博士学位論文

# KMS States on Operator Algebras Associated with Self-Similar Groups
（自己相似群に付随する作用素環上の KMS 状態）

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# KMS STATES ON OPERATOR ALGEBRAS ASSOCIATED WITH SELF-SIMILAR GROUPS

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Abstract. We study operator algebras arising from self-similar groups which are some kinds of subgroups of the group of homeomorphisms on the Cantor space. In particular, it is our main interest to observe KMS states on the operator algebras. Let $\mu$ be the Bernoulli measure on the Cantor space and $G$ be a countable self-similar group. We show that if $\mu$-almost every point in the Cantor space is $G$-regular then a certain simple C*-algebra associated with $G$ admits a unique KMS state. After that, we consider von Neumann algebras on the GNS spaces of the unique KMS states and compute the Murray–von Neumann–Connes types of them. The KMS states are also used to prove the simplicity of C*-algebras associated with self-similar groups in some cases.

## 1. Introduction

It has been one of the main theme of operator algebra theory to study correspondence between "non-commutative objects" and operator algebras arising from them. Such operator algebras have been studied since the first half of 20th century when operator algebras were introduced by F. J. Murray and J. von Neumann. A non-abelian group is one of the most interesting "non-commutative objects". Operator algebras associated to free groups are known to be the oldest examples. It is still interesting even today to study operator algebras arising from discrete non-abelian groups.

Dynamical systems also provide operator algebras. From a group action on a topological (or measure) space, one can construct an operator algebra generated by the group and continuous (or $L^\infty$) functions on the space. Multiplications between elements in the group and functions are defined by relations given by the group action. These multiplications are typically non-commutative even if the group is abelian. In this sense, operator algebras associated to the group actions might be regarded as the ones arising from non-commutative relations.

In fact, it is not rare to consider operator algebras generated by non-commutative relations. Examples of such operator algebras include the Cuntz algebras [11]. For a natural number $n \geq 2$, the Cuntz algebra $\mathcal{O}_n$ is the universal unital C*-algebra generated by $S_1, \cdots, S_n$ satisfying

$$\sum_{i=1}^{n} S_i S_i^* = 1,\ S_j^* S_j = 1 \tag{1.1}$$

for any $j$. Although $\mathcal{O}_n$ is the universal C*-algebra, it is known to be simple. Thus, $\mathcal{O}_n$ is isomorphic to any nonzero C*-algebra generated by elements with the relations (1.1). One can construct operators with relations (1.1) from shift maps on the Cantor space. Fix notations to observe this. Let $X$ denote the finite set $\{1, \cdots, n\}$ and $X^\omega$ be the set of unilateral infinite words over $X$. Note that $X^\omega$ is homeomorphic to the Cantor space. Define the shift map $T_x$ on $X^\omega$ by

$$T_x(w) = xw\ (w \in X^\omega)$$

for each $x \in X$. Write $T_x^* \colon T_x(X^\omega) \to X^\omega$ for the local inverse of $T_x$. We regard each $T_x$ as an operator on a Hilbert space in a canonical way. Then the operators satisfy the equations (1.1). Thus, we often identify $S_x$ with $T_x$ and also write $S_x$ for each shift map.

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In this paper, we discuss subgroups of the group of homeomorphisms on $X^\omega$ compatible with shift maps. These groups are introduced by V. V. Nekrashevych. A subgroup $G$ of the group of homeomorphisms on $X^\omega$ is said to be a *self-similar group over* $X$ if for every $g \in G$ and $x \in X$ there exist $h \in G$ and $v \in X$ with

$$g(xw) = vh(w) \tag{1.2}$$

for any $w \in X^\omega$.

The Grigorchuk group introduced in [15] is a typical example of self-similar groups (see Example 3.5 for the definition). The Grigorchuk group is a key object in geometric group theory since it is the first example of finitely generated groups of intermediate growth (see [15] for details). In the last section, we observe a generalization of the Grigorchuk group introduced in [30].

Iterated monodromy groups are also important self-similar groups. They provide useful techniques to the study of some complex dynamical systems (see [2] for details). In addition, an operator algebraic approach to iterated monogromy groups gives a $K$-theoretic invariant for complex dynamical systems (see [26]).

Self-similar groups can be characterized by the relations between shift maps and group elements. For $x \in X, g \in G$, using $v \in X, h \in G$ in the equation (1.2), we write $g(x) := v$ and $g|_x := h$. Then the equation (1.2) implies that we have

$$gS_x = S_{g(x)}g|_x \tag{1.3}$$

for any $g \in G, x \in X$. For $u = x_1 \cdots x_n \in X^n$ we write $S_u := S_{x_1} \cdots S_{x_n}$. In [26], V. V. Nekrashevych provides representations of self-similar groups on Hilbert spaces. More precisely, he has considered the universal C$^*$-algebra generated by $G$ and $\{S_x : x \in X\}$ satisfying relations (1.1) and (1.3). In this paper, we write $\mathcal{O}_{G_{\max}}$ for this universal C$^*$-algebra. It is the main topic of this paper to study the properties of $\mathcal{O}_{G_{\max}}$ and its quotients. The studies in [5, 6, 24] may be listed as previous researches on $\mathcal{O}_{G_{\max}}$ closely related to this paper. However, we pay more attention to a certain simple quotient $\mathcal{O}_{G_{\min}}$ of $\mathcal{O}_{G_{\max}}$ than $\mathcal{O}_{G_{\max}}$ itself. This simple quotient is also found by V. V. Nekrashevych in [25]. In particular, we discuss states on $\mathcal{O}_{G_{\min}}$. To mention the main results, we fix some notations. Let $\mu$ be the product measure of the uniform probability measures on $X$'s (we identify $X^\omega$ with $X^{\mathbb{N}}$). We often refer to the measure $\mu$ as the Bernoulli measure. For $g \in G$, let $(X^\omega)_g$ be the set of fixed points of $g$. We write $e$ for the unit of $G$. We give the overview of each section.

In the second section, we review the definitions, theorems and facts related to operator algebras. In particular, we recall some topics on Cuntz–Pimsner algebras since $\mathcal{O}_{G_{\min}}$ is a Cuntz–Pimsner algebra. The Cuntz algebra $\mathcal{O}_n$ is also a Cuntz–Pimsner algebra. In later sections, We check whether $\mathcal{O}_{G_{\min}}$ and $\mathcal{O}_{G_{\max}}$ have analogous properties as ones of the Cuntz algebras or not.

In the third section, we discuss a measure theory with self-similar groups. We recall an important subset $(X^\omega)_{G\text{-reg}}$ of $X^\omega$ (see Definition 3.6). The main result of this section is described as the following theorem (see Theorem 3.13).

**Main Theorem A.** *For any countable self-similar group $G$ over $X$, we have either $\mu((X^\omega)_{G\text{-reg}}) = 1$ or $\mu((X^\omega)_{G\text{-reg}}) = 0$.*

Assume that $(X^\omega)_g$ is a $\mu$-null set for any $g \in G\backslash\{e\}$. Then we have $\mu((X^\omega)_{G\text{-reg}}) = 1$. In addition, the Grigorchuk group satisfies $\mu((X^\omega)_{G\text{-reg}}) = 1$. To see this, at the last of the third section, we recall a kind of finite generation property called contracting. It is not hard to show that the Grigorchuk group is contracting. Moreover, we show that a countable contracting self-similar group $G$ satisfies $\mu((X^\omega)_{G\text{-reg}}) = 1$.

In the fourth section, under the assumption $\mu((X^\omega)_{G\text{-reg}}) = 1$, we discuss the KMS states on $\mathcal{O}_{G_{\min}}$. See Definition 2.10 for the definition of KMS states. In the first half of the section, we recall the construction of $\mathcal{O}_{G_{\min}}$ and related topics. After that, we consider the following group action of $\mathbb{R}$ on $\mathcal{O}_{G_{\min}}$. Let $A_{G_{\min}}$ be the C$^*$-subalgebra of $\mathcal{O}_{G_{\min}}$ generated by $G$. Define the canonical gauge action

$\Gamma\colon \mathbb{R} \curvearrowright \mathcal{O}_{G_{\min}}$ by

$$\Gamma_t(a) := a \text{ and } \Gamma_t(S_x) := \exp(it)S_x$$

where $a \in A_{G_{\min}}$, $t \in \mathbb{R}$ and $x \in X$. We also define a function $\psi_0$ on $G$ given by $\psi_0(g) := \mu((X^\omega)_g)$ for $g \in G$. In the second half of the section, we observe that $\psi_0$ extends to a unique $(\log|X|, \Gamma)$-KMS state on $\mathcal{O}_{G_{\min}}$. Stating this as a theorem, we have the following one (see Theorems 4.10 and 4.11).

**Main Theorem B.** *Assume that $G$ is a countable self-similar group over $X$ with $\mu((X^\omega)_{G\text{-reg}}) = 1$. Then there exists a unique $(\log|X|, \Gamma)$-KMS state on $\mathcal{O}_{G_{\min}}$.*

The Cuntz algebra $\mathcal{O}_n$ has the unique KMS state for the gauge action (see [12]). The main theorem is similar to this fact. Note that the uniqueness of the similar KMS state on $\mathcal{O}_{G_{\max}}$ has been shown in [5]. However, our uniqueness in Theorem 4.11 is stronger because this implies the uniqueness of tracial states on the gauge invariant subalgebra. Moreover, since we discuss an at least formally smaller norm than $\mathcal{O}_{G_{\max}}$ we need more arguments to show the continuity of linear functionals.

In the fifth section, we observe von Neumann algebras on the GNS spaces of the unique KMS states. Let $\mathcal{O}_G''$ be the double commutant of $\mathcal{O}_{G_{\min}}$ in the GNS space. The uniqueness of KMS states implies that $\mathcal{O}_G''$ is a factor (namely $\mathcal{O}_G'' \cap (\mathcal{O}_G'')' = \mathbb{C}1$). Combining Theorem 4.11 and a classification theory for factors (see [31]), one can determine an invariant of the von Neumann algebra $\mathcal{O}_G''$ as follows (see Theorem 5.1).

**Main Theorem C.** *If $G$ is a countable self-similar group over $X$ with $\mu((X^\omega)_{G\text{-reg}}) = 1$, then $\mathcal{O}_G''$ is a type $\mathrm{III}_{|X|^{-1}}$ factor.*

The above theorem is an analogue of the Cuntz algebra case (see [17]). Moreover, if $\mathcal{O}_G''$ has a property called approximately finite dimensional (AFD in short), then $\mathcal{O}_G''$ is isomorphic to the von Neumann algebra constructed from the Cuntz algebra in the same way. Hence, we consider when $\mathcal{O}_G''$ is an AFD factor in the second half of the fifth section.

In the last section, we discuss when the universal C$^*$-algebra $\mathcal{O}_{G_{\max}}$ is simple. For this purpose, we recall a topological groupoid $[G, X]$ arising from a self-similar group. It was shown in [26] by V. V. Nekrashevych that $\mathcal{O}_{G_{\max}}$ coincides with the full groupoid C$^*$-algebra $C^*([G, X])$ (see [13] for more general case). We consider the case where $C^*([G, X])$ and the reduced groupoid C$^*$-algebra $C^*_r([G, X])$ are isomorphic. In this situation, if $C^*_r([G, X])$ is isomorphic to $\mathcal{O}_{G_{\min}}$ then $\mathcal{O}_{G_{\max}}$ is simple. We aim to construct an isomorphism between $C^*_r([G, X])$ and $\mathcal{O}_{G_{\min}}$ in the case $G$ is a multispinal group. A multispinal group is a contracting self-similar group arising from two finite groups $A, B$ and a map $\Psi\colon X \to \mathrm{Aut}(A) \cup \mathrm{Hom}(A, B)$ (see [30]). Here $\mathrm{Aut}(A)$ is the set of automorphisms on $A$ and $\mathrm{Hom}(A, B)$ is the set of homomorphisms from $A$ to $B$. We do not observe the definition here, see sixth section for it. The Grigorchuk group is known to be a multispinal group generated by two finite groups $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_2$.

To construct an isomorphism, we compare two Hilbert spaces. The reduced C$^*$-algebra $C^*_r([G, X])$ acts on one of the Hilbert spaces and $\mathcal{O}_{G_{\min}}$ acts on the other. For the comparison, we consider the full group C$^*$-algebra $C^*(A)$ of $A$. We identify $C^*(A)$ with C$^*$-subalgebra of $\mathcal{O}_{G_{\max}}$ generated by $A$. As a consequence, we obtain the following theorem (see Theorem 6.13).

**Main Theorem D.** *Let $G = G(A, B, \Psi)$ be a multispinal group over $X$. Assume that $[G, X]$ is amenable. Then the following conditions are equivalent.*

(i) *The restriction of $\psi_{\max}$ to $C^*(A)$ is faithful.*
(ii) *The $|A| \times |A|$ matrix $[\psi_0(a_1^{-1}a_2)]_{a_1, a_2 \in A}$ is invertible.*
(iii) *The universal C$^*$-algebra $\mathcal{O}_{G_{\max}}$ is simple.*

Studies in the third, fourth and fifth sections are based on the author's published paper [33]. The paper includes Main Theorems A to C. Studies in the sixth section are based on the author's preprint [34] which includes Main Theorem D.

## 2. Preliminaries

We recall definitions, theorems and facts related to operator algebras. Throughout the paper, we only consider unital C*-algebras. The main C*-algebras we observe in this paper have the property called purely infinite simple. Thus, we first recall the definition of it.

**Definition 2.1** ([10])**.** A unital C*-algebra $D$ is said to be *purely infinite simple* if for any nonzero element $d \in D$, there exist $x, y \in D$ with

$$xdy = 1.$$

It is trivial that if $D$ is a purely infinite simple C*-algebra then $D$ is simple. The definition of the pure infiniteness for non-simple C*-algebras is known (see [21]) but we avoid seeing it here because we do not discuss the pure infiniteness. The simplicity is more important in this paper.

We also see the following finite dimensional approximation property for C*-algebras. See [4] for related topics.

**Definition 2.2.** A unital C*-algebra $D$ is said to be *nuclear* if there exist nets of unital completely positive maps

$$\phi_k \colon D \to M_{n(k)}(\mathbb{C}), \quad \psi_k \colon M_{n(k)}(\mathbb{C}) \to D$$

with $\lim_k \psi_k(\phi_k(d)) = d$ in norm for any $d \in D$.

**Definition 2.3.** A *Kirchberg algebra* is a unital separable purely infinite simple nuclear C*-algebra .

It may be the most important fact on Kirchberg algebras that they are classifiable by $K$-groups [20, 27]. We review examples of Kirchberg algebras playing an essential role in [20, 27].

**Definition 2.4** ([11])**.** Let $n \geq 2$ be a natural number. The *Cuntz algebra* $\mathcal{O}_n$ is the unital universal C*-algebra generated by the $\{S_1, \cdots, S_n\}$ such that

$$\sum_{i=1}^{n} S_i S_i^* = 1, \ S_j^* S_j = 1$$

for any $j$.

The Cuntz algebra $\mathcal{O}_n$ is clearly separable. In fact, it is purely infinite simple ([10, 11]) and nuclear (see Theorem 2.9). Thus as a consequence, we obtain the following theorem.

**Theorem 2.5.** *The Cuntz algebra $\mathcal{O}_n$ is a Kirchberg algebra.*

Some other examples of Kirchberg algebras are constructed from Hilbert bimodules. Here we revisit the definition of Hilbert bimodules (see [4] for details).

**Definition 2.6.** Let $D$ be a unital C*-algebra and consider a right $D$-module $\mathcal{H}$ with a sesquilinear form (linear in the the second variable)

$$\langle \cdot, \cdot \rangle \colon \mathcal{H} \times \mathcal{H} \to D.$$

If $\langle \cdot, \cdot \rangle$ satisfies the following conditions for any $\xi, \eta \in \mathcal{H}, d \in D$, then $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is said to be *Hilbert right $D$-module*:

1. $\langle \xi, \xi \rangle \geq 0$,
2. $\langle \xi, \xi \rangle = 0$ implies $\xi = 0$,
3. $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$,
4. $\langle \xi, \eta \rangle d = \langle \xi, \eta d \rangle$,
5. $\mathcal{H}$ is complete with respect to the norm given by $\|\xi\| := \|\langle \xi, \xi \rangle\|^{\frac{1}{2}}$.

If there is no fear of confusion, we may simply write $\mathcal{H}$ for a Hilbert right $D$-module $(\mathcal{H}, \langle \cdot, \cdot \rangle)$.

**Definition 2.7.** A $D$-linear operator $T$ on a Hilbert right $D$-module $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is said to be *adjointable* if there exists a linear operator $S$ on $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ with

$$\langle T\xi, \eta\rangle = \langle \xi, S\eta\rangle$$

for any $\xi, \eta \in \mathcal{H}$.

The above $S$ is unique. We define $T^* := S$. Note that adjointable operators are bounded. The set $\mathbb{B}(\mathcal{H})$ of adjointable operators forms a C*-algebra in the same way as the set of bounded operators on a Hilbert space.

**Definition 2.8.** Consider a Hilbert right $D$-module $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. Assume that there exists a unital embedding $\phi$ of $D$ into $\mathbb{B}(\mathcal{H})$. Then $(\mathcal{H}, \langle\cdot,\cdot\rangle, \phi)$ is said to be a *Hilbert $D$-bimodule.*

In the above situation, we have a left action of $D$ by the embedding. This explains the name. From a Hilbert $D$-bimodule $(\mathcal{H}, \langle\cdot,\cdot\rangle, \phi)$, one can construct the universal C*-algebra

$$\mathcal{O}(\mathcal{H}) = C^*(D \cup \{S_\xi : \xi \in \mathcal{H}\})$$

given by the relations

$$S_{\alpha\xi+\eta} = \alpha S_\xi + S_\eta,\ S_{\phi(c)\xi d} = cS_\xi d,\ S_\xi^* S_\eta = \langle \xi, \eta\rangle$$

for any $\alpha \in \mathbb{C}, \xi, \eta \in \mathcal{H}, c, d \in D$. We omit the details of construction but one can find it in [4, chapter 4]. The C*-algebra $\mathcal{O}(\mathcal{H})$ is said to be the Cuntz–Pimsner algebra of $\mathcal{H}$. The Cuntz algebra $\mathcal{O}_n$ is the Cuntz–Pimsner algebra in the case $D = \mathbb{C}$, $\mathcal{H} = \mathbb{C}^n$.

As a fact related to the nuclearity, the following theorem is known.

**Theorem 2.9** ([4])**.** *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle, \phi)$ be a Hilbert $D$-bimodule. Assume that $D$ is nuclear. Then $\mathcal{O}(\mathcal{H})$ is nuclear.*

One can find Kirchberg algebras arising from the Cuntz–Pimsner construction in [25, 26] and [19]. Note that a Kirchberg algebra does not admit a tracial state. Sometimes, other special state called KMS state plays a nice role. See [3] for details.

**Definition 2.10.** Let $D$ be a C*-algebra. Fix a group action $\alpha\colon \mathbb{R} \curvearrowright D$. Assume that the map $t \mapsto \alpha_t(c)$ defined on $\mathbb{R}$ is norm-continuous for any $c \in D$. An element $d \in D$ is said to be *$\alpha$-analytic* if the map $t \mapsto \alpha_t(d)$ extends to an analytic map $z \mapsto \alpha_z(d)$ on $\mathbb{C}$.

In addition, fix a nonzero real number $\beta$. A state $\varphi$ on $D$ is said to be a *$(\beta, \alpha)$-KMS state* if

$$\varphi(dc) = \varphi(c\alpha_{i\beta}(d))$$

for any $c \in C$ and any $\alpha$-analytic element $d \in C$.

Under the assumption in Definition 2.10, we fix a notation.

$$D^\alpha := \{d \in D : \alpha_t(d) = d \text{ for any } t \in \mathbb{R}\}.$$

It is trivial that $D^\alpha$ is a C*-subalgebra of $D$.

A Cuntz–Pimsner algebra $\mathcal{O}(\mathcal{H})$ admits the canonical gauge action $\Gamma\colon \mathbb{R} \curvearrowright \mathcal{O}(\mathcal{H})$ given by

$$\Gamma_t(d) = d,\ \Gamma_t(S_\xi) = e^{it} S_\xi$$

for $d \in D, \xi \in \mathcal{H}, t \in \mathbb{R}$. It is more natural to regard $\Gamma$ as an action of $\mathbb{T}$ than $\mathbb{R}$. However, we regard it as an action of $\mathbb{R}$ to study KMS states and Murray–von Neumann–Connes types of associated von Neumann algebras. The C*-subalgebra $\mathcal{O}(\mathcal{H})^\Gamma$ of $\mathcal{O}(\mathcal{H})$ is said to be the *gauge invariant subalgebra.* We recall a fact related to the gauge action $\Gamma$.

**Lemma 2.11.** ([28, Lemma 3.2]) *Fix a C*-algebra $D$ with a tracial state $\tau$. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle, \phi)$ be a Hilbert $D$-bimodule and $\gamma > 0$. Assume that $\mathcal{H}$ is a finitely generated right $D$-module and there exists a basis $\{\xi_1, \cdots \xi_n\}$ with*

$$\tau\left(\sum_{i=1}^{n} \langle \xi_i, \phi(d)\xi_i\rangle\right) = \gamma\tau(d)$$

*for any $d \in D$. Then $\tau$ extends uniquely to a $(\log\gamma, \Gamma)$-KMS state on $\mathcal{O}(\mathcal{H})$ where $\Gamma$ is the canonical gauge action.*

Arguments on KMS states for the gauge actions are the main subjects of this paper. States on operator algebras are "non-commutative measures". In the next section, we observe measure theory as preparation of arguments on KMS states.

## 3. The Bernoulli measure and $G$-regular points

We begin with notations used in this paper.

**Notation 3.1.** Let $X$ be a finite set $X$ with $|X| \geq 2$. In this paper, $X^*$ denotes the set of finite words over the alphabet $X$. In other words, $X^* = \bigsqcup_{n\in\mathbb{N}} X^n$ (we define $X^0 := \{\emptyset\}$). Write $X^\omega$ for the set of unilateral infinite words over $X$. For $w \in X^\omega$ and $n \in \mathbb{N}$, $w^{(n)} \in X^n$ denotes the first $n$ letters of $w$.

For a finite word $u \in X^n$, we write $|u|$ for the length of the word $u$, namely $|u| = n$. Take a subset $P \subset X^\omega$. We define

$$uP := \{uw : w \in P\} \subset X^\omega.$$

We identify $X^{\mathbb{N}}$ with $X^\omega$. Thus, it is equipped with the product topology of the discrete sets $X$ Note that $X^\omega$ is homeomorphic to the Cantor space. Write $\mathrm{Homeo}(X^\omega)$ for the group of homeomorphisms on $X^\omega$.

**Definition 3.2.** ([25, Definition 2.1]) A subgroup $G$ of $\mathrm{Homeo}(X^\omega)$ is said to be a *self-similar group over* $X$ if for every $g \in G$ and $x \in X$ there exist $h \in G$ and $v \in X$ with

$$g(xw) = vh(w) \tag{3.1}$$

for any $w \in X^\omega$. A faithful group action $\alpha\colon H \curvearrowright X^\omega$ is said to be *self-similar* if $\alpha(H)$ is a self-similar group over $X$.

In this paper, $G$ always denotes a self-similar group over $X$ where $X$ is a finite set with at least 2 elements.

**Remark 3.3.** Using the equation (3.1) several times, we see that for every $n \in \mathbb{N}$, $g \in G$ and $u \in X^n$ there exist $h \in G$ and $v \in X^n$ with

$$g(uw) = vh(w)$$

for any $w \in X^\omega$. A direct calculation shows that $h$ and $v$ are uniquely determined by $g$ and $u$. We write $h = g|_u$ and $v = g(u)$.

For more details of self-similar groups and self-similar actions, see [26].

**Example 3.4.** Let $X = \{0, 1\}$. Define homeomorphisms $a$ and $b$ on $X^\omega$ by

$$a(0w) = 1w,\ a(1w) = 0w,$$
$$b(0w) = 0a(w),\ b(1w) = 1b(w)$$

for $w \in X^\omega$. The subgroup generated by $a$ and $b$ is a self-similar group. A straightforward calculation shows that $a^2 = b^2 = \mathrm{id}_{X^\omega}$. Moreover, it is not hard to show that $\mathbb{Z}_2 * \mathbb{Z}_2$ is isomorphic to this self-similar group. Hence, the homomorphisms $a$ and $b$ provides a self-similar action of $\mathbb{Z}_2 * \mathbb{Z}_2$.

The following finitely generated group was first introduced in [15].

**Example 3.5.** ([25, Example 2.5]) Let $X = \{0, 1\}$. Consider homeomorphisms $a, b, c, d$ on $X^\omega$ given by

$$a(0w) = 1w,\ a(1w) = 0w,$$
$$b(0w) = 0a(w),\ b(1w) = 1c(w),$$
$$c(0w) = 0a(w),\ c(1w) = 1d(w),$$
$$d(0w) = 0w,\ d(1w) = 1b(w)$$

for $w \in X^\omega$. Let $G$ be the subgroup of $\mathrm{Homeo}(X^\omega)$ generated by $a, b, c, d$. The above relations imply that $G$ is a self-similar group. The group $G$ is called the *Grigorchuk group*.

For more examples, see [24, 26]. We often keep the following regularity in our mind in the study of self-similar groups and their operator algebras.

**Definition 3.6.** ([25, Definition 4.1]) Let $G$ be a self-similar group over $X$ and fix $g \in G$. An element $w \in X^\omega$ is said to be *$g$-regular* if either $g(w) \neq w$ or there exists an open neighborhood of $w$ consisting of fixed points of $g$. Let $(X^\omega)_{g\text{-reg}}$ be the set of all $g$-regular points. Write $(X^\omega)_{G\text{-reg}} := \bigcap_{g \in G} (X^\omega)_{g\text{-reg}}$. An element $w \in (X^\omega)_{G\text{-reg}}$ is said to be *$G$-regular.*

Recall notations. Take any $u \in X^*$ and let $T_u$ be the shift map on $X^\omega$ given by $w \mapsto uw$. Let $T_u^*$ denote the local inverse map of $T_u$ defined on the range of $T_u$. Write

$$\langle G, X\rangle := \{T_{u_1} g T_{u_2}^* : u_1, u_2 \in X^*, g \in G\}.$$

**Definition 3.7.** ([25, Definition 9.1]) Let $G$ be a self-similar group over $X$ and fix $f \in \langle G, X\rangle$. An element $w \in X^\omega$ is said to be *$f$-regular* if either $f$ is not defined on $w$ or does not fix $w$ or there exists an open neighborhood of $w$ consisting of fixed points of $f$. We write $(X^\omega)_{f\text{-reg}}$ for the set of $f$-regular points. Also write $(X^\omega)^{\mathrm{S}}_{G\text{-reg}} := \bigcap_{f \in \langle G, X\rangle} (X^\omega)_{f\text{-reg}}$. An element in $(X^\omega)^{\mathrm{S}}_{G\text{-reg}}$ is said to be *strictly $G$-regular.*

From now on, we always assume that $G$ is a countable self-similar group. The following remark tells us why we need the countability.

**Remark 3.8.** It is not hard to show that $(X^\omega)_{f\text{-reg}}$ is an open dense subset of $X^\omega$ for any $f \in \langle G, X\rangle$. Thus, the Baire category theorem implies that $(X^\omega)^{\mathrm{S}}_{G\text{-reg}}$ is dense in $X^\omega$ by the countability of $G$. Note that $(X^\omega)^{\mathrm{S}}_{G\text{-reg}}$ is a $\langle G, X\rangle$-invariant set.

For $f \in \langle G, X\rangle$, let $(X^\omega)_f$ be the set of fixed points of $f$. By definition, $X^\omega \backslash (X^\omega)_{f\text{-reg}} \subset (X^\omega)_f$.

Recall that the Bernoulli measure on $X^\omega$ is denoted by $\mu$. In other words, $\mu$ is the product measure of uniform probability measures on the finite set $X$. Note that we have $\mu(uX^\omega) = |X|^{-|u|}$ for any $u \in X^*$.

**Remark 3.9.** Take $f = T_{u_1} g T_{u_2}^* \in \langle G, X\rangle$. Assume that $|u_1| > |u_2|$ and there exists a fixed point $w$ of $f$. Then a direct calculation shows that $w \in u_1 X^\omega$ and there exists $u \in X^*$ with $u_1 = u_2 u$. Hence, we have a $w_1 \in X^\omega$ with $w = u_1 w_1$. The equation $T_{u_1} g T_{u_2}^*(u_2 u w_1) = u_2 u w_1$ implies $g(uw_1) = w_1$. From the self-similarity, we have $g(u) g|_u w_1 = w_1$. This shows $w_1 \in g(u) X^\omega$. Thus, there exists an infinite word $w_2 \in X^\omega$ with $w_1 = g(u) w_2$. Combining $g(u) g|_u w_1 = w_1$ and $w_1 = g(u) w_2$, we have $g|_u (g(u) w_2) = w_2$. Therefore $w_2 \in g|_u(g(u)) X^\omega$. Repeating this, one can uniquely determine $w$ from $u_1$, $u_2$ and $g$. Hence the number of fixed points of $f$ is at most one.

Assume that $|u_1| < |u_2|$ and there exists a fixed point $w$ of $f$. Then we have $T_{u_2} g^{-1} T_{u_1}^* w = w$. Thus, the above argument shows that the number of fixed points of $f$ is at most one. Consequently, the set of fixed points of $f$ is a $\mu$-null set if $|u_1| \neq |u_2|$.

The following lemmas are elementary but important for our study.

**Lemma 3.10.** *Let $G$ be a countable self-similar group over $X$. Then the following conditions are equivalent.*

(i) $\mu((X^\omega)_{G\text{-reg}}) = 1$.
(ii) $\mu((X^\omega)_{g\text{-reg}}) = 1$ *for any* $g \in G$.
(iii) $\mu((X^\omega)^{\mathrm{S}}_{G\text{-reg}}) = 1$.
(iv) $\mu((X^\omega)_{f\text{-reg}}) = 1$ *for any* $f \in \langle G, X\rangle$.

*Proof.* A standard measure theoretical argument provides equivalences (i) $\Leftrightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv). Since $(X^\omega)^{\mathrm{S}}_{G\text{-reg}}$ is a subset of $(X^\omega)_{G\text{-reg}}$, (iii) implies (i).

At last, we assume (ii) and prove (iv). Take any $f \in \langle G, X\rangle$. We show that $\mu((X^\omega)_{f\text{-reg}}) = 1$. We write $f = T_{u_1} g T_{u_2}^*$ for some $u_1, u_2 \in X^*$ and $g \in G$. By Remark 3.9, we may assume that $|u_1| = |u_2|$. If $u_1 \neq u_2$ then the set of fixed points of $f$ is empty so we further assume $u_1 = u_2$. In this case we have $(X^\omega)_f = u_1(X^\omega)_g$ and $(X^\omega)_f \cap (X^\omega)_{f\text{-reg}} = u_1((X^\omega)_g \cap (X^\omega)_{g\text{-reg}})$. The assumption implies that $\mu((X^\omega)_g) = \mu((X^\omega)_g \cap (X^\omega)_{g\text{-reg}})$. Hence, we have $\mu((X^\omega)_f) = \mu((X^\omega)_f \cap (X^\omega)_{f\text{-reg}})$. This implies that $\mu(X^\omega \backslash (X^\omega)_{f\text{-reg}}) = \mu((X^\omega)_f \backslash (X^\omega)_{f\text{-reg}}) = 0$. Thus, we finished the proof. □

**Lemma 3.11.** *Let $G$ be a self-similar group over $X$. Then for any $g \in G$, the following equations hold:*

$$\lim_{n\to\infty} |X|^{-n} |\{u \in X^n : g(u) = u\}| = \mu((X^\omega)_g). \tag{3.2}$$

$$\lim_{n\to\infty} |X|^{-n} |\{u \in X^n : g|_u \neq e, g(u) = u\}| = \mu(X^\omega \backslash (X^\omega)_{g\text{-reg}}). \tag{3.3}$$

*Proof.* Note that

$$\bigcup_{\{u\in X^n : g(u)=u\}} uX^\omega = \{w \in X^\omega : g(w^{(n)}) = w^{(n)}\}$$

for any $n \in \mathbb{N}$. In addition, the sequence of the open sets $\{w \in X^\omega : g(w^{(n)}) = w^{(n)}\}$ is monotonically decreasing and

$$\bigcap_{n\in\mathbb{N}} \{w \in X^\omega : g(w^{(n)}) = w^{(n)}\} = (X^\omega)_g.$$

Thus the first equation holds. Similarly, the second one does. □

**Lemma 3.12.** *Let $G$ be a self-similar group over $X$. Then for any $g \in G$, we have the following equations:*

$$\mu((X^\omega)_g) = |X|^{-1} \sum_{x\in X} \delta_{x,g(x)} \mu((X^\omega)_{g|_x}). \tag{3.4}$$

$$\mu((X^\omega)_g \cap (X^\omega)_{g\text{-reg}}) = |X|^{-1} \sum_{x\in X} \delta_{x,g(x)} \mu((X^\omega)_{g|_x} \cap (X^\omega)_{g|_x\text{-reg}}). \tag{3.5}$$

*Here $\delta$ is the Kronecker's $\delta$.*

*Proof.* One can check that

$$(X^\omega)_g = \bigcup_{\{x\in X : g(x)=x\}} x((X^\omega)_{g|_x})$$

and

$$(X^\omega)_g \cap (X^\omega)_{g\text{-reg}} = \bigcup_{\{x\in X : g(x)=x\}} x((X^\omega)_{g|_x} \cap (X^\omega)_{g|_x\text{-reg}}).$$

We consider the Bernoulli measure $\mu$ on the above equations to finish the proof. □

The following theorem provides a dichotomy for self-similar groups.

**Theorem 3.13.** *For any countable self-similar group $G$ over $X$, one has either $\mu((X^\omega)_{G\text{-reg}}) = 1$ or $\mu((X^\omega)_{G\text{-reg}}) = 0$.*

*Proof.* If $\mu((X^\omega)_{G\text{-reg}}) \neq 1$, then $\mu((X^\omega)_{g\text{-reg}}) \neq 1$ for some $g \in G$ by Lemma 3.10. Combining Lemma 3.12 and $X^\omega \backslash (X^\omega)_{g\text{-reg}} = (X^\omega)_g \backslash ((X^\omega)_g \cap (X^\omega)_{g\text{-reg}})$, we have

$$\mu(X^\omega \backslash (X^\omega)_{g\text{-reg}}) = |X|^{-1} \sum_{x\in X} \delta_{x,g(x)} \mu(X^\omega \backslash (X^\omega)_{g|_x\text{-reg}}).$$

Note that $(X^\omega)_{e\text{-reg}} = X^\omega$ where $e$ is the unit of $G$. Then we have

$$\mu(X^\omega \backslash (X^\omega)_{g\text{-reg}}) = |X|^{-1} \sum_{\{x\in X : g|_x \neq e\}} \delta_{x,g(x)} \mu(X^\omega \backslash (X^\omega)_{g|_x\text{-reg}}). \tag{3.6}$$

Using (3.6) repeatedly, we have

$$\mu(X^\omega\backslash(X^\omega)_{g\text{-reg}}) = |X|^{-n} \sum_{\{u\in X^n: g|_u\neq e\}} \delta_{u,g(u)}\mu(X^\omega\backslash(X^\omega)_{g|_u\text{-reg}}) \tag{3.7}$$

for any $n\in\mathbb{N}$. For each $n\in\mathbb{N}$ choose $g_n\in G$ with

$$\mu(X^\omega\backslash(X^\omega)_{g_n\text{-reg}}) = \max\{\mu(X^\omega\backslash(X^\omega)_{g|_u\text{-reg}}) : u\in X^n \text{ with } g(u)=u \text{ and } g|_u\neq e\}.$$

One can take such $g_n$ because for any $n\in\mathbb{N}$ there exists at least one $u\in X^n$ with $g(u)=u$ and $g|_u\neq e$. Indeed, if we have a natural number $n\in\mathbb{N}$ such that for any $u\in X^n$ either $g(u)\neq u$ or $g|_u = e$, then $\mu(X^\omega\backslash(X^\omega)_{g\text{-reg}}) = 0$ by (3.7). This is a contradiction. Using (3.7), we have

$$\mu(X^\omega\backslash(X^\omega)_{g\text{-reg}}) \le |X|^{-n} \sum_{\{x\in X^n: g|_u\neq e\}} \delta_{u,g(u)}\mu(X^\omega\backslash(X^\omega)_{g_n\text{-reg}})$$

for any $n\in\mathbb{N}$. Applying Lemma 3.11 and taking the limit, we obtain

$$\begin{aligned}\mu(X^\omega\backslash(X^\omega)_{g\text{-reg}}) &\le \liminf_{n\to\infty} (\mu(X^\omega\backslash(X^\omega)_{g_n\text{-reg}}) \lim_{n\to\infty} |X|^{-n}|\{x\in X^n : g|_u\neq e, g(u)=u\}| \\ &= \mu(X^\omega\backslash(X^\omega)_{g\text{-reg}}) \liminf_{n\to\infty} \mu(X^\omega\backslash(X^\omega)_{g_n\text{-reg}}).\end{aligned}$$

Note that $\mu(X^\omega\backslash(X^\omega)_{g\text{-reg}})$ is nonzero and $\limsup_{n\to\infty}\mu(X^\omega\backslash(X^\omega)_{g_n\text{-reg}})$ is at most one. Then we have $\lim_{n\to\infty}\mu(X^\omega\backslash(X^\omega)_{g_n\text{-reg}}) = 1$. This shows $\mu((X^\omega)_{G\text{-reg}}) = 0$. □

As a direct corollary of the previous theorem and Lemma 3.10, we have the following one.

**Corollary 3.14.** *For any countable self-similar group $G$ over $X$, one has either $\mu((X^\omega)^{\mathrm{S}}_{G\text{-reg}}) = 1$ or $\mu((X^\omega)^{\mathrm{S}}_{G\text{-reg}}) = 0$.*

In the next section, we observe that a KMS state nicely behaves in the case $\mu((X^\omega)_{G\text{-reg}}) = 1$. Many countable self-similar groups satisfy the condition $\mu((X^\omega)_{G\text{-reg}}) = 1$. For example, if $\mu((X^\omega)_g) = 0$ for any $g\neq e$ then we have $\mu((X^\omega)_{G\text{-reg}}) = 1$. For the other case, we recall the definition of a property called contracting.

**Definition 3.15.** ([26, Definition 2.2]) A self-similar group $G$ over $X$ is said to be *contracting* if there exists a finite set $\mathcal{N}\subset G$ with the following condition:

For any $g\in G$ there exists $n\in\mathbb{N}$ satisfying $g|_v\in\mathcal{N}$ for any $v\in X^*$ with $|v|>n$.

If a self-similar group $G$ is contracting, the smallest finite set of $G$ satisfying the above condition is said to be the *nucleus of* $G$.

**Example 3.16.** Let $G$ be the Grigorchuk group (see Example 3.5). Then $G$ is contracting. The nucleus of $G$ is $\{e,a,b,c,d\}$.

One can find the following proposition in the proof of [24, Theorem 7.3 (3)] but for reader's convenience we prove it here.

**Proposition 3.17.** *Let $G$ be a contracting countable self-similar group over $X$. Then we have $\mu((X^\omega)_{G\text{-reg}}) = 1$.*

*Proof.* Take any $g\in G\backslash\{e\}$. We show $\mu(X^\omega\backslash(X^\omega)_{g\text{-reg}}) = 0$. Since the action is contracting, the set $\{g|_u\in G : u\in X^*\}$ is finite. Hence, one can find $m\in\mathbb{N}$ such that for every $u\in X^*$ with $g|_u\neq e$ there exists $u'\in X^m$ with $g|_u(u')\neq u'$. Using an inductive argument, we next show that

$$|\{u\in X^{mn} : g(u)=u, g|_u\neq e\}| \le (|X|^m-1)^n \tag{3.8}$$

for any $n\in\mathbb{N}, g\in G\backslash\{e\}$. If $n=1$, then

$$|\{u\in X^m : g(u)=u\}| \le |X|^m - 1$$

by the choice of $m$. Next we assume that (3.8) holds for a given $n\in\mathbb{N}$. Take any $u\in X^{m(n+1)}$ with $g(u)=u$ and $g|_u\neq e$. We divide $u$ into two finite words $u_0\in X^{mn}$ and $u_1\in X^m$ with $u=u_0u_1$. Then we have $g(u_0) = u_0$, $g|_{u_0} \neq e$, $g|_{u_0}(u_1) = u_1$ and $g|_{u_0 u_1} \neq e$. Since $g|_{u_0}(u') \neq u'$ for some $u' \in X^\omega$, we get

$$|\{u' \in X^m : g|_{u_0}(u') = u', g|_{u_0 u'} \neq e\}| \leq (|X|^m - 1)$$

Hence,

$$\begin{aligned} |\{u \in X^{m(n+1)} : g(u) = u, g|_u \neq e\}| &\leq (|X|^m - 1)|\{u \in X^{mn} : g(u) = u, g|_u \neq e\}| \\ &\leq (|X|^m - 1)^{n+1} \end{aligned}$$

by the inductive assumption. Consequently, We get (3.8). Therefore, we have

$$\begin{aligned} \mu(X^\omega \backslash (X^\omega)_{g\text{-reg}}) &= \lim_{n\to\infty} |X|^{-mn}|\{u \in X^{mn} : g(u) = u, g|_u \neq e\}| \\ &\leq \lim_{n\to\infty} (|1 - |X|^{-m})^n = 0. \end{aligned}$$

□

## 4. KMS states on Cuntz–Pimsner algebras associated with self-similar groups

At first, we review the Cuntz–Pimsner algebras introduced by V. V. Nekrashevych constructed from self-similar groups in [25, 26]. We always assume that $G$ is a countable self-similar group over $X$.

**Definition 4.1.** ([25, Definition 3.1]) Define $\mathcal{O}_{G_{\max}}$ to be the universal C*-algebra generated by $G$ (we assume that every relation in $G$ is preserved) and $\{S_x : x \in X\}$ satisfying the following relations for any $x \in X$ and $g \in G$:

$$g^* g = g g^* = 1, \tag{4.1}$$

$$S_x^* S_x = 1, \tag{4.2}$$

$$\sum_{y \in X} S_y S_y^* = 1, \tag{4.3}$$

$$g S_x = S_{g(x)} g|_x. \tag{4.4}$$

The universal C*-algebra $\mathcal{O}_{G_{\max}}$ admits a nonzero representation. We recall the following representation.

**Remark 4.2.** For each $w \in X^\omega$, $\langle G, X\rangle(w)$ denotes the $\langle G, X\rangle$-orbit of $w$, i.e.

$$\langle G, X\rangle(w) := \{f(w) : f \in \langle G, X\rangle, w \in \mathrm{Dom} f\}.$$

Let $l^2(\langle G, X\rangle(w))$ be the set of $l^2$-functions on $\langle G, X\rangle(w)$. Write $\delta_{w_1}$ for the characteristic function of $w_1 \in \langle G, X\rangle(w)$. We naturally identify each shift map $T_x$ with an isometry on $l^2(\langle G, X\rangle(w))$ given by $T_x(\delta_{w_1}) := \delta_{x w_1}$. Similarly, we identify each $g \in G$ with a unitary on $l^2(\langle G, X\rangle(w))$ given by $g(\delta_{w_1}) = \delta_{g(w_1)}$. Thanks to the universality of $\mathcal{O}_{G_{\max}}$, we get a representation $\pi_w$ of $\mathcal{O}_{G_{\max}}$ on $l^2(\langle G, X\rangle(w))$ given by

$$\pi_w(S_x) = T_x,\ \pi_w(g) = g$$

for any $x \in X, g \in G$. In particular, $\mathcal{O}_{G_{\max}}$ is nonzero.

For $u = x_1 x_2 \cdots x_n \in X^n$, we write $S_u := S_{x_1} \cdots S_{x_n}$. Equations (4.2), (4.3) and Remark 2.13 imply that $\mathcal{O}_{G_{\max}}$ contains the Cuntz algebra $\mathcal{O}_{|X|}$.

We recall the following result due to V. V. Nekrashevych.

**Theorem 4.3.** ([26, Theorem 3.3]) *Let $\rho$ be a unital representation of $\mathcal{O}_{G_{\max}}$ on a nonzero Hilbert space. Then for any $w \in (X^\omega)^{\mathrm{S}}_{G\text{-reg}}$ and $a \in \mathcal{O}_{G_{\max}}$, we have*

$$\|\pi_w(a)\| \leq \|\rho(a)\|.$$

The above theorem implies that the C*-algebra $\mathcal{O}_{G_{\min}} := \pi_w(\mathcal{O}_{G_{\max}})$ does not depend on the choice of $w \in (X^\omega)^{\mathrm{S}}_{G\text{-reg}}$ up to canonical isomorphisms. Let $A_{G_{\min}}$ be the C*-subalgebra of $\mathcal{O}_{G_{\min}}$ generated by $G$. We discuss when $\pi_w \colon \mathcal{O}_{G_{\max}} \to \mathcal{O}_{G_{\min}}$ is an isomorphism in section 6. From now on, we also write $S_x$ and $g$ for $\pi_w(S_x)$ and $\pi_w(g)$, respectively.

To understand $\mathcal{O}_{G_{\min}}$, we consider other algebras arising from self-similar groups. For this purpose, we observe a Hilbert $A_{G_{\min}}$-bimodule. Define a subspace $\Phi$ of $\mathcal{O}_{G_{\min}}$ by

$$\Phi := \left\{ \sum_{x\in X} S_x a_x \;\middle|\; a_x \in A_{G_{\min}} \right\}.$$

We regard $\Phi$ as a right $A_{G_{\min}}$-module with the basis $\{S_x\}_{x\in X}$. Define an $A_{G_{\min}}$-valued inner product on $\Phi$ by

$$\langle \sum_{x\in X} S_x a_x, \sum_{x\in X} S_x b_x \rangle := \sum_{x\in X} a_x^* b_x$$

where $a_x, b_x \in A_{G_{\min}}$ for any $x \in X$.

Consider the left action of $A_{G_{\min}}$ on $\mathcal{O}_{G_{\min}}$ arising from the multiplication. The left action provides a left $A_{G_{\min}}$-module structure on $\Phi$. Thus, we obtain a Hilbert $A_{G_{\min}}$-bimodule. We also write $\Phi$ for this Hilbert $A_{G_{\min}}$-bimodule.

The following definition is almost the same as [25, Definition 6.1].

**Definition 4.4.** ([25, Definition 6.1]) Define $\mathcal{O}'_{G_{\min}}$ to be the universal C*-algebra generated by $A_{G_{\min}}$ and $\{S_x : x \in X\}$ satisfying equations (4.2), (4.3) and the following relation for any $a \in A_{G_{\min}}$ and $x \in X$:

$$aS_x = \sum_{y\in X} S_y \langle S_y, aS_x \rangle. \tag{4.5}$$

Here the inner product is the $A_{G_{\min}}$-valued one defined as above.

The same argument as [25, Theorem 8.3] implies the simplicity of $\mathcal{O}'_{G_{\min}}$.

**Theorem 4.5.** ([25, Theorem 8.3]) *The universal C*-algebra $\mathcal{O}'_{G_{\min}}$ is unital, purely infinite and simple.*

In fact, $\mathcal{O}'_{G_{\min}}$ is isomorphic to the Cuntz–Pimsner algebra $\mathcal{O}(\Phi)$ and also isomorphic to $\mathcal{O}_{G_{\min}}$. Indeed, using the simplicity from Theorem 4.5, one can construct injective universal surjections from $\mathcal{O}'_{G_{\min}}$ onto both $\mathcal{O}_{G_{\min}}$ and $\mathcal{O}(\Phi)$. Hence $\mathcal{O}'_{G_{\min}}$ is isomorphic to $\mathcal{O}_{G_{\min}}$ and $\mathcal{O}(\Phi)$.

Let $\Gamma$ be the canonical gauge action of $\mathbb{R}$ on $\mathcal{O}_{G_{\max}}$ given by

$$\Gamma_t(g) := g \text{ and } \Gamma_t(S_x) := \exp(it)S_x$$

for $g \in G$, $t \in \mathbb{R}$ and $x \in X$. We also define the canonical gauge action $\Gamma$ on $\mathcal{O}_{G_{\min}}$ (we use the same symbol as there is no confusion) by

$$\Gamma_t(a) := a \text{ and } \Gamma_t(S_x) := \exp(it)S_x$$

for $a \in A_{G_{\min}}$, $t \in \mathbb{R}$ and $x \in X$. Using the universalities, one can check the above gauge actions are well-defined. Let $K_\beta(\mathcal{O}_{G_{\min}})$ and $K_\beta(\mathcal{O}_{G_{\max}})$ be the sets of $(\beta, \Gamma)$-KMS states on $\mathcal{O}_{G_{\min}}$ and $\mathcal{O}_{G_{\max}}$, respectively. Note that if $\varphi \in K_\beta(\mathcal{O}_{G_{\min}})$ (or $K_\beta(\mathcal{O}_{G_{\max}})$), then

$$\varphi(S_{u_1} g S_{u_2}^*) = \exp(-\beta|u_1|)\varphi(g S_{u_2}^* S_{u_1})$$

for any $g \in G$ and $u_1, u_2 \in X^*$. Combining the above and equation (4.3), we have

$$1 = \varphi\left(\sum_{x\in X} S_x S_x^*\right) = |X| \exp(-\beta).$$

Hence there is no $(\beta, \Gamma)$-KMS state if $\beta \neq \log|X|$. From now on, we only consider the case $\beta = \log|X|$ and let KMS states mean $(\log|X|, \Gamma)$-KMS states.

Let us recall the notation that $G$ is a countable self-similar group over $X$ and $e$ is the unit of $G$.

**Definition 4.6.** A positive definite function $\varphi$ on $G$ (we assume $\varphi(e) = 1$) is said to be a *pre-KMS function* if

$$\varphi(g) = |X|^{-1} \sum_{x \in X} \delta_{x,g(x)} \varphi(g|_x) \tag{4.6}$$

for any $g \in G$.

**Remark 4.7.** The restriction of every $\varphi \in K_{\log|X|}(\mathcal{O}_{G_{\min}})$ (or $K_{\log|X|}(\mathcal{O}_{G_{\max}})$) to $G$ is a pre-KMS function. To check this, note that for any $g \in G$ we have the following equation:

$$g = g \sum_{x \in X} S_x S_x^* = \sum_{x \in X} S_{g(x)} g|_x S_x^*.$$

Then the definition of KMS states implies the equation (4.6).

From the above remark, we should study pre-KMS functions to find KMS states. Lemma 3.12 implies that $g \mapsto \mu((X^\omega)_g)$ and $g \mapsto \mu((X^\omega)_g \cap (X^\omega)_{g\text{-reg}})$ are pre-KMS functions on $G$ (we prove the positive definiteness for the first function later). In the case $\mu((X^\omega)_{G\text{-reg}}) = 1$, the two functions coincide by Lemma 3.10. We show the uniqueness of pre-KMS functions on $G$ in the case $\mu((X^\omega)_{G\text{-reg}}) = 1$. The following proposition was already proved in [5, Proposition 8.3] but for the reader's convenience we prove it here without using more general terminology in [5].

**Proposition 4.8.** *If $\mu((X^\omega)_{G\text{-reg}}) = 1$ and $\varphi$ is a pre-KMS function on $G$, then $\varphi(g) = \mu((X^\omega)_g)$ for any $g \in G$.*

*Proof.* For any $n \in \mathbb{N}$ and $g \in G$, we have

$$\begin{aligned}\varphi(g) &= |X|^{-n} |\{u \in X^n : g|_u = e, g(u) = u\}| \\ &\quad + |X|^{-n} \sum_{\{u \in X^n | g|_u \neq e\}} \delta_{u,g(u)} \varphi(g|_u)\end{aligned}$$

from the definition of pre-KMS function. Note that we have $|\varphi(g)| \leq 1$ for any $g \in G$ by the positive definiteness. Hence,

$$\begin{aligned}&|(\varphi(g) - |X|^{-n} |\{u \in X^n : g|_u = e, g(u) = u\}|)| \\ &\qquad \leq |X|^{-n} |\{u \in X^n : g|_u \neq e, g(u) = u\}|.\end{aligned}$$

Applying Lemma 3.11 and taking the limit, we have

$$|\varphi(g) - \mu((X^\omega)_g \cap (X^\omega)_{g\text{-reg}})| \leq \mu(X^\omega \backslash (X^\omega)_{g\text{-reg}}).$$

The right side of the above inequality is 0 by the assumption and Lemma 3.10. Hence

$$\varphi(g) = \mu((X^\omega)_g \cap (X^\omega)_{g\text{-reg}}) = \mu((X^\omega)_g).$$

□

Put $\psi_0(g) := \mu((X^\omega)_g)$ for $g \in G$. As Proposition 4.8 shows, $\psi_0$ is the unique pre-KMS function on $G$ in the case $\mu((X^\omega)_{G\text{-reg}}) = 1$. We write $\mathbb{C}\langle G, X\rangle$ for the $*$-subalgebra of $\mathcal{O}_{G_{\max}}$ generated by $\{S_x\}_{x \in X}$ and $G$. Let $\psi$ be a linear functional on $\mathbb{C}\langle G, X\rangle$ given by

$$\psi(S_{u_1} g S_{u_2}^*) = \delta_{u_1,u_2} |X|^{-|u_1|} \psi_0(g) \tag{4.7}$$

where $g \in G, u_1, u_2 \in X^*$.

We show that $\psi$ provides the unique KMS state on $\mathcal{O}_{G_{\min}}$. For the positive definiteness of $\psi_0$, we need the next lemma. The lemma might be proven somewhere as a corollary of [14, Theorem 2] but for the reader's convenience we put a proof here.

**Lemma 4.9.** *Let $Y$ be a Hausdorff space and $\nu$ be a Borel probability measure on $Y$. Assume that $G$ is a countable subgroup of the group of measure preserving homeomorphisms on $Y$. Let $Y_g$ be the set of fixed points of $g \in G$. Then the map $g \mapsto \nu(Y_g)$ is a positive definite function.*

*Proof.* Let $R := \{(x,y) \in Y^2 : y = g(x) \text{ for some } g \in G\}$. Then $R$ defines an equivalence relation on $Y$. For $g \in G$, we define $\mathrm{Graph}(g) := \{(x, g(x)) : x \in Y\} \subset R$. Moreover, consider the projection $\pi_l \colon Y^2 \to Y$ given by $\pi_l((x,y)) := x$. We denote by $\nu_l$ the left counting measure on $R$ (see [14, Theorem 2]). For any $g, h \in G$, we have

$$\nu_l(\mathrm{Graph}(g) \cap \mathrm{Graph}(h)) = \int_Y |\pi_l(x)^{-1} \cap \mathrm{Graph}(g) \cap \mathrm{Graph}(h)| d\nu = \nu(Y_{g^{-1}h}).$$

Take $f \in \mathbb{C}G$ and write $f = \sum_{g \in F} \alpha_g g$ where $F$ is a finite subset of $G$. Set $f' := \sum_{g \in F} \alpha_g 1_{\mathrm{Graph}(g)}$. Then

$$\sum_{g,h \in F} \overline{\alpha_g} \alpha_h \nu(Y_{g^{-1}h}) = \int_R |f'|^2 d\nu_l \geq 0.$$

Thus we get the conclusion. □

**Theorem 4.10.** *Assume that $G$ is a countable self-similar group over $X$ with $\mu((X^\omega)_{G\text{-reg}}) = 1$. Then there exists a unique $(\log|X|, \Gamma)$-KMS state on $\mathcal{O}_{G_{\min}}$.*

*Proof.* Take any strictly $G$-regular point $w_0$. We define a linear functional $\psi'$ on $\pi_w(\mathbb{C}G) \subset A_{G_{\min}}$ to be

$$\psi'(\pi_{w_0}(f)) := \psi(f).$$

for $f \in \mathbb{C}G$. First, we check that $\psi'$ is well-defined. Take any $f' \in \mathbb{C}G$ with $\pi_{w_0}(f') = 0$. We prove $\psi(f') = 0$. Write

$$f' = \sum_{g \in G} \gamma_g g$$

where $\gamma_g = 0$ for all but finitely many $g$. Note that

$$\psi(g_1^{-1} g_2) = \mu((X^\omega)_{g_1^{-1} g_2}) = \int_{X^\omega} \langle g_1(\delta_w), g_2(\delta_w) \rangle d\mu(w)$$

for any $g_1, g_2 \in G$. Each inner product above is defined on $l^2(\langle G, X \rangle(w))$. The assumption and Lemma 3.10 imply $\mu((X^\omega)^{\mathrm{S}}_{G\text{-reg}}) = 1$. Then a calculation shows

$$\begin{aligned}\psi(f) = \sum_{g \in G} \gamma_g \int_{X^\omega} \langle \delta_w, g(\delta_w) \rangle d\mu(w) &= \sum_{g \in G} \gamma_g \int_{(X^\omega)^{\mathrm{S}}_{G\text{-reg}}} \langle \delta_w, g(\delta_w) \rangle d\mu(w) \\ &= \int_{(X^\omega)^{\mathrm{S}}_{G\text{-reg}}} \langle \delta_w, (\pi_w(f))(\delta_w) \rangle d\mu(w) = 0\end{aligned}$$

since $\pi_w(f) = 0$ for any $w \in (X^\omega)^{\mathrm{S}}_{G\text{-reg}}$. This proves the claim.

Lemma 4.9 shows $\psi'(\pi_{w_0}(f^* f)) = \psi(f^* f) \geq 0$ for any $f \in \mathbb{C}G$. Next, We prove the continuity of $\psi'$ on $\pi_{w_0}(\mathbb{C}G)$ with respect to the norm on $A_{G_{\min}}$. Take an arbitrary element $f \in \mathbb{C}G$. Write

$$f = \sum_{g \in G} \alpha_g g$$

where $\alpha_g = 0$ for all but finitely many $g$. From the definition of $A_{G_{\min}}$, we get

$$\begin{aligned}\psi'(\pi_{w_0}(f^* f)) = \psi(f^* f) &= \sum_{g_1, g_2 \in G} \overline{\alpha_{g_1}} \alpha_{g_2} \int_{X^\omega} \langle g_1(\delta_w), g_2(\delta_w) \rangle d\mu(w) \\ &= \int_{X^\omega} \langle f(\delta_w), f(\delta_w) \rangle d\mu(w) \\ &= \int_{(X^\omega)^{\mathrm{S}}_{G\text{-reg}}} \langle f(\delta_w), f(\delta_w) \rangle d\mu(w) \leq \|f\|^2_{A_{G_{\min}}}.\end{aligned}$$

We finished proving the continuity. Hence, $\psi'$ extends to a state on $A_{G_{\min}}$. It is not hard to check that we have $\psi_0(g_1 g_2) = \psi_0(g_2 g_1)$ for any $g_1, g_2 \in G$. Thus, $\psi'$ defines a tracial state on $A_{G_{\min}}$. Moreover, the tracial state satisfies the assumption of Lemma 2.11. Therefore $\psi'$ extends uniquely to a KMS state on $\mathcal{O}_{G_{\min}}$.

By Proposition 4.8, no other tracial state on $A_{G_{\min}}$ satisfies the assumption of Lemma 2.11. This proves the uniqueness part. □

The above theorem implies that there exists a KMS state on $\mathcal{O}_{G_{\max}}$. From now on, we also write $\psi$ for $\psi'$. This notation does not cause the confusion in the case $\mu((X^\omega)_{G\text{-reg}}) = 1$. Furthermore, we write $\psi_{\min}$ and $\psi_{\max}$ for the KMS states given by $\psi$ on $\mathcal{O}_{G_{\min}}$ and $\mathcal{O}_{G_{\max}}$, respectively. Note that the restrictions of $\psi_{\min}$ and $\psi_{\max}$ provide tracial states on the gauge invariant subalgebras of $\mathcal{O}_{G_{\min}}$ and $\mathcal{O}_{G_{\max}}$, respectively. We show that they are unique ones.

**Theorem 4.11.** *If $G$ is a countable self-similar group over $X$ with $\mu((X^\omega)_{G\text{-reg}}) = 1$, then there exists a unique tracial state on the gauge invariant subalgebras of $\mathcal{O}_{G_{\min}}$ and $\mathcal{O}_{G_{\max}}$.*

*Proof.* Take a tracial state $\varphi$ on the gauge invariant subalgebra of $\mathcal{O}_{G_{\min}}$ or $\mathcal{O}_{G_{\max}}$. It is sufficient to prove $\varphi(S_{u_1}gS_{u_2}^*) = |X|^{-n}\delta_{u_1,u_2}\psi(g)$ for any $g \in G$ and any pair of finite words $u_1, u_2 \in X^*$ with same length $n$. Note that $\varphi(S_{u_1}gS_{u_2}^*) = 0$ if $u_1 \neq u_2$ and $\varphi(S_{u_1}S_{u_1}^*) = |X|^{-n} = \psi(S_{u_1}S_{u_1}^*)$ for any $u_1 \in X^*$. Hence we get

$$\varphi(S_{u_1}gS_{u_1}^*) - \psi(S_{u_1}gS_{u_1}^*) = \sum_{\{u \in X^m : g(u)=u, g|_u \neq e\}} (\varphi - \psi)(S_{u_1}S_{g(u)}g|_u S_u^* S_{u_1}^*) \tag{4.8}$$

for any $m \in \mathbb{N}$. For each $u \in X^m$ with $g(u) = u$, define $\varphi_u(x) := \varphi(S_{u_1}S_{g(u)}xS_u^*S_{u_1}^*)$. Note that $\varphi_u$ is a positive linear functional with $\|\varphi_u\| = \varphi(S_{u_1}S_{g(u)}S_u^*S_{u_1}^*)$. Then the equation (4.8) shows

$$\begin{aligned}
|\varphi(S_{u_1}gS_{u_1}^*) - \psi(S_{u_1}gS_{u_1}^*)| \leq & \sum_{\{u \in X^m : g(u)=u, g|_u \neq e\}} \varphi(S_{u_1}S_{g(u)}S_u^*S_{u_1}^*) \\
&+ \sum_{\{u \in X^m : g(u)=u, g|_u \neq e\}} \psi(S_{u_1}S_{g(u)}S_u^*S_{u_1}^*) \\
&\leq 2|X|^{-n-m}|\{u \in X^m : g(u) = u, g|_u \neq e\}|.
\end{aligned}$$

By the assumption, we have $\mu(X^\omega \backslash (X^\omega)_{g\text{-reg}}) = 0$ so Lemma 3.11 and the above inequality implies $\varphi(S_{u_1}gS_{u_1}^*) = \psi(S_{u_1}gS_{u_1}^*)$. □

From the above theorem, one can also show that the uniqueness part of Theorem 4.10 without using Proposition 4.8. We also obtain the following theorem as a corollary of the above theorem. However, the following theorem was already proved in [5] from arguments on KMS states on Toeplitz type C*-algebra associated to right LCM monoids.

**Theorem 4.12.** *Assume that $G$ is a countable self-similar group over $X$ with $\mu((X^\omega)_{G\text{-reg}}) = 1$. Then there exists a unique* $(\log|X|, \Gamma)$*-KMS state on $\mathcal{O}_{G_{\max}}$.*

## 5. von Neumann algebraic approaches

In this section we consider GNS representations of KMS states which we have discussed in the previous section. We always assume that $G$ is a countable self-similar group over $X$ with $\mu((X^\omega)_{G\text{-reg}}) = 1$. Let a pair $(\pi_{\psi_{\min}}, H_{\psi_{\min}})$ denote the GNS representation of $\psi_{\min}$. Similarly, we write $(\pi_{\psi_{\max}}, H_{\psi_{\max}})$ for the GNS representation of $\psi_{\max}$. Moreover let $\rho$ be the canonical quotient map from $\mathcal{O}_{G_{\max}}$ onto $\mathcal{O}_{G_{\min}}$. Then one obtains $\psi_{\max} = \psi_{\min} \circ \rho$. This implies that $H_{\psi_{\max}}$ and $H_{\psi_{\min}}$ are isomorphic via the canonical unitary. The canonical unitary provides the unitarily equivalence between $(\pi_{\psi_{\max}}, H_{\psi_{\max}})$ and $(\pi_{\psi_{\min}} \circ \rho, H_{\psi_{\min}})$. In particular, $\pi_{\psi_{\max}}(\mathcal{O}_{G_{\max}})''$ is isomorphic to $\pi_{\psi_{\min}}(\mathcal{O}_{G_{\min}})''$. We write simply $\mathcal{O}_G''$ for this von Neumann algebra. In this section, we only consider $\mathcal{O}_{G_{\min}}$ and $(\pi_{\psi_{\min}}, H_{\psi_{\min}})$. Hence we use a simpler symbol $\psi$ instead of $\psi_{\min}$. From the simplicity of $\mathcal{O}_{G_{\min}}$, we obtain the faithfulness of $\pi_\psi$. Thus, $\psi$ is a faithful KMS state (see [3, Corollary 5.3.9]). In addition, $\psi$ and $\Gamma$ extend to a normal faithful state and a group action on $\mathcal{O}_G''$, respectively (see [32, Proposition 8.1.4]). We also write $\psi$ and $\Gamma$ for these extensions. Note that the extended state $\psi$ is a $(\log|X|, \Gamma)$-KMS state on $\mathcal{O}_G''$. The uniqueness of $\psi$ in Theorem 4.10 implies that the von Neumann algebra $\mathcal{O}_G''$ is a factor. We obtain the Murray–von Neumann–Connes type of $\mathcal{O}_G''$ as a corollary of the uniqueness of tracial states in Theorem 4.11. We observe that $\mathcal{O}_G''$ is an AFD $\mathrm{III}_{|X|^{-1}}$ factor in some cases. One can find details of type III factors in textbooks [31, 32].

**Theorem 5.1.** *If $G$ is a countable self-similar group over $X$ with $\mu((X^\omega)_{G\text{-reg}}) = 1$, then $\mathcal{O}_G''$ is a type $\mathrm{III}_{|X|^{-1}}$ factor.*

*Proof.* Let $\mathcal{O}_{G_{\min}}^\Gamma$ be the gauge invariant subalgebra of $\mathcal{O}_{G_{\min}}$ and let $\tau$ be the restriction of $\psi$ to $\mathcal{O}_{G_{\min}}^\Gamma$. In addition, let the GNS representation of $\tau$ be denoted by $(\pi_\tau, H_\tau)$. Note that $\pi_\psi(\mathcal{O}_{G_{\min}}^\Gamma)''$ is isomorphic to $\pi_\tau(\mathcal{O}_{G_{\min}}^\Gamma)''$ by [17, Lemma 4.1]. Theorem 4.11 implies that $\pi_\tau(\mathcal{O}_{G_{\min}}^\Gamma)''$ is a factor, so is $\pi_\psi(\mathcal{O}_{G_{\min}}^\Gamma)''$.

We write $\sigma^\psi$ for the modular automorphism group of $\psi$. Recall the uniqueness of the one parameter automorphism group in the sense of [32, Theorem 8.1.2]. Then we obtain $\Gamma_{-t\log|X|} = \sigma_t^\psi$ for any $t \in \mathbb{R}$. Using the periodicity of $\Gamma$, we have the equation

$$(\mathcal{O}_G'')^{\sigma^\psi} = (\mathcal{O}_G'')^\Gamma = \pi_\psi(\mathcal{O}_{G_{\min}}^\Gamma)''$$

where $(\mathcal{O}_G'')^{\sigma^\psi}$ is the invariant sublagebra of $\sigma^\psi$. Thus $(\mathcal{O}_G'')^{\sigma^\psi}$ is a factor. Therefore, the Connes spectrum of $\sigma^\psi$ coincides with the Arveson spectrum of $\sigma^\psi$ (see [31, 16.1]).

We write $\mathrm{sp}(\sigma^\psi)$ for the the Arveson spectrum of $\sigma^\psi$. We show $\mathrm{sp}(\sigma^\psi) = (\log|X|^{-1})\mathbb{Z}$. The equation $\sigma_t^\psi(S_x) = e^{-it\log|X|}S_x$ implies $-\log|X| \in \mathrm{sp}(\sigma^\psi)$ (see [31, 14.5]). Since $\mathrm{sp}(\sigma^\psi)$ coincides with the Connes spectrum of $\sigma_\psi$, it is a subgroup of $\mathbb{R}$. Thus, we have $(\log|X|^{-1})\mathbb{Z} \subset \mathrm{sp}(\sigma^\psi)$.

Conversely, assume $\lambda \in \mathrm{sp}(\sigma^\psi)$. Then for any $\varepsilon > 0$ and $t \in \mathbb{R}$, there exists a norm one element $a \in \mathcal{O}_G''$ with

$$\|\sigma_t^\psi(a) - e^{i\lambda t}a\| < \varepsilon$$

by a proposition in [31, 14.5]. Note that we have

$$\sigma^\psi_{\frac{2\pi}{\log|X|}} = \mathrm{id}_{\mathcal{O}_G''}.$$

This implies

$$e^{i\lambda\frac{2\pi}{\log|X|}} = 1.$$

Hence, we obtain $\lambda \in (\log|X|^{-1})\mathbb{Z}$. Consequently, we get $\mathrm{sp}(\sigma^\psi) = (\log|X|^{-1})\mathbb{Z}$. Thus, we conclude that $\mathcal{O}_G''$ is a type $\mathrm{III}_{|X|^{-1}}$ factor. □

The similar result for Cuntz algebras is known (see [17]). More precisely, the von Neumann algebra $\pi_\psi(\mathcal{O}_{|X|})''$ is an AFD (approximately finite dimensional) type $\mathrm{III}_{|X|^{-1}}$ factor. Note that an AFD type $\mathrm{III}_\lambda$ factor is unique up to isomorphisms for each $0 < \lambda < 1$ by Connne's classification theorem [7]. Hence, in the rest part of this section, we consider when the factor $\mathcal{O}_G''$ is AFD.

**Proposition 5.2.** *If $G$ is a countable amenable self-similar group over $X$ with $\mu((X^\omega)_{G\text{-reg}}) = 1$, then $\mathcal{O}_G''$ is the AFD type $\mathrm{III}_{|X|^{-1}}$ factor.*

*Proof.* We only show that $\mathcal{O}_G''$ is AFD. If $G$ is amenable, then its full group C*-algebra is nuclear, so is $A_{G_{\min}}$. Thus, Theorem 2.9 implies that $\mathcal{O}_{G_{\min}}$ is nuclear. Consequently, $\mathcal{O}_G''$ is AFD by [4, Corollary 3.8.6]. □

In some cases, we also show that $\mathcal{O}_G''$ is an AFD $\mathrm{III}_{|X|^{-1}}$ without assuming the amenability of $G$. For any $g \in G$ and $n \in \mathbb{N}$, write $Y_g^n := \{w \in X^\omega : g|_{w^{(n)}} = e\}$.

**Theorem 5.3.** *Assume that $G$ is a countable self-similar group over $X$. If $\mu(\bigcup_{n\in\mathbb{N}} Y_g^n) = 1$ for any $g \in G$, then $\mathcal{O}_G''$ is an AFD type $\mathrm{III}_{|X|^{-1}}$factor.*

*Proof.* We first check $\mu((X^\omega)_{G\text{-reg}}) = 1$. By definition, we have

$$\left(\bigcup_{n\in\mathbb{N}} Y_g^n\right) \cap (X^\omega)_g = (X^\omega)_{g\text{-reg}} \cap (X^\omega)_g.$$

for any $g \in G$. Thus, the assumption implies

$$\mu((X^\omega)_g) = \mu\left(\left(\bigcup_{n\in\mathbb{N}} Y_g^n\right) \cap (X^\omega)_g\right) = \mu((X^\omega)_{g\text{-reg}} \cap (X^\omega)_g).$$

Hence, $\mu((X^\omega)_{G\text{-reg}}) = 1$ by Lemma 3.10.

We show that $G$ is contained in the strong operator closure of $\mathrm{span}\{S_{u_1}S_{u_2}^* : u_1, u_2 \in X^*\}$. Take $g \in G$. The self-similarity provides the equation

$$g = \sum_{u\in X^n} S_{g(u)} g|_u S_u^* = \sum_{\{u\in X^n : g(u)=e\}} S_{g(u)} S_u^* + \sum_{\{u\in X^n : g(u)\neq e\}} S_{g(u)} g|_u S_u^*.$$

Consider the sequence $\{a_n\}_{n\in\mathbb{N}}$ defined by

$$a_n := \sum_{\{u\in X^n : g(u)=e\}} S_{g(u)} S_u^* \in \mathrm{span}\{S_{u_1}S_{u_2}^* : u_1, u_2 \in X^*\}.$$

Note that $u \mapsto g(u)$ is a bijective map from $X^n$ onto $X^n$ for any $g \in G$ since there exists an inverse map arising from $g^{-1}$. Then we compute

$$\begin{aligned}\psi((g-a_n)^*(g-a_n)) &= \sum_{\{u\in X^n : g(u)\neq e\}} \psi(S_u S_u^*)\\ &= |X|^{-n}|\{u \in X^n : g(u) \neq e\}| \to 0 \ (n \to \infty).\end{aligned}$$

Note that $\{g - a_n\}_{n\in\mathbb{N}}$ is a norm bounded sequence since range projections of $S_{g(u)}$'s are mutually orthogonal. Recall that $\psi$ is a $(\log|X|, \Gamma)$-KMS state. Then, for any $u_1, u_2 \in X^*$ and $g_1 \in G$, we have

$$\begin{aligned}\psi(((g-a_n)S_{u_1}g_1S_{u_2}^*)^*((g-a_n)S_{u_1}g_1S_{u_2}^*)) &= |X|^{|u_2|-|u_1|}\psi((g-a_n)S_{u_1}S_{u_1}^*(g-a_n)^*)\\ &\leq |X|^{|u_2|-|u_1|}\psi((g-a_n)^*(g-a_n)).\end{aligned}$$

Thus, for any $a \in \mathrm{span}\{S_{u_1}g_1S_{u_2}^* : u_1, u_2 \in X^* \text{ and } g_1 \in G\}$,

$$\psi(((g-a_n)a)^*(g-a_n)a) \to 0 \ (n \to \infty).$$

Consequently, $\{a_n\}_{n\in\mathbb{N}}$ converges to $g$ in the strong operator topology thanks to the norm boundedness of $\{g - a_n\}_{n\in\mathbb{N}}$ and the density of $\mathrm{span}\{S_{u_1}g_1S_{u_2}^* : u_1, u_2 \in X^*, g_1 \in G\}$ in $H_\psi$. Thus, we have $G \subset \overline{\mathrm{span}\{S_{u_1}S_{u_2}^* : u_1, u_2 \in X^*\}}$ with respect to the strong operator topology. Therefore, we obtain

$$\mathcal{O}_G'' = \pi_\psi(\mathcal{O}_{|X|})''.$$

Combining [17, Lemma 4.1, Theorem 4.7], we conclude that $\pi_\psi(\mathcal{O}_{|X|})''$ is an AFD type $\mathrm{III}_{|X|^{-1}}$ factor, so is $\mathcal{O}_G''$. □

In the next proposition, we give a condition providing the assumption of Theorem 5.3.

**Proposition 5.4.** *Assume that $G$ is a contracting countable self-similar group over $X$ with the nucleus $\mathcal{N}$. If for any $g \in \mathcal{N}$ there exists $u \in X^*$ with $g|_u = e$, then $\mu(\bigcup_{n\in\mathbb{N}} Y_g^n) = 1$ for any $g \in G$.*

*Proof.* It is sufficient to show $\mu(\bigcup_{n\in\mathbb{N}} Y_g^n) = 1$ for any $g \in \mathcal{N}$. We use a similar argument to Proposition 3.17. By assumption, there exists $m \in \mathbb{N}$ such that for any $g' \in \mathcal{N}$ there exist $u \in X^m$ with $g'|_u = e$. We show that we have

$$|\{u \in X^{mn} : g|_u \neq e\}| \leq (|X|^m - 1)^n \tag{5.1}$$

for any $n \in \mathbb{N}, g \in \mathcal{N}$ by an inductive argument. From the choice of $m$, it is trivial that (5.1) holds in case $n = 1$. For the inductive step, we assume that (5.1) holds for a given $n$. For $u \in X^{m(n+1)}$, we have a division $u = u_1u_2$ where $u_1 \in X^{mn}$ and $u_2 \in X^m$. Take any $g \in \mathcal{N}$. Note that $g|_{u'} \in \mathcal{N}$ for any $u' \in X^*$. Then the choice of $m$ implies

$$|\{u_2 \in X^m : g|_{u_1u_2} \neq e\}| \leq |X|^m - 1.$$

Hence we get

$$|\{u \in X^{m(n+1)} : g|_u \neq e\}| \leq |\{u_1 \in X^{mn} : g|_{u_1} \neq e\}|(|X|^m - 1).$$

By the inductive assumption, (5.1) holds for $n+1$ and hence we have proved the equation (5.1). Thus we get

$$\mu(X^\omega \setminus (\bigcup_{n\in\mathbb{N}} Y_g^n)) = \lim_{n\to\infty} |X|^{-mn}|\{u \in X^{mn} : g|_u \neq e\}| \leq \lim_{n\to\infty}(1 - |X|^{-m})^n = 0.$$

Therefore we finished the proof. □

At the last of this section, we compute the type of the von Neumann algebra arising from the Grigorchuk group. For the definition of the Grigorchuk group, see Example 3.5.

**Example 5.5.** Recall that the Grigorchuk group is contracting and its nucleus is $\{e, a, b, c, d\}$ (see [24, Proposition 2.7]). It is not hard to check that the assumption in the above proposition holds. Thus we conclude that the von Neumann algebra associated with the Grigorchuk group is an AFD factor of type $\mathrm{III}_{\frac{1}{2}}$. Using the fact that the Grigorchuk group is amenable (see [15]), we get the same result.

## 6. Groupoid approaches and simplicity of $\mathcal{O}_{G_{\max}}$

In this section, we observe the groupoids arising from self-simlar groups and their groupoid C*-algebras. For some contracting self-similar groups, we discuss the simplicity of $\mathcal{O}_{G_{\max}}$ through the groupoid C*-algebras.

6.1. **Groupoids arising from self-similar groups.** As the first part of this section, we recall the definitions and fix notations. A small category $\mathcal{G}$ whose any morphism has the inverse is said to be a *groupoid*. We identify a groupoid $\mathcal{G}$ with the set of morphisms and identify the objects with the identity morphisms on them. We write the set of objects $\mathcal{G}^{(0)} \subset \mathcal{G}$ and refer to it as the *unit space of* $\mathcal{G}$. Define two maps $s, r$ from $\mathcal{G}$ onto $\mathcal{G}^{(0)}$ by

$$s(\gamma) = \gamma^{-1}\gamma \text{ and } r(\gamma) = \gamma\gamma^{-1}$$

for $\gamma \in \mathcal{G}$. The map $s$ is called the *source map* and $r$ is called the *range map*.

A *topological groupoid* is a groupoid with a topology such that the multiplication operator and the inverse operator are continuous. The source and range maps are continuous on any topological groupoid. A topological groupoid is said to be *étale* (or $r$-discrete) if the unit space is locally compact with respect to the relative topology of $\mathcal{G}$ and the source map is a local homeomorphism.

A groupoid of germs of local homeomorphisms on a topological space is a good example of a groupoid. We observe the groupoids of germs arising from self-similar groups. Such groupoids have been introduced in [25].

Let $(\langle G, X\rangle \times X^\omega)'$ be the subset of $\langle G, X\rangle \times X^\omega$ given by

$$(\langle G, X\rangle \times X^\omega)' := \{(f, w) \in \langle G, X\rangle \times X^\omega : w \in \mathrm{Dom} f\}$$

We define an equivalence relation on $(\langle G, X\rangle \times X^\omega)'$ as follows. Pairs $(f_1, w_1), (f_2, w_2)$ in $(\langle G, X\rangle \times X^\omega)'$ are equivalent if $w_1 = w_2$ and $f_1 = f_2$ on a neighborhood of $w_1 = w_2$. The quotient set is denoted by $[G, X]$. We write $[f, w]$ for the equivalence class represented by $(f, w)$.

The set $[G, X]$ forms a groupoid under the following operations. The multiplication is given by

$$[f_1, w_1] \cdot [f_2, w_2] := [f_1f_2, w_2]$$

when $w_1 = f_2(w_2)$. The inverse is given by

$$[f, w]^{-1} := [f^{-1}, f(w)].$$

We also equip $[G, X]$ with the topology generated by sets of the form:

$$\mathcal{U}_{U,f} = \{[f, w] : w \in U\}$$

where $f \in \langle G, X\rangle$ and $U$ is an open subset of Dom$f$. It is not hard to check that $[G, X]$ is étale.

The topological groupoid $[G, X]$ might not be Hausdorff. For example, let $X = \{0, 1\}, G = \langle a, b, c, d\rangle$ be the Grigorchuk group (see Example 3.5 for definition). Define

$$1^\infty := 11111 \cdots \in X^\omega.$$

Then $[e, 1^\infty] \neq [d, 1^\infty]$ by definition but no open sets separate them. To check this, note that $d = e$ on $1^{3m}0X^\omega$ for any nonnegative integer $m$. For any open neighborhood $U, V$ of $1^\infty$, there exists $n \in \mathbb{N}$ with $1^{3n}X^\omega \subset U$ and $1^{3n}X^\omega \subset V$. We have

$$\mathcal{U}_{1^{3n}0X^\omega,e} \subset \mathcal{U}_{1^{3n}X^\omega,e} \subset \mathcal{U}_{U,e}$$

and

$$\mathcal{U}_{1^{3n}0X^\omega,d} \subset \mathcal{U}_{1^{3n}X^\omega,d} \subset \mathcal{U}_{V,d}.$$

In addition, we also have $\mathcal{U}_{1^{3n}0X^\omega,e} = \mathcal{U}_{1^{3n}0X^\omega,d} \neq \emptyset$. Therefore, $\mathcal{U}_{U,e} \cap \mathcal{U}_{V,d} \neq \emptyset$. This proves the claim.

Next we review the groupoid C*-algebra. In this context, one often assumes that groupoids are Hausdorff. In this paper, however, we do NOT assume the Hausdorffness to treat examples in subsection 6.2. The definitions of groupoid C*-algebras without Hausdorffness are introduced by A. Connes in [8].

We again consider a (not necessarily Hausdorff) étale groupoid $\mathcal{G}$. Let $\mathfrak{U} \subset \mathcal{G}$ be an open Hausdorff subset. The set of continuous functions on $\mathfrak{U}$ with compact support is denoted by $C_c(\mathfrak{U})$. Let Funct$(\mathcal{G})$ be the set of all functions on $\mathcal{G}$. For $\eta \in C_c(\mathfrak{U}), \gamma \notin \mathfrak{U}$, we define $\eta(\gamma) = 0$. This gives an embedding of $C_c(\mathfrak{U})$ into Funct$(\mathcal{G})$. We then define

$$C(\mathcal{G}) := \operatorname{span} \bigcup_{\mathfrak{U}} C_c(\mathfrak{U}) \subset \operatorname{Funct}(\mathcal{G})$$

where the union is taken over all Hausdorff open subsets $\mathfrak{U}$ of $\mathcal{G}$. Note that a function in $C(\mathcal{G})$ might not be continuous on $\mathcal{G}$.

We define the multiplication and involution operators on $C(\mathcal{G})$ by

$$\eta^*(\gamma) := \overline{\eta(\gamma^{-1})}, \quad (\eta_1 * \eta_2)(\gamma) := \sum_{\gamma_1\gamma_2=\gamma} \eta_1(\gamma_1)\eta_2(\gamma_2)$$

for any $\eta, \eta_1, \eta_2 \in C(\mathcal{G})$. To introduce a C*-norm on $C(\mathcal{G})$, we fix the following notation. Let $\alpha \in \mathcal{G}^{(0)}$ and define

$$\mathcal{G}_\alpha := \{\gamma \in \mathcal{G} : s(\gamma) = \alpha\}.$$

Consider the $*$-representation $\lambda_\alpha$ of $C(\mathcal{G})$ on $l^2(\mathcal{G}_\alpha)$ given by

$$(\lambda_\alpha(\eta)\zeta)(\gamma) = \sum_{\gamma_1\gamma_2=\gamma} \eta(\gamma_1)\zeta(\gamma_2)$$

for $\gamma \in \mathcal{G}_\alpha$. It is not hard to check that

$$\|\eta\|_{\mathrm{red}} = \sup_{\alpha\in\mathcal{G}^{(0)}} \|\lambda_\alpha(\eta)\|$$

is a C*-norm on $C(\mathcal{G})$. The completion of $C(\mathcal{G})$ with respect to $\|\cdot\|_{\mathrm{red}}$ is denoted by $C^*_r(\mathcal{G})$. The C*-algebra $C^*_r(\mathcal{G})$ is said to be the *reduced groupoid C*-algebra of* $\mathcal{G}$.

For $\eta \in C(\mathcal{G})$, we define the universal norm by

$$\|\eta\|_u := \sup\{\|\rho(\eta)\| : \rho \text{ is a } *\text{-representation of } C(\mathcal{G}) \text{ on a Hilbert space}\}.$$

This indeed defines a C*-norm on $C(\mathcal{G})$ (see [23, Lemma 1.2.3]). The completion of $C(\mathcal{G})$ with respect to $\|\cdot\|_u$ is denoted by $C^*(\mathcal{G})$. The C*-algebra $C^*(\mathcal{G})$ is said to be the *full groupoid C*-algebra of* $\mathcal{G}$.

We again consider the groupoid of germs arising from a self-similar group $G$ over a finite set $X$. The universality of $\mathcal{O}_{G_{\max}}$ provides a surjection from $\mathcal{O}_{G_{\max}}$ onto $C^*([G,X])$ given by

$$\langle G, X\rangle \ni f \mapsto 1_{\mathcal{U}_{\mathrm{Dom}f,f}} \in C([G,X]) \subset C^*([G,X])$$

where $1_{\mathcal{U}_{\mathrm{Dom}f,f}}$ is the characteristic function on $\mathcal{U}_{\mathrm{Dom}f,f}$. It was shown by V. V. Nekrashevych that this surjection is in fact an isomorphism.

**Theorem 6.1.** ([25, Theorem 5.1], [13, Corollary 6.4]) *The full groupoid C*-algebra $C^*([G,X])$ is isomorphic to $\mathcal{O}_{G_{\max}}$.*

Thanks to the theorem, one can identify $f$ with $1_{\mathcal{U}_{\mathrm{Dom}f,f}}$. Moreover, by identifying $C^*([G,X])$ with $\mathcal{O}_{G_{\max}}$, we also consider the gauge action $\Gamma$ defined in section 4 on $C^*([G,X])$ and its quotient $C^*_r([G,X])$. Let

$$[G,X]^\Gamma := \{[S_u g S_v^*, w] \in [G,X] : |u| = |v|\}$$

then the restriction of the isomorphism in Theorem 6.1 gives the following identification.

**Theorem 6.2.** ([25, Theorem 5.3]) $C^*([G,X]^\Gamma) \simeq C^*([G,X])^\Gamma \simeq \mathcal{O}^\Gamma_{G_{\max}}$.

At the rest of this section, we discuss the simplicity of $C^*([G,X])$ and $C^*_r([G,X])$ rather than $\mathcal{O}_{G_{\max}}$. If $C^*([G,X])$ and $C^*_r([G,X])$ are not canonically isomorphic, then $C^*([G,X])$ is not simple. Thus, we only consider the case they are isomorphic.

It is a well known fact that the full and reduced C*-algebra of an amenable groupoid are canonically isomorphic. One can find this fact for Hausdorff groupoids in [1] or [29]. The same argument provides the isomorphism in the non-Hausdorff case. We recall sufficient conditions of the amenability of $[G,X]$ in [13] and [25].

**Theorem 6.3.** ([13, Corollary 10.18]) *If a self-similar group $G$ over $X$ is amenable as a discrete group, then $[G,X]$ is amenable.*

For the other sufficient condition, we recall the following definition.

**Definition 6.4.** ([25, Definition 2.3]) A self-similar group $G$ over $X$ is said to be *self-replicating* if for any $x, y \in X, h \in G$ there exists $g \in G$ with $g(x) = y$ and $g|_x = h$.

**Theorem 6.5.** ([25, Theorem 5.6]) *If a self-similar group $G$ over $X$ is contracting and self-replicating, then $[G,X]$ is amenable.*

If $[G,X]$ is amenable and Hausdorff, then $C^*([G,X])$ is simple by standard arguments (see [25]). However, there exists a self-similar group $G$ over $X$ such that $[G,X]$ is non-Hausdorff amenable groupoid but the $C^*([G,X])$ is not simple. We observe this later. In the next subsection, we consider a class of self-similar groups whose groupoids are not Hausdorff. We discuss the simplicity of their C*-algebras.

6.2. **Multispinal groups.** From now on we consider a class of self-similar groups called multispinal groups introduced in [30]. We first recall the construction.

Let $A, B$ be finite groups and let $B$ act freely on a finite set $X$. Write $\mathrm{Aut}(A)$ and $\mathrm{Hom}(A,B)$ for the set of automorphisms of $A$ and the set of homomorphisms from $A$ to $B$, respectively. Consider a map $\Psi$ from $X$ to $\mathrm{Aut}(A) \cup \mathrm{Hom}(A,B)$. Define

$$\mathcal{A} := \Psi(X) \cap \mathrm{Aut}(A), \ \mathcal{B} := \Psi(X) \cap \mathrm{Hom}(A,B).$$

Set

$$\mathcal{B}\cdot\mathcal{A} := \mathcal{B} \cup \bigcup_{n\geq 2} \{\lambda_1 \circ \lambda_2 \circ \cdots \circ \lambda_n : \lambda_1 \in \mathcal{B}, \lambda_i \in \mathcal{A} \text{ for } 2 \leq i \leq n\} \subset \mathrm{Hom}(A,B).$$

We assume $\mathcal{A} \neq \emptyset, \mathcal{B} \neq \emptyset$ and

$$\bigcap_{\lambda \in \mathcal{B}\cdot\mathcal{A}} \ker\lambda = \{1_A\}.$$

Here $1_A$ is the unit of $A$. We define actions of $A$ and $B$ on $X^\omega$ by

$$a(xw) := x(\Psi(x)(a))(w),\ b(xw) := b(x)w$$

where $a \in A, b \in B, x \in X, w \in X^\omega$. The assumption implies that these two actions are faithful. Hence, we identify two finite groups $A, B$ with their images in $\mathrm{Homeo}(X^\omega)$, respectively. Let $G$ be the subgroup of $\mathrm{Homeo}(X^\omega)$ generated by $A$ and $B$. The group $G$ is said to be a *multispinal group* over $X$. Let $G = G(A, B, \Psi)$ denote the multispinal group arising from two finite groups $A$, $B$ and a map $\Psi$. Put

$$Y := \Psi^{-1}(\mathrm{Hom}(A, B)) \subset X.$$

**Remark 6.6.** Let $G = G(A, B, \Psi)$ be a multispinal group over $X$. By definition, the multispinal group $G$ is a self-similar group. In fact, the multispinal group $G$ is always contracting and the nucleus is contained in $A \cup B$ [30, Proposition 7.1]. Assume that the action of $B$ on $X$ is transitive and

$$\bigcup_{y \in Y} \Psi(y)(A) = B.$$

Then $G$ is self-replicating (thus $G$ is infinite) and the nucleus of $G$ coincides with $A \cup B$ (See section 7 of [30]). Therefore, $[G, X]$ is amenable by Theorem 6.5.

The Grigorchuk group is an example of a multispinal group.

**Example 6.7.** Let $A = \mathbb{Z}_2 \times \mathbb{Z}_2, B = X = \mathbb{Z}_2$. Consider the left translation action $B \curvearrowright X$. We define $\Psi(0) \in \mathrm{Hom}(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2)$ and $\Psi(1) \in \mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ to be

$$\Psi(0)(x, y) = y,\ \Psi(1)(x, y) = (y, x + y).$$

Then the multispinal group $G(A, B, \Psi)$ coincides with the Grigorchuk group generated by $a, b, c, d$ (we use the same symbols as Example 3.5). To see this, it is sufficient to identify $a$ with the generator $1 \in \mathbb{Z}_2$ and identify $b, c, d$ with $(0, 1), (1, 1), (1, 0) \in \mathbb{Z}_2 \times \mathbb{Z}_2$, respectively.

**Lemma 6.8.** *Let $G = G(A, B, \Psi)$ be a multispinal group over $X$. Take $w \in X^\omega, g, g' \in A \cup B \subset G$ with $g(w) = g'(w)$. We write*

$$w = x_1 x_2 x_3 \cdots ; \quad x_1, x_2, x_3, \ldots \in X.$$

*Assume that $w \in X^\omega$ uses an alphabet in $Y$ and define*

$$m := \min\{i : x_i \in Y\}.$$

*Then $g$ and $g'$ coincide on the cylinder set $w^{(m)} X^\omega$. In particular, we obtain $[g, w] = [g', w]$.*

*Proof.* First we assume that $g \in B \backslash \{e\}$. Then by $g(w) = g'(w)$, we have $g = g'$ (otherwise the first alphabet of $g(w)$ and $g'(w)$ do not coincide).

Second, let $g \in A$. By the first step, we may assume $g' \in A$. Thus, it is sufficient to show that $g$ and $e$ coincide on $w^{(m)} X^\omega$ for $g \in A$ with $g(w) = w$. By definition, $x_m \in Y$ and $x_i \notin Y$ for $1 \leq i \leq m - 1$. Hence,

$$(\Psi(x_m) \circ \Psi(x_{m-1}) \circ \cdots \circ \Psi(x_1))(g) \in B.$$

Then, since $g(w) = w$, we obtain

$$g(w^{(m)}) = w^{(m)},\ g|_{w^{(m)}} = (\Psi(x_m) \circ \Psi(x_{m-1}) \circ \cdots \circ \Psi(x_1))(g) = e.$$

This shows the claim. □

**Lemma 6.9.** *Let $G = G(A, B, \Psi)$ be a multispinal group over $X$. Then we have $(X \backslash Y)^\omega \subset X^\omega \backslash (X^\omega)_{G\text{-reg}}$. More specifically, we have*

$$a(w) = a'(w),\ [a, w] \neq [a', w]$$

*for any $w \in (X \backslash Y)^\omega, a, a' \in A$ with $a \neq a'$.*

*Proof.* It suffices to show that

$$a(w) = w, \ [a, w] \neq [e, w]$$

for $w \in (X \backslash Y)^\omega, a \in A \backslash \{e\}$. We write

$$w = x_1 x_2 x_3 \cdots ; \quad x_1, x_2, x_3, \ldots \in X.$$

By the definition of the action $A \curvearrowright X^\omega$, $a$ does not change $x_1$. The assumption $w \in (X \backslash Y)^\omega$ implies $x_1 \in X \backslash Y$. Thus, $\Psi(x_1)(a) \in A$. This shows that $\Psi(x_1)(a)$ does not change $x_2 \in X \backslash Y$. Repeating this, we have $a(w) = w$.

For each $i$, $\Psi(x_i)$ is an automorphism on $A$. Therefore, we obtain

$$a|_{w^{(n)}} = (\Psi(x_n) \circ \Psi(x_{n-1}) \circ \cdots \circ \Psi(x_1))(a) \neq e$$

for any $n \in \mathbb{N}$. Thus $[a, w] \neq [e, w]$. □

Let $G = G(A, B, \Psi)$ be a multispinal group over $X$. Recall that $G$ is contracting [30, Proposition 7.1]. Hence we have a state $\psi_{\min}$ on $\mathcal{O}_{G_{\min}}$ by Proposition 3.17 and Theorem 4.10. The state $\psi_{\min}$ is the extension of a positive definite function $\psi_0$ on $G$ given by

$$\psi_0(g) := \mu((X^\omega)_g)$$

for $g \in G$. We write

$$\pi_{\psi_{\min}} : \mathcal{O}_{G_{\min}} \to B(H_{\psi_{\min}}), \quad \pi_{\psi_{\max}} : \mathcal{O}_{G_{\max}} \to B(H_{\psi_{\max}})$$

for the GNS representations of $\psi_{\min}$ and $\psi_{\max}$, respectively. Let $w$ be an arbitrary strictly $G$-regular point. Recall that $(\pi_{\psi_{\min}} \circ \pi_w, H_{\psi_{\min}})$ and $(\pi_{\psi_{\max}}, H_{\psi_{\max}})$ are unitarily equivalent via the canonical unitary. In addition, $\pi_{\psi_{\min}}$ is faithful since $\mathcal{O}_{G_{\min}}$ is simple by Theorem 4.5. Thus, we have

$$\|\pi_w(\eta)\| = \|\pi_{\psi_{\min}}(\pi_w(\eta))\| = \|\pi_{\psi_{\max}}(\eta)\|$$

for any $\eta \in \mathcal{O}_{G_{\max}}$.

Write $C^*(A)$ for the full group C*-algebra of $A$. Since $A$ is finite, the canonical inclusion

$$A \to \mathcal{O}_{G_{\max}}$$

extends to an embedding

$$C^*(A) \to \mathrm{span}(A) \subset \mathcal{O}_{G_{\max}}.$$

From now on, we identify $C^*(A)$ with the finite dimensional C*-subalgebra span$(A)$ of $\mathcal{O}_{G_{\max}}$.

**Lemma 6.10.** *Let $G = G(A, B, \Psi)$ be a multispinal group over $X$. Assume that the restriction of $\psi_{\max}$ to $C^*(A)$ is faithful. Then there exists an orthonormal system $\{\overline{a}\}_{a \in A}$ in $H_{\psi_{\max}}$ with $\pi_{\psi_{\max}}(a)\overline{a'} = \overline{aa'}$ for any $a, a' \in A$.*

*Proof.* Let $\tau$ be the canonical tracial state on $C^*(A)$:

$$\tau(a) = \delta_{a,e}$$

for any $a \in A$. Here $\delta$ is the Kronecker delta. Recall that $\tau$ is faithful. In addition, the restriction of $\psi_{\max}$ to $C^*(A)$ is a faithful tracial state on $C^*(A)$ by assumption. Note that $C^*(A)$ is finite dimensional. Therefore, the non-commutative Radon–Nikodym theorem provides a positive invertible element $h$ in $C^*(A)$ with

$$\tau(k) = \psi_{\max}(hk)$$

for any $k \in C^*(A)$ (see [32, Corollary 3.3.6]). Define

$$\overline{a} := ah^{\frac{1}{2}}$$

for each $a \in A$. Note that the restriction of the canonical map $\mathcal{O}_{G_{\max}} \to H_{\psi_{\max}}$ to $C^*(A)$ is injective by assumption. We regard each $\overline{a}$ as a unit vector in $H_{\psi_{\max}}$ via the canonical map. Clearly,

$$\pi_{\psi_{\max}}(a)\overline{a'} = \overline{aa'}$$

for any $a, a' \in A$. Considering the inner product $\langle \cdot, \cdot \rangle$ on $H_{\psi_{\max}}$, we have

$$\langle \overline{a}, \overline{e} \rangle = \psi_{\max}(ah) = \psi_{\max}(ha) = \tau(a) = \delta_{a,e}$$

for any $a \in A$. This shows the claim. □

We fix notations for the proof of the main theorem of this section. We observe the reduced groupoid C*-algebra $C_r^*([G, X]^\Gamma)$. Consider the map

$$w \mapsto [e, w]$$

from $X^\omega$ to $[G, X]^\Gamma$. It is not hard to show that the map provides a homeomorphism from $X^\omega$ onto the unit space $([G, X]^\Gamma)^{(0)}$. For $[e, w] \in ([G, X]^\Gamma)^{(0)}$, we write $w$ instead of $[e, w]$. Then the C*-norm on $C_r^*([G, X]^\Gamma)$ is given by

$$\sup_{w \in X^\omega} \|\lambda_w^\Gamma(\,\cdot\,)\|.$$

We write

$$\langle G, X \rangle^\Gamma := \{S_u g S_v^* \in \langle G, X \rangle : |u| = |v|\}.$$

Let $\mathbb{C}\langle G, X \rangle^\Gamma$ be the $*$-subalgebra of $\mathcal{O}_{G_{\max}}$ generated by $\langle G, X \rangle^\Gamma$. Note that for each $w \in X^\omega$, we have

$$([G, X]^\Gamma)_w = \{[f, w] : f \in \langle G, X \rangle^\Gamma, w \in \mathrm{Dom} f\}.$$

Let $\delta_{[f,w]} \in l^2(([G, X]^\Gamma)_w)$ be the characteristic function of $[f, w] \in ([G, X]^\Gamma)_w$. Then we have

$$\lambda_w^\Gamma(f_1)\delta_{[f_2,w]} = \begin{cases} \delta_{[f_1 f_2, w]} & \text{if } f_2(w) \in \mathrm{Dom} f_1 \\ 0 & \text{otherwise} \end{cases}$$

for any $f_1 \in \langle G, X \rangle^\Gamma$ and $[f_2, w] \in [G, X]^\Gamma$.

**Remark 6.11.** Let $G$ be a self-similar group over $X$. For any $S_u g S_v^* \in \langle G, X \rangle^\Gamma$ with $|u| = |v| < n_0$, we have

$$g = g \sum_{u' \in X^{n_0 - |u|}} S_{u'} S_{u'}^* = \sum_{u' \in X^{n_0 - |u|}} S_{g(u')} g|_{u'} S_{u'}^*. \tag{6.1}$$

Consider any finite sum

$$\eta = \sum_{g,u,v,|u|=|v|} \alpha_{u,v}^g S_u g S_v^* \in \mathbb{C}\langle G, X \rangle^\Gamma. \tag{6.2}$$

Using (6.1), we may assume that the sum is taken over $u, v \in X^n$ for some fixed $n \in \mathbb{N}$.

Next, we further assume that $G$ is a contracting self-similar group with the nucleus $\mathcal{N}$. Then for any finite subset $F' \subset G$, there exists a natural number $n_1 \in \mathbb{N}$ satisfying $g'|_{u'} \in \mathcal{N}$ for any $u' \in X^*$ with $|u'| > n_1$ and $g' \in F'$. Thus, for any $g' \in F', n' > n_1$, we have

$$g' = g' \sum_{u' \in X^{n'}} S_{u'} S_{u'}^* = \sum_{u' \in X^{n'}} S_{g'(u')} g'|_{u'} S_{u'}^*.$$

This shows that one can further assume that the sum in (6.2) is taken over $g \in \mathcal{N}, u, v \in X^n$ for some fixed (larger) $n \in \mathbb{N}$.

For finite words $u, v \in X^*$, $uv$ is also a finite word. For short, we temporary write

$$u^{-1}(uv) := v.$$

Let $w \in X^\omega$ and $n, m$ be natural numbers with $n > m$. We write

$$w^{-(m)+(n)} := (w^{(m)})^{-1} w^{(n)} \in X^{n-m}.$$

**Remark 6.12.** Let $G$ be a self-similar group over $X$. Take any $w \in X^\omega$. For any $[f, w] \in ([G,X]^\Gamma)_w$, there exist $u \in X^*, g \in G$ and $m \in \mathbb{N}$ with $|u| = m, f = S_u g S^*_{w(m)}$. For any natural number $n$ with $n > m$, we have

$$[S_u g S^*_{w(m)}, w] = [S_u g S^*_{w(m)} S_{w(n)} S^*_{w(n)}, w] = [S_u S_{g(w^{-(m)+(n)})} g|_{w^{-(m)+(n)}} S^*_{w(n)}, w]. \tag{6.3}$$

Assume that $G$ is a contracting self-similar group with the nucleus $\mathcal{N}$. Take finitely supported function $\xi \in l^2(([G,X]^\Gamma)_w)$. Then, by the equation (6.3) and similar arguments to Remark 6.11, one can write

$$\xi = \sum_{u' \in X^{n'}, g' \in \mathcal{N}} \beta^{g'}_{u'} \delta_{[S_{u'} g' S^*_{w(n')}, w]} \tag{6.4}$$

for some fixed natural number $n'$ and complex numbers $\beta^{g'}_{u'}$. Note that one can replace $n'$ to be arbitrarily large.

In the following theorem, we assume that $[G,X]$ is amenable. See Theorem 6.3 and Remark 6.6 for the sufficient condition of the amenability of $[G,X]$.

**Theorem 6.13.** *Let $G = G(A,B,\Psi)$ be a multispinal group over $X$. Assume that $[G,X]$ is amenable. Then the following conditions are equivalent.*

(i) *The restriction of $\psi_{\max}$ to $C^*(A)$ is faithful.*
(ii) *The $|A| \times |A|$ matrix $[\psi_0(a_1^{-1}a_2)]_{a_1,a_2 \in A}$ is invertible.*
(iii) *The universal C*-algebra $\mathcal{O}_{G_{\max}}$ is simple.*

*Proof.* The equivalence between (i) and (ii) is given by standard arguments.

If we assume (iii) then $\pi_{\psi_{\max}}$ is faithful by the simplicity of $\mathcal{O}_{G_{\max}}$. Recall that $\psi_{\max}$ is a KMS state on $\mathcal{O}_{G_{\max}}$. Therefore, $\psi_{\max}$ is faithful by [3, Corollary 5.3.9]. This proves (iii) $\Rightarrow$ (i).

We show (i) $\Rightarrow$ (iii). Combining the amenability of $[G,X]$ and Theorem 6.1, we have the identifications

$$\mathcal{O}_{G_{\max}} \simeq C^*([G,X]) \simeq C^*_r([G,X]).$$

We prove $\mathcal{O}_{G_{\min}} \simeq C^*_r([G,X])$. Recall that $\mathcal{O}_{G_{\min}}$ is a simple quotient of $C^*_r([G,X])$ (Theorems 4.3, 4.5). It is sufficient to show the injectivity of this quotient map. Since the gauge action $\Gamma$ is periodic, we only check the injectivity on the gauge invariant subalgebra $C^*_r([G,X])^\Gamma \simeq C^*_r([G,X]^\Gamma)$ (see [4, Proposition 4.5.1]). We do this by the norm estimate.

Take any $w \in X^\omega$ and any nonzero element $\eta \in \mathbb{C}\langle G, X\rangle^\Gamma$. The goal of this proof is to show

$$\|\lambda^\Gamma_w(\eta)\| \le \|\pi_{\psi_{\max}}(\eta)\| = \|\pi_{w_0}(\eta)\|.$$

Here $w_0$ is a strictly $G$-regular point which we choose later. Recall that we have $\|\pi_{\psi_{\max}}(\eta)\| = \|\pi_{w'}(\eta)\|$ for any $w' \in (X^\omega)^{\mathrm{S}}_{G\text{-reg}}$. From now on, we write $\psi$ for $\psi_{\max}$.

By the definition of operator norm, for any $\varepsilon > 0$ one can choose a nonzero finitely supported function $\xi \in l^2([G,X]^\Gamma_w)$ with

$$\|\lambda^\Gamma_w(\eta)\xi\| \ge (1-\varepsilon)\|\lambda^\Gamma_w(\eta)\|\|\xi\|.$$

Write

$$\eta = \sum_{u,v,g,|u|=|v|} \alpha^g_{u,v} S_u g S^*_v, \quad \xi = \sum_{u',g'} \beta^{g'}_{u'} \delta_{[S_{u'} g' S^*_{w(|u'|)}, w]}.$$

Here $\delta_{[S_{u'} g' S^*_{w(|u'|)}, w]}$ is the characteristic function on $[S_{u'} g' S^*_{w(|u'|)}, w]$. Note that $G$ is contracting [30, Proposition 7.1]. We write $\mathcal{N}$ for the nucleus of $G$, namely

$$\mathcal{N} := A \cup \bigcup_{y \in Y} \Psi(y)(A) \subset A \cup B.$$

By Remarks 6.11, 6.12, we may assume that the finite sums are taken over $g, g' \in \mathcal{N}$ and $u, v, u' \in X^n$ for some fixed $n \in \mathbb{N}$. We write

$$w = x_1 x_2 x_3 \cdots; \quad x_1, x_2, x_3, \ldots \in X.$$

We divide the proof into two cases. For the first case, assume that $w$ uses infinitely many alphabets in $Y$. Then we have a natural number $n'$ with $n' > n$ such that there exists $i \in \mathbb{N}$ with $n < i < n'$ and $x_i \in Y$. Note that we have

$$bS_x = S_{b(x)}, \ aS_{yx} = S_y S_{(\Psi(y)(a))(x)}$$

for any $x \in X, y \in Y, a \in A, b \in B$. This shows

$$\begin{aligned}[S_{u'} g' S^*_{w^{(n)}}, w] &= [S_{u'} g' S^*_{w^{(n)}} S_{w^{(n')}} S^*_{w^{(n')}}, w] \\ &= [S_{u'} S_{g'(w^{-(n)+(n')})} g'|_{w^{-(n)+(n')}} S^*_{w^{(n')}}, w] \\ &= [S_{u'} S_{g'(w^{-(n)+(n')})} S^*_{w^{(n')}}, w]\end{aligned}$$

for any $u' \in X^n, g' \in \mathcal{N} \subset A \cup B$. Now replace $n$ by $n'$ and rewrite

$$\eta = \sum_{g \in \mathcal{N}, u, v \in X^n} \alpha^g_{u,v} S_u g S^*_v, \quad \xi = \sum_{u' \in X^n} \beta_{u'} \delta_{[S_{u'} S^*_{w^{(n)}}, w]}.$$

Then, we compute

$$\lambda^\Gamma_w(\eta)\xi = \sum_{g \in \mathcal{N}, u, v, u' \in X^n} \alpha^g_{u,v} \beta_{u'} \lambda^\Gamma_w(S_u g S^*_v) \delta_{[S_{u'} S^*_{w^{(n)}}, w]} = \sum_{g,u,v} \alpha^g_{u,v} \beta_v \delta_{[S_u g S^*_{w^{(n)}}, w]}.$$

Let $m$ be the smallest natural number with $x_{n+m} \in Y$. Using the density of $(X^\omega)_{G\text{-reg}}$ in $X^\omega$, we choose a $G$-regular point $w_0 \in w^{(n+m)} X^\omega$. Define a vector $\tilde{\xi} \in l^2(\langle G, X \rangle (w_0))$ by

$$\tilde{\xi} := \sum_{u' \in X^n} \beta_{u'} \delta_{(S_{u'} S^*_{w^{(n)}}(w_0))}.$$

A calculation shows

$$\pi_{w_0}(\eta)\tilde{\xi} = \sum_{g \in \mathcal{N}, u, v, u' \in X^n} \alpha^g_{u,v} \beta_{u'} \pi_{w_0}(S_u g S^*_v) \delta_{(S_{u'} S^*_{w^{(n)}}(w_0))} = \sum_{g,u,v} \alpha^g_{u,v} \beta_v \delta_{(S_u g S^*_{w^{(n)}}(w_0))}.$$

Note that $\left\{\delta_{[S_{u'} S^*_{w^{(n)}}, w]}\right\}_{u' \in X^n}$ and $\left\{\delta_{(S_{u'} S^*_{w^{(n)}}(w_0))}\right\}_{u' \in X^n}$ are orthonormal systems in $l^2(([G, X]^\Gamma)_w)$ and $l^2(\langle G, X \rangle (w_0))$, respectively. Thus, we have $\|\xi\| = \|\tilde{\xi}\|$. To compare the norms $\|\lambda^\Gamma_w(\eta)\xi\|$ and $\|\pi_{w_0}(\eta)\tilde{\xi}\|$, we prove the following claim.

**Claim.** *Take any $u_1, u_2 \in X^n, g_1, g_2 \in A \cup B$. Then the following conditions are equivalent.*

(a) $[S_{u_1} g_1 S^*_{w^{(n)}}, w] = [S_{u_2} g_2 S^*_{w^{(n)}}, w]$.
(b) $S_{u_1} g_1 S^*_{w^{(n)}}(w_0) = S_{u_2} g_2 S^*_{w^{(n)}}(w_0)$.

*Proof of Claim.* First, we assume (a). Then we have $u_1 = u_2$ and $g_1 S^*_{w^{(n)}}(w) = g_2 S^*_{w^{(n)}}(w)$. Hence, Lemma 6.8 implies that $g_1 S^*_{w^{(n)}}$ and $g_2 S^*_{w^{(n)}}$ coincide on $w^{(n+m)} X^\omega$ from the choice of $m$. Since $w_0 \in w^{(n+m)} X^\omega$, we obtain (b).

Conversely, we assume (b). Then $u_1 = u_2$ and $g_1 S^*_{w^{(n)}}(w_0) = g_2 S^*_{w^{(n)}}(w_0)$. Hence, Lemma 6.8 shows that $g_1 S^*_{w^{(n)}}$ and $g_2 S^*_{w^{(n)}}$ coincide on $w_0^{(n+m)} X^\omega = w^{(n+m)} X^\omega$. This implies (a). □

The claim implies $\|\lambda^\Gamma_w(\eta)\xi\| = \|\pi_{w_0}(\eta)\tilde{\xi}\|$. Thus,

$$\|\pi_{w_0}(\eta)\| \|\tilde{\xi}\| \geq \|\pi_{w_0}(\eta)\tilde{\xi}\| = \|\lambda^\Gamma_w(\eta)\xi\| \geq (1-\varepsilon)\|\lambda^\Gamma_w(\eta)\| \|\xi\| = (1-\varepsilon)\|\lambda^\Gamma_w(\eta)\| \|\tilde{\xi}\|.$$

Since $\varepsilon$ is arbitrary, we have $\|\pi_{w_0}(\eta)\| \geq \|\lambda^\Gamma_w(\eta)\|$.

For the second case, we assume that $w$ uses at most finitely many alphabets in $Y$. Choose a natural number $n'$ with $S^*_{w^{(n')}}(w) \in (X \backslash Y)^\omega$ and $n' > n$. Then we have

$$\begin{aligned}[S_{u'} g' S^*_{w^{(n)}}, w] &= [S_{u'} g' S^*_{w^{(n)}} S_{w^{(n'+2)}} S^*_{w^{(n'+2)}}, w] \\ &= [S_{u'} S_{g'(w^{-(n)+(n'+2)})} g'|_{w^{-(n)+(n'+2)}} S^*_{w^{(n'+2)}}, w]\end{aligned}$$

and $g'(x_{n'+2}) \in X \backslash Y, g'|_{w^{-(n)+(n'+2)}} \in A$ for any $g' \in \mathcal{N} \subset A \cup B$. For each $l \in \mathbb{N}$, define

$$Z_l := \{u = z_1 z_2 \cdots z_l \in X^l : z_l \in X \backslash Y\}.$$

Note that for any $g_0 \in A \cup B, u_0 \in Z_l$, we have $g_0|_{u_0} \in A$. Thus, one can rewrite

$$\xi = \sum_{u' \in Z_{n'+2}, g' \in A} \beta_{u'}^{g'} \delta_{[S_{u'} g' S^*_{w(n'+2)}, w]}.$$

Take an orthonormal system $\{\overline{g}\}_{g \in A}$ given in Lemma 6.10. Define a vector $\tilde{\xi} \in H_\psi$ by

$$\tilde{\xi} := \sum_{u' \in Z_{n'+2}, g' \in A} \beta_{u'}^{g'} \pi_\psi(S_{u'}) \overline{g'}.$$

Take any $g_1', g_2' \in A, u_1', u_2' \in Z_{n'+2}$. By Lemma 6.9, $[S_{u_1'} g_1' S^*_{w(n'+2)}, w] = [S_{u_2'} g_2' S^*_{w(n'+2)}, w]$ if and only if $S_{u_1'} = S_{u_2'}, g_1' = g_2'$. Thus, $\|\xi\| = \|\tilde{\xi}\|$.

For each $u' = z_1' z_2' \cdots z_{n'+2}' \in Z_{n'+2}$ we define

$$u'^{(n)} := z_1' z_2' \cdots z_n' \in X^n.$$

Then for any $g \in \mathcal{N}, g' \in A, u, v \in X^n, u' \in Z_{n'+2}$ with $u'^{(n)} = v$, we have

$$\begin{aligned} \lambda_w^\Gamma(S_u g S_v^*) \delta_{[S_{u'} g' S^*_{w(n'+2)}, w]} &= \sum_{v' \in X^{n'+2-n}} \lambda_w^\Gamma(S_u g S_{v'} S_{v'}^* S_v^*) \delta_{[S_{u'} g' S^*_{w(n'+2)}, w]} \\ &= \delta_{[S_{ug(v^{-1}u')} g|_{v^{-1}u'} g' S^*_{w(n'+2)}, w]}. \end{aligned}$$

Note that $g|_{v^{-1}u'} \in A$ since $v^{-1}u' \in Z_{n'+2-n}$. Similarly, we obtain

$$\pi_\psi(S_u g S_v^*) \left(\pi_\psi(S_{u'}) \overline{g'}\right) = \pi_\psi(S_{ug(v^{-1}u')}) \overline{g|_{v^{-1}u'} g'}.$$

In addition, for any $g \in \mathcal{N}, g' \in A, u, v \in X^n, u' \in Z_{n'+2}$ with $u'^{(n)} \neq v$, we have

$$\lambda_w^\Gamma(S_u g S_v^*) \delta_{[S_{u'} g' S^*_{w(n'+2)}, w]} = 0, \quad \pi_\psi(S_u g S_v^*) \left(\pi_\psi(S_{u'}) \overline{g'}\right) = 0.$$

Hence, we get

$$\lambda_w^\Gamma(\eta)\xi = \sum_{u, v, u', g, g'} \alpha_{u,v}^g \beta_{u'}^{g'} \delta_{[S_{ug(v^{-1}u')} g|_{v^{-1}u'} g' S^*_{w(n'+2)}, w]}.$$

and

$$\pi_\psi(\eta)\tilde{\xi} = \sum_{u, v, u', g, g'} \alpha_{u,v}^g \beta_{u'}^{g'} \pi_\psi(S_{ug(v^{-1}u')}) \overline{g|_{v^{-1}u'} g'}.$$

Here the sums are taken over $g \in \mathcal{N}, g' \in A, u, v \in X^n, u' \in Z_{n'+2}$ with $u'^{(n)} = v$.

We claim $\|\lambda_w^\Gamma(\eta)\xi\| = \|\pi_\psi(\eta)\tilde{\xi}\|$. Define

$$T := \{S_{v'} h : h \in A, v' \in Z_{n'+2}\} \subset \langle G, X \rangle.$$

Take any $S_{v_1'} h_1, S_{v_2'} h_2 \in T$. By Lemma 6.9, $[S_{v_1'} h_1 S^*_{w(n'+2)}, w] = [S_{v_2'} h_2 S^*_{w(n'+2)}, w]$ if and only if $S_{v_1'} = S_{v_2'}, h_1 = h_2$. Hence, $[S_{v_1'} h_1 S^*_{w(n'+2)}, w] = [S_{v_2'} h_2 S^*_{w(n'+2)}, w]$ if and only if $\pi_\psi(S_{v_1'}) \overline{h_1} = \pi_\psi(S_{v_2'}) \overline{h_2}$. Now, we rewrite

$$\lambda_w^\Gamma(\eta)\xi = \sum_{S_{v'} h \in T} \left( \sum_{S_{ug(v^{-1}u')} g|_{v^{-1}u'} g' = S_{v'} h} \alpha_{u,v}^g \beta_{u'}^{g'} \right) \delta_{[S_{v'} h S^*_{w(n'+2)}, w]}, \tag{6.5}$$

$$\pi_\psi(\eta)\tilde{\xi} = \sum_{S_{v'} h \in T} \left( \sum_{S_{ug(v^{-1}u')} g|_{v^{-1}u'} g' = S_{v'} h} \alpha_{u,v}^g \beta_{u'}^{g'} \right) \pi_\psi(S_{v'}) \overline{h}. \tag{6.6}$$

The equations (6.5, 6.6) shows the claim. Consequently, we have $\|\pi_\psi(\eta)\| \geq \|\lambda_w^\Gamma(\eta)\|$ from the same argument as the first case. Thus we finished the proof. □

**Corollary 6.14.** *Let $G = G(A, B, \Psi)$ be a multispinal group over $X$. If one of the equivalent conditions of Theorem 6.13 holds, then the universal $C^*$-algebra $\mathcal{O}_{G_{\max}}$ is a Kirchberg algebra.*

*Proof.* Since the assumption implies that $\mathcal{O}_{G_{\max}}$ is simple, we have

$$\mathcal{O}_{G_{\max}} \simeq \mathcal{O}_{G_{\min}}.$$

Thus, $\mathcal{O}_{G_{\max}}$ is purely infinite simple by Theorem 4.5. Note that $[G, X]$ is amenable by Theorem 6.4. Then as $C^*([G, X])$ is nuclear, so is $\mathcal{O}_{G_{\max}}$. Trivially, $\mathcal{O}_{G_{\max}}$ is separable and unital. Hence, $\mathcal{O}_{G_{\max}}$ is a Kirchberg algebra. □

For the other corollary, we fix notations. Consider a multispinal group $G = G(A, B, \Psi)$. We write $C^*(B)$ for the full group C*-algebra of the finite group $B$. In the same way as $C^*(A)$, we identify $C^*(B)$ with a finite dimensional C*-subalgebra of $\mathcal{O}_{G_{\max}}$. For each $\lambda \in \mathrm{Hom}(A, B)$, define a $*$-homomorphism $\tilde{\lambda}\colon C^*(A) \to C^*(B)$ to be

$$\tilde{\lambda}\left(\sum_{a\in A} \alpha_a a\right) := \sum_{a\in A} \alpha_a \lambda(a).$$

Here each $\alpha_a$ is a complex number. We recall a spacial case of [30, Theorem 7.5].

**Theorem 6.15.** ([30, Theorem 7.5]) *Let $G = G(A, B, \Psi)$ be a multispinal group over $X$. Then $\mathbb{C}\langle G, X\rangle$ is simple if and only if*

$$\bigcap_{\lambda\in\mathcal{B}\cdot\mathcal{A}} \ker\tilde{\lambda} = \{0\}.$$

Combining Theorems 6.13, 6.15, we have the following corollary.

**Corollary 6.16.** *Let $G = G(A, B, \Psi)$ be a multispinal group over $X$. Assume that $[G, X]$ is amenable. Then $\mathcal{O}_{G_{\max}}$ is simple if and only if $\mathbb{C}\langle G, X\rangle$ is simple.*

*Proof.* First, we show the simplicity of $\mathbb{C}\langle G, X\rangle$ from the simplicity of $\mathcal{O}_{G_{\max}}$. In sections 2, 4 of [30], it was shown that $\mathbb{C}\langle G, X\rangle$ is isomorphic to the Steinberg $\mathbb{C}$-algebra associated to $[G, X]$. Thus, the simplicity of $\mathcal{O}_{G_{\max}} \simeq C^*([G, X])$ implies the simplicity of $\mathbb{C}\langle G, X\rangle$ by [6, Corollary 4.12].

Next, we prove the simplicity of $\mathcal{O}_{G_{\max}}$ from the simplicity of $\mathbb{C}\langle G, X\rangle$. We check the condition (i) of Theorem 6.13. Take any nonzero positive element $\eta \in C^*(A)$. Write

$$\eta = \sum_{a\in A} \alpha_a a.$$

Since $\eta \neq 0$, there exists a homomorphism $\lambda \in \mathcal{B}\cdot\mathcal{A}$ with $\tilde{\lambda}(\eta) \neq 0$ by Theorem 6.15. In the case $\lambda \notin \mathcal{B}$, write

$$\lambda = \Psi(x_n) \circ \Psi(x_{n-1}) \circ \cdots \circ \Psi(x_1),\ u = x_1 x_2 \cdots x_n.$$

Here $n \geq 2, x_n \in Y$ and $x_i \in X\backslash Y$ for any $1 \leq i \leq n-1$. In the case $\lambda \in \mathcal{B}$, we write $\lambda = \Psi(x_1)$ and $u = x_1 \in Y$. For each $a \in A$, we have

$$aS_u = aS_{x_1}S_{x_2}\cdots S_{x_n} = S_{x_1}\Psi(x_1)(a)S_{x_2}\cdots S_{x_n} = \cdots = S_u\lambda(a).$$

This shows that

$$\eta S_u S_u^* = \left(\sum_{a\in A}\alpha_a a\right) S_u S_u^* = S_u\left(\sum_{a\in A}\alpha_a\lambda(a)\right)S_u^* = S_u\tilde{\lambda}(\eta)S_u^*. \tag{6.7}$$

Note that $\tilde{\lambda}(\eta)$ is a nonzero positive element in $C^*(B)$. We write

$$(\tilde{\lambda}(\eta))^{\frac{1}{2}} = \sum_{b\in B}\beta_b b.$$

Here $\beta_{b_0} \neq 0$ for some $b_0 \in B$. Since $\psi_{\max}(b) = \mu((X^\omega)_b) = 0$ for any $b \neq e$, we have

$$\psi_{\max}(\tilde{\lambda}(\eta)) = \psi_{\max}\left(\left(\sum_{b\in B}\beta_b b\right)^*\left(\sum_{b\in B}\beta_b b\right)\right) = \sum_{b\in B}|\beta_b|^2 > 0. \tag{6.8}$$

Recall that the restriction of $\psi_{\max}$ to $\mathcal{O}_{G_{\max}}^{\Gamma}$ is tracial. Then a calculation shows

$$\psi_{\max}(\eta S_u S_u^*) = \psi_{\max}(\eta^{\frac{1}{2}} S_u S_u^* \eta^{\frac{1}{2}}) \leq \psi_{\max}(\eta^{\frac{1}{2}}\eta^{\frac{1}{2}}) = \psi_{\max}(\eta). \tag{6.9}$$

Combining (6.7), (6.8), (6.9) and Theorem 4.10, we obtain

$$\psi_{\max}(\eta) \geq \psi_{\max}(\eta S_u S_u^*) = \psi_{\max}(S_u \tilde{\lambda}(\eta) S_u^*) = |X|^{-|u|}\psi_{\max}(\tilde{\lambda}(\eta)) > 0.$$

This proves the claim. □

Using Theorem 6.13, we show the simplicity of $\mathcal{O}_{G_{\max}}$ in the case that $G$ is the Grigorchuk group. The simplicity of $\mathcal{O}_{G_{\max}}$ of the Grigorchuk group has been proved in [6, Theorem 5.22]. The proof in [6] is based on its main theorem. The main theorem provides a relevance between supports of functions on a non-Hausdorff groupoid and the simplicity of its reduced groupoid C*-algebra (see [6, Theorem 4.10]). We observe a different proof in the following example.

**Example 6.17.** We regard the Grigorchuk group as a multispinal group (see Example 6.7). In this example, we use the same symbols as Example 3.5 for generators. In other words, we write

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{e, b, c, d\},\ \mathbb{Z}_2 = \{e, a\}.$$

Note that we have

$$b^2 = c^2 = d^2 = e, bc = cb = d, bd = db = c, cd = dc = b.$$

We determine the values $\psi(b), \psi(c), \psi(d)$ to check the condition (ii) in Theorem 6.13. This is done if one solves the following simultaneous equation given by the self-similarity:

$$\psi(b) = \frac{\psi(a) + \psi(c)}{2} = \frac{\psi(c)}{2},$$
$$\psi(c) = \frac{\psi(a) + \psi(d)}{2} = \frac{\psi(d)}{2},$$
$$\psi(d) = \frac{\psi(e) + \psi(b)}{2} = \frac{1 + \psi(b)}{2}.$$

As the solution of these equations, we have $\psi(b) = \frac{1}{7}, \psi(c) = \frac{2}{7}, \psi(d) = \frac{4}{7}$. Thus,

$$\begin{pmatrix} \psi(e) & \psi(b) & \psi(c) & \psi(d) \\ \psi(b) & \psi(e) & \psi(d) & \psi(c) \\ \psi(c) & \psi(d) & \psi(e) & \psi(b) \\ \psi(d) & \psi(c) & \psi(b) & \psi(e) \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 7 & 1 & 2 & 4 \\ 1 & 7 & 4 & 2 \\ 2 & 4 & 7 & 1 \\ 4 & 2 & 1 & 7 \end{pmatrix}.$$

Compute the determinant of the matrix

$$\det \begin{pmatrix} 7 & 1 & 2 & 4 \\ 1 & 7 & 4 & 2 \\ 2 & 4 & 7 & 1 \\ 4 & 2 & 1 & 7 \end{pmatrix} = 896.$$

By Theorem 6.13, the universal C*-algebra $\mathcal{O}_{G_{\max}}$ is simple.

In the next example, we observe a non-simple case. Combining [6, Corollary 4.12] and [30, Example 7.6], one can get the same result.

**Example 6.18.** Let $A = \mathbb{Z}_2 \times \mathbb{Z}_2, B = X = \mathbb{Z}_2$. Consider the left translation action $B \curvearrowright X$. We define $\Psi(0) \in \mathrm{Hom}(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2)$ and $\Psi(1) \in \mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ to be

$$\Psi(0)(x, y) := y,\ \Psi(1)(x, y) := (y, x).$$

Respecting the Grigorchuk group, we write $b := (0, 1), c := (1, 1), d := (1, 0)$ and let $a$ denote the generator of $\mathbb{Z}_2$. Then one obtains

$$a(0w) = 1w,\ a(1w) = 0w,$$

$$b(0w) = 0a(w),\ b(1w) = 1d(w),$$
$$c(0w) = 0a(w),\ c(1w) = 1c(w),$$
$$d(0w) = 0w,\ d(1w) = 1b(w)$$

for any $w \in X^\omega$. These equations give

$$\psi(b) = \frac{\psi(a) + \psi(d)}{2} = \frac{\psi(d)}{2},$$
$$\psi(c) = \frac{\psi(a) + \psi(c)}{2} = \frac{\psi(c)}{2},$$
$$\psi(d) = \frac{\psi(e) + \psi(b)}{2} = \frac{1 + \psi(b)}{2}.$$

Then we get $\psi(b) = \dfrac{1}{3}, \psi(c) = 0, \psi(d) = \dfrac{2}{3}$. Thus, we have

$$\det \begin{pmatrix} \psi(e) & \psi(b) & \psi(c) & \psi(d) \\ \psi(b) & \psi(e) & \psi(d) & \psi(c) \\ \psi(c) & \psi(d) & \psi(e) & \psi(b) \\ \psi(d) & \psi(c) & \psi(b) & \psi(e) \end{pmatrix} = \det \left( \frac{1}{3} \begin{pmatrix} 3 & 1 & 0 & 2 \\ 1 & 3 & 2 & 0 \\ 0 & 2 & 3 & 1 \\ 2 & 0 & 1 & 3 \end{pmatrix} \right) = 0.$$

Theorem 6.13 implies that the universal C*-algebra $\mathcal{O}_{G_{\max}}$ is not simple in this case.

**Example 6.19.** Next we observe the case $A = \mathbb{Z}_3 \times \mathbb{Z}_3, B = X = \mathbb{Z}_3$. The action of $B$ on $X$ is the left translation action. We regard the elements in $A$ as $\mathbb{Z}_3$-valued row vectors. We define $\Psi(0), \Psi(1) \in \mathrm{Aut}(\mathbb{Z}_3 \times \mathbb{Z}_3)$ and $\Psi(2) \in \mathrm{Hom}(\mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_3)$ to be

$$\Psi(0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ \Psi(1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ \Psi(2) = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

Let

$$a_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ a_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ a_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ a_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ a_4 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

By the self-similarity, we have

$$\psi(a_1) = \frac{\psi(a_1) + \psi(a_3) + 1}{3},$$
$$\psi(a_2) = \frac{\psi(a_3) + \psi(a_2)}{3},$$
$$\psi(a_3) = \frac{2\psi(a_4)}{3},$$
$$\psi(a_4) = \frac{\psi(a_2) + \psi(a_1)}{3}.$$

From these equations, we obtain

$$\psi(a_1) = \psi(a_1^{-1}) = \frac{4}{7},$$
$$\psi(a_2) = \psi(a_2^{-1}) = \frac{1}{14},$$
$$\psi(a_3) = \psi(a_3^{-1}) = \frac{1}{7},$$
$$\psi(a_4) = \psi(a_4^{-1}) = \frac{3}{14}.$$

Hence we obtain the matrix

$$[\psi(a_i^{-1}a_j)]_{0\le i,j\le 8} = \frac{1}{14}\begin{pmatrix} 14 & 1 & 8 & 2 & 3 & 1 & 8 & 2 & 3 \\ 1 & 14 & 3 & 8 & 2 & 1 & 2 & 3 & 8 \\ 8 & 3 & 14 & 1 & 1 & 2 & 8 & 3 & 2 \\ 2 & 8 & 1 & 14 & 1 & 3 & 3 & 2 & 8 \\ 3 & 2 & 1 & 1 & 14 & 8 & 2 & 8 & 3 \\ 1 & 1 & 2 & 3 & 8 & 14 & 3 & 8 & 2 \\ 8 & 2 & 8 & 3 & 2 & 3 & 14 & 1 & 1 \\ 2 & 3 & 3 & 2 & 8 & 8 & 1 & 14 & 1 \\ 3 & 8 & 2 & 8 & 3 & 2 & 1 & 1 & 14 \end{pmatrix}.$$

Here $a_5 = a_1^{-1}, a_6 = a_2^{-1}, a_7 = a_3^{-1}, a_8 = a_4^{-1}$. The MDETERM function of Excel provides the result of the calculation as follows:

$$\det\begin{pmatrix} 14 & 1 & 8 & 2 & 3 & 1 & 8 & 2 & 3 \\ 1 & 14 & 3 & 8 & 2 & 1 & 2 & 3 & 8 \\ 8 & 3 & 14 & 1 & 1 & 2 & 8 & 3 & 2 \\ 2 & 8 & 1 & 14 & 1 & 3 & 3 & 2 & 8 \\ 3 & 2 & 1 & 1 & 14 & 8 & 2 & 8 & 3 \\ 1 & 1 & 2 & 3 & 8 & 14 & 3 & 8 & 2 \\ 8 & 2 & 8 & 3 & 2 & 3 & 14 & 1 & 1 \\ 2 & 3 & 3 & 2 & 8 & 8 & 1 & 14 & 1 \\ 3 & 8 & 2 & 8 & 3 & 2 & 1 & 1 & 14 \end{pmatrix} = 634894848.$$

As a consequence, we get the simplicity of $\mathcal{O}_{G_{\max}}$ in this case.

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