X \setminus E) = 0$.

Then we show beneficial theorems under the assumptions of conservativeness and ergodicity for a given Markov operator. The following well-known Chacon–Ornstein theorem is the generalization of the Birkhoff ergodic theorem (see Theorem E in Chapter III in [22]).

Theorem 2.9 (The Chacon–Ornstein theorem)**.** *Let P be a conservative and ergodic Markov operator over a probability space $(X, \mathscr{B}, m)$. Then we have for any $f, g \in L^1(X, m)$ with $g > 0$ almost everywhere, that*

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} P^i f}{\sum_{i=0}^{n-1} P^i g}(x) = \frac{\int_X f dm}{\int_X g dm}$$

for m-almost every $x \in X$. Consequently, if m is an ergodic invariant probability measure for P then we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} P^i f(x) = \int_X f dm$$

for any $f \in L^1(X, m)$ and almost every $x \in X$.

The following theorem states the uniqueness of an absolutely continuous σ-finite invariant measure for a conservative and ergodic Markov operator.

Theorem 2.10 (Theorem A in Chapter VI [22])**.** *Let P be a Markov operator over a probability space $(X, \mathscr{B}, m)$. If the system (P, m) is conservative and ergodic, then there can exist at most one absolutely continuous σ-finite invariant measure (up to a multiplicative constant).*

Remark 7. The above theorem no longer holds for totally dissipative and ergodic, non-invertible transformations. In fact, Aaronson and Meyerovitch showed in [5] that if T is a non-invertible dissipative and ergodic σ-finite measure preserving transformation on a standard and non-atomic space, then there exist uncountably many non-propotional absolutely continuous σ-finite invariant measures for T.

2.3 Banach limits

In this subsection, we introduce Banach limits which are generalization of the conventional limit of sequences of real numbers.

Definition 2.11. A `Banach limit` is a linear functional LIM on ℓ^∞ the set of all bounded sequences of real numbers such that

(1) for $\{x_n\}_{n\in\mathbb{N}} \in \ell^\infty$, if $x_n \geq 0$ for any $n \in \mathbb{N}$, then $\mathrm{LIM}(\{x_n\}_n) \geq 0$;

(2) for any $\{x_n\}_{n\in\mathbb{N}} \in \ell^\infty$, $\mathrm{LIM}(\{x_n\}_n) = \mathrm{LIM}(\{x_{n+1}\}_n)$; and

(3) for $\mathbf{1} = \{1, 1, \dots\} \in \ell^\infty$, $\mathrm{LIM}(\mathbf{1}) = 1$.

Theorem 2.12 (Theorem (α) and (β) in [65])**.** *Banach limits exist and any Banach limit* LIM *satisfies*

$$\lim_{n\to\infty} \left(\inf_{j\geq 0} \frac{1}{n} \sum_{i=0}^{n-1} x_{i+j} \right) \leq \mathrm{LIM}\left(\{x_n\}_n\right) \leq \lim_{n\to\infty} \left(\sup_{j\geq 0} \frac{1}{n} \sum_{i=0}^{n-1} x_{i+j} \right) \quad \textit{for } \{x_n\}_n \in \ell^\infty.$$

Moreover, left hand side and right hand side of the above inequality are also Banach limits.

Since the construction of Banach limits is based on the Hahn–Banach extension theorem, a Banach limit is not necessarily unique. The following properties on Banach limits are easy to see and we omit the proof.

Proposition 2.13. *Any Banach limit* LIM *satisfes the following properties:*

(1) *for any* $\{x_n\}$ *and* $\{y_n\} \in \ell^\infty$ *with* $x_n \geq y_n$ *for all* $n \in \mathbb{N}$, $\mathrm{LIM}(\{x_n\}_n) \geq \mathrm{LIM}(\{y_n\}_n)$*; and*

(2) LIM *is continuous.*

3 Absolutely continuous finite invariant measures

3.1 Weak almost periodicity and mean ergodicity

We recall important concepts of weak almost periodicity and mean ergodicity of Markov operators in order to construct an absolutely continuous finite invariant measure for a given Markov operator.

Recall that a set $\mathcal{F} \subset L^1(X, m)$ is called `weakly precompact` if for any subset $\{f_n\}_{n\geq 1} \subset \mathcal{F}$, there exists a further subsequence $\{f_{n_k}\}_{k\geq 1}$ such that a weak limit of f_{n_k} exists. Weak precompactness of $\mathcal{F} \subset L^1(X, m)$ is known to be equivalent to (the Dunford–Pettis theorem, see [18, 55])

(1) there is $M < \infty$ such that $\|f\|_{L^1(m)} \leq M$ for any $f \in \mathcal{F}$; and

(2) for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any $f \in \mathcal{F}$, we have

$$m(A) < \delta \text{ implies } \int_A |f|\, dm < \varepsilon.$$

The second condition is called `uniform integrability` of $\mathcal{F}$ and is fundamental to estimate the existence of an absolutely continuous finite invariant measure which will be shown in subsection 3.2. When one considers the Perron–Frobenius operator P of some non-singular transformation T over $L^1(X, m)$ and $\mathcal{F} = \{P^n 1_X\}_{n\geq 1}$, uniform integrability of $\mathcal{F}$ is equivalent to the condition that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $n \geq 1$ we have

$$m(A) < \delta \text{ implies } m \circ T^{-n}(A) < \varepsilon, \tag{3}$$

which is called `equi-uniform absolute continuity` of $\{m \circ T^{-n}\}_n$ (see [26, 19]).

The definition of weak almost periodicity, which turns out to be necessary and sufficient for the existence an absolutely continuous finite invariant measure with a certain support condition shown in subsection 3.2, is the following.

Definition 3.1. A Markov operator P over $L^1(X, m)$ is said to be `weakly almost periodic` if for any $f \in L^1(X, m)$ it holds that $\{P^n f\}_{n\geq 1}$ is weakly precompact. If $\{P^n f\}_{n\geq 1}$ in the above condition is precompact in strong, then P is called `almost periodic`.

The following theorem clarifies the meaning of the name "weak almost periodicity." See Theorem 1.1.4 in [21] or §2.4 in [37] for more details.

Theorem 3.2 (The Jacobs–de Leeuw–Glicksberg splitting theorem)**.** *Let* $\bar{P}$ *be the complexification of a Markov operator* P *over* $\mathbb{C}$*-valued* L^1*-space denoted by* $\widehat{L^1}(X, m)$*. If* $\bar{P}$ *is weakly almost periodic then* $\widehat{L^1}(X, m)$ *can be decomposed into the direct sum* $\widehat{L^1}(X, m) = \widehat{L^1}_{uds}(\bar{P}) \oplus \widehat{L^1}_{fl}(\bar{P})$*, where*

$$\widehat{L^1}_{uds}(\bar{P}) = \overline{\mathrm{span}}\left\{f \in \widehat{L^1}(X, m) : \bar{P}f = \lambda f \text{ for some } \lambda \in \mathbb{C} \text{ with } |\lambda| = 1\right\}$$

and

$$\widehat{L^1}_{fl}(\bar{P}) = \left\{ f \in \widehat{L^1}(X,m) : \text{w-}\lim_{k\to\infty} \bar{P}^{n_k} f = 0 \textit{ for some } \{n_k\}_k \subset \mathbb{N} \right\}.$$

Moreover, if $\bar{P}$ is almost periodic then

$$\widehat{L^1}_{fl}(\bar{P}) = \left\{ f \in \widehat{L^1}(X,m) : \lim_{n\to\infty} \left\| \bar{P}^n f \right\|_{L^1(m)} = 0 \right\}.$$

Here, "uds" and "fl" stand for unimodular discrete spectrum and flight, respectively.

Definition 3.3. A Markov operator P is called `mean ergodic` if for any $f \in L^1(X,m)$ the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} P^i f$$

exists in strong.

An immediate consequence of mean ergodicity of P is the existence of a finite invariant density

$$h = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} P^i 1_X.$$

The following theorem owes to Yosida and Kakutani in 1941 (Theorem 1 in [76]) and plays a key role in showing relations between weak almost periodicity, mean ergodicity and the existence of absolutely continuous finite invariant measures in the next subsection (see Proposition 3.11).

Theorem 3.4 (Mean ergodic theorem)**.** *Let P be a Markov operator over $L^1(X,m)$. If P is weakly almost periodic then P is mean ergodic.*

3.2 The existence of absolutely continuous finite invariant measures

In this subsection, we consider necessary and sufficient conditions for the existence of fixed points of Markov operators. Weak almost periodicity of a Markov operator plays an important role in considering invariant measures. In the latter of this subsection, we consider necessary and sufficient conditions for the existence of equivalent finite invariant measures for non-singular transformations which are the generalization of the result of [26] to the case of not necessarily invertible transformations.

The following theorem gives equivalent conditions for the existence of a finite invariant density function with the maximal support.

Theorem 3.5. *Let $(X, \mathscr{B}, m)$ be a probability space and $P : L^1(X,m) \to L^1(X,m)$ be a Markov operator. Then the following statements are equivalent*:

(I) *There exists an absolutely continuous finite invariant measure μ for P such that*

$$\lim_{n\to\infty} P^{*n} 1_{\operatorname{supp}\mu}(x) = 1 \ \textit{ for m-almost every } x \in X; \tag{4}$$

(II) *P is weakly almost periodic*;

(III) *$\{P^n 1_X\}_n$ is weakly precompact*;

(*IV*) *For any Banach limit* LIM*, a set function* μ *on* $(X, \mathscr{B})$ *defined by*

$$\mu(E) = \mathrm{LIM}\left(\left\{\int_E P^n 1_X dm\right\}_n\right) \quad \text{for } E \in \mathscr{B}$$

is an absolutely continuous finite invariant measure for P satisfying the equation (4).

Before proving this theorem, we give some remark and corollary.

Remark 8. (i) The equation (4) in Theorem 3.5 (I) is indeed weaker condition that (P, m) is ergodic. (See also Remark 9.) Thus the condition of maximal support can be regarded as a natural requirement for an invariant density.

(ii) As long as we can construct an absolutely continuous finite invariant measure via Banach limits method which is originally due to [26], the resulting measure automatically satisfies the maximal support condition (4).

By complexification of a Markov operator P and by Theorem 3.2, we obtain the following corollary.

Corollary 3.6. *Let* $(X, \mathscr{B}, m)$ *be a probability space and* $P : L^1(X, m) \to L^1(X, m)$ *be a Markov operator. If one of the conditions in Theorem 3.5 holds, then the following assertions are true.*

(1) $\bar{P}$ *on* $\widehat{L^1}(X, m)$ *is weakly almost periodic;*

(2) $\widehat{L^1}(X, m)$ *is the direct sum of the closed invariant subspaces:*

$$\widehat{L^1}(X, m) = \widehat{L^1}_{uds}(\bar{P}) \oplus \widehat{L^1}_{fl}(\bar{P})$$

where $\widehat{L^1}_{uds}(\bar{P})$ *is the set of eigenfunctions of unimodular eigenvalue and* $\widehat{L^1}_{fl}(\bar{P})$ *is the set of flight functions.*

We prepare a sequence of lemmas to prove the above theorem.

Lemma 3.7. *If* $\{P^n 1_X\}_n$ *is weakly precompact then a set function on* $(X, \mathscr{B})$ *by*

$$\mu(E) = \mathrm{LIM}\left(\left\{\int_E P^n 1_X dm\right\}_n\right) \quad \text{for } E \in \mathscr{B}.$$

satisfies countable additivity.

Proof. We have to show countable additivity of μ, that is, $\mu\left(\bigcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty \mu(A_i)$ for any mutually disjoint sets $\{A_i\}_i$. By assumption of weak precompactness of $\{P^n 1_X\}_n$, equivalently uniform integrability of $\{P^n 1_X\}_n$, we have that for each $\varepsilon > 0$, there is $\delta > 0$ such that if $m(A) < \delta$ then $\int_A P^n 1_X dm < \varepsilon$. For showing countable additivity of μ, we prepare the following assertion:

$$\lim_{N\to\infty}\left\{\sum_{i=1}^N \int_{A_i} P^n 1_X dm\right\}_n = \left\{\sum_{i=1}^\infty \int_{A_i} P^n 1_X dm\right\}_n \tag{5}$$

in ℓ^∞ norm where $\{A_i\}_i$ are mutually disjoint measurable sets. Since $\{A_i\}_i$ are mutually disjoint and m is a probability, we can find $N_0 \in \mathbb{N}$ for which $m\left(\bigcup_{i=N+1}^\infty A_i\right) < \delta$ for $N \geq N_0$. Write for $N \geq N_0$

$$\sup_{n\geq 1}\left\{\sum_{i=N+1}^\infty \int_{A_i} P^n 1_X dm\right\}_n = \sup_{n\geq 1}\left\{\int_{\bigcup_{i=N+1}^\infty} P^n 1_X dm\right\}_n < \varepsilon$$

and hence we have the equation (5).

Then, by equality (5), for any mutually disjoint measurable sets $\{A_i\}_i$

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mathrm{LIM}\left(\left\{\int_{\bigcup_{i=1}^{\infty} A_i} P^n 1_X dm\right\}_n\right) = \mathrm{LIM}\left(\sum_{i=1}^{\infty} \left\{\int_{A_i} P^n 1_X dm\right\}_n\right).$$

By Proposition 2.13, (2) it follows

$$\mathrm{LIM}\left(\sum_{i=1}^{\infty} \left\{\int_{A_i} P^n 1_X dm\right\}_n\right) = \sum_{i=1}^{\infty} \mathrm{LIM}\left(\left\{\int_{A_i} P^n 1_X dm\right\}_n\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

Thus μ satisfies countable additivity. □

Lemma 3.8. *For any locally integrable and non-negative measurable function f, it holds that*

$$P^* 1_{X \setminus \operatorname{supp} Pf} \leq 1_{X \setminus \operatorname{supp} f}.$$

In particular, if h is an absolutely continuous σ-finite invariant measure for P then we have

$$P^* 1_{\operatorname{supp} h} \geq 1_{\operatorname{supp} h}.$$

Proof. We firstly show $\int_{\operatorname{supp} f} P^* 1_{X \setminus \operatorname{supp} Pf} dm = 0$. Note that for any $\varepsilon > 0$, there exists $C > 0$ such that $m(N_C) < \varepsilon$ where $N_C = \{x \in \operatorname{supp} f : f(x) < C\}$. Since $1_{\operatorname{supp} f}(x) \leq f(x)/C$ for $x \in \operatorname{supp} f \setminus N_C$, we have

$$\begin{aligned}
\int_{\operatorname{supp} f} P^* 1_{X \setminus \operatorname{supp} Pf} dm &= \int_{X \setminus \operatorname{supp} Pf} P 1_{\operatorname{supp} f} dm \\
&= \int_{X \setminus \operatorname{supp} Pf} P(1_{N_C} + 1_{\operatorname{supp} f \setminus N_C}) dm \\
&\leq \int_X P 1_{N_C} dm + \int_{X \setminus \operatorname{supp} Pf} \frac{Pf}{C} dm \\
&< \varepsilon.
\end{aligned}$$

Now ε is arbitrary small and we get $\int_{\operatorname{supp} f} P^* 1_{X \setminus \operatorname{supp} Pf} dm = 0$. This implies $\operatorname{supp} P^* 1_{X \setminus \operatorname{supp} Pf} \subset X \setminus \operatorname{supp} f$ and by $P^* 1_X = 1_X$ we obtain

$$P^* 1_{X \setminus \operatorname{supp} Pf} \leq 1_{\operatorname{supp} P^* 1_{X \setminus \operatorname{supp} Pf}} \leq 1_{X \setminus \operatorname{supp} f}$$

as desired. □

Remark 9. Lemma 3.8 implies that the support of an absolutely continuous σ-finite invariant measure for a Markov operator P spreads under the process by P. In the case when the system is ergodic then an absolutely continuous σ-finite invariant measure μ of P always has the maximal support in the sense of (4) (when one considers a non-singular transformation T it means that $\bigcup_{n \geq 0} T^{-n} (\operatorname{supp} \mu) = X \pmod{m}$).

The following Lemma 3.9 was obtained in [15] and it plays an important role in proving weak almost periodicity of a Markov operator.

Lemma 3.9 (Proposition 3 in [15])**.** *If there exists an almost everywhere positive fixed point of P in $L^1(X, m)$, then $\{P^n g\}_n$ is weakly precompact for any $g \in L^1_+(X, m)$.*

Proof of Theorem 3.5. (I)$\Rightarrow$(II): Recall that in a probability space, for any $f \in L^1(X,m)$, $\{P^n f\}_n$ is weakly precompact if and only if $\{P^n f\}_n$ is uniformly integrable from subsection 3.1 (2). Since Markov property of P leads to $\int_A |P^n f| dm \leq \int_A P^n |f| dm$ for each $f \in L^1(X,m)$ and $A \in \mathscr{B}$, we are to show that $\{P^n g\}_n$ is uniformly integrable only for $g \in L^1_+(X,m)$.

We consider the following measure ν and operator $\tilde{P}$: $\nu(E) = m(E \cap \operatorname{supp}\mu)$ for $E \in \mathscr{B}$ and $\tilde{P} f = P\left(f 1_{\operatorname{supp}\mu}\right)$. Then $\tilde{P}$ is a Markov operator over $L^1(\operatorname{supp}\mu, \nu)$ and $\tilde{P} h = h$ where $h = d\mu/dm$ an invariant density. Hence by Lemma 3.9 we have that for any $g \in L^1_+(\operatorname{supp}\mu, \nu)$ and $\varepsilon > 0$, there exists $\delta_1 > 0$ for which $\nu(A) < \delta_1$ implies

$$\sup_{n\geq 1} \int_A \tilde{P}^n g d\nu < \varepsilon.$$

By Lemma 3.8, one can see that $\operatorname{supp} \tilde{P} g \subset \operatorname{supp}\mu \pmod m$ for any $g \in L^1_+(X,m)$. Then for any $g \in L^1_+(X,m)$ and $\varepsilon > 0$, there exists $\delta_1 > 0$ such that we have

$$\sup_{n\geq 1} \int_A P^n \left(g 1_{\operatorname{supp}\mu}\right) dm < \varepsilon \quad \text{for } A \in \mathscr{B} \cap \operatorname{supp}\mu \text{ with } m(A) < \delta_1.$$

Now we will show $\sup_{n\geq 1} \int_A P^n \left(g 1_{X\setminus\operatorname{supp}\mu}\right) dm$ is dominated by an arbitrary small $\varepsilon > 0$ to conclude

$$\sup_{n\geq 1} \int_A P^n g dm \leq \sup_{n\geq 1} \int_A P^n \left(g 1_{\operatorname{supp}\mu}\right) dm + \sup_{n\geq 1} \int_A P^n \left(g 1_{X\setminus\operatorname{supp}\mu}\right) dm < 2\varepsilon$$

for $A \in \mathscr{B} \cap \operatorname{supp}\mu$ small enough in the sense of measure. By assumption that μ has the maximal support in the sense of (4) and by the fact that $\operatorname{supp} f_1 \subset \operatorname{supp} f_2$ implies $\operatorname{supp} P f_1 \subset \operatorname{supp} P f_2$ for any $f_1, f_2 \in L^1_+(X,m)$, it holds that for any $g \in L^1_+(X,m)$ and $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $\int_{X\setminus\operatorname{supp}\mu} P^{N_0} \left(g 1_{X\setminus\operatorname{supp}\mu}\right) dm < \varepsilon$. Thus, for any $g \in L^1_+(X,m)$ and $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ and $\delta_1 > 0$ such that for any $A \in \mathscr{B} \cap \operatorname{supp}\mu$ with $m(A) < \delta_1$, we have

$$\begin{aligned}
&\sup_{n\geq N_0+1} \int_A P^n \left(g 1_{X\setminus\operatorname{supp}\mu}\right) dm \\
=&\sup_{n\geq 1} \int_A P^{n+N_0} \left(g 1_{X\setminus\operatorname{supp}\mu}\right) dm \\
\leq&\sup_{n\geq 1} \int_A P^n \left(P^{N_0} \left(g 1_{X\setminus\operatorname{supp}\mu}\right) 1_{\operatorname{supp}\mu}\right) dm + \sup_{n\geq 1} \int_A P^n \left(P^{N_0} \left(g 1_{X\setminus\operatorname{supp}\mu}\right) 1_{X\setminus\operatorname{supp}\mu}\right) dm \\
<&2\varepsilon.
\end{aligned}$$

For $\varepsilon > 0$, $g \in L^1_+(X,m)$ and $N_0 \in \mathbb{N}$ as above, we can find $\delta_2 > 0$ such that for any $n \in \{1, \dots, N_0 - 1\}$ and $A \in \mathscr{B} \cap \operatorname{supp}\mu$ with $m(A) < \delta_2$ we have $\int_A P^n g dm < \varepsilon$. Then, for any $\varepsilon > 0$ and $g \in L^1_+(X,m)$, there exists $\delta = \min\{\delta_1, \delta_2\} > 0$ such that for each $A \in \mathscr{B} \cap \operatorname{supp}\mu$ with $m(A) < \delta$ we have

$$\sup_{n\geq 1} \int_A P^n g dm < \varepsilon.$$

On the other hand, by Lemma 3.8 and assumption (4), $\{P^{*n} 1_{X\setminus\operatorname{supp}\mu}\}_n$ is decreasing to 0 monotonically and from the Lebesgue convergence theorem, it holds that for any $g \in L^1(X,m)$,

$$\lim_{n\to\infty} \int_{X\setminus\operatorname{supp}\mu} P^n g dm = \int_X g \lim_{n\to\infty} P^{*n} 1_{X\setminus\operatorname{supp}\mu} dm = 0.$$

This shows that for any $g \in L^1_+(X, m)$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for each $A \in \mathscr{B} \cap (X \setminus \operatorname{supp}\mu)$ with $m(A) < \delta$ we have

$$\sup_{n \geq 1} \int_A P^n g dm < \varepsilon.$$

Therefore, P is weakly almost periodic.

(III)⇒(IV): Define a linear functional on $L^\infty(X, m)$ by

$$\mu(g) = \mathrm{LIM}\left(\left\{\int_X P^{*n} g dm\right\}_n\right)$$

and set $\mu(A) := \mu(1_A)$ for $A \in \mathscr{B}$. Then the assumption of weak precompactness of $\{P^n 1_X\}_n$ implies that μ is an absolutely continuous probability measure on $(X, \mathscr{B})$ by Lemma 3.7 and for $g \in L^\infty(X, m)$ we have

$$\mu(g) = \int_X g d\mu.$$

Indeed, for a function $g = \sum_{i \geq 1} \alpha_i 1_{A_i}$ for some $\alpha_i \in \mathbb{R}$ and (mutually disjoint) $A_i \in \mathscr{B}$, we have

$$\begin{aligned}
\mu(g) &= \mathrm{LIM}\left(\left\{\int_X P^{*n}\left(\sum_i \alpha_i 1_{A_i}\right) dm\right\}_n\right) \\
&= \sum_i \alpha_i \,\mathrm{LIM}\left(\left\{\int_X P^{*n}\left(1_{A_i}\right) dm\right\}_n\right) \\
&= \sum_i \alpha_i \mu(A_i)
\end{aligned}$$

by uniform integrability of $\{P^n 1_X\}_n$. Simple function approximation shows the equality holds for any $g \in L^\infty(X, m)$. Applying this to $P^* 1_A \in L^\infty(X, m)$ for any $A \in \mathscr{B}$, we obtain

$$\int_X P^* 1_A d\mu = \mathrm{LIM}\left(\left\{\int_X P^{*n}(P^* 1_A) dm\right\}_n\right) = \mathrm{LIM}\left(\left\{\int_X P^{*n} 1_A dm\right\}_n\right) = \int_X 1_A d\mu$$

or

$$\int_A P\left(\frac{d\mu}{dm}\right) dm = \int_A \frac{d\mu}{dm} dm.$$

Therefore μ is an absolutely continuous finite invariant measure for P.

The rest is to show $\lim_{n\to\infty} \int_{\operatorname{supp}\mu} P^n 1_X dm = 1$. To show this we prepare the following inequality:

$$\int_{\operatorname{supp}\mu} P^n 1_X dm \leq \int_{\operatorname{supp}\mu} P^{n+1} 1_X dm \quad \text{for } n \in \mathbb{N}.$$

This inequality is true from Lemma 3.8 and $\int_X P^{*n}(P^* 1_{\operatorname{supp}\mu} - 1_{\operatorname{supp}\mu}) dm \geq 0$. Therefore, the Banach limit of $\left\{\int_{\operatorname{supp}\mu} P^n 1_X dm\right\}_n$ coincides with the conventional limit and we have

$$\lim_{n\to\infty} \int_{\operatorname{supp}\,\mu} P^n 1_X dm = 1$$

by $\mu(\operatorname{supp}\mu) = 1$ and the definition of μ. □

If one considers the Perron–Frobenius operator corresponding to a non-singular transformation T, the following corollary of Theorem 3.5 is immediately obtained since weak precompactness of $\{P^n 1_X\}_n$ is equivalent to equi-uniform absolute continuity of $\{m \circ T^{-n}\}_n$ with respect to m in the sense of (3).

Corollary 3.10. *Let $(X, \mathscr{B}, m)$ be a probability measure space and $T : X \to X$ be a non-singular transformation. Then the following are equivalent*:

(I) *There exists an absolutely continuous finite invariant measure μ with respect to m such that*

$$\bigcup_{n=0}^{\infty} T^{-n} (\operatorname{supp} \mu) = X \pmod{m}; \tag{6}$$

(II) $\{m \circ T^{-n}\}_{n \in \mathbb{N}}$ *is equi-uniformly absolutely continuous with respect to m*;

(III) *For any Banach limit* LIM*, a set function*

$$\mu(E) = \mathrm{LIM}\left(\{m \circ T^{-n}(E)\}_n\right) \quad \textit{for } E \in \mathscr{B}$$

is an absolutely continuous finite invariant measure for T satisfying the equation (6).

Remark 10. Let us look at the following condition:

(II') There exists $\delta > 0$, and $\alpha \in (0, 1)$ such that $m(A) < \delta$ implies $m(T^{-n}A) < \alpha$ for any $n \geq 1$.

If we replace the condition (II) in Corollary 3.10 by the above weaker condition (II'), we may not obtain an absolutely continuous finite invariant measure with the maximal support in the sense of (6). We recall that Straube showed in [62] that the condition (II') is equivalent to the existence of an absolutely continuous finite invariant measure (which does not necessarily have the maximal support).

Theorem 3.4 from section 3.1 gives a sufficient condition for all Banach limits to coincide with Césaro average. Moreover, it plays a key role in establishing the following result.

Proposition 3.11. *Let $(X, \mathscr{B}, m)$ be a probability space and P be a Markov operator over $L^1(X, m)$. Then each of the following conditions is equivalent to the existence of an absolutely continuous finite invariant measure for P with the maximal support (4) (and hence all of the conditions in Theorem 3.5 hold):*

(V) *P is mean ergodic*;

(VI) *For any $A \in \mathscr{B}$, the following limit exists*:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \int_A P^i 1_X dm.$$

Remark 11. As a related result to Proposition 3.11, equivalent conditions for Perron–Frobenius operators being weakly almost periodic were given in [20]. It reads that the following are equivalent for the Perron–Frobenius operator P:

(1) P is weakly almost periodic;

(2) P is mean ergodic;

(3) There exists $f_0 \in D(X, m)$ such that for any $f \in D(X, m)$ we have $\limsup_{n\to\infty} \|P^n f - f_0\|_{L^1(m)} < 2$.

Further, in [33], Ito showed the equivalence of mean ergodicity of a Markov operator P and weak precompactness of $\left\{\frac{1}{n} \sum_{i=0}^{n-1} P^i 1_X\right\}_n$. Therefore, our result is the extension of their result.

Proof of Proposition 3.11. By Theorem 3.4 (the Kakutani–Yosida mean ergodic theorem), we only show the implications of conditions in Theorem 3.5 and Proposition 3.11 (V)⇒(VI) and (VI)⇒(I).

(V)⇒(VI): Suppose mean ergodicity of P. Then it follows that for any $A \in \mathscr{B}$,

$$\left| \int_A \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} P^i f - \frac{1}{N} \sum_{i=0}^{N-1} \int_A P^i f dm \right| \leq \int_A \left| \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} P^i f - \frac{1}{N} \sum_{i=0}^{N-1} P^i f \right| dm$$
$$\to 0$$

as $N \to \infty$ and hence the condition (VI) is valid.

(VI)⇒(I): From the assumption, we can set, for each $E \in \mathscr{B}$,

$$\mu(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \int_E P^i 1_X dm$$

and the Vitali–Hahn–Saks theorem guarantees that μ is an absolutely continuous probability measure with respect to m. It is also easy to see that μ is invariant under P. Further, the fact that $P^{*n} 1_{\operatorname{supp}\mu}$ is monotonic increasing by Lemma 3.8 implies that

$$\mu(\operatorname{supp}\mu) = \lim_{n\to\infty} \int_{\operatorname{supp}\mu} P^n 1_X dm$$

and μ has the maximal support. The proof is completed. □

In particular, if P is the Perron–Frobenius operator corresponding to a non-singular transformation $T : X \to X$, the following corollary is valid.

Corollary 3.12. *The equivalent condition for all of the conditions in Corollary 3.10 is that for any $A \in \mathscr{B}$, the following limit exists:*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} m \circ T^{-i}(A).$$

In the rest of this subsection, we consider necessary and sufficient conditions for the existence of an equivalent finite invariant measure for a non-singular transformation on a probability space. First of all, we prepare the definition to state Theorem 3.14.

Definition 3.13. Let $(X, \mathscr{B}, m)$ be a probability space and T be a measurable transformation. A set $W \in \mathscr{B}$ is called a `weakly wandering set` for T (with a `weakly wandering sequence` $\{n_i\}_i$) if there exists $\{n_i\}_{i=1}^\infty \subset \mathbb{N}$ such that $T^{-n_i} W \cap T^{-n_j} W = \emptyset \pmod m$, for any $i, j \in \mathbb{N}$ with $i \neq j$.

Measures on $(X, \mathscr{B})$, $\{\nu_n\}_{n\in\mathbb{N}}$ and μ, are called `uniformly equivalent` if $\{\nu_n\}_n$ is equi-uniformly absolutely continuous with respect to μ and, at the same time, it holds that for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\inf_{n\geq 1} \nu_n(A) < \delta$ implies $\mu(A) < \varepsilon$.

In [19] and [26], for an invertible transformation, one can see an equivalent condition for the existence of an equivalent finite invariant measure (I) as the conditions (IV) in the next Theorem 3.14. Further, they proved the equivalence of (I) and equi-uniform absolute continuity of $\{m \circ T^n\}_{n\in\mathbb{Z}}$ with respect to m instead of (II) for invertible case. We extend their results to not necessarily invertible cases.

Theorem 3.14. *Let $(X, \mathscr{B}, m)$ be a probability space and $T : X \to X$ be non-singular with respect to m. Then the following are equivalent*:

(*I*) *There exists an equivalent T-invariant probability measure μ with respect to m*;

(II) $\{m \circ T^{-n}\}_{n\in\mathbb{N}}$ *and m are uniformly equivalent*;

(III) *For any Banach limit* LIM, *a set function*

$$\mu(E) = \mathrm{LIM}\left(\{m \circ T^{-n}(E)\}_n\right) \quad \textit{for } E \in \mathscr{B}$$

is an equivalent T-invariant probability measure with respect to m;

(IV) *Let $A, B \in \mathscr{B}$ such that there exist $\{A_i\}_{i\in\mathbb{N}}$ and $\{B_i\}_{i\in\mathbb{N}}$ partitions of A and B, and exist $\{n_i\}_{i\in\mathbb{N}}$ satisfying $T^{-n_i}A_i = B_i$ for any $i \in \mathbb{N}$. If $A \supset B$ and $m(A) > 0$, then $m(A \setminus B) = 0$.*

Remark 12. For an invertible transformation T, two measurable sets A and B such that there are $\{n_i\}_i$, partitions $\{A_i\}_i$ and $\{B_i\}_i$ with $T^{n_i}A_i = B_i \pmod{m}$ for each i are called `Hopf equivalent`. The condition that if A and B are Hopf equivalent and $A \supset B$ then $m(A \setminus B) = 0$ is called `Hopf incompressible` and is originally from [28]. Thus the condition (IV) in the above Theorem 3.14 can be considered as the definition of Hopf incompressible for the non-invertible case. Now let us recall the definition of Type III for non-singular transformations. A transformation is said to be of `Type III` if there is no equivalent σ-finite invariant measure (recall that a transformation is said to be of `Type II`$_1$ or `Type II`$_\infty$ if it admits an equivalent finite or σ-finite infinite invariant measure, respectively). Then Theorem 3.14 states that a non-singular transformation is of Type II$_1$ if and only if X itself cannot be Hopf equivalent with any proper subset of X. However it is known that for an ergodic invertible Type III transformation, any two positive measure sets are Hopf equivalent (see [73]). We will investigate non-invertible generalization of classical results in [73] as a future work.

To prove Theorem 3.14, we prepare the following two lemmas.

Lemma 3.15. *Assume that there exists an equivalent T-invariant probability measure μ with respect to m. Then $\{m \circ T^{-n}\}_n$ is equi-uniformly absolutely continuous with respect to m.*

Proof. We suppose that $\{m \circ T^{-n}\}_n$ is not equi-uniformly absolutely continuous with respect to m, i.e., there exists $\varepsilon_0 > 0$, $\{n_k\}_k \subset \mathbb{N}$ and $\{A_k\}_k \subset \mathscr{B}$ such that $m(A_k) < 1/2^k$ and $m \circ T^{-n_k}(A_k) \geq \varepsilon_0$ for any $k \in \mathbb{N}$. We set $B_k = \bigcup_{i=k}^{\infty} A_i$. Then we have

$$\lim_{k\to\infty} m(B_k) = 0, \tag{7}$$

$$m(T^{-n_k}B_k) \geq \varepsilon_0 \quad (\text{for } k \in \mathbb{N}). \tag{8}$$

By the equation (7) and absolute continuity of μ with respect to m, we have that for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ and $N_0 = N_0(\delta(\varepsilon)) \in \mathbb{N}$, such that for any $k \geq N_0$ we have $m(B_k) < \delta$ so that $\mu(B_k) < \varepsilon$. Since μ is T-invariant, for any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for each $k \geq N_0$ it holds $\mu(T^{-n_k}B_k) < \varepsilon$. By absolute continuity of m with respect to μ, the above assertion $\mu(T^{-n_k}B_k) < \varepsilon$ contradicts to the inequality (8). □

The following result was proven in [66] for not necessarily invertible case.

Lemma 3.16 (Theorem 1 in [66])**.** *There exists an equivalent finite invariant measure if and only if there is no weakly wandering set with positive m-measure.*

Proof of Theorem 3.14. (II)⇒(III): From Lemma 3.7, we only have to show that the condition of uniform equivalent of $\{m \circ T^{-n}\}_n$ and m implies $\mu(E) = \mathrm{LIM}\left(\{m \circ T^{-n}(E)\}_n\right)$ satisfies absolute continuity of m with respect to μ. We suppose not i.e., there exists $A \in \mathscr{B}$ such that $\mu(A) = 0$ and $m(A) = c$ for some $c > 0$. By the property of Banach limits (see Theorem 2.12),

$$\lim_{n\to\infty}\left(\inf_{j\geq 0} \frac{1}{n} \sum_{i=j}^{n-1+j} m \circ T^{-i}(A)\right) = 0.$$

Thus, for any $\varepsilon > 0$ fixed, there exists $N_0 \in \mathbb{N}$ such that for any $n > N_0$ we have

$$\inf_{j\geq 0} \frac{1}{n} \sum_{i=j}^{n-1+j} m \circ T^{-i}(A) < \varepsilon.$$

It follows that for this ε there exists $j_0 \geq 0$ such that

$$\frac{1}{n} \sum_{i=j_0}^{n-1+j_0} m \circ T^{-i}(A) - \varepsilon < \inf_{j\geq 0} \frac{1}{n} \sum_{i=j}^{n-1+j} m \circ T^{-i}(A).$$

Thus we can find $n_0 \in \{j_0, j_0+1, \dots, j_0+n-1\}$ such that

$$m \circ T^{-n_0}(A) < 2\varepsilon. \tag{9}$$

On the other hand, by the assumption (II), for $c > 0$, there exists $\delta > 0$, such that if $\inf_{n\geq 1} m \circ T^{-n}(A) < \delta$ then $m(A) < c$ holds. We set $\varepsilon = \delta/2$. Then from the inequality (9), we have that $m(A) < c$ and this contradicts our assumption. Therefore, the condition (III) holds.

(III)$\Rightarrow$(I): It is obvious.

(I)$\Rightarrow$(II): From Lemma 3.15, we only have to show that if there exists an equivalent T-invariant probability measure μ then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\inf_{n\geq 1} \nu(A) < \delta$ implies $\mu(A) < \varepsilon$. We assume not, i.e., there exists $\varepsilon_0 > 0$ and $\{A_k\}_k \subset \mathscr{B}$ such that $\inf_{n\geq 1} m \circ T^{-n}(A_k) < 1/2^k$ and $m(A_k) \geq \varepsilon_0$ for $k \geq 1$. Then there exists $\varepsilon_0 > 0$, $\{A_k\}_k \subset \mathscr{B}$ and $\{n_k\}_k$ such that $m \circ T^{-n_k}(A_k) < 1/2^{k-1}$ and $m(A_k) \geq \varepsilon_0$ for $k \in \mathbb{N}$. We set $B_k = \bigcup_{i=k}^{\infty} A_i$ and

$$\lim_{k\to\infty} m \circ T^{-n_k}(B_k) = 0 \tag{10}$$

$$m(B_k) \geq \varepsilon_0 \quad \text{(for each } k \in \mathbb{N}). \tag{11}$$

By the equation (10) and absolute continuity of μ with respect to m, for any $\varepsilon > 0$ there exists $\delta > 0$ and $N_0 \in \mathbb{N}$ such that for each $k \geq N_0$ we have $m(T^{-n_k} B_k) < \delta$ so that $\mu(T^{-n_k} B_k) < \varepsilon$. Since m is also absolutely continuous with respect to μ for $\varepsilon_0 > 0$ we can find $\delta_0 > 0$ such that $\mu(B_k) < \delta_0$ implies $m(B_k) < \varepsilon_0$. This contradicts to the inequality (11) and the condition (II) holds.

(I)$\Rightarrow$(IV): Assume that there exists such A and B satisfying that $\bigcup_{i=1}^{\infty} T^{-n_i} A_i = \bigcup_{i=1}^{\infty} B_i$ (disj.) for some $\{n_i\}_i$. Then, by assumption of the existence of finite invariant measure μ,

$$\mu(A \setminus B) = \mu\left(\bigcup_{i=1}^{\infty} A_i \Big\backslash \bigcup_{i=1}^{\infty} T^{-n_i} A_i\right) = \sum_{i=1}^{\infty} (\mu(A_i) - \mu(T^{-n_i} A_i)) = 0.$$

Absolute continuity of m with respect to μ implies $m(A \setminus B) = 0$.

(IV)$\Rightarrow$(I): From Lemma 3.16, we have to show the condition (IV) implies that there is no weakly wandering set with positive measure. We suppose there exists a weakly wandering set W with a weakly wandering sequence $\{n_i\}_i$ with $n_1 = 0$ with $m(W) > 0$. We set $A_i = T^{-n_i} W$, $B_i = T^{-n_{i+1}} W$, $A = \bigcup_{i=1}^{\infty} A_i$ (disj.), and $B = \bigcup_{i=1}^{\infty} B_i$ (disj.). Clearly it holds that $T^{-(n_{i+1}-n_i)} A_i = B_i$ for $i \geq 1$ and $A \supset B$. By assumption (4), we have that $m(A \setminus B) = 0$. This contradicts to $m(A \setminus B) = m(T^{-n_1} W) = m(W) > 0$. □

3.3 Asymptotic periodicity of Markov operators

In this subsection, we study constrictive Markov operators, which always have invariant densities (see Proposition 3.18 and Remark 13 below for the invariant density). Many important examples of Perron–Frobenius operators or Markov operators are constrictive. Otherwise, jump operators for Markov operators with respect to suitable sweep-out sets are constrictive. We refer to [39, 40, 42, 43, 44, 56, 57] for interesting properties of asymptotic behavior of constrictive Markov operators.

Definition 3.17. Let $(X, \mathscr{B}, m)$ be a probability space and P be a Markov operator over $L^1(X, m)$. A set $\mathcal{F} \subset L^1(X, m)$ is called a `constrictor` (or an `attractor`) for P if for any $h \in D(X, m)$,

$$\lim_{n\to\infty} \inf_{f\in\mathcal{F}} \|P^n h - f\|_{L^1(m)} = 0.$$

and P is called `constrictive` (resp. `weakly constrictive`) if there exists a compact (resp. weakly compact) constrictor $\mathcal{F} \subset L^1(X, m)$ for P.

A Markov operator P is called `smoothing` if there exists $\delta > 0$ such that for any $A \in \mathscr{B}$ with $m(A) < \delta$ and $f \in D(X, m)$ it holds

$$\limsup_{n\to\infty} \int_A P^n f dm < 1.$$

The next result is well-known spectral decomposition theorem for constrictive Markov operators and given in [39, 40, 44].

Proposition 3.18 (The spectral decomposition theorem)**.** *Let $(X, \mathscr{B}, m)$ be a probability space. Suppose that $P : L^1(X, m) \to L^1(X, m)$ is a weakly constrictive or smoothing Markov operator. Then there exist $r \in \mathbb{N}$, two sequences of non-negative functions $g_i \in D(X, m)$ and $k_i \in L^\infty(X, m)$ for $i = 1, 2, \ldots, r$, and an operator $Q : L^1(X, m) \to L^1(X, m)$ such that for any $f \in L^1(X, m)$, Pf is written in the form*

$$Pf = \sum_{i=1}^{r} \lambda_i(f) g_i + Qf \quad \textit{where} \quad \lambda_i(f) = \int_X f \cdot k_i dm.$$

The functions g_i and operator Q have the following properties:

(1) *Functions g_i have disjoint supports (i.e.,* $\operatorname{supp} g_i \cap \operatorname{supp} g_j = \emptyset$ *for all $i \neq j$)*;

(2) *There exists a permutation α on $\{1, \ldots, r\}$ such that $Pg_i = g_{\alpha(i)}$ for $i = 1, \ldots, r$*;

(3) $\|P^n Qf\|_{L^1(m)} \to 0$ *as $n \to \infty$ for any $f \in L^1(X, m)$.*

The property of P in Proposition 3.18 is called `asymptotically periodic`.

Remark 13. A constrictive Markov operator P always admits a finite invariant density of the form

$$h = \frac{1}{r} \sum_{i=1}^{r} g_i.$$

The following result gives the converse of Proposition 3.18.

Proposition 3.19. *Let $(X, \mathscr{B}, m)$ be a probability space and $P : L^1(X, m) \to L^1(X, m)$ be a Markov operator. Then the following are equivalent*:

(*I*) *P is weakly constrictive*;

(*II*) *P is asymptotically periodic, namely the assertion of proposition 3.18 holds*;

(*III*) *P is constrictive*;

(*IV*) *P is smoothing.*

Proof. (I)⇒(II), (IV)⇒(II): It follows from Proposition 3.18.

(II)⇒(III): We show that there exists a compact constrictor for P if the spectral decomposition theorem holds. We will show that the subset of $L^1(X,m)$,

$$\mathcal{F} = \left\{ \sum_{i=1}^{r} a_i g_i : |a_i| \le \max_{1\le j\le r} \left\| k_j \right\|_{L^\infty(m)} \text{ for } i = 1,\cdots,r \right\}$$

is a compact constrictor for P where $\{g_i\}_{i=1}^r$ and $\{k_i\}_{i=1}^r$ are given as in Proposition 3.18. First of all, we show that for any $h \in D(X,m)$, $P^n h$ approaches to $\mathcal{F}$ asymptotically. Indeed, by assumption (II), for each $h \in D(X,m)$,

$$P^{n+1}h = \sum_{i=1}^{r} \left(\int_X h \cdot k_i dm \right) g_{\alpha^n(i)} + P^n Q h$$

and $\left| \int_X h \cdot k_i dm \right| \le \|hk_i\|_{L^1(m)} \le \|k_i\|_{L^\infty(m)}$ hold. Thus it follows from

$$\sum_{i=1}^{r} \left(\int_X h \cdot k_i dm \right) g_{\alpha^n(i)} \in \mathcal{F}$$

that

$$\begin{aligned}\inf_{f\in\mathcal{F}} \left\| P^{n+1}h - f \right\|_{L^1(m)} &\le \inf_{f\in\mathcal{F}} \left\| \sum_{i=1}^{r} \left(\int_X h \cdot k_i dm \right) g_{\alpha^n(i)} - f \right\|_{L^1(m)} + \|P^n Q h\|_{L^1(m)} \\ &= \|P^n Q h\|_{L^1(m)} \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Next we show that the set $\mathcal{F}$ is compact. For each sequence $\{f_n\}_n$ in $\mathcal{F}$, f_n can be written in the form $f_n = \sum_{i=1}^r a_i^{(n)} g_i$ where $\left|a_i^{(n)}\right| \le \max_j \left\|k_j\right\|_{L^\infty(m)}$ for $i = 1,\ldots,r$. Then there exists $\{n_k\}_k$ such that

$$a_i^* := \lim_{k\to\infty} a_i^{(n_k)}$$

for $i = 1,\ldots,r$ and the function $f^* = \sum_{i=1}^r a_i^* g_i$ satisfies $f^* = \lim_{k\to\infty} f_{n_k} \in \mathcal{F}$.

(III)⇒(IV): We assume that there exists a compact constrictor $\mathcal{F}$. We first show $\mathcal{F}_D = \mathcal{F} \cap D(X,m)$ is also a compact constrictor. $\mathcal{F}_D$ is a closed subset of a compact set and hence compact. Since an element of a constrictor belongs to the closure of $\{P^n h\}_n$ for some $h \in D(X,m)$ and $D(X,m)$ is closed, $\mathcal{F}_D$ the restriction of the constrictor $\mathcal{F}$ on $D(X,m)$ is also a constrictor. Then the rest is to show P is smoothing. Since $\mathcal{F}_D$ is compact, $\delta := \inf_{f\in\mathcal{F}_D} m(\operatorname{supp} f)/2 > 0$. Hence, $d := \sup_{A\in\mathscr{B}, m(A)<\delta} \int_A f dm < 1$ holds for any $f \in \mathcal{F}_D$. Therefore, for any $A \in \mathscr{B}$ with $m(A) < \delta$, it holds that for any $h \in D(X,m)$ and $f \in \mathcal{F}_D$,

$$\begin{aligned}\int_A P^n h dm &\le \int_A f dm + \|P^n h - f\|_{L^1(m)} \\ &\le d + \|P^n h - f\|_{L^1(m)}\end{aligned}$$

and $\limsup_{n\to\infty} \int_A P^n h dm < 1$. That is, P is smoothing. □

We recall the definition of mixing and exactness for Markov operators.

Definition 3.20. A Markov operator P is called `mixing` if for any $f \in L^1(X,m)$ with $\int_X f dm = 0$, it holds that

$$\lim_{n\to\infty} \int_X P^n f \cdot g dm = 0 \quad \text{for any } g \in L^\infty(X,m)$$

and is called `exact` if for any $f \in L^1(X,m)$ with $\int_X h dm = 0$, it holds that

$$\lim_{n\to\infty} \|P^n f\|_{L^1(m)} = 0.$$

The following theorem gives a simple sufficient condition for a Markov operator to be constrictive via exactness (which is the generalization of Example 1.3.10 in [21]). Inoue and Ishitani in [32] also showed a similar result only for the Perron–Frobenius operators. We also refer to [35] as a related result. From the following result, in case of an invertible non-singular transformation, the Perron–Frobenius operator cannot be constrictive.

Proposition 3.21. *Let $(X, \mathscr{B}, m)$ be a probability space and $P : L^1(X, m) \to L^1(X, m)$ be a Markov operator. If there exists $k \in \mathbb{N}$ such that P^k has an invariant density and exact, then P is constrictive. Conversely, if P is constrictive then P admits an invariant density and there exists $k \in \mathbb{N}$ such that $P \mid^k_{\mathrm{supp} g_i}$ is exact for $i = 1, \dots, r$.*

Proof. Assume there is $h_0 \in D$ such that $P^k h_0 = h_0$ and P^k is exact. Then $h = \frac{1}{k}\sum_{i=0}^{k-1} P^i h_0$ is an invariant density for P and we will show P is smoothing which is equivalent to constrictive according to Proposition 3.19. For each $f \in D(X, m)$, $A \in \mathscr{B}$ and $N \in \mathbb{N}$,

$$\begin{aligned}\sup_{n \geq kN} \int_A P^n f dm &= \sup_{l \geq N} \left\{ \int_A P^{kl} f dm, \int_A P^{kl+1} f dm, \dots, \int_A P^{kl+k-1} f dm \right\} \\ &\leq \sup_{l \geq N} \left\{ \left\| P^{kl} f - h \right\|_{L^1(m)} + \int_A h dm, \dots, \left\| P^{kl+k-1} f - h \right\|_{L^1(m)} + \int_A h dm \right\}.\end{aligned}$$

since $\{n : n \geq kN\} = \{kl + j : j = 0, \dots, k-1,\ l \geq N\}$. Note that $\left\| P^{kn+j} f - h \right\|_{L^1(m)} \to 0$ as $n \to \infty$ for $j = 0, \dots, k-1$ since $P^j f - h = P^j(f - h)$ is zero-average and P^k is exact. This implies the above supremum is bounded above by 1 for A small enough (with respect to m) and P is smoothing.

Next, conversely, we suppose P is constrictive. We only have to show $P \mid^k_{\mathrm{supp} g_i}$ is exact for some k, $(i = 1, \dots, r)$. It follows from asymptotic periodicity of P and $P g_i = g_{\alpha(i)}$ that, for any $f \in D(X, m)$ supported on $\mathrm{supp} g_i$, $\lambda_{\alpha(i)}(f) = 1$ and $\lambda_j(f) = 0$ for each $j \neq \alpha(i)$. Now let k be the smallest number with $P^k g_i = g_i$ for all $i = 1, \dots, r$. Then we have $P^k f = P^{k-1}(g_{\alpha(i)} + Qf) = g_i + P^{k-1} Q f$ and

$$\lim_{n \to \infty} \left\| P^{kn} f - g_i \right\|_{L^1(m)} \leq \lim_{n \to \infty} \left\| P^{kn-1} Q f \right\|_{L^1(m)} = 0.$$

This means $P \mid^k_{\mathrm{supp} g_i}$ is exact and completes the proof. □

Remark 14. From the above Proposition 3.21, we can say the difference between the decomposition of a constrictive Markov operator and that of a quasi-compact operator. Recall that quasi-compact bounded linear operators on complex Banach spaces (see definition in [16, 18]) also admit the decomposition as in Proposition 3.18. Hence it seems that the spectral decomposition for constrictive Markov operators is just "real" version of the decomposition for quasi-compact operators on complex Banach spaces. However we remark the decomposition of a constrictive Markov operator and that of a quasi-constrictive operator are essentially different. Namely, the decay of $\{P^n Q f\}_n$ in Proposition 3.18 is not always exponentially fast although the decay for a quasi-constrictive operator is always exponentially fast. Indeed, for an intermittent map $T : [0, 1] \to [0, 1]$

$$Tx = \begin{cases} x + 2^\alpha x^{1+\alpha} & x \in [0, 1/2) \\ 2x - 1 & x \in [1/2, 1] \end{cases}$$

with $0 < \alpha < 1$, because T admits a Lebesgue-absolutely continuous finite invariant measure for which T is exact (see [67, 68]), the corresponding Perron–Frobenius operator is constrictive by Proposition 3.21. But the decay of correlation for this T is not exponential as the speed of convergence of the iterated Perron–Frobenius operator is not exponentially fast (see [41, 45, 47]) whereas the decay for quasi-compact operators is always exponentially fast.

The next result gives a relation between Theorem 3.5 (weak almost periodicity) and Proposition 3.18 (the spectral decomposition theorem).

Proposition 3.22. *Let $(X, \mathscr{B}, m)$ be a probability space and $P : L^1(X, m) \to L^1(X, m)$ be a weakly constrictive Markov operator. Then P is almost periodic so that each condition of Theorem 3.5 is valid.*

Proof. We will show that $\{P^n f\}_n$ is precompact for any fixed $f \in L^1(X, m)$. By Proposition 3.18, for $f \in L^1(X, m)$, $P^{n+1} f = \sum_{i=1}^{r} \lambda_i(f) g_{\alpha(i)} + P^n Q f$ and note that $\alpha^{r!}(i) = i$ for $i = 1, \cdots, r$ since α is permutation. Then it holds that for any $\{n_k\}_k \subset \mathbb{N}$ there exists $\{k_j\}_j \subset \mathbb{N}$ such that $n_{k_j} = N_0 + m_j r!$ for some $N_0 \in \mathbb{N}$ and $\{m_j\}_j \subset \mathbb{N}$. Since

$$
\begin{aligned}
P^{n_{k_j}} f &= P^{N_0 + m_j r! - 1} \left(\sum_{i=1}^{r} \lambda_i(f) g_{\alpha(i)} + Q f \right) \\
&= \sum_{i=1}^{r} \lambda_i(f) g_{\alpha^{N_0 - 1}(i)} + P^{n_{k_j} - 1} Q f
\end{aligned}
$$

and the operator Q satisfies that $\lim_{n \to \infty} \|P^n Q f\|_{L^1(m)} = 0$, we have

$$
P^{n_{k_j}} f \to \sum_{i=1}^{r} \lambda_i(f) g_{\alpha^{N_0 - 1}(i)}
$$

strongly as $j \to \infty$. Therefore, P is almost periodic. □

The following proposition shows that in the class of constrictive Markov operator, mixing property is equivalent to exactness. The following proposition was shown in [42] under the assumption that $P 1_X = 1_X$. We prove the following assertion without this assumption.

Proposition 3.23. *Let $(X, \mathscr{B}, m)$ be a probability space and $P : L^1(X, m) \to L^1(X, m)$ be a constrictive Markov operator. Then the following are equivalent*:

(*I*) *P is exact*;

(*II*) *P is mixing*;

(*III*) $r = 1$ *in representation of Proposition 3.18.*

Proof. (I)⇒(II): It is obvious.

(II)⇒(III): We assume that $r > 1$ and $h = \frac{1}{r} \sum_{i=1}^{r} g_i$ is fixed point of P. Set $f = g_1$ and $A_i = \operatorname{supp} g_i$, then

$$
\begin{aligned}
\int_X P^n f 1_{A_1} dm &= \int_X g_{\alpha^n(1)} 1_{A_1} dm \\
&= \begin{cases} 1 & (\alpha^n(1) = 1) \\ 0 & (\alpha^n(1) \neq 1). \end{cases}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\left| \int_X (P^n f - h) 1_{A_1} dm \right| &= \left| \int_X P^n f 1_{A_1} dm - \int_{A_1} h dm \right| \\
&= \begin{cases} \left| 1 - \int_{A_1} h dm \right| & (\alpha^n(1) = 1) \\ \left| \int_{A_1} h dm \right| & (\alpha^n(1) \neq 1) \end{cases} \\
&> 0
\end{aligned}
$$

and this contradicts to mixing property of P since $\int_X (f - f_0)dm = 0$.

(III)⇒(I): Assume that $r = 1$ i.e., for each $f \in L^1(X, m)$, $Pf = \lambda(f)h_0 + Qf$. Since $\lambda(f) = 1$ for each $f \in D(X, m)$, $\int_X fk_0 dm = 1$. Hence, taking $f = 1_A/m(A)$ for any $A \in \mathscr{B}$ with $m(A) > 0$, we have $k_0(x) = 1_X(x)$ almost every $x \in X$. Therefore, for any $f \in L^1(X, m)$ with $\int_X fdm = 0$, it holds that $\lambda(f + h_0) = \int_X (f + h_0)k_0 dm = 1$ and

$$\begin{aligned}
\|P^n f\|_{L^1(m)} &= \|P^n(f + g_0) - g_0\|_{L^1(m)} \\
&= \left\|\lambda(f + g_0)g_0 + P^{n-1}Q(f + g_0) - g_0\right\|_{L^1(m)} \\
&= \left\|P^{n-1}Q(f + g_0)\right\|_{L^1(m)} \\
&\to 0
\end{aligned}$$

as $n \to \infty$. □

We recall the Jacobs–de Leeuw–Glicksberg splitting theorem and we consider the converse of Proposition 3.22. A Markov operator P is called `quasi-constrictive` if the closed subspace

$$\mathcal{X}_0(P) := \left\{f \in L^1(X, m) : \lim_{n\to\infty} \|P^n f\|_{L^1(m)} = 0\right\}$$

has finite codimension. It is easy to verify that a constrictive Markov operator is also quasi-constrictive. We remark relation between constrictive operators and weakly almost periodic operators. It is known that (see [21]) if a quasi-constrictive Markov operator is weakly almost periodic, it is constrictive. For our future work, we ask further equivalent conditions for a Markov operator being constrictive by using Theorem 3.5 and Corollary 3.6.

Example 4 (Example 3, Revised)**.** We revisit additive noisy type random dynamical systems considered in Example 3. According to Theorem 2.8 in [34], the Markov operator P corresponding to this random dynamical system given by the equation (2) is constrictive, for any non-singular transformation T on the unit and any distribution of noise g. Namely, P is asymptotically periodic and in particular weakly almost periodic by Proposition 3.22. Thus, we conclude that this random dynamical system always admits an absolutely continuous finite invariant measure with the maximal condition in the sense of (4).

4 Absolutely continuous σ-finite invariant measures

In this section, we consider σ-finite invariant densities for Markov operators. We use the induced operator and the jump operator for a Markov operator which are the generalization of the induced transformation and the jump transformation respectively. The induced operator and its use for finding a σ-finite invariant measure were suggested first by Halmos in 1947 ([27]) and matured by Foguel in 1969 ([22]). The method of the jump transformation was given in [58] or [67]. In this section, we generalize the jump transformation to the jump operator for a Markov operator. We use these methods for giving equivalent conditions for the existence of a σ-finite invariant density for an eventually conservative Markov operator with nice support condition.

Recall that $\mathcal{M}_+^\sigma$ denotes (the quotient space of) the set of all non-negative measurable functions f over $(X, \mathscr{B}, m)$ such that the measure m_f defined by $m_f(A) = \int_A fdm$ for $A \in \mathscr{B}$ is σ-finite. Then a Markov operator on $L^1(X, m)$ can be naturally extended to the positive linear operator on the space $\mathcal{M}_+^\sigma$ as in Remark 1. Therefore, we can consider a fixed point of a Markov operator in $\mathcal{M}_+^\sigma$ as the density of a σ-finite (infinite) invariant measure.

4.1 Induced operators and jump operators

First of all, we recall the definition of induced transformations.

Definition 4.1. Let T be a non-singular transformation on a probability space $(X, \mathscr{B}, m)$. Assume that there exists a set $E \in \mathscr{B}$ of positive measure such that $\bigcup_{n\geq 0} T^{-n}E = X \pmod m$. Then the `hitting time` (or the `first return time` if $x \in E$) of $A \in \mathscr{B}$, φ_E, is defined m-almost everywhere by $\varphi_E(x) = \min\{n \in \mathbb{N} : T^n x \in E\}$ and we define the `induced transformation` on E, $T_E : X \to E$, by $T_E x := T^{\varphi_E(x)} x$ (which is defined m-almost everywhere).

Remark 15. The conventional definition of the induced transformation is given by $T_E|_E : E \to E$. But for our purpose of having the maximal support property of the invariant density, the induced transformation should be defined almost everywhere.

The generalization of the induced transformation for a point transformation given in [22, 23] is the following.

Definition 4.2. Let $(X, \mathscr{B}, m)$ be a probability space and $P : L^1(X, m) \to L^1(X, m)$ be a Markov operator. We define the `induced operator` (by P) on $E \in \mathscr{B}$ by

$$P_E = (I_E P) \sum_{n=0}^{\infty} (I_{X\setminus E} P)^n,$$

or the adjoint operator of P_E by

$$P_E^* = \sum_{n=0}^{\infty} (P^* I_{X\setminus E})^n (P^* I_E)$$

where I_E is the restriction operator on E: $I_E f = 1_E f$ for any measurable function f.

Next, we recall the definition of jump transformations given in [58, 67].

Definition 4.3. Let T be a non-singular transformation on a probability space $(X, \mathscr{B}, m)$. Assume that there exists a set $E \in \mathscr{B}$ of positive measure such that $\bigcup_{n\geq 0} T^{-n}E = X \pmod m$. Then the `first entry time` e is defined m-almost everywhere by $e(x) = \min\{n \geq 0 : T^n x \in E\}$ and we define the `jump transformation` with respect to E, $T^* : X \to X$, by $T^* x = T^{e(x)+1} x$ (which is defined m-almost everywhere).

Now we define the generalization of the jump transformation as follows.

Definition 4.4. Let $(X, \mathscr{B}, m)$ be a probability space and $P : L^1(X, m) \to L^1(X, m)$ be a Markov operator. We define `the jump operator` (by P with respect to $E \in \mathscr{B}$) by

$$\widehat{P} = \widehat{P}_E = (P I_E) \sum_{n=0}^{\infty} (P I_{X\setminus E})^n,$$

or the adjoint operator of $\widehat{P}$ by

$$\widehat{P}^* = \widehat{P}_E^* = \sum_{n=0}^{\infty} (I_{X\setminus E} P^*)^n (I_E P^*).$$

Proposition 4.5. *Let P be the Perron–Frobenius operator corresponding to a non-singular transformation T on a probability space $(X, \mathscr{B}, m)$. Then the Perron–Frobenius operator corresponding to the induced transformation T_E is the induced operator P_E.*

Proof. Note that $T_E^{-1}A = \bigcup_{n=1}^{\infty} (\{x \in X : \varphi_E(x) = n\} \cap T^{-n}A) \pmod m$ and the Perron–Frobenius operator P_E corresponding to T_E satisfies for any $A \in \mathscr{B}$ and $f \in L^1(X, m)$,

$$\int_A P_E f dm = \int_{T_E^{-1}A} f dm = \sum_{n=1}^{\infty} \int_{\{x \in X : \varphi_E(x) = n\} \bigcap T^{-n}A} f dm.$$

Since

$$\{x \in X : \varphi_E = n\} = \begin{cases} T^{-1}E & (n = 1) \\ T^{-n}E \bigcap \bigcap_{i=1}^{n-1} T^{-i}(X \setminus E) & (n \geq 2), \end{cases}$$

it holds that

$$\int_A P_E f dm = \int_{T^{-1}(A \cap E)} f dm + \sum_{n=2}^{\infty} \int_{T^{-n}(A \cap E) \cap \bigcap_{i=1}^{n-1} T^{-i}(X \setminus E)} f dm.$$

One can see that for $n \geq 2$,

$$\begin{aligned} \int_{T^{-n}(A \cap E) \cap \bigcap_{i=1}^{n-1} T^{-i}(X \setminus E)} f dm &= \int_{T^{-n+1}(A \cap E) \cap \bigcap_{i=1}^{n-2} T^{-i}(X \setminus E)} I_{X \setminus E} P f dm \\ &= \cdots = \int_{T^{-1}(A \cap E)} (I_{X \setminus E} P)^{n-1} f dm \\ &= \int_A I_E P (I_{X \setminus E} P)^{n-1} f dm \end{aligned}$$

and

$$\begin{aligned} \int_A P_E f dm &= \int_A I_E P f dm + \sum_{n=2}^{\infty} \int_A I_E P (1_{X \setminus E} P)^{n-1} f dm \\ &= \sum_{n=0}^{\infty} \int_A (I_E P)(I_{X \setminus E} P)^n f dm. \end{aligned}$$

This equality and the monotone convergence theorem imply our assertion. □

By the same way of proving the above proposition, we have the following proposition about the relation of the jump operator and the jump transformation.

Proposition 4.6. *Let P be the Perron–Frobenius operator corresponding to a non-singular transformation T on a probability space $(X, \mathscr{B}, m)$. Then the Perron–Frobenius operator corresponding to the jump transformation T^* with respect to E is the jump operator with respect to E.*

Even if we do not have any non-singular transformation, under proper assumptions, we may still obtain a well-defined induced operator or jump operator as a Markov operator on $L^1(X, m)$. The following lemma is given in [22] (Lemma B in Chapter VI) only for the induced operator but obviously we have the same statement for the jump operator and we omit the proof.

Lemma 4.7. *Let $(X, \mathscr{B}, m)$ be a probability space and $P : L^1(X, m) \to L^1(X, m)$ be a Markov operator. For $E \in \mathscr{B}$, if $\lim_{n\to\infty} (P^* I_{X \setminus E})^n 1_X = 0$, then P_E the induced operator and $\widehat{P}_E$ the jump operator with respect to E are well-defined Markov operators on $L^1(X, m)$.*

From the above lemma, we naturally give the following definition of sweep-out sets. See [61] for more precise properties of sweep-out sets.

Definition 4.8. A set $E \in \mathscr{B}$ is called a (P-)`sweep-out set` (with respect to m) if $P_E^* 1_X(x) = 1$ for m-almost every $x \in X$ or equivalently, P_E and $\widehat{P_E}$ are well-defined Markov operators on $L^1(X, m)$.

Remark 16. (i) A set E being a sweep-out set means that almost all points will visit E sooner or later under the process P. It is obvious that E is a sweep-out set if and only if it holds

$$\lim_{n\to\infty} (P^* I_{X\setminus E})^n 1_X = 0$$

or equivalently

$$\lim_{n\to\infty} (I_{X\setminus E} P^*)^n 1_X = 0.$$

(ii) By the consequence of Proposition 3.5 in [61], if the system (P, m) is conservative and ergodic, then any set of positive measure is a sweep-out set.

Now, recall the Hopf decomposition theorem (Theorem 2.6) and the definition of the conservative part $\mathfrak{C}$ and the dissipative part $\mathfrak{D}$ (Definition 2.7) for a given Markov operator over a probability space. One can see other characterization of the conservative part and the dissipative part in [22, 37], that is, for any $g \in L_+^\infty(X, m)$, we have

$$\sum_{n=0}^{\infty} P^{*n} g = 0 \ \text{ or } \ \infty$$

on $\mathfrak{C}$ and there exists $g_{\mathfrak{D}} \in L_+^\infty(X, m)$ such that $\operatorname{supp} g_{\mathfrak{D}} = \mathfrak{D}$ and

$$\sum_{n=0}^{\infty} P^{*n} g_{\mathfrak{D}} \leq 1.$$

Further it holds that $P^* 1_{\mathfrak{C}} \geq 1_{\mathfrak{C}}$ and $P^* 1_{\mathfrak{D}} \leq 1_{\mathfrak{D}}$. Therefore, we have the following decomposition of X (mod m):

$$X = \left\{ x \in X : \lim_{n\to\infty} P^{*n} 1_{\mathfrak{C}}(x) > 0 \right\} \cup \left\{ x \in X : \lim_{n\to\infty} P^{*n} 1_{\mathfrak{C}}(x) = 0 \right\}.$$

Then we are in a position to give a new definition of eventual conservativeness for a system, which makes the definition of conservativeness and dissipation slightly finer.

Definition 4.9. For a Markov operator P over $L^1(X, m)$, we call the system (P, m) `eventually conservative` if $\lim_{n\to\infty} P^{*n} 1_{\mathfrak{C}}(x) = 1$ m-almost every $x \in X$.

Remark 17. It is obvious by the definition that if a system is ergodic and not totally dissipative, then the system is eventually conservative.

4.2 The existence of absolutely continuous σ-finite invariant measures

Now, we establish our main result in this thesis on the existence of a σ-finite invariant density for a Markov operator.

Theorem 4.10. *Let $(X, \mathscr{B}, m)$ be a probability space and $P : L^1(X, m) \to L^1(X, m)$ be a Markov operator. Then the following are equivalent:*

(I) *There exists a σ-finite invariant density $h \in \mathcal{M}_+^\sigma$ for P and there exists a sweep-out set $E \subset$* $\operatorname{supp} h \cap \mathfrak{C}$ *such that $\int_E hdm < \infty$*;

(II) *There exists a sweep-out set $E \in \mathscr{B}$ such that P_E the induced operator on E admits a finite invariant density $h_0 \in L_+^1(X, m)$ whose support* $\operatorname{supp} h_0$ *is a sweep-out set*;

(III) *There exists a sweep-out set $E \in \mathscr{B}$ such that P_E admits a finite invariant density $h_1 \in L_+^1(X, m)$ with* $\operatorname{supp} h_1 = E$;

(IV) *There exists a sweep-out set $E \in \mathscr{B}$ such that $\widehat{P}$ the jump operator with respect to E is weakly almost periodic.*

Before proving Theorem 4.10, we give some remarks and immediate corollary.

Remark 18. (i) We can apply Theorem 3.5 to the condition (IV) in Theorem 4.10 via the jump operator. Further, if $E = X$ in the condition (IV), then P is weakly almost periodic and h is a finite invariant density for P in the condition (I). Hence Theorem 3.5 is a special case of Theorem 4.10 with $A = \operatorname{supp} h$ in the condition (I).

(ii) The condition (I) implies that the system is eventually conservative. Indeed $\mathfrak{D} \subset X \setminus A$ and we have

$$\lim_{n\to\infty} P^{*n} 1_{\mathfrak{D}}(x) \le \lim_{n\to\infty} (P^* I_{X\setminus A})^n 1_X(x) = 0.$$

Further, from Lemma 3.8, the condition (I) implies that $\lim_{n\to\infty} P^{*n} 1_{X\setminus \operatorname{supp} h} = 0$ holds so that P has a σ-finite invariant density in $\mathcal{M}_+^\sigma$ with the maximal support.

In particular, if we consider the Perron–Frobenius operator corresponding to a non-singular transformation then the following corollary is valid.

Corollary 4.11. *Let $(X, \mathscr{B}, m)$ be a probability space and $T : X \to X$ be a non-singular transformation. Then the following are equivalent*:

(I) *There exist an absolutely continuous σ-finite T-invariant measure μ and $A \subset$* $\operatorname{supp} \mu \cap \mathfrak{C}$ *with $\mu(A) < \infty$ and $\bigcup_{n\ge1} T^{-n} A = X \pmod{m}$*;

(II) *There exists a set $E \in \mathscr{B}$ such that T_E the induced transformation on E admits an absolutely continuous finite T_E-invariant measure μ_E which is equivalent to $m|_E$*;

(III) *There exists a set $E \in \mathscr{B}$ such that T^* the jump transformation with respect to E admits an absolutely continuous finite T^*-invariant measure ν such that $\bigcup_{n\ge0} (T^*)^{-n} (\operatorname{supp} \nu) = X \pmod{m}$.*

Remark 19. If one would like to use the usual definition of the induced transformation $T_E|_E$, one has an equivalent condition for Corollary 4.11 as

(II') There exists an equivalent finite invariant measure for the induced system $(E, \mathscr{B} \cap E, m|_E, T_E|_E)$ for some $E \in \mathscr{B}$ with $\bigcup_{n\ge0} T^{-n} E = X \pmod{m}$.

To prove Theorem 4.10, we prepare a sequence of lemmas.

Lemma 4.12. *If there exists $h^* \in L_+^1(X, m)$ a fixed point of P, then* $\operatorname{supp} h^* \subset \mathfrak{C}$.

Proof. The conservative part $\mathfrak{C}$ is characterized as $\mathfrak{C} = \left\{x \in X : \sum_{n=0}^\infty P^n u(x) = \infty\right\}$ where u is an arbitrary strictly positive L^1-function. Taking $u = h^* + 1_{X\setminus \operatorname{supp} h^*}$ and for any $A \subset \operatorname{supp} h^*$ of positive m-measure, it holds that

$$\int_A P^n u dm = \int_A \left(h^* + P^n 1_{X\setminus \operatorname{supp} h^*}\right) dm \ge \int_A h^* dm > 0$$

for each $n \geq 0$. Hence, $\sum_{n=0}^{\infty} P^n u(x) = \infty$ for m-almost every $x \in A$ and $A \subset \mathfrak{C}$. This implies $\operatorname{supp} h^* \subset \mathfrak{C}$ (mod m). □

The following key lemma for the Theorem 4.10 was proven in [22] with the assumption that the whole space is equal to the conservative part $\mathfrak{C}$ modulo m and a Markov operator is ergodic. We replace this assumption by weaker one but the proof is essentially same as in [22] and we omit it.

Lemma 4.13. *Assume that there exists $h^* \in L^1_+(X, m)$ such that $P_E h^* = h^*$ for P_E the induced operator for some sweep-out set E. Then there exists $h \in \mathcal{M}^{\sigma}_+$ such that $Ph = h$ given by*

$$h = \sum_{n=0}^{\infty} (I_{X\setminus E} P)^n h^*. \tag{12}$$

We also have the following formula of the invariant density via the jump operator. The proof is almost same as Lemma 4.13 and omitted.

Lemma 4.14. *Assume that there exists $h^* \in L^1_+$ such that $\widehat{P} h^* = h^*$ for $\widehat{P}$ the jump operator for some sweep-out set E. Then there exists $h \in \mathcal{M}^{\sigma}_+$ such that $Ph = h$ given by*

$$h = \sum_{n=0}^{\infty} (P I_{X\setminus E})^n h^*. \tag{13}$$

Proof of Theorem 4.10. (I)⇒(III): Suppose (I) holds and let $E \subset \operatorname{supp} h$ be the set as in the condition (I). Since $\lim_{n\to\infty} \left\| (P^* I_{X\setminus E})^n 1_X \right\|_{L^\infty(m)} = 0$, P_E is a Markov operator. Then we show that P_E has a finite invariant density $h 1_E \in L^1_+(X, m)$ with the maximal support. Write for any $B \in \mathscr{B}$ with $\int_B h dm < \infty$,

$$\begin{aligned}
\int_B P_E(h 1_E) dm &= \sum_{n=0}^{\infty} \int_B I_E P (I_{X\setminus E} P)^n I_E h dm \\
&= \sum_{n=0}^{\infty} \left(\int_B I_E (P I_{X\setminus E})^n h dm - \int_B I_E (P I_{X\setminus E})^{n+1} h dm \right) \\
&= \int_B I_E h dm - \lim_{n\to\infty} \int_B I_E (P I_{X\setminus E})^n h dm \\
&= \int_B h 1_E dm - \lim_{n\to\infty} \int_X h I_{X\setminus E} (P^* I_{X\setminus E})^{n-1} P^* 1_{E\cap B} dm \\
&= \int_B h 1_E dm
\end{aligned}$$

since $\int_B I_E (P I_{X\setminus E})^n h dm \leq \int_E h dm < \infty$ and $\lim_{n\to\infty} (P^* I_{X\setminus E})^n 1_X = 0$. Therefore, we have $P_E(h 1_E) = h 1_E$ and the support of this invariant density for P_E equals to E.

(III)⇒(II): It is obvious.

(II)⇒(I): Assume that P_E the induced operator is a Markov operator for some $E \in \mathscr{B}$ and P_E has an invariant density h^* with $\lim_{n\to\infty} (P^* I_{X\setminus \operatorname{supp} h^*})^n 1_X = 0$. From the equation (12) in Lemma 4.13, P has a σ-finite invariant density $h \in \mathcal{M}^{\sigma}_+$. Set $A = \operatorname{supp} h^*$. Then $A \subset \operatorname{supp} h \cap \mathfrak{C}$ (see the conservative part of the induced operator for [61]) and it follows from $A \subset E$ that

$$\begin{aligned}
\int_A h dm = \int_A P h dm &= \int_A P \sum_{n=0}^{\infty} (I_{X\setminus E} P)^n h^* dm \\
&\leq \int_X I_E P \sum_{n=0}^{\infty} (I_{X\setminus E} P)^n h^* dm = \int_X h^* dm \\
&< \infty.
\end{aligned}$$

(III)⇒(IV): We recall that the condition

(IV') For $\widehat{P}$, there exists $h_1 \in L^1_+(X,m)$ such that $\widehat{P}h_1 = h_1$ and $\lim_{n\to\infty} \widehat{P}^{*n} 1_{X\setminus \operatorname{supp} h_1}(x) = 0$ for m-almost every $x \in X$.

is one of the equivalent condition for the condition (IV) by Theorem 3.5 and so we prove the implication (III) to (IV'). We note that by definition of the induced operator and the jump operator it holds $\widehat{P}P = PP_E$. Then if $P_E h_0 = h_0$ for some $h_0 \in L^1_+(X,m)$ we have also $\widehat{P}(Ph_0) = Ph_0$. Now we assume $\operatorname{supp} h_0 = E$ as in the condition (III) and write

$$\widehat{P}^* 1_{X\setminus \operatorname{supp} Ph_0} = \sum_{n=0}^{\infty} (I_{X\setminus E} P^*)^n I_E P^* 1_{X\setminus \operatorname{supp} Ph_0} \le \sum_{n=0}^{\infty} (I_{X\setminus E} P^*)^n I_E 1_{X\setminus \operatorname{supp} h_0} = 0$$

by using Lemma 3.8. This means weak almost periodicity of $\widehat{P}$.

(IV)⇒(I): Since $\widehat{P}$ is weakly almost periodic, there exists $h_1 \in L^1_+(X,m)$ a finite invariant density of $\widehat{P}$ such that $\lim_{n\to\infty} \widehat{P}^{*n} 1_{X\setminus \operatorname{supp} h_1} = 0$. Put

$$h = \sum_{n=0}^{\infty} (PI_{X\setminus E})^n h_1$$

and this is a σ-finite invariant density for P in $\mathcal{M}^{\sigma}_{+}$ as in Lemma 4.14 and the above proof of the implication (I)⇒(II). Hence the rest of the proof is to show $E \cap \operatorname{supp} h$ is a sweep-out set, namely, $P^*_{E\cap \operatorname{supp} h} 1_X = 1_X$. Proposition 2.2 in [61] implies that

$$\begin{aligned} P^*_{E\cap \operatorname{supp} h} 1_X &= \sum_{n=0}^{\infty} \left(P^* I_{X\setminus(E\cap \operatorname{supp} h)}\right)^n P^* I_{E\cap \operatorname{supp} h} 1_X \\ &= \sum_{k=0}^{\infty} \left(\sum_{l=0}^{\infty} (P^* I_{X\setminus E})^l P^* I_{E\cap(X\setminus \operatorname{supp} h)} \right)^k \sum_{l=0}^{\infty} (P^* I_{X\setminus E})^l P^* I_E (I_{\operatorname{supp} h} 1_X) \\ &= \sum_{k=0}^{\infty} \left(P^*_E I_{X\setminus \operatorname{supp} h}\right)^k P^*_E I_{\operatorname{supp} h} 1_X. \end{aligned}$$

The last term equals to 1_X if and only if $\lim_{n\to\infty} (P^*_E I_{X\setminus \operatorname{supp} h})^n 1_X = 0$. From weak almost periodicity of $\widehat{P}$, $P^*\widehat{P}^* = P^*_E P^*$, and $P^* 1_{X\setminus \operatorname{supp} h} \le 1_{X\setminus \operatorname{supp} h}$, we have

$$\left\| \left(P^*_E I_{X\setminus \operatorname{supp} h}\right)^{n+1} 1_X \right\|_{L^\infty(m)} \le \left\| (P^*_E)^n P^* 1_{X\setminus \operatorname{supp} h} \right\|_{L^\infty(m)} \le \left\| P^* \widehat{P}^{*n} 1_{X\setminus \operatorname{supp} h_1} \right\|_{L^\infty(m)} \to 0$$

as $n \to \infty$. Therefore, $E \cap \operatorname{supp} h$ is a sweep-out set and the proof is completed. □

Remark 20. By the proof of implication (I) to (II), we can also add the following equivalent condition to Theorem 4.10:

(I') There exist $h \in \mathcal{M}^{\sigma}_{+}$ a σ-finite invariant density for P and $A \subset \operatorname{supp} h$ a sweep-out set such that $\int_A h dm < \infty$.

This means that for a σ-finite invariant density h, if there exists a sweep-out set of finite measure with respect to this invariant density contained in $\operatorname{supp} h$, then the system (P, m) is eventually conservative. Hence it holds

$$\int_{\mathfrak{D}} h dm = \int_X h \cdot P^{*n} 1_{\mathfrak{D}} dm \to 0$$

as $n \to \infty$ and $\operatorname{supp} h \subset \mathfrak{C} \pmod m$. That is, the existence of a sweep-out set of finite measure contained in the support of a σ-finite invariant density is necessary for the system being eventually conservative.

4.3 Ergodic structure of random iterations of weakly expanding transformations

In this subsection we investigate some ergodic properties of Markov operators through the induced operators or the jump operators. As an application, we show some ergodic property for random iterations of non-uniformly expanding maps. For uniformly expanding maps on average, Pelikan studied the number of ergodic components in [53]. We also refer to [24, 48] for ergodic properties of random dynamics.

The following proposition reveals the relation between the sub-σ-algebras of invariant sets for a Markov operator and its induced operator or jump operator. For deterministic case (i.e., for a non-singular transformation), the following results were already known in [3, 58].

Proposition 4.15. *For a Markov operator P over $L^1(X, m)$, let P_E and $\widehat{P}_E$ be the induced operator and the jump operator with respect to some sweep-out set $E \in \mathscr{B}_+$. Then the following relations on Σ_i the sub-σ-algebras of invariant sets defined in Definition 2.8 are true:*

$$\Sigma_i(P^*) \subset \Sigma_i(P_E^*) \pmod m \quad \text{and} \quad \Sigma_i(P^*) \subset \Sigma_i(\widehat{P}_E^*) \pmod m.$$

Consequently, if P_E or $\widehat{P}_E$ is ergodic with respect to m then (P, m) is ergodic.

Proof. Let $f \in L^\infty(X, m)$ such that $P^* f = f$. Write $P^* I_E f = f - P^* I_{X\setminus E} f$ and

$$P_E^* f = \sum_{n\geq 0} (P^* I_{X\setminus E})^n P^* I_E f = \sum_{n\geq 0} (P^* I_{X\setminus E})^n (f - P^* I_{X\setminus E} f) = f$$

since $\lim_{n\to\infty}(P^* I_{X\setminus E})^n f = 0$. To show $\widehat{P}_E^* f = f$ for any $f = P^* f$ is almost same way as above and we omit it. □

From now on, we focus on random iterations of maps on the unit interval $X = [0, 1]$ which satisfy expanding property on average except finitely many indifferent fixed points in the following sense.

Definition 4.16. A random iteration $\{T_i, dp(i) : i \in I\}$ is said to satisfy `expanding property on average except finitely many indifferent fixed points` if

(1) each T_i is a piecewise C^1-invertible, namely, there exist at most countable partitions into subintervals $\{X_{n,i}\}_n$ such that $T_i |_{X_{n,i}} : X_{n,i} \to T X_{n,i}$ is a C^1 and invertible map and

(2) there are finitely many common fixed points $x_1, \ldots, x_N$ for all T_i with $\left|T_i'(x_n)\right| = 1$ such that

$$\sup_{x\in X\setminus U} \int_I \frac{1}{|T_i'(x)|} dp(i) < 1 \tag{14}$$

(whenever derivative can be defined) where $U = \bigcup_{n=1}^N U_n$ for U_n any small open neighborhood of x_n. Note that x_i does not necessarily belong to $X_{n,i}$.

Remark 21. (i) We recall that the definition of expanding property on average. Under the setting of (1) in Definition 4.16, a random iteration $\{T_i, dp(i) : i \in I\}$ is said to satisfy expanding property on average if there is no common indifferent fixed point in Definition 4.16 and the inequality (14) holds where the supremum is taken over X the whole space. For a random iteration satisfying expanding property on average, the existence of an absolutely continuous finite invariant measure and many statistical properties were well-studied (see [7, 17, 24, 25, 53, 30]).

(ii) We also remark that for a random iteration which satisfies expanding property on average except an indifferent fixed point, sufficient conditions for the existence of an absolutely continuous σ-finite invariant measure and further statistical properties were studied. For example, we refer to [9, 10, 11, 31, 51, 52].

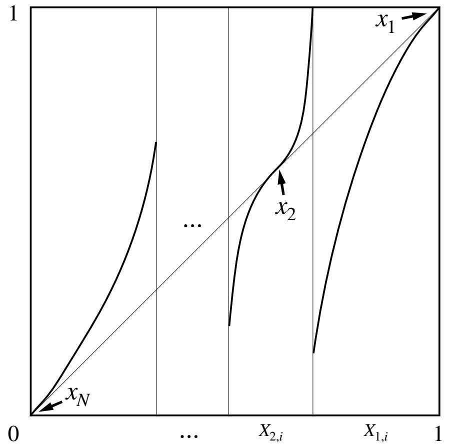

Figure 1: A graph of T_i for some $i \in I$ in Definition 4.16

For a random iteration satisfying expanding property on average except indifferent fixed points in the sense of (14), we can show that the sub-σ-algebra of ergodic components is purely atomic and has only finitely many atoms under certain assumptions.

Using Proposition 4.15, we give sufficient conditions for that random iterations of maps satisfying expanding property on average except indifferent fixed points have at most finitely many ergodic components. The existence of an absolutely continuous σ-finite invariant measure for the setting in the following Corollary was already shown in [31] via the induced system. However, in the following Corollary, the alternative proof by the jump operator will be shown.

Corollary 4.17. *Let $(I, \mathscr{F}, p)$ be a probability space and $\{T_i, dp(i) : i \in I\}$ be a random iteration of non-singular transformations on $[0,1]$ with Lebesgue measure λ, which are piecewise C^1 with (possibly countable) partition $\{X_{n,i}\}_n$ for each $i \in I$. Suppose that*

- (i) *$\{T_i, dp(i) : i \in I\}$ satisfies expanding property on average except finitely many indifferent fixed points $x_1, \ldots, x_N$;*
- (ii) *The average reciprocal of derivative is of bounded variation: there exists a constant $M > 0$ such that for each $i \in I$, $\bigvee_{[0,1]} g_i < M$ holds where*

$$g_i(x) = \begin{cases} \frac{1}{|T_i'|} & x \in \bigcup_n \mathrm{int}(X_{n,i}), \\ 0 & x \in X \setminus \bigcup_n \mathrm{int}(X_{n,i}); \end{cases}$$

- (iii) *There exist open neighborhoods of $x_1, \ldots, x_N$, $\{U_n\}_n$, such that $T_i \mid_{\bigcup U_n} = T_j \mid_{\bigcup U_n}$ for any $i, j \in I$;*
- (iv) *$E = [0,1] \setminus \bigcup_n U_n$ is a P-sweep-out set with respect to λ where P is the Markov operator corresponding to the random iteration $\{T_i, dp(i) : i \in I\}$.*

Then P admits an absolutely continuous σ-finite invariant measure and has at most finitely many ergodic components.

Proof. For a Markov operator P corresponding to the random iteration $\{T_i, dp(i) : i \in I\}$, we firstly show that the jump operator $\widehat{P_E}$ with respect to E corresponds to the random iteration $\{T_i^*, dp(i) : i \in I\}$ where

T_i^* denotes the jump transformation with respect to E. Indeed, by using the Perron–Frobenius operator P_i corresponding to T_i, for each $f \in L^1(X, m)$,

$$\widehat{P}_E f(x) = PI_E \sum_{n\geq 0} (PI_{X\setminus E})^n f(x) = \int_I P_i I_E \sum_{n\geq 0} (PI_{X\setminus E})^n f(x) dp(i) = \int_I \widehat{P}_i f(x) dp(i)$$

where $\widehat{P}_i$ is the Perron–Frobenius operator associated to the jump transformation T_i^*. The last equality follows from $T_i\mid_{\bigcup U_n} = T_j\mid_{\bigcup U_n}$ for $i, j \in I$. Note that $\{T_i^*, dp(i) : i \in I\}$ satisfies expanding on average:

$$\sup_{x\in[0,1]} \int_I \frac{1}{\left|T_i^{*\prime}(x)\right|} dp(i) < 1.$$

It also follows from expanding property of $\{T_i^* : i \in I\}$ on $\bigcup_n U_n$ and the assumption (ii) that the average reciprocal of derivative of the jump system is of bounded variation. Thus, the result of [30] and Ionescu-Tulcea and Marinescu theorem (see also [7, 31]) implies that $\widehat{P}_E$ is quasi-compact or asymptotically periodic. Hence the jump operator $\widehat{P}_E$ admits an absolutely continuous finite invariant measure and the original operator P has an absolutely continuous σ-finite invariant measure by Theorem 4.10. Also, this implies $\Sigma_i(\widehat{P}_E^*)$ consists of only finitely many atoms and the proof is completed by Proposition 4.15. □

Example 5. Let I be a closed interval and $\alpha > 0$. Define $\{T_i : i \in I\}$ on the unit interval (cf. LSV map defined in [45]) given by

$$T_i x = \begin{cases} x + 2^\alpha x^{1+\alpha} & x \in [0, 1/2] \\ f_i & x \in (1/2, 1] \end{cases}$$

where $f_i : (1/2, 1] \to f_i((1/2, 1])$ is a C^1-invertible map with $|f'| > 1$ uniformly and f' is monotone. Then, from Corollary 4.17 for any probability measure p on I, the random iteration $\{T_i, dp(i) : i \in I\}$ admits an absolutely continuous σ-finite invariant measure and has at most finitely many ergodic components with respect to Lebesgue measure on $[0, 1]$.

5 Random iterations of weakly expanding maps with uniformly contractive parts

In this section, apart from the last subsection 4.3, we focus on random iterations which do not satisfy expanding property on average except indifferent fixed points. Namely, in this section we deal with random iterations which admit both the common indifferent fixed point and uniformly contractive part. We will define our model of random iterations which do not satisfy expanding property on average except an indifferent fixed point in the sense of (14) and study its absolutely continuous σ-finite invariant measure. We will show the existence of an absolutely continuous σ-finite invariant measure for our model and give the criterion of the invariant measure to be a finite or infinite measure. Throughout this section the phase space is $X = [0, 1]$, $\mathscr{B}$ is the σ-algebra of Borel sets and λ is Lebesgue measure.

5.1 The definition of the model

We give the concrete definition of our model in this subsection. Set arbitrary $\alpha > 0$. We define the partition of $X = [0, 1]$ into $\mathscr{Q}_\alpha = \left\{X_n = \left((n+1)^{-1/\alpha}, n^{-1/\alpha}\right]\right\}_{n\geq 1}$. Then we have $\lambda(X_n)/\lambda(X_{n+1})$ monotonically decrease to 1 as $n \to \infty$ where λ is Lebesgue measure on X. Let I be a countable set or a closed interval in $\mathbb{R}$ and for each $i \in I$, J_i be a (non-empty) subset of $\mathbb{N}$. We consider a family of transformations $\{T_i : i \in I\}$ on X (see Figure 2 below for an example) which are piecewise monotone and piecewise linear on the partition $\mathscr{Q}_\alpha$ satisfying (a-1) For each $i \in I$, $T_i|_{X_n} : X_n \to X_{n-1}$, for any $n \geq 2$, monotonically increasing given by

$$T_i|_{X_n}(x) = \frac{(n-1)^{-1/\alpha} - n^{-1/\alpha}}{n^{-1/\alpha} - (n+1)^{-1/\alpha}} x - \frac{(n+1)^{-1/\alpha}(n-1)^{-1/\alpha} - n^{-2/\alpha}}{n^{-1/\alpha} - (n+1)^{-1/\alpha}},$$

namely, all T_i are identical on $X \setminus X_1$;

(a-2) $T_i|_{X_1} : X_1 \to \bigcup_{k \in J_i} X_k$, monotonically increasing and surjective which is piecewise linear in the sense

$$T_i'|_{X_1} = \frac{\sum_{k \in J_i} \lambda(X_k)}{\lambda(X_1)}$$

whenever the derivative can be defined, for each $i \in I$.

We call a family of transformations $\{T_i : i \in I\}$ with the above conditions (a-1) and (a-2) `piecewise linear intermittent Markov maps` (with the index $\{J_i\}_{i \in I}$).

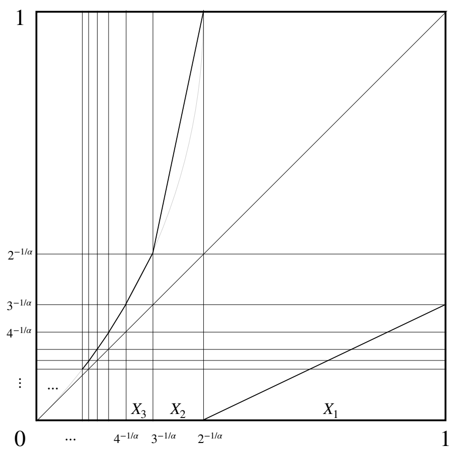

Figure 2: A graph of T_i for some $i \in I$ of piecewise linear intermittent Markov maps

Then, for a given Borel probability measure p on I, we consider a `random iteration of piecewise linear intermittent Markov maps with uniformly contractive part` such that

(b-1) $\{T_i : i \in I\}$ is piecewise linear intermittent Markov maps and

(b-2) on X_1, the random iteration $\{T_i, dp(i) : i \in I\}$ is uniformly contractive on average:

$$\int_I \frac{1}{\left|T_i'|_{X_1}\right|} dp(i) = \int_I \frac{\lambda(X_1)}{\sum_{k \in J_i} \lambda(X_k)} dp(i) > 1.$$

By definition, random iterations of piecewise linear intermittent Markov maps with uniformly contractive part do not satisfy expanding on average in the sense (14).

5.2 The existence and estimation of σ-finite invariant densities

In this subsection, we show the existence of an absolutely continuous σ-finite invariant measure for any random iteration of piecewise linear intermittent Markov maps with uniformly contractive part defined in the last subsection. For weakly expanding case in the sense of (14), sufficient conditions for the existence of an absolutely continuous σ-finite invariant measure was already shown in [31] via the inducing scheme. The main result in this section is as follows.

Theorem 5.1. *Any random iteration $\{T_i, dp(i) : i \in I\}$ of piecewise linear intermittent Markov maps with uniformly contractive part, which satisfies (b-1) and (b-2), admits an absolutely continuous σ-finite invariant measure.*

Proof. We first note that the jump operator with respect to X_1 represents the random iteration of $\{T_i^*, dp(i) : i \in I\}$ where each T_i^* is the jump transformation of T_i with respect to X_1. This follows from exactly same manner as in the proof of Corollary 4.17. By the properties (a-1) and (a-2) for each T_i, we have

$$T_i^* : [0,1] \to \bigcup_{k\in J_i} X_k \text{ satisfying } T_i^{*\prime} \mid_{X_n} = \frac{\sum_{j\in J_i} \lambda(X_j)}{\lambda(X_n)}.$$

Using $\phi_n^{(i)} : \bigcup_{j\in J_i} X_j \to X_n$ the inverse branch of $T_i^* \mid_{X_n}$ for $i \in I$, the Perron–Frobenius operator $\widehat{P}_i$ corresponding to the jump transformation T_i^* with respect to X_1 is given by

$$\widehat{P}_i f(x) = \sum_{\substack{n\geq 1 \\ x\in\bigcup_{j\in J_i} X_j}} f\left(\phi_n^{(i)} x\right) \frac{\lambda(X_n)}{\sum_{j\in J_i} \lambda(X_j)}. \tag{15}$$

Next we show that the Markov operator $\widehat{P}_{X_1}$ corresponding to the random iteration $\{T_i^*, dp(i) : i \in I\}$ obtained above is weakly almost periodic. By using the formula of (15), we have

$$\widehat{P}_{X_1} 1_X = \int_I \sum_{n\geq 1} 1_{\bigcup_{j\in J_i} X_j} \frac{\lambda(X_n)}{\sum_{j\in J_i} \lambda(X_j)} dp(i) = \int_I \frac{1_{\bigcup_{j\in J_i} X_j}}{\sum_{j\in J_i} \lambda(X_j)} dp(i)$$

and

$$\widehat{P}_k 1_{\bigcup_{j\in J_i} X_j} = \sum_{j\in J_i} \widehat{P}_k 1_{X_j} = \frac{\sum_{j\in J_i} \lambda(X_j)}{\sum_{l\in J_k} \lambda(X_l)} 1_{\bigcup_{s\in J_k} X_s}.$$

Hence, by commuting $\widehat{P}_{X_1}$ with integral in Bochner integral (for Bochner integration theory, see [75] for instance), we have

$$\begin{aligned}
\widehat{P}_{X_1}^2 1_X &= \int_I \frac{1}{\sum_{j\in J_i} \lambda(X_j)} \widehat{P}_{X_1} 1_{\bigcup_{j\in J_i} X_j} dp(i) \\
&= \int_I \frac{1}{\sum_{j\in J_i} \lambda(X_j)} \left(\int_I \frac{\sum_{j\in J_i} \lambda(X_j)}{\sum_{l\in J_k} \lambda(X_l)} 1_{\bigcup_{s\in J_k} X_s} dp(k) \right) dp(i) \\
&= \int_I \frac{1}{\sum_{l\in J_k} \lambda(X_l)} 1_{\bigcup_{s\in J_k} X_s} dp(k) \\
&= \widehat{P}_{X_1} 1_X
\end{aligned}$$

and $\left\{\widehat{P}_{X_1}^n 1_X\right\}_n$ is weakly precompact in $L^1(X, \lambda)$. This implies the existence of an absolutely continuous σ-finite invariant measure for P from Theorem 4.10 and completes the proof. □

For a random iteration of piecewise linear intermittent Markov maps with uniformly contractive part, we can calculate explicit formula of the invariant density.

Corollary 5.2. *For any random iteration of piecewise linear intermittent Markov maps with uniformly contractive part $\{T_i, dp(i) : i \in I\}$, the invariant density $d\mu/d\lambda$ is given by the following formula:*

$$\frac{d\mu}{d\lambda} = \int_I \frac{1}{\sum_{j\in J_i} \lambda(X_j)} \sum_{n\geq 0} \sum_{\substack{j\in J_i\\ j>n}} \frac{\lambda(X_j)}{\lambda(X_{j-n})} 1_{X_{j-n}} dp(i). \tag{16}$$

Consequently, $\mu(X) < \infty$ if and only if

$$\int_I \frac{1}{\sum_{j\in J_i} \lambda(X_j)} \sum_{n\geq 0} \sum_{\substack{j\in J_i\\ j>n}} \lambda(X_j) dp(i) < \infty.$$

Proof. By the proof of Theorem 5.1, $h_0 = \int_I \frac{1}{\sum_{j\in J_i}\lambda(X_1)} 1_{\bigcup_{j\in J_i} X_j} dp(i)$ is an invariant density for $\widehat{P}_{X_1}$. For the original Markov operator P, an invariant density is given by $h = \sum_{n\geq 0} (PI_{X\setminus X_1})^n h_0$ according to Theorem 4.10. Therefore, for an absolutely continuous σ-finite invariant measure μ of $\{T_i, dp(i) : i \in I\}$, we have

$$\frac{d\mu}{d\lambda} = \sum_{n\geq 0} (PI_{X\setminus X_1})^n \int_I \frac{1}{\sum_{j\in J_i} \lambda(X_1)} 1_{\bigcup_{j\in J_i} X_j} dp(i).$$

Write for each $i \in I$ and $n \geq 0$

$$\begin{aligned}
(PI_{X\setminus X_1})^n \, 1_{\bigcup_{j\in J_i} X_j} &= \sum_{j\in J_i} (PI_{X\setminus X_1})^n \, 1_{X_j} \\
&= \sum_{\substack{j\in J_i\\ j>n}} \frac{\lambda(X_j)}{\lambda(X_{j-1})} \frac{\lambda(X_{j-1})}{\lambda(X_{j-2})} \cdots \frac{\lambda(X_{j-n+1})}{\lambda(X_{j-n})} 1_{X_{j-n}} \\
&= \sum_{\substack{j\in J_i\\ j>n}} \frac{\lambda(X_j)}{\lambda(X_{j-n})} 1_{X_{j-n}}.
\end{aligned}$$

Thus, we have the formula of the invariant density (16). □

Remark 22. For the case of nonlinear random maps like LSV map or Manneville-Pomeau map with uniformly contractive part, we also may be able to show the existence of absolutely continuous σ-finite invariant measures and to estimate the invariant densities if the branches on X_1 is linear in the sense (a-2). Otherwise, it would be possible to estimate the invariant density under certain distortion property on X_1. That will be studied in the other paper.

In the rest of this subsection, we give several examples of random iterations of piecewise linear intermittent Markov maps with uniformly contractive part.

The first example is a random iteration of piecewise linear intermittent Markov maps with "thin branches" which return to the indifferent fixed point. The absolutely continuous σ-finite invariant measure of the following example varies at $\alpha = 1$ from finite to infinite, which is same as the deterministic case [67].

Example 6. Let $\alpha > 0$, $I = \{1, 2\}$ and $J_i = \{(10i)n : n \geq 1\}$ for $i \in I$. Set $p_1 = p$ (and $p_2 = 1 - p$). (See Figure 3 below for the corresponding map.) Then the random iteration of piecewise linear intermittent Markov maps with uniformly contractive part $\{T_i; p_i : i \in I\}$ admits an absolutely continuous σ-finite invariant measure μ by Theorem 5.1. Corollary 5.2 tells us that

$$\mu(X) = \sum_{i=1,2} \frac{p_i}{\sum_{j\in J_i} \lambda(X_j)} \sum_{n\geq 0} \sum_{\substack{j\in J_i\\ j>n}} \lambda(X_j).$$

For $i = 1$, we have

$$\sum_{n\geq 0}\sum_{\substack{j\in J_1\\ j>n}}\lambda(X_j) = \sum_{n\geq 0}\sum_{\substack{k\geq 1\\ 10k>n}}\left(\frac{1}{(10k)^{1/\alpha}} - \frac{1}{(10k+1)^{1/\alpha}}\right)$$
$$=10\sum_{n\geq 0}\sum_{k\geq n+1}\left(\frac{1}{(10k)^{1/\alpha}} - \frac{1}{(10k+1)^{1/\alpha}}\right).$$

since $\lambda(X_j) = j^{-1/\alpha} - (j+1)^{-1/\alpha}$. For enough large n, the last summation in the above equality is of order $(n+1)^{-1/\alpha}$. This is also the case for $i = 2$ for sufficient large n. Therefore, we conclude $\mu(X) < \infty$ if and only if $\alpha < 1$ same as deterministic transformation case.

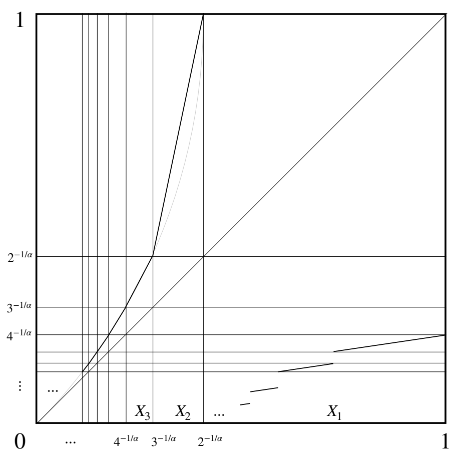

Figure 3: A graph of T_1 in Example 6

The following example is a random iteration which always admits an absolutely continuous finite invariant measure for any α of the order of tangency at the indifferent fixed point, because all points will never return to a neighborhood of the indifferent fixed point.

Example 7. Let $\alpha > 0$, $I = \mathbb{N}$ and J_i satisfy that $\bigcup_{i\in I} J_i$ is a finite set. Then, from Corollary 5.2, for any probability vector $\{p_i\}_{i\in I}$, the random iteration of $\{T_i; p_i : i \in I\}$ admits an absolutely continuous finite invariant measure. Further, the invariant density for this random iteration is bounded above even around the point 0.

The following example also admits an absolutely continuous finite invariant measure, although many points of positive measure will return to an enough small neighborhood of the indifferent fixed point with positive probability.

Example 8. Let $\alpha > 0$, $I = \mathbb{N}$ and $J_i = \{2, 3, \ldots, i+1\}$ for $i \in I$. See Figure 4 for the maps $\{T_i : i \in I\}$. If we put $p_i = 1/2^i$ for $i \in I$, then the random iteration $\{T_i, p_i : i \in I\}$ admits an absolutely continuous

σ-finite invariant measure μ by Theorem 5.1 and

$$\begin{aligned}
\mu(X) &= \sum_{n=1}^{\infty} \frac{p_n}{\sum_{j\in\{2,\dots,n+1\}} \lambda(X_j)} \sum_{k\geq 0} \sum_{\substack{j\in\{2,\dots,n+1\} \\ j>k}} \lambda(X_j) \\
&= \sum_{n=1}^{\infty} \frac{1}{2^n \sum_{j=2}^{n+1} \lambda(X_j)} \left(\sum_{j\in\{2,\dots,n+1\}} \lambda(X_j) + \sum_{k=1}^{n} \sum_{k<j\leq n+1} \lambda(X_j) \right) \\
&= \sum_{n=1}^{\infty} \frac{1}{2^n} \left\{ 1 + \frac{1}{\sum_{j=2}^{n+1} \lambda(X_j)} \left(\sum_{k=1}^{n} \frac{1}{(k+1)^{1/\alpha}} - \frac{n}{(n+1)^{1/\alpha}} \right) \right\}
\end{aligned}$$

since $\lambda(X_j) = j^{-1/\alpha} - (j+1)^{-1/\alpha}$. Therefore, since $\sum_{k=1}^{n}(k+1)^{-1/\alpha}$ is at most of order n, we can see $\mu(X) < \infty$ for any $\alpha > 0$. That is, the invariant measure μ is always finite.

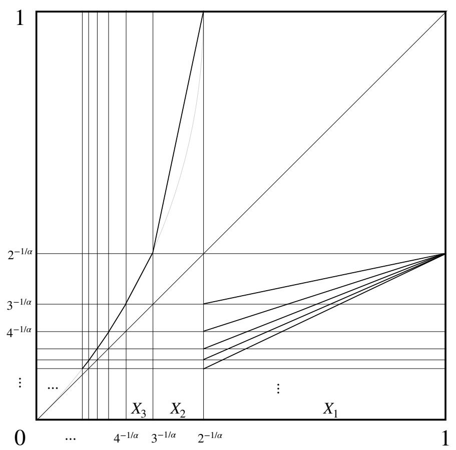

Figure 4: A graph of transformations in Example 8 and 9 (infinitely many branches on X_1 correspond to $T_i\,|_{X_1}$, $i \in I$, respectively)

We modify Example 8 and we show that the modified random dynamics admit both finite and σ-finite infinite invariant measure depending on the parameter α. The critical point of α is quite different from the deterministic case.

Example 9. Let $\alpha > 0$ and $k \geq 2$ a natural number. Set $I = \mathbb{N}$ and $p_i = 1/2^i$. Then we put $J_i = \{2, 3, \dots, 2^{k(i+1)}\}$. As the same way in Example 8, we have the invariant measure for this random iteration $\mu(X)$ is of order $\sum_{n\geq 1} 2^{kn(1-1/\alpha)}/2^n$ for $\alpha \neq 1$. For $\alpha = 1$, one can see that $\mu(X)$ is finite. Thus, we conclude that μ is an absolutely continuous finite invariant measure if and only if $\alpha < k/(k-1)$.

In the last two examples, most of points will rarely return to the indifferent fixed point 0 so that the absolutely continuous invariant measures hard to become infinite measures. Then, conversely, we will see that the following example makes absolutely continuous invariant measures tend to become infinite measures.

Example 10. Let $\alpha > 0$, $I = \mathbb{N}$ and $k(i+1) \geq k(i) \geq 2$ for each $i \in I$. We now set $J_i = \{j \in \mathbb{N} : j \geq k(i)\}$.

Then for any probability vector $\{p_i\}_{i\in I}$, the invariant measure μ for this random iteration forms

$$\mu(X) = \sum_{i\in\mathbb{N}} \frac{p_i}{\sum_{j\in J_i}\lambda(X_j)} \sum_{n\geq 0}\sum_{\substack{j\in J_i\\ j>n}} \lambda(X_j) = \sum_{i\in\mathbb{N}} \frac{p_i}{k(i)^{-1/\alpha}} \sum_{n\geq 0}\sum_{\substack{j\in J_i\\ j>n}} \lambda(X_j).$$

Here write

$$\sum_{n\geq 0}\sum_{\substack{j\in J_i\\ j>n}} \lambda(X_j) = \sum_{n=0}^{k(i)-1} \sum_{j\in\{k(i),\dots\}} \lambda(X_j) + \sum_{n=k(i)}^{\infty}\sum_{j>n}\lambda(X_j) = k(i)\cdot k(i)^{-\frac{1}{\alpha}} + \sum_{n=k(i)}^{\infty} n^{-\frac{1}{\alpha}}$$

and we have

$$\mu(X) = \sum_{i=1}^{\infty} p_i k(i) + \sum_{i=1}^{\infty} \frac{p_i \sum_{n\geq k(i)} n^{-1/\alpha}}{k(i)^{-1/\alpha}}.$$

If $\alpha \geq 1$ then the right hand side of the above equality diverges and if $\alpha < 1$ then we have that $\mu(X)$ is of order $\sum_{i=1}^{\infty} p_i k(i)$. Therefore, we conclude that μ is finite if and only if $\alpha < 1$ and $\sum_i p_i k(i) < \infty$. Thus, for example, if $p_i = 6/(i^2\pi^2)$ and $k(i) = 2 + [i^\gamma]$ for some $\gamma > 0$, then μ is finite if and only if $\alpha < 1$ and $0 < \gamma < 1$.

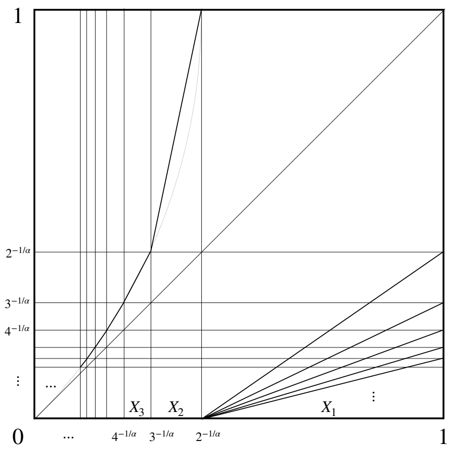

Figure 5: A graph of transformations in Example 10 (infinitely many branches on X_1 correspond to $T_i|_{X_1}$, $i \in I$, respectively)

Remark 23. For random iterations of non-uniformly expanding maps as Example 9 and 10, the critical value of α where the invariant measure varies from finite to infinite can be different from deterministic case ([67]). Similar example which is expanding on average except an indifferent fixed point can be seen in Example 6.2 of [31].

5.3 Concluding remarks

In this subsection, we will give some remarks on our results presented in this thesis and raise some future direction.

First of all, we give a remark on Theorem 5.1 and its applications Examples 6–10. Non-uniformly expanding random maps (intermittent random systems as in [31, 51]) and contractive random maps (dissipative random systems as in [12, 14]) were recently studied separately. However, for random iterations which have both non-uniformly expanding part and uniformly contractive branches like our model, statistical properties including the existence of absolutely continuous σ-finite invariant measures are still not known. Therefore, our result could be interpreted as the first step toward the direction of non-uniformly expanding random maps with uniformly contractive part.

Secondly, we give a comment on dissipative systems comparing Theorem 3.5 and Theorem 4.10. As mentioned in Remark 20, a non-eventually conservative system (i.e., the dissipative part remains asymptotically of positive measure) has no sweep-out set of finite measure with respect to a σ-finite invariant measure. On non-eventually conservative systems, we may not hope that the method of the induced operator or the jump operator still works. Therefore, we will have to construct new method to get σ-finite (infinite!) invariant measures for dissipative systems. We may head for the direction of dissipative systems based on [13].

Finally, we mention the difference between annealed type random dynamical systems and quenched type random dynamical systems, and comment our future works. All random dynamical systems dealt with in this thesis are so called *annealed* type random dynamical systems but typical random dynamical systems in ergodic theory are of the skew-product form as follows. Let $(X, \mathscr{B}, m)$ and $(\Omega, \mathscr{F}, \mathbb{P})$ be probability spaces, interpreted as a phase space and a parameter space respectively. Let $\sigma : \Omega \to \Omega$ a $\mathbb{P}$-preserving transformation and $T_\omega : X \to X$ a non-singular transformation for each $\omega \in \Omega$. Then a random dynamical system (in the sense of skew-product transformation) $\Theta : X \times \Omega \to X \times \Omega$ is defined by

$$\Theta(x, \omega) = (T_\omega x, \sigma\omega).$$

Although we can deal with numerous transformations $\{T_\omega : \omega \in \Omega\}$ through this type of random dynamical systems, Θ is a somewhat "deterministic" over $X \times \Omega$ whenever we observe this system via the reference measure $m \times \mathbb{P}$. However, recent progress in the theory of random dynamical system make it possible to handle the statistical properties of "fiber-wise" or "sample-wise" random dynamical system $T_{\sigma^n(\omega)} \circ \cdots \circ T_{\sigma(\omega)} \circ T_\omega$ for $\mathbb{P}$-almost every $\omega \in \Omega$. Such systems in view of fiber-wise observation are called *quenched* type random dynamical systems. We refer to [7, 11]. For the direction of quenched type random dynamical systems, we recently introduced *Markov operator cocycles* and obtained some results on their mixing properties in [50].

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