Doctoral Thesis

HOKKAIDO UNIVERSITY

Studies on combinatorics of discriminantal arrangement

（判別的配置とその組合せ論的構造）

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March 2021

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Abstract

For a generic arrangement of n hyperplanes $\mathcal{A} = \{H_1^0, \ldots, H_n^0\}$ in $\mathbb{C}^k$, $k < n$ consider the space $\mathbb{S} \simeq \mathbb{C}^n$ of n-tuples $(H_1^{x_1}, \ldots, H_n^{x_n})$, $x_i \in \mathbb{C}$ of parallel translations of $(H_1^0, \ldots, H_n^0)$. The closed subset of $\mathbb{S}$ formed by translations of hyperplanes in $\mathcal{A}$ that fail to form a generic arrangement defines an arrangement of hyperplanes. This arrangement $\mathcal{B}(n, k, \mathcal{A})$ is defined as a generalization of raid arrangement by Manin and Schechtman, which they called the *discriminantal arrangement*. In particular, $\mathcal{B}(n, 1, \mathcal{A})$ coincides with the braid arrangement.

The discriminantal arrangements have several beautiful relations with diverse problems such as the Zamolodchikov equation with its relation to higher category theory, vanishing of cohomology of bundles on toric varieties, the representations of higher braid groups and naturally, with combinatorics. It is well known that there exists an open Zariski set $\mathcal{Z}$ in the space of (central) generic arrangements of n hyperplanes in $\mathbb{C}^k$, such that the intersection lattice of the discriminantal arrangement $\mathcal{B}(n, k, \mathcal{A})$ is independent from the choice of the arrangement $\mathcal{A} \in \mathcal{Z}$. The combinatorial structure of the discriminantal arrangements for very generic one is well-known, while for non very generic one lesser is known. In 2018 Libgober and Settepanella gave a description of combinatorics of discriminantal arrangement of rank 2 intersections.

In this thesis firstly, we begin with a review of definition of discriminantal arrangement and basic results on its combinatorics up to work of Libgober and Settepanella.
Secondly, we see some interesting applications which are based on the result given by Libgober and Settepanella. The content includes a simpler polynomial expression $\tilde{p}_{\mathbb{T}}(a_{ij})$ of a polynomial $p_{\mathbb{T}}(a_{ij})$ introduced by Athanasiadis, relation with a hypersurface in complex Grassmannian $Gr(3, \mathbb{C}^n)$, a restatement and a new proof of classical Pappus's hexagon theorem and retrieval the Hesse configuration of lines.
Finally, we provide a sufficient condition for an arrangement to be non very generic. The condition given here is nothing else but a generalization of dependency condition given by Libgober and Set-

tepanella. In particular, we introduce notions of (r, s)-dependency and $K_{\mathbb{T}}$-vector condition, and by using $K_{\mathbb{T}}$-vector condition we also provide numerical examples of non very generic arrangement, which would be helpful to understand $\mathcal{L}(\mathcal{B}(n, k, \mathcal{A}^0))$ for $\mathcal{A}^0$ a non very generic more deeply.

Contents

Chapter 1

Introduction

In 1989, Manin and Schechtman ([15]) introduced a family of arrangements of hyperplanes generalizing classical braid arrangements, which they called the *discriminantal arrangements* (p.209 [15]). Such an arrangement $\mathcal{B}(n, k, \mathcal{A}^0), n, k \in \mathbb{N}$ for $k \geq 2$ depends on a choice $\mathcal{A}^0 = \{H_1^0, \ldots, H_n^0\}$ of a collection of hyperplanes in the general position in $\mathbb{C}^k$, i.e., such that $\dim \bigcap_{i\in K,|K|=k} H_i^0 = 0$. It consists of parallel translations of $H_1^{x_1}, \ldots, H_n^{x_n}, (x_1, \ldots, x_n) \in \mathbb{C}^n$ which fail to form a general position arrangement in $\mathbb{C}^k$. $\mathcal{B}(n, k, \mathcal{A}^0)$ can be viewed as a generalization of the braid arrangement ([17]) with which $\mathcal{B}(n, 1) = \mathcal{B}(n, 1, \mathcal{A}^0)$ coincides.

These arrangements have several beautiful relations with diverse problems such as the Zamolodchikov equation with its relation to higher category theory (see Kapranov-Voevodsky [11], see also [9],[10]) the vanishing of cohomology of bundles on toric varieties (see [18]), the representations of higher braid groups (see [12]) and, naturally, with combinatorics. The latter is the connection we are mainly interested in and it goes from matroids to special configurations of points, from fiber polytopes to higher Bruhat orders. As for fiber polytopes and higher Bruhat orders, Manin and Schechtman introduced discriminantal arrangements as higher braid arrangements in order to introduce higher Bruhat orders which model the set of minimal path through a discriminantal arrangement.

From a different perspective, unknown in the literature of discriminantal arrangement until Athanasiadis pointed it out in 1999 (see [1]), Crapo introduced for the first time in 1985 ([4]) what he called *geometry of circuits* and which is the matroid $M(n, k, C)$ of circuits of the configuration C of n generic points in $\mathbb{R}^k$. The geometric lattice of flats of $M(n, k, C)$ is the intersection lattice of $\mathcal{B}(n,k,\mathcal{A}^0)$, $\mathcal{A}^0$ arrangement of n hyperplanes in $\mathbb{R}^k$ orthogonal to the vectors joining the origin with the n points in C (for further development see [5]).

Both Manin-Schecthman ([15]) and Crapo (see [4]) were mainly interested in arrangements $\mathcal{B}(n,k,\mathcal{A}^0)$ (resp. matroids $M(n,k,C)$) for which the intersection lattice is constant when $\mathcal{A}^0$ varies within a Zariski open set $\mathcal{Z}$ in the space of general position arrangements. Crapo shows that, in this case, the matroid $M(n,k)$ is isomorphic to the Dilworth completion of the k-th lower truncation of the Boolean algebra of rank n. More recently in [1], Athanasiadis proved a conjecture by Bayer and Brandt (see [3]) providing a full description of combinatorics of $\mathcal{B}(n,k,\mathcal{A}^0)$ when $\mathcal{A}^0$ belongs to $\mathcal{Z}$. Following [1] (more precisely Bayer and Brandt), we call arrangements $\mathcal{A}^0$ in $\mathcal{Z}$ *very generic* and non very generic otherwise.
However [15] does not describe the set $\mathcal{Z}$ of very generic arrangements explicitly, which, in time, lead to the misunderstanding that the combinatorial type of $\mathcal{B}(n,k,\mathcal{A}^0)$ was independent from the arrangement $\mathcal{A}^0$ (see for instance, [16], sect. 8, [17] or [13]). Neither [4] doesn't provide a description of $\mathcal{Z}$ even if Crapo presents an example of non very generic arrangement of 6 lines in $\mathbb{R}^2$. This corresponds to the case of 6 lines in generic position which admits a translation such that the 6 lines are respectively sides and diagonals of a quadrilateral as in Figure 1.1 (Crapo calls it a quadrilateral set). Few years later in 1996, Falk provided an higher dimensional example of non very generic arrangement of 6 planes in $\mathbb{R}^3$ (see [7]). Similar to Crapo's example, Falk's example too turned out to be related to a special configuration of lines, this time in projective plane (see [19],[20]).

In 2018 the first general result on non very generic arrangements is provided. In [14] Libgober and Settepanella described a necessary and sufficient *geometric* condition on arrangement $\mathcal{A}^0$ to be non very generic. This condition assures that $\mathcal{B}(n,k,\mathcal{A}^0)$ admits codimension 2 intersections of multiplicity 3 which do not exist in very generic case and it is given in terms of a notion of *dependency* for the arrangement $\mathcal{A}_\infty$ in $\mathbb{P}^{k-1}$ of hyperplanes $H_{\infty,1},\dots,H_{\infty,n}$ which are the intersections of projective closures of $H^0_1,\dots,H^0_n \in \mathcal{A}^0$ with the hyperplane at infinity. Their main result shows that $\mathcal{B}(n,k,\mathcal{A}^0), k > 1$ admits a codimension 2 intersections of multiplicity 3 if and only if $\mathcal{A}_\infty$ is an arrangement in $\mathbb{P}^{k-1}$ admitting a restriction[1] which is a dependent arrangement. This construction generalizes Falk's example which corresponds to the case $n = 6, k = 3$ and which has been object of study in two subsequent papers [19], [20], by Sawada, Settepanella and the candidate (see Section 3.2 and Chapter 4 respectively). In those papers authors proved how the arrangement $\mathcal{A}^0$ of 6 planes in $\mathbb{R}^3$ (resp. $\mathbb{C}^3$) for which the rank 2 intersections of $\mathcal{B}(6,2,\mathcal{A}^0)$ are in minimal number

[1] Here restriction is the standard restriction of arrangements to subspaces as defined in Section 2.1. See also [17].

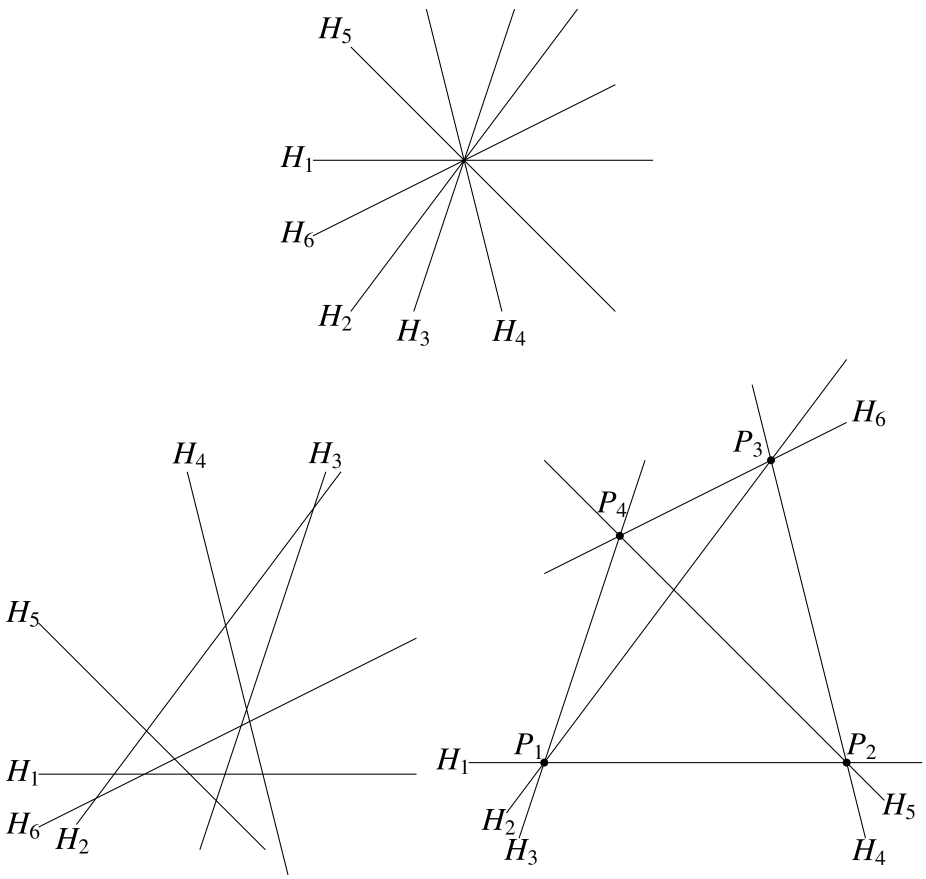

Figure 1.1: Six central lines on the above figure, their generic translation on the left figure and their non very generic translation on the right figure.

corresponds to Pappus's (resp. Hesse's) configuration providing a main example of what already conjectured by Crapo that the intersection lattice of discriminantal arrangement represents a combinatorial way to encode special configurations of points in the space. Notice that in [19] authors connected the non very generic arrangements $\mathcal{A}^0$ of n planes in $\mathbb{C}^3$ to well defined hypersurfaces in Grassmannian $Gr(3, \mathbb{C}^n)$.

In this thesis based on [19] we present a simplified Athanasiadis polynomial $\tilde{p}_{\mathbb{T}}(a_{ij})$ and hypersuface in Grassmannian. Based on [20] we present a new proof of Pappus's theorem. These results are both based on results of description of codimension 2 intersections of $\mathcal{L}(\mathcal{B}(n,k,\mathcal{A}))$ given by Libgober and Settepanella in [14]. We also advance the study of non very generic arrangements and generalize the dependency condition given in [14] providing a necessary and sufficient condition for non very generic intersections, i.e., intersections which doesn't exist in $\mathcal{B}(n,k,\mathcal{A}^0), \mathcal{A}^0 \in \mathcal{Z}$, to appear in rank $r \geq 2$ (see Chapter 5). In particular, we call an intersection of r hyperplanes in $\mathcal{B}(n, k, \mathcal{A}^0)$ which satisfies the property that the arrangement $\mathcal{A}^0$ is very generic *simple* if and only if all intersections of multiplicity r are of rank r (that is they are r hyperplanes intersecting transversally). Then we provide both geometric and algebraic necessary and sufficient conditions for existence of simple intersections of multiplicity r in rank strictly lower than r. The main interest of this result is firstly to connect configurations of non very generic points (i.e. points associated to non very generic arrangements) to a special family of graphs (called $K_{\mathbb{T}}$-configurations) which turns out to be very helpful in understanding $\mathcal{B}(n, k, \mathcal{A}^0)$ for $\mathcal{A}^0 \notin \mathcal{Z}$ as conjectured by Crapo in [4]. Secondly to reduce the geometric problem of the existence of special (non very generic) configurations of points to a combinatorial problem on the numerical properties that r subsets of indices $L_i \subset \{1, \ldots, n\}$ of cardinality $k + 1$ have to satisfy in order for a $K_{\mathbb{T}}$-configuration, $\mathbb{T} = \{L_1, \ldots, L_r\}$, to exists. This latter problem is left open together with the problem of necessary and sufficient conditions for existence of intersections in $\mathcal{B}(n, k, \mathcal{A}^0)$ which are nor simple nor very generic (i.e. intersections which can only exist in non very generic case).

The content of the thesis is organized as following. In Chapter 2 we recall the definition of discriminantal arrangement and basic results on intersection lattice of discriminantal arrangement up to work of Libgober and Settepanella. In Chapter 3 we see results given in [19] and [20], which are based on work of Libgober and Settepanella in [14]. In particular, as an application of description of codimension 2 intersections of $\mathcal{L}(\mathcal{B}(n, k, \mathcal{A}_\infty))$ given in [14] the authors in [19] showed that the set of generic arrangements $\mathcal{A}$ of n hyperplanes in $\mathbb{C}^3$ that contains a dependent subarrangement is the set of points in an hypersurface in Grassmannian $Gr(3, \mathbb{C}^n)$ such that each component is intersection of Grassmannian with a quadric. This is explained in Section 3.2. Moreover in [20] the authors gave a restatement and a new proof of classical Pappus's theorem, and study intersections of higher numbers of quadrics and Hesse configuration. These are explained in Chapter 4. In Chapter 5 firstly, we introduce the notion of $K_{\mathbb{T}}$-translation and $K_{\mathbb{T}}$-configuration providing a geometric condition for an arrangement $\mathcal{A}^0$ to be non very generic (Theorem 5.1.6). Secondly, we define the $K_{\mathbb{T}}$-vector condition for a set of vectors belonging to hyperplanes in $\mathcal{A}^0$. We prove that the existence of a finite number of sets of vectors, that we call $K_{\mathbb{T}}$-vector sets, which satisfy $K_{\mathbb{T}}$-vector condition is sufficient condition for $\mathcal{A}^0$ to be non very generic (Theorem 5.2.5). Finally we provide numerical examples of non very generic arrangements obtained by imposing conditions stated in Theorem 5.2.5.

Chapter 2

Preliminaries

2.1 Definition of discriminantal arrangement

Let $H_i^0, i = 1, \dots, n$ be a central arrangement in $\mathbb{C}^k, k < n$ which is generic[1], i.e., any m hyperplanes intersect in codimension m for any $m \leq k$. We will call such an arrangement a *central generic arrangement*. Space of parallel translations $\mathbb{S}(H_1^0, \dots, H_n^0)$ (or simply $\mathbb{S}$ when dependence on H_i^0 is clear or not essential) is the space of n-tuples of translations $H_1, \dots, H_n$ such that either $H_i \cap H_i^0 = \emptyset$ or $H_i = H_i^0$ for any $i = 1, \dots, n$.
One can identify $\mathbb{S}$ with n-dimensional affine space $\mathbb{C}^n$ in such a way that $(H_1^0, \dots, H_n^0)$ corresponds to the origin. In particular, an ordering of hyperplanes in $\mathcal{A}$ determines the coordinate system in $\mathbb{S}$ (see [14]).
Given a central generic arrangement $\mathcal{A}$ in $\mathbb{C}^k$ formed by hyperplanes $H_i, i = 1, \dots, n$ *the trace at infinity*, denoted by $\mathcal{A}_\infty$, is the arrangement formed by hyperplanes $H_{\infty,i} = \bar{H}_i^0 \cap H_\infty$ in the space $H_\infty \simeq \mathbb{P}^{k-1}(\mathbb{C})$, where $\bar{H}_i^0$ are hyperplanes in the compactification $\mathbb{P}^k(\mathbb{C})$ of $\mathbb{C}^k \simeq \mathbb{P}^k(\mathbb{C}) \setminus H_\infty$, projective closures of affine hyperplanes H_i^0. Notice that condition of genericity is equivalent to $\bigcup_i H_{\infty,i}^0$ being a normal crossing divisor in $\mathbb{P}^{k-1}(\mathbb{C})$, i.e., $\mathcal{A}_\infty$ is a generic arrangement.
The trace $\mathcal{A}_\infty$ of an arrangement $\mathcal{A}$ determines the space of parallel translations $\mathbb{S}$ (as a subspace in the space of n-tuples of hyperplanes in $\mathbb{P}^k$). Fixed a generic central arrangement $\mathcal{A}$, consider the closed subset of $\mathbb{S}$ formed by those collections which fail to form a generic arrangement. This

[1]Notice that, in general, generic, referred to an arrangement of hyperplanes, has a slightly different meaning. With an abuse of notation, we use the word *generic* in this case since the defined property is equivalent to the existence of a translated of the given central arrangement which is generic in the classical sense.

subset of $\mathbb{S}$ is a union of hyperplanes $D_L \subset \mathbb{S}$ (see [15]). Each hyperplane D_L corresponds to a subset $L = \{i_1, \dots, i_{k+1}\} \subset [n] := \{1, \dots, n\}$ and it consists of n-tuples of translations of hyperplanes $H_1^0, \dots, H_n^0$ in which translations of $H_{i_1}^0, \dots, H_{i_{k+1}}^0$ fail to form a generic arrangement. The arrangement $\mathcal{B}(n, k, \mathcal{A})$ of hyperplanes D_L is called *discriminantal arrangement* and has been introduced by Manin and Schechtman in [15] [2]. Notice that $\mathcal{B}(n, k, \mathcal{A})$ depends on the trace at infinity $\mathcal{A}_\infty$ hence it is sometimes more properly denoted by $\mathcal{B}(n, k, \mathcal{A}_\infty)$.

It is possible to define the discriminantal arrangement using normal vectors as follows (see [3]).

Definition 2.1.1 (Definition 2.2, [3]). *Let $\mathcal{A}$ be a generic arrangement of n hyperplanes in $\mathbb{C}^k$ with normal vectors $\alpha_1, \dots, \alpha_n$. The discriminantal arrangement $\mathcal{B}(n, k, \mathcal{A})$ is the arrangement in $\mathbb{C}^n$ of hyperplanes with normal vectors the distinct, nonzero vectors of the form*

$$\alpha_L = \sum_{i=1}^{k+1} (-1)^i \det(\alpha_{s_1}, \dots, \hat{\alpha_{s_i}}, \dots, \alpha_{s_{k+1}}) e_{s_i}, \tag{2.1}$$

where $\{s_1 < \dots < s_{k+1}\} \subset [n]$ and $\{e_j\}_{1 \leq j \leq n}$ is the standard basis of $\mathbb{C}^n$.

For a subset $\mathcal{A}' \subset \mathcal{A}$, let us denote by $X_{\mathcal{A}'} = \bigcap_{H \in \mathcal{A}'} H$ the intersection of its hyperplanes. The arrangement

$$\mathcal{A}^{X_{\mathcal{A}'}} = \{H \cap X_{\mathcal{A}'} | H \in \mathcal{A} \setminus \mathcal{A}', H \cap X_{\mathcal{A}'} \neq \emptyset\} \tag{2.2}$$

is called *a restriction* of $\mathcal{A}$ to $X_{\mathcal{A}'}$. Restrictions of $\mathcal{A}$ are in one-to-one correspondence with the splits $\mathcal{A} = \mathcal{A}' \cup \mathcal{A}''$ of the set of hyperplanes in $\mathcal{A}$ into a disjoint union. If $\mathcal{A}' = \emptyset$, then the restriction arrangement coincides with $\mathcal{A}$.

2.2 Discriminantal arrangement and its combinatorics

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement in $\mathbb{C}^k$. To study the combinatorial structure of $\mathcal{A}$ we introduce the notion of *intersection poset* $\mathcal{L}(\mathcal{A})$. $\mathcal{L}(\mathcal{A})$ is the poset of all nonempty intersections of hyperplanes, including V itself as intersection with empty, and ordered by reverse inclusion. That is $\mathcal{L}(\mathcal{A})$ is a poset defined as following:

$$\mathcal{L}(\mathcal{A}) = \{\textstyle\bigcap_{H \in \mathcal{B}} H \neq \emptyset | \mathcal{B} \subset \mathcal{A}\},$$
$$x < y \iff x \supset y \text{ for any } x, y \in \mathcal{L}(\mathcal{A}).$$

[2] Notice that Manin and Schechtman defined the discriminantal arrangement starting from a generic arrangement instead of its central translated as we do in this thesis. For our purpose the latter is a more convenient choice.

The poset $\mathcal{L}(\mathcal{A})$ has a semilattice structure and moreover if $\mathcal{A}$ is a central arrangement that is an intersection of all hyperplanes is nonempty, $\mathcal{L}(\mathcal{A})$ also has a lattice structure. Since in this thesis we assume $\mathcal{A}$ to be central, we can assume that the poset $\mathcal{L}(\mathcal{A})$ has a lattice structure.

Now, let us see brief history of researches on combinatorics of discriminantal arrangement.

In 1985 Crapo [4] introduced the matroid $M(n,k,C)$ of circuits of configuration C of n points in $\mathbb{R}^k$. The geometric lattice of $M(n,k,C)$ is the intersection lattice of $\mathcal{B}(n,k,\mathcal{A}^0)$, where $\mathcal{A}^0$ arrangement of n hyperplanes in $\mathbb{R}^k$ orthogonal to the vectors joining the origin with the n points in C. He proved that the intersection lattice of $\mathcal{B}(n,k,\mathcal{A}), \mathcal{A} \in \mathcal{Z}$ is isomorphic to the Dilworth completion of the k-times lower-truncated Boolean algebra B_n (see Theorem 2. page 149, [4]). This work is the first appearance of discriminantal arrangement and its combinatorics (for further development see [4]).

Independently of Crapo's work in 1989 Manin-Schechtman [15] defined discriminantal arrangement as a generalization of braid arrangement. They gave a description of elements of rank 2 intersections in $\mathcal{L}(\mathcal{B}(n,k,\mathcal{A}^0))$ for the case in which $\mathcal{L}(\mathcal{B}(n,k,\mathcal{A}^0))$ is constant when $\mathcal{A}^0$ varies within an open Zariski set $\mathcal{Z}$ (see Proposition 4 [15]). However they did neither complete the description of combinatorics nor describe $\mathcal{Z}$ explicitly.

In 1994 Falk [7] remarks that the discriminantal arrangement of n hyperplanes in $\mathbb{R}^k$ is associated to a Zonotope $\hat{Z}$, i.e., a fiber polytope (see Remark 2.3 in [7]) which associated graph $G(\hat{Z})$ turned out to be included in the diagram $B(n,k)$ of higher Bruhat order. Moreover he showed that, contrary to what was frequently stated (see for instance [16], [17] or [13]), the combinatorial structure of $\mathcal{B}(n,k,\mathcal{A}^0)$ indeed depends on the fixed generic arrangement $\mathcal{A}^0$. He provided a counter example such that the combinatorial structure would change when $\mathcal{A}^0$ varies outside of the Zariski open set $\mathcal{Z}$.

In 1997 Bayer and Brandt in [3] call the arrangements $\mathcal{A} \in \mathcal{Z}$ *very generic* and the ones which are not in $\mathcal{Z}$, *non very generic*. We use their terminology in the rest of this thesis. The name very generic comes from the fact that in this case the number of intersections of hyperplanes in $\mathcal{B}(n,k,\mathcal{A})$ is the largest possible for any (central) generic arrangement $\mathcal{A}$ of n hyperplanes in $\mathbb{C}^k$. A more precise description of this lattice is due to Athanasiadis who proved in [1] a conjecture by Bayer and Brandt which stated that the intersection lattice of the discriminantal arrangement in very generic case is isomorphic to the collection of all sets $\{S_1,\ldots,S_m\}$, where S_i are subsets of

$[n] = \{1, \ldots, n\}$, each of cardinality at least $k+1$, such that

$$| \bigcup_{i \in I} S_i | > k + \sum_{i \in I} (| S_i | - k) \text{ for all } I \subset [m] = \{1, \ldots, m\}, | I | \geq 2 \quad . \tag{2.3}$$

The isomorphism is the natural one which associate to the set S_i the space $D_{S_i} = \bigcap_{L \subset S_i, |L| = k+1} D_L, D_L \in \mathcal{B}(n, k, \mathcal{A})$ of all translated of $\mathcal{A}$ having hyperplanes indexed in S_i intersecting in a not empty space. In particular, $\{S_1, \ldots, S_m\}$ will correspond to the intersection $\bigcap_{i=1}^m D_{S_i}$.

If $\mathcal{A}$ is very generic and the condition in equation (2.3) is satisfied, this implies that the subspaces $D_{S_i}, i = 1, \ldots, m$ intersect transversally or, equivalently, that

$$\operatorname{rank} \bigcap_{i=1}^{m} D_{S_i} = \sum_{i=1}^{m} (| S_i | - k) \tag{2.4}$$

being rank $D_{S_i} =| S_i | - k$ (see Corollary 3.6 in [1]). Notice that the condition in equation (2.3) implies (see also [1]) that

$$\bigcap_{i \in I} D_{L_i} \neq D_S, | S | > k + 1 \text{ for any } I \subset [r] = \{1, \ldots, r\}, | I | \geq 2. \tag{2.5}$$

The fact that for all sets $\{S_1, \ldots, S_m\}$ which satisfy condition in equation (2.5) the condition in equation (2.4) is also satisfied corresponds to the definition provided by Crapo in [4] for discriminantal arrangement in very generic case (which he called geometry of circuits[3]). From those considerations we can get the following remark.

Remark 2.2.1. *Let $\mathcal{A}'$ be a translated of a central generic arrangement $\mathcal{A}$ such that the hyperplanes in $\mathcal{A}'$ indexed in subsets $L_i \subset [n]$, $|L_i| = k+1, i = 1, \ldots, r$ intersect in a point, i.e., $P_i = \bigcap_{p \in L_i} H_p \neq \emptyset$, and $\bigcap_{p \in L_i \cup \{t\}} H_p = \emptyset$. Then $\mathcal{A}'$ is an element in the intersection $\bigcap_{i=1}^r D_{L_i}$ and $\mathcal{A}' \notin D_{S_i}$ for any $S_i \supset L_i$, i.e. $\bigcap_{i \in I} D_{L_i} \neq D_S, | S | > k+1$ for any $I \subset [r] = \{1, \ldots, r\}, | I | \geq 2$. It follows that if $\mathcal{A}$ is very generic then D_{L_i} are r hyperplanes intersecting transversally, i.e.* rank $\bigcap_{i=1}^r D_{L_i} = r$.

We will use Remark 2.2.1 which essentially states that if a central generic arrangement $\mathcal{A}$ admits a translated $\mathcal{A}'$ such that the hyperplanes in $\mathcal{A}'$ indexed in subsets $L_i \subset [n], |L_i| = k+1, i = 1, \ldots, r$ intersect in a point, i.e. $P_i = \bigcap_{p \in L_i} H_p \neq \emptyset$, $\bigcap_{p \in L_i \cup \{t\}} H_p = \emptyset$, and the rank $\bigcap_{i=1}^r D_{L_i} < r$, then $\mathcal{A}$ is non very generic. The following definition will be very useful in Chapter 5.

[3]Here Crapo followed the preference of his advisor Rota who rarely used the name matroid.

Definition 2.2.2. *An element X in the intersection lattice of the discriminantal arrangement* $\mathcal{B}(n, k, \mathcal{A})$ *is said to be* ***simple*** *intersection if* $X = \bigcap_{i=1}^{r} D_{L_i}, |L_i| = k+1$ *and* $\bigcap_{i \in I} D_{L_i} \neq D_S, \mid S \mid > k+1$ *for any* $I \subset [r], \mid I \mid \geq 2$. *We call the number r of hyperplanes intersecting in X the multiplicity of the simple intersection X.*

The description of intersection lattice $\mathcal{L}(\mathcal{B}(n, k, \mathcal{A}^0))$ for $\mathcal{A}^0$ to be very generic is now well-understood.

Contrary to the very generic case, very few is known about the non very generic case. It is recent work that condition for $\mathcal{A}^0$ to be non very generic is given by Libgober and Settepanella [14]. In particular, the condition gives a full description for rank 2 elements of intersection lattice $L(\mathcal{B}(n, k, \mathcal{A}^0))$. Their condition is based on *dependency* of trace at infinity.

Lemma 2.2.3 (Lemma 3.1 [14]). *Let* $s \geq 2$, $n = 3s$, $k = 2s - 1$ *and* $\mathcal{A}$ *be a generic arrangement of n hyperplanes in* $\mathbb{C}^k$. *Given a set* $\mathbb{T} = \{L_1, L_2, L_3\}$ *of* $[n] = [3s]$, *consider the triple of codimension s subspaces* $H_{\infty,i,j} = \bigcap_{t \in L_i \cap L_j} H_{\infty,t}$ *of the hyperplane at infinity* H_∞. *Then* $H_{\infty,i,j}$ *span a proper subspace in* H_∞ *if and only if the codimension of* $D_{L_1} \cap D_{L_2} \cap D_{L_3}$ *is 2.*

This lemma motivates the notion of *dependency* as following.

Definition 2.2.4 (Definition 3.3 [14]). *An arrangement* $\mathcal{A}^0$ *of* $3s$ *hyperplanes in* $\mathbb{C}^{2s-1}$, $s \geq 2$ *is dependent if there exists a set* $\mathbb{T} = \{L_1, L_2, L_3\}$ *of subsets* $L_i \subset [3s]$ *such that* $\mid L_i \mid = 2s$, $\mid L_i \cap L_j \mid = s$, $\mid \bigcup_{i=1}^{3} L_i \mid = 3s$ *and spaces* $H_{i,j} = \bigcap_{p \in L_i \cap L_j} H_p$ *span a subspace of dimension* $2s - 2$ *in* $\mathbb{C}^{2s-1}$.

The main result in [14] is stated as following.

Theorem 2.2.5 (Theorem 3.9, [14]). *Let* $\mathcal{A}_\infty$ *be a generic arrangement of hyperplanes in* $\mathbb{P}^{k-1}$.
1. The arrangement $\mathcal{B}(n, k, \mathcal{A}_\infty)$ *has* $\binom{n}{k+2}$ *rank 2 element of multiplicity* $k + 2$.
2. There is a one to one correspondence between
(a) restriction arrangements of $\mathcal{A}_\infty$ *which are dependent, and*
(b) triples of hyperplanes in $\mathcal{B}(n, k, \mathcal{A}_\infty)$ *for which the rank of their intersection is equal to 2.*
3. There are no rank 2 element having multiplicity 4 unless $k = 3$. *All rank 2 elements of* $\mathcal{B}(n, k, \mathcal{A}_\infty)$ *not listed in part 1, have multiplicity either 2 or 3 (the latter corresponding to triples of hyperplanes in (b)).*
4. Combinatorial type of $\mathcal{B}(n, 2, \mathcal{A}_\infty)$ *is independent of* $\mathcal{A}$.

Until now the description of codimension 2 intersections is completed. In Chapter 5 we generalize this idea to obtain conditions for $\mathcal{A}^0$ to have r multiple intersections in rank $r - 1$.

Chapter 3

Polynomial $\widetilde{p}_{\mathbb{T}}(a_{ij})$ and hypersurface in Grassmannian $Gr(3, \mathbb{C}^n)$

In this chapter we present results given in [19]. In particular, as an application of description of codimension 2 intersections of $\mathcal{L}(\mathcal{B}(n, k, \mathcal{A}_\infty))$ given in [14], the authors in [19] showed that the set of generic arrangements $\mathcal{A}$ of n hyperplanes in $\mathbb{C}^3$ that contains a dependent subarrangement is the set of points in a hypersurface in Grassmannian $Gr(3, \mathbb{C}^n)$ such that each component is intersection of Grassmannian with a quadric. This is explained in Section 3.2.

3.1 Grassmannian $Gr(k, \mathbb{C}^n)$

To explain results in Section 3.2 and Chapter 4 we recall the notion of Grassmannian in this section. The *Grassmannian* $Gr(k, \mathbb{C}^n)$ is the set of k-dimensional subspaces of $\mathbb{C}^n$. Consider the map

$$\begin{aligned} \gamma : Gr(k, \mathbb{C}^n) &\to \mathbb{P}(\bigwedge^k \mathbb{C}^n) \\ < v_1, \dots, v_k > &\mapsto [v_1 \wedge \cdots \wedge v_k], \end{aligned} \tag{3.1}$$

where $v_1, \dots, v_k$ is a basis for the k-dimensional subspace.

This map is well-defined since if we take another basis as $v'_1, \dots, v'_k$, by the change of basis we can write $v'_i = \sum_{j=1}^k a_{ij} v_j$ for $a_{ij} \in \mathbb{C}$, thus we get $v'_1 \wedge \cdots \wedge v'_k = \det(a_{ij}) v_1 \wedge \cdots \wedge v_k$ and therefore $[v'_1 \wedge \cdots \wedge v'_k] = [v_1 \wedge \cdots \wedge v_k]$.

When $x \in \bigwedge^k \mathbb{C}^n$ is of the form $x = v_1 \wedge \cdots \wedge v_k$ for some $v_1, \ldots, v_k \in \mathbb{C}^n$, x is called *totally decomposable*. Then the following lemma holds (see, for instance, p.64 in [8]).

Lemma 3.1.1. *For $x \in \bigwedge^k \mathbb{C}^n$ and $x \neq 0$ let $\varphi_x : \mathbb{C}^n \to \bigwedge^{k+1} \mathbb{C}^n$ be a linear map defined by $\varphi_x(v) = v \wedge x$. Then* dim*ker $(\varphi_x) \leq k$ and* dim*ker $(\varphi_x) = k$ if and only if x is totally decomposable. If $x = v_1 \wedge \cdots \wedge v_k$ then ker $(\varphi_x) = < v_1, \ldots, v_k >$.*

Using Lemma 3.1.1 we can see that the map γ is injective and $\gamma(Gr(k,\mathbb{C}^n))$ is closed in $\mathbb{P}(\bigwedge^k \mathbb{C}^n)$ thus $Gr(k,\mathbb{C}^n)$ can be regarded as a projective variety in $\mathbb{P}(\bigwedge^k \mathbb{C}^n) = \mathbb{P}^{\binom{n}{k}-1}$.

γ maps $< v_1, \ldots, v_k >$ to $[v_1 \wedge \cdots \wedge v_k]$ by definition, and by Lemma 3.1.1 when $x = v_1 \wedge \cdots \wedge v_k$ is given then $< v_1, \ldots, v_k > =$ ker (φ_x). Thus we can construct inverse image by sending $[x]$ to ker (φ_x), that is we can recover k-dimensional vector subspace $< v_1, \ldots, v_k >$ from x, and so we get injection of γ.

The map γ is called the *Plücker embedding*. We assume the identification $Gr(k,n)$ with $\gamma(Gr(k,\mathbb{C}^n)) \subset \mathbb{P}(\bigwedge^k \mathbb{C}^n)$ from now on.

By Lemma 3.1.1 it follows that $[x] \in \mathbb{P}(\bigwedge^k \mathbb{C}^n)$ is in $Gr(k,\mathbb{C}^n)$ if and only if dim$ker(\varphi_x) = k$ i.e. ker $\varphi_x =< v_1, \ldots, v_k >$. If $e_1, \ldots, e_n$ is a basis of $\mathbb{C}^n$ then $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$, $I = \{i_1, \ldots, i_k\} \subset [n], i_1 < \cdots < i_k$, is a basis for $\bigwedge^k \mathbb{C}^n$ and $x \in \bigwedge^k \mathbb{C}^n$ can be written uniquely as

$$x = \sum_{\substack{I \subseteq [n] \\ |I|=k}} \beta_I e_I = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \beta_{i_1 \ldots i_k} (e_{i_1} \wedge \cdots \wedge e_{i_k}) \tag{3.2}$$

where β_I, called the Plücker coordinates of x, are homogeneous coordinates on $\mathbb{P}(\bigwedge^k \mathbb{C}^n) = \mathbb{P}^{\binom{n}{k}-1}$ associated to the ordered basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$. With this choice of basis for $\mathbb{C}^n$ the matrix $M_x = (b_{ij})$ associated to φ_x is the $\binom{n}{k+1} \times n$ matrix with rows indexed by ordered subsets $I = \{i_1, \ldots, i_{k+1}\} \subseteq [n]$, and entries

$$b_{ij} = \begin{cases} (-1)^l \beta_{I \setminus \{j\}} & \text{if } j = i_l \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Plücker relations, i.e conditions for dim(ker φ_x) $= k$, are vanishing conditions of all $(n-k+1) \times (n-k+1)$ minors of M_x. Thus image of Plücker embedding can be expressed as the set of common zero locus of collection of homogeneous polynomials of degree $n-k+1$ in the β_I and so the image is closed.

It is well known (see for instance [8]) that Plücker relations are degree 2 relations and they can also be written as

$$\sum_{l=0}^{k}(-1)^l\beta_{i_1\ldots i_{k-1}j_l}\beta_{j_0\ldots\hat{j}_l\ldots j_k} = 0 \tag{3.3}$$

for any $2k$-tuple $(i_1, \ldots, i_{k-1}, j_0, \ldots, j_k)$.

3.2 Polynomial $\widetilde{p}_{\mathbb{T}}(a_{ij})$ and hypersurfaces in Grassmannian $Gr(3, \mathbb{C}^n)$

Let $\alpha_i = (a_{i1}, \ldots, a_{ik})$ be the normal vectors of hyperplanes H_i^0, $1 \leq i \leq n$, in the generic arrangement $\mathcal{A}$ in $\mathbb{C}^k$. Normal here is intended with respect to the unitary inner product. Then the normal vectors to hyperplanes D_L, $L = \{s_1 < \cdots < s_{k+1}\} \subset [n]$ in $\mathbb{S} \simeq \mathbb{C}^n$ are nonzero vectors of the form

$$\alpha_L = \sum_{i=1}^{k+1}(-1)^i \det(\alpha_{s_1}, \ldots, \hat{\alpha_{s_i}}, \ldots, \alpha_{s_{k+1}})e_{s_i}, \tag{3.4}$$

where $\{e_j\}_{1\leq j\leq n}$ is the standard basis of $\mathbb{C}^n$ (see Definition 2.1.1).

Let $\mathcal{P}_{k+1}([n]) = \{L \subset [n] \mid |L| = k + 1\}$ be the set of cardinality $k + 1$ subsets of $[n]$, we denote by

$$A(\mathcal{A}_\infty) = (\alpha_L)_{L\in\mathcal{P}_{k+1}([n])} \tag{3.5}$$

the matrix having in each row the entries of vectors α_L normal to hyperplanes D_L and by $A_{\mathbb{T}}(\mathcal{A}_\infty)$ the submatrix of $A(\mathcal{A}_\infty)$ with rows α_L, $L \in \mathbb{T}$, $\mathbb{T} \subset \mathcal{P}_{k+1}([n])$ of cardinality m.

The construction naturally holds also in real case, i.e. $\mathcal{A}$ arrangement in $\mathbb{R}^k$. In this case Athanasiadis (see [1]) defined the polynomial

$$p_{\mathbb{T}}(a_{ij}) = \sum_{\substack{J\subset[n] \\ |J|=m}} \det[A_{\mathbb{T},J}(\mathcal{A}_\infty)]^2 \tag{3.6}$$

in the variable a_{ij} given by the sum of the squares of determinants of the $m \times m$ submatrices $A_{\mathbb{T},J}$ of $A_{\mathbb{T}}(\mathcal{A}_\infty)$ obtained considering columns $j \in J$.

Following [19], [20] we introduce here the notion of good $3s$-partition.

Definition 3.2.1. *Given $s \geq 2$ and $n \geq 3s$, consider the set $\mathbb{T} = \{L_1, L_2, L_3\}$, with L_i subsets of $[n]$ such that $|L_i| = 2s$, $|L_i \cap L_j| = s (i \neq j)$, $L_1 \cap L_2 \cap L_3 = \emptyset$ (in particular $|\bigcup L_i| = 3s$) with a choice $L_1 = \{i_1, \ldots, i_{2s}\}$, $L_2 = \{i_{s+1}, \ldots, i_{3s}\}$, $L_3 = \{i_1, \ldots, i_s, i_{2s+1}, \ldots, i_{3s}\}$. We call the set $\mathbb{T} = \{L_1, L_2, L_3\}$ a good $3s$-partition.*

Notice that if $\mathcal{A}$ is a generic arrangement in $\mathbb{R}^k$, if $\mathbb{T} = \{L_1, L_2, L_3\}$ is a good $3s$-partition then the condition in Lemma 2.2.3 is equivalent to $p_{\mathbb{T}}(a_{ij}) = 0$.

Remark 3.2.2. *Notice that vectors α_L in equation (3.4) normal to hyperplanes D_L correspond to rows $I = L$ in the Plücker matrix M_x, that is*

$$A(\mathcal{A}_\infty) = M_x.$$

For this reason, following [19], we will call $A(\mathcal{A}_\infty)$ Plücker matrix. Notice that, in particular, $\det(\alpha_{s_1}, \ldots, \hat{\alpha_{s_i}}, \ldots, \alpha_{s_{k+1}})$ *is the Plücker coordinate $\beta_I, I = \{s_1, \ldots, s_{k+1}\} \backslash \{s_i\}$.*

In the following section we give an example to illustrate the general Theorem in Subsection 3.2.2 (see also [19]). This example appears also in [7], [14] and, in the context of oriented matroids, in [2].

3.2.1 Main example $\mathcal{B}(6, 3, \mathcal{A}_\infty)$ in a real case

Consider $\mathcal{A} = \{H_1^0, H_2^0, \ldots, H_6^0\}$ be a generic arrangement of hyperplanes in $\mathbb{R}^3$ with normal vectors $\alpha_i = (a_{i1}, a_{i2}, a_{i3})$, $1 \leq i \leq 6$ and $H_i^{t_i}$ be hyperplane obtained by translating H_i^0 along direction α_i, i.e., $H_i^{t_i} = H_i^0 + t_i\alpha_i$, $t_i \in \mathbb{R}$. Let $\mathbb{T} = \{L_1, L_2, L_3\}$ be the good 6-partition with $L_1 = \{1, 2, 3, 4\}, L_2 = \{1, 2, 5, 6\}$ and $L_3 = \{3, 4, 5, 6\}$, then

$$A_{\mathbb{T}}(\mathcal{A}_\infty) = \begin{pmatrix} \alpha_{L_1} \\ \alpha_{L_2} \\ \alpha_{L_3} \end{pmatrix} = \begin{pmatrix} -\beta_{234} & \beta_{134} & -\beta_{124} & \beta_{123} & 0 & 0 \\ -\beta_{256} & \beta_{156} & 0 & 0 & -\beta_{126} & \beta_{125} \\ 0 & 0 & -\beta_{456} & \beta_{356} & -\beta_{346} & \beta_{345} \end{pmatrix}, \quad \beta_{ijk} = \det \begin{pmatrix} a_{i1} & a_{j1} & a_{k1} \\ a_{i2} & a_{j2} & a_{k2} \\ a_{i3} & a_{j3} & a_{k3} \end{pmatrix}$$

is a submatrix of the Plücker matrix $A(\mathcal{A}_\infty)$.

Let $\alpha_i \times \alpha_j$ be the cross product of α_i, α_j corresponding to the direction orthogonal to both α_i and α_j and denote by $(\alpha_i \times \alpha_{i+1})$ the matrix $\begin{pmatrix} \alpha_1 \times \alpha_2 \\ \alpha_3 \times \alpha_4 \\ \alpha_5 \times \alpha_6 \end{pmatrix}$. Then $\alpha_i \times \alpha_j$ is the direction of the line $H_i \cap H_j$, since

α_i and α_j are, respectively, directions orthogonal to H_i and H_j and rank$A_{\mathbb{T}}(\mathcal{A}_\infty) = 2$ if and only if rank$(\alpha_i \times \alpha_{i+1}) = 2$. Indeed rank$(A_{\mathbb{T}}(\mathcal{A}_\infty)) = 2$ is equivalent to codim $(D_{L_1} \cap D_{L_2} \cap D_{L_3}) = 2$, hence by Lemma 2.2.3, the points $\bigcap_{i\in L_1\cap L_2} \bar{H}_i^{t_i} \cap H_\infty = \bar{H}_3^{t_3} \cap \bar{H}_4^{t_4} \cap H_\infty$, $\bigcap_{i\in L_1\cap L_3} \bar{H}_i^{t_i} \cap H_\infty = \bar{H}_1^{t_1} \cap \bar{H}_2^{t_2} \cap H_\infty$, and $\bigcap_{i\in L_2\cap L_3} \bar{H}_i^{t_i} \cap H_\infty = \bar{H}_5^{t_5} \cap \bar{H}_6^{t_6} \cap H_\infty$ are collinear, that is directions of $H_i^{t_i} \cap H_{i+1}^{t_{i+1}}$ are dependent and hence rank$(\alpha_i \times \alpha_{i+1}) = 2$ (see Fig. 3.1).

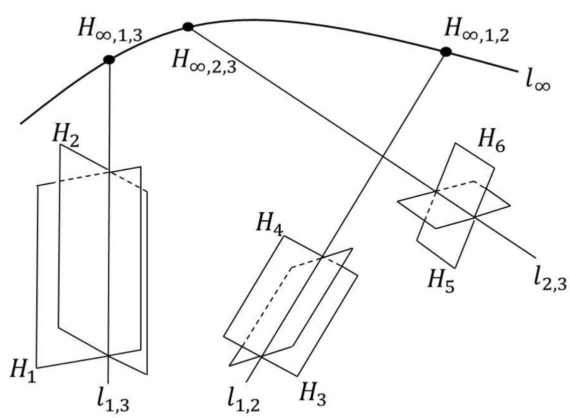

Figure 3.1: Picture of case $\mathcal{B}(6, 3, \mathcal{A}_\infty^0)$

The rank of $A_{\mathbb{T}}(\mathcal{A}_\infty)$ is equal to 2 if and only if all 3×3 minors of $A_{\mathbb{T}}(\mathcal{A}_\infty)$ are equal to 0.

Therefore the rank of $A_{\mathbb{T}}(\mathcal{A}_\infty)$ is equal to 2 if and only if β_{ijk} are solutions of the system: [1]

$$(I)\begin{cases}-\beta_{456}(\beta_{134}\beta_{256}-\beta_{234}\beta_{156})=0\\ \beta_{356}(\beta_{134}\beta_{256}-\beta_{234}\beta_{156})=0\\ -\beta_{346}(\beta_{134}\beta_{256}-\beta_{234}\beta_{156})=0\\ \beta_{345}(\beta_{134}\beta_{256}-\beta_{234}\beta_{156})=0\\ -\beta_{256}(\beta_{124}\beta_{356}-\beta_{123}\beta_{456})=0\\ \beta_{156}(\beta_{124}\beta_{356}-\beta_{123}\beta_{456})=0\\ -\beta_{126}(\beta_{124}\beta_{356}-\beta_{123}\beta_{456})=0\\ \beta_{125}(\beta_{124}\beta_{356}-\beta_{123}\beta_{456})=0\\ -\beta_{234}(\beta_{125}\beta_{346}-\beta_{126}\beta_{345})=0\\ \beta_{134}(\beta_{125}\beta_{346}-\beta_{126}\beta_{345})=0\\ -\beta_{124}(\beta_{125}\beta_{346}-\beta_{126}\beta_{345})=0\\ \beta_{123}(\beta_{125}\beta_{346}-\beta_{126}\beta_{345})=0\end{cases} \quad\text{and}\quad (II)\begin{cases}\beta_{234}\beta_{126}\beta_{456}+\beta_{124}\beta_{256}\beta_{346}=0\\ -(\beta_{234}\beta_{125}\beta_{456}+\beta_{124}\beta_{256}\beta_{345})=0\\ -(\beta_{234}\beta_{126}\beta_{356}+\beta_{123}\beta_{256}\beta_{346})=0\\ \beta_{234}\beta_{125}\beta_{356}+\beta_{123}\beta_{256}\beta_{345}=0\\ -(\beta_{134}\beta_{126}\beta_{456}+\beta_{124}\beta_{156}\beta_{346})=0\\ \beta_{134}\beta_{125}\beta_{456}+\beta_{124}\beta_{156}\beta_{345}=0\\ \beta_{134}\beta_{126}\beta_{356}+\beta_{123}\beta_{156}\beta_{346}=0\\ -(\beta_{134}\beta_{125}\beta_{356}+\beta_{123}\beta_{156}\beta_{345})=0\end{cases} \tag{3.7}$$

and polynomial $p_{\mathbb{T}}(a_{ij})$ is

$$p_{\mathbb{T}}(a_{ij})=\sum_{\substack{J\subset[6]\\|J|=3}}\det(A_{\mathbb{T},J})^2=(\beta_{134}\beta_{256}-\beta_{234}\beta_{156})^2(\sum_{\substack{I_1\subset\{3,4,5,6\}\\|I_1|=3}}\beta^2_{I_1})+(\beta_{124}\beta_{356}-\beta_{123}\beta_{456})^2(\sum_{\substack{I_2\subset\{1,2,5,6\}\\|I_2|=3}}\beta^2_{I_2})$$
$$+(\beta_{125}\beta_{346}-\beta_{126}\beta_{345})^2(\sum_{\substack{I_3\subset\{1,2,3,4\}\\|I_3|=3}}\beta^2_{I_3})+\sum_{\substack{i=5,6\\j=3,4}}(\beta_{234}\beta_{12i}\beta_{j56}+\beta_{12j}\beta_{256}\beta_{34i})^2+\sum_{\substack{i=5,6\\j=3,4}}(\beta_{134}\beta_{12i}\beta_{j56}+\beta_{12j}\beta_{156}\beta_{34i})^2.$$

On the other hand the condition $\mathrm{rank}(\alpha_i\times\alpha_{i+1})=2$ is simply $\det(\alpha_i\times\alpha_{i+1})=0$ and if we define

$$\widetilde{p}_{\mathbb{T}}(a_{ij})=[\det(\alpha_i\times\alpha_{i+1})]^2=\{(a_{12}a_{23}-a_{13}a_{22})\Delta_{11}+(a_{11}a_{23}-a_{13}a_{21})\Delta_{12}+(a_{11}a_{22}-a_{12}a_2)\Delta_{13}\}^2, \tag{3.8}$$

Δ_{1l} cofactors of $(\alpha_i\times\alpha_{i+1})$, then $p_{\mathbb{T}}(a_{ij})=0$ if and only if $\widetilde{p}_{\mathbb{T}}(a_{ij})=0$. That is polynomial $\widetilde{p}_{\mathbb{T}}(a_{ij})$ is a polynomial of, in general, lower degree than $p_{\mathbb{T}}(a_{ij})$ with the same set of zeros.

3.2.2 Polynomial $\widetilde{p}_{\mathbb{T}}(a_{ij})$ in $\mathcal{B}(n,k,\mathcal{A}_\infty)$

Let $\mathcal{A}=\{H_1,\dots,H_n\}$ be a generic arrangement of hyperplanes in $\mathbb{R}^k$ and $\mathbb{T}=\{L_1,L_2,L_3\}$ be a good $3s$-partition of indices in $[n]$. If α_τ are normal vectors to $H_\tau\in\mathcal{A}$, $\tau=1,\dots,n$, $T=\{j_1,\cdots,j_t\}$

[1] See [21] for more detailed computations.

a subset of $[n]$ which has empty intersection with $L_1 \cup L_2 \cup L_3$, define vector spaces

$$U_{i,j}^{\perp} = \{v \in \mathbb{R}^k \mid v \cdot \alpha_\tau = 0, \tau \in L_i \cap L_j\},$$

where $v \cdot \alpha_\tau$ is the scalar product of v and α_τ, and

$$W_T = \begin{cases} \mathbb{R}^k & (T = \emptyset) \\ \{v \in \mathbb{R}^k \mid v \cdot \alpha_\tau = 0, \tau \in T\} & (T \neq \emptyset) \end{cases} . \tag{3.9}$$

Then W_T is the vector space associated to $\bigcap_{\tau \in T} H_\tau$ and

$$U_{i,j}^{\perp} \cap W_T = \{v \in \mathbb{R}^k \mid v \cdot \alpha_\tau = 0, \tau \in (L_i \cap L_j) \cup T\}$$

is a vector space of dimension $k - (s + t)$, where s and t are, respectively, cardinalities of $L_i \cap L_j$ and T. With above notations, define the polynomial

$$\widetilde{p}_{\mathbb{T},T}(a_{ij}) = \sum_{U \in \mathbb{U}_{\mathbb{T},T}} [\det U]^2,$$

where $\mathbb{U}_{\mathbb{T},T}$ is the set of all $k \times k$ submatrices of the $3(k-s-t) \times k$ matrix having as rows the vectors spanning $U_{i,j}^{\perp} \cap W_T$.

If $k = 2s-1$ and $n = 3s$, $s \geq 2$, we have $T = \emptyset$ and hence $U_{i,j}^{\perp} \cap W_T = U_{i,j}^{\perp}$ is a space of dimension $\dim U_{i,j}^{\perp} = s-1$. $\mathbb{U}_{\mathbb{T},\emptyset}$ is the set of all $(2s-1) \times (2s-1)$ submatrices of the $3(s-1) \times (2s-1)$ matrix having as rows the vectors spanning $U_{i,j}^{\perp}$ and the following lemma equivalent to Lemma 2.2.3 holds.

Lemma 3.2.3 (Lemma 4.3, [19])**.** *Let $s \geq 2$, $n = 3s$, $k = 2s-1$, i.e. $T = \emptyset$, and $\mathcal{A}$ be a generic arrangement of n hyperplanes in $\mathbb{R}^k$. Given a good $3s$-partition $\mathbb{T} = \{L_1, L_2, L_3\}$ of $[3s] = [n]$, $U_{i,j}^{\perp}$ span a proper subspace of $\mathbb{R}^k$ if and only if the rank of $A_{\mathbb{T}}(\mathcal{A}_\infty)$ is 2. That is $\widetilde{p}_{\mathbb{T},\emptyset}(a_{ij}) = 0$ if and only if $p_{\mathbb{T}}(a_{ij}) = 0$.*

Notice that if $s = 2$, i.e. case $\mathcal{B}(6, 3, \mathcal{A}_\infty)$, $\widetilde{p}_{\mathbb{T},\emptyset}(a_{ij})$ coincides with $\widetilde{p}_{\mathbb{T}}(a_{ij})$ defined in Subsection 3.2.1. In this case 1-dimensional subspaces $U_{1,2}^{\perp}$, $U_{1,3}^{\perp}$ and $U_{2,3}^{\perp}$ are spanned, respectively, by $\alpha_1 \times \alpha_2, \alpha_3 \times \alpha_4$ and $\alpha_5 \times \alpha_6$, that is they are the lines drawn in Figure 3.1.
Analogously to [14] we call a generic arrangement $\mathcal{A} = \{W_1, \cdots, W_{3s}\}$ in $\mathbb{R}^{2s-1}$, $s \geq 2$, dependent if there exists a good $3s$-partition such that $U_{i,j}^{\perp}$ span a proper subspace of $\mathbb{R}^{2s-1}$. With this notation, by Lemma 2.2.3 and Theorem 2.2.5, the following theorem holds.

Theorem 3.2.4 (Theorem 4.4, [19]). *Let $\mathcal{A}$ be a generic arrangement of n hyperplanes in $\mathbb{R}^k$, $\mathbb{T}$ a good $3s$-partition, $3s \leq n$, and $T = [n] \setminus \cup_{L\in\mathbb{T}} L$. If W_T is the vector space defined in equation (3.9), then rank of $A_{\mathbb{T}}(\mathcal{A}_\infty)$ is equal to 2 if and only if the restriction arrangement*

$$\mathcal{A}^{W_T} = \{H \cap \bigcap_{\tau\in T} H_\tau \mid H \in \mathcal{A}\backslash\{H_i\}_{i\in T}\}$$

is dependent. With this choice of T and $\mathbb{T}$ we get that $p_{\mathbb{T}}(a_{ij}) = 0$ if and only if $\widetilde{p}_{\mathbb{T},T}(a_{ij}) = 0$.

Remark 3.2.5. *For a fixed good $3s$-partition $\mathbb{T}$, equation $p_{\mathbb{T}}(a_{ij}) = 0$ corresponds to $\binom{n}{3s}\binom{3s}{s}$ non-linear relations on Plücker coordinates β_I, $(2s-1)\times(2s-1)$ minors of the matrix $A = (a_{ij})$. On the other hand $\widetilde{p}_{\mathbb{T},T}(a_{ij}) = 0$ is equivalent to vanishing of $(2s-1)\times(2s-1)$ minors of the matrix with rows given by solutions of system $A_I \cdot x = 0$, $A_I = (a_{ij})_{i\in I}$, i.e. $\binom{n}{3s}\binom{3s-3}{2s-1}$ equations on a_{ij}. That is $\widetilde{p}_{\mathbb{T},T}(a_{ij}) = 0$ is reduced form of $p_{\mathbb{T}}(a_{ij}) = 0$.*

3.2.3 Hypersurface in Grassmannian $Gr(3,\mathbb{C}^n)$

Let now $\mathcal{A}$ be a generic arrangement of 6 hyperplanes in $\mathbb{C}^3$ (i.e. the example in Subsection 3.2.1 in $\mathbb{C}^3$ instead of $\mathbb{R}^3$) and

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \vdots & \vdots & \vdots \\ a_{61} & a_{62} & a_{63} \end{pmatrix} \tag{3.10}$$

be the matrix having in each row normal vectors α_i to hyperplanes $H_i^0 \in \mathcal{A}$. Since $\mathcal{A}$ is generic, columns of A are independent vectors in $\mathbb{C}^6$ and they span a subspace of dimension 3 in $\mathbb{C}^6$, i.e. an element in the Grassmannian $Gr(3,\mathbb{C}^6)$. The non null 3×3 minors of A are Plücker coordinates β_{ijk} and the matrix $A(\mathcal{A}_\infty)$ is the matrix of the map

$$\varphi_x : \mathbb{C}^6 \to \bigwedge^4 \mathbb{C}^6$$
$$v \mapsto v \wedge x,$$

where $x = \sum_{1\leq i<j<k\leq n} \beta_{ijk}(e_i \wedge e_j \wedge e_k)$. If $\mathcal{A}_\infty$ is dependent then β_{ijk} have to satisfy both, classical Plücker relations and relations in equation (3.7) (notice that since relations in equation (3.7) come directly from condition rank$A_{\mathbb{T}}(\mathcal{A}_\infty) = 2$, we get exactly same relations in real and complex case) . The latter can be simplified as:

$$(I):\begin{cases}(a):\beta_{134}\beta_{256}-\beta_{234}\beta_{156}=0\\(b):\beta_{124}\beta_{356}-\beta_{123}\beta_{456}=0\\(c):\beta_{125}\beta_{346}-\beta_{126}\beta_{345}=0\end{cases}\quad\text{and}\quad(II):\begin{cases}(d):\beta_{234}\beta_{126}\beta_{456}+\beta_{124}\beta_{256}\beta_{346}=0\\(e):\beta_{234}\beta_{125}\beta_{456}+\beta_{124}\beta_{256}\beta_{345}=0\\(f):\beta_{234}\beta_{126}\beta_{356}+\beta_{123}\beta_{256}\beta_{346}=0\\(g):\beta_{234}\beta_{125}\beta_{356}+\beta_{123}\beta_{256}\beta_{345}=0\\(h):\beta_{134}\beta_{126}\beta_{456}+\beta_{124}\beta_{156}\beta_{346}=0\\(i):\beta_{134}\beta_{125}\beta_{456}+\beta_{124}\beta_{156}\beta_{345}=0\\(j):\beta_{134}\beta_{126}\beta_{356}+\beta_{123}\beta_{156}\beta_{346}=0\\(k):\beta_{134}\beta_{125}\beta_{356}+\beta_{123}\beta_{156}\beta_{345}=0\end{cases}.$$

Actually all these 11 equations are all equivalent via Plücker relations and so we get $\beta_{134}\beta_{256}-\beta_{234}\beta_{156}=0$ (see [19] for more details).

That is relations in equation (3.7) are all equivalent and we are left with only one independent relation

$$(a)=0:\quad \beta_{134}\beta_{256}-\beta_{234}\beta_{156}=0. \tag{3.11}$$

The above computations are a direct consequence of the following more general Lemma in [19].

Lemma 3.2.6 (Lemma 5.1, [19])**.** *Let $A(\mathcal{A}_\infty)$ be the Plücker matrix associated to a generic arrangement $\mathcal{A}$ of n hyperplanes in $\mathbb{C}^3$ and $\mathbb{T}$ a good 6-partition of indices $i_1,\dots,i_6\in[n]$. If entries β_I of the matrix $A(\mathcal{A}_\infty)$ satisfy Plücker relations, then* rank$A_{\mathbb{T}}(\mathcal{A}_\infty)=2$ *if and only if one of its 3 × 3 minor vanishes.*

Remark 3.2.7. *Recall that if $\mathcal{A}$ is an arrangement of n hyperplanes in $\mathbb{C}^3$ then the matrix $A(\mathcal{A}_\infty)$ is an $\binom{n}{4}\times n$ matrix such that for any $L=\{s_1<s_2<s_3<s_4\}$, entries $(x_1,\dots,x_n)$ of row vector α_L are all zeros except $x_{i_j}=(-1)^j\beta_{I_j}, I_j=L\setminus\{s_j\}, j=1,\dots,4$. Hence for any fixed 6 indices $s_1<\dots<s_6\in[n]$ we get a $\binom{6}{4}\times 6$ submatrix of $A(\mathcal{A}_\infty)$ obtained considering all rows α_L, $L\subset\{s_1,\dots,s_6\},\mid L\mid=4$ and columns $\{s_1,\dots,s_6\}$ (all columns $j\notin\{s_1,\dots,s_6\}$ of the matrix $(\alpha_L)_{L\subset\{s_1,\dots,s_6\},|L|=4}$ are zero). It follows that the general case of n hyperplanes in $\mathbb{C}^3$ essentially reduce to the case $n=6$.*
On the other hand it is an easy remark that, if $s_1<\dots<s_6\in[n]$ are 6 fixed indices and $\mathbb{T}=\{\{s_1,s_2,s_3,s_4\},\{s_1,s_2,s_5,s_6\},\{s_3,s_4,s_5,s_6\}\}$ (analogous of good 6-partition $\{\{1,2,3,4\},\{1,2,5,6\},\{3,4,5,6\}\}$ of indices $\{1,\dots 6\}$), then any other good 6-partition on indices $\{s_1,\dots,s_6\}$ is of the form

$$\sigma.\mathbb{T}=\{\{i_1,i_2,i_3,i_4\},\{i_1,i_2,i_5,i_6\},\{i_3,i_4,i_5,i_6\}\} \tag{3.12}$$

where $i_j = \sigma(s_j)$, $\sigma \in \mathbf{S_6}$, $\mathbf{S_6}$ being the group of all permutations of indices $\{s_1, \dots, s_6\}$. Notice that in general i_j are not ordered and we can have $i_j > i_{j+1}$.

The following Lemma holds.

Lemma 3.2.8 (Lemma 5.3,[19]). *Let $\mathcal{A}$ be an arrangement of n hyperplanes in $\mathbb{C}^3$ and $\sigma.\mathbb{T} = \{\{i_1, i_2, i_3, i_4\}, \{i_1, i_2, i_5, i_6\}, \{i_3, i_4, i_5, i_6\}\}$, a good 6-partition of indices $s_1 < \dots < s_6 \in [n]$ such that* $\operatorname{rank} A_{\sigma.\mathbb{T}}(\mathcal{A}_\infty) = 2$ *then $\mathcal{A}$ is a point in the hypersurface*

$$\beta_{i_1 i_3 i_4}\beta_{i_2 i_5 i_6} - \beta_{i_2 i_3 i_4}\beta_{i_1 i_5 i_6} = 0 \quad . \tag{3.13}$$

By Remark 3.2.7 and Lemma 3.2.8, the following main Theorem follows.

Theorem 3.2.9 (Theorem 5.4, [19]). *The set of generic arrangements $\mathcal{A}$ of n hyperplanes in $\mathbb{C}^3$ that contains a dependent sub-arrangement is the set of points in an hypersurface in Grassmannian $Gr(3, n)$ such that each component is intersection of Grassmannian with a quadric.*

Chapter 4

Discriminantal arrangement and Pappus variety

In this chapter based on [20] we restate and reprove Pappus's hexagon theorem, and we also study intersections of quadrics defined in previous chapter in complex case retrieving Hesse configuration of lines.

4.1 Motivating example

In classical projective geometry the following theorem is known as Pappus's theorem or Pappus's hexagon theorem.

Theorem 4.1.1 (*Pappus*)**.** *On a projective plane, consider two lines l_1 and l_2, and a couple of triple points A, B, C and A', B', C' which are on l_1 and l_2 respectively. Let X, Y, Z be points of $AB' \cap A'B$, $AC' \cap A'C$ and $BC' \cap B'C$ respectively. Then there exists a line l_3 passing through the three points X, Y, Z (see Figure 4.1).*

This theorem was originally stated by Pappus of Alexandria around 290-350 A.D. .
In this section, we restate this classical theorem in terms of quadrics in the Grassmannian. Indeed the six lines $AB', A'B$, $BC', B'C$, $AC', A'C \in \mathbb{P}^2(\mathbb{C})$ correspond to lines in the trace at infinity $\mathcal{A}_\infty$ of a generic arrangement $\mathcal{A}$ in $\mathbb{C}^3$ and lines l_1, l_2 and l_3 correspond to collinearity conditions for intersection points of lines in $\mathcal{A}_\infty$.

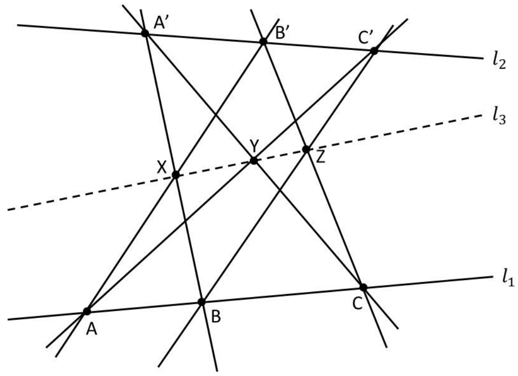

Figure 4.1: Original Pappus's Theorem

Consider a generic arrangement $\mathcal{A} = \{H_1, \ldots, H_6\}$ of 6 hyperplanes in $\mathbb{C}^3$, $\mathcal{A}_\infty$ its trace at infinity and $\mathbb{T} = \{L_1, L_2, L_3\}$ the good 6-partition defined by $L_1 = \{1,2,3,4\}$, $L_2 = \{1,2,5,6\}$, $L_3 = \{3,4,5,6\}$. By Lemma3.2.8 we get that the triple points $\bigcap_{i \in L_1 \cap L_2} \bar{H}_i \cap H_\infty$, $\bigcap_{i \in L_1 \cap L_3} \bar{H}_i \cap H_\infty$, $\bigcap_{i \in L_2 \cap L_3} \bar{H}_i \cap H_\infty$ are collinear if and only if $\mathcal{A}$ is a point of the quadric

$$\mathrm{Q}_1 : \beta_{134}\beta_{256} - \beta_{234}\beta_{156} = 0$$

in $Gr(3, \mathbb{C}^6)$.

Analogously if $\mathbb{T}' = \{L_1', L_2', L_3'\}$, $L_1' = \{4,6,2,5\}$, $L_2' = \{4,6,1,3\}$, $L_3' = \{2,5,1,3\}$ and $\mathbb{T}'' = \{L_1'', L_2'', L_3''\}$, $L_1'' = \{2,4,1,6\}$, $L_2'' = \{2,4,3,5\}$, $L_3'' = \{1,6,3,5\}$ are different good 6-partitions then triple points $\bigcap_{i \in L_1' \cap L_2'} \bar{H}_i \cap H_\infty$, $\bigcap_{i \in L_1' \cap L_3'} \bar{H}_i \cap H_\infty$, $\bigcap_{i \in L_2' \cap L_3'} \bar{H}_i^{t_i} \cap H_\infty$ and $\bigcap_{i \in L_1'' \cap L_2''} \bar{H}_i \cap H_\infty$, $\bigcap_{i \in L_1'' \cap L_3''} \bar{H}_i \cap H_\infty$, $\bigcap_{i \in L_2'' \cap L_3''} \bar{H}_i \cap H_\infty$ are collinear if and only if $\mathcal{A}$ is, respectively, a point of quadrics

$$\mathrm{Q}_2 : \beta_{425}\beta_{613} - \beta_{625}\beta_{413} = 0 \quad \text{and}$$

$$\mathrm{Q}_3 : \beta_{216}\beta_{435} - \beta_{416}\beta_{235} = 0 \quad .$$

With above remarks and notations we can restate Pappus's theorem as follows (see Figure 4.2).

Theorem 4.1.2. *(Pappus's theorem) Let $\mathcal{A} = \{H_1, \ldots, H_6\}$ be a generic arrangement of hyperplanes in $\mathbb{C}^3$. If $\mathcal{A}$ is a point of two of three quadrics $\mathrm{Q}_1, \mathrm{Q}_2$ and Q_3 in the Grassmannian $Gr(3, \mathbb{C}^6)$, then $\mathcal{A}$ is also a point of the third. In other words*

$$\mathrm{Q}_{i_1} \cap \mathrm{Q}_{i_2} = \bigcap_{i=1}^{3} \mathrm{Q}_i, \quad \{i_1, i_2\} \subset [3].$$

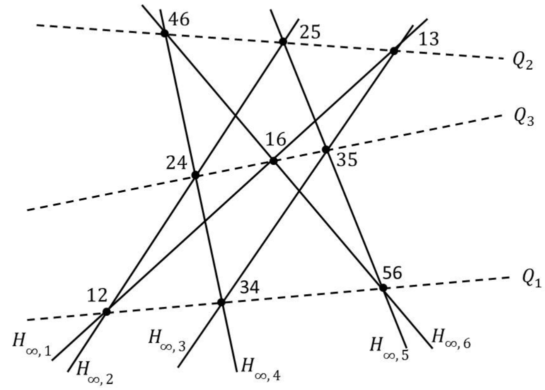

Figure 4.2: Trace at infinity of $\mathcal{A} \in \bigcap_{i=1}^{3} \mathrm{Q}_i$. ij denotes $H_{\infty,i} \cap H_{\infty,j}$.

We develop this argument in the following sections providing in Theorem 4.3.3 a general statement on quadrics in the Grassmannian which implies Pappus hexagon Theorem in the projective plane.

4.2 Pappus Variety

In this section, we consider a generic arrangement $\{H_1, \dots, H_n\}$ in $\mathbb{C}^3$ ($n \geq 6$). Let's introduce basic notations that we will use in the rest of this section.

Notation 4.2.1. *Let $\{s_1, \dots, s_6\}$ be a subset of indices $\{1, \dots, n\}$ and $\mathbb{T} = \{L_1, L_2, L_3\}$ be the good 6-partition given by $L_1 = \{s_1, s_2, s_3, s_4\}$, $L_2 = \{s_1, s_2, s_5, s_6\}$ and $L_3 = \{s_3, s_4, s_5, s_6\}$. Then for any permutation $\sigma \in \boldsymbol{S}_6$ we denote by $\sigma.\mathbb{T} = \{\sigma.L_1, \sigma.L_2, \sigma.L_3\}$ the good 6-partition given by subsets $\sigma.L_1 = \{i_1, i_2, i_3, i_4\}, \sigma.L_2 = \{i_1, i_2, i_5, i_6\}, \sigma.L_3 = \{i_3, i_4, i_5, i_6\}$ with $i_j = s_{\sigma(j)}$. Accordingly, we denote by Q_σ the quadric in $Gr(3, \mathbb{C}^n)$ of equation*

$$\mathrm{Q}_\sigma : \beta_{i_1 i_3 i_4} \beta_{i_2 i_5 i_6} - \beta_{i_2 i_3 i_4} \beta_{i_1 i_5 i_6} = 0 \quad .$$

The following lemma holds.

Lemma 4.2.2 (Lemma 4.1, [20])**.** *Let $\sigma, \sigma' \in \boldsymbol{S}_6$ be distinct permutations, then $\mathrm{Q}_\sigma = \mathrm{Q}_{\sigma'}$ if and only if there exists $\tau \in \boldsymbol{S}_3$ such that $\sigma.L_i \cap \sigma.L_j = \sigma'.L_{\tau(i)} \cap \sigma'.L_{\tau(j)}$ $(1 \leq i < j \leq 3)$.*

Proof. By definition of good 6-partition we have that

$$L_1 = (L_1 \cap L_2) \cup (L_1 \cap L_3) \quad ,$$

$$L_2 = (L_2 \cap L_1) \cup (L_2 \cap L_3) \quad ,$$
$$L_3 = (L_3 \cap L_1) \cup (L_3 \cap L_2) \quad .$$

Then there exists $\tau \in \mathbf{S}_3$ such that σ and σ' satisfy $\sigma.L_i \cap \sigma.L_j = \sigma'.L_{\tau(i)} \cap \sigma'.L_{\tau(j)}$ $(1 \leq i < j \leq 3)$ if and only if $\sigma.L_l = \sigma'.L_{\tau(l)}$ for $l = 1, 2, 3$, that is $A_{\sigma'.\mathbb{T}}(\mathcal{A}_\infty)$ is obtained by permuting rows of $A_{\sigma.\mathbb{T}}(\mathcal{A}_\infty)$. It follows that $\mathrm{rank} A_{\sigma.\mathbb{T}}(\mathcal{A}_\infty) = 2$ if and only if $\mathrm{rank} A_{\sigma'.\mathbb{T}}(\mathcal{A}_\infty) = 2$ and hence by Lemma 3.2.8 this is equivalent to $\mathrm{Q}_\sigma \cap N_{s_1,\ldots,s_6} = \mathrm{Q}_{\sigma'} \cap N_{s_1,\ldots,s_6}$, where $N_{s_1,\ldots,s_6} = \{x = \sum_{\substack{I \subseteq [n] \\ |I|=3}} \beta_I e_I \mid \beta_I \neq 0 \text{ for any } I \subset \{s_1, \ldots, s_6\}\}$. Since $N_{s_1,\ldots,s_6}$ is dense open set in $\gamma(Gr(3, \mathbb{C}^n))$, $\mathrm{Q}_\sigma \cap N_{s_1,\ldots,s_6} = \mathrm{Q}_{\sigma'} \cap N_{s_1,\ldots,s_6}$ if and only if $\mathrm{Q}_\sigma = \mathrm{Q}_{\sigma'}$. Viceversa if $\mathrm{Q}_\sigma \cap N_{s_1,\ldots,s_6} = \mathrm{Q}_{\sigma'} \cap N_{s_1,\ldots,s_6}$, then any generic arrangement $\mathcal{A}$ corresponding to a point in $\mathrm{Q}_\sigma \cap N_{s_1,\ldots,s_6}$ corresponds to a point in $\mathrm{Q}_{\sigma'} \cap N_{s_1,\ldots,s_6}$, that is $\mathrm{rank} A_{\sigma.\mathbb{T}}(\mathcal{A}_\infty) = 2$ if and only if $\mathrm{rank} A_{\sigma'.\mathbb{T}}(\mathcal{A}_\infty) = 2$. It follows that $A_{\sigma.\mathbb{T}}(\mathcal{A}_\infty)$ and $A_{\sigma'.\mathbb{T}}(\mathcal{A}_\infty)$ are submatrices of $A(\mathcal{A}_\infty)$ defined by the same three rows , i.e. $\sigma.L_l = \sigma'.L_{\tau(l)}$ for $l = 1, 2, 3$. □

Definition 4.2.3. *For any 6 fixed indices $T = \{s_1, \cdots, s_6\} \subset [n]$ the Pappus Variety is the hypersurface in $Gr(3, \mathbb{C}^n)$ given by*

$$\mathcal{P}_T = \bigcup_{\sigma \in \mathbf{S}_6} \mathrm{Q}_\sigma \quad .$$

Notice that all the content of this section and the following section is based on the choice of six indices $\{s_1 < \cdots < s_6\} \subset [n]$. This is related to result in Theorem 3.8 in [14] and, consequently, Lemma 5.3 in [20] (Lemma 3.2.8 in this thesis). Indeed Theorem 3.8 in [14] states that in order to study special configurations of n lines in $\mathbb{P}^2$, that is non very generic arrangements of n lines in $\mathbb{P}^2$, it is sufficient to study subsets of six lines out of n. On the other hand since Pappus variety can be defined inside $Gr(3, \mathbb{C}^n)$, we decided to keep the discussion more general picking six indices $\{s_1 < \cdots < s_6\} \subset [n]$ instead of simply study the case $Gr(3, \mathbb{C}^6)$ (see also Remark 4.4.7).

For $\sigma, \sigma' \in \mathbf{S}_6$ we define the equivalence relation $\sigma.\mathbb{T} \sim \sigma'.\mathbb{T}$ corresponding to $\mathrm{Q}_\sigma = \mathrm{Q}_{\sigma'}$ as following:

$$\sigma.\mathbb{T} \sim \sigma'.\mathbb{T} \Leftrightarrow \exists\, \tau \in \mathbf{S}_3 \text{ such that } \sigma.L_i \cap \sigma.L_j = \sigma'.L_{\tau(i)} \cap \sigma'.L_{\tau(j)} (1 \leq i < j \leq 3) \quad .$$

We denote by $[\sigma]$ the equivalence class containing $\sigma.\mathbb{T}$ and by Q_σ the corresponding quadric (notice that σ in the notation Q_σ can be any representative of $[\sigma]$). By Lemma 4.2.2 $[\sigma]$ only depends on couples $L_i \cap L_j$ hence for each class $[\sigma]$ we can choice a representative $\tilde{\sigma}.\mathbb{T}_0 = \{\{j_1, j_2, j_3, j_4\},$

$\{j_1, j_2, j_5, j_6\}$, $\{j_3, j_4, j_5, j_6\}\}$ such that $j_1 < j_2, j_3 < j_4, j_5 < j_6$ and $j_1 < j_3 < j_5$ and we can equivalently define

$$[\sigma] = \{\{j_1, j_2\}, \{j_3, j_4\}, \{j_5, j_6\}\} \quad .$$

Since the number of choices of $[\sigma]$ is $\dfrac{\binom{6}{2}\binom{4}{2}\binom{2}{2}}{3!} = 15$, Pappus variety is composed by 15 quadrics. Finally remark that $[\sigma] = \{\{j_1, j_2\}, \{j_3, j_4\}, \{j_5, j_6\}\}$ and $[\sigma'] = \{\{j'_1, j'_2\}, \{j'_3, j'_4\}, \{j'_5, j'_6\}\}$ are *disjoint*, i.e. $[\sigma] \cap [\sigma'] = \emptyset$, if and only if $\{j_{2l-1}, j_{2l}\} \neq \{j'_{2l'-1}, j'_{2l'}\}$ for any $1 \leq l, l' \leq 3$.

Definition 4.2.4. *(Pappus configuration) Let* $[\sigma_1], [\sigma_2]$ *and* $[\sigma_3]$ *be disjoint classes, a Pappus configuration is a set* $\{\mathrm{Q}_{\sigma_1}, \mathrm{Q}_{\sigma_2}, \mathrm{Q}_{\sigma_3}\}$ *of quadrics in* $Gr(3, \mathbb{C}^n)$ *such that*

$$\mathrm{Q}_{\sigma_{i_1}} \cap \mathrm{Q}_{\sigma_{i_2}} = \bigcap_{i=1}^{3} \mathrm{Q}_{\sigma_i}$$

for any $\{i_1, i_2\} \subset [3]$.

Quadrics $\mathrm{Q}_{\sigma_1}, \mathrm{Q}_{\sigma_2}, \mathrm{Q}_{\sigma_3}$ are said to be in Pappus configuration if $\{\mathrm{Q}_{\sigma_1}, \mathrm{Q}_{\sigma_2}, \mathrm{Q}_{\sigma_3}\}$ is a Pappus configuration.

Remark 4.2.5. *Fixed a class of good 6-partition* $[\sigma] = \{\{j_1, j_2\}, \{j_3, j_4\}, \{j_5, j_6\}\}$*, we shall count the number of disjoint classes.*
First let's count the number of classes $[\sigma'] = \{\{j'_1, j'_2\}, \{j'_3, j'_4\}, \{j'_5, j'_6\}\}$ *not disjoint and distinct from* $[\sigma]$*. Since* $[\sigma]$ *and* $[\sigma']$ *are distinct, only one couple* $\{j'_l, j'_{l+1}\}$ *is contained in* $[\sigma]$*. Without lost of generality we can assume* $\{j_l, j_{l+1}\} = \{j'_1, j'_2\}$ *(l is either* $1, 3$ *or* 5*) then pairs* $\{j'_3, j'_4\}$ *and* $\{j'_5, j'_6\}$ *are not in the same set, i.e. we have two possibilities:*

$$\{j'_3, j'_5\} \text{ and } \{j'_4, j'_6\} \in [\sigma] \quad ,$$
$$\text{or}$$
$$\{j'_3, j'_6\} \text{ and } \{j'_4, j'_5\} \in [\sigma] \quad .$$

Hence there are $2 \cdot 3 + 1 = 7$ *not disjoint classes from* $[\sigma]$ *and, since the number of all classes is* 15*, we get that any fixed* $[\sigma]$ *admits exactly* $15 - 7 = 8$ *disjoint classes.*

4.3 Pappus's Theorem

In this section we restate Pappus's Theorem for quadrics in $Gr(3, \mathbb{C}^n)$ by using notation introduced in the previous section. For a fixed class $[\sigma] = \{\{j_1, j_2\}, \{j_3, j_4\}, \{j_5, j_6\}\}$ let's denote by $G_{[\sigma]}$

the free group generated by permutations of elements in each subset of $[\sigma]$, that is

$$G_{[\sigma]} = \langle (j_{2l-1}\ j_{2l}) \in \mathbf{S}_6 \,|\, l = 1, 2, 3 \rangle \quad ,$$

and, for any class, $[\sigma']$ let's define the set

$$orbit_{G_{[\sigma]}}([\sigma']) = \{\tau[\sigma'] \,|\, \tau \in G_{[\sigma]}\}$$

where τ acts naturally as permutation of entries of each set in $[\sigma']$.

Remark 4.3.1. *The action of $G_{[\sigma]}$ on class $[\sigma']$ disjoint from $[\sigma]$ is faithful. Indeed let $\tau, \tau' \in G_{[\sigma]}$ be such that $\tau[\sigma'] = \tau'[\sigma']$ then $\tau^{-1}\tau'[\sigma'] = [\sigma']$, i.e. $\tau^{-1}\tau' \in G_{[\sigma']}$. Thus we get $\tau^{-1}\tau' \in G_{[\sigma]} \cap G_{[\sigma']}$. Since $[\sigma]$ and $[\sigma']$ are disjoint, $G_{[\sigma]} \cap G_{[\sigma']} = \{e\}$, i.e., $\tau = \tau'$. Remark that $|orbit_{G_{[\sigma]}}([\sigma'])| = |G_{[\sigma]}| = 8$ and $\tau[\sigma] = [\sigma]$ for any $\tau \in G_{[\sigma]}$.*

Lemma 4.3.2 (Lemma 5.2, [20])**.** *Let $[\sigma]$ and $[\sigma']$ be disjoint classes, then*

$$orbit_{G_{[\sigma]}}([\sigma']) = \{[\sigma''] \,|\, [\sigma] \cap [\sigma''] = \emptyset\} \quad .$$

Proof. First we prove that $orbit_{G_{[\sigma]}}([\sigma']) \subset \{[\sigma''] \,|\, [\sigma] \cap [\sigma''] = \emptyset\}$. Let $[\sigma] = \{\{j_1, j_2\}, \{j_3, j_4\}, \{j_5, j_6\}\}$ and $[\sigma'] = \{\{j'_1, j'_2\}, \{j'_3, j'_4\}, \{j'_5, j'_6\}\}$ be disjoint, then $|\{j_{2l-1}, j_{2l}\} \cap \{j'_{2m-1}, j'_{2m}\}| \leq 1$. Since $\tau \in G_{[\sigma]}$ permutes only j_{2l-1} and j_{2l} then $\tau[\sigma'] \cap [\sigma] = \emptyset$, that is $\tau[\sigma']$ is disjoint from $[\sigma]$, i.e. $\tau[\sigma'] \in \{[\sigma''] \,|\, [\sigma] \cap [\sigma''] = \emptyset\}$. Since $G_{[\sigma]}$ is faithful, $|orbit_{G_{[\sigma]}}([\sigma'])| = 8$ and, by calculations in the Remark 4.2.5, $|\{[\sigma''] \,|\, [\sigma] \cap [\sigma''] = \emptyset\}| = 8$, it follows that $orbit_{G_{[\sigma]}}([\sigma']) = \{[\sigma''] \,|\, [\sigma] \cap [\sigma''] = \emptyset\}$. □

The following theorem holds.

Theorem 4.3.3 (Theorem 5.3, [20])**.** *(Pappus's Theorem) For any disjoint classes $[\sigma]$ and $[\sigma']$, there exists a unique class $[\sigma'']$ disjoint from $[\sigma]$ and $[\sigma']$ such that $\{\mathrm{Q}_{\sigma}, \mathrm{Q}_{\sigma'}, \mathrm{Q}_{\sigma''}\}$ is a Pappus configuration.*

Remark 4.3.4. *Let $[\sigma_1]$ and $[\sigma_2]$, $[\sigma_i] = \{\{j_{1,i}, j_{2,i}\}, \{j_{3,i}, j_{4,i}\}, \{j_{5,i}, j_{6,i}\}\}$, $i = 1, 2$ be classes of indices in $\{1, \ldots, 6\}$. Recall the following facts (see Lemma 3.2.8):*

i) If $x = \sum_{\substack{I \subseteq [6] \\ |I|=3}} \beta_I e_I, \beta_I \neq 0$ is a point in Q_{σ_i} then any arrangement $\mathcal{A} \in \mathbb{C}^3$ such that $A(\mathcal{A}_\infty) = M_x$ is an arrangement of 6 planes in general position in $\mathbb{C}^3$ with lines in $\mathcal{A}_\infty$ such that points $H_{\infty, j_{1,i}} \cap H_{\infty, j_{2,i}}$, $H_{\infty, j_{3,i}} \cap H_{\infty, j_{4,i}}$ and $H_{\infty, j_{5,i}} \cap H_{\infty, j_{6,i}}$ are collinear.

ii) Viceversa if $\mathcal{A}$ is an arrangement of 6 lines in general position in $\mathbb{C}^3$ with the intersection points $H_{\infty,j_{1,i}} \cap H_{\infty,j_{2,i}}$, $H_{\infty,j_{3,i}} \cap H_{\infty,j_{4,i}}$ and $H_{\infty,j_{5,i}} \cap H_{\infty,j_{6,i}}$ collinear, then any point $x = \sum_{\substack{I\subseteq[6]\\|I|=3}} \beta_I e_I$ such that $M_x = A(\mathcal{A}_\infty)$ verifies $\beta_I \neq 0$ and $x \in \mathrm{Q}_{\sigma_i}$.

From ii) it follows that if $\mathcal{A}_\infty$ is an arrangement of 6 lines in general position in $\mathbb{P}^2$ such that $H_{\infty,j_{1,i}} \cap H_{\infty,j_{2,i}}$, $H_{\infty,j_{3,i}} \cap H_{\infty,j_{4,i}}$ and $H_{\infty,j_{5,i}} \cap H_{\infty,j_{6,i}}$ are collinear for $i = 1, 2$, then any point $x = \sum_{\substack{I\subseteq[6]\\|I|=3}} \beta_I e_I$ such that $M_x = A(\mathcal{A}_\infty)$ belongs to $\mathrm{Q}_{\sigma_1} \cap \mathrm{Q}_{\sigma_2}$. Moreover $[\sigma_1]$ and $[\sigma_2]$ are disjoint classes. By Theorem 4.3.3 there exists a third class $[\sigma_3] = \{\{j_{1,3}, j_{2,3}\}, \{j_{3,3}, j_{4,3}\}, \{j_{5,3}, j_{6,3}\}$ such that $\{\mathrm{Q}_{\sigma_1}, \mathrm{Q}_{\sigma_2}, \mathrm{Q}_{\sigma_3}\}$ is a Pappus configuration. Then $x \in \bigcap_{1\leq i\leq 3} \mathrm{Q}_{\sigma_i}$ which implies, by [i)] that also $H_{\infty,j_{1,3}} \cap H_{\infty,j_{2,3}}$, $H_{\infty,j_{3,3}} \cap H_{\infty,j_{4,3}}$ and $H_{\infty,j_{5,3}} \cap H_{\infty,j_{6,3}}$ have to be collinear. That is Theorem 4.3.3 implies Pappus hexagon Theorem in the plane.
Notice that Theorem 4.3.3 is slightly more general than Pappus hexagon Theorem since it also applies to the case in which some $\beta_I = 0$.

Proof of Theorem 4.3.3. Following example in section 4.1, for any class $[\omega_1] = \{\{j_1, j_2\}, \{j_3, j_4\}, \{j_5, j_6\}\}$ let's consider disjoint classes $[\omega_2] = \{\{j_1, j_3\}, \{j_2, j_5\}, \{j_4, j_6\}\}$ and $[\omega_3] = \{\{j_1, j_6\}, \{j_2, j_4\}, \{j_3, j_5\}\}$.
The corresponding quadrics have equations:

$$\mathrm{Q}_{\omega_1} : \beta_{j_1 j_3 j_4}\beta_{j_2 j_5 j_6} - \beta_{j_2 j_3 j_4}\beta_{j_1 j_5 j_6} = 0 \quad ,$$

$$\mathrm{Q}_{\omega_2} : \beta_{j_4 j_2 j_5}\beta_{j_6 j_1 j_3} - \beta_{j_6 j_2 j_5}\beta_{j_4 j_1 j_3} = 0 \quad ,$$

$$\mathrm{Q}_{\omega_3} : \beta_{j_5 j_1 j_6}\beta_{j_3 j_2 j_4} - \beta_{j_3 j_1 j_6}\beta_{j_5 j_2 j_4} = 0 \quad .$$

By definition of β_{ijk}, equations of Q_{ω_2} and Q_{ω_3} can equivalently be written as

$$\mathrm{Q}_{\omega_2} : \beta_{j_2 j_4 j_5}\beta_{j_1 j_3 j_6} + \beta_{j_2 j_5 j_6}\beta_{j_1 j_3 j_4} = 0 \quad ,$$

$$\mathrm{Q}_{\omega_3} : \beta_{j_1 j_5 j_6}\beta_{j_2 j_3 j_4} + \beta_{j_1 j_3 j_6}\beta_{j_2 j_4 j_5} = 0 \quad .$$

If we denote left side of defining equations of Q_{ω_i} by P_{ω_i} then

$$P_{\omega_2} - P_{\omega_1} = P_{\omega_3} \quad ,$$

that is zeros of any two polynomials $P_{\omega_{i_1}}, P_{\omega_{i_2}}$ are zeros of $P_{\omega_{i_3}}$, $\{i_1, i_2, i_3\} = \{1, 2, 3\}$. We get

$$\mathrm{Q}_{\omega_{i_1}} \cap \mathrm{Q}_{\omega_{i_2}} = \bigcap_{i=1}^{3} \mathrm{Q}_{\omega_i}$$

for any $\{i_1, i_2\} \subset [3]$, i.e. $Q_{\omega_1}, Q_{\omega_2}$ and Q_{ω_3} are in Pappus configuration.
By Lemma 4.3.2, since $[\omega_1] \cap [\omega_2] = \emptyset$, the set of disjoint classes from $[\omega_1]$ is given by

$$\{[\sigma_0] \mid [\omega_1] \cap [\sigma_0] = \emptyset\} = \{\tau_0[\omega_2] \mid \tau_0 \in G_{[\omega_1]}\} \quad .$$

Then if $[\sigma']$ is disjoint from $[\omega_1]$, there exists a unique element $\tau \in G_{[\omega_1]}$ such that $[\sigma'] = \tau[\omega_2]$. That is, for a generic class $[\omega_1]$, any disjoint couple $([\omega_1], [\sigma'])$ is of the form $([\omega_1], \tau[\omega_2]) = (\tau[\omega_1], \tau[\omega_2])$ and we have

$$Q_{\omega_1} = Q_{\tau\omega_1} : \beta_{\tau(j_1)\tau(j_3)\tau(j_4)}\beta_{\tau(j_2)\tau(j_5)\tau(j_6)} - \beta_{\tau(j_2)\tau(j_3)\tau(j_4)}\beta_{\tau(j_1)\tau(j_5)\tau(j_6)} = 0 \quad ,$$
$$Q_{\sigma'} = Q_{\tau\omega_2} : \beta_{\tau(j_4)\tau(j_2)\tau(j_5)}\beta_{\tau(j_6)\tau(j_1)\tau(j_3)} - \beta_{\tau(j_6)\tau(j_2)\tau(j_5)}\beta_{\tau(j_4)\tau(j_1)\tau(j_3)} = 0 \quad .$$

By antisymmetric property of indices of β_{ijk}, if we denote by P_{ω_1} and $P_{\sigma'}$ the left side of above equations, i.e.

$$P_{\omega_1} = \beta_{\tau(j_1)\tau(j_3)\tau(j_4)}\beta_{\tau(j_2)\tau(j_5)\tau(j_6)} - \beta_{\tau(j_2)\tau(j_3)\tau(j_4)}\beta_{\tau(j_1)\tau(j_5)\tau(j_6)} \quad ,$$
$$P_{\sigma'} = \beta_{\tau(j_4)\tau(j_2)\tau(j_5)}\beta_{\tau(j_6)\tau(j_1)\tau(j_3)} - \beta_{\tau(j_6)\tau(j_2)\tau(j_5)}\beta_{\tau(j_4)\tau(j_1)\tau(j_3)}$$

then

$$P_{\sigma''} := P_{\sigma'} - P_{\omega_1} = \beta_{\tau(j_5)\tau(j_1)\tau(j_6)}\beta_{\tau(j_3)\tau(j_2)\tau(j_4)} - \beta_{\tau(j_3)\tau(j_1)\tau(j_6)}\beta_{\tau(j_5)\tau(j_2)\tau(j_4)}$$

is the defining polynomial of $Q_{\tau\omega_3}$. That is $[\sigma'']$ is uniquely determined by disjoint couple $([\omega_1], [\sigma'])$.
□

From proof of Theorem 4.3.3 we get that for any class $[\omega_1] = \{\{j_1, j_2\}, \{j_3, j_4\}, \{j_5, j_6\}\}$ if we denote $[\omega_2] = \{\{j_1, j_3\}, \{j_2, j_5\}, \{j_4, j_6\}\}$ and $[\omega_3] = \{\{j_1, j_6\}, \{j_2, j_4\}, \{j_3, j_5\}\}$, then all Pappus configurations are of the form $\{Q_{\tau\omega_1}, Q_{\tau\omega_2}, Q_{\tau\omega_3}\}$, $\tau \in G_{[\omega_1]}$ and the following Corollary holds.

Notice that the proof of Theorem 4.3.3 only uses equations of quadrics Q_σ and hence provides alternative proof to Pappus hexagon Theorem (see [20]). In particular, it is also alternative to classical proof based on Grassmann-Plücker relations. Indeed the latter proof uses the fact that points in Pappus configurations verify the Grassmann-Plücker relations while, in our cases, quadrics Q_σ are proper quadrics in the Grassmannian, i.e. equations of quadrics Q_σ are not Grassmann-Plücker relations.

Corollary 4.3.5 (Corollary 5.5, [20]). *The number of Pappus configurations* $\{Q_\sigma, Q_{\sigma'}, Q_{\sigma''}\}$ *in* $Gr(3, \mathbb{C}^6)$ *is* 20.

Proof. By Remark 4.2.5 the number of $[\sigma]$ is 15 and by Lemma 4.3.2 each fixed class $[\sigma]$ admits 8 disjoint classes. By Theorem 4.3.3 if $[\sigma]$ and $[\sigma']$ are fixed, $[\sigma'']$ is uniquely determined, thus the number of the sets $\{[\sigma], [\sigma'], [\sigma'']\}$ is $15 \times 8/3! = 20$. □

Corollary 4.3.5 establishes that for any given 6 lines in $\mathbb{P}^2$ there are 20 possible combinations of their intersections that give rise to a Pappus's configuration. like the one in Figure 4.2.

4.4 Intersections of quadrics

In this section we study intersections of quadrics in $Gr(3, \mathbb{C}^n)$. In particular, we are interested in the intersection of sets

$$\mathrm{Q}^\circ_\sigma = \mathrm{Q}_\sigma \cap \{x = \sum_{\substack{I \subset [n] \\ |I|=3}} \beta_I e_I | \beta_I \neq 0, \text{ for any } I \subset \{s_1, \ldots, s_6\}\}$$

of points in quadrics Q_σ that correspond to arrangements of lines in $\mathbb{P}^2(\mathbb{C})$ with subarrangement $\{H_{s_1}, \ldots, H_{s_6}\}$ generic. The following lemmas hold.

Lemma 4.4.1 (Lemma 6.1, [20])**.** *If* $[\sigma_1], [\sigma_2], [\sigma_3]$ *are distinct and pairwise not disjoint classes then* $\mathrm{Q}^\circ_{\sigma_1} \cap \mathrm{Q}^\circ_{\sigma_2} \cap \mathrm{Q}^\circ_{\sigma_3} = \emptyset$*.*

Proof. If $[\sigma_1], [\sigma_2], [\sigma_3]$ are not disjoint then either

(1) $|[\sigma_1] \cap [\sigma_2] \cap [\sigma_3]| = 1$ or

(2) $|[\sigma_{i_1}] \cap [\sigma_{i_2}]| = 1\,(1 \leq i_1 < i_2 \leq 3)$ and $[\sigma_1] \cap [\sigma_2] \cap [\sigma_3] = \emptyset$.

(1) Assume $[\sigma_1] \cap [\sigma_2] \cap [\sigma_3] = \{i_1, i_2\}$. Let $[\sigma_1] = \{\{i_1, i_2\}, \{i_3, i_4\}, \{i_5, i_6\}\}$, $[\sigma_2] = \{\{i_1, i_2\}, \{i_3, i_5\}, \{i_4, i_6\}\}$, and $[\sigma_3] = \{\{i_1, i_2\}, \{i_3, i_6\}, \{i_4, i_5\}\}$ then we obtain the following quadrics

$$\begin{aligned}
\mathrm{Q}_{\sigma_1} &: \beta_{i_1 i_3 i_4}\beta_{i_2 i_5 i_6} - \beta_{i_2 i_3 i_4}\beta_{i_1 i_5 i_6} = 0 \quad , \\
\mathrm{Q}_{\sigma_2} &: \beta_{i_1 i_3 i_5}\beta_{i_2 i_4 i_6} - \beta_{i_2 i_3 i_5}\beta_{i_1 i_4 i_6} = 0 \quad , \\
\mathrm{Q}_{\sigma_3} &: \beta_{i_1 i_3 i_6}\beta_{i_2 i_4 i_5} - \beta_{i_2 i_3 i_6}\beta_{i_1 i_4 i_5} = 0 \quad .
\end{aligned}$$

Any point $x \in \mathrm{Q}^\circ_{\sigma_1} \cap \mathrm{Q}^\circ_{\sigma_2}$ belongs to $Gr(3, \mathbb{C}^n)$, that is x satisfies Plücker relations in (3.3). In particular, $x \in Pl_1 \cap Pl_2$ where Pl_1 and Pl_2 are the quadrics:

$$Pl_1 : \beta_{i_1i_3i_2}\beta_{i_4i_5i_6} - \beta_{i_1i_3i_4}\beta_{i_2i_5i_6} + \beta_{i_1i_3i_5}\beta_{i_2i_4i_6} - \beta_{i_1i_3i_6}\beta_{i_2i_4i_5} = 0 \quad ,$$
$$Pl_2 : \beta_{i_2i_3i_1}\beta_{i_4i_5i_6} - \beta_{i_2i_3i_4}\beta_{i_1i_5i_6} + \beta_{i_2i_3i_5}\beta_{i_1i_4i_6} - \beta_{i_2i_3i_6}\beta_{i_1i_4i_5} = 0 \quad .$$

Notice that Pl_1 and Pl_2 can be obtained from equations in (3.3) considering the 6-tuples $(p_1, p_2, q_0, q_1, q_2, q_3) = (i_1, i_3, i_2, i_4, i_5, i_6)$ and $(i_2, i_3, i_1, i_4, i_5, i_6)$ respectively. We get

$$Q_{\sigma_2} - Q_{\sigma_1} - Pl_1 + Pl_2 : \beta_{i_1i_3i_6}\beta_{i_2i_4i_5} - \beta_{i_2i_3i_6}\beta_{i_1i_4i_5} + 2(\beta_{i_1i_2i_3}\beta_{i_4i_5i_6}) = 0 \quad .$$

Since $\beta_{i_1i_2i_3} \neq 0$ and $\beta_{i_4i_5i_6} \neq 0$ then $\beta_{i_1i_2i_3}\beta_{i_4i_5i_6} \neq 0$ and hence $\beta_{i_1i_3i_6}\beta_{i_2i_4i_5} - \beta_{i_2i_3i_6}\beta_{i_1i_4i_5} \neq 0$, that is $x \notin Q^\circ_{\sigma_3}$.

(2) Assume $[\sigma_1] \cap [\sigma_2] = \{i_1, i_2\}$, $[\sigma_1] \cap [\sigma_3] = \{i_3, i_4\}$ and $[\sigma_2] \cap [\sigma_3] = \{i_5, i_6\}$ and name $P_1 = \{i_1, i_2\}$, $P_2 = \{i_3, i_4\}$, $P_3 = \{i_5, i_6\}$. To any point $x \in Q^\circ_{\sigma_1} \cap Q^\circ_{\sigma_2} \cap Q^\circ_{\sigma_3}$ corresponds the existence of an arrangement with a generic sub-arrangement indexed by $\{i_1, \ldots, i_6\}$ which trace at infinity $\{H_{\infty,i_1}, \ldots, H_{\infty,i_6}\}$ satisfyies collinearity conditions as in Figure 4.3. That is there exist couples $P_4 \in [\sigma_1], P_5 \in [\sigma_2]$ and $P_6 \in [\sigma_3]$ that correspond, respectively, to intersection points p_4, p_5 and p_6 of lines in $\{H_{\infty,i_1}, \ldots, H_{\infty,i_6}\}$ (see Figure 4.3).

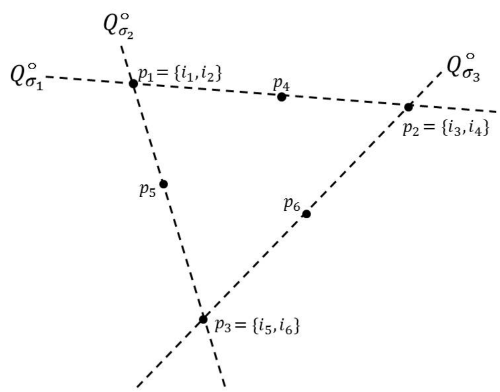

Figure 4.3: Case (2) trace at infinity of $\mathcal{A} \in \bigcap_{i=1}^3 Q^\circ_{\sigma_i}$, $\{i, j\}$ corresponds to $H_{\infty,i} \cap H_{\infty,j}$.

By definition of P_1, P_2 and P_3 we have

$$P_3 = \{i_5, i_6\} \in (\{i_1, \ldots, i_6\} \backslash P_1) \cap (\{i_1, \ldots, i_6\} \backslash P_2) \quad .$$

On the other hand, if P_4 is different from P_1 and P_2 in $Q^\circ_{\sigma_1}$ then $P_4 = (\{i_1, \ldots, i_6\} \backslash P_1) \cap (\{i_1, \ldots, i_6\} \backslash P_2)$. Thus we get $P_3 = P_4$ and, similarly, $P_5 = P_2$ and $P_6 = P_1$, that is $Q^\circ_{\sigma_1} = Q^\circ_{\sigma_2} = Q^\circ_{\sigma_3}$ which contradict hypothesis. □

Lemma 4.4.2 (Lemma 6.2, [20]). *For any three pairwise disjoint classes* $[\sigma_1], [\sigma_2], [\sigma_3]$*, either* $\{Q_{\sigma_1}, Q_{\sigma_2}, Q_{\sigma_3}\}$ *is a Pappus configuration or* $\bigcap_{i=1}^{3} Q^{\circ}_{\sigma_i} = \emptyset$.

Proof. By Pappus's Theorem, for any two disjoint classes $[\sigma_i], [\sigma_j]$, there exists $[\sigma_{ij}]$ such that $\{Q_{\sigma_i}, Q_{\sigma_j}, Q_{\sigma_{ij}}\}$ is Pappus configuration. If $[\sigma_{ij}] = [\sigma_k]$ for some $k \in [3]$, then $\{Q_{\sigma_1}, Q_{\sigma_2}, Q_{\sigma_3}\}$ is a Pappus configuration. Thus assume all $[\sigma_{ij}] \neq [\sigma_k]$ for any $k = 1, 2, 3$. Moreover $[\sigma_{12}], [\sigma_{13}], [\sigma_{23}]$ are distinct since if $[\sigma_{ij}] = [\sigma_{ik}]$ then $[\sigma_j] = [\sigma_k]$.
If $[\sigma_{12}] \cap [\sigma_{13}] \neq \emptyset$, $[\sigma_{12}] \cap [\sigma_{23}] \neq \emptyset$ and $[\sigma_{13}] \cap [\sigma_{23}] \neq \emptyset$, then $\bigcap_{1 \leq l_1 < l_2 \leq 3} Q^{\circ}_{\sigma_{l_1 l_2}} = \emptyset$ by Lemma 4.4.1 and $\bigcap_{i=1}^{3} Q^{\circ}_{\sigma_i} = (\bigcap_{i=1}^{3} Q^{\circ}_{\sigma_i}) \cap (\bigcap_{1 \leq l_1 < l_2 \leq 3} Q^{\circ}_{\sigma_{l_1 l_2}}) = \emptyset$.
Otherwise assume $[\sigma_{12}] \cap [\sigma_{13}] = \emptyset$, we get a new Pappus configuration. Since the number of disjoint classes is finite, iterating the process, we will eventually get 3 classes $[\sigma_{l_1}], [\sigma_{l_2}]$, $[\sigma_{l_3}]$ pairwise not disjoint and $\bigcap_{i=1}^{3} Q^{\circ}_{\sigma_i} = (\bigcap_{i=1}^{3} Q^{\circ}_{\sigma_i}) \cap Q^{\circ}_{\sigma_{l_1}} \cap Q^{\circ}_{\sigma_{l_2}} \cap Q^{\circ}_{\sigma_{l_3}} = \emptyset$. □

Lemma 4.4.3 (Lemma 6.3, [20]). *If* $[\sigma_1], [\sigma_2], [\sigma_3]$ *are distinct classes such that* $[\sigma_1] \cap [\sigma_2] \neq \emptyset$ *and* $[\sigma_i] \cap [\sigma_3] = \emptyset$ *for* $i = 1, 2$*, then* $\bigcap_{i=1}^{3} Q^{\circ}_{\sigma_i} = \emptyset$.

Proof. Since $[\sigma_1], [\sigma_3]$ and $[\sigma_2], [\sigma_3]$ are disjoint, there exist $[\sigma_4]$ and $[\sigma_5]$ such that $\{Q_{\sigma_1}, Q_{\sigma_3}, Q_{\sigma_4}\}$ and $\{Q_{\sigma_2}, Q_{\sigma_3}, Q_{\sigma_5}\}$ are Pappus configurations and

$$[\sigma_1] \cap [\sigma_5] \neq \emptyset,\ [\sigma_2] \cap [\sigma_4] \neq \emptyset,\ [\sigma_4] \cap [\sigma_5] \neq \emptyset \quad .$$

Indeed if one of them is empty, we obtain 3 disjoint classes not in Pappus configuration and by Lemma 4.4.2, it follows $\bigcap_{i=1}^{3} Q^{\circ}_{\sigma_i} = \bigcap_{i=1}^{5} Q^{\circ}_{\sigma_i} = \emptyset$. Since $[\sigma_1] \cap [\sigma_2] \neq \emptyset$, we can assume $\{i_1, i_2\} = [\sigma_1] \cap [\sigma_2]$ and we can set

$$[\sigma_1] = \{\{i_1, i_2\}, \{i_3, i_4\}, \{i_5, i_6\}\},\ [\sigma_2] = \{\{i_1, i_2\}, \{i'_3, i'_4\}, \{i'_5, i'_6\}\},\ [\sigma_3] = \{\{j_1, j_2\}, \{j_3, j_4\}, \{j_5, j_6\}\} \quad .$$

To any point $x \in \bigcap_{i=1}^{3} Q^{\circ}_{\sigma_i} \neq \emptyset$ corresponds an arrangement $\mathcal{A}$ with generic subarrangement $\{H_{i_1}, \ldots, H_{i_6}\}$ with trace at infinity $\{H_{\infty, i_1}, \ldots, H_{\infty, i_6}\}$ intersecting as in Figures 4.4 and 4.5 (up to rename). It follows that $\{j_4, j_6\} \in [\sigma_4]$ and since $\{j_3, j_5\} = \{i_1, i_2\} \in [\sigma_1]$ and $[\sigma_1] \cap [\sigma_4] = \emptyset$ (see Figure 4.4), there are two possibilities:

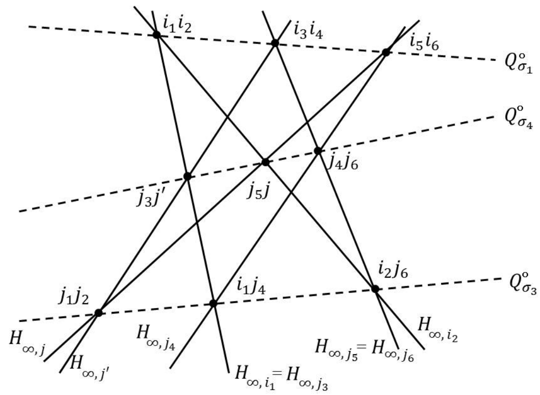

Figure 4.4: each j, j' is j_1 or j_2.

$$[\sigma_4] = \{\{j_4, j_6\}, \{j_1, j_3\}, \{j_2, j_5\}\}$$
$$\text{or}$$
$$[\sigma_4] = \{\{j_4, j_6\}, \{j_1, j_5\}, \{j_2, j_3\}\}.$$

Analogously (see Figure 4.5) class $[\sigma_5]$ is of the form

$$[\sigma_5] = \{\{j_4, j_6\}, \{j_1, j_3\}, \{j_2, j_5\}\}$$
$$\text{or}$$
$$[\sigma_5] = \{\{j_4, j_6\}, \{j_1, j_5\}, \{j_2, j_3\}\}.$$

Since $[\sigma_1] \cap [\sigma_5] \neq \emptyset$ and $[\sigma_5] \not\ni \{j_3, j_5\} = \{i_1, i_2\}$, we deduce that $\{j_4, j_6\} = \{i_3, i_4\}$ or $\{i_5, i_6\}$, which is not possible by $[\sigma_1] \cap [\sigma_4] = \emptyset$. Hence $\bigcap_{i=1}^{3} \mathrm{Q}^\circ_{\sigma_i} = \emptyset$. □

Notice that the Hesse arrangement in $\mathbb{P}^2(\mathbb{C})$ (see Figure 4.6) can be regarded as a generic arrangement of 6 lines which intersection points satisfy 6 collinearity conditions.

Definition 4.4.4. *(Hesse configuration) Let* $[\sigma_i], 1 \leq i \leq 6$ *be distinct classes, we call Hesse configuration a set* $\{\mathrm{Q}_{\sigma_1}, \ldots, \mathrm{Q}_{\sigma_6}\}$ *of quadrics in* $Gr(3, \mathbb{C}^n)$ *such that there exist disjoint sets* $I, J \subset [6]$, $|I| = |J| = 3$ *such that* $\{\mathrm{Q}_{\sigma_i}\}_{i \in I}, \{\mathrm{Q}_{\sigma_j}\}_{j \in J}$ *are Pappus configurations and* $[\sigma_i] \cap [\sigma_j] \neq \emptyset$ *for any* $i \in I$, $j \in J$.

With above notations, the following classification Theorem holds.

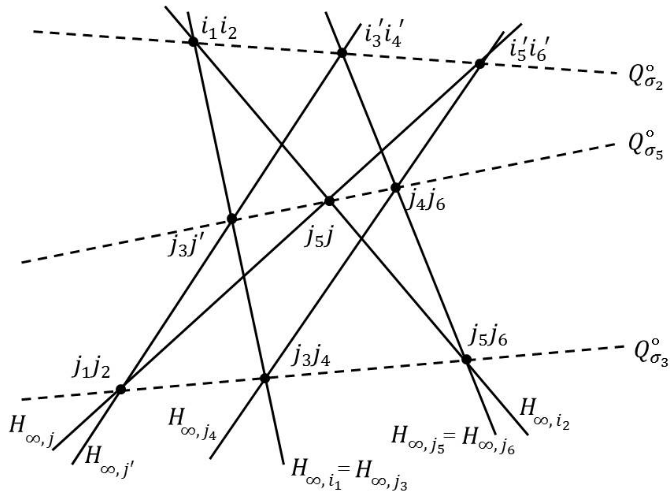

Figure 4.5: each j, j' is j_1 or j_2.

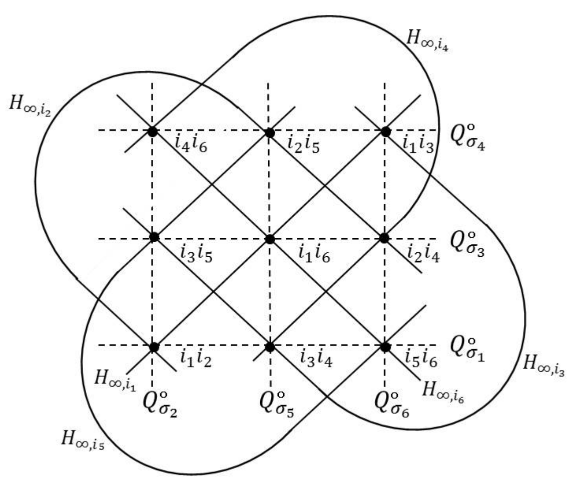

Figure 4.6: Hesse arrangement with $H_{\infty i_1}, \dots H_{\infty i_6}$ and $\bigcap_{i=1}^{6} Q^\circ_{\sigma_i} \neq \emptyset$.

Theorem 4.4.5 (Theorem 6.5, [20]). *For any choice of indices $\{s_1, \ldots, s_6\} \subset [n]$ sets Q°_σ, $\sigma \in \boldsymbol{S}_6$, in the Grassmannian $Gr(3, \mathbb{C}^n)$ intersect as follows.*

(1) For any disjoint classes $[\sigma_1]$ and $[\sigma_2]$, there exist $[\sigma_3], \ldots, [\sigma_6]$ such that $\{Q_{\sigma_1}, \ldots, Q_{\sigma_6}\}$ is an Hesse configuration for $I = \{1, 2, 3\}$, $J = \{4, 5, 6\}$ and

$$\bigcap_{i=1}^{2} Q^\circ_{\sigma_i} = \bigcap_{i=1}^{3} Q^\circ_{\sigma_i} \supsetneq \bigcap_{i=1}^{4} Q^\circ_{\sigma_i} \supsetneq \bigcap_{i=1}^{6} Q^\circ_{\sigma_i} \supsetneq \emptyset \quad .$$

(2) For any not disjoint classes $[\sigma_1]$ and $[\sigma_2]$, there exist $[\sigma_3], \ldots, [\sigma_6]$ such that $\{Q_{\sigma_1}, \ldots, Q_{\sigma_6}\}$ is an Hesse configuration for $I = \{1, 3, 4\}$, $J = \{2, 5, 6\}$ and

$$\bigcap_{i=1}^{2} Q^\circ_{\sigma_i} \supsetneq \bigcap_{i=1}^{3} Q^\circ_{\sigma_i} = \bigcap_{i=1}^{4} Q^\circ_{\sigma_i} \supsetneq \bigcap_{i=1}^{6} Q^\circ_{\sigma_i} \supsetneq \emptyset \quad .$$

All other intersections are empty.

Remark 4.4.6. *Notice that, since Hesse configuration only exists in the complex case, in $Gr(3, \mathbb{C}^n)$ we can find 6 quadrics $\{Q_{\sigma_1}, \ldots, Q_{\sigma_6}\}$ such that*

$$\bigcap_{i=1}^{6} Q^\circ_{\sigma_i} \supsetneq \emptyset \quad ,$$

while in $Gr(3, \mathbb{R}^n)$,

$$\bigcap_{j \in J \subset [6],\, |J| > 4} Q^\circ_{\sigma_j} = \emptyset \quad .$$

It follows that in the real case, for any choice of indices $\{s_1, \ldots, s_6\} \subset [n]$, we have at most 4 collinearity conditions (see Figure 4.7) corresponding to 15 hyperplanes in the discriminantal arrangement with 4 multiplicity 3 intersections in codimension 2 (see Figure 4.8). While in the complex case Hesse configuration (see Figure 4.6) gives rise to a discriminantal arrangement containing 15 hyperplanes intersecting in 6 multiplicity 3 spaces in codimension 2.
This remark allows a better understanding of differences in the combinatorics of discriminantal arrangement in the real and complex cases. Indeed the existence of a discriminantal arrangement of 15 hyperplanes intersecting in 6 multiplicity 3 spaces in codimension 2 in $\mathbb{C}$ but not in $\mathbb{R}$ implies that there exist combinatorics of discriminantal arrangements that cannot be realised in any field. This is especially interesting since in the case known until now, i.e. in the case of very generic arrangements $\mathcal{A}$, the combinatorics of discriminantal arrangement $\mathcal{B}(n, k, \mathcal{A})$ is independent from the field (see [1]).

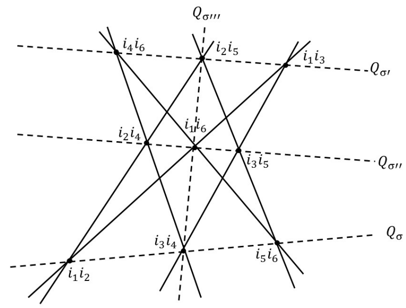

Figure 4.7: Generic arrangement $\mathcal{A}$ in $\mathbb{R}^3$ containing 6 lines satisfying 4 collinearity conditions.

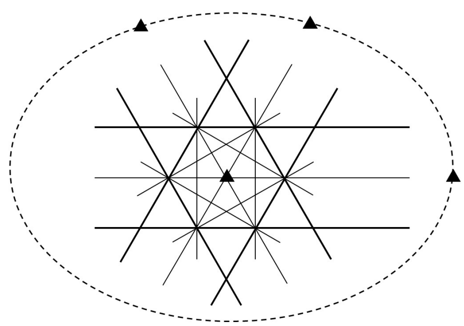

Figure 4.8: Codimension 2 intersections of 15 hyperplanes in $\mathcal{B}(n,3,\mathcal{A}_\infty)$ indexed in $\{s_1,\ldots,s_6\} \subset [n]$ with 4 multiplicity 3 points ▲ corresponding to intersections $\bigcap_{i=1}^3 D_{\sigma.L_i}$, $\bigcap_{i=1}^3 D_{\sigma'.L_i}$, $\bigcap_{i=1}^3 D_{\sigma''.L_i}$ and $\bigcap_{i=1}^3 D_{\sigma'''.L_i}$, $\sigma,\sigma',\sigma'',\sigma'''$ as in Figure 4.7.

Remark 4.4.7. *Finally Theorem 4.4.5 implies that the maximum number of intersections of multiplicity* 3 *in codimension* 2 *in the complex case is strictly higher than the one in the real case. This agrees with results on maximum number of triple points in an arrangement of lines in* $\mathbb{P}^2$ *(see [6] for a discussion on line arrangements with maximal number of triple points over arbitrary fields). Those observations suggest that special configurations of lines in the projective plane intersecting in a big number of triple points could be understood by studying discriminantal arrangements with maximum number of multiplicity 3 intersections in codimension 2. Indeed each multiplicity 3 intersection in codimension 2 of* $\mathcal{B}(n,3,\mathcal{A}_\infty)$ *corresponds to a collinearity condition for lines in* $\mathcal{A}_\infty$ *which is equivalent to the possibility to add a line that gives rise to "higher" number of triple points. It seems hence interesting to study exact number of intersections of type (1) and (2) in Theorem 4.4.5 in the Grassmannian* $Gr(3,\mathbb{C}^n)$*. This will be object of further studies.*

Chapter 5

A geometric condition for non very genericity

In this chapter we provide a necessary and sufficient condition on a central generic arrangement $\mathcal{A}^0$ in $\mathbb{C}^k$ for the existence of a simple intersection X of multiplicity r and rank$X < r$ in the intersection lattice of the discriminantal arrangement $\mathcal{B}(n, k, \mathcal{A}^0)$. In particular, we give a notion of an (r, s)-dependency defined in Definition 5.1.3), which is nothing else but a generalization of a notion of dependency given in Definition 2.2.4 (see also [14]). The relation between dependency and 3-dependency which is a special case of (r, s)-dependency is explained in Remark 5.1.4. Furthermore we give an equivalent condition for the (r, s)-dependency, which we call $K_{\mathbb{T}}$-vector condition. The condition gives rise to an algebraic condition for $\mathcal{A}^0$ to be non very generic as stated in Theorem 5.2.5. Finally, as an application of Theorem 5.2.5 we give some numerical constructions of non very generic arrangements $\mathcal{A}^0$.

Notation 5.0.1. *To begin with let us fix some notations we will use throughout this chapter.*

- $\mathcal{A}^0$ *is a central generic arrangement of n hyperplanes in* $\mathbb{C}^k$

- *For each subset L of* $\{1, \dots, n\}$ *with* $|L| = k + 1$*,* $D_L \subset \mathbb{C}^n$ *will denote the hyperplane in* $\mathcal{B}(n, k, \mathcal{A}^0)$ *corresponding to the subset L.*

- *Fixed a set* $\mathbb{T} = \{L_1, \dots, L_r\}$ *of subsets* $L_i \subset [n]$*,* $|L_i| = k + 1$*, for any arrangement* $\mathcal{A} =$

$\{H_1,\ldots,H_n\}$ translated of $\mathcal{A}^0$ we will denote by $P_i = \bigcap_{p\in L_i} H_p$ and $H_{i,j} = \bigcap_{p\in L_i\cap L_j} H_p$. Notice that P_i is a point if and only if $\mathcal{A}\in D_{L_i}$, it is empty otherwise.

5.1 $\mathbf{K}_{\mathbb{T}}$-translated and $\mathbf{K}_{\mathbb{T}}$-configurations

Let $\mathcal{A}^0 = \{H_1^0,\ldots,H_n^0\}$ be a central generic arrangement in $\mathbb{C}^k$, $\mathbb{T} = \{L_1,\ldots,L_r\}$ fixed as in Notation 5.0.1 and such that the conditions

$$\bigcup_{i=1}^{r} L_i = \bigcup_{i\in I\subset[r],|I|=r-1} L_i \quad \text{and} \quad L_i\cap L_j \neq \emptyset \tag{5.1}$$

are satisfied for any subset $I\subset[r], |I| = r-1$. In the rest of this chapter a set $\mathbb{T}$ which satisfies those properties will be called an r-**set**.

A translated $\mathcal{A} = \{H_1,\ldots,H_n\}$ of $\mathcal{A}^0$ will be called $\mathbf{K}_{\mathbb{T}}$ or $\mathbf{K}_{\mathbb{T}}$-translated if $\bigcap_{p\in L_i} H_p \neq \emptyset$ and $\bigcap_{p\in L_i\cup\{t\}} H_p = \emptyset$ for any $i\in[r]$ and $t\notin L_i$. The complete graph (as depicted in Figure 5.1) having the points $P_i = \bigcap_{p\in L_i} H_p$ as vertices and the vectors P_iP_j joining P_i and P_j as edges will be called $K_{\mathbb{T}}$-**configuration** and denoted by $K_{\mathbb{T}}(\mathcal{A})$ (examples of graphs $K_{\mathbb{T}}(\mathcal{A})$ for $|\mathbb{T}| = 3,4,5$ are represented in Figure 5.2). Notice that $P_iP_j \in H_{i,j} = \bigcap_{p\in L_i\cap L_j} H_p \neq \emptyset$ for any $1\leq i<j\leq r$.

$\mathcal{A}$ will be called **almost-** $\mathbf{K}_{\mathbb{T}}$ if it is $\mathbf{K}_{\mathbb{T}}$ but for one hyperplane, i.e. if there exists an hyperplane $H_l\in\mathcal{A}, l\in\bigcup_{i=1}^r L_i\setminus\bigcap_{i=1}^r L_i$, such that $\bigcap_{p\in L_i\setminus\{l\}} H_p\neq\emptyset$[1] for any $1\leq i<j\leq r$ but there exists L_i such that $l\in L_i$ and $\bigcap_{p\in L_i} H_p = \emptyset$. If we keep the notation $P_i = \bigcap_{p\in L_i\setminus\{l\}} H_p$, the complete graph having P_i as vertices and P_iP_j as edges will be called **almost** $K_{\mathbb{T}}$-**configuration**.

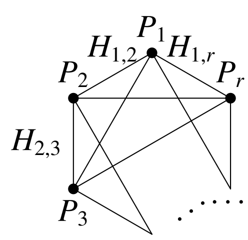

Figure 5.1: $K_{\mathbb{T}}$-configuration for $|\mathbb{T}| = r$

[1]Notice that $\bigcap_{p\in L_i\setminus\{l\}} H_p = \bigcap_{p\in L_i} H_p$ for any L_i such that $l\notin L_i$

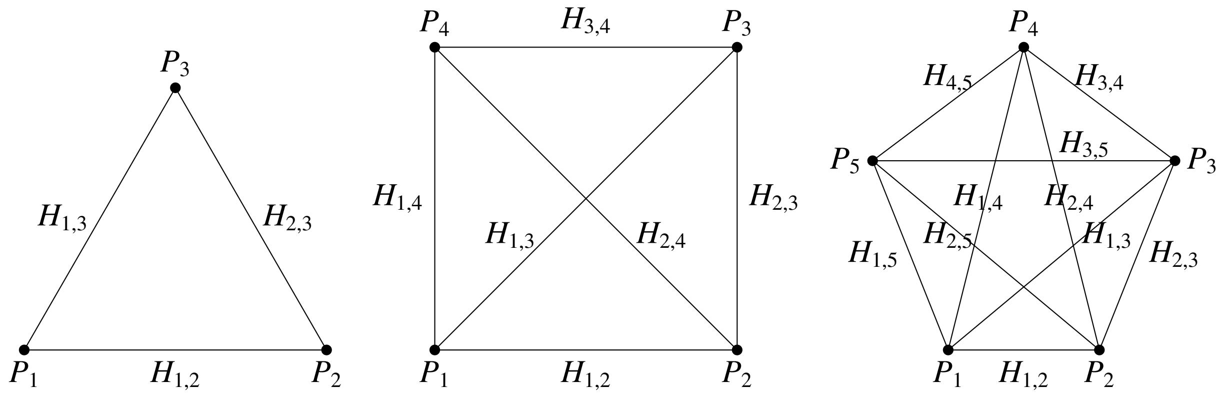

Figure 5.2: Left: $|\mathbb{T}| = 3$, Center: $|\mathbb{T}| = 4$, Right: $|\mathbb{T}| = 5$

Remark 5.1.1. *Notice that as soon as $L_i \cap L_j \neq \emptyset$ then $H_{i,j} = \bigcap_{p \in L_i \cap L_j} H_p \neq \emptyset$ for any translated $\mathcal{A}$ of $\mathcal{A}^0$. Indeed $H_{i,j}$ is an affine space having as underlining vector space $H^0_{i,j} = \bigcap_{p \in L_i \cap L_j} H^0_p$ of codimension* $\text{Card}(L_i \cap L_j)$*. In particular for any translated $\mathcal{A}$ such that $P_i = \bigcap_{p \in L_i} H_p \neq \emptyset$ there is one and only one vector $v_{i,j} \in H^0_{i,j}$ such that $v_{i,j}$ applied to point P_i has exactly P_j as ending point. This is very relevant fact as this essentially provide, in non very generic case, what Gale called affine dependency.*

Remark 5.1.2. *Similarly to the $K_{\mathbb{T}}$-configuration we could define the $\Delta_{\mathbb{T}}$-configuration as the simplicial complex having as t-face $P_{i_1} \dots P_{i_{t+1}} \in \bigcap_{p \in \cap_{j=1}^{t+1} L_{i_j}} H_p \neq \emptyset$. Notice that in general, the intersection $\bigcap_{p \in \cap_{j=1}^{t+1} L_{i_j}} H_p$ can be empty, that is $\Delta_{\mathbb{T}}$ is not a simplex. As pointed out by Crapo in [4], this simplicial complex may play a fundamental role in the study of non very generic arrangements.*

With the notations introduced above, we provide the following main definition.

Definition 5.1.3. *A central generic arrangement $\mathcal{A}^0$ of n hyperplanes in $\mathbb{C}^k$ is called **(r,s)-dependent** if there exists an r-set $\mathbb{T} = \{L_1, \dots, L_r\}$ such that any almost $K_{\mathbb{T}}$-configuration becomes a $K_{\mathbb{T}}$-configuration for any translation H_l of the hyperplane H^0_l which satisfies $\bigcap_{p \in L_i} H_p \cap H_l \neq \emptyset$ for any $L_i \in S \subsetneq \{L_i \in \mathbb{T} \mid l \in L_i\}, \mid S \mid = s$. If $s = \text{Card}\{L_i \in \mathbb{T} \mid l \in L_i\} - 1$ then we call $\mathcal{A}^0$ simply **r-dependent**.*

The definition of (r, s)-dependency generalizes the definition of dependency given in Definition 2.2.4 (see also Definition 3.3 in [14]). Indeed we have the following remark.

Remark 5.1.4 (Dependency and 3-dependency)**.** *Let's focus on the case in which $\mathbb{T} = \{L_1, L_2, L_3\}$ is a set of cardinality 3 to show that, in fact, the definition of r-dependency is a generalization of the* *definition of dependency given in [14]. In order to do that it is enough to show that both conditions, i.e. 3-dependency and dependency, are equivalent to the condition that the space $H_{i,j}$ is a subspace of $H_{i,k} + H_{k,j}$.*

Dependency. *Recall that an arrangement $\mathcal{A}^0$ of $3s$ hyperplanes in $\mathbb{C}^{2s-1}$, $s \geq 2$ is dependent if there exists a set $\mathbb{T} = \{L_1, L_2, L_3\}$ of subsets $L_i \subset [3s]$ such that $| L_i | = 2s$, $| L_i \cap L_j | = s$, $| \bigcup_{i=1}^3 L_i | = 3s$ and spaces $H_{i,j} = \bigcap_{p \in L_i \cap L_j} H_p$ span a subspace of dimension $2s-2$ in $\mathbb{C}^{2s-1}$. The condition $| L_i \cap L_j | = s$ implies that $H_{i,j}$ are spaces of codimension s, that is of dimension $s-1$ in $\mathbb{C}^{2s-1}$. Moreover $| \bigcup_{i=1}^3 L_i | = 3s$ implies that the $\bigcup_{i=1}^3 L_i$ is disjoint union of the three sets $L_i \cap L_j$, that is any two subspaces $H_{i,j}$ are in direct sum, i.e. $H_{i,j} + H_{j,k}$ span a space of dimension $2s-2$. Hence dependency condition is equivalent to the fact that $H_{i,k}$ belongs to the space in $H_{i,j} + H_{j,k}$.*

3***-dependency*** *First of all notice that $K_{\mathbb{T}}$-configuration when $| \mathbb{T} | = 3$ is equivalent to the fact that $P_iP_j = P_iP_k + P_kP_j$ (see Figure 5.1) and that since $\bigcup_{i=1}^3 L_i = \bigcup_{i \in I \subset [r], |I|=2} L_i$ then any index $l \in \bigcup_{i=1}^3 L_i \setminus \bigcap_{i=1}^3 L_i$ belongs to exactly two different subsets L_i and L_j. The 3-dependency condition is then equivalent to the fact that any time the vertex $P_i = \bigcap_{p \in L_i} H_p \neq \emptyset$ exists then $P_j = \bigcap_{p \in L_j} H_p \neq \emptyset$ exists and $P_iP_j = P_iP_k + P_kP_j$ for any $P_i, P_j \in H_{i,j}$, that is $H_{i,j}$ is a subspace of $H_{i,k} + H_{k,j}$.*

Notice that the condition of r-dependency is non trivial one. Indeed by $\bigcup_{i=1}^r L_i = \bigcup_{i \in I \subset [r], |I|=r-1} L_i$ it follows that any index $l \in \bigcup_{i=1}^r L_i$ has to belong to at least two different subsets L_i's. Hence if $L_i \neq L_j$ are two different subsets containing the index l the fact that H_l is a translation of H_l^0 for which $\bigcap_{p \in L_i \setminus \{l\}} H_p \cap H_l \neq \emptyset$ does not imply, in general, that $\bigcap_{p \in L_j \setminus \{l\}} H_p \cap H_l \neq \emptyset$. In particular (r,s)-dependency is always non trivial for any $1 \leq s < \text{Card}\{L_i \in \mathbb{T} \mid l \in L_i\}$.

It is also a simple remark that $\mathcal{A}^0$ is (r,s)-dependent if and only if any translated arrangement $\mathcal{A} \in \bigcap_{L_i \in \mathbb{T}, l \notin L_i} D_{L_i} \cap \bigcap_{L_i \in S} D_{L_i}$, where S is the set in Definition 5.1.3, satisfies $\mathcal{A} \in \bigcap_{L_i \in \mathbb{T}, l \in L_i} D_{L_i}$. That is $\mathcal{A}^0$ is (r,s)-dependent if and only if there exists a set $\mathbb{T} = \{L_1, \ldots, L_r\}$ such that

$$\bigcap_{L_i \in \mathbb{T}, l \notin L_i} D_{L_i} \cap \bigcap_{L_i \in S} D_{L_i} = \bigcap_{L_i \in \mathbb{T}} D_{L_i} \quad , \tag{5.2}$$

where $X = \bigcap_{L_i \in \mathbb{T}} D_{L_i}$ is a simple intersection. By equality in (5.2) it follows that $\text{rank}X < r$ and the following lemma is proved.

Lemma 5.1.5. *The discriminantal arrangement $\mathcal{B}(n,k,\mathcal{A}^0)$ admits a simple intersection X of multiplicity r and* rank$X < r$ *if and only if $\mathcal{A}^0$ is (r,s)-dependent for some $s \geq 1$.*

An immediate consequence of Lemma 5.1.5 and the remarks in Section 2.2 is the following main theorem.

Theorem 5.1.6. *If a central generic arrangement $\mathcal{A}^0$ of n hyperplanes in $\mathbb{C}^k$ is (r, s)-dependent for some $s \geq 1$ then $\mathcal{A}^0$ is non very generic.*

Let's remark that while Theorem 5.1.6 provides an example of a special configuration which gives rise to non very genericity, it is quite hard to verify if an arrangement satisfies such condition. In the rest of the section we will focus on providing an equivalent condition for (r, s)-dependency which can be computationally verified and which allows to build non very generic arrangements. In order to do this we will need to introduce the following vector condition.

Let $\mathbb{T}$ be an r-set, $Diag([r]) = \{(i, j) \in \mathbb{Z}_r \times \mathbb{Z}_r | i + 1 < j\}$ be the set of not adjacent pairs of integers mod r and for any (i, j) in $Diag([r])$, $v_{i,j}$ vectors defined as linear combination of the form

$$v_{i,j} := \sum_{p=i}^{j-1} v_{p,p+1} = -\sum_{p=j}^{i-1} v_{p,p+1}, \quad v_{p,p+1} \in H^0_{p,p+1} \tag{5.3}$$

where the right summand is intended from j to the first representative $h > j$ such that $h \equiv_r i - 1$ (see Figure 5.3). Notice that up to re-order of indices, the vectors $P_p + v_{p,p+1}$, i.e. $v_{p,p+1}$ applied to points $P_p = \bigcap_{i \in L_p} H_i$, H_i's translation of H^0_i, can be regarded as sides of an r-gon having vertices the applications points P_p with direction given by counter clockwise order as depicted in Figure 5.3.

We say that vectors $v_{i,j}$ satisfy the **$K_{\mathbb{T}}$-vector condition** if there exists a hyperplane $H^0_l \in \mathcal{A}^0, l \in \bigcup_{i=1}^r L_i \setminus \bigcap_{i=1}^r L_i$ and a subset $S \subsetneq \{L_i \in \mathbb{T} \mid l \in L_i\}$ such that if $v_{i,j} \in H^0_{i,j} = \bigcap_{p \in L_i \cap L_j} H^0_p, l \notin L_i \cap L_j$ and $v_{i,j} \in \bigcap_{p \in L_i \cap L_j} H^0_p, L_i, L_j \in S$, then $v_{i,j} \in H^0_{i,j}$ for any $(i, j) \in Diag([r])$.

By definition each vector $v_{i,j} \in H^0_{i,j}$ is a vector that applied to P_i has P_j as ending point, i.e. the edge P_iP_j in the $K_{\mathbb{T}}$-configuration $K_{\mathbb{T}}(\mathcal{A})$, $\mathcal{A}$ translate of $\mathcal{A}^0$, is a vector of the form $v_{i,j}$ applied to P_i or, equivalently

$$P_i + v_{i,j} = P_j \in \bigcap_{p \in L_j} H_p$$

as described in Figure 5.3. As a consequence of this remark we get the following lemma.

Lemma 5.1.7. *A central generic arrangement $\mathcal{A}^0$ of n hyperplanes in $\mathbb{C}^k$ is (r, s)-dependent if and only if there exists a set $\mathbb{T} = \{L_1, \ldots, L_r\}$ such that any set of vectors $\{v_{i,j}\}$ defined as in equation (5.3) satisfy the $K_{\mathbb{T}}$-vector condition for some subset S of cardinality s.*

In the next section we will use the $K_{\mathbb{T}}$-vector condition to simplify the condition for (r, s)-dependency. In particular we will show that it is not needed, as stated in Lemma 5.1.7, that any set of vectors $\{v_{i,j}\}$ defined as in equation (5.3) satisfy the $K_{\mathbb{T}}$-vector condition for $\mathcal{A}^0$ to be (r, s)-dependent, but it is sufficient that the $K_{\mathbb{T}}$-vector condition is satisfied by just a finite subset of them.

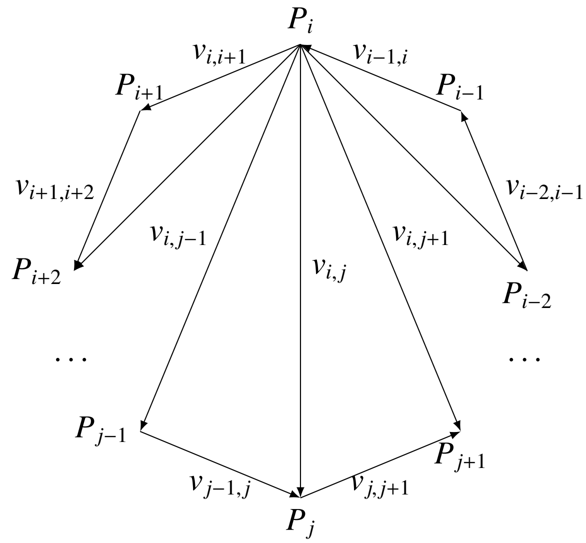

Figure 5.3: Diagonal vectors $v_{i,j}$ can be written as a sum of side vectors $v_{p,p+1}$.

5.2 A linear condition for non very genericity

In this section $\mathcal{A}^t = \{H_1^{x_1}, \ldots, H_n^{x_n}\}$ will denote the translation of the central generic arrangement $\mathcal{A}^0 = \{H_1^0, \ldots, H_n^0\}$ in $\mathbb{C}^k$ by the vector $t = (x_1, \ldots, x_n) \in \mathbb{C}^n$, i.e. $H_i^{x_i} = H_i^0 + \alpha_i x_i$ with α_i unitary vector normal to H_i^0. Notice that this notation gives rise to a natural identification of the space $\mathbb{S} = \mathbb{S}[\mathcal{A}^0]$ of parallel translations of $\mathcal{A}^0$ with $\mathbb{C}^n$ in such a way that the arrangement $\mathcal{A}^0$ corresponds to the origin.
The discriminantal arrangement $\mathcal{B}(n, k, \mathcal{A}^0)$ is not essential arrangement of center $D_{[n]} = \bigcap_{L \subset n, |L|=k+1} D_L \simeq \mathbb{C}^k$ given by all translated $\mathcal{A}^t$ of $\mathcal{A}^0$ which are central arrangements. Hence we can consider its essentialization $ess(\mathcal{B}(n, k, \mathcal{A}^0))$ in $\mathbb{C}^{n-k} \simeq \mathbb{S}/D_{[n]}$. An element $\mathcal{A}^t \in ess(\mathcal{B}(n, k, \mathcal{A}^0))$ will corresponds uniquely to a translation $t \in \mathbb{C}^n/C \simeq \mathbb{C}^{n-k}$, $C = \{t \in \mathbb{C}^n \mid \mathcal{A}^t \text{ is central}\}$. The following Proposition arises naturally.

Proposition 5.2.1. *Let $\mathcal{A}^0$ be a generic central arrangement of n hyperplanes in $\mathbb{C}^k$. Translations $\mathcal{A}^{t_1}, \ldots, \mathcal{A}^{t_d}$ of $\mathcal{A}^0$ are linearly independent vectors in $\mathbb{S}/D_{[n]} \simeq \mathbb{C}^{n-k}$ if and only if $t_1, \ldots, t_d$ are linearly independent vectors in $\mathbb{C}^n/C$.*

Given $\mathcal{A}^{t_1}, \ldots, \mathcal{A}^{t_d}$ of $\mathcal{A}^0$ we will say that the $K_{\mathbb{T}}$-configurations $K_{\mathbb{T}}(\mathcal{A}^{t_i})$ are independent if $\mathcal{A}^{t_i}, i = 1, \ldots, d$ are.

Let's consider vectors $\{v_{i,j}\}$ introduced in equation (5.3) associated to a set $\mathbb{T}$. We can remark the following three facts:

1. To each $K_{\mathbb{T}}$-configuration $K_{\mathbb{T}}(\mathcal{A}^t)$ of a translated arrangement $\mathcal{A}^t$, $t = (x_1, \dots, x_n)$ corresponds a unique family $\{v^t_{i,j}\}$ of vectors such that $P^t_i + v^t_{i,j} = P^t_j = \bigcap_{p \in L_j} H^{x_p}_p$. In the rest of the section we will denote by $\{v^t_{i,j}\}$ the family of vectors associated to $K_{\mathbb{T}}(\mathcal{A}^t)$. Notice that the converse is not uniquely defined since two different $K_{\mathbb{T}}$-configurations can define the same family $\{v_{i,j}\}$.

2. By construction vectors $\{v^t_{i,j}\}$ satisfy the property that $v^t_{k,l} = v^t_{i.l} - v^t_{i,k}$ (this can be easily seen looking at translated of vectors $v_{i,j}$'s represented in Figure 5.3). Then the set $\{v^t_{i,j}\}$ is uniquely determined by any subset of the form $\{v^t_{j,i_0}, v^t_{i_0,j}\}_{j \neq i_0}$ for a fixed index $i_0 \in [r]$. For simplicity in the rest of the section we will use the set $\{v^t_{j,i_0}, v^t_{i_0,j}\}_{j \neq i_0}$ instead of $\{v_{i,j}\}$ and we will call it $K_{\mathbb{T}}$-**vector set**.

3. Any family of vectors $\{v^t_{i,j}\}$ associated to a $K_{\mathbb{T}}$-configuration $K_{\mathbb{T}}(\mathcal{A}^t)$ satisfies, by construction, the $K_{\mathbb{T}}$-vector condition and, consequently, any family of vectors $\{v^t_{i,j}\}$ builded from a $K_{\mathbb{T}}$-vector set satisfies the $K_{\mathbb{T}}$-vector condition.

Given a $K_{\mathbb{T}}$-vector set we can naturally define operation of multiplication by a scalar

$$a\{v^t_{j,i_0}, v^t_{i_0,j}\}_{j \neq i_0} = \{av^t_{j,i_0}, av^t_{i_0,j}\}_{j \neq i_0}, \quad a \in \mathbb{C}$$

and sum of two different $K_{\mathbb{T}}$-vector sets

$$\{v^{t_1}_{j,i_0}, v^{t_1}_{i_0,j}\}_{j \neq i_0} + \{v^{t_2}_{j,i_0}, v^{t_2}_{i_0,j}\}_{j \neq i_0} = \{v^{t_1}_{j,i_0} + v^{t_2}_{j,i_0}, v^{t_1}_{i_0,j} + v^{t_2}_{i_0,j}\}_{j \neq i_0} \quad .$$

With above notations and operations, we have the following definition.

Definition 5.2.2. *For a fixed set $\mathbb{T}$, d different $K_{\mathbb{T}}$-vector sets $\{\{v^{t_h}_{j,i_0}, v^{t_h}_{i_0,j}\}_{j \neq i_0}\}_{h=1,\dots,d}$ are linearly independent if and only if for any $a_1, \dots, a_d \in \mathbb{C}$ such that*

$$\sum_{h=1}^{d} a_h \{v^{t_h}_{j,i_0}, v^{t_h}_{i_0,j}\}_{j \neq i_0} = 0, \tag{5.4}$$

then $a_1 = \ldots = a_d = 0$.

The following remark is a key point to prove the connection between linearly independence of $K_{\mathbb{T}}$-configurations and linearly independence of associated $K_{\mathbb{T}}$-vector sets.

Remark 5.2.3. *Let $K_{\mathbb{T}}(\mathcal{A}^t)$ be the $K_{\mathbb{T}}$-configuration of the arrangement $\mathcal{A}^t$ translation of $\mathcal{A}^0$. Then for any $c \in \mathbb{C}$, the $K_{\mathbb{T}}$-configuration $K_{\mathbb{T}}(\mathcal{A}^{ct})$ is an "expansion" by c of $K_{\mathbb{T}}(\mathcal{A}^t)$, that is $v^{ct}_{i,j} = cv^t_{i,j}$ (Figure 5.4 shows an example of expansion in the case of $|\ \mathbb{T}\ |= 4$). This is consequence of the fact that for any $i \in [r]$ the vector OP^{ct}_i joining the origin with the points P^{ct}_i satisfies $OP^{ct}_i = cOP^t_i$ by definition of translation. Hence $P^{ct}_i P^{ct}_j = cP^t_i P^t_j$, i.e.*

$$v^{ct}_{i,j} = cv^t_{i,j} \quad .$$

Analogously we have that, if $t_1, t_2 \in \mathbb{C}^n$ are two translations then

$$v^{t_1}_{i,j} + v^{t_2}_{i,j} = v^{t_1+t_2}_{i,j} \quad .$$

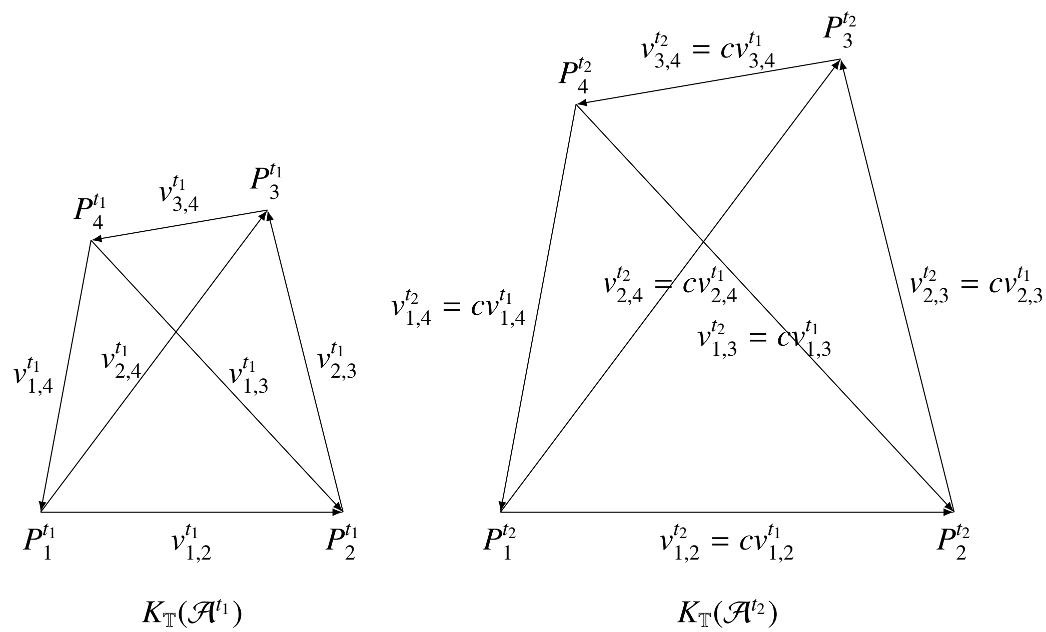

Figure 5.4: All vectors $v^{t_2}_{i,j}$ is obtained by multiplying $v^{t_1}_{i,j}$ by c.

We can now prove the main lemma of this section.

Lemma 5.2.4. *Let $\mathcal{A}^0$ be a central generic arrangement of n hyperplanes in $\mathbb{C}^k$ and $\mathbb{T} = \{L_1, \ldots, L_r\}$ be an r-set such that $[n] = \bigcup_{i=1}^r L_i$. The $K_{\mathbb{T}}$-translated arrangements $\mathcal{A}^{t_1}, \ldots, \mathcal{A}^{t_d}$ of $\mathcal{A}^0$ are linearly independent if and only if their associated $K_{\mathbb{T}}$-vector sets $\{\{v^{t_h}_{j,i_0}, v^{t_h}_{i_0,j}\}_{j\neq i_0}\}_{h=1,\ldots,d}$ are independent.*

Proof. By definition $\mathcal{A}^{t_1}, \ldots, \mathcal{A}^{t_d}$ are linearly independent if and only if translations $t_1, \ldots, t_d$ are linearly independent vectors in $\mathbb{C}^n/C$. Let's consider a linear combination $\sum_{h=1}^d a_h t_h$ of vectors t_h

and translated arrangements $\mathcal{A}^{a_h t_h}$. By Remark 5.2.3 we have that $K_{\mathbb{T}}$-vector sets associated to $\mathcal{A}^{a_h t_h}$ verify

$$\sum_{h=1}^{d} a_h \{v^{t_h}_{j,i_0}, v^{t_h}_{i_0,j}\}_{j\neq i_0} = \{v^{\sum_{h=1}^{d} a_h t_h}_{j,i_0}, v^{\sum_{h=1}^{d} a_h t_h}_{i_0,j}\}_{j\neq i_0} \quad .$$

Since $v^t_{i,j}$ is, by definition, the vector such that $P^t_i + v^t_{i,j} = P^t_j$ then $v^t_{i,j} = 0$ if and only if t is a translation such that points $P^t_i \equiv P^t_j$ coincides. Hence the condition that $\sum_{h=1}^{d} a_h \{v^{t_h}_{j,i_0}, v^{t_h}_{i_0,j}\}_{j\neq i_0} = 0$ is equivalent to the fact that $\sum_{h=1}^{d} a_h t_h \in C$. Indeed by $\sum_{h=1}^{d} a_h \{v^{t_h}_{j,i_0}, v^{t_h}_{i_0,j}\}_{j\neq i_0} = \{v^{\sum_{h=1}^{d} a_h t_h}_{j,i_0}, v^{\sum_{h=1}^{d} a_h t_h}_{i_0,j}\}_{j\neq i_0} = 0$ we get that $\sum_{h=1}^{d} a_h t_h$ is a translation such that all intersection points P_j coincide with the same point P_{i_0}, i.e. $P_{i_0} = \bigcap_{p\in\bigcup_{i=1}^{r} L_i} H_p^{\sum_{h=1}^{d} a_h t_h}$ center of the translated arrangement $\mathcal{A}^{\sum_{h=1}^{d} a_h t_h}$ of $[n] = \bigcup_{i=1}^{r} L_i$ hyperplanes. The proof of the statement follows. □

Let us remark that the assumption that the r-set $\mathbb{T} = \{L_1, \ldots, L_r\}$ satisfies the condition $\bigcup_{i=1}^{r} L_i = [n]$ in Lemma 5.2.4 is equivalent to consider, in the more general case in which $\bigcup_{i=1}^{r} L_i \subset [n]$, a subset $\mathcal{A}'^{,0} \subset \mathcal{A}^0$ which only contains the hyperplanes indexed in $\bigcup_{i=1}^{r} L_i$. On the other hand if a (central) generic arrangement $\mathcal{A}^0$ contains a subarrangement $\mathcal{A}'^{,0}$ which is non very generic then $\mathcal{A}^0$ is non very generic (this simply comes from the fact that non very genericity is a local property on the intersection lattice of the Discriminatal arrangement). Analogously if $\mathcal{A}^0$ admits a non very generic restriction arrangement $\mathcal{A}^{Y_{\mathcal{A}'}} = \{H \cap Y_{\mathcal{A}'} | H \in \mathcal{A} \setminus \mathcal{A}', H \cap Y_{\mathcal{A}'} \neq \emptyset\}, Y_{\mathcal{A}'} = \bigcap_{H\in\mathcal{A}'} H$ for some $\mathcal{A}' \subset \mathcal{A}$, then $\mathcal{A}^0$ is non very generic and the following main Theorem of this section follows.

Theorem 5.2.5. *Let $\mathcal{A}^0$ be a central generic arrangement of n hyperplanes in $\mathbb{C}^k$. If there exists a set $\mathbb{T} = \{L_1, \ldots, L_r\}$, $\mathrm{Card}\bigcup_{i=1}^{r} L_i = m$, $y = \mathrm{rank}\bigcap_{p\in\bigcap_{i=1}^{r} L_i} H_p$, which admits $m - y - k - r'$ independent $K_{\mathbb{T}}$-vector sets for some $r' < r$, then $\mathcal{A}^0$ is non very generic.*

Proof. Let's consider the subarrangement $\mathcal{A}'$ of $\mathcal{A}^0$ given by hyperplanes indexed in $m = \bigcup_{i=1}^{r} L_i$ and its essentialization, i.e. the restriction arrangement $\mathcal{A}'^Y$, $Y = \bigcap_{p\in\bigcap_{i=1}^{r} L_i} H_p$. If $y = \mathrm{rank}Y$ then the arrangement $\mathcal{A}'^Y$ is a central essential arrangement in $\mathbb{C}^{m-y}$. By Lemma 5.2.4, if $\mathcal{A}'^{t_1}, \ldots, \mathcal{A}'^{t_{m-y-k-r'}}$ are $K_{\mathbb{T}}$-translated of $\mathcal{A}'^Y$ associated to the $m-y-k-r'$ independent $\mathbb{K}_{\mathbb{T}}$-vector sets, then $\mathcal{A}^{t_1}, \ldots, \mathcal{A}^{t_{m-y-k-r'}}$ are linearly independent vectors in $\mathbb{S}[\mathcal{A}'^Y]/D_{[m]} \simeq \mathbb{C}^{m-y-k}$. That is $\mathcal{A}^{t_1}, \ldots, \mathcal{A}^{t_{m-y-k-r'}}$ span a subspace of dimension $m - y - k - r'$. On the other hand, by construction, $\mathcal{A}^{t_j}$ are $K_{\mathbb{T}}$-translated, i.e. $\mathcal{A}^{t_j} \in ess(X), X = \bigcap_{i=1}^{r} D_{L_i}$ for any $j = 1, \ldots, m - y - k - r'$, that is the space spanned by $\mathcal{A}^{t_1}, \ldots, \mathcal{A}^{t_{m-y-k-r'}}$ is included in $ess(X)$. This implies that the simple intersection $ess(X)$ has dimension $d \geq m-y-k-r' > m-y-k-r$ that is its codimension is smaller than r, i.e. $\mathrm{rank}\,ess(X) < r$ and hence $\mathrm{rank}X < r$. This implies that $\mathcal{A}'$ is non very generic and hence $\mathcal{A}^0$ is non very generic. □

Theorem 5.2.5 allows to build non very generic arrangements simply imposing linear dependency conditions on vectors $v_{i,j} \in H^0_{i,j}$ and, viceversa, to check wether an arrangement is non very generic checking opportunely defined linear dependencies.
Still two main questions are left open. One from geometric point of view and the other one combinatorial.

1. While we provided a geometric/algebraic necessary and sufficient condition for a simple intersection X of multiplicity r to be of rank$X < r$, it is still open the problem on non simple intersections. That is, is it possible to have intersections $\bigcap_{i=1}^m D_{S_i}, \mid S_i \mid > k$ such that $\bigcap_{i \in I} D_{S_i} \neq D_K$ for any subset $I \subset [m], \mid I \mid \geq 2$ and rank $\bigcap_{i=1}^m D_{S_i} < \sum_i^m (\mid S_i \mid -k)$? Does any such intersection contain a simple intersection of rank strictly lesser than its multiplicity?

2. Which are the numerical conditions on the sets L_i's for an intersection X to be simple in a non very generic arrangement?

In the next section we will provide non trivial examples of how to build non very generic arrangements by means of Theorem 5.2.5.

5.3 Examples of non very generic arrangements

In this section we present few examples to illustrate how to use Theorem 5.2.5 to construct non very generic arrangements.

Example 5.3.1 ($\mathscr{B}(12, 8, \mathcal{A}^0)$ with an intersection of multiplicity 4 in rank 3)**.** *Let* $L_1 = [12] \setminus \{10, 11, 12\}, L_2 = [12] \setminus \{7, 8, 9\}, L_3 = [12] \setminus \{4, 5, 6\}, L_4 = [12] \setminus \{1, 2, 3\}$ *be subsets of* $[12]$ *of* $k+1 = 9$ *indices. It is an easy computation that the set* $\mathbb{T} = \{L_1, L_2, L_3, L_4\}$ *is a* 4*-set. Let's consider a central generic arrangement* $\mathcal{A}^0$ *of* 12 *hyperplanes in* $\mathbb{C}^8$ *and* $\mathcal{A}^t$ *an almost* $K_{\mathbb{T}}$*-translated, i.e.* $K_{\mathbb{T}}$ *but for the hyperplane* H_{12}*. In this case* $m = n = 12$ *and* $m - k - r = 12 - 8 - 4 = 0$*, hence, by Theorem 5.2.5 in order for* $\mathcal{A}^0$ *to be non very generic it is enough the existence of just one* $K_{\mathbb{T}}$*-vector set* $\{v_{1,2}, v_{2,3}, v_{2,4}\}$ *such that vectors* $v_{2,3} \in \bigcap_{p \in L_2 \cap L_3 \setminus \{12\}} H^0_p$ *and* $v_{2,4} \in \bigcap_{p \in L_2 \cap L_4 \setminus \{12\}} H^0_p$ *belong to* H^0_{12} *(see Figure 5.5). Notice that since* $v_{3,4} = v_{2,4} - v_{2,3}, v_{3,4} \in \bigcap_{p \in L_3 \cap L_4 \setminus \{12\}} H^0_p$*, if* $v_{2,3}, v_{2,4} \in H^0_{12}$*, it automatically follows that* $v_{3,4} \in H^0_{12}$*. That is all hyperplanes in* $\mathcal{A}^0$ *can be chosen freely, but* H_{12} *which has to contain vectors* $v_{2,3}, v_{2,4}$*.*

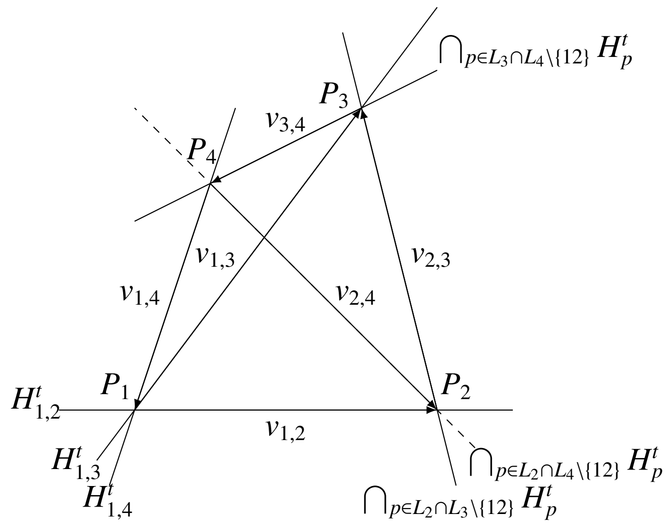

Figure 5.5: $K_{\mathbb{T}}$-configuration $K_{\mathbb{T}}(\mathcal{A}^t)$ of $\mathcal{B}(12, 8, \mathcal{A}^0)$. $v_{i,j}$ is a vector in $H^0_{i,j}$.

Let's see a numerical example. Fix a $K_{\mathbb{T}}$-vector set

$$\{v_{1,2}, v_{1,3}, v_{1,4}\} = \{(1, 0, 0, 0, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0, 0), (0, 0, -1, 0, 0, 0, 0, 0)\} \quad ,$$

then the other vectors are obtained as

$$v_{2,3} = v_{1,3} - v_{1,2} = (-1, 1, 0, 0, 0, 0, 0, 0),$$
$$v_{2,4} = -(v_{1,2} + v_{1,4}) = (-1, 0, 1, 0, 0, 0, 0, 0),$$
$$v_{3,4} = -(v_{1,3} + v_{1,4}) = (0, -1, 1, 0, 0, 0, 0, 0).$$

Notice that since $v_{i,j}$ are vectors in the intersection $\bigcap_{p\in L_i\cap L_j} H^0_p$, the normals to hyperplanes H^0_p, $p \in L_i \cap L_j$ have to be orthogonal to $v_{i,j}$. Thus, for instance we can fix equations of hyperplanes $H^0_1, \ldots, H^0_{11}$ as following:

$$H^0_1 : x_3 + x_4 + x_6 - x_7 + x_8 = 0,$$
$$H^0_2 : x_4 + x_5 + x_6 + x_7 - x_8 = 0,$$
$$H^0_3 : -x_3 + x_7 + x_8 = 0,$$
$$H^0_4 : x_2 + x_4 + x_5 + x_6 + x_8 = 0,$$
$$H^0_5 : 2x_2 - x_4 - x_5 + x_7 - x_8 = 0,$$
$$H^0_6 : -x_2 + 2x_4 + +x_5 - x_6 - x_7 + x_8 = 0,$$
$$H^0_7 : x_1 + x_4 - x_6 - x_7; x_8 = 0,$$

$$H_8^0 : -x_1 + 2x_5 + x_6 + x_7 + x_8 = 0,$$
$$H_9^0 : -x_1 + x_5 - x_6 + x_7 + x_8 = 0,$$
$$H_{10}^0 : x_1 + x_2 + x_3 - x_4 - x_5 - x_6 - x_7 + x_8 = 0,$$
$$H_{11}^0 : x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + 3x_7 + x_8 = 0.$$

Then in order for $\mathcal{A}^0$ to be non very generic H_{12}^0 has to contain $v_{2,3}, v_{2,4}$. Thus we can choose H_{12}^0, for instance, as

$$H_{12}^0 : 2x_1 + 2x_2 + 2x_3 + 3x_4 + 4x_5 + 5x_6 + 6x_7 + 7x_8 = 0.$$

Example 5.3.2 ($\mathcal{B}(16, 11, \mathcal{A}^0)$ with an intersection of multiplicity 4 in rank 3)**.** *Let $\mathbb{T} = \{L_1, L_2, L_3, L_4\}$ be a partition with a choice $L_1 = [16] \setminus \{13, 14, 15, 16\}, L_2 = [16] \setminus \{9, 10, 11, 12\}, L_3 = [16] \setminus \{5, 6, 7, 8\}, L_4 = [16] \setminus \{1, 2, 3, 4\}$ be subsets of $[16]$ of $k + 1 = 12$ indices. This set is a 4-set. Let's consider a central generic arrangement $\mathcal{A}^0$ of 16 hyperplanes in $\mathbb{C}^{11}$ and $\mathcal{A}^t$ be almost $K_{\mathbb{T}}$-translated with removed hyperplane H_{16}. In this case $m = n = 16$ and $m - k - r = 16 - 11 - 4 = 1$, hence, by Theorem 5.2.5 in order for $\mathcal{A}^0$ to be non very generic we need two linearly independent $K_{\mathbb{T}}$-vector sets $\{v_{1,2}^1, v_{1,3}^1, v_{1,4}^1\}$ and $\{v_{1,2}^2, v_{1,3}^2, v_{1,4}^2\}$ such that vectors $v_{2,3}^k \in \bigcap_{p\in L_2\cap L_3\setminus\{16\}} H_p^0$ and $v_{2,4}^k \in \bigcap_{p\in L_2\cap L_4\setminus\{16\}} H_p^0$, $k = 1, 2$, belong to H_{16}^0 (see Figure 5.6). Notice that since $v_{3,4}^k = v_{2,4}^k - v_{2,3}^k$, $v_{3,4}^k \in \bigcap_{p\in L_2\cap L_4\setminus\{16\}} H_p^0$, if $v_{2,3}^k, v_{2,4}^k \in H_{16}^0$, it automatically follows that $v_{3,4}^k \in H_{16}^0$. That is all hyperplanes in $\mathcal{A}^0$ can be chosen freely, but H_{16}^0 which has to contain vectors $v_{2,3}^k, v_{2,4}^k$, $k = 1, 2$.*

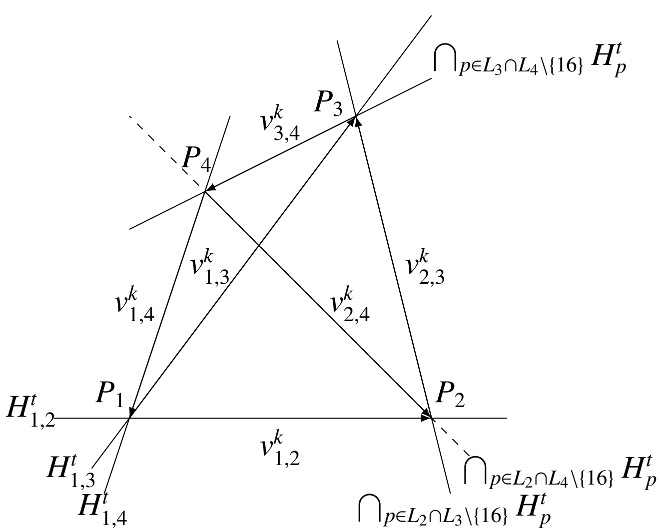

Figure 5.6: $K_{\mathbb{T}}$-configuration $K_{\mathbb{T}}(\mathcal{A}^t)$ of $\mathcal{B}(16, 11, \mathcal{A}^0)$. $v_{i,j}^k$ is a vector in $H_{i,j}^0$.

Let's see a numerical example. Fix two $K_{\mathbb{T}}$-vector sets

$$\{v^1_{1,2}, v^1_{1,3}, v^1_{1,4}\} = \{(1,0,0,0,0,0,0,0,0,0,0),(0,1,0,0,0,0,0,0,0,0,0),(0,0,-1,0,0,0,0,0,0,0,0)\},$$
$$\{v^2_{1,2}, v^2_{1,3}, v^2_{1,4}\} = \{(0,0,0,1,0,0,0,0,0,0,0),(0,0,0,0,1,0,0,0,0,0,0),(0,0,0,0,0,-1,0,0,0,0,0)\},$$

then the other vectors are obtained as

$$\begin{aligned}
v^1_{2,3} &= v^1_{1,3} - v^1_{1,2} = (-1,1,0,0,0,0,0,0,0,0,0),\\
v^1_{2,4} &= -(v^1_{1,2} + v^1_{1,4}) = (-1,0,1,0,0,0,0,0,0,0,0),\\
v^1_{3,4} &= -(v^1_{1,3} + v^1_{1,4}) = (-1,-1,0,0,0,0,0,0,0,0,0),\\
v^2_{2,3} &= v^2_{1,3} - v^2_{1,2} = (0,0,0,-1,1,0,0,0,0,0,0),\\
v^2_{2,4} &= -(v^2_{1,2} + v^2_{1,4}) = (0,0,0,-1,0,1,0,0,0,0,0),\\
v^2_{3,4} &= -(v^2_{1,3} + v^2_{1,4}) = (0,0,0,0,-1,1,0,0,0,0,0).
\end{aligned}$$

Notice that since $v^k_{i,j}$, $k = 1,2$ are vectors in the intersection $\bigcap_{p\in L_i\cap L_j} H^0_p$, the normals to hyperplanes H^0_p, $p \in L_i \cap L_j$ have to be orthogonal to $v^k_{i,j}$. Thus, for instance we can fix equations of hyperplanes $H^0_1, \dots, H^0_{15}$ as following:

$$\begin{aligned}
&H^0_1 : x_3 + x_6 + x_{10} - x_{11} = 0,\\
&H^0_2 : -x_3 + x_6 + x_7 + x_8 + x_9 - x_{10} = 0,\\
&H^0_3 : 2x_3 + x_6 + x_7 + x_9 + x_{10} = 0,\\
&H^0_4 : x_3 + x_6 + x_7 + x_{11} = 0,\\
&H^0_5 : -x_2 + x_5 + x_7 + x_8 + x_9 - x_{10} = 0,\\
&H^0_6 : x_2 + 2x_5 - x_8 - x_9 + x_{11} = 0,\\
&H^0_7 : 2x_2 - x_5 - x_7 + x_{10} + x_{11} = 0,\\
&H^0_8 : -x_2 + 2x_5 + x_7 + x_8 + x_9 = 0,\\
&H^0_9 : x_1 - 3x_4 - x_7 - x_8 + x_9 + x_{10} + x_{11} = 0,\\
&H^0_{10} : 2x_1 + 5x_4 + x_7 - x_8 - x_9 + x_{10} + x_{11} = 0,\\
&H^0_{11} : 3x_1 + x_4 + x_7 - x_8 + 2x_9 + x_{11} = 0,\\
&H^0_{12} : x_1 + 5x_4 + x_7 + x_9 + x_{10} = 0,\\
&H^0_{13} : x_1 + x_2 + x_3 - 3x_4 - 3x_5 - 3x_6 - x_7 - 3x_8 + 2x_9 - 2x_{10} - x_{11} = 0,\\
&H^0_{14} : x_1 + x_2 + x_3 - 2x_7 + x_8 - 8x_9 + x_{10} + x_{11} = 0,\\
&H^0_{15} : -5x_4 - 5x_5 - 5x_6 + x_7 + 2x_8 - 3x_9 - 4x_{10} + 7x_{11} = 0.
\end{aligned}$$

Then in order for $\mathcal{A}^0$ to be non very generic H^0_{16} has to contain $v^k_{2,3}, v^k_{2,4}$, $k = 1, 2$. Thus we can choose H^0_{16}, for instance, as

$$H^0_{16} : x_1 + x_2 + x_3 - 2x_4 - 2x_5 - 2x_6 + 5x_7 + 6x_8 + 7x_9 + 8x_{10} + 9x_{11} = 0.$$

Example 5.3.3 ($\mathcal{B}(10, 3, \mathcal{A}^0)$ with an intersection of multiplicity 5 in rank 4)**.** *Let $\mathbb{T} = \{L_1, L_2, L_3, L_4, L_5\}$ be a partition with a choice $L_1 = \{1, 2, 3, 4\}, L_2 = \{1, 5, 6, 7\}, L_3 = \{2, 5, 8, 9\}, L_4 = \{3, 6, 8, 10\}, L_5 = \{4, 7, 9, 10\}$ be subsets of $[10]$ of $k + 1 = 4$ indices. This set is a 5-set. Let's consider a central generic arrangement $\mathcal{A}^0$ of 10 hyperplanes in $\mathbb{C}^3$ and $\mathcal{A}^t$ be almost $K_{\mathbb{T}}$-translated with removed hyperplanes H_9, H_{10}. In this case $m = n = 10$ and $m - k - r = 10 - 3 - 5 = 2$, hence, by Theorem 5.2.5 in order for $\mathcal{A}^0$ to be non very generic we need three linearly independent $K_{\mathbb{T}}$-vector sets $\{v^1_{1,2}, v^1_{1,3}, v^1_{1,4}, v^1_{1,5}\}$, $\{v^2_{1,2}, v^2_{1,3}, v^2_{1,4}, v^2_{1,5}\}$ and $\{v^3_{1,2}, v^3_{1,3}, v^3_{1,4}, v^3_{1,5}\}$ such that vectors $v^k_{3,5}$ and $v^k_{4,5}$, $k = 1, 2, 3$, belong to H^0_9 and H^0_{10} respectively (see Figure 5.7). Notice that one of vectors $v^k_{3,5}$ and one of vectors $v^k_{4,5}$ has to be linearly dependent respectively. In this case 8 hyperplanes in $\mathcal{A}^0$ can be chosen freely, but H^0_9, H^0_{10} have to contain vectors $v^k_{3,5}$ and $v^k_{4,5}$, $k = 1, 2, 3$, respectively.*

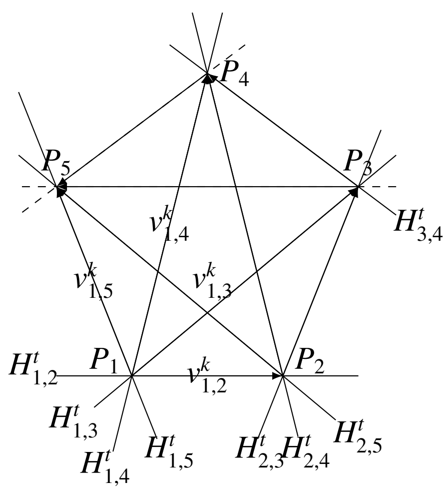

Figure 5.7: $K_{\mathbb{T}}$-configuration $K_{\mathbb{T}}(\mathcal{A}^t)$ of $\mathcal{B}(10, 3, \mathcal{A}^0)$. $v^k_{i,j}$ is a vector in $H^0_{i,j}$.

Let's see a numerical example. Fix three $K_{\mathbb{T}}$-vector sets

$$\{v^1_{1,2}, v^1_{1,3}, v^1_{1,4}, v^1_{1,5}\} = \{(1, -3, 10), (\frac{9}{2}, \frac{21}{2}, 10), (\frac{9}{2}, 3, \frac{25}{2}), (-\frac{77}{9}, \frac{77}{3}, -\frac{125}{9})\},$$
$$\{v^2_{1,2}, v^2_{1,3}, v^2_{1,4}, v^2_{1,5}\} = \{(-2, 6, -20), (-9, -47, -20), (-3, -2, -27), (-\frac{2}{3}, 2, -\frac{50}{3})\},$$
$$\{v^3_{1,2}, v^3_{1,3}, v^3_{1,4}, v^3_{1,5}\} = \{(-3, 3, -10), (-\frac{9}{2}, -\frac{2391}{80}, -10), (-\frac{1467}{1040}, -\frac{489}{520}, \frac{16151}{1040}), (-\frac{4}{3}, 4, -\frac{71}{6})\}.$$

then the other vectors are obtained as

$$v^1_{2,3} = v^1_{1,3} - v^1_{1,2} = (\frac{7}{2}, \frac{27}{2}, 0), \qquad v^1_{2,4} = v^1_{1,4} - v^1_{1,2} = (\frac{7}{2}, 6, \frac{5}{2}),$$
$$v^1_{2,5} = v^1_{1,5} - v^1_{1,2} = (-\frac{86}{9}, \frac{86}{3}, -\frac{215}{9}), \qquad v^1_{3,4} = v^1_{1,4} - v^1_{1,3} = (0, -\frac{15}{2}, \frac{5}{2}),$$
$$v^1_{3,5} = v^1_{1,5} - v^1_{1,3} = (-\frac{235}{18}, \frac{91}{6}, -\frac{215}{9}), \qquad v^1_{4,5} = v^1_{1,5} - v^1_{1,4} = (-\frac{235}{18}, \frac{68}{3}, -\frac{475}{18}),$$
$$v^2_{2,3} = v^2_{1,3} - v^2_{1,2} = (-7, -53, 0), \qquad v^2_{2,4} = v^2_{1,4} - v^2_{1,2} = (-1, -8, -7),$$
$$v^2_{2,5} = v^2_{1,5} - v^2_{1,2} = (\frac{4}{3}, -4, \frac{10}{3}), \qquad v^2_{3,4} = v^2_{1,4} - v^2_{1,3} = (6, 45, -7),$$
$$v^2_{3,5} = v^2_{1,5} - v^2_{1,3} = (\frac{25}{3}, 49, \frac{10}{3}), \qquad v^2_{4,5} = v^2_{1,5} - v^2_{1,4} = (\frac{7}{3}, 4, \frac{31}{3}),$$
$$v^3_{2,3} = v^3_{1,3} - v^3_{1,2} = (-\frac{3}{2}, -\frac{2631}{80}, 0), \qquad v^3_{2,4} = v^3_{1,4} - v^3_{1,2} = (\frac{1653}{1040}, -\frac{2049}{520}, -\frac{5751}{1040}),$$
$$v^3_{2,5} = v^3_{1,5} - v^3_{1,2} = (\frac{85}{3}, 1, -\frac{11}{6}), \qquad v^3_{3,4} = v^3_{1,4} - v^3_{1,3} = (\frac{3213}{1040}, \frac{6021}{208}, -\frac{5751}{1040}),$$
$$v^3_{3,5} = v^3_{1,5} - v^3_{1,3} = (\frac{19}{6}, \frac{2711}{80}, -\frac{11}{6}), \qquad v^3_{4,5} = v^3_{1,5} - v^3_{1,4} = (\frac{241}{3120}, \frac{2569}{520}, \frac{11533}{3120}),$$

Notice that since $v^k_{i,j}$, $k = 1, 2, 3$ are vectors in the intersection $\bigcap_{p \in L_i \cap L_j} H^0_p$, the normals to hyperplanes H^0_p, $p \in L_i \cap L_j$ have to be orthogonal to $v^k_{i,j}$. Thus, for instance we can fix equations of hyperplanes $H^0_1, \dots, H^0_8$ as following:

$$H^0_1 : 10x_2 + 3x_3 = 0,$$
$$H^0_2 : 20x_1 - 9x_3 = 0,$$
$$H^0_3 : 2x_1 - 3x_2 = 0,$$
$$H^0_4 : 3x_1 + x_2 = 0,$$
$$H^0_5 : x_3 = 0,$$
$$H^0_6 : x_1 - x_2 + x_3 = 0,$$
$$H^0_7 : x_1 + 2x_2 + 2x_3 = 0,$$
$$H^0_8 : 4x_1 - x_2 - 3x_3 = 0,$$

Then in order for $\mathcal{A}^0$ to be non very generic H^0_9, H^0_{10} has to contain $v^k_{3,5}, v^k_{4,5}$, $k = 1, 2, 3$ respectively. Thus we can choose H^0_9, H^0_{10}, for instance, as

$$H^0_9 : 314x_1 + 40x_2 + 197x_3 = 0,$$
$$H^0_{10} : 139x_1 + 30x_2 - 43x_3 = 0.$$

Remark 5.3.4. *Notice that Example 5.3.3 is slightly different from other examples for two reasons. Firstly, it uses a different combinatorics. In Example 5.3.1 and Example 5.3.2 the 4-sets* $\mathbb{T} = \{L_1, L_2, L_3, L_4\}$ *are of the form* $L_i = [n] \setminus K_i$ *as given in these examples. While Example 5.3.3* L_i*'s are not defined of the same form. Secondly, in Example 5.3.1 and Example 5.3.2 to obtain non very generic arrangement we impose* $K_{\mathbb{T}}$*-vector condition on one hyperplane while in Example 5.3.3 we impose the condition on two hyperplanes instead of one hyperplane. In particular, in this example we have to consider 3 independent* $K_{\mathbb{T}}$*-vector sets while the dimension of hyperplanes is 2, so we can choose 8 hyperplanes but not 9 hyperplanes freely. This is understood as following. Let us consider fixed 8 hyperplanes, say* $H^0_1, \ldots, H^0_8$ *freely but* H^0_9, H^0_{10} *in terms of vectors* $v^k_{i,j}$.

First of all we can fix three vectors $v^k_{1,2}$, $k = 1, 2, 3$ *freely in such a way that* H^0_1 *is determined, and consider vectors* $v^k_{1,3}, v^k_{1,4}, v^k_{1,5}$, $k = 1, 2, 3$ *with unknown entries. Notice that the number of unknowns would be* $3 \times 3 \times 3 = 27$*. As a next step let us count the number of equations to determine the left hyperplanes, which are coming from* $K_{\mathbb{T}}$*-vector condition.*

Since $v^k_{1,i}$, $k = 1, 2, 3$ $i = 3, 4, 5$*, determine a hyperplane* H^0_{i-1}*, for each i we have three equations of the form*

$$v^3_{1,i} = \alpha v^1_{1,i} + \beta v^2_{1,i}, \tag{5.5}$$

and thus we have $3 \times 3 = 9$ *equations of this form. Moreover, since* $v^k_{i,j} = v^k_{1,j} - v^k_{1,i}$, $k = 1, 2, 3$*, to determine hyperplanes* $H^0_5, \ldots, H^0_8$ *we have equations of the form*

$$v^3_{i,j} = \alpha' v^1_{i,j} + \beta' v^2_{i,j}. \tag{5.6}$$

The LHS of (5.6) can be written as

$$v^3_{i,j} = v^3_{1,j} - v^3_{1,i} = (\alpha'' v^1_{1,j} + \beta'' v^2_{1,j}) - (\alpha v^1_{1,i} + \beta v^2_{1,i}), \tag{5.7}$$

and the RHS of (5.6) can be written as

$$\alpha' v^1_{i,j} + \beta' v^2_{i,j} = \alpha'(v^1_{1,j} - v^1_{1,i}) + \beta'(v^2_{1,j} - v^2_{1,i}). \tag{5.8}$$

Thus to determine hyperplanes $H^0_5, \ldots, H^0_8$ *for* $i = 3, 4, 5$ *we have three equations of the form*

$$(\alpha'' - \alpha')v^1_{1,j} + (\beta'' - \beta')v^2_{1,j} - (\alpha - \alpha')v^1_{1,i} - (\beta - \beta')v^2_{1,i} = 0. \tag{5.9}$$

Since we fixed 4 hyperplanes here, we have $3 \times 4 = 12$ *equations of this form.*

Thus we obtain totally 9+12 = 21 equations to fix $H^0_1, \ldots, H^0_8$*, then we can write entries of vectors* $v^k_{1,i}$, $i = 3, 4, 5$ *with 27-21 = 6 parameters.*

Now, by the assumption of $K_{\mathbb{T}}$-vector condition we also have to consider the following 6 equations of the form

$$\begin{aligned} v^3_{3,5} &= \gamma(v^1_{1,5} - v^1_{1,3}) + \delta(v^2_{1,5} - v^2_{1,3}), \textit{and} \\ v^3_{4,5} &= \gamma'(v^1_{1,5} - v^1_{1,4}) + \delta'(v^2_{1,5} - v^2_{1,4}), \end{aligned} \tag{5.10}$$

By solving the above 21+6=27 equations we can determine $v^k_{1,i}$, $i = 3, 4, 5$ $k = 1, 2, 3$ uniquely, thus $v^k_{3,5}$ and $v^k_{4,5}$, $k = 1, 2, 3$ are determined uniquely. In particular, H^0_9 and H^0_{10} are determined uniquely.

If the number of hyperplanes which is fixed freely is 9, the number of unknowns in entries of vectors which are not fixed is $3 \times 2 \times 3 = 18$ while the number of equations coming from $K_{\mathbb{T}}$-vector condition is 21, which is not solvable simultaneous equations in general.

Remark 5.3.5. *Notice that the above examples are just very special cases of non very generic r-partitions. How many non very generic r-partitions can be provided and how to describe them are still open problems.*

Acknowledge

First and foremost, I am grateful to professor Settepanella for useful comments, encouragement and persistent help as my advisor.

Professor Libgober kindly accepted me to work with during spring semester of University of Illinois at Chicago (UIC) in 2020. He really kindly took care of me not only mathematics but also daily life. Discussion with him was always suggestive, which I believe that led to my growth as a researcher even for a short stay because of COVID19. I could also enjoy lectures at UIC. The life at Chicago was also irreplaceable experience for my life. I'd like to thank to him for helping my staying and mathematical works.

This research project is partially supported by Hokkaido University "Ambitious Leader's Program" and Japan Society for the Promotion of Science (JSPS). Funding from Ambitious Leader's Program and JSPS is gratefully acknowledged.

At last I would like to grateful to my family who supported my daily life, without their supporting I couldn't accomplish my research completely.

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