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博士学位論文

The Theory of Pseudo-Fan and Toric Construction of Moduli Space of Quasi Maps from $\mathbb{P}^1$ with Two Marked Points to $\mathbb{P}^1 \times \mathbb{P}^1$

（擬扇の理論および２点付き$\mathbb{P}^1$から$\mathbb{P}^1 \times \mathbb{P}^1$への擬写像のモジュライ空間のトーリック構成）

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The Theory of Pseudo-Fan and Toric Construction of Moduli Space of Quasi Maps from $\mathbb{P}^1$ with Two Marked Points to $\mathbb{P}^1 \times \mathbb{P}^1$

Kohki Matsuzaka

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1 Introduction

1.1 Background: Classical Mirror Symmetry

Mirror symmetry is some symmetry between Calabi-Yau variety X (A-model: Kähler structure) and another Calabi-Yau variety X° (B-model: complex structure). Mirror symmetry comes from superstring theory. In superstring theory, a particle in spacetime M is considered as a string and its trajection is described as a map from Riemann surface Σ to M. By conformal invariance, M must have 10-dimensions. Then M is considered as a product of 4-dimensional spacetime $M^{3,1}$ and (complex) 3-dimensional Kähler manifold K whose Ricci curvature vanishes (namely, Calabi-Yau threefold), and K is small to have escape ditection in ordinary our observation. This is the compactification of superstring theory. Moreover, the theory which describes the motion of a string in Calabi-Yau threefold K is described by $N = 2$ superconformal field theory (SCFT). $N = 2$ SCFT suggests the symmetry: *for Calabi-Yau threefold X, we can take another Calabi-Yau threefold X° such that $h^{p,q}(X) = h^{3-p,q}(X^\circ)$*, where $h^{p,q}(X)$ is the Hodge number of X. By the way, it is convenient to introduce the *Hodge diamond* of 3-dimensional compact Kähler manifold X:

$$\begin{array}{ccccccc}
 & & & h^{3,3}(X) & & & \\
 & & h^{3,2}(X) & & h^{2,3}(X) & & \\
 & h^{3,1}(X) & & h^{2,2}(X) & & h^{1,3}(X) & \\
h^{3,0}(X) & & h^{2,1}(X) & & h^{1,2}(X) & & h^{0,3}(X) \\
 & h^{2,0}(X) & & h^{1,1}(X) & & h^{0,2}(X) & \\
 & & h^{1,0}(X) & & h^{0,1}(X) & & \\
 & & & h^{0,0}(X) & & &
\end{array}$$

Since $h^{p,q}(X) = h^{q,p}(X) = h^{3-p,3-q}(X) = h^{3-q,3-p}(X)$ and $h^{0,0}(X) = 1, h^{1,0}(X) = 0, h^{2,0}(X) = 0, h^{3,0}(X) = 1$ for Calabi-Yau threefold X, it is sufficient to consider $h^{1,1}(X)$ and $h^{2,1}(X)$. Then the Hodge diamond of X is

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1}(X) & & 0 & \\
1 & & h^{2,1}(X) & & h^{2,1}(X) & & 1 \\
 & 0 & & h^{1,1}(X) & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

and by using "mirror symmetry" $h^{2,1}(X^\circ) = h^{1,2}(X^\circ) = h^{1,1}(X), h^{1,1}(X^\circ) = h^{2,2}(X^\circ) = h^{2,1}(X)$, the Hodge diamond of X° is

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{2,1}(X) & & 0 & \\
1 & & h^{1,1}(X) & & h^{1,1}(X) & & 1 \\
 & 0 & & h^{2,1}(X) & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

Thus, *the Hodge diamond of X° is a 45° line reversal of the Hodge diamond of X*. This is the origin of the word "mirror" of mirror symmetry.

In general, $h^{2,1}(X)$ equals to the dimension of complex moduli space (moduli space of complex structure) of X and $h^{1,1}(X)$ equals to the dimension of Kähler moduli space (moduli space of Kähler structure) of X. By "mirror symmetry" $h^{1,1}(X) = h^{2,1}(X^\circ)$, we can give the conjecture about moduli:

Kähler moduli space of X coincides with complex moduli space of X°. Then we can describe the classical mirror symmetry. Now, we consider the case that X is a quintic threefold $V(z_1^5+z_2^5+z_3^5+z_4^5+z_5^5-5\psi z_1z_2z_3z_4z_5)\subset\mathbb{P}^4$ with one parameter ψ. Then $h^{1,1}(X)=1, h^{2,1}(X)=101$ and thus Kähler moduli space of X is described by one parameter t. The mirror X° of X is defined as the quotient space $X^\circ:=V(z_1^5+z_2^5+z_3^5+z_4^5+z_5^5-5\psi z_1z_2z_3z_4z_5)/G$ ($G\simeq(\mathbb{Z}/5\mathbb{Z})^3$), where ψ is the parameter (or coordinate) of complex moduli space of X°. Then $h^{2,1}(X^\circ)=1, h^{1,1}(X^\circ)=101$, and it is compatible with the fact that only one parameter ψ appears in X°. Here, in physics, the quantity called *Yukawa coupling* is important. The B-model Yukawa coupling $K_{\psi\psi\psi}$ of X° can be computed by using *periods* which satisfy *Picard-Fuchs equation*; as a result, $K_{\psi\psi\psi}=C\psi^2/(1-\psi^5)$, where C is some constant. In contrast, the direct computation of A-model Yukawa coupling K_{ttt} of X is difficult because instanton corrections appears in K_{ttt}. But, by Candelas, de la Ossa, Green and Parkes [2], K_{ttt} is computed by using the moduli conjecture, namely, the equivalence of Kähler moduli space of X and complex moduli space of X°; to compute K_{ttt}, they used the *mirror map* $\psi=\psi(t)$. A mirror map is the coordinate transformation between Kähler moduli space of X and complex moduli space of X°. The equivalence of two moduli spaces suggests that K_{ttt} can be computed by using $K_{\psi\psi\psi}$ via the mirror map. In [2], K_{ttt} was finally computed as follows:

$$K_{ttt}=5+\sum_{d=1}^{\infty}d^3N_dq^d=5+2875q+4876875q^2+\cdots, \tag{1}$$

where $q:=e^{2\pi\sqrt{-1}t}$. This is a surprising result. The number 2875 is nothing but the number of the degree-1 rational curves in X and 4876875 relates to the number 609250 of the degree-2 rational curves in X as $4876875=2^3\cdot 609250+1^3\cdot 2875$. In this way, mirror symmetry is useful to the enumeration of the rational curves on varieties. The number $\langle\mathcal{O}_h,\mathcal{O}_h,\mathcal{O}_h\rangle_{0,d}=d^3N_d$ is called *Gromov-Witten invariant* (GW invariant). Mathematically, mirror symmetry for quintic surfaces was proved by Givental [7], and Lian, Liu and Yau [12].

In general case, mirror symmetry conjecture is described as follows: *There exists a pair of Calabi-Yau manifolds (X,X°) such that A-model (Kähler structure of X) and B-model (complex structure of X°) are equivalence via the mirror map. In classical case, A-model Yukawa coupling on X "coincides" with B-model Yukawa coupling on X° via the mirror map.* A-model Yukawa coupling in general case is

$$\langle \omega_1, \omega_2, \omega_3 \rangle = \sum_{\beta \in H_2(X,\mathbb{Z})} \langle \omega_1, \omega_2, \omega_3 \rangle_\beta \, q^\beta, \tag{2}$$

where $\omega_1, \omega_2, \omega_3 \in H^*(X, \mathbb{Q})$ and q^β is a formal symbol and $\langle \omega_1, \omega_2, \omega_3 \rangle_\beta$ is the (three point) *Gromov-Witten invariant*:

$$\langle \omega_1, \omega_2, \omega_3 \rangle_\beta := \int_{[\overline{M}_{0,3}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\omega_1)\mathrm{ev}_2^*(\omega_2)\mathrm{ev}_3^*(\omega_3). \tag{3}$$

Moreover, $[\overline{M}_{0,3}(X,\beta)]^{\mathrm{vir}}$ is the virtual fundamental class, and $\overline{M}_{0,3}(X,\beta)$ is the moduli space of 3-pointed genus 0 stable maps in X with homology class β, and $\mathrm{ev}_i : \overline{M}_{0,3}(X,\beta) \to X$ $(i = 1, 2, 3)$ is the evaluation map at the i-th marked point. We can also define the GW invariants as the intersection number of $\overline{M}_{0,3}(V,\beta)$, where V is an ambient space of X (for example, if X is the quintic threefold in $\mathbb{P}^4$, then the ambient space of X is $\mathbb{P}^4$). In general, the direct computation of GW invariants needs the localization technique that is more complicated. Classical mirror symmetry gives us the easy way of the computation of GW invariants.

For more detail about mirror symmetry, see [3, 8, 10].

1.2 Motivation: Another Compactification of Moduli Space

As mentioned above, GW invariants of Calabi-Yau variety X is defined as the intersection number of $\overline{M}_{0,3}(V,\beta)$, where V is an ambient space of X. $\overline{M}_{0,3}(V,\beta)$ is compactification of the moduli space of 3-pointed holomorphic maps from $\mathbb{P}^1$ to V with homology class $\beta \in H_2(V,\mathbb{Z})$. But, in this paper, we treat the another compactification $\widetilde{Mp}_{0,2}(V, \boldsymbol{d})$ when V is a toric variety (especially, $\mathbb{P}^1 \times \mathbb{P}^1$), which was introduced by Jinzenji [9] (see Section 4). This is the compactification of the moduli space $Mp_{0,2}(V, \boldsymbol{d})$ of multi degree-$\boldsymbol{d}$ quasi maps from $\mathbb{P}^1$ with two marked points $0 := [1 : 0], \infty := [0 : 1]$ to V which is well-defined at $0, \infty$. The moduli space of quasi maps is simpler than the moduli space of stable maps. Indeed, it is expected that $\widetilde{Mp}_{0,2}(V, \boldsymbol{d})$ is written as a toric variety. Then we can easily compute Chow ring of $\widetilde{Mp}_{0,2}(V, \boldsymbol{d})$. Moreover, as the analysis of gauged linear sigma model by Morrison and Plesser [14] tell us, we can expect that $\widetilde{Mp}_{0,2}(V, \boldsymbol{d})$ has more information of B-model. For example, for $V = \mathbb{P}^{N-1}, \boldsymbol{d} = d$, the generating function of intersection numbers

$$w(\mathcal{O}_{h^\alpha}, \mathcal{O}_{h^\beta})_{0,d} := \int_{\widetilde{Mp}_{0,2}(\mathbb{P}^{N-1},d)} \mathrm{ev}_1^*(h^\alpha) \wedge \mathrm{ev}_2^*(h^\beta) \wedge c_{top}(\mathcal{E}_d^k), \tag{4}$$

where h is a hypersurface class of $\mathbb{P}^{N-1}$ and $\mathrm{ev}_i : \widetilde{Mp}_{0,2}(\mathbb{P}^{N-1}, d) \to \mathbb{P}^{N-1}$ $(i = 1, 2)$ is the evaluation map at the i-th marked point and $\mathcal{E}_d^k$ is the rank $kd + 1$ vector bundle on $\widetilde{Mp}_{0,2}(\mathbb{P}^{N-1}, d)$ corresponding to the condition that the images of quasi maps are contained in the hypersurface $V(z_1^k + \cdots + z_N^k)$ in $\mathbb{P}^{N-1}$, gives the mirror map in the computaions of classical mirror symmetry of surfaces in $\mathbb{P}^{N-1}$ (see [9]). For the above reasons, we study about $\widetilde{Mp}_{0,2}(V, \boldsymbol{d})$ for a toric variety V.

Histrically (about the toric construction of $\widetilde{Mp}_{0,2}(V, \boldsymbol{d})$), Jinzenji [9] conjectured the existence of the moduli space $\widetilde{Mp}_{0,2}(V, \boldsymbol{d})$ and explicitly constructed the $\widetilde{Mp}_{0,2}(V, \boldsymbol{d})$ by using the homogeneous coordinates in the case of $V = \mathbb{P}^{N-1}, \mathbb{P}^1 \times \mathbb{P}^1$. But concrete toric construction of $\widetilde{Mp}_{0,2}(V, \boldsymbol{d})$ for these cases were not given in [9]. In [15], Saito gave the toric construction of $\widetilde{Mp}_{0,2}(N, d) := \widetilde{Mp}_{0,2}(\mathbb{P}^{N-1}, d)$. In [11], Jinzenji and Saito gave the toric construction of $\widetilde{Mp}_{0,2}(\mathbb{P}(1, 1, 1, 3), d)$. In [13], the author gave the toric construction of $\widetilde{Mp}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$.

1.3 Main Results and Outline of This Paper

As mentioned above, in [9], moduli space $\widetilde{Mp}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$ is defined as a quotient space, but its toric construction is not given. As the case of $V = \mathbb{P}^{n-1}, \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}(1, 1, 1, 3)$, we can expect that $\widetilde{Mp}_{0,2}(V, \boldsymbol{d})$ is written as a quotient space when V is a toric variety in general. In this paper, we discuss general theory of pseudo-fan and method of toric construction of a quotient space by using pseudo-fan. Moreover, as an application of it, we give the toric construction of $\widetilde{Mp}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$ according to [13]. First similar discussion is expanded in specific cases in [11, 15], and after, in [13], the author generalized their methods about toric construction by introducing *pseudo-fan*.

This paper consists of two parts. The first part is development of general theory of *pseudo-fan* and quotient space. In Section 2, we review the basics of toric geometry used in this paper. In Section 3, we discuss the toric construction of quotient space and computing Chow ring. To do toric construction, in Subsection 3.1, we introduce pseudo-fan and *min-value condition* (MVC), and we prove that pseudo-fan satisfying (MVC) is actually complete and simplicial fan (Lemma 4). In Subsection 3.2, we explain the construction of pseudo-fan from a quotient space. we also see that (MVC) can be rewritten as *alternative min-value condetion* (AMVC) in the case of quotient space (Theorem 4). (AMVC) is easier than (MVC) because (AMVC) is just min-value linear system. In Subsection 3.3, we compute the action of pseudo-fan satisfying (AMVC) and compare it and the action of former quotient space. In conclusion, it is shown that a quotient space satisfying some conditions is complete and simplicial toric variety (Theorem 5). It is one of our main theorem. In Subsection 3.4, we compute Chow ring of quotient space by using its weight matrix (Theorem 6).

The second part is discussion about $\widetilde{Mp}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$. In Subection 4.1, we give the definition of quasi map and its moduli space. In Subsection 4.2, we give the explicit definition of $\widetilde{Mp}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$ according to [9]. The remaining subsections in Section 4 are assigned to toric construction of $\widetilde{Mp}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$ according to the toric construction program that is discussed in Section 3. More specifically, we give pseudo-fan of $\widetilde{Mp}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$ and confirm that it has a *vertex matrix* and satisfies (AMVC) (Theorem 7). In conclusion, one of our main theorem (Theorem 8; $\widetilde{Mp}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$ is a complete and simplicial toric variety) is obtained. In Section 5, we compute Chow ring of $\widetilde{Mp}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$ (Theorem 9) and give two conjectures for its Poincaré polynomials (Conjecture 1 and Conjecture 2).

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Part I
General Theory of Pseudo-Fan and Its Geometry

2 Reviews in Toric Geometry

In this section, we review standard definitions and basic facts in toric geometry which we will use in this paper. For more details, see [4] or [6].

2.1 Fan and Toric Variety

Let M be a free Abelian group of rank n (i.e., $M \simeq \mathbb{Z}^n$) and $N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})(\simeq \mathbb{Z}^n)$ be its dual. We denote $M \otimes \mathbb{R}$ and $N \otimes \mathbb{R}$ by $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$, respsctively. Similarly, we denote that $M_{\mathbb{Q}} := M \otimes \mathbb{Q}$ and $N_{\mathbb{Q}} := N \otimes \mathbb{Q}$. Then we have a natural pairing:

$$\langle \bullet, \bullet \rangle : M \times N \longrightarrow \mathbb{Z}. \tag{5}$$

A *rational polyhedral cone* (hereinafter, reffered to as the cone) is a subset σ of $N_{\mathbb{R}}$ that is written as

$$\sigma = \langle v_1, \ldots, v_l \rangle_{\geq 0} := \{\lambda_1 v_1 + \cdots + \lambda_l v_l \mid \lambda_1, \ldots, \lambda_l \geq 0\} \tag{6}$$

for some $v_1, \ldots, v_l \in N$. A cone σ is called a *strongly convex cone* if it satisfies

$$\sigma \cap (-\sigma) = \{0\}. \tag{7}$$

A *Dimension* of a cone σ is defined as

$$\dim \sigma := \dim_{\mathbb{R}} \mathrm{span}(\sigma), \tag{8}$$

where $\mathrm{span}(\sigma)$ is a minimal $\mathbb{R}$–linear subspace including the cone σ. A *Dual cone* of a cone σ is defined by

$$\check{\sigma} := \{m \in M_{\mathbb{R}} \mid \langle m, v \rangle \geq 0 \text{ for any } v \in \sigma\}. \tag{9}$$

Let τ be a subset of σ. If we can take $m \in M \cap \check{\sigma}$ such that

$$\tau = \{v \in \sigma \mid \langle m, v \rangle = 0\}, \tag{10}$$

then we call τ a *face* of σ. Note that a cone $\sigma := \langle v_1, \ldots, v_l \rangle_{\geq 0}$ generated by $\mathbb{R}$–linearly independent vectors $v_1, \ldots, v_l$ is strongly convex.

Then we give basic properties of cones without proof:

Lemma 1

(i) *A face of a cone is also a cone.*

(ii) *Let σ be a cone and τ be its face. Then every face of τ is a face of σ.*

(iii) *If σ is a cone, then $\check{\sigma} \cap M$ is a finitely generated semigroup (Gordan's lemma).*

(iv) *If ρ is a cone with* $\dim \rho = 1$, *then there exists the unique generator v_ρ of the semigroup $\rho \cap N$.*

From (iv) of above lemma, we identify the 1–dimensional cone ρ with its generator v_ρ in this paper. The following lemma is used in Section 3:

Lemma 2 *Let $\{v_1, \ldots, v_l\}$ be a subset of N and S be a subset of $\{v_1, \ldots, v_l\}$. If $v_1, \ldots, v_l$ are linearly independent as vectors of $N_\mathbb{R}$, then $\langle S \rangle_{\geq 0}$ is a face of $\sigma := \langle v_1, \ldots, v_l \rangle_{\geq 0}$. Moreover, every face of σ can be written as $\sigma = \langle T \rangle_{\geq 0}$ for some $T \subset \{v_1, \ldots, v_l\}$.*

Proof. Since σ is a face of σ and a face of a face is a face, we can assume that $S = \{v_1, \ldots, v_{l-1}\}$. It is clear that $v_1, \ldots, v_l$ are linearly independent as vectors in $N_\mathbb{Q}$. We define two subspaces V_1, V_2 of $M_\mathbb{Q}$ as

$$V_1 := \{m \in M_\mathbb{Q} \bigm| \langle m, v_1 \rangle = \cdots = \langle m, v_{l-1} \rangle = 0\}, \tag{11}$$

$$V_2 := \{m \in M_\mathbb{Q} \bigm| \langle m, v_1 \rangle = \cdots = \langle m, v_l \rangle = 0\}. \tag{12}$$

Then $\dim_\mathbb{Q} V_1 = n-(l-1)$ and $\dim_\mathbb{Q} V_2 = n-l$. This follows that $V_1 \supsetneq V_2$ and we can take some element m_0 of V_1 such that $m_0 \notin V_2$. By multiplying m_0 by some integer if necessary, m_0 can be made into an element of M satisfying $\langle m_0, v_l \rangle > 0$. Thus the following equality holds:

$$\langle S \rangle_{\geq 0} = \{v \in \sigma \bigm| \langle m_0, v \rangle = 0\}, \tag{13}$$

and therefore, $\langle S \rangle_{\geq 0}$ is a face of σ.

Let τ be a face of σ. By the definition of a face, τ is written as $\tau = \{v \in \sigma \bigm| \langle m, v \rangle = 0\}$ for some $m \in M \cap \check{\sigma}$. Now, we put $T := \{v_i \bigm| \langle m, v_i \rangle = 0\}$

and $I(T) := \{i \mid \langle m, v_i \rangle = 0\}$. Let $w := \lambda_1 v_1 + \dots \lambda_l v_l$ $(\lambda_1, \dots, \lambda_l \geq 0)$ be an element of τ. Then

$$0 = \langle m, w \rangle = \sum_{i=1}^{l} \lambda_i \langle m, v_i \rangle = \sum_{i \notin I(T)} \lambda_i \langle m, v_i \rangle. \tag{14}$$

Since m belongs to $\check{\sigma}$, pairings $\langle m, v_i \rangle$ $(i \in \{1, \dots, l\} \setminus I(T))$ are positive, and therefore, λ_i $(i \in \{1, \dots, l\} \setminus I(T))$ must be 0; hence

$$w = \sum_{i \in I(T)} \lambda_i v_i \in \langle T \rangle_{\geq 0} \tag{15}$$

and therefore $\tau = \langle T \rangle_{\geq 0}$. □

Now, we give the definition of a *fan*, which is the fundamental material of the construction of toric varieties:

Definition 1 *A set Σ which consists of strongly convex cones is called a* fan *if*

(i) Σ is a finite set.

(ii) for any $\sigma \in \Sigma$, every face of σ is also cone of Σ.

(iii) for any $\sigma, \tau \in \Sigma$, $\sigma \cap \tau$ is a face of σ and τ.

Moreover, the set $|\Sigma|$ denotes $\bigcup_{\sigma \in \Sigma} \sigma$ and we define the set

$$\Sigma(k) := \{\sigma \in \Sigma \mid \dim \sigma = k\}. \tag{16}$$

Especially, we also denote by $\Sigma(1)$ a set of generators of 1-dimensional cones in Σ.

A fan Σ defines a toric variety $X(\Sigma)$ which is obtained by gluing *affine toric varieties* $X_\sigma := \mathrm{Spec}(\mathbb{C}[M \cap \check{\sigma}])$ $(\sigma \in \Sigma)$, where $\mathbb{C}[M \cap \check{\sigma}]$ is a $\mathbb{C}$-algebra generated by χ^m $(m \in M \cap \check{\sigma})$ with relations $\chi^{m_1} \cdot \chi^{m_2} = \chi^{m_1 + m_2}$. Then $X_{\{0\}} = \mathrm{Spec}(\mathbb{C}[M])$ is an algebraic torus $T_N := N \otimes \mathbb{C}^\times \simeq (\mathbb{C}^\times)^n$. Recall that the character χ^m $(m \in M)$ is a rational function on $X(\Sigma)$.

At this stage, we introduce a fundamental theorem in toric geometry which we use in this paper. First, we give the geometrical properties of $X(\Sigma)$ using terminology of cones and fans:

Theorem 1 *Let Σ be a fan.*

(i) $X(\Sigma)$ is simplicial, namely, an orbifold if and only if Σ is simplicial, *namely, the generators of σ are linearly independent as vectors of $N_{\mathbb{R}}$ for any $\sigma \in \Sigma$.*

(ii) $X(\Sigma)$ is complete, namely, compact if and only if Σ is complete*, namely, $|\Sigma| = N_{\mathbb{R}}$.*

If the generators of 1–dimentional cones of Σ span $N_{\mathbb{R}}$, then the following sequence of $\mathbb{Z}$–modules is exact:

$$0 \longrightarrow M \xrightarrow{\varphi} \mathrm{Div}_{T_N}(X(\Sigma)) \xrightarrow{\psi} A_{n-1}(X(\Sigma)) \longrightarrow 0, \quad (17)$$

where $\mathrm{Div}_{T_N}(X(\Sigma))$ is the group of T_N-invariant Weil divisors and $A_{n-1}(X(\Sigma))$ is Chow group of $X(\Sigma)$. Moreover, $\varphi(m) := \mathrm{div}(\chi^m)$, and $\psi(D)$ is the divisor class of D. Recall that 1–dimensional cone ρ of Σ corresponds to T_N–invariant divisor D_ρ on $X(\Sigma)$ and $\mathrm{div}(\chi^m) = \sum_\rho \langle m, v_\rho \rangle D_\rho$.

As is well known in the case of complex projective space, toric variety $X(\Sigma)$ can be written as a quotient space when the fan Σ is complete and simplicial (see [5]). We briefly explain outline of this construction. First, we define a closed subspace of $\mathbb{C}^{\Sigma(1)}$ by

$$Z(\Sigma) := \left\{ (x_\rho)_{\rho \in \Sigma(1)} \in \mathbb{C}^{\Sigma(1)} \;\middle|\; \prod_{\rho \not\subset \sigma} x_\rho = 0 \text{ for all } \sigma \in \Sigma \right\}, \quad (18)$$

and an abelian group that acts on $\mathbb{C}^{\Sigma(1)}$ by

$$G = G(\Sigma) := \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X(\Sigma)), \mathbb{C}^\times). \quad (19)$$

Then the group action of G is given by

$$G \times \mathbb{C}^{\Sigma(1)} \ni (g, (x_\rho)_\rho) \longmapsto (g([D_\rho])x_\rho)_\rho \in \mathbb{C}^{\Sigma(1)}. \quad (20)$$

Under the above preparations, we can state the following theorem that is due to Cox (see [4, 5]):

Theorem 2 *Let Σ be a fan. If the generators of 1–dimentional cones of Σ span $N_{\mathbb{R}}$, then $X(\Sigma)$ is the geometrical quotient of $\mathbb{C}^{\Sigma(1)} - Z(\Sigma)$ by G if and only if Σ is simplicial.*

In order to describe the closed set $Z(\Sigma)$ in a simpler manner, we introduce *primitive collection* of Σ:

Definition 2 ([1]) *Let Σ be a simplicial fan. A subset S of $\Sigma(1)$ is called a* primitive collection *of Σ if*

(i) S does not generate a cone in Σ.

(ii) every proper subset of S generates a cone in Σ.

Moreover, we denote the set of primitive collections of Σ by $\mathrm{PC}(\Sigma)$.

If Σ is simplicial, $Z(\Sigma)$ takes the following form using primitive collections of Σ:

Lemma 3 ([1]) *If Σ is a simplicial fan, then $Z(\Sigma)$ is rewriten as*

$$Z(\Sigma) = \bigcup_{S \in \mathrm{PC}(\Sigma)} \{(x_\rho) \in \mathbb{C}^{\Sigma(1)} \mid x_\rho = 0 \text{ for any } \rho \in S\}. \tag{21}$$

2.2 Chow Ring and Cohomology Ring of Toric Varieties

Let Σ be an n–dimensional fan. Then Σ defines a toric variety $X(\Sigma)$ as above. In order to compute Chow ring and cohomology ring (over $\mathbb{Q}$) of $X(\Sigma)$, we define two ideals of polynomial ring $\mathbb{Q}[x_\rho | \rho \in \Sigma(1)]$. The first ideal, denoted by $I(\Sigma)$, is defined as an ideal generated by degree–1 homogeneous polynomials

$$\sum_{\rho \in \Sigma(1)} \langle m, v_\rho \rangle \cdot x_\rho \ (m \in M). \tag{22}$$

By taking m as canonical basis e_i $(i = 1, \cdots, n)$ of M, this ideal is rewritten as

$$\left\{ \sum_{\rho \in \Sigma(1)} v_{i\rho} x_\rho \middle| i = 1, \dots, n \right\}, \tag{23}$$

where $v_\rho = {}^t(v_{1\rho}, \dots, v_{n\rho})$. The second ideal, called *Stanley–Reisner ideal*, is defined by

$$SR(\Sigma) := \left\{ \prod_{v_\rho \in S} x_\rho \middle| S \in \mathrm{PC}(\Sigma) \right\}. \tag{24}$$

Then Chow ring, cohomology ring, and Betti numbers of complete simplicial toric variety $X(\Sigma)$ can be computed as follows (see [3, 4]):

Theorem 3 *Let Σ be a complete and simplicial fan.*

(i)

$$\begin{aligned} A^*_{\mathbb{Q}}(X(\Sigma)) &\simeq H^*(X(\Sigma), \mathbb{Q}) \\ &\simeq \mathbb{Q}[x_\rho | \rho \in \Sigma(1)]/(I(\Sigma) + SR(\Sigma)). \end{aligned} \tag{25}$$

(ii)

$$b_{2k}(X(\Sigma)) = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} |\Sigma(n-i)| \quad (\textit{for all } k), \tag{26}$$

where $b_{2k}(X(\Sigma)) := \dim H^{2k}(X(\Sigma), \mathbb{Q})$ is Betti number of $X(\Sigma)$.

(iii)

$$b_{2k+1}(X(\Sigma)) = 0 \quad (\textit{for all } k). \tag{27}$$

(iv)

$$b_{2k}(X(\Sigma)) = b_{2n-2k}(X(\Sigma)) \quad (\textit{for all } k). \tag{28}$$

From this theorem, computation of Poincaré polynomial $P(t) := \sum_{i=0}^{2n} b_{2i} t^i$ of complete simplicial toric variety is replaced with counting the number of cones in its fan.

3 Pseudo-Fan and Quotient Space

In previous section, we saw that a complete and simplicial toric variety can be represented using homogeneous coordinates. Thus, conversely, we discuss the method of finding the fan of quotient spaces in this section. The key of this method is the notion of pseudo-fan and min-value condition (MVC) (or alternative min-value condition (AMVC)).

Before the discussion, we treat a simpler example, projective space $\mathbb{P}^2$. We define a fan $\Sigma_{\mathbb{P}^2}$ as

$$\Sigma_{\mathbb{P}^2} := \{0\} \cup \{\langle u_1\rangle_{\geq 0}, \langle u_2\rangle_{\geq 0}, \langle u_3\rangle_{\geq 0}\} \cup \{\langle u_1, u_2\rangle_{\geq 0}, \langle u_1, u_3\rangle_{\geq 0}, \langle u_2, u_3\rangle_{\geq 0}\}, \tag{29}$$

where $u_1 := {}^t(1,0), u_2 := {}^t(0,1), u_3 := {}^t(-1,-1)$. This is clearly complete and simplicial. Moreover, it has the only primitive collection $\{u_1, u_2, u_3\}$. Thus we can apply the Theorem 2 and firstly obtain

$$Z(\Sigma_{\mathbb{P}^2}) = \{(0,0,0)\}. \tag{30}$$

To check the action of $G = G(\Sigma_{\mathbb{P}^2})$, we must find the relation among $[D_1]$, $[D_2]$, and $[D_3]$, where D_i is the divisor corresponding to u_i, and calculate Chow group of $X(\Sigma_{\mathbb{P}^2})$. For $m = (m_1, m_2) \in \mathbb{Z}^2 (\simeq M)$, the map φ in (17) is

$$\varphi(m) = \mathrm{div}(\chi^m) = m_1 D_1 + m_2 D_2 + (-m_1 - m_2) D_3, \tag{31}$$

where D_i is the corresponding divisor to u_i. Therefore, a divisor $D = n_1 D_1 + n_2 D_2 + n_3 D_3$ ($n_1, n_2, n_3 \in \mathbb{Z}$) is an element of the image of φ if and only if $n_1 + n_2 + n_3 = 0$, and $[D_3]$ generates Chow group $A_1(X(\Sigma_{\mathbb{P}^2}))$ since $[\mathrm{div}(\chi^{(1,0)})] = [D_1] - [D_3] = 0$ and $[\mathrm{div}(\chi^{(0,1)})] = [D_2] - [D_3] = 0$. Thus $A_1(X(\Sigma_{\mathbb{P}^2})) \simeq \mathbb{Z}$, and by (20), $G = \mathrm{Hom}_{\mathbb{Z}}(A_1(X(\Sigma_{\mathbb{P}^2})), \mathbb{C}^\times)$ acts on $\mathbb{C}^3 - Z(X(\Sigma_{\mathbb{P}^2})) = \mathbb{C}^3 - \{(0,0,0)\}$ as

$$\begin{aligned} &G \times (\mathbb{C}^3 - \{(0,0,0)\}) \ni (g, (z_1, z_2, z_3)) \\ &\qquad\qquad \longmapsto (\lambda z_1, \lambda z_2, \lambda z_3) \in \mathbb{C}^3 - \{(0,0,0)\}, \end{aligned} \tag{32}$$

where $\lambda := g([D_1]) = g([D_2]) = g([D_3])$. Because we can identify G with $\mathbb{C}^\times$ by the following correspondence:

$$\begin{array}{ccc} G & & \mathbb{C}^\times \\ \cup & & \cup \\ g & \longmapsto & g([D_3]) \\ g_\lambda & \longleftarrow & \lambda, \end{array} \tag{33}$$

where $g_\lambda([D_3]) := \lambda$, the action of G is equivalent to the ordinary action in $\mathbb{P}^2$, and therefore, $X(\Sigma_{\mathbb{P}^2}) \simeq \mathbb{P}^2$ by using Theorem 2. Note that $1 \cdot u_1 + 1 \cdot u_2 + 1 \cdot u_3 = 0$ and the action (32) can be written as $(g, (z_1, z_2, z_3)) \mapsto (\lambda^1 z_1, \lambda^1 z_2, \lambda^1 z_3)$; that is, coefficients $(1,1,1)$ of the relation $u_1 + u_2 + u_3 = 0$ corresponds to the powers $(1,1,1)$ of $\mathbb{C}^\times$ action (namely, λ) in (32).

Conversely, we can predict the fan of $\mathbb{P}^2$ from its quotient form. As a quotient space, $\mathbb{P}^2$ is defined as

$$\mathbb{P}^2 := (\mathbb{C}^3 - \{(0,0,0)\})/\mathbb{C}^\times, \tag{34}$$

where the group $\mathbb{C}^\times$ acts as

$$\mathbb{C}^\times \times (\mathbb{C}^3 - \{(0,0,0)\}) \ni (\lambda, (z_1, z_2, z_3)) \\ \longmapsto (\lambda z_1, \lambda z_2, \lambda z_3) \in \mathbb{C}^3 - \{(0,0,0)\}. \quad (35)$$

Then, using 1×3 matrix $W_{\mathbb{P}^2} = (w_{ij}) := (1\ 1\ 1)$ (*weight matrix*), we can interpret this action as

$$\mathbb{C}^\times \times (\mathbb{C}^3 - \{(0,0,0)\}) \ni (\lambda, (z_1, z_2, z_3)) \\ \longmapsto (\lambda^{w_{11}} z_1, \lambda^{w_{12}} z_2, \lambda^{w_{13}} z_3) \in \mathbb{C}^3 - \{(0,0,0)\}. \quad (36)$$

From the calculation of the action in Theorem 2 (see above calculation), we notice that we should solve the following $\mathbb{Z}$-linear equation to find 1–dimensional cones of the fan of $\mathbb{P}^2$:

$$W_{\mathbb{P}^2} \cdot \mathbf{x} = \mathbf{0}. \quad (37)$$

The solutions of this equation are $\mathbf{x}_1 = {}^t(-1,1,0), \mathbf{x}_2 = {}^t(-1,0,1)$. We define the 2×3 matrix $V_{\mathbb{P}^2}$ (*vertex matrix*) as

$$V_{\mathbb{P}^2} = (v_1, v_2, v_3) := \begin{pmatrix} {}^t\mathbf{x}_1 \\ {}^t\mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}. \quad (38)$$

Then $v_1 = u_1, v_2 = u_2, v_3 = u_3$, and from the numerator of $\mathbb{P}^2$ in (34), we can easily predict the set of primitive collection of $\mathbb{P}^2$ as $\{v_1, v_2, v_3\}$. Therefore we should define the fan $\Sigma'_{\mathbb{P}^2}$ of $\mathbb{P}^2$ as follows:

$$\Sigma'_{\mathbb{P}^2} := \{\langle S\rangle_{\geq 0} \mid S \subset \{v_1, v_2, v_3\}, \text{ and } \{v_1, v_2, v_3\} \text{ is not contained in } S\ \}. \quad (39)$$

Then $\Sigma'_{\mathbb{P}^2} = \Sigma_{\mathbb{P}^2}$, and of course, $\Sigma'_{\mathbb{P}^2}$ is a complete and simplicial fan which has the only primitive collection $\{u_1, u_2, u_3\}$.

In the case of general weight matrix, it is expected that obtained vertex matrix as above is more complicated than the case of $\mathbb{P}^2$ and it is difficult to check that obtained "fan" as above is really complete and simplicial fan. To deal with this problem, we will introduce the notion of *pseudo-fan* and *min–value condition* in next subsection.

3.1 Pseudo-Fan and Min–Value Condition

Definition 3 *Let V be a finite subset of $N \simeq \mathbb{Z}^d$ and $P_1, \ldots, P_l$ be non-empty subsets of V. Moreover, we put $\Pi := \{P_1, \ldots, P_l\}$ and*

$$\Sigma = \Sigma(V, \Pi) := \{\langle S \rangle_{\geq 0} \mid S \subset V, \text{ and } P_1, \ldots, P_l \text{ are not contained in } S \}. \tag{40}$$

*Then we call a triplet (Σ, V, Π) d–*dimensional pseudo-fan *if it satisfies the following conditions:*

(i) $P_1, \ldots, P_l$ span V i.e.,

$$P_1 \cup \cdots \cup P_l = V. \tag{41}$$

(ii) If $i \neq j$, then $P_i \not\subset P_j$.

Now, we introduce the condition (MVC) that provides a basis for our discussion. (MVC) gives sufficient condition for a pseudo-fan to be a complete and simplical fan.

Definition 4 *Let (Σ, V, Π) be a d–dimensional pseudo-fan. Then we say that (Σ, V, Π) satisfies* Min–Value Condition (MVC) *if for any $\beta \in \mathbb{R}^d$, there exists the unique $\alpha = (\alpha_i) \in \mathbb{R}^r$ $(r := |V|)$ such that*

(i)

$$\beta = \sum_{i=1}^{r} \alpha_i v_i, \tag{42}$$

where $V = \{v_1, \ldots, v_r\}$.

(ii)

$$\min\{\alpha_i \mid v_i \in P\} = 0 \quad (\text{for any } P \in \Pi). \tag{43}$$

Lemma 4 *Let (Σ, V, Π) be a d–dimensional pseudo-fan satisfying (MVC). Then Σ is a d–dimentional complete and simplicial fan with* $\mathrm{PC}(\Sigma) = \Pi$.

Proof. Let $\sigma := \langle u_1, \ldots, u_l \rangle_{\geq 0}$ $(u_1, \ldots, u_l \in V)$ be any element of Σ. If $u_1, \ldots, u_l$ are linearly dependent over $\mathbb{R}$, there exists $\alpha_1, \ldots, \alpha_l \in \mathbb{R}$ such that $\alpha_1 u_1 + \cdots + \alpha_l u_l = 0$ and $(\alpha_1, \ldots, \alpha_l) \neq (0, \ldots, 0)$. Now we can assume that $\alpha_1, \ldots, \alpha_p \geq 0$, $\alpha_{p+1}, \ldots, \alpha_l < 0$ $(1 \leq p \leq l)$. Since every $P \in \Pi$ is not contained in $\{u_1, \dots, u_l\}$ (see (40)), we can take some element v of V such that $v \notin \{u_1, \dots, u_l\}$ and $v \in P$ for each $P \in \Pi$. Then both representations

$$\alpha_1 u_1 + \cdots + \alpha_p u_p + \sum_{v \in V \setminus \{u_1, \dots, u_p\}} 0 \cdot v$$
$$= (-\alpha_{p+1}) u_{p+1} + \cdots + (-\alpha_l) u_l + \sum_{v \in V \setminus \{u_{p+1}, \dots, u_l\}} 0 \cdot v \quad (44)$$

satisfy (i), (ii) of (MVC). This follows that $(\alpha_1, \dots, \alpha_l) = (0, \dots, 0)$ by uniqueness of α, which contradicts our assumption, and therefore $u_1, \dots, u_l$ are linearly independent. In particular, every cone in Σ is strongly convex.

Let $\sigma := \langle S \rangle_{\geq 0}$, $\tau := \langle T \rangle_{\geq 0}$ ($S, T \subset V$) be any elements of Σ. Then we put $S \cap T = \{u_1, \dots, u_l\}$, $S \setminus T = \{w_1, \dots, w_p\}$, and $T \setminus S = \{w'_1, \dots, w'_q\}$. It is clear that $\langle S \cap T \rangle_{\geq 0} \subset \sigma \cap \tau$. Conversely, if $u \in \sigma \cap \tau$, we can write

$$u = \sum_{i=1}^{l} \alpha_i u_i + \sum_{j=1}^{p} \gamma_j w_j + \sum_{v \in S} 0 \cdot v = \sum_{i=1}^{l} \alpha'_i u_i + \sum_{k=1}^{q} \gamma'_k w'_k + \sum_{v \in T} 0 \cdot v \quad (45)$$

for some constants $\alpha_i, \alpha'_i, \gamma_j, \gamma'_k \geq 0$. Since these representations satisfy (i), (ii) of (MVC), one can see that $\alpha_i = \alpha'_i$ and $\gamma_j, \gamma'_k = 0$ as above. Thus $\sigma \cap \tau$ coincides with $\langle S \cap T \rangle_{\geq 0}$ and Σ satisfies (iii) of Definition 1 by Lemma 2. The condition (i) of Definition 1 is obvious by (40). Also, Σ satisfies (ii) of Definition 1. Indeed, for every subset S' of S, $S' \not\supset P$ for any $P \in \Pi$ since $S \not\supset P$ for any $P \in \Pi$, and this means that $\langle S' \rangle_{\geq 0} \in \Sigma$. From Lemma 2, every face of $\sigma = \langle S \rangle_{\geq 0}$ is written as the form $\langle S' \rangle_{\geq 0}$ for some subset S' of S, and therefore we have confirmed (ii) of Definition 1. From the above, Σ is a simplicial fan.

Next, we prove the completeness of Σ. Let β be any element of $N_{\mathbb{R}} \simeq \mathbb{R}^d$. By (i), (ii) of (MVC), it can uniquely be written as $\beta = \sum_{i=1}^{l} \alpha_i v_i$ with $\min\{\alpha_i \mid v_i \in P\} = 0$ for any $P \in \Pi$. To prove $\beta \in |\Sigma|$, we define two sets $S^+ := \{i \mid \alpha_i > 0\}$, $S^0 := \{i \mid \alpha_i = 0\}$. It is clear that $S^+ \cap S^0 = \emptyset$. Also, we can easily see that $S^+ \cup S^0 = V$ by (i) in Definition 3. Since there exists $v_i \in P$ such that $i \in S^0$ for each $P \in \Pi$, every $P \in \Pi$ is not contained in the set $S := \{v_i \mid i \in S^+\}$ and this follows that $\langle S \rangle_{\geq 0} \in \Sigma$. Then $\beta = \sum_{i \in S^+} \alpha_i v_i \in \langle S \rangle_{\geq 0}$ and hence, $\beta \in |\Sigma|$. This means that Σ is complete.

Finally, we prove thet $\mathrm{PC}(\Sigma)$ coincides with Π. By (40), every member P of Π does not generate a cone in Σ. But by (ii) in Definition 3, every proper subset of P generates a cone in Σ. Hence every member of Π is a primitive collection of Σ i.e., $\Pi \subset \mathrm{PC}(\Sigma)$. Next, let S be in $\mathrm{PC}(\Sigma)$. Here, suppose that any member of Π is not contained in S. Then S generates a cone in Σ by (40), which contradicts that S is a primitive collection of Σ. Thus we can take a set $P \in \Pi$ such that P is a subset of S. Here, if P is a proper subset of S, then P generates a cone in Σ since S is a primitive collection of Σ (see Definition 2). However, P is a primitive collection of Σ since we have already shown $\Pi \subset \mathrm{PC}(\Sigma)$. Thus S must be equal to P i.e., $S \in \Pi$, and this means that $\mathrm{PC}(\Sigma) \subset \Pi$. $\square$

Example. $\Sigma'_{\mathbb{P}^2}$ gives a 2–dimensional pseudo-fan $(\Sigma'_{\mathbb{P}^2}, \{v_1, v_2, v_3\}, \{\{v_1, v_2, v_3\}\})$ (see (39)) and it satisfies (MVC). Indeed, for arbitrary $\beta = {}^t(\beta_1, \beta_2) \in \mathbb{R}^2$, the (i) of (MVC) is

$$\begin{aligned} &\beta = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \\ &\Longleftrightarrow \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 - \alpha_3 \\ \alpha_2 - \alpha_3 \end{pmatrix} \\ &\Longleftrightarrow \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} t + \beta_1 \\ t + \beta_2 \\ t \end{pmatrix}, \end{aligned} \tag{46}$$

where $t := \alpha_3$. Then (ii) of (MVC) is a single equation for t:

$$\min\{t, t + \beta_1, t + \beta_2\} = 0. \tag{47}$$

Since $\min\{t, t + \beta_1, t + \beta_2\} = t - \max\{0, -\beta_1, -\beta_2\}$, The unique solution is

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} \beta_1 + \max\{0, -\beta_1, -\beta_2\} \\ \beta_2 + \max\{0, -\beta_1, -\beta_2\} \\ \max\{0, -\beta_1, -\beta_2\} \end{pmatrix} = \begin{pmatrix} \max\{\beta_1, 0, \beta_1 - \beta_2\} \\ \max\{\beta_2, \beta_2 - \beta_1, 0\} \\ \max\{0, -\beta_1, -\beta_2\} \end{pmatrix}. \tag{48}$$

Thus $(\Sigma'_{\mathbb{P}^2}, \{v_1, v_2, v_3\}, \{\{v_1, v_2, v_3\}\})$ satisfies (MVC). This result is not contradict to the fact that $\mathbb{P}^2 \simeq X(\Sigma'_{\mathbb{P}^2})$ is complete and simplicial.

3.2 Weight Matrix of Quorient Space and Alternative Min-Value Condition

Let Y be a quotient space:

$$Y := (\mathbb{C}^r - Z)/(\mathbb{C}^\times)^m \ (m < r), \tag{49}$$

where Z is *axial*, namely, a closed subspace of $\mathbb{C}^r$ with the following form:

$$Z = \bigcup_{j=1}^{l} \{(x_i) \in \mathbb{C}^r | x_i = 0 \text{ for any } i \in S_j\}, \tag{50}$$

where $S_1, \ldots, S_l$ are a subset of $\{1, \ldots, r\}$ with $S_1 \cup \cdots \cup S_l = \{1, \ldots, r\}$ and $S_{j_1} \not\subset S_{j_2}$ for any $j_1, j_2 \in \{1, \ldots, r\}$ such that $j_1 \neq j_2$. Moreover, m actions of $\mathbb{C}^\times$ are given by

$$\begin{aligned} \mathbb{C}^\times \times (\mathbb{C}^r - Z) \ni (\lambda_j, (z_1, \ldots, z_r)) \\ \longmapsto (\lambda_i^{w_{i1}} z_1, \ldots, \lambda_i^{w_{ir}} z_r) \in \mathbb{C}^r - Z \;(i = 1, \ldots, m), \end{aligned} \tag{51}$$

where $w_{ij} \in \mathbb{Z}$. Then we call $m \times r$ integer matrix $W_Y := (w_{ij})$ *weight matrix* of Y. In this paper, we assume that there exists $v_1, \ldots, v_r \in \mathbb{Z}^{r-m}$ such that a following sequence is exact:

$$0 \longrightarrow \mathbb{Z}^n \xrightarrow{{}^tV_Y} \mathbb{Z}^r \xrightarrow{W_Y} \mathbb{Z}^m \longrightarrow 0, \tag{52}$$

where $n := r - m$ and $V_Y := (v_1, \ldots, v_r)$ is the $n \times r$ matrix. We call this *vertex matrix* of Y. Then we can construct a pseudo-fan from Y; we define two sets Ver_Y and Π_Y as

$$\mathrm{Ver}_Y := \{v_1, \ldots, v_r\} \tag{53}$$

$$\Pi_Y := \{\{v_i | i \in S_k\} | k = 1, \ldots, l\}. \tag{54}$$

Then $(\Sigma(\mathrm{Ver}_Y, \Pi_Y), \mathrm{Ver}_Y, \Pi_Y)$ is an n-dimensional pseudo-fan. It is hard to show that $(\Sigma(\mathrm{Ver}_Y, \Pi_Y), \mathrm{Ver}_Y, \Pi_Y)$ satisfies the condition (MVC) in general, but we can rewrite (MVC) by using weight matrix W_Y and it is more simple than (MVC):

Theorem 4 *If* $(\Sigma(\mathrm{Ver}_Y, \Pi_Y), \mathrm{Ver}_Y, \Pi_Y)$ *satisfies the following condition* Alternative Min-Value Condition (AMVC)*:*

(AMVC) *For any* $y = (y_j) \in \mathbb{R}^r$*, there exists the unique* $x = (x_i) \in \mathbb{R}^m$ *such that*

$$\min \left\{ -y_j + \sum_{i=1}^{m} w_{ij} x_i \;\middle|\; j \in S_k \right\} = 0 \quad (k = 1, \ldots, l), \tag{55}$$

then $(\Sigma(\mathrm{Ver}_Y, \Pi_Y), \mathrm{Ver}_Y, \Pi_Y)$ *satisfies (MVC) and therefore* $\Sigma(\mathrm{Ver}_Y, \Pi_Y)$ *is an* n*-dimensional complete and simplicial fan with* $\mathrm{PC}(\Sigma(\mathrm{Ver}_Y, \Pi_Y)) = \Pi_Y$.

Proof. First, we prove the existence of $\alpha = (\alpha_j)$ satisfying (i), (ii) of (MVC). From the exact sequence (52), the map $V_Y : \mathbb{Z}^r \to \mathbb{Z}^n$ is surjective and $V_Y{}^tW_Y = {}^t(W_Y{}^tV_Y) = O$. Then for any $\beta \in \mathbb{R}^n$ and $x = (x_i) \in \mathbb{R}^m$, there exists $y = (y_j) \in \mathbb{R}^r$ such that

$$\begin{aligned} \beta &= V_Y \cdot (-y + {}^tW_Y \cdot x) \\ &= \sum_{j=1}^{r} \left(-y_j + \sum_{i=1}^{m} w_{ij} x_i \right) v_j. \end{aligned} \tag{56}$$

Hence we should put

$$\alpha_j = -y_j + \sum_{i=1}^{m} w_{ij} x_i \ (j = 1, \ldots, r). \tag{57}$$

Moreover, by the condition (AMVC), one can find x satisfying the following equality:

$$\min\{\alpha_j \mid j \in S_k\} = \min\left\{ -y_j + \sum_{i=1}^{m} w_{ij} x_i \;\middle|\; j \in S_k \right\} = 0 \tag{58}$$

for $k = 1, \ldots, l$. Thus α satisfying (i), (ii) of (MVC) exists. Next, we prove the uniqueness of α in (MVC). We assume that $\beta \in \mathbb{R}^n$ can be written as

$$\beta = \sum_{j=1}^{r} \alpha_j v_j = \sum_{j=1}^{r} \gamma_j v_j \tag{59}$$

such that

$$\min\{\alpha_j \mid j \in S_k\} = \min\{\gamma_j \mid j \in S_k\} = 0 \ \ (k = 1, \ldots, l) \tag{60}$$

for some $\alpha = (\alpha_j), \gamma = (\gamma_j) \in \mathbb{R}^r$. By acting $\mathrm{Hom}_{\mathbb{Z}}(\bullet, \mathbb{R})$ to (52), we obtain the following exact sequence:

$$0 \longrightarrow \mathbb{R}^m \xrightarrow{\ {}^tW_Y\ } \mathbb{R}^r \xrightarrow{\ V_Y\ } \mathbb{R}^n \longrightarrow 0\,. \tag{61}$$

Since $V_Y(\alpha-\gamma)=\sum_{j=1}^r(\alpha_j-\gamma_j)v_j=\beta-\beta=0$, there exists $x=(x_i)\in\mathbb{R}^m$ such that $\alpha-\gamma={}^tW_Y\cdot x$, i.e.,

$$\alpha_j=-(-\gamma_j)+\sum_{i=1}^m w_{ij}x_i \quad (j=1,\ldots,r) \tag{62}$$

by the above exact sequence (61). Then

$$\min\left\{-(-\gamma_j)+\sum_{i=1}^m w_{ij}x_i \mid j\in S_k\right\}=\min\{\alpha_j \mid j\in S_k\}=0 \tag{63}$$

by (60) and (62). On the other hand,

$$\min\left\{-(-\gamma_j)+\sum_{i=1}^m w_{ij}\cdot 0 \mid j\in S_k\right\}=\min\{\gamma_j \mid j\in S_k\}=0. \tag{64}$$

Hence x must be 0 by uniqueness of x in (AMVC), and therefore, $\alpha=\gamma$ by (62). □

Example. In this example, we show that $(\Sigma'_{\mathbb{P}^2},\{v_1,v_2,v_3\},\{\{v_1,v_2,v_3\}\})$ satisfies (AMVC). Since the weight matrix of $\mathbb{P}^2$ is $W_{\mathbb{P}^2}=(1\ 1\ 1)$, (AMVC) is a single equation for $x_1=x$:

$$\min\{-y_1+x,-y_2+x,-y_3+x\}=0, \tag{65}$$

and this is easily solved as $x=\max\{y_1,y_2,y_3\}$. Thus $(\Sigma'_{\mathbb{P}^2},\{v_1,v_2,v_3\},\{\{v_1,v_2,v_3\}\})$ satisfies (AMVC) and therefore $\Sigma'_{\mathbb{P}^2}$ is a complete and simplicial fan. From the above two examples, we have confirmed that (AMVC) is more simple than (MVC) in the case of $\mathbb{P}^2$.

3.3 Quotient Construction of $X(\Sigma(\mathrm{Ver}_Y,\Pi_Y))$

In this subsection, we fix the notations in previous subsection and assume that $(\Sigma(\mathrm{Ver}_Y,\Pi_Y),\mathrm{Ver}_Y,\Pi_Y)$ satisfies (AMVC). This means that $\Sigma(\mathrm{Ver}_Y,\Pi_Y)$ is a complete and simplicial fan with $\mathrm{PC}(\Sigma(\mathrm{Ver}_Y,\Pi_Y))=\Pi_Y$ by Theorem 4. The exact sequence (17) in the case of $\Sigma_Y:=\Sigma(\mathrm{Ver}_Y,\Pi_Y)$ is

$$0\longrightarrow M\xrightarrow{\ \varphi\ }\mathrm{Div}_{T_N}(X(\Sigma_Y))\xrightarrow{\ \psi\ }A_{n-1}(X(\Sigma_Y))\longrightarrow 0. \tag{66}$$

From the exact sequence (52), we can take the $\mathbb{Z}$–basis $\beta_1, \ldots, \beta_m \in \mathbb{Z}^r$ satisfying $W_Y\beta_i = e_j^{(m)}$ $(i = 1, \ldots, m)$, where $e_i^{(l)}$ is an i–th canonical base of $\mathbb{Z}^l$. Let $E_i := \psi(\tilde{\beta}_i)$ be a divisor class, where $\tilde{\beta}_i$ is a T_N–invariant divisor corresponding to $\beta_i \in \mathbb{Z}^r$ (remark that $\mathrm{Div}_{T_N}(X(\Sigma_Y)) \simeq \mathbb{Z}^{X(\Sigma_Y)(1)} = \mathbb{Z}^{\mathrm{Ver}_Y} \simeq \mathbb{Z}^r$). Then

$$\begin{aligned} W_Y e_j^{(r)} &= \sum_{i=1}^m w_{ij} e_i^{(m)} \\ &= \sum_{i=1}^m w_{ij} W_Y \beta_i \\ &= W_Y \left(\sum_{i=1}^m w_{ij}\beta_i \right) \end{aligned} \tag{67}$$

(recall that w_{ij} is the (i, j)–component of weight matrix W_Y) and therefore $e_j^{(r)} - \sum_{i=1}^m w_{ij}\beta_i \in \mathrm{Ker}\, W_Y$. Since $e_j^{(r)}$ corresponds to the divisor D_j corresponding to $v_j \in \mathrm{Ver}_Y$, $e_j^{(r)} - \sum_{i=1}^m w_{ij}\beta_i$ corresponds to $D_i - \sum_{i=1}^m w_{ij}\tilde{\beta}_i$, and it belongs to $\mathrm{Ker}\,\psi$ since ${}^tV_Y$ essential equals to φ as a map by these definitions and $\mathrm{Im}\, {}^tV_Y = \mathrm{Ker}\, W_Y$, $\mathrm{Im}\,\varphi = \mathrm{Ker}\,\psi$. This means that $[D_j] = \sum_{i=1}^m w_{ij}E_i$. Moreover, $E_1, \ldots, E_m$ are $\mathbb{Z}$–basis of $A_{n-1}(X(\Sigma_Y))$. Indeed, it is clear that $E_1, \ldots, E_m$ generate $A_{n-1}(X(\Sigma_Y))$, and if $\sum_{i=1}^m c_iE_i = 0$ for some $c_1, \ldots, c_m \in \mathbb{Z}$, then

$$\psi\left(\sum_{i=1}^m c_i\tilde{\beta}_i \right) = \sum_{i=1}^m c_iE_i = 0, \tag{68}$$

and therefore $\sum_{i=1}^m c_i\beta_i \in \mathrm{Ker}\, W_Y$. By $W_Y\beta_i = e_i^{(m)}$, this means that $\sum_{i=1}^m c_ie_i^{(m)} = 0$. Thus $c_1, \ldots, c_m$ must be equal to 0. From the above, we obtain the following two lemmas:

Lemma 5

$$A_{n-1}(X(\Sigma_Y)) \simeq \mathbb{Z}^m. \tag{69}$$

In particular, the group $G = \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X(\Sigma_Y)), \mathbb{C}^\times)$ *is isomorphic to* $(\mathbb{C}^\times)^m$.

Lemma 6 *We can choose $\mathbb{Z}$–bases $E_1, \ldots, E_m$ of $A_{n-1}(X(\Sigma_Y))$ such that*

$$[D_j] = \sum_{i=1}^m w_{ij}E_i \quad (j = 1, \ldots, r). \tag{70}$$

Next, we confirm the action of $G = \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X(\Sigma_Y)), \mathbb{C}^\times)$. From (20), G acts on $\mathbb{C}^r$ as follows:

$$G \times \mathbb{C}^r \ni (g, (x_j)_j) \longmapsto (g([D_j])x_j)_j \in \mathbb{C}^r. \tag{71}$$

From Lemma 6, we obtain

$$\begin{aligned} g([D_j]) &= g\left(\sum_{i=1}^{m} w_{ij} E_i\right) \\ &= \prod_{i=1}^{m} g(E_i)^{w_{ij}}, \end{aligned} \tag{72}$$

and since G is identified with $(\mathbb{C}^\times)^m$ (Lemma 5), we obtain the one of our main theorem by Lemma 3, Theorem 2 and Theorem 4:

Theorem 5

$$X(\Sigma_Y) \simeq Y, \tag{73}$$

and Y is an $n(= r - m)$–dimensional complete and simplicial toric variety. In topological terminology, Y is a compact orbifold.

Summarize the above results, in order to prove that quotient space $Y = (\mathbb{C}^r - Z)/(\mathbb{C}^\times)^m$ $(m < r)$ is a complete and simplicial toric variety, we should confirm the following conditions:

1. Z is axial (see (50)).

2. The m actions of $\mathbb{C}^\times$ in Y is given by weight matrix W_Y (see (51)) and W_Y has the vertex matrix V_Y; that is, there exists the $n \times r$ integer matrix V_Y such that the following sequence is exact:

   $$0 \longrightarrow \mathbb{Z}^n \xrightarrow{\ {}^tV_Y\ } \mathbb{Z}^r \xrightarrow{\ W_Y\ } \mathbb{Z}^m \longrightarrow 0,$$

   where $n := r - m$.

3. Pseudo-fan $(\Sigma(\mathrm{Ver}_Y, \Pi_Y), \mathrm{Ver}_Y, \Pi_Y)$ from Y (see Definition 3 and (53), (54)) satisfies (AMVC): *For any $y = (y_j) \in \mathbb{R}^r$, there exists the unique $x = (x_i) \in \mathbb{R}^m$ such that*

   $$\min\left\{ -y_j + \sum_{i=1}^{m} w_{ij} x_i \;\middle|\; j \in S_k \right\} = 0 \quad (k = 1, \ldots, l).$$

3.4 Chow Ring of Quotient Space

In this subsection, we compute Chow ring and of quotient space $Y = X(\Sigma_Y)$ treated in Subsection 3.2 and 3.3, namely, $X(\Sigma_Y)$ is obtained from the pseudo-fan $(\Sigma(\mathrm{Ver}_Y, \Pi_Y), \mathrm{Ver}_Y, \Pi_Y)$ satisfying (AMVC).

Theorem 6

$$\begin{aligned} A^*_{\mathbb{Q}}(X(\Sigma_Y)) &\simeq H^*(X(\Sigma_Y), \mathbb{Q}) \\ &\simeq \mathbb{Q}[h_1, \ldots, h_m] \Big/ \left\langle \prod_{j \in S_k} (w_{1j}h_1 + \cdots + w_{mj}h_m) \middle| k = 1, \ldots, l \right\rangle. \end{aligned} \tag{74}$$

Proof. First, we prove that $\mathbb{Q}[x_1, \ldots, x_r]/I(\Sigma_Y) \simeq \mathbb{Q}[h_1, \ldots, h_m]$ with identification

$$x_i \text{ ``}=\text{''} \sum_{k=1}^{m} w_{ki}h_k. \tag{75}$$

for $i = 1, \ldots, r$. We define a $\mathbb{Q}$-algebra homomorphism θ from $\mathbb{Q}[x_1, \ldots, x_r]$ to $\mathbb{Q}[h_1, \ldots, h_m]$ as

$$x_i \longmapsto \sum_{k=1}^{m} w_{ki}h_k. \tag{76}$$

By exact sequence (52), $V_Y{}^tW_Y = O$, and θ induces a $\mathbb{Q}$-algebra homomorphism $\overline{\theta}$ from $\mathbb{Q}[x_1, \ldots, x_r]/I(\Sigma_Y)$ to $\mathbb{Q}[h_1, \ldots, h_m]$. By acting $\mathrm{Hom}_{\mathbb{Z}}(\bullet, \mathbb{Q})$ to (52), we can obtain the following exact sequence

$$0 \longrightarrow \mathbb{Q}^m \xrightarrow{{}^tW_Y} \mathbb{Q}^r \xrightarrow{V_Y} \mathbb{Q}^n \longrightarrow 0, \tag{77}$$

and it is also exact as a sequence of $\mathbb{Q}$-modules. Since this sequence splits (i.e., $\mathbb{Q}^r \simeq \mathbb{Q}^m \oplus \mathbb{Q}^n$), we can take two matrices $A = (a_{ki}) \in M(m, r; \mathbb{Q}), B = (b_{ij}) \in M(r, n; \mathbb{Q})$ satisfying

$${}^tW_Y A + BV_Y = I_r, \tag{78}$$

$$A{}^tW_Y = I_m, \tag{79}$$

and

$$V_Y B = I_n. \tag{80}$$

By using matrix A, we can define a $\mathbb{Q}$-algebra homomorphism τ from $\mathbb{Q}[h_1, \ldots, h_m]$ to $\mathbb{Q}[x_1, \ldots, x_r]/I(\Sigma_Y)$ as

$$h_k \longmapsto \left[\sum_{i=1}^{r} a_{ki} x_i \right]. \tag{81}$$

By (78) and definition of the ideal $I(\Sigma)$, we can compute $\tau \circ \overline{\theta}([x_i])$ as follows:

$$\begin{aligned} \tau \circ \overline{\theta}([x_i]) &= \sum_{k=1}^{m} w_{ki} \sum_{l=1}^{r} a_{kl} [x_l] \\ &= \sum_{l=1}^{r} ({}^t W_Y A)_{il} [x_l] \\ &= \sum_{l=1}^{r} (I_r - B V_Y)_{il} [x_l] \\ &= [x_i] - \sum_{j=1}^{n} b_{ij} \left[\sum_{l=1}^{r} v_{jl} x_l \right] \\ &= [x_i], \end{aligned} \tag{82}$$

where v_{ij} and $(C)_{ij}$ are the (i, j)–component of V_Y and C, respectively. Similarly, we can also show that $\overline{\theta} \circ \tau(h_k) = h_k$ by using (79). Therefore, $\overline{\theta}$ is an isomorphism. By the identification (75) and definition of Stanley–Reisner ideal (see (24)), we have proved our goal (74). $\square$

Example. Since the weight matrix of $\mathbb{P}^2$ is (1 1 1) and $\mathrm{PC}(\Sigma_{\mathbb{P}^2}) = \{\{u_1, u_2, u_3\}\}$, Chow ring (or cohomology ring) of $\mathbb{P}^2$ is

$$A^*_{\mathbb{Q}}(\mathbb{P}^2) = H^*(\mathbb{P}^2, \mathbb{Q}) = \mathbb{Q}[h]/(h^3). \tag{83}$$

Part II

Toric Construction of Moduli Space $\widetilde{Mp}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$ and Its Chow Ring

4 Moduli Space of Quasi Maps from $\mathbb{P}^1$ with Two Marked Points to $\mathbb{P}^1 \times \mathbb{P}^1$

In this section, we give definition and toric construction of moduli space $\widetilde{Mp}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$. Definition of $\widetilde{Mp}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$ is due to Jinzenji [9], and toric construction of $\widetilde{Mp}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$ is done according to previous discussion.

4.1 Definition of $Mp_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$

Firstly, we give the definition of moduli space of quasi maps from $\mathbb{P}^1$ with two marked points to $\mathbb{P}^1 \times \mathbb{P}^1$. A *quasi map* p of bidegree (d_1, d_2) $(d_1, d_2 > 0)$ from $\mathbb{P}^1$ to $\mathbb{P}^1 \times \mathbb{P}^1$ is given by

$$p([s:t]) = \left(\left[\sum_{i=0}^{d_1} \boldsymbol{a}_i s^i t^{d_1-i} \right], \left[\sum_{j=0}^{d_2} \boldsymbol{b}_j s^j t^{d_2-j} \right] \right)$$
$$:= \left(\left[\sum_{i=0}^{d_1} a_i^0 s^i t^{d_1-i} : \sum_{i=0}^{d_1} a_i^1 s^i t^{d_1-i} \right], \left[\sum_{j=0}^{d_2} b_j^0 s^j t^{d_2-j} : \sum_{j=0}^{d_2} b_j^1 s^j t^{d_2-j} \right] \right), \tag{84}$$

where $\boldsymbol{a}_i := (a_i^0, a_i^1) \in \mathbb{C}^2$ and $\boldsymbol{b}_j := (b_j^0, b_j^1) \in \mathbb{C}^2$. If two polynomials $\sum_{i=0}^{d_1} a_i^0 s^i t^{d_1-i}$ and $\sum_{i=0}^{d_1} a_i^1 s^i t^{d_1-i}$ (or $\sum_{j=0}^{d_2} b_j^0 s^j t^{d_2-j}$ and $\sum_{j=0}^{d_2} b_j^1 s^j t^{d_2-j}$) are divisible by some common polynomial with degree > 0, p has non-defined points which are the zero loci of the common polynomial. Then, in physics, it is called a *freckled instanton*. In this paper, we fix the two marked points $0 := [1:0]$ and $\infty := [0:1]$ of $\mathbb{P}^1$ and we only consider the quasi maps that are well-defined at marked points 0 and ∞. This means that $\boldsymbol{a}_0, \boldsymbol{a}_{d_1}, \boldsymbol{b}_0, \boldsymbol{b}_{d_2} \neq \boldsymbol{0}$

in (84). To define the moduli space $Mp_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$ of quasi maps from $\mathbb{P}^1$ with two marked points to $\mathbb{P}^1 \times \mathbb{P}^1$, we identify two quasi maps under the automorphisms of $\mathbb{P}^1$ which fixes 0 and ∞ i.e. $[s : t] \mapsto [s : \mu t]$ ($\mu \in \mathbb{C}^\times$). Then $Mp_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$ is defined by

$$\begin{aligned} &Mp_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2)) \\ &\qquad := \{(\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{d_1}, \boldsymbol{b}_0, \ldots, \boldsymbol{b}_{d_2}) \mid \boldsymbol{a}_0, \boldsymbol{a}_{d_1}, \boldsymbol{b}_0, \boldsymbol{b}_{d_2} \neq \boldsymbol{0}\}/(\mathbb{C}^\times)^3, \end{aligned} \quad (85)$$

where three $\mathbb{C}^\times$ actions are given by

$$(\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{d_1}, \boldsymbol{b}_0, \ldots, \boldsymbol{b}_{d_2}) \mapsto (\mu_1 \boldsymbol{a}_0, \ldots, \mu_1 \boldsymbol{a}_{d_1}, \boldsymbol{b}_0, \ldots, \boldsymbol{b}_{d_2}), \quad (86)$$

$$(\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{d_1}, \boldsymbol{b}_0, \ldots, \boldsymbol{b}_{d_2}) \mapsto (\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{d_1}, \mu_2 \boldsymbol{b}_0, \ldots, \mu_2 \boldsymbol{b}_{d_2}), \quad (87)$$

$$(\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{d_1}, \boldsymbol{b}_0, \ldots, \boldsymbol{b}_{d_2}) \mapsto (\boldsymbol{a}_0, \mu_3^1 \boldsymbol{a}_1, \ldots, \mu_3^{d_1} \boldsymbol{a}_{d_1}, \boldsymbol{b}_0, \mu_3^1 \boldsymbol{b}_1, \ldots, \mu_3^{d_2} \boldsymbol{b}_{d_2}), \quad (88)$$

where $\mu_1, \mu_2, \mu_3 \in \mathbb{C}^\times$. The first and second actions come from the $\mathbb{C}^\times$ action of first and second factors of target space $\mathbb{P}^1 \times \mathbb{P}^1$ of quasi maps, respectively, and the third action comes from the automorphisms of source space $\mathbb{P}^1$ fixing two marked points 0 and ∞. The point $[\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{d_1}, \boldsymbol{b}_0, \ldots, \boldsymbol{b}_{d_2}] \in Mp_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$ corresponds to bidegree-(d_1, d_2) quasi map $p([s : t]) = \left(\left[\sum_{i=0}^{d_1} \boldsymbol{a}_i s^i t^{d_1-i}\right], \left[\sum_{j=0}^{d_2} \boldsymbol{b}_j s^j t^{d_2-j}\right]\right)$. By definition of $Mp_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$, it is clear that $Mp_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2)) \simeq Mp_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_2, d_1))$. After this, we assume that $d_1 \geq d_2$ in this section.

4.2 Compactification of $Mp_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$

To give compactification $\widetilde{Mp}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$ of $Mp_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$, we must determine its weight matrix $W_{(d_1,d_2)}$. $W_{(d_1,d_2)}$ is $((d_1+1)(d_2+1)+1) \times ((d_1+3)(d_2+3)-6)$ integer matrix. Compactification $\widetilde{Mp}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$ is given as follows:

Definition 5

$$\begin{aligned} &\widetilde{Mp}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2)) \\ &:= \{(\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{d_1}, \boldsymbol{b}_0, \ldots, \boldsymbol{b}_{d_2}, \widetilde{\boldsymbol{u}_0}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_{d_1-1}, \widetilde{\boldsymbol{u}_{d_1}}) \in \mathbb{C}^{(d_1+3)(d_2+3)-6} \mid \\ &\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{d_1}, \boldsymbol{b}_0, \ldots, \boldsymbol{b}_{d_2} \in \mathbb{C}^2, \widetilde{\boldsymbol{u}_0}, \widetilde{\boldsymbol{u}_{d_1}} \in \mathbb{C}^{d_2}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_{d_1-1} \in \mathbb{C}^{d_2+1}, \\ &\boldsymbol{a}_0, \boldsymbol{a}_{d_1}, \boldsymbol{b}_0, \boldsymbol{b}_{d_2} \neq \boldsymbol{0}, \\ &(\boldsymbol{a}_i, u_{(i,0)} \cdot \cdots \cdot u_{(i,d_2)}) \neq \boldsymbol{0} \ (1 \leq i \leq d_1 - 1), \end{aligned}$$

$(\boldsymbol{b}_j, u_{(0,j)} \cdot \cdots \cdot u_{(d_1,j)}) \neq \mathbf{0}$ $(1 \leq j \leq d_2 - 1)$,
$(u_{(i_1,i_2)}, u_{(j_1,j_2)}) \neq (0,0)$
if $0 \leq i_1 < j_1 \leq d_1$ *and* $0 \leq j_2 < i_2 \leq d_2$
$\}/(\mathbb{C}^\times)^{(d_1+1)(d_2+1)+1}$, (89)

where $\boldsymbol{a}_i = (a_i^1, a_i^2)$ $(i = 0, \ldots, d_1)$, $\boldsymbol{b}_j = (b_j^1, b_j^2)$ $(j = 0, \ldots, d_2)$, $\widetilde{\boldsymbol{u}_0} = (u_{(0,1)}, u_{(0,2)}, \cdots, u_{(0,d_2)})$, $\boldsymbol{u}_i = (u_{(i,0)}, u_{(i,1)}, \cdots, u_{(i,d_2)})$ $(i = 1, 2, \cdots, d_1 - 1)$, $\widetilde{\boldsymbol{u}_{d_1}} = (u_{(d_1,0)}, u_{(d_1,1)}, \cdots, u_{(d_1,d_2-1)})$.
Let us write

$$(x_1, x_2, \ldots, x_{(d_1+3)(d_2+3)-6}) = (\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{d_1}, \boldsymbol{b}_0, \ldots, \boldsymbol{b}_{d_2}, \widetilde{\boldsymbol{u}_0}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_{d_1-1}, \widetilde{\boldsymbol{u}_{d_1}}). \tag{90}$$

Then $((d_1+1)(d_2+1)+1)$ *torus actions are given by*

$$(x_1, \ldots, x_{(d_1+3)(d_2+3)-6}) \to \left(\prod_{j=1}^{(d_1+1)(d_2+1)+1} \lambda_j^{w_{(j,1)}} x_1, \ldots, \prod_{j=1}^{(d_1+1)(d_2+1)+1} \lambda_j^{w_{(j,(d_1+3)(d_2+3)-6)}} x_{(d_1+3)(d_2+3)-6} \right), \tag{91}$$

where $w_{(i,j)}$ *is the* (i,j)*–component of* $W_{(d_1,d_2)}$.

According to [9], $W_{(d_1,d_2)}$ is determined by following steps:

1. Give the labeling of column and row of $W_{(d_1,d_2)}$ as follows:

$$W_{(d_1,d_2)} = \begin{array}{c} \\ z \\ w \\ f_1 \\ f_2 \\ \vdots \\ f_{d_1-1} \\ f_{d_1} \end{array} \begin{array}{c} \begin{array}{ccccccccc} a^1 & a^2 & b^1 & b^2 & \widetilde{u_0} & u_1 & \cdots & u_{d_1-1} & \widetilde{u_{d_1}} \end{array} \\ \left(\begin{array}{ccccccccc} & & & & & & \cdots & & \\ & & & & & & \cdots & & \\ & & & & & & \cdots & & \\ & & & & & & \cdots & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ & & & & & & \cdots & & \\ & & & & & & \cdots & & \end{array} \right) \end{array}, \tag{92}$$

   where we label rows and columns of these matrices by using the following column and row vectors:

column vectors used for labeling row vectors of $W_{(d_1,d_2)}$:

$$z = {}^t(z_0, z_1, \dots, z_{d_1}),$$

$$w = {}^t(w_0, w_1, \dots, w_{d_2}),$$

$$f_i = {}^t(f_{(i,1)}, f_{(i,2)}, \dots, f_{(i,d_2)})\ (i = 1, 2, \dots, d_1),$$

row vectors used for labeling column vectors of $W_{(d_1,d_2)}$:

$$a^i = (a^i_0, a^i_1, \dots, a^i_{d_1})\ (i = 1, 2),$$

$$b^i = (b^i_0, b^i_1, \dots, b^i_{d_2})\ (i = 1, 2),$$

$$\widetilde{u_0} = (u_{(0,1)}, u_{(0,2)}, \cdots, u_{(0,d_2)}),$$

$$u_i = (u_{(i,0)}, u_{(i,1)}, \dots, u_{(i,d_2)})\ (i = 1, 2, \dots, d_1 - 1),$$

$$\widetilde{u_{d_1}} = (u_{(d_1,0)}, u_{(d_1,1)}, \dots, u_{(d_1,d_2-1)}).$$

Before proceeding, we introduce the following two sets which will be used in after subsections:

$$\mathrm{Row}(d_1, d_2) := \{z_0, \dots, z_{d_1}, w_0, \dots, w_{d_2}, f_{(1,1)}, \dots, f_{(d_1,d_2)}\},$$

and

$$\mathrm{Col}(d_1, d_2) := \{a_0, \dots a_{d_1}, b_0, \dots, b_{d_2}, u_{(0,1)}, \dots u_{(0,d_2)}, u_{(1,0)}, \dots \\ \dots, u_{(1,d_2)}, \dots, u_{(d_1,0)}, \dots, u_{(d_1,d_2-1)}\},$$

where $a_i := (a^1_i, a^2_i), b_j := (b^1_j, b^2_j)$ $(i = 0, \dots, d_1, j = 0, \dots, d_2)$. Moreover, we denote the canonical $\mathbb{Z}$–bases e_i, e_{d_1+1+j}, and $e_{d_1+d_2+2+(i-1)d_2+j}$ of $\mathbb{Z}^{(d_1+1)(d_2+1)+1}$ by e_{z_i}, e_{w_j}, and $e_{f_{(i,j)}}$, respsctively. In the rest of this paper, we fix the numbers n, r as $n := 2d_1 + 2d_2 + 1$ and $r := |\Sigma_{(d_1,d_2)}(1)| = (d_1+3)(d_2+3)-6$. Then $m := r-n = (d_1+1)(d_2+1)+1$. We remark that we have taken the order of columns of $W_{(d_1,d_2)}$ in (92) that differs from (89) for convenience sake.

2. a^i and b^i-th columns of $W_{(d_1,d_2)}$ is given as follows:

$$W_{(d_1,d_2)} = \begin{array}{c} \\ z \\ w \\ f_1 \\ f_2 \\ \vdots \\ f_{d_1-1} \\ f_{d_1} \end{array} \begin{array}{c} \begin{array}{ccccccccc} a^1 & a^2 & b^1 & b^2 & \widetilde{u_0} & u_1 & \cdots & u_{d_1-1} & \widetilde{u_{d_1}} \end{array} \\ \left(\begin{array}{ccccccccc} I_{d_1+1} & I_{d_1+1} & O & O & & & \cdots & & \\ O & O & I_{d_2+1} & I_{d_2+1} & & & \cdots & & \\ O & O & O & O & & & \cdots & & \\ O & O & O & O & & & \cdots & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ O & O & O & O & & & \cdots & & \\ O & O & O & O & & & \cdots & & \end{array} \right) \end{array}, \tag{93}$$

where O is zero matrix and I_N $(N \in \mathbb{N})$ is identity matrix of size N.

3. $u_{(0,j)}$–th column vector $(j = 1, \ldots, d_2 - 1)$ is given as follows: w_{j-1} component is -1, w_j component is 1, $f_{(1,j)}$ component is 1, $f_{(1,j+1)}$ component is -1, and the other components are 0.

4. $u_{(i,0)}$–th column vector $(i = 1, \ldots, d_1 - 1)$ is given as follows: z_{i-1} component is -1, z_i component is 1, $f_{(i,1)}$ component is 1, $f_{(i+1,1)}$ component is -1, and the other components are 0.

5. $u_{(d_1,j)}$–th column vector $(j = 1, \ldots, d_2 - 1)$ is given as follows: w_j component is 1, w_{j+1} component is -1, $f_{(d_1,j)}$ component is -1, $f_{(d_1,j+1)}$ component is 1, and the other components are 0.

6. $u_{(i,d_2)}$–th column vector $(i = 1, \ldots, d_1 - 1)$ is given as follows: z_i component is 1, z_{i+1} component is -1, $f_{(i,d_2)}$ component is -1, $f_{(i+1,d_2)}$ component is 1, and the other components are 0.

7. $u_{(i,j)}$–th column vector $(i = 1, \ldots, d_1 - 1, j = 1, \ldots, d_2 - 1)$ is given as follows: $f_{(i,j)}$ component is -1, $f_{(i+1,j)}$ component is 1, $f_{(i,j+1)}$ component is 1, $f_{(i+1,j+1)}$ component is -1, and the other components are 0.

8. $u_{(0,d_2)}$–th column vector is given as follows: z_1 component is -1, w_{d_2-1} component is -1, $f_{(1,d_2)}$ component is 1, and the other components are 0.

9. $u_{(d_1,0)}$–th column vector is given as follows: z_{d_1-1} component is -1, w_1 component is -1, $f_{(d_1,1)}$ component is 1, and the other components are 0.

As a result, $W_{(d_1,d_2)}$ is given by

$$
W_{(d_1,d_2)}
= \begin{array}{c} \\ z \\ w \\ f_1 \\ f_2 \\ \vdots \\ f_{d_1-1} \\ f_{d_1} \end{array}
\begin{array}{c}
\begin{array}{ccccccccccc} a^1 & a^2 & b^1 & b^2 & \widetilde{u_0} & u_1 & u_2 & \cdots & u_{d_1-2} & u_{d_1-1} & \widetilde{u_{d_1}} \end{array} \\
\begin{pmatrix}
I_{d_1+1} & I_{d_1+1} & O & O & U_0 & K_1 & K_2 & \cdots & K_{d_1-2} & K_{d_1-1} & U_{d_1} \\
O & O & I_{d_2+1} & I_{d_2+1} & L_0 & O & O & \cdots & O & O & L_{d_1} \\
O & O & O & O & \widetilde{J}_d & J_u & O & \cdots & O & O & O \\
O & O & O & O & O & J_d & J_u & \cdots & O & O & O \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
O & O & O & O & O & O & O & \cdots & J_d & J_u & O \\
O & O & O & O & O & O & O & \cdots & O & J_d & \widetilde{J}_u
\end{pmatrix}
\end{array},
\tag{94}
$$

where other matrices are defined as follows:

$$
U_0 := \left(\begin{array}{c|c} & 0 \\ & -1 \\ \mathbf{0} & 0 \\ & \vdots \\ & 0 \end{array} \right) \in M(d_1+1, d_2; \mathbb{Z}), \tag{95}
$$

$$
U_{d_1} := \left(\begin{array}{c|c} 0 & \\ \vdots & \\ 0 & \mathbf{0} \\ -1 & \\ 0 & \end{array} \right) \in M(d_1+1, d_2; \mathbb{Z}), \tag{96}
$$

$$
K_i := \left(\begin{array}{c|c|c}
0 & & 0 \\
\vdots & & \vdots \\
0 & & 0 \\
-1 & & 0 \\
1 & \text{\Large 0} & 1 \\
0 & & -1 \\
0 & & 0 \\
\vdots & & \vdots \\
0 & & 0
\end{array}\right) < z_i \ \in M(d_1{+}1, d_2{+}1; \mathbb{Z}), \quad (i = 1, \dots, d_1{-}1), \tag{97}
$$

$$
L_0 := \left(\begin{array}{cccccc|c}
-1 & & & & & & 0 \\
1 & -1 & & & \text{\Large 0} & & 0 \\
 & 1 & -1 & & & & 0 \\
 & & & \ddots & & & \vdots \\
 & \text{\Large 0} & & & 1 & -1 & 0 \\
 & & & & & 1 & -1 \\
\hline
0 & 0 & 0 & \dots & 0 & 0 & 0
\end{array}\right) \in M(d_2+1, d_2; \mathbb{Z}), \tag{98}
$$

$$
L_{d_1} := \left(\begin{array}{c|cccccc}
0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\hline
-1 & 1 & & & & & \\
0 & -1 & 1 & & & \text{\Large 0} & \\
\vdots & & & \ddots & & & \\
0 & & & & -1 & 1 & \\
0 & & \text{\Large 0} & & & -1 & 1 \\
0 & & & & & & -1
\end{array}\right) \in M(d_2+1, d_2; \mathbb{Z}), \tag{99}
$$

$$
\widetilde{J}_d := \begin{pmatrix}
1 & & & & & \\
-1 & 1 & & & \text{\Large 0} & \\
 & -1 & 1 & & & \\
 & & & \ddots & & \\
 & \text{\Large 0} & & & 1 & \\
 & & & & -1 & 1
\end{pmatrix} \in M(d_2, d_2; \mathbb{Z}), \tag{100}
$$

$$\widetilde{J}_u := \begin{pmatrix} 1 & -1 & & & & \\ & 1 & -1 & & \text{\Large 0} & \\ & & 1 & & & \\ & & & \ddots & & \\ & \text{\Large 0} & & & 1 & -1 \\ & & & & & 1 \end{pmatrix} \in M(d_2, d_2; \mathbb{Z}), \tag{101}$$

$$J_d := \begin{pmatrix} -1 & 1 & & & & \\ & -1 & 1 & & \text{\Large 0} & \\ & & -1 & & & \\ & & & \ddots & & \\ & \text{\Large 0} & & & 1 & \\ & & & & -1 & 1 \end{pmatrix} \in M(d_2, d_2 + 1; \mathbb{Z}), \tag{102}$$

$$J_u := \begin{pmatrix} 1 & -1 & & & & \\ & 1 & -1 & & \text{\Large 0} & \\ & & 1 & & & \\ & & & \ddots & & \\ & \text{\Large 0} & & & -1 & \\ & & & & 1 & -1 \end{pmatrix} \in M(d_2, d_2 + 1; \mathbb{Z}). \tag{103}$$

In the above, we denoted the set of $p \times q$ integer matrices by $M(p, q; \mathbb{Z})$.

Example.

$$W_{(1,1)} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \tag{104}$$

$$W_{(2,1)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \end{pmatrix}. \tag{105}$$

Following lemma is used in Subsection 4.4. We can easily prove it by definitions of wight matrix $W_{(d_1,d_2)}$ of $\widetilde{Mp}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$:

Lemma 7

(i)

$$\sum_{j=0}^{d_2} \mathbf{w}_{u_{(i,j)}} = -e_{z_{i-1}} + 2e_{z_i} - e_{z_{i+1}} \ (i = 1, \ldots, d_1 - 1), \tag{106}$$

(ii)

$$\sum_{i=0}^{d_1} \mathbf{w}_{u_{(i,j)}} = -e_{w_{j-1}} + 2e_{w_j} - e_{w_{j+1}} \ (j = 1, \ldots, d_2 - 1), \tag{107}$$

(iii)

$$\left(\sum_{p=0}^{i-1} \sum_{q=j+1}^{d_2} \mathbf{w}_{u_{(p,q)}} \right)_{f_{(s,t)}} = \left(e_{f_{(i,j+1)}} \right)_{f_{(s,t)}} \ \left(\begin{array}{c} s = 1, \ldots, d_1, t = 1 \ldots, d_2, \\ i = 1, \ldots, d_1, j = 0, \ldots, d_2 - 1 \end{array} \right), \tag{108}$$

(iv)

$$\left(\sum_{p=i+1}^{d_1} \sum_{q=0}^{j-1} \mathbf{w}_{u_{(p,q)}} \right)_{f_{(s,t)}} = \left(e_{f_{(i+1,j)}} \right)_{f_{(s,t)}} \ \left(\begin{array}{c} s = 1, \ldots, d_1, t = 1 \ldots, d_2, \\ i = 0, \ldots, d_1 - 1, j = 1, \ldots, d_2 \end{array} \right), \tag{109}$$

where $\mathbf{w}_{u_{(i,j)}}$ *is the* $u_{(i,j)}$*–th columun vector of* $W_{(d_1,d_2)}$ *and* $(\mathbf{x})_{f_{(s,t)}}$ *is the* $f_{(s,t)}$*–component of a vector*

$$\mathbf{x} = (x_{z_0}, \ldots, x_{z_{d_1}}, x_{w_0} \ldots, x_{w_{d_2}}, x_{f_{(1,1)}}, \ldots, x_{f_{(1,d_2)}}, \ldots, x_{f_{(d_1,1)}}, \ldots, x_{f(d_1,d_2)}). \tag{110}$$

4.3 Toric Construction of $\widetilde{Mp}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$

In order to give toric construction of $X_{(d_1,d_2)} := \widetilde{Mp}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$, first, we must check that $X_{(d_1,d_2)}$ has vertex matrix.

Lemma 8 *There exists an integer matrix* $V_{(d_1,d_2)}$ *such that the following sequence*

$$0 \longrightarrow \mathbb{Z}^n \xrightarrow{{}^tV_{(d_1,d_2)}} \mathbb{Z}^r \xrightarrow{W_{(d_1,d_2)}} \mathbb{Z}^m \longrightarrow 0 \tag{111}$$

is exact.

Proof. We define the $n \times r$ matrix $V_{(d_1,d_2)}$ as follows:

$$V_{(d_1,d_2)}$$
$$:= \begin{array}{c} \\ z \\ w \\ G_{d_1} \\ G \\ G_{d_2} \end{array} \begin{array}{c} \begin{array}{ccccccccccc} a^1 & a^2 & b^1 & b^2 & \widetilde{u_0} & u_1 & u_2 & \cdots & u_{d_1-2} & u_{d_1-1} & \widetilde{u_{d_1}} \end{array} \\ \begin{pmatrix} I_{d_1+1} & -I_{d_1+1} & O & O & O & O & O & \cdots & O & O & O \\ O & O & I_{d_2+1} & -I_{d_2+1} & O & O & O & \cdots & O & O & O \\ O & \widetilde{A_{d_1}} & O & O & O & \widetilde{e_1} & \widetilde{e_2} & \cdots & \widetilde{e_{d_1-2}} & \widetilde{e_{d_1-1}} & O \\ O & G_a & O & G_b & -1^{d_2} & e_1 & e_1 & \cdots & e_1 & e_1 & e_1^- \\ O & O & O & \widetilde{A_{d_2}} & I_{d_2}^R & I_{d_2}^{LR} & I_{d_2}^{LR} & \cdots & I_{d_2}^{LR} & I_{d_2}^{LR} & I_{d_2}^L \end{pmatrix} \end{array}, \tag{112}$$

where new row labelings G_{d_1}, G_{d_2}, G are given as follows:

$$G_{d_1} = {}^t(g_1^{(d_1)}, g_2^{(d_1)}, \ldots, g_{d_1-1}^{(d_1)}),$$

$$G_{d_2} = {}^t(g_1^{(d_2)}, g_2^{(d_2)}, \ldots, g_{d_2-1}^{(d_2)}),$$

$$G = {}^t(g),$$

and e_1 and e_1^- are first canonical bases of $\mathbb{Z}^{d_2+1}$ and $\mathbb{Z}^{d_2}$, respectivery. Moreover,

$$\widetilde{A_N} := \begin{pmatrix} 1 & -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & \text{\Large 0} & \\ & & 1 & -2 & 1 & & & \\ & \text{\Large 0} & & & & \ddots & & \\ & & & & & 1 & -2 & 1 \end{pmatrix} \in M(N-1, N+1; \mathbb{Z}), \tag{113}$$

$$G_a := (1\ \ -1\ 0 \cdots 0) \in M(1, d_1+1; \mathbb{Z}), \tag{114}$$

$$G_b := (-1\ 1\ 0 \cdots 0) \in M(1, d_2+1; \mathbb{Z}), \tag{115}$$

$$-1^{d_2} := (-1 \ -1 \cdots -1) \in M(1, d_2; \mathbb{Z}), \tag{116}$$

$$\widetilde{e_i} := \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} < g_i^{(d_1)} \ \in M(d_1 - 1, d_2 + 1; \mathbb{Z}) \quad (i = 1, \dots, d_1), \tag{117}$$

$$I_{d_2}^R := \left(\begin{array}{ccccc|c} 1 & & & & & 0 \\ & 1 & & \text{\Large 0} & & 0 \\ & & \ddots & & & \vdots \\ & \text{\Large 0} & & 1 & & 0 \\ & & & & 1 & 0 \end{array}\right) \in M(d_2 - 1, d_2; \mathbb{Z}), \tag{118}$$

$$I_{d_2}^{LR} := \left(\begin{array}{c|ccccc|c} 0 & 1 & & & & & 0 \\ 0 & & 1 & & \text{\Large 0} & & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & & \text{\Large 0} & & 1 & & 0 \\ 0 & & & & & 1 & 0 \end{array}\right) \in M(d_2 - 1, d_2 + 1; \mathbb{Z}), \tag{119}$$

$$I_{d_2}^L := \left(\begin{array}{c|ccccc} 0 & 1 & & & & \\ 0 & & 1 & & \text{\Large 0} & \\ \vdots & & & \ddots & & \\ 0 & & \text{\Large 0} & & 1 & \\ 0 & & & & & 1 \end{array}\right) \in M(d_2 - 1, d_2; \mathbb{Z}). \tag{120}$$

Note that the center of $-\widetilde{A_N}$ is nothing but Cartan matrix A_{N-1}. Then we can easily see that the sequence (111) is exact by elementary row operation for $W_{(d_1,d_2)}$ as an integer matrix. $\square$

Example.

$$V_{(1,1)} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & -1 & 1 \end{pmatrix} \tag{121}$$

$$V_{(2,1)} = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 1 \end{pmatrix} \tag{122}$$

Next, we define the pseudo-fan corresponding to moduli space $X_{(d_1,d_2)}$.

Definition 6 *Let ν, ν_1, ν_2 be members of $\mathrm{Col}(d_1, d_2)$. Then we define P_ν and $Q_{(\nu_1,\nu_2)}$ as*

$$P_\nu := \begin{cases} \{v_{a_0^1}, v_{a_0^2}\} & (\nu = a_0) \\ \{v_{a_{d_1}^1}, v_{a_{d_1}^2}\} & (\nu = a_{d_1}) \\ \{v_{b_0^1}, v_{b_0^2}\} & (\nu = b_0) \\ \{v_{b_{d_2}^1}, v_{b_{d_2}^2}\} & (\nu = b_{d_2}) \\ \emptyset & (otherwise), \end{cases} \tag{123}$$

$$Q_{(\nu_1,\nu_2)} := \begin{cases} \{v_{a_i^1}, v_{a_i^2}, v_{u_{(i,j)}}\} & ((\nu_1,\nu_2) = (a_i, u_{(i,j)})\ (i = 1\ldots, d_1 - 1, j = 0\ldots, d_2)) \\ \{v_{b_j^1}, v_{b_j^2}, v_{u_{(i,j)}}\} & ((\nu_1,\nu_2) = (b_j, u_{(i,j)})\ (j = 1\ldots, d_2 - 1, i = 0\ldots, d_1)) \\ \{v_{u_{(i,j)}}, v_{u(k,l)}\} & ((\nu_1,\nu_2) = (u_{(i,j)}, u_{(k,l)})\ (0 \le i < k \le d_1, 0 \le l < j \le d_2)) \\ \emptyset & (otherwise), \end{cases} \tag{124}$$

where v_ν ($\nu \in \mathrm{Col}(d_1, d_2)$) is the ν–th column vector of $V_{(d_1,d_2)}$ in the proof of Lemma 8 (see (112)). Moreover, we define three sets

$$\mathrm{Ver}_{(d_1,d_2)}; = \mathrm{Ver}_{X_{(d_1,d_2)}} = \{v_\nu \mid \nu \in \mathrm{Col}(d_1, d_2)\}, \tag{125}$$

$$\Pi_{(d_1,d_2)} := \Pi_{X_{(d_1,d_2)}} = \big(\{P_\nu | \nu \in \mathrm{Col}(d_1,d_2)\} \cup \{Q_{(\nu_1,\nu_2)} | \nu_1,\nu_2 \in \mathrm{Col}(d_1,d_2)\}\big) \setminus \{\emptyset\}, \tag{126}$$

and

$$\Sigma_{(d_1,d_2)} := \Sigma(\mathrm{Ver}_{(d_1,d_2)}, \Pi_{(d_1,d_2)}). \tag{127}$$

Then the one of our main results in this paper can be stated as follows:

Theorem 7 *Psuedo-fan* $(\Sigma_{(d_1,d_2)}, \mathrm{Ver}_{(d_1,d_2)}, \Pi_{(d_1,d_2)})$ *of* $X_{(d_1,d_2)}$ *satisfies (AMVC).*

We will prove this theorem in next subsection. By Theorem 4 and Theorem 5, the following main theorem is obtained:

Theorem 8 $X_{(d_1,d_2)} = \widetilde{Mp}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1,d_2))$ *is a complete and simplicial toric variety.*

4.4 Proof of Theorem 7

We denote the set $(\{0,\ldots,d_1\} \times \{0,\ldots,d_2\}) \setminus \{(0,0),(d_1,d_2)\}$, namely, the set of indices of $u_{(i,j)}$ by I and define

$$J^+_{(i,j)} := \{(k,l) \in \{0,\ldots,d_1\} \times \{0,\ldots,d_2\} \bigm| k = i+1,\ldots,d_1, l = 0,\ldots,j-1\}, \tag{128}$$

$$J^-_{(i,j)} := \{(k,l) \in \{0,\ldots,d_1\} \times \{0,\ldots,d_2\} \bigm| k = 0,\ldots,i-1, l = j+1,\ldots,d_2\}, \tag{129}$$

and

$$J_{(i,j)} := J^+_{(i,j)} \cup J^-_{(i,j)} \tag{130}$$

for $(i,j) \in I$.

Before solving the system (55) in the case of $X_{(d_1,d_2)}$, we prove the following key lemma. It is very simple:

Lemma 9 *Let* $a, b_1, \ldots, b_m$ *be real numbers. Then* $\min\{a, b_i\} = 0$ *for all* $i = 1, \ldots, m$ *if and only if* $\min\{a, b_1 + \cdots + b_m\} = 0$ *and* $b_1, \ldots, b_m \geq 0$.

Proof. In the case of $a = 0$, it is obvious. Hence we assume that $a \neq 0$. If $\min\{a, b_i\} = 0$ for all $i = 1, \ldots, m$, then $a > 0$ and $b_1, \ldots, b_m = 0$ i.e., $b_1 + \cdots + b_m = 0$. Conversely, if $\min\{a, b_1 + \cdots + b_m\} = 0$ and $b_1, \ldots, b_m \geq 0$, then $a > 0$ by our assumption and $b_1 + \cdots + b_m = 0$. Since $b_1, \ldots, b_m$ are non-negative, $b_1 \ldots, b_m$ must be equal to 0. □

Now, we solve the system (55) using this lemma and Lemma 7. In the case of $P_{a_0}, P_{a_{d_1}}, P_{b_0}, P_{b_{d_2}}$, the corresponding min–value equations are

$$\begin{cases} \min\{-y_{a_0^1} + x_{z_0}, -y_{a_0^2} + x_{z_0}\} = 0 \quad (P_{a_0}) \\ \min\{-y_{a_{d_1}^1} + x_{z_{d_1}}, -y_{a_{d_1}^2} + x_{z_{d_1}}\} = 0 \quad (P_{a_{d_1}}) \\ \min\{-y_{b_0^1} + x_{w_0}, -y_{b_0^2} + x_{w_0}\} = 0 \quad (P_{b_0}) \\ \min\{-y_{b_{d_2}^1} + x_{w_{d_2}}, -y_{b_{d_2}^2} + x_{w_{d_2}}\} = 0 \quad (P_{b_{d_2}}), \end{cases} \tag{131}$$

and we can solve these equations as follows:

$$\begin{cases} x_{z_0} = \max\{y_{a_0^1}, y_{a_0^2}\} \\ x_{z_{d_1}} = \max\{y_{a_{d_1}^1}, y_{a_{d_1}^2}\} \\ x_{w_0} = \max\{y_{b_0^1}, y_{b_0^2}\} \\ x_{w_{d_2}} = \max\{y_{b_{d_2}^1}, y_{b_{d_2}^2}\}. \end{cases} \tag{132}$$

By (106) (in Lemma 7), Lemma 9, and the equalities

$$\min\{a, b, c\} = \min\{\min\{a, b\}, c\}, \tag{133}$$

$$\min\{-a + x, -b + x\} = x - \max\{a, b\}, \tag{134}$$

for each $i = 1, \dots, d_1 - 1$, equations corresponding to $Q_{(a_i, u_{(i,j)})}$ ($j = 0, \dots, d_2$) are combined into the following conditions consisting of one min–value equation and $d_2 + 1$ inequalities:

$$\begin{cases} \min\{-y_{a_i} + x_{z_i}, -y_{(i,0)} - \cdots - y_{(i,d_2)} - x_{z_{i-1}} + 2x_{z_i} - x_{z_{i+1}}\} = 0, \\ -y_{(i,j)} + \sum_{\mu \in \mathrm{Row}(d_1,d_2)} w_{(\mu, u_{(i,j)})} x_\mu \geq 0 \quad (j = 0, \dots, d_2). \end{cases} \tag{135}$$

Note here that we use the abbreviation: $y_{(i,j)} := y_{u_{(i,j)}}$, $y_{a_i} := \max\{y_{a_i^1}, y_{a_i^2}\}$. Since the inequalities in the second line of (135) appear later again (see (137) and the second line of (140) and (141)), it is enough to consider the following system consisting of $d_1 - 1$ min–value equations given in the first lines of (135), namely,

$$\begin{cases} \min\{-y_{a_1} + x_{z_1}, -y_{(1,0)} - \cdots - y_{(1,d_2)} - x_{z_0} + 2x_{z_1} - x_{z_2}\} = 0 \\ \min\{-y_{a_2} + x_{z_2}, -y_{(2,0)} - \cdots - y_{(2,d_2)} - x_{z_1} + 2x_{z_2} - x_{z_3}\} = 0 \\ \cdots \\ \min\{-y_{a_{d_1-1}} + x_{z_{d_1-1}}, -y_{(d_1-1,0)} - \cdots - y_{(d_1-1,d_2)} - x_{z_{d_1-2}} + 2x_{z_{d_1-1}} - x_{z_{d_1}}\} = 0, \end{cases} \tag{136}$$

where we recall that $x_{z_0} = \max\{y_{a_0^1}, y_{a_0^2}\}$ and $x_{z_{d_1}} = \max\{y_{a_{d_1}^1}, y_{a_{d_1}^2}\}$. However, it is already shown by Saito [15] that this system has unique solution of $x_{z_0}, \ldots, x_{z_{d_1}}$ for fixed y's by using the theory of piecewise linear maps. Thus we can find unique solution of $x_{w_1}, \ldots, x_{w_{d_2-1}}$ from the system corresponding to $Q_{(b_j, u_{(i,j)})}$ ($j = 1, \ldots, d_2 - 1, i = 0, \ldots, d_1$). Now, we denote these solutions by x'_{z_i}, x'_{w_j}, and define $y'_{(i,j)}$ ($(i,j) \in I$) as

$$y'_{(i,j)} := y_{(i,j)} - \sum_{p=0}^{d_1} w_{(z_p, u_{(i,j)})} x'_{z_p} - \sum_{q=0}^{d_2} w_{(w_q, u_{(i,j)})} x'_{w_q} \quad ((i,j) \in I). \tag{137}$$

The second line of (135) gives

$$\sum_{j=0}^{d_2} y'_{(i,j)} \leq 0 \quad (i = 1, \ldots, d_1 - 1). \tag{138}$$

Inequalities corresponding to $Q_{(b_j, u_{(i,j)})}$ ($j = 1, \ldots, d_2 - 1, i = 0, \ldots, d_1$) also gives

$$\sum_{i=0}^{d_1} y'_{(i,j)} \leq 0 \quad (j = 1, \ldots, d_2 - 1). \tag{139}$$

We see later that these inequalities are necessary for solvability of the system (55).

Next, we find the solution $x_{(p,q)} := x_{f_{(p,q)}}$ ($(p,q) \in \{1, \ldots, d_1\} \times \{1, \ldots, d_2\}$). Let (i,j) be fixed element of I. As above, from (109) (in Lemma 7) and Lemma 9, equations corresponding to $Q_{(u_{(i,j)}, u_{(k,l)})}$ ($(k,l) \in J^+_{(i,j)}$) are combined into

$$\begin{cases} \min\left\{-y'_{(i,j)} + \sum_{p=1}^{d_1} \sum_{q=1}^{d_2} w_{(f_{(p,q)}, u_{(i,j)})} x_{(p,q)}, -\sum_{p=i+1}^{d_1} \sum_{q=0}^{j-1} y'_{(p,q)} + x_{(i+1,j)}\right\} = 0, \\ -y'_{(k,l)} + \sum_{p=1}^{d_1} \sum_{q=1}^{d_2} w_{(f_{(p,q)}, u_{(k,l)})} x_{(p,q)} \geq 0 \quad (\text{for all } (k,l) \in J^+_{(i,j)}), \end{cases} \tag{140}$$

and the ones corresponding to $Q_{(u_{(i,j)}, u_{(k,l)})}$ ($(k,l) \in J^-_{(i,j)}$) are combined into

$$\begin{cases} \min\left\{-y'_{(i,j)} + \sum_{p=1}^{d_1} \sum_{q=1}^{d_2} w_{(f_{(p,q)}, u_{(i,j)})} x_{(p,q)}, -\sum_{p=0}^{i-1} \sum_{q=j+1}^{d_1} y'_{(p,q)} + x_{(i,j+1)}\right\} = 0, \\ -y'_{(k,l)} + \sum_{p=1}^{d_1} \sum_{q=1}^{d_2} w_{(f_{(p,q)}, u_{(k,l)})} x_{(p,q)} \geq 0 \quad (\text{for all } (k,l) \in J^-_{(i,j)}), \end{cases} \tag{141}$$

where we put $x_{(p,q)} := 0$ $((p,q) \notin \{1,\ldots,d_1\} \times \{1,\ldots,d_2\})$ and $y'_{(s,t)} := 0$ $((s,t) \notin I)$. Since

$$w_{(f_{(p,q)},u_{(s,t)})} = \begin{cases} -1 \ \ ((p,q) = (s,t),(s+1,t+1)) \\ 1 \ \ ((p,q) = (s+1,t),(s,t+1)) \\ 0 \ \ (otherwise) \end{cases} \tag{142}$$

for $p = 1,\ldots,d_1$,$q = 1\ldots,d_2$ and $(s,t) \in I$, (140) and (141) are further combined into

$$\begin{cases} \min\{-y'_{(i,j)} - x_{(i,j)} + x_{(i+1,j)} + x_{(i,j+1)} - x_{(i+1,j+1)}, \\ \qquad -\Delta^{(d_1,j-1)}_{(i+1,0)} - \Delta^{(i-1,d_2)}_{(0,j+1)} + x_{(i+1,j)} + x_{(i,j+1)}\} = 0, \\ -y'_{(k,l)} - x_{(k,l)} + x_{(k+1,l)} + x_{(k,l+1)} - x_{(k+1,l+1)} \geq 0 \ \ (\text{for all } (k,l) \in J_{(i,j)}), \end{cases} \tag{143}$$

where we introduce the following notation:

$$\Delta^{(s,t)}_{(p,q)} := \begin{cases} \sum_{k=p}^{s} \sum_{l=q}^{t} y'_{(p,q)} \ \ (p \leq s, q \leq t) \\ 0 \ \ (otherwise). \end{cases} \tag{144}$$

Since the l.h.s. of the inequalities in the second line of (140) or (141) corresponding to $Q_{(u_{(i,j)},u_{(k,l)})}$ also appear in the min–value equations in the first line of (140) or (141) corresponding to another $Q_{(u_{(i,j)},u_{(k,l)})}$, we can omit the second lines of (140) and (141). Therefore, the following system of min–value equations is equivalent to equations corresponding to $Q_{(u_{(i,j)},u_{(k,l)})}$ $((k,l) \in J_{(i,j)})$ in (55) about $X_{(d_1,d_2)}$:

$$\begin{aligned} &\min\{-y'_{(i,j)} - x_{(i,j)} + x_{(i+1,j)} + x_{(i,j+1)} - x_{(i+1,j+1)}, \\ &\qquad - \Delta^{(d_1,j-1)}_{(i+1,0)} - \Delta^{(i-1,d_2)}_{(0,j+1)} + x_{(i+1,j)} + x_{(i,j+1)}\} = 0 \ \ ((i,j) \in I). \end{aligned} \tag{145}$$

By (134), the above system is further rewritten as

$$x_{(i+1,j)} + x_{(i,j+1)} = \max\{y'_{(i,j)} + x_{(i,j)} + x_{(i+1,j+1)}, \Delta^{(d_1,j-1)}_{(i+1,0)} + \Delta^{(i-1,d_2)}_{(0,j+1)}\} \ \ ((i,j) \in I). \tag{146}$$

Lemma 10 *The unique solution of (146) is given by*

$$x_{(p,q)} = \max\{\Delta^{(d_1,q-1)}_{(p,0)}, \Delta^{(p-1,d_2)}_{(0,q)}\} \ \ ((p,q) \in \{1,\ldots,d_1\} \times \{1,\ldots,d_2\}). \tag{147}$$

Remark. The index (i, j) is used in $u_{(i,j)}$ and (p, q) is used in $f_{(p,q)}$. Index (i, j) of $x_{(i,j)}$ in (143), (145) and (146) comes from the index of $u_{(i,j)}$ via (142).

Proof. First we check that (147) satisfies the equation (146). Now, we assume that $i \neq 0, d_1$ and $j \neq 0, d_2$ in (146). We remark that

$$\begin{aligned}\max\{a, b\} + \max\{c, d\} &= \max\{a + \max\{c, d\}, b + \max\{c, d\}\} \\ &= \max\{\max\{a + c, a + d\}, \max\{b + c, b + d\}\} \\ &= \max\{a + c, a + d, b + c, b + d\}. \qquad (148)\end{aligned}$$

Then we get

$$\begin{aligned}x_{(i+1,j)} + x_{(i,j+1)} &= \max\{\Delta^{(d_1,j-1)}_{(i+1,0)}, \Delta^{(i,d_2)}_{(0,j)}\} + \max\{\Delta^{(d_1,j)}_{(i,0)}, \Delta^{(i-1,d_2)}_{(0,j+1)}\} \\ &= \max\{\Delta^{(d_1,j-1)}_{(i+1,0)} + \Delta^{(d_1,j)}_{(i,0)}, \Delta^{(d_1,j-1)}_{(i+1,0)} + \Delta^{(i-1,d_2)}_{(0,j+1)}, \\ &\quad \Delta^{(i,d_2)}_{(0,j)} + \Delta^{(d_1,j)}_{(i,0)}, \Delta^{(i,d_2)}_{(0,j)} + \Delta^{(i-1,d_2)}_{(0,j+1)}\}. \qquad (149)\end{aligned}$$

By (138) and (139), the inequality

$$\begin{aligned}\Delta^{(i,d_2)}_{(0,j)} + \Delta^{(d_1,j)}_{(i,0)} &= \Delta^{(d_1,j-1)}_{(i+1,0)} + \Delta^{(i-1,d_2)}_{(0,j+1)} + \sum_{j=0}^{d_2} y'_{(i,j)} + \sum_{i=0}^{d_1} y'_{(i,j)} \\ &\leq \Delta^{(d_1,j-1)}_{(i+1,0)} + \Delta^{(i-1,d_2)}_{(0,j+1)}, \qquad (150)\end{aligned}$$

holds. Therefore, we have

$$x_{(i+1,j)}+x_{(i,j+1)} = \max\{\Delta^{(d_1,j-1)}_{(i+1,0)}+\Delta^{(d_1,j)}_{(i,0)}, \Delta^{(d_1,j-1)}_{(i+1,0)}+\Delta^{(i-1,d_2)}_{(0,j+1)}, \Delta^{(i,d_2)}_{(0,j)}+\Delta^{(i-1,d_2)}_{(0,j+1)}\}. \qquad (151)$$

On the other hand, the r.h.s. of (146) can be computed as follows:

$$\begin{aligned}&\max\{y'_{(i,j)} + x_{(i,j)} + x_{(i+1,j+1)}, \Delta^{(d_1,j-1)}_{(i+1,0)} + \Delta^{(i-1,d_2)}_{(0,j+1)}\} \\ &= \max\big\{y'_{(i,j)} + \max\{\Delta^{(d_1,j-1)}_{(i,0)}, \Delta^{(i-1,d_2)}_{(0,j)}\} + \max\{\Delta^{(d_1,j)}_{(i+1,0)}, \Delta^{(i,d_2)}_{(0,j+1)}\}, \\ &\quad \Delta^{(d_1,j-1)}_{(i+1,0)} + \Delta^{(i-1,d_2)}_{(0,j+1)}\big\} \\ &= \max\big\{y'_{(i,j)} + \max\{\Delta^{(d_1,j-1)}_{(i,0)} + \Delta^{(d_1,j)}_{(i+1,0)}, \Delta^{(d_1,j-1)}_{(i,0)} + \Delta^{(i,d_2)}_{(0,j+1)}, \\ &\quad \Delta^{(i-1,d_2)}_{(0,j)} + \Delta^{(d_1,j)}_{(i+1,0)}, \Delta^{(i-1,d_2)}_{(0,j)} + \Delta^{(i,d_2)}_{(0,j+1)}\}, \Delta^{(d_1,j-1)}_{(i+1,0)} + \Delta^{(i-1,d_2)}_{(0,j+1)}\big\} \\ &= \max\big\{y'_{(i,j)} + \Delta^{(d_1,j-1)}_{(i,0)} + \Delta^{(d_1,j)}_{(i+1,0)}, y'_{(i,j)} + \Delta^{(d_1,j-1)}_{(i,0)} + \Delta^{(i,d_2)}_{(0,j+1)},\end{aligned}$$

$$y'_{(i,j)} + \Delta^{(i-1,d_2)}_{(0,j)} + \Delta^{(d_1,j)}_{(i+1,0)}, y'_{(i,j)} + \Delta^{(i-1,d_2)}_{(0,j)} + \Delta^{(i,d_2)}_{(0,j+1)}, \Delta^{(d_1,j-1)}_{(i+1,0)} + \Delta^{(i-1,d_2)}_{(0,j+1)}\}$$

$$= \max\Bigg\{ \Delta^{(d_1,j)}_{(i,0)} + \Delta^{(d_1,j-1)}_{(i+1,0)}, \Delta^{(d_1,j-1)}_{(i+1,0)} + \Delta^{(i-1,d_2)}_{(0,j+1)} + \sum_{j=0}^{d_2} y'_{(i,j)},$$

$$\Delta^{(i-1,d_2)}_{(0,j+1)} + \Delta^{(d_1,j-1)}_{(i+1,0)} + \sum_{i=0}^{d_1} y'_{(i,j)}, \Delta^{(i,d_2)}_{(0,j)} + \Delta^{(i-1,d_2)}_{(0,j+1)}, \Delta^{(d_1,j-1)}_{(i+1,0)} + \Delta^{(i-1,d_2)}_{(0,j+1)} \Bigg\}$$

$$= \max\{\Delta^{(d_1,j)}_{(i,0)} + \Delta^{(d_1,j-1)}_{(i+1,0)}, \Delta^{(i,d_2)}_{(0,j)} + \Delta^{(i-1,d_2)}_{(0,j+1)}, \Delta^{(d_1,j-1)}_{(i+1,0)} + \Delta^{(i-1,d_2)}_{(0,j+1)}\}. \quad (152)$$

Hence $x_{(p,q)} = \max\{\Delta^{(d_1,q-1)}_{(p,0)}, \Delta^{(p-1,d_2)}_{(0,q)}\}$ satisfy the system (146) in the case of $i \neq 0, d_1$ and $j \neq 0, d_2$. The cases $i = 0$ or $i = d_1$ or $j = 0$ or $j = d_2$ are similar to above but more simple. Thus we have proved existence of solution of the system (146). On the other hand, uniqueness of the solution is obvious. Indeed, $x_{(1,d_2)}$ is uniquely determined by the equation for $(i,j) = (0,d_2)$:

$$x_{(1,d_2)} = \max\{y'_{(0,d_2)}, \Delta^{(d_1,d_2-1)}_{(1,0)}\} \quad (153)$$

Then $x_{(1,d_2-1)}$ and $x_{(2,d_2)}$ are uniquely determined by the equations

$$x_{(1,d_2-1)} = \max\{y'_{(0,d_2-1)} + x_{(1,d_2)}, \Delta^{(d_1,d_2-2)}_{(1,0)}\} \ ((i,j) = (0, d_2-1)), \quad (154)$$

and

$$x_{(2,d_2)} = \max\{y'_{(1,d_2)} + x_{(1,d_2)}, \Delta^{(d_1,d_2-1)}_{(2,0)}\} \ ((i,j) = (1, d_2)). \quad (155)$$

Therefore, $x_{(2,d_2-1)}$ is uniquely determined by the equation for $(i,j) = (1, d_2-1)$:

$$x_{(2,d_2-1)} + x_{(1,d_2)} = \max\{y'_{(1,d_2-1)} + x_{(1,d_2-1)} + x_{(2,d_2)}, \Delta^{(d_1,d_2-2)}_{(2,0)} + \Delta^{(0,d_2)}_{(0,d_2)}\}. \quad (156)$$

By repeating it, the remaining $x_{(p,q)}$'s are inductively determined by remaining equations in (146). $\Box$

In this way, we have constructed unique solution of the system (55) about $X_{(d_1,d_2)}$ for arbitrary $y = (y_\rho)_{\rho \in \mathrm{Ver}_{(d_1,d_2)}} \in \mathbb{R}^r$, and therefore have completed the proof of Theorem 7.

5 Chow Ring of $\widetilde{Mp}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$ and Computation of Its Poincaré Polynomial

In this section, we give Chow ring and compute Poincaré polynomial of $X_{(d_1,d_2)} = \widetilde{Mp}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$ using Theorem 6 and Theorem 3.

5.1 Chow Ring of $\widetilde{Mp}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$

By Theorem 6, we can compute Chow ring of $X_{(d_1,d_2)}$:

Theorem 9

$$\begin{aligned} A^*_{\mathbb{Q}}(X_{(d_1,d_2)}) &\simeq H^*(X_{(d_1,d_2)}, \mathbb{Q}) \\ &\simeq \mathbb{Q}[h_1, \ldots, h_m] \Big/ \Big\langle \prod_{v_j \in S} (w_{(1,j)}h_1 + \cdots + w_{(m,j)}h_m) \Big| S \in \Pi_{(d_1,d_2)} \Big\rangle. \end{aligned} \tag{157}$$

Example. $((d_1, d_2) = (1, 1))$

$$A^*_{\mathbb{Q}}(X_{(1,1)}) \simeq \mathbb{Q}[h_1, \ldots, h_5]/\langle h_1^2, h_2^2, h_3^2, h_4^2, (-h_2 - h_3 + h_5)(-h_1 - h_4 + h_5)\rangle \tag{158}$$

Example. $((d_1, d_2) = (2, 1))$

$$\begin{aligned} A^*_{\mathbb{Q}}(X_{(2,1)}) \simeq &\mathbb{Q}[h_1, \ldots, h_7]/\langle h_1^2, h_3^2, h_4^2, h_5^2, h_2^2(-h_1 + h_2 + h_6 - h_7), \\ &h_2^2(h_2 - h_3 - h_6 + h_7), (-h_2 - h_4 + h_6)(-h_1 + h_2 + h_6 - h_7), \\ &(-h_2 - h_4 + h_6)(h_2 - h_3 - h_6 + h_7), \\ &(h_2 - h_3 - h_6 + h_7)(-h_2 - h_5 + h_7)\rangle \end{aligned} \tag{159}$$

5.2 Poincaré Polynomial of $\widetilde{Mp}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$

Next, we compute Poincaré Polynomial $P_{(d_1,d_2)}(t)$ of $X_{(d_1,d_2)}$ in the case of $(d_1, d_2) = (1, 1), (2, 1)$ by using Theorem 3 and give some conjecture for general (d_1, d_2). As mentioned above, computation of Poincaré polynomial of complete simplicial toric variety is replaced with counting the number of cones in its fan. More specifically, we must count the number of cones in $\Sigma_{(d_1,d_2)}(k)$ for $k = 0, \ldots, n$. But actually, thanks to the facts (iii) and (iv) in Theorem 3, we only have to count it for $k = 0, \ldots, (n-1)/2 = d_1 + d_2$. Moreover, $|\Sigma_{(d_1,d_2)}(k)|$ equals to the number

$$\Big|\{S \mid S \subset \mathrm{Ver}_{(d_1,d_2)}, |S| = k, \text{ and all } P \in \Pi_{(d_1,d_2)} \text{ are not contained in } S\}\Big| \tag{160}$$

by definition of $\Sigma_{(d_1,d_2)}$ (see (40) and (127)). Under the above preparations, we compute $P_{(1,1)}(t)$ and $P_{(2,1)}(t)$:

(i) $((d_1, d_2) = (1, 1))$ Since $|\Sigma_{(1,1)}(0)| = 1, |\Sigma_{(1,1)}(1)| = 10, |\Sigma_{(1,1)}(2)| = 40$, Betti numbers are

$$\begin{aligned} & b_0(X_{(1,1)}) = b_{10}(X_{(1,1)}) = |\Sigma_{(1,1)}(0)| = 1, \\ & b_2(X_{(1,1)}) = b_8(X_{(1,1)}) = |\Sigma_{(1,1)}(1)| - 5|\Sigma_{(1,1)}(0)| = 5, \\ & b_4(X_{(1,1)}) = b_6(X_{(1,1)}) \\ & = |\Sigma_{(1,1)}(2)| - 4|\Sigma_{(1,1)}(1)| + 10|\Sigma_{(1,1)}(0)| = 10, \\ & b_1(X_{(1,1)}) = b_3(X_{(1,1)}) = b_5(X_{(1,1)}) = b_7(X_{(1,1)}) = b_9(X_{(1,1)}) = 0. \end{aligned} \tag{161}$$

Therefore, Poincaré polynomial of $X_{(1,1)}$ is given by

$$\begin{aligned} P_{(1,1)}(t) &= 1 + 5t^2 + 10t^4 + 10t^6 + 5t^8 + t^{10} \\ &= (1 + t^2)^5. \end{aligned} \tag{162}$$

(ii) $((d_1, d_2) = (2, 1))$ Since $|\Sigma_{(2,1)}(0)| = 1, |\Sigma_{(2,1)}(1)| = 14, |\Sigma_{(2,1)}(2)| = 84, |\Sigma_{(2,1)}(3)| = 280$, Betti numbers are

$$\begin{aligned} & b_0(X_{(2,1)}) = b_{14}(X_{(2,1)}) = |\Sigma_{(2,1)}(0)| = 1, \\ & b_2(X_{(2,1)}) = b_{12}(X_{(2,1)}) = |\Sigma_{(2,1)}(1)| - 7|\Sigma_{(2,1)}(0)| = 7, \\ & b_4(X_{(2,1)}) = b_{10}(X_{(2,1)}) \\ & = |\Sigma_{(2,1)}(2)| - 6|\Sigma_{(2,1)}(1)| + 21|\Sigma_{(2,1)}(0)| = 21, \\ & b_6(X_{(2,1)}) = b_8(X_{(2,1)}) \\ & = |\Sigma_{(2,1)}(3)| - 5|\Sigma_{(2,1)}(2)| + 15|\Sigma_{(2,1)}(1)| - 35|\Sigma_{(2,1)}(0)| = 35, \\ & b_1(X_{(2,1)}) = b_3(X_{(2,1)}) = \cdots = b_{11}(X_{(2,1)}) = b_{13}(X_{(2,1)}) = 0. \end{aligned} \tag{163}$$

Therefore, Poincaré polynomial of $X_{(2,1)}$ is computed as

$$\begin{aligned} P_{(2,1)}(t) &= 1 + 7t^2 + 21t^4 + 35t^6 + 35t^8 + 21t^{10} + 7t^{12} + t^{14} \\ &= (1 + t^2)^7. \end{aligned} \tag{164}$$

Remark. From the above results, we are tempted to expect that $P_{(d_1,1)}(t) = (1 + t^2)^{2d_1+3}$, but it is not true. Indeed, from $|\Sigma_{(d_1,1)}(0)| = 1, |\Sigma_{(d_1,1)}(1)| = 4d_1 + 6, |\Sigma_{(d_1,1)}(2)| = (15d_1^2 + 43d_1 + 22)/2, \ldots$, lower cases of Betti numbers of $X_{(d_1,1)}$ are given by

$$\begin{aligned} & b_0(X_{(d_1,1)})(= b_{4d_1+6}(X_{(d_1,1)})) = 1, \\ & b_2(X_{(d_1,1)})(= b_{4d_1+4}(X_{(d_1,1)})) = 2d_1 + 3, \\ & b_4(X_{(d_1,1)})(= b_{4d_1+2}(X_{(d_1,1)})) = \frac{3d_1^2 + 13d_1 + 4}{2}, \\ & \ldots \end{aligned} \tag{165}$$

Hence, Poincaré polynomial $P_{(d_1,1)}(t)$ of $X_{(d_1,1)}$ is given by

$$\begin{aligned} P_{(d_1,1)}(t) =&1 + (2d_1+3)t^2 + \frac{3d_1^2+13d_1+4}{2}t^4 \\ &+\cdots+\frac{3d_1^2+13d_1+4}{2}t^{4d_1+2} + (2d_1+3)t^{4d_1+4} + t^{4d_1+6}. \end{aligned} \tag{166}$$

However, it is not equal to the following:

$$\begin{aligned} (1+t^2)^{2d_1+3} =&1 + (2d_1+3)t^2 + (d_1+1)(2d_1+3)t^4 \\ &+\cdots+(d_1+1)(2d_1+3)t^{4d_1+2} + (2d_1+3)t^{4d_1+4} + t^{4d_1+6}. \end{aligned} \tag{167}$$

By using computer, we can compute the Poincaré polynomials in higher bidegrees as follows:

$$\begin{aligned} P_{(2,2)}(t) &= (1+t^2)^5(1+5t^2+7t^4+5t^6+t^8) \\ &= 1+10t^2+42t^4+100t^6+151t^8+151t^{10}+100t^{12} \\ &+42t^{14}+10t^{16}+t^{18}, \end{aligned} \tag{168}$$

$$\begin{aligned} P_{(3,1)}(t) &= (1+t^2)^5(1+t^2+t^4)(1+3t^2+t^4) \\ &= 1+9t^2+35t^4+79t^6+116t^8+116t^{10}+79t^{12}+35t^{14} \\ &+9t^{16}+t^{18}, \end{aligned} \tag{169}$$

$$\begin{aligned} P_{(3,2)}(t) &= (1+t^2)^5(1+8t^2+19t^4+26t^6+19t^8+8t^{10}+t^{12}) \\ &= 1+13t^2+69t^4+211t^6+424t^8+594t^{10}+594t^{12} \\ &+424t^{14}+211t^{16}+69t^{18}+13t^{20}+t^{22}, \end{aligned} \tag{170}$$

$$\begin{aligned} P_{(3,3)}(t) &= (1+t^2)^5(1+12t^2+43t^4+88t^6+109t^8+88t^{10} \\ &+43t^{12}+12t^{14}+t^{16}) \\ &= 1+17t^2+113t^4+433t^6+1104t^8+2004t^{10}+2680t^{12} \\ &+2680t^{14}+2004t^{16}+1104t^{18}+433t^{20}+113t^{22}+17t^{24} \\ &+t^{26}, \end{aligned} \tag{171}$$

$$\begin{aligned} P_{(4,1)}(t) &= (1+t^2)^5(1+t^2+t^4)^2(1+4t^2+t^4) \\ &= 1+11t^2+52t^2+146t^6+277t^8+377t^{10}+377t^{12} \\ &+277t^{14}+146t^{16}+52t^{18}+11t^{20}+t^{22}, \end{aligned} \tag{172}$$

$$\begin{aligned} P_{(4,2)}(t) &= (1+t^2)^5(1+t^2+t^4)(1+10t^2+26t^4 \\ &+37t^6+26t^8+10t^{10}+t^{12}) \end{aligned}$$

$$
\begin{aligned}
&= 1 + 16t^2 + 102t^4 + 378t^6 + 939t^8 + 1674t^{10} + 2218t^{12} \\
&+ 2218t^{14} + 1674t^{16} + 939t^{18} + 378t^{20} + 102t^{22} + 16t^{24} \\
&+ t^{26}, \qquad (173)\\
P_{(4,3)}(t) &= (1+t^2)^5(1 + 16t^2 + 77t^4 + 210t^6 + 366t^8 + 440t^{10} \\
&+ 366t^{12} + 210t^{14} + 77t^{16} + 16t^{18} + t^{20}) \\
&= 1 + 21t^2 + 167t^4 + 765t^6 + 2351t^8 + 5221t^{10} + 8727t^{12} \\
&+ 11227t^{14} + 11227t^{16} + 8727t^{18} + 5221t^{20} + 2351t^{22} \\
&+ 765t^{24} + 167t^{26} + 21t^{28} + t^{30}, \qquad (174)\\
P_{(4,4)}(t) &= (1+t^2)^5(1 + 21t^2 + 132t^4 + 462t^6 + 1053t^8 + 1692t^{10} \\
&+ 1973t^{12} + 1692t^{14} + 1053t^{16} + 462t^{18} + 132t^{20} + 21t^{22} \\
&+ t^{24}) \\
&= 1 + 26t^2 + 247t^4 + 1342t^6 + 4898t^8 + 13003t^{10} \\
&+ 26264t^{12} + 41449t^{14} + 51890t^{16} + 51890t^{18} + 41449t^{20} \\
&+ 26264t^{22} + 13003t^{24} + 4898t^{26} + 1342t^{28} + 247t^{30} \\
&+ 26t^{32} + t^{34}. \qquad (175)
\end{aligned}
$$

We remark that $P_{(d_1,d_2)}(t) = P_{(d_2,d_1)}(t)$ since $X_{(d_1,d_2)} \simeq X_{(d_2,d_1)}$. From the above, we can give the first conjecture:

Conjecture 1 *The Poincaré polynomial $P_{(d_1,d_2)}(t)$ of moduli space $\widetilde{Mp}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))$ is divisible by polynomial $(1+t^2)^5$ and if $d_1 - d_2 > 1$, $P_{(d_1,d_2)}(t)$ is divisible by $(1+t^2+t^4)^{d_1-d_2-1}$.*

Hence, we set

$$p_{(d_1,d_2)}(t) := P_{(d_1,d_2)}(t)(1+t^2)^{-5}(1+t^2+t^4)^{-d_1+d_2+1}. \qquad (176)$$

At least, $p_{(d_1,d_2)}(t)$ is a polynomial in the case of $1 \le d_1 \le 4, 1 \le d_2 \le 4$. To find the relations among $P_{(d_1,d_2)}(t)$'s, we fix the second indice d_2 $(1 \le d_2 \le 3)$ and compute the differences

$$\delta^{(1)}_{(d_1,d_2)}(t) := p_{(d_1+1,d_2)}(t) - p_{(d_1,d_2)}(t) \qquad (177)$$

for $d_1 = 1, 2, 3$:

($d_2 = 1$)

$$\delta^{(1)}_{(1,1)}(t) = t^2, \tag{178}$$
$$\delta^{(1)}_{(2,1)}(t) = t^2, \tag{179}$$
$$\delta^{(1)}_{(3,1)}(t) = t^2. \tag{180}$$

($d_2 = 2$)

$$\delta^{(1)}_{(1,2)}(t) = 2t^2 + 5t^4 + 7t^6 + 5t^8 + 2t^{10}, \tag{181}$$
$$\delta^{(1)}_{(2,2)}(t) = 2t^2 + 6t^4 + 9t^6 + 6t^8 + 2t^{10}, \tag{182}$$
$$\delta^{(1)}_{(3,2)}(t) = 2t^2 + 7t^4 + 11t^6 + 7t^8 + 2t^{10}. \tag{183}$$

($d_2 = 3$)

$$\begin{aligned}\delta^{(1)}_{(1,3)}(t) &= 3t^2 + 15t^4 + 40t^6 + 68t^8 + 81t^{10} \\ &\quad + 68t^{12} + 40t^{14} + 15t^{16} + 3t^{18},\end{aligned} \tag{184}$$
$$\begin{aligned}\delta^{(1)}_{(2,3)}(t) &= 3t^2 + 18t^4 + 53t^6 + 95t^8 + 115t^{10} \\ &\quad + 95t^{12} + 53t^{14} + 18t^{16} + 3t^{18},\end{aligned} \tag{185}$$
$$\begin{aligned}\delta^{(1)}_{(3,3)}(t) &= 3t^2 + 21t^4 + 67t^6 + 126t^8 + 155t^{10} \\ &\quad + 126t^{12} + 67t^{14} + 21t^{16} + 3t^{18}.\end{aligned} \tag{186}$$

Furthermore, we consider differences

$$\delta^{(2)}_{(d_1,d_2)}(t) := \delta^{(1)}_{(d_1,d_2)}(t) - \delta^{(1)}_{(d_1,d_2)}(t), \tag{187}$$

and

$$\delta^{(3)}_{(d_1,d_2)}(t) := \delta^{(2)}_{(d_1,d_2)}(t) - \delta^{(2)}_{(d_1,d_2)}(t). \tag{188}$$

Then we can compute as

($d_2 = 2$)

$$\delta^{(2)}_{(1,2)}(t) = t^4(1 + t^2)^2, \tag{189}$$
$$\delta^{(2)}_{(2,2)}(t) = t^4(1 + t^2)^2. \tag{190}$$

$(d_2 = 3)$

$$\delta^{(2)}_{(1,3)}(t) = 3t^4 + 13t^6 + 27t^8 + 34t^{10} + 27t^{12} + 13t^{14} + 3t^{16}, \quad (191)$$

$$\delta^{(2)}_{(2,3)}(t) = 3t^4 + 14t^6 + 31t^8 + 40t^{10} + 31t^{12} + 14t^{14} + 3t^{16}, \quad (192)$$

$$\delta^{(3)}_{(1,3)}(t) = t^6(1+t^2)^4. \quad (193)$$

$$\quad (194)$$

Note that

$$\delta^{(1)}_{(d_1,d_2)}(t) = p_{(d_1+1,d_2)}(t) - p_{(d_1,d_2)}(t), \quad (195)$$

$$\delta^{(2)}_{(d_1,d_2)}(t) = p_{(d_1+2,d_2)}(t) - 2p_{(d_1+1,d_2)}(t) + p_{(d_1,d_2)}(t), \quad (196)$$

$$\delta^{(3)}_{(d_1,d_2)}(t) = p_{(d_1+3,d_2)}(t) - 3p_{(d_1+2,d_2)}(t) + 3p_{(d_1+1,d_2)}(t) - p_{(d_1,d_2)}(t). \quad (197)$$

Thus, we can give the second conjecture:

Conjecture 2 *The polynomial $p_{(d_1,d_2)}(t)$ satisfies the folowing relation:*

$$\sum_{k=0}^{d_2} (-1)^{d_2-k} \binom{d_2}{k} p_{(k+d_1,d_2)}(t) = t^{2d_2}(1+t^2)^{2(d_2-1)}. \quad (198)$$

In particular, for $d_2 = 1$,

$$p_{(d_1,1)}(t) = 1 + d_1 t^2 + t^4; \quad (199)$$

$$P_{(d_1,1)}(t) = (1+t^2)^5(1+t^2+t^4)^{d_1-2}(1+d_1t^2+t^4). \quad (200)$$

If we assume these conjectures and that $P_{(1,1)}(t), P_{(2,2)}(t), \ldots$ are already obtained, we can compute Poincaré polynomials $P_{(d_1,d_2)}(t)$ (or $p_{(d_1,d_2)}(t)$) for all (d_1, d_2) by following algorithm:

1. From Conjecture 2, we can determine the polynomials $p_{(l,1)}(t)$ ($l = 1, 2, \ldots$). Indeed, (198) for $d_2 = 1$ is

   $$p_{(l+1,1)}(t) - p_{(l,1)}(t) = t^2 \ (l = 1, 2, \ldots), \quad (201)$$

   and since $p_{(1,1)}(t)$ is already obatined, $p_{(2,1)}(t), p_{(3,1)}(t), \ldots$ are inductively determined by (201).

2. Since $P_{(1,2)}(t) = P_{(2,1)}(t)$, $p_{(1,2)}(t)$ can be determined by (176).

3. The equality (198) for $d_2 = 2$ is

$$p_{(l+2,2)}(t) - 2p_{(l+1,2)}(t) + p_{(l,2)}(t) = t^4(1+t^2)^2 \ (l = 1, 2, \dots). \quad (202)$$

   Since we already computed $p_{(1,2)}(t)$ in previous step and $p_{(2,2)}(t)$ was already obtained by our assumptions, $p_{(3,2)}(t), p_{(4,2)}(t), \dots$ are inductively determined by (202).

4. Since $P_{(1,3)}(t) = P_{(3,1)}(t)$ and $P_{(2,3)}(t) = P_{(3,2)}(t)$, $p_{(1,3)}(t)$ and $p_{(2,3)}(t)$ can be determined by (176).

5. Repeat the above processes. Then we can obtain $p_{(d_1,d_2)}(t)$ (and $P_{(d_1,d_2)}(t)$) for all (d_1, d_2).

From the above, we must expect the explicit form of $p_{(d,d)}(t)$ for all d. Thus, The first our future goal about studying Poincaré polynomials of $X_{(d_1,d_2)}$ is to give the conjecture such that determines $p_{(d,d)}(t)$ for all d.

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