THE UNIQUENESS OF INVERSE PROBLEMS FOR A FRACTIONAL EQUATION WITH A SINGLE MEASUREMENT

Yavar KIAN
Aix-Marseille Université, Université de Toulon, CNRS, CPT, Marseille, France.
yavar.kian@univ-amu.fr

Zhiyuan LI
School of Mathematics and Statistics, Shandong University of Technology
Zibo, Shandong 255049, China.
zyli@sdut.edu.cn

Yikan LIU
Research Center of Mathematics for Social Creativity, Research Institute for Electronic Science,
Hokkaido University, N12W7, Kita-Ward, Sapporo 060-0812, Japan.
ykliu@es.hokudai.ac.jp

Masahiro YAMAMOTO
Graduate School of Mathematical Sciences, The University of Tokyo
3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan;
Honorary Member of Academy of Romanian Scientists
Splaiul Independentei Street, No. 54, 050094 Bucharest, Romania;
Peoples' Friendship University of Russia (RUDN University)
6 Miklukho-Maklaya Street, Moscow 117198, Russian Federation.
myama@ms.u-tokyo.ac.jp

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ABSTRACT

This article is concerned with an inverse problem on simultaneously determining some unknown coefficients and/or an order of derivative in a multidimensional time-fractional evolution equation either in a Euclidean domain or on a Riemannian manifold. Based on a special choice of the Dirichlet boundary input, we prove the unique recovery of at most two out of four x-dependent coefficients (possibly with an extra unknown fractional order) by a single measurement of the partial Neumann boundary output. Especially, both a vector-valued velocity field of a convection term and a density can also be uniquely determined. The key ingredient turns out to be the time-analyticity of the decomposed solution, which enables the construction of Dirichlet-to-Neumann maps in the frequency domain and thus the application of inverse spectral results.

Keywords Coefficient inverse problem · Fractional diffusion-wave equation · Single boundary measurement · Time-analyticity · Uniqueness

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1 Introduction

Let $T \in \mathbb{R}_+ := (0,\infty)$ and $\alpha \in (0,2)$. By ∂_t^α we denote the α-th order Caputo derivative with respect to t defined by

$$\partial_t^\alpha f(t) := \begin{cases} \dfrac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\displaystyle\int_0^t \dfrac{f^{(\lceil\alpha\rceil)}(s)}{(t-s)^{\alpha-\lfloor\alpha\rfloor}}\,\mathrm{d}s, & \alpha\in(0,2)\setminus\{1\},\\ f'(t), & \alpha=1, \end{cases}$$

where $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor and the ceiling functions, respectively. Let $\Omega\subset\mathbb{R}^d$ $(d=2,3,\dots)$ be a bounded connected domain with a $C^{1,1}$ boundary $\partial\Omega$. Let $\rho\in L^\infty(\Omega)$, $a\in C^1(\overline{\Omega})$, $\boldsymbol{B}=(B_1,\dots,B_d)\in(L^\infty(\Omega))^d$, $c\in L^\infty(\Omega)$ and assume that

$$\underline{\rho}\le\rho\le\overline{\rho}\text{ in }\Omega,\quad a>0\text{ on }\overline{\Omega},\quad c\ge 0\text{ in }\Omega, \tag{1}$$

where $\underline{\rho},\overline{\rho}\in\mathbb{R}_+$ are constants. The first object of this paper is the following initial-boundary value problem for a time-fractional diffusion(-wave) equation with a nonhomogeneous Dirichlet boundary condition

$$\begin{cases} \rho\,\partial_t^\alpha u-\operatorname{div}(a\nabla u)+\boldsymbol{B}\cdot\nabla u+c\,u=0 & \text{in }(0,T)\times\Omega,\\ \partial_t^m u=0\ (m=0,\lceil\alpha\rceil-1) & \text{in }\{0\}\times\Omega,\\ u=\Phi & \text{on }(0,T)\times\partial\Omega. \end{cases} \tag{2}$$

Meanwhile, we will also consider the same problem as (2) on manifolds. To this end, let $(\mathcal{M},g)$ be a smooth compact connected Riemannian manifold of dimensions $d\ge 2$ with boundary $\partial\mathcal{M}$. Let $\mu\in C^\infty(\mathcal{M})$ and $c\in L^\infty(\mathcal{M})$ such that $\mu>0$ and $c\ge 0$ on $\mathcal{M}$. We introduce the weighted Laplace-Beltrami operator

$$\triangle_{g,\mu} := \mu^{-1}\operatorname{div}_g\mu\nabla_g,$$

where div_g and ∇_g denote divergence and gradient operators on $(\mathcal{M},g)$ respectively. The second object of this paper is the following initial-boundary value problem on the manifold $\mathcal{M}$

$$\begin{cases} \partial_t^\alpha u-\triangle_{g,\mu}u+c\,u=0 & \text{in }(0,T)\times\mathcal{M},\\ \partial_t^m u=0\ (m=0,\lceil\alpha\rceil-1) & \text{in }\{0\}\times\mathcal{M},\\ u=\Phi & \text{on }(0,T)\times\partial\mathcal{M}. \end{cases} \tag{3}$$

This article is concerned with the following inverse problem for (2) and (3).

Problem 1.1. *Let u satisfy (2) (or (3)) and $\Gamma_{\rm in},\Gamma_{\rm out}$ be open subsets of $\partial\Omega$ (or $\partial\mathcal{M}$). For a suitably chosen Dirichlet boundary input Φ supported on the sub-boundary $[0,T]\times\Gamma_{\rm in}$, determine as many coefficients as possible among the set $(\rho,a,\boldsymbol{B},c)$ (or $(c,\mu,\mathcal{M})$) and/or α from a single measurement of $a\,\partial_{\boldsymbol{\nu}}u$ on the sub-boundary $(0,T)\times\Gamma_{\rm out}$, where $\boldsymbol{\nu}$ denotes the outward unit normal vector to $\partial\Omega$ (or $\partial\mathcal{M}$).*

Time-fractional diffusion(-wave) equations of the forms (2) and (3) describe diffusion in different kinds of physical phenomena. For $\alpha\ne 1$, (2) and (3) describe the anomalous diffusion of substances in heterogeneous media, diffusion in inhomogeneous anisotropic porous media, turbulent plasma, diffusion in a turbulent flow, a percolation model in porous media, several biological and financial problems (see [9]). For instance, it is known (see [1]) that in several cases, the classical diffusion-advection equation is not a suitable model for describing field data of diffusion of substances in the soil. Time-fractional diffusion(-wave) equations can be regarded as an alternative model. Note also that time-fractional diffusion(-wave) equations are derived from continuous-time random walk (see [38,42]). Due to their modeling feasibility, time-fractional differential equations have received great attention in the last decades. Without being exhaustive we refer to [37,39,41,44] for further details. Especially, the well-posedness for problem (2) has been studied by [28,43] for $\boldsymbol{B}=\boldsymbol{0}$ and by [22] for $\boldsymbol{B}\ne\boldsymbol{0}$.

On the other hand, the above inverse problem addressed in the present paper corresponds to the determination of several coefficients describing the diffusion of some physical quantities. The convection term $\boldsymbol{B}$ is associated with the velocity field of the moving quantities, while the coefficients (ρ,a,c) can be associated to the density of the medium. For instance, our inverse problem can be stated as the determination of the velocity field and the density of the medium in the diffusion process of a contaminant in soil from a single measurement on $\Gamma_{\rm out}$.

In retrospect, many authors considered inverse problems for (2) when $\alpha = 1$. Without being exhaustive, we can refer to [7,10,15,16,24]. In particular, we mention the paper [3], where the recovery of the coefficient c has been addressed from a single boundary measurement. The idea of [3] is to use a suitable input Φ that allows to recover the associated elliptic Dirichlet-to-Neumann map from a single measurement of the solution of the associated parabolic equation on the lateral boundary $(0,T) \times \partial\Omega$. This idea has been extended by [13] to the recovery of the convection term $\boldsymbol{B}$ in the case of $d = 2$. Here the authors applied the results in [12,14] related to the recovery of a convection term appearing in a stationary convection diffusion equation from the associated Dirichlet-to-Neumann map.

In contrast to $\alpha = 1$, inverse problems associated with (2) when $\alpha \in (0,2) \setminus \{1\}$ have received less attention. For $d = 1$, [11] determined uniquely the fractional order α and a coefficient from Dirichlet boundary measurements. In [18, 43], the authors considered the problem of stably determining a time-dependent coefficient appearing in a time-fractional diffusion equation. In [32], the authors uniquely determined coefficients by means of the Dirichlet-to-Neumann map associated with the system applied to Dirichlet boundary conditions taking the form $\lambda(t)\Phi(\boldsymbol{x})$, where λ is given suitably. In [26], the authors considered the recovery of a general class of coefficients on a Riemannian manifold and a Riemannian metric from the partial Dirichlet-to-Neumann map, associated with the fractional diffusion equation under consideration, taken at one arbitrarily fixed time. In the above work, the authors started by proving the unique recovery of boundary spectral data associated with the elliptic part of their equations. Then the uniqueness results were obtained by using inverse spectral results and related inverse problems stated in [7,8,23,31]. For variable order and distributed order fractional diffusion equations, we refer to the papers [27, 33], where results related to inverse problems for these equations have been stated. In [29], the authors proved the unique reconstruction of source terms and the stable recovery of some class of zeroth order time dependent coefficients appearing in fractional diffusion equations in a cylindrical domain. In a recent paper [21], the recovery of a Riemannian manifold without boundary was proved from a single internal measurement of the solution of a fractional diffusion equation with a suitable internal source. Finally, we refer to the review articles [34–36] as summaries on the recent progress of inverse problems for time-fractional evolution equations. In the one-dimensional case, we mention also the work of [17], where the recovery of a conductivity coefficient appearing in a parabolic equation from a single measurement at one point was considered. Here the author connected the inverse problem stated for a parabolic equation with an inverse Sturm-Liouville problem.

The remainder of this paper is organized as follows. In Sect. 2, we recall the necessary ingredients to treat Problem 1.1 and state the main uniqueness results along with some explanations and remarks. As a key to establishing the main results, in Sect. 3 we prove the time-analyticity of the solutions to the forward problems under consideration. Finally, proofs of the main results are provided in Sect. 4.

2 Preliminaries and main results

We start with fixing the notations and terminology used in the sequel. Throughout this paper, by $\mathbb{N} := \{1, 2, \dots\}$ we denote the positive natural numbers. Let $H^m(\Omega)$, $H^{m-1/2}(\partial\Omega)$, $W^{m,q}(\Omega)$, etc. ($m \in \mathbb{Z}$, $q \in [1,\infty]$) denote the usual Sobolev spaces (see Adams [2]). Given Banach spaces X and Y, by $\mathcal{B}(X,Y)$ we denote the family of bounded linear operators from X to Y. For a connected set $\mathcal{O} \subset \mathbb{R}$ or $\mathbb{C}$, we denote by $C^\omega(\mathcal{O}; X)$ the family of analytic functions in $\mathcal{O}$ taking values in X.

2.1 Statements of the Main results

We first deal with the initial-boundary value problem (2) in a bounded domain $\Omega \subset \mathbb{R}^d$. We denote the inner products of $L^2(\Omega)$ and $L^2(\partial\Omega)$ by $(\,\cdot\,,\,\cdot\,)$ and $\langle\,\cdot\,,\,\cdot\,\rangle$, respectively, that is,

$$(f_1, f_2) := \int_\Omega f_1(\boldsymbol{x}) f_2(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \quad f_1, f_2 \in L^2(\Omega); \qquad \langle f_3, f_4\rangle := \int_{\partial\Omega} f_3(\boldsymbol{y}) f_4(\boldsymbol{y})\,\mathrm{d}\sigma(\boldsymbol{y}), \quad f_3, f_4 \in L^2(\partial\Omega).$$

For $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, we denote its Laplace transform as

$$\widehat{f}(\xi) = (\mathcal{L}f)(\xi) := \int_0^\infty \mathrm{e}^{-\xi t} f(t)\,\mathrm{d}t.$$

Recall that, according to [28,43], we can define the weak solutions of (2) in the following way.

Definition 2.1. *Let $F \in L^1(0,T;L^2(\Omega))$. We say that the problem*

$$\begin{cases}\rho\,\partial_t^\alpha u - \mathrm{div}\,(a\nabla u) + c\,u = F & \text{in } (0,T)\times\Omega,\\ \partial_t^m u = 0\ (m = 0, \lceil\alpha\rceil - 1) & \text{in } \{0\}\times\Omega,\\ u = 0 & \text{on } (0,T)\times\partial\Omega\end{cases} \tag{4}$$

admits a weak solution u if there exists $v \in L^1_{\mathrm{loc}}(\mathbb{R}_+;L^2(\Omega))$ such that

(i) $v|_{(0,T)\times\Omega} = u$ *and* $\inf\{\varepsilon > 0 \mid \mathrm{e}^{-\varepsilon t}v(t,\,\cdot\,) \in L^1(\mathbb{R}_+;L^2(\Omega))\} = 0$,

(ii) *for all $\xi > 0$, the Laplace transform $\widehat{v}(\xi,\,\cdot\,)$ of $v(t,\,\cdot\,)$ solves*

$$\begin{cases}-\mathrm{div}(a\nabla\widehat{v}(\xi,\,\cdot\,)) + (\xi^\alpha\rho + c)\widehat{v}(\xi,\,\cdot\,) = \displaystyle\int_0^{\mathrm{T}} \mathrm{e}^{-\xi t}F(t,\,\cdot\,)\,\mathrm{d}t & \text{in } \Omega,\\ \widehat{v}(\xi,\,\cdot\,) = 0 & \text{on } \partial\Omega.\end{cases}$$

Following [27, 28, 43], there exists $S \in L^1(0,T;\mathcal{B}(L^2(\Omega),H^1(\Omega)))$ such that the solution of (4) takes the form

$$u(t,\,\cdot\,) = \int_0^t S(t-s)F(s,\,\cdot\,)\,\mathrm{d}s.$$

Therefore, regarding the advection term $\boldsymbol{B}\cdot\nabla u$ as an additional source term, we define the solution of the problem

$$\begin{cases}\rho\,\partial_t^\alpha u - \mathrm{div}\,(a\nabla u) + \boldsymbol{B}\cdot\nabla u + c\,u = F & \text{in } (0,T)\times\Omega,\\ \partial_t^m u = 0\ (m = 0, \lceil\alpha\rceil - 1) & \text{in } \{0\}\times\Omega,\\ u = 0 & \text{on } (0,T)\times\partial\Omega\end{cases}$$

in the mild sense as a solution of the integral equation

$$u(t,\,\cdot\,) = \int_0^t S(t-s)F(s,\,\cdot\,)\,\mathrm{d}s - \int_0^t S(t-s)\boldsymbol{B}\cdot\nabla u(s,\,\cdot\,)\,\mathrm{d}s.$$

For the unique existence of solutions of (2) in the above sense as well as classical properties of solutions of such problems, we refer to [22, 27, 28, 43]. In the case $\alpha = 1$, the solution of (2) corresponds to the classical variational solution of this parabolic equation lying in $H^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H^1(\Omega))$.

Now we turn to the investigation of Problem 1.1. For the choice of open sub-boundaries Γ_{in} and Γ_{out}, we assume that

$$\Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out}} = \partial\Omega, \quad \Gamma_{\mathrm{in}} \cap \Gamma_{\mathrm{out}} \neq \emptyset. \tag{5}$$

This condition will be relaxed later in the framework of smooth Riemannian manifolds with smooth coefficients.

Next we specify the choice of the Dirichlet input Φ, which plays an essential role in the consideration of Problem 1.1. Let $\chi \in C^\infty(\partial\Omega)$ satisfy $\mathrm{supp}\,\chi \subset \Gamma_{\mathrm{in}}$ and $\chi = 1$ on Γ'_{in}, where Γ'_{in} is an open subset of $\partial\Omega$ such that $\Gamma'_{\mathrm{in}} \cup \Gamma_{\mathrm{out}} = \partial\Omega$ and $\Gamma'_{\mathrm{in}} \cap \Gamma_{\mathrm{out}} \neq \emptyset$. We fix $T_0 \in (0,T]$ and choose a strictly increasing sequence $\{t_k\}_{k=0}^\infty$ such that $t_0 = 0$ and $\lim_{k\to\infty} t_k = T_0$. Consider a sequence $\{p_k\}_{k=0}^\infty \subset [0,+\infty)$ and a sequence $\{\psi_k\}_{k\in\mathbb{N}} \subset C^\infty([0,+\infty);[0,+\infty))$ of not identically vanishing functions such that

$$\psi_k = \begin{cases}0 & \text{on } [0,t_{2k-2}],\\ p_k & \text{on } [t_{2k-1},\infty).\end{cases} \tag{6}$$

We fix also a sequence $\{b_k\}_{k=0}^\infty$ of $\mathbb{R}_+$ such that

$$\sum_{k=1}^\infty b_k\|\psi_k\|_{W^{2,\infty}(\mathbb{R}_+)} < \infty.$$

Finally, we select a sequence $\{\eta_k\}_{k\in\mathbb{N}} \subset H^{3/2}(\partial\Omega)$ such that $\operatorname{span}\{\eta_k\}$ is dense in $H^{3/2}(\partial\Omega)$ and $\|\eta_k\|_{H^{3/2}(\partial\Omega)} = 1$ ($k \in \mathbb{N}$). Now we can construct the input $\Phi \in C^2([0,+\infty); H^{3/2}(\partial\Omega))$ as

$$\Phi(t,\boldsymbol{x}) := \chi(\boldsymbol{x}) \sum_{k=1}^{\infty} b_k\, \psi_k(t)\eta_k(\boldsymbol{x}). \tag{7}$$

Note that clearly, we have $\operatorname{supp}\Phi \subset [0,+\infty) \times \Gamma_{\mathrm{in}}$. Using this definition of the input Φ we can now state our first two main results in the Euclidean case.

First, we fix $\alpha \in (0,2)$, $\boldsymbol{B} = \boldsymbol{0}$ and consider the recovery of the coefficients (ρ, a, c). Actually, due to the obstructions described in Sect. 2.2, we can only consider the recovery of any two among the three coefficients ρ, a, c. Our first main result can be stated as follows.

Theorem 2.2. *Let $\alpha \in (0,2)$ be fixed, $\Gamma_{\mathrm{in}}, \Gamma_{\mathrm{out}} \subset \partial\Omega$ satisfy* (5) *and the triples (ρ^j, a^j, c^j) $(j = 1,2)$ fulfill* (1). *Suppose that either of the following conditions is satisfied:*

(i) $\rho^1 = \rho^2$ *and*
$$\nabla a^1 = \nabla a^2 \quad \text{on } \partial\Omega. \tag{8}$$

(ii) $a^1 = a^2$ *and*
$$\exists\, C > 0,\ |\rho^1(\boldsymbol{x}) - \rho^2(\boldsymbol{x})| \le C \operatorname{dist}(\boldsymbol{x}, \partial\Omega)^2, \quad \boldsymbol{x} \in \Omega. \tag{9}$$

(iii) $c^1 = c^2$ *and* (8)–(9) *hold true simultaneously.*

Let u^j $(j = 1,2)$ be the solutions to (2) *with Φ given by* (7), *$\boldsymbol{B} = \boldsymbol{0}$ and $(\rho, a, c) = (\rho^j, a^j, c^j)$. Then the condition*

$$a^1 \partial_{\boldsymbol{\nu}} u^1 = a^2 \partial_{\boldsymbol{\nu}} u^2 \quad \text{on } (0,T_0) \times \Gamma_{\mathrm{out}} \tag{10}$$

implies $(\rho^1, a^1, c^1) = (\rho^2, a^2, c^2)$.

Second, we focus our attention on the simultaneous recovery of the convection term $\boldsymbol{B}$, the weight ρ and the fractional order α under the assumption that $a = 1$, $c = 0$. Our second main result can be stated as follows.

Theorem 2.3. *Let $d \ge 3$, $\alpha^j \in (0,2)$, $\rho^j \in L^\infty(\Omega)$ satisfy* (1) *and $\boldsymbol{B}^j \in (C^\kappa(\overline{\Omega}))^d$ with $\kappa > 2/3$ $(j = 1,2)$. Let u^j $(j = 1,2)$ be the solutions to* (2) *with $a \equiv 1$, $c \equiv 0$, $(\alpha, \rho, \boldsymbol{B}) = (\alpha^j, \rho^j, \boldsymbol{B}^j)$ and for Φ given by* (7) *with $\chi \equiv 1$. Then the condition*

$$\partial_{\boldsymbol{\nu}} u^1 = \partial_{\boldsymbol{\nu}} u^2 \quad \text{on } (0,T_0) \times \partial\Omega$$

implies $(\alpha^1, \rho^1, \boldsymbol{B}^1) = (\alpha^2, \rho^2, \boldsymbol{B}^2)$.

Now we turn to the simultaneous recovery (modulo suitable invariance) of the manifold $(\mathcal{M}, g)$ and the coefficients (μ, c) from a single measurement of the solution of (3) on Γ_{out} in some suitable sense.

Similarly to that in the Euclidean case, we select $\Gamma'_{\mathrm{in}} \subset \partial\mathcal{M}$ and $\chi \in C^\infty(\partial\mathcal{M})$ such that $\operatorname{supp}\chi \subset \Gamma_{\mathrm{in}}$ and $\chi = 1$ in Γ'_{in}. Then we define the Dirichlet input Ψ in (3) exactly the same as (7), where $\{\eta_k\}_{k\in\mathbb{N}}$ is a sequence of $H^{3/2}(\partial\mathcal{M})$ such that $\operatorname{span}\{\eta_k\}$ is dense in $H^{3/2}(\partial\mathcal{M})$ and $\|\eta_k\|_{H^{3/2}(\partial\mathcal{M})} = 1$ ($k \in \mathbb{N}$).

We will state three different extensions of Theorem 2.2, among which the first one can be stated as follows.

Corollary 2.4. *For $j = 1,2$, let $(\mathcal{M}^j, g^j)$ be two compact and smooth connected Riemannian manifolds of dimensions $d \ge 2$ with the same boundary, and let $\mu^j \in C^\infty(\mathcal{M}^j)$ and $c^j \in C^\infty(\mathcal{M}^j)$ satisfy $\mu^j > 0$ and $c^j \ge 0$ on $\mathcal{M}^j$. Suppose*

$$\Gamma_{\mathrm{in}} = \Gamma_{\mathrm{out}} \subset \partial\mathcal{M}^1, \quad g^1 = g^2,\ \mu^1 = \mu^2 = 1,\ \partial_{\boldsymbol{\nu}}\mu^1 = \partial_{\boldsymbol{\nu}}\mu^2 = 0 \text{ on } \partial\mathcal{M}^1.$$

Denote by u^j $(j = 1,2)$ the solutions of (3) *with Φ given by* (7) *and $(\mathcal{M}, g, \mu, c) = (\mathcal{M}^j, g^j, \mu^j, c^j)$. Then the condition*

$$\partial_{\boldsymbol{\nu}} u^1 = \partial_{\boldsymbol{\nu}} u^2 \quad \text{in } (0,T_0) \times \Gamma_{\mathrm{out}} \tag{11}$$

implies that $(\mathcal{M}^1, g^1)$ and $(\mathcal{M}^2, g^2)$ are isometric. Moreover, (11) *implies that there exist $\psi \in C^\infty(\mathcal{M}^2; \mathcal{M}^1)$ and $\kappa \in C^\infty(\mathcal{M}^2)$ satisfying*

$$\kappa = 1,\ \partial_{\boldsymbol{\nu}}\kappa = 0 \quad \text{on } \partial\mathcal{M}^2 \tag{12}$$

such that

$$\mu^2 = \kappa^{-2}\mu^1 \circ \psi, \quad c^2 = c^1 \circ \psi - \kappa \triangle_{g^2,\mu^1} \kappa^{-1}. \tag{13}$$

We can also extend our results to the case where the excitation and the measurements are disjoint. To this end, we need the following definition.

Definition 2.5. *Consider the initial-boundary value problem for a hyperbolic equation with Dirichlet data* $\Psi \in C_0^\infty((0,\infty)\times\partial\mathcal{M})$

$$\begin{cases}(\partial_t^2 - \triangle_g + c)u = 0 & \text{in } (0,\infty)\times\mathcal{M},\\ u = \partial_t u = 0 & \text{in } \{0\}\times\mathcal{M},\\ u = \Psi & \text{on } (0,\infty)\times\partial\mathcal{M},\end{cases} \tag{14}$$

where $\triangle_g = \triangle_{\mu,g}$ *with* $\mu = 1$*. We say that* (14) *is exactly controllable from a sub-boundary* $\Gamma \subset \partial\mathcal{M}$ *if there exists* $T > 0$ *such that the map*

$$L^2((0,T)\times\Gamma) \ni \Psi \longmapsto (u(T,\,\cdot\,), \partial_t u(T,\,\cdot\,)) \in L^2(\mathcal{M})\times H^{-1}(\mathcal{M})$$

is surjective.

We refer to [4] for geometrical conditions that guarantee the exact controllability of (14). We can now state the following two results with data on disjoint sets.

Corollary 2.6. *Let* $(\mathcal{M}^j, g^j)$ $(j=1,2)$ *be two compact and smooth connected Riemannian manifolds of dimensions* $d \ge 2$ *with the same boundary. Denote by* u^j $(j=1,2)$ *the solutions of* (3) *with* Φ *given by* (7)*,* $(\mathcal{M}, g) = (\mathcal{M}^j, g^j)$ $(j=1,2)$ *and* $\mu = 1,\ c = 0$ *on* $\mathcal{M}$*. In addition, we assume that* (14)*, with* $(\mathcal{M}, g) = (\mathcal{M}^j, g^j)$ $(j=1,2)$*, is exactly controllable from* Γ'_{in} *and* $\Gamma_{\mathrm{in}}\cap\Gamma_{\mathrm{out}} = \emptyset$*. Then* (11) *implies that* $(\mathcal{M}^1, g^1)$ *and* $(\mathcal{M}^2, g^2)$ *are isometric.*

Corollary 2.7. *Let* $(\mathcal{M}, g)$ *be a compact and smooth connected Riemannian manifold of dimensions* $d \ge 2$*. Let* $\mu = 1$ *on* $\mathcal{M}$ *and* $c^j \in C^\infty(\mathcal{M})$ $(j=1,2)$ *be non-negative. We assume that* (14) *is exactly controllable from* Γ'_{in}*, and* Γ_{out} *is strictly convex,* $\Gamma_{\mathrm{in}}\cap\Gamma_{\mathrm{out}} = \emptyset$*. Then* (11) *implies that there exists a neighborhood* U *of* Γ_{out} *in* $\mathcal{M}$ *such that* $c^1 = c^2$ *in* U*.*

2.2 Obstructions to the simultaneous recovery of three coefficients

In our main results, we consider only the simultaneous recovery of two coefficients appearing in problem (2), additionally with the order α in Theorem 2.3. This limitation of our results is due to some natural obstructions against the simultaneous recovery of some third coefficient appearing in (2). For instance, fix $\alpha \in (0,2)$, $\rho^1, c^1 \in L^\infty(\Omega)$ satisfying (1) and $\boldsymbol{B}^1 \in (C^\kappa(\overline{\Omega}))^d$. Then, let $\psi \in W^{2,\infty}(\Omega)$ be a not identically vanishing real-valued function satisfying the following conditions

$$\psi|_{\partial\Omega} = \partial_{\boldsymbol{\nu}}\psi|_{\partial\Omega} = 0, \quad \triangle\psi - |\nabla\psi|^2 - \boldsymbol{B}^1\cdot\nabla\psi \ge -c^1, \tag{15}$$

and let $(\rho^2, \boldsymbol{B}^2, c^2)$ be given by

$$\rho^2 = \rho^1, \quad \boldsymbol{B}^2 = \boldsymbol{B}^1 + 2\nabla\psi, \quad c^2 = \triangle\psi - |\nabla\psi|^2 - \boldsymbol{B}^1\cdot\nabla\psi + c^1. \tag{16}$$

Note that in particular condition (15) implies that $\rho^2, c^2 \in L^\infty(\Omega)$ satisfy (1). In view of (15), for the solution u^j $(j=1,2)$ of (2) with any suitable Dirichlet data Φ and $(\rho, \boldsymbol{B}, a, c) = (\rho^j, \boldsymbol{B}^j, 1, c^j)$, one can verify that the identity

$$u^2(t,\boldsymbol{x}) = \mathrm{e}^{\psi(\boldsymbol{x})}u^1(t,\boldsymbol{x}), \quad (t,\boldsymbol{x}) \in (0,T)\times\Omega$$

holds true. Then, applying again (15), we deduce that

$$\partial_{\boldsymbol{\nu}}u^1(t,\boldsymbol{x}) = \partial_{\boldsymbol{\nu}}u^2(t,\boldsymbol{x}), \quad (t,\boldsymbol{x}) \in (0,T)\times\partial\Omega$$

but $\boldsymbol{B}^1 \ne \boldsymbol{B}^2$ since $\nabla\psi \not\equiv \boldsymbol{0}$. Therefore, for $a \equiv 1$, the boundary measurement of Theorem 2.3 is invariant by the gauge transformation

$$(\rho, \boldsymbol{B}, c) \longmapsto (\rho, \boldsymbol{B} + 2\nabla\psi, \triangle\psi - |\nabla\psi|^2 - \boldsymbol{B}\cdot\nabla\psi + c)$$

parameterized by a function $\psi \in W^{2,\infty}(\Omega)$ satisfying (15) with $c^1 = c$. Note that when $c^1 = c^2$, conditions (15)–(16) imply that $\psi \in W^{2,\infty}(\Omega)$ solves the boundary value problem

$$\begin{cases}-\triangle\psi + \boldsymbol{B}^3\cdot\nabla\psi = 0 & \text{in } \Omega,\\ \psi = 0 & \text{on } \partial\Omega\end{cases}$$

with $\boldsymbol{B}^3 = \boldsymbol{B}^1 + \nabla\psi \in (W^{1,\infty}(\Omega))^d$. Then the uniqueness of this boundary value problem (see e.g. [19, Theorem 8.3]) implies that $\psi = 0$. Therefore, the obstruction given by the gauge invariance (16) does not hold when $c^1 = c^2$. This explains why it is possible to prove the simultaneous recovery of $(\rho, \boldsymbol{B})$ as stated in Theorem 2.3.

The second example of the obstruction in the simultaneous recovery of three coefficients that we can mention is given by the simultaneous recovery of the set of coefficients (ρ, a, c) when $\boldsymbol{B} \equiv 0$. Namely, fix $\alpha \in (0, 2)$ and let (ρ^1, a^1, c^1) be defined in a similar way to that in Theorem 2.2. For any positive function $\kappa \in C^2(\overline{\Omega}) \setminus \{1\}$ satisfying

$$\kappa|_{\partial\Omega} = 1, \quad \partial_{\boldsymbol{\nu}}\kappa|_{\partial\Omega} = 0, \quad -\kappa^{-1}\mathrm{div}(a^1\nabla\kappa) \geq -c^1, \tag{17}$$

we assume that ρ^2, a^2, c^2 are given by

$$\rho^2 = \kappa^2\rho^1, \quad a^2 = \kappa^2 a^1, \quad c^2 = c^1\kappa^2 - \kappa\,\mathrm{div}(a^1\nabla\kappa).$$

Then, for the solution u^j $(j = 1, 2)$ of (2) with $(\rho, \boldsymbol{B}, a, c) = (\rho^j, 0, a^j, c^j)$, one can verify that $u^1 = \kappa u^2$. Since $\kappa \not\equiv 1$, this means that

$$a^1(\boldsymbol{x})\partial_{\boldsymbol{\nu}}u^1(t, \boldsymbol{x}) = a^2(\boldsymbol{x})\partial_{\boldsymbol{\nu}}u^2(t, \boldsymbol{x}), \quad (t, \boldsymbol{x}) \in (0, T) \times \partial\Omega$$

but $(\rho^1, a^1, c^1) \neq (\rho^2, a^2, c^2)$. Therefore, the boundary measurements are invariant under the gauge transformation

$$(\rho, a, c) \longmapsto (\kappa^2\rho, \kappa^2 a, c\,\kappa^2 - \kappa\,\mathrm{div}(a\nabla\kappa))$$

parameterized by a positive function $\kappa \in C^\infty(\overline{\Omega})$ satisfying (17) with $c^1 = c$. Again, this gauge invariance fails as long as one of the three coefficients ρ, a, c is fixed. This explains why the recovery of two out of the three coefficients ρ, a, c is the best result that one can expect for this problem.

2.3 Comments about our results

To the best of our knowledge, Theorems 2.2 and 2.3 are the first results on the recovery of coefficients appearing in a fractional diffusion-wave equation in a general bounded domain $\Omega \subset \mathbb{R}^d$ with $d \geq 2$ from a single boundary measurement. For existing literature using other types of observation data with spatial dimensions $d \geq 2$, we refer to [26, 27, 32] for an infinite number of boundary measurements, [21] for interior excitation and observation, and [29] for a single measurement in a cylindrical domain. In particular, Theorem 2.3 seems to be the first result of unique recovery of a convection term appearing in a fractional diffusion-wave equation.

Our approach is based on a special choice (7) of the boundary input Φ inspired by [3, 13] and suitable time-analyticity properties of the solutions to (2) and (3). More precisely, the key ingredient in our approach comes from the results in Propositions 3.1 and 3.2 that, for $k \in \mathbb{N}$ and $\varepsilon_k \in (0, (t_{2k} - t_{2k-1})/2)$, the restriction of the solutions of (18) and (20) stated later to $(t_{2k-1} + \varepsilon_k, \infty)$ are analytic in time as functions taking values in $H^{2\gamma}(\Omega)$ for some $\gamma \in (3/4, 1]$. In the case of interior excitation and observation considered by [21], one can deduce these results by applying properties of time-analyticity of solutions to fractional diffusion equations with time independent coefficients and source terms compactly supported in time. Similar results can be deduced in the case $\alpha = 1$ and non-homogeneous boundary condition by using a classical lifting argument. However, for $\alpha \neq 1$, due to the presence of the nonlocal operator ∂_t^α, this approach becomes more difficult. Instead, we use a new representation of the solution to (18) which, combined with some decay properties of the Mittag-Leffler functions, allows us to prove an analytic extension in time of the solution to (18) into a conical neighborhood of $(t_{2k-1} + \varepsilon_k, \infty)$. For (20), we employ Proposition 3.1 to build a sequence of analytic functions on a conical neighborhood of $(t_{2k-1} + \varepsilon_k, \infty)$ taking values in $H^{2\gamma}(\Omega)$, and then show its convergence to an extension of the solution to (20). Once these results are proved, we complete the proof of Theorems 2.2 and 2.3 by transforming our inverse problem into an inverse boundary value problem stated for a family of elliptic equations. Our approach not only simplifies and extends the result of [3, 13] to fractional diffusion-wave equations, but also improves the result in [3] even for $\alpha = 1$ since Theorem 2.2 is stated with partial boundary measurements and for more general type of coefficients.

In contrast to [3, 13], the result of Theorem 2.2 is not based on recovering the coefficients appearing in elliptic equations from the associated Dirichlet-to-Neumann map. Instead, we prove Theorem 2.2 by applying some inverse spectral results of [8]. Namely, in a similar way to [26], we prove the recovery of the boundary spectral data associated with the elliptic operator appearing in (2). This approach allows us to consider a general class of coefficients as well as the sub-boundaries $\Gamma_{\rm in}, \Gamma_{\rm out}$ subjected only to the conditions (5). In the framework of a smooth Riemannian manifold and smooth coefficients, we even prove in Corollary 2.4 that the condition on $\Gamma_{\rm in}$ and $\Gamma_{\rm out}$ can be reduced to another type of condition $\Gamma_{\rm in} = \Gamma_{\rm out}$ for the recovery of coefficients as well as the Riemannian manifold up to an isometry. Indeed, similarly to [26, Sect. 5, pp. 1165–1166], we prove Corollary 2.4 by applying [23] based on the boundary control method where the condition $\Gamma'_{\rm in} \cup \Gamma_{\rm out} = \partial\mathcal{M}$ is not required. In Corollaries 2.6 and 2.7, we further consider the recovery of a manifold and some local recovery of coefficients when $\Gamma_{\rm in} \cap \Gamma_{\rm out} = \emptyset$. The results of Corollaries 2.4, 2.6 and 2.7 are based on the strategy of Theorem 2.2, combined with the results of [23, 25, 31] based on the boundary control method initiated by the works [5, 6].

Let us remark that we allow some freedom in the choice of several parameters appearing in the input Φ taking the form (7). Namely, the values $\{p_k\}_{k=0}^\infty$ can be arbitrary chosen in $[0, +\infty)$ and the sequence $\{\psi_k\}_{k\in\mathbb{N}} \subset C^\infty(\mathbb{R}_+; [0, +\infty))$ are only subjected to (6) and a not identically vanishing condition. This allows to extend the class of input under consideration in [3, 13] which might be useful in the context of some experiments.

Next, we observe that the results of Theorem 2.2 and Corollaries 2.4, 2.6 and 2.7 are similar to that in [26] with different type of measurements. More precisely, [26] used an infinite number of measurements, whereas in Theorem 2.2 and Corollaries 2.4, 2.6 and 2.7 we state our results with a single measurement. However, the measurements in [26] are taken at one fix time, while the measurements in Theorem 2.2 and Corollaries 2.4, 2.6 and 2.7 are taken in a time interval $(0, T_0)$. In such a sense, Theorem 2.2 and Corollaries 2.4, 2.6 and 2.7 can be regarded as a supplementation to that in [26] because the amounts of data information are essentially the same.

Finally, let us remark that by applying [12, 14], without any difficulty we can extend Theorem 2.3 to the case $d = 2$. However, since the result in that case will require a different definition of the input Φ, we do not consider it in the present paper.

3 Time-analyticity of solutions

In this section, we investigate the analyticity of the solutions to (2) and (3) in time. Without loss of generality, we only deal with the Euclidean case (2) and basically assume (1) for the coefficients (ρ, a, c). To this end, for any $k \in \mathbb{N}$ we fix $\varepsilon_k \in (0, (t_{2k} - t_{2k-1})/2)$ and define a conical domain $D_{k,\theta} := \{t_{2k-1} + \varepsilon_k + r\,\mathrm{e}^{\mathrm{i}\beta} \mid \beta \in (-\theta, \theta)\} \subset \mathbb{C}$ with some $\theta \in (0, \pi \min(\frac{1}{\alpha} - \frac{1}{2}, \frac{1}{2}))$. For better understanding of the readers, we illustrate a typical choice of $D_{k,\theta}$ together with the graph of $\psi_k(t)$ in Figure 1.

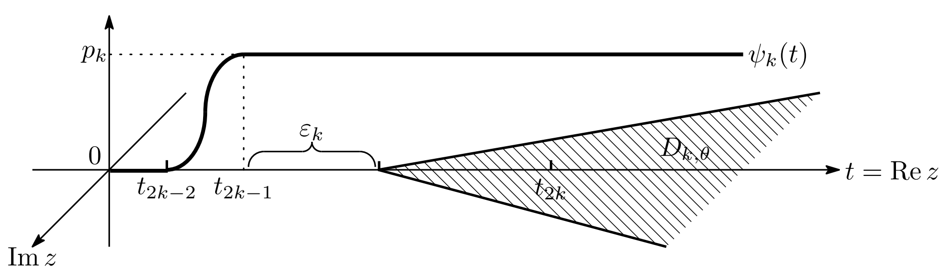

Figure 1: An illustration of the conical domain $D_{k,\varepsilon}$ and the function $\psi_k(t)$.

Due to the technical difference, we first treat the non-advection case $\boldsymbol{B} = \boldsymbol{0}$ of (2) in Sect. 3.1, and then proceed to the general situation in Sect. 3.2.

3.1 The case of $B = 0$

Let $k \in \mathbb{N}$. We consider the initial-boundary value problem

$$\begin{cases} \rho\,\partial_t^\alpha u_k - \mathrm{div}(a\nabla u_k) + c\,u_k = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ \partial_t^m u_k = 0\ (m = 0, \lceil\alpha\rceil - 1) & \text{in } \{0\} \times \Omega, \\ u_k = b_k\,\chi\,\psi_k\,\eta_k & \text{on } \mathbb{R}_+ \times \partial\Omega. \end{cases} \tag{18}$$

Then we state the following intermediate result.

Proposition 3.1. *The solution of* (18) *restricted to* $(t_{2k-1}+\varepsilon_k,\infty)\times\Omega$ *can be extended uniquely to a function* $\widetilde u_k\in C^1(\overline{D_{k,\theta}};H^2(\Omega))\cap C^\omega(D_{k,\theta};H^2(\Omega))$.

Proof. We denote by $\mathcal A$ the operator defined by

$$\mathcal A f:=\frac{-\mathrm{div}(a\nabla f)+c\,f}{\rho},\quad f\in\mathcal D(\mathcal A)$$

with its domain $\mathcal D(\mathcal A)=\{f\in H_0^1(\Omega)\mid \mathrm{div}(a\nabla f)\in L^2(\Omega)\}$ acting on $L^2(\Omega;\rho\,\mathrm d\boldsymbol x)$. It is well known that such an operator is self-adjoint with a spectrum consisting of the non-decreasing sequence of strictly positive eigenvalues $\{\lambda_\ell\}_{\ell\in\mathbb N}$ and an associated orthonormal basis of eigenfunctions $\{\varphi_\ell\}_{\ell\in\mathbb N}$. Then, fixing $\ell\ge1$, taking the scalar product of (18) with φ_ℓ and integrating by parts, we deduce that $u_{k,\ell}(t):=(u_k(t,\,\cdot\,),\rho\,\varphi_\ell)$ solves the fractional ordinary differential equation

$$\begin{cases}\partial_t^\alpha u_{k,\ell}(t)+\lambda_\ell u_{k,\ell}(t)=-b_k\,\psi_k(t)\langle\chi\,\eta_k,a\,\partial_{\boldsymbol\nu}\varphi_\ell\rangle, & t>0,\\ \partial_t^m u_{k,\ell}(0)=0\ (m=0,\lceil\alpha\rceil-1).\end{cases}$$

It follows that

$$u_{k,\ell}(t)=-b_k\langle\chi\,\eta_k,a\,\partial_{\boldsymbol\nu}\varphi_\ell\rangle\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_\ell(t-s)^\alpha)\psi_k(s)\,\mathrm ds.$$

Using the fact that $\psi_k(0)=0$, integrating by parts and applying [43, Lemma 3.2] yield

$$u_{k,\ell}(t)=\frac{b_k\langle\chi\,\eta_k,a\,\partial_{\boldsymbol\nu}\varphi_\ell\rangle}{\lambda_\ell}\left(-\psi_k(t)+\int_0^t E_{\alpha,1}(-\lambda_\ell(t-s)^\alpha)\psi_k'(s)\,\mathrm ds\right).$$

Setting

$$v_k(t,\,\cdot\,):=-b_k\,\psi_k(t)\sum_{\ell=1}^\infty\frac{\langle\chi\,\eta_k,a\,\partial_{\boldsymbol\nu}\varphi_\ell\rangle}{\lambda_\ell}\varphi_\ell,$$

$$w_k(t,\,\cdot\,):=b_k\sum_{\ell=1}^\infty\frac{\langle\chi\,\eta_k,a\,\partial_{\boldsymbol\nu}\varphi_\ell\rangle}{\lambda_\ell}\left(\int_0^t E_{\alpha,1}(-\lambda_\ell(t-s)^\alpha)\psi_k'(s)\,\mathrm ds\right)\varphi_\ell$$

for $t>0$, we see $u_k=v_k+w_k$. Therefore, it suffices to prove that the restrictions of v_k,w_k to $(t_{2k-1}+\varepsilon_k,\infty)\times\Omega$ can be extended to the functions $\widetilde v_k,\widetilde w_k\in C^1(\overline{D_{k,\theta}};H^2(\Omega))\cap C^\omega(D_{k,\theta};H^2(\Omega))$. For v_k, we claim that $v_k=b_k\,\psi_k\,G_k$, where G_k solves

$$\begin{cases}-\mathrm{div}(a\nabla G_k)+c\,G_k=0 & \text{in }\Omega,\\ G_k=\chi\,\eta_k & \text{on }\partial\Omega.\end{cases}$$

To see this, it is enough to take the scalar product of G_k with φ_ℓ and integrate by parts to find

$$\begin{aligned}\lambda_\ell(G_k,\rho\,\varphi_\ell)&=(G_k,-\mathrm{div}(a\nabla\varphi_\ell)+c\,\varphi_\ell)\\&=-\langle\chi\,\eta_k,a\,\partial_{\boldsymbol\nu}\varphi_\ell\rangle+(-\mathrm{div}(a\nabla G_k)+c\,G_k,\varphi_\ell)\\&=-\langle\chi\,\eta_k,a\,\partial_{\boldsymbol\nu}\varphi_\ell\rangle.\end{aligned}$$

Thus, using the fact that Ω is a $C^{1,1}$ domain and $\eta_k\in H^{3/2}(\partial\Omega)$, we deduce $G_k\in H^2(\Omega)$ and $\|G_k\|_{H^2(\Omega)}\le C\|\eta_k\|_{H^{3/2}(\partial\Omega)}$ (see e.g. [20, Theorem 2.2.2.3]). Moreover, from the definition of ψ_k, we deduce that

$$v_k=b_k\,p_k\,G_k\quad\text{in }[t_{2k-1},\infty)\times\Omega.$$

This clearly proves that the restriction of v_k to $(t_{2k-1}+\varepsilon_k,\infty)\times\Omega$ can be extended uniquely to $\widetilde v_k\in C^1(\overline{D_{k,\theta}};H^2(\Omega))\cap C^\omega(D_{k,\theta};H^2(\Omega))$.

Now let us consider w_k. Note first that thanks to [41, Theorem 1.6] as well as the facts that $\psi_k'=0$ on (t_{2k-1},∞) and

$$\sum_{\ell=1}^\infty\left|\frac{\langle\chi\,\eta_k,a\,\partial_{\boldsymbol\nu}\varphi_\ell\rangle}{\lambda_\ell}\right|^2=\|G_k\|_{L^2(\Omega)}^2<\infty,\tag{19}$$

we have $w_k \in C([0,\infty);L^2(\Omega))$ and

$$w_k(t,\,\cdot\,) := b_k \sum_{\ell=1}^{\infty} \frac{\langle \chi\,\eta_k, a\,\partial_{\boldsymbol\nu}\varphi_\ell\rangle}{\lambda_\ell}\left(\int_0^{t_{2k-1}} E_{\alpha,1}(-\lambda_\ell(t-s)^\alpha)\psi_k'(s)\,\mathrm{d}s\right)\varphi_\ell, \quad t\in(t_{2k-1},\infty).$$

For any $\ell \ge 1$, we introduce

$$\left.\begin{aligned} h_\ell(z) &:= \frac{b_k\langle \chi\,\eta_k, a\,\partial_{\boldsymbol\nu}\varphi_\ell\rangle}{\lambda_\ell}\int_0^{t_{2k-1}} E_{\alpha,1}(-\lambda_\ell(z-s)^\alpha)\psi_k'(s)\,\mathrm{d}s,\\ w_{k,N}(z) &:= \sum_{\ell=1}^{N} h_\ell(z)\varphi_\ell, \end{aligned}\right. \qquad z\in D_{k,\theta}.$$

Let $z_* \in \overline{D_{k,\theta}}$ and K be a compact neighborhood of z_* with respect to the topology induced by $\overline{D_{k,\theta}}$. It is clear that

$$\mathrm{Re}(z-s) \ge t_{2k-1}+\varepsilon_k - t_{2k-1} = \varepsilon_k, \quad z\in K,\ s\in[0,t_{2k-1}].$$

Thus, for all $s\in[0,t_{2k-1}]$, the function $z\longmapsto (z-s)^\alpha$ is holomorphic in K and we deduce that $w_{k,N}$ is a holomorphic function in K taking values in $\mathcal{D}(\mathcal{A})$. On the other hand, we have

$$-(z-s)^\alpha \in \{r\,\mathrm{e}^{\mathrm{i}\beta} \mid r>\varepsilon_k^\alpha,\ \beta\in(\pi-\alpha\theta,\pi+\alpha\theta)\}, \quad z\in K,\ s\in[0,t_{2k-1})$$

and $\alpha\theta\in(0,\pi-\frac{\pi\alpha}{2})$. Therefore, applying [41, Theorem 1.6], we deduce that

$$\begin{aligned} |h_\ell(z)| &\le \frac{|\langle \chi\,\eta_k, a\,\partial_{\boldsymbol\nu}\varphi_\ell\rangle|}{\lambda_\ell}\int_0^{t_{2k-1}} \frac{C|\psi_k'(s)|}{1+\lambda_\ell|z-s|^\alpha}\,\mathrm{d}s \le \frac{|\langle \chi\,\eta_k, a\,\partial_{\boldsymbol\nu}\varphi_\ell\rangle| C\,t_{2k-1}\|\psi_k\|_{W^{1,\infty}(\mathbb{R})}}{\lambda_\ell(1+\varepsilon_k^\alpha\lambda_\ell)}\\ &\le C_k\lambda_\ell^{-2}|\langle \chi\,\eta_k, a\,\partial_{\boldsymbol\nu}\varphi_\ell\rangle|, \quad z\in K,\ \ell\ge 1, \end{aligned}$$

where C_k is a constant independent of z and ℓ. Thus, applying (19) yields

$$\sup_{z\in K}\sum_{\ell=1}^{\infty}|\lambda_\ell h_\ell(z)|^2 \le C_k\sum_{\ell=1}^{\infty}\left|\frac{\langle \chi\,\eta_k, a\,\partial_{\boldsymbol\nu}\varphi_\ell\rangle}{\lambda_\ell}\right|^2 < \infty.$$

This proves that $w_{k,N}$ converges to the function

$$\widetilde{w}_k(z) = \sum_{\ell=1}^{\infty} h_\ell(z)\varphi_\ell$$

uniformly with respect to $z\in K$ as a function taking values in $\mathcal{D}(\mathcal{A})$. Since z_* is arbitrarily chosen in $\overline{D_{k,\theta}}$, we deduce that $\widetilde{w}_k \in C^1(\overline{D_{k,\theta}}\,;\mathcal{D}(\mathcal{A}))\cap C^\omega(D_{k,\theta};\mathcal{D}(\mathcal{A}))$. On the other hand, since Ω is $C^{1,1}$, by the regularity of the operator $\mathcal{A}$ (see [20, Theorem 2.2.2.3]), we deduce that $\mathcal{D}(\mathcal{A})$ is embedded continuously into $H^2(\Omega)$ and $\widetilde{w}_k \in C^1(\overline{D_{k,\theta}}\,;H^2(\Omega))\cap C^\omega(D_{k,\theta};H^2(\Omega))$. Combining this with the fact that

$$w_k(t) = \widetilde{w}_k(t), \quad t\in(t_{2k-1}+\varepsilon_k,\infty),$$

we deduce that $\widetilde{u}_k = \widetilde{v}_k+\widetilde{w}_k \in C^1(\overline{D_{k,\theta}}\,;H^2(\Omega))\cap C^\omega(D_{k,\theta};H^2(\Omega))$ is the unique holomorphic extension of u_k restricted to $(t_{2k-1}+\varepsilon_k,\infty)$. □

3.2 The case of $B\neq 0$

In this subsection, we restrict $a=1$, $c=0$ in (2) and assume $(\rho,\boldsymbol{B})\in(L^\infty(\Omega))^{d+1}$ with ρ satisfying (1). Parallel to the previous subsection, for $k\in\mathbb{N}$ we consider the initial-boundary value problem

$$\begin{cases} \rho\,\partial_t^\alpha v_k - \triangle v_k = -\boldsymbol{B}\cdot\nabla v_k & \text{in } \mathbb{R}_+\times\Omega,\\ \partial_t^m v_k = 0\ (m=0,\lceil\alpha\rceil-1) & \text{in } \{0\}\times\Omega,\\ v_k = b_k\,\chi\,\psi_k\,\eta_k & \text{on } \mathbb{R}_+\times\partial\Omega. \end{cases} \tag{20}$$

Regarding the advection term as a new source, we see that (20) admits a unique solution $v_k \in C^1([0,\infty); H^{2\gamma}(\Omega))$ ($s \in (3/4,1)$) taking the form of

$$v_k(t,\,\cdot\,) = u_k(t,\,\cdot\,) + \sum_{\ell=1}^{\infty} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_\ell (t-s)^\alpha)(\boldsymbol{B}\cdot\nabla v_k(s,\,\cdot\,),\varphi_\ell)\,\mathrm{d}s\,\varphi_\ell,$$

where u_k is the solution of (18). Moreover, for all $T>0$, one can prove the following estimate

$$\|v_k(t,\,\cdot\,)\|_{H^{2\gamma}(\Omega)} \le C\|\eta_k\|_{H^{3/2}(\partial\Omega)}, \quad 0<t\le T.$$

We will show that the restriction of v_k to $(t_{2k-1}+\varepsilon_k,\infty)\times\Omega$ admits a holomorphic extension as a function taking its value in $H^{2\gamma}(\Omega)$.

Proposition 3.2. *Let $s\in(3/4,1)$. Then the solution of* (20) *restricted to $(t_{2k-1}+\varepsilon_k,\infty)\times\Omega$ can be extended to $\widetilde{v}_k \in C^1(\overline{D_{k,\theta}}\,; H^{2\gamma}(\Omega)) \cap C^\omega(D_{k,\theta}; H^{2\gamma}(\Omega))$.*

Proof. Let us fix $v_k^0 = 0$ and, for $n\in\mathbb{N}$ and $z\in\overline{D_{k,\theta}}$, define

$$\begin{aligned} v_k^n(z,\,\cdot\,) :=\ & \widetilde{u}_k(z,\,\cdot\,) + \widetilde{y}_k(z,\,\cdot\,) \\ & + \sum_{\ell=1}^{\infty}\left(\int_0^{z-t_{2k-1}-\varepsilon_k} \zeta^{\alpha-1} E_{\alpha,\alpha}(-\lambda_\ell\zeta^\alpha)\left(\boldsymbol{B}\cdot\nabla v_k^{n-1}(z-\zeta,\,\cdot\,),\varphi_\ell\right)\mathrm{d}\zeta\right)\varphi_\ell, \end{aligned} \tag{21}$$

where

$$\widetilde{y}_k(z,\,\cdot\,) = \sum_{\ell=1}^{\infty}\left(\int_0^{t_{2k-1}+\varepsilon_k} (z-\zeta)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_\ell (z-\zeta)^\alpha)(\boldsymbol{B}\cdot\nabla v_k(\zeta,\,\cdot\,),\varphi_\ell)\,\mathrm{d}\zeta\right)\varphi_\ell.$$

We divide the proof into three steps. We start by proving that $v_k^n \in C^1(\overline{D_{k,\theta}}\,; H^{2\gamma}(\Omega)) \cap C^\omega(D_{k,\theta}; H^{2\gamma}(\Omega))$ for all $n\in\mathbb{N}$. Since we have $\widetilde{u}_k \in C^1(\overline{D_{k,\theta}}\,; H^{2\gamma}(\Omega)) \cap C^\omega(D_{k,\theta}; H^{2\gamma}(\Omega))$ in view of Proposition 3.1, it suffices to show that $\widetilde{y}_k \in C^1(\overline{D_{k,\theta}}\,; H^{2\gamma}(\Omega)) \cap C^\omega(D_{k,\theta}; H^{2\gamma}(\Omega))$. Once this is proved, we can complete the proof by showing that $\{v_k^n\}_{n\in\mathbb{N}}$ converges uniformly to $\widetilde{v}_k$ on any compact subset of $\overline{D_{k,\theta}}$, which coincides with v_k in $(t_{2k-1}+\varepsilon_k,\infty)$.

Step 1 We prove that $\widetilde{y}_k \in C^1(\overline{D_{k,\theta}}\,; H^{2\gamma}(\Omega)) \cap C^\omega(D_{k,\theta}; H^{2\gamma}(\Omega))$. For this purpose, we fix $\delta\in(0,t_{2k-1}+\varepsilon_k)$ and consider

$$\widetilde{y}_{k,\delta}(z,\,\cdot\,) = \sum_{\ell=1}^{\infty}\left(\int_0^{t_{2k-1}+\varepsilon_k-\delta} (z-\zeta)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_\ell (z-\zeta)^\alpha)(\boldsymbol{B}\cdot\nabla v_k(\zeta,\,\cdot\,),\varphi_\ell)\,\mathrm{d}\zeta\right)\varphi_\ell, \quad z\in\overline{D_{k,\theta}}\,.$$

Repeating the arguments used in the proof of Proposition 3.1, one can verify that $\widetilde{y}_{k,\delta} \in C^1(\overline{D_{k,\theta}}\,; H^{2\gamma}(\Omega)) \cap C^\omega(D_{k,\theta}; H^{2\gamma}(\Omega))$. Moreover, for any compact set $K\subset\overline{D_{k,\theta}}$, applying [41, Theorem 1.6], we obtain

$$\begin{aligned} \|(\widetilde{y}_k-\widetilde{y}_{k,\delta})(z,\,\cdot\,)\|_{H^{2\gamma}(\Omega)} &\le C\|(\widetilde{y}_k-\widetilde{y}_{k,\delta})(z,\,\cdot\,)\|_{\mathcal{D}(\mathcal{A}^\gamma)} \\ &\le C\|v_k\|_{L^\infty(0,T;H^1(\Omega))}\left(\int_{t_{2k-1}+\varepsilon_k-\delta}^{t_{2k-1}+\varepsilon_k} \zeta^{\alpha(1-\gamma)-1}\,\mathrm{d}\zeta\right) \\ &\le C\left\{(t_{2k-1}+\varepsilon_k)^{\alpha(1-\gamma)} - (t_{2k-1}+\varepsilon_k-\delta)^{\alpha(1-\gamma)}\right\}, \quad z\in K, \end{aligned}$$

where $C>0$ is a constant independent of z. This proves that $\widetilde{y}_{k,\delta}$ converges to $\widetilde{y}_k$ uniformly for $z\in K$ in $H^{2\gamma}(\Omega)$ as $\delta\to0$. From this result and the fact that $v_k\in C^1([0,\infty);H^{2\gamma}(\Omega))$, we deduce that $\widetilde{y}_k \in C^1(\overline{D_{k,\theta}}\,; H^{2\gamma}(\Omega)) \cap C^\omega(D_{k,\theta}; H^{2\gamma}(\Omega))$.

Step 2 We show by induction that $v_k^n \in C^1(\overline{D_{k,\theta}}\,; H^{2\gamma}(\Omega)) \cap C^\omega(D_{k,\theta}; H^{2\gamma}(\Omega))$ for all $n\in\mathbb{N}$.

It is clear that this property is true for $n=0$. Now suppose that for some $n\in\mathbb{N}$, there holds $v_k^j \in C^1(\overline{D_{k,\theta}}\,; H^{2\gamma}(\Omega)) \cap C^\omega(D_{k,\theta}; H^{2\gamma}(\Omega))$ for $j=0,\dots,n-1$. Consider v_k^n defined on $z\in\overline{D_{k,\theta}}$ by (21) and

$$w_k^n(z) := \sum_{\ell=1}^{\infty}\left(\int_0^{z-t_{2k-1}-\varepsilon_k} \zeta^{\alpha-1} E_{\alpha,\alpha}(-\lambda_\ell\zeta^\alpha)\left(\boldsymbol{B}\cdot\nabla v_k^{n-1}(z-\zeta,\,\cdot\,),\varphi_\ell\right)\mathrm{d}\zeta\right)\varphi_\ell.$$

According to the above discussion, one can complete the proof by showing that $w_k^n \in C^1(\overline{D_{k,\theta}}\,; H^{2\gamma}(\Omega)) \cap C^\omega(D_{k,\theta}; H^{2\gamma}(\Omega))$. To this end, let us consider

$$r_\ell^n(z) := \int_0^{z-t_{2k-1}-\varepsilon_k} \zeta^{\alpha-1} E_{\alpha,\alpha}(-\lambda_\ell \zeta^\alpha) \left(\boldsymbol{B} \cdot \nabla v_k^{n-1}(z-\zeta,\,\cdot\,), \varphi_\ell\right) \mathrm{d}\zeta, \quad \ell \in \mathbb{N}.$$

Since $v_k^{n-1} \in C^1(\overline{D_{k,\theta}}\,; H^{2\gamma}(\Omega))$, one can easily verify that $r_\ell^n \in C^1(\overline{D_{k,\theta}})$. Moreover, for any $z \in D_{k,\theta}$ and $\tau \in \mathbb{C}$ such that $|\tau|$ is sufficiently small, we have

$$\frac{r_\ell^n(z+\tau) - r_\ell^n(z)}{\tau} =: \mathrm{I} + \mathrm{II},$$

where

$$\begin{aligned}
\mathrm{I} &:= \frac{1}{\tau} \int_0^{z+\tau-t_{2k-1}-\varepsilon_k} \zeta^{\alpha-1} E_{\alpha,\alpha}(-\lambda_\ell \zeta^\alpha) \left(\boldsymbol{B} \cdot \nabla v_k^{n-1}(z-\zeta,\,\cdot\,), \varphi_\ell\right) \mathrm{d}\zeta \\
&\quad - \frac{1}{\tau} \int_0^{z-t_{2k-1}-\varepsilon_k} \zeta^{\alpha-1} E_{\alpha,\alpha}(-\lambda_\ell \zeta^\alpha) \left(\boldsymbol{B} \cdot \nabla v_k^{n-1}(z-\zeta,\,\cdot\,), \varphi_\ell\right) \mathrm{d}\zeta \\
\mathrm{II} &:= \int_0^{z+\tau-t_{2k-1}-\varepsilon_k} \zeta^{\alpha-1} E_{\alpha,\alpha}(-\lambda_\ell \zeta^\alpha) \left(\boldsymbol{B} \cdot \nabla \left(\frac{v_k^{n-1}(z+\tau-\zeta,\,\cdot\,) - v_k^{n-1}(z-\zeta,\,\cdot\,)}{\tau}\right), \varphi_\ell\right) \mathrm{d}\zeta.
\end{aligned}$$

Fix $\delta \in (0, \operatorname{Re} z - t_{2k-1} - \varepsilon_k)$. Since $v_k^{n-1} \in C^\omega(D_{k,\theta}; H^{2\gamma}(\Omega))$, one can find a neighborhood $\mathcal{O}$ of $[\delta, z - t_{2k-1} - \varepsilon_k] \cup [\delta, z + \tau - t_{2k-1} - \varepsilon_k]$ such that

$$\zeta \longmapsto \zeta^{\alpha-1} E_{\alpha,\alpha}(-\lambda_\ell \zeta^\alpha)(\boldsymbol{B} \cdot \nabla v_k^{n-1}(z-\zeta,\,\cdot\,), \varphi_\ell) \in C^\omega(\mathcal{O}; \mathbb{C}).$$

Therefore, we have

$$\begin{aligned}
&\int_\delta^{z+\tau-t_{2k-1}-\varepsilon_k} \zeta^{\alpha-1} E_{\alpha,\alpha}(-\lambda_\ell \zeta^\alpha) \left(\boldsymbol{B} \cdot \nabla v_k^{n-1}(z-\zeta,\,\cdot\,), \varphi_\ell\right) \mathrm{d}\zeta \\
&\quad - \int_\delta^{z-t_{2k-1}-\varepsilon_k} \zeta^{\alpha-1} E_{\alpha,\alpha}(-\lambda_\ell \zeta^\alpha) \left(\boldsymbol{B} \cdot \nabla v_k^{n-1}(z-\zeta,\,\cdot\,), \varphi_\ell\right) \mathrm{d}\zeta \\
&= \int_{z-t_{2k-1}-\varepsilon_k}^{z+\tau-t_{2k-1}-\varepsilon_k} \zeta^{\alpha-1} E_{\alpha,\alpha}(-\lambda_\ell \zeta^\alpha) \left(\boldsymbol{B} \cdot \nabla v_k^{n-1}(z-\zeta,\,\cdot\,), \varphi_\ell\right) \mathrm{d}\zeta.
\end{aligned}$$

Using the fact that $v_k^{n-1} \in C^1(\overline{D_{k,\theta}}\,; H^{2\gamma}(\Omega))$, we can let $\delta \to 0$ and deduce

$$\mathrm{I} = \frac{1}{\tau} \int_{z-t_{2k-1}-\varepsilon_k}^{z+\tau-t_{2k-1}-\varepsilon_k} \zeta^{\alpha-1} E_{\alpha,\alpha}(-\lambda_\ell \zeta^\alpha) \left(\boldsymbol{B} \cdot \nabla v_k^{n-1}(z-\zeta,\,\cdot\,), \varphi_\ell\right) \mathrm{d}\zeta. \tag{22}$$

Combining this with the fact that $v_k^{n-1} \in C^1(\overline{D_{k,\theta}}\,; H^{2\gamma}(\Omega))$, we deduce that

$$\lim_{\tau \to 0} \mathrm{I} = z^{\alpha-1} E_{\alpha,\alpha}(-\lambda_\ell z^\alpha) \left(\boldsymbol{B} \cdot \nabla v_k^{n-1}(t_{2k-1} + \varepsilon_k,\,\cdot\,), \varphi_\ell\right).$$

In the same way, for $\delta > 0$ small enough, using the fact that $v_k^{n-1} \in C^\omega(D_{k,\theta}; H^{2\gamma}(\Omega))$, we have

$$\begin{aligned}
&\lim_{\tau \to 0} \int_\delta^{z+\tau-t_{2k-1}-\varepsilon_k} \zeta^{\alpha-1} E_{\alpha,\alpha}(-\lambda_\ell \zeta^\alpha) \left(\boldsymbol{B} \cdot \nabla \left(\frac{v_k^{n-1}(z+\tau-\zeta,\,\cdot\,) - v_k^{n-1}(z-\zeta,\,\cdot\,)}{\tau}\right), \varphi_\ell\right) \mathrm{d}\zeta \\
&= \int_\delta^{z-t_{2k-1}-\varepsilon_k} \zeta^{\alpha-1} E_{\alpha,\alpha}(-\lambda_\ell \zeta^\alpha) \left(\boldsymbol{B} \cdot \nabla \partial_z v_k^{n-1}(z-\zeta,\,\cdot\,), \varphi_\ell\right) \mathrm{d}\zeta.
\end{aligned}$$

Combining this with the fact that $v_k^{n-1} \in C^1(\overline{D_{k,\theta}}\,; H^{2\gamma}(\Omega))$ and letting $\delta \to 0$, we obtain

$$\lim_{\tau \to 0} \mathrm{II} = \int_0^{z-t_{2k-1}-\varepsilon_k} \zeta^{\alpha-1} E_{\alpha,\alpha}(-\lambda_\ell \zeta^\alpha) \left(\boldsymbol{B} \cdot \nabla \partial_z v_k^{n-1}(z-\zeta,\,\cdot\,), \varphi_\ell\right) \mathrm{d}\zeta.$$

Combining this with (22), we deduce that $r_\ell^n \in C^1(\overline{D_{k,\theta}}) \cap C^\omega(D_{k,\theta};\mathbb{C})$. In addition, for any $m_1, m_2 \in \mathbb{N}$ satisfying $m_1 < m_2$ and a compact subset K of $D_{k,\theta}$, applying [41, Theorem 1.6], we obtain

$$\begin{aligned}\left\|\sum_{\ell=m_1}^{m_2} r_\ell^n(z)\varphi_\ell\right\|_{H^{2\gamma}(\Omega)}^2 &\le C\left\|\sum_{\ell=m_1}^{m_2} r_\ell^n(z)\varphi_\ell\right\|_{\mathcal{D}(\mathcal{A}^\gamma)}^2 \\ &\le C\sup_{z\in K_1}\sum_{\ell=m_1}^{m_2}\left|\left\langle \boldsymbol{B}\cdot\nabla v_k^{n-1}(z,\,\cdot\,),\varphi_\ell\right\rangle\right|^2\left(\int_0^1 s^{\alpha(1-\gamma)-1}\,\mathrm{d}s\right)^2 \\ &\le C\sup_{z\in K_1}\sum_{\ell=m_1}^{m_2}\left|\left\langle \boldsymbol{B}\cdot\nabla v_k^{n-1}(z,\,\cdot\,),\varphi_\ell\right\rangle\right|^2,\end{aligned}$$

where

$$K_1 := \{z-\zeta \mid z\in K,\ \zeta\in[0, z-t_{2k-1}-\varepsilon_k]\}$$

is a compact subset of $\overline{D_{k,\theta}}$. Combining this with the fact that $v_k^{n-1} \in C^1(\overline{D_{k,\theta}}; H^{2\gamma}(\Omega))$, we deduce that the series $\sum_{\ell=1}^\infty r_\ell^n(z)\varphi_\ell$ converges uniformly with respect to $z\in K$ to w_k^n as functions taking values in $H^{2\gamma}(\Omega)$. This proves that $w_k^n \in C^1(\overline{D_{k,\theta}}; H^{2\gamma}(\Omega)) \cap C^\omega(D_{k,\theta}; H^{2\gamma}(\Omega))$ and by the same way that $v_k^n \in C^1(\overline{D_{k,\theta}}; H^{2\gamma}(\Omega)) \cap C^\omega(D_{k,\theta}; H^{2\gamma}(\Omega))$. By induction, it follows that this property holds true for all $n\in\mathbb{N}$.

Step 3 Now we complete the proof by showing that the sequence $\{v_k^n\}_{n\in\mathbb{N}}$ converges uniformly on any compact set $K\subset\overline{D_{k,\theta}}$ to $\widetilde{v}_k \in C^1(\overline{D_{k,\theta}}; H^{2\gamma}(\Omega)) \cap C^\omega(D_{k,\theta}; H^{2\gamma}(\Omega))$, which coincides with v_k restricted to $(t_{2k-1}+\varepsilon_k,\infty)$. To see this, let us first remark that v_k^n can be rewritten as

$$\begin{aligned}v_k^n(z,\,\cdot\,) &= \widetilde{u}_k(z,\,\cdot\,) + \widetilde{y}_k(z,\,\cdot\,) \\ &\quad + \sum_{\ell=1}^\infty\left(\int_{t_{2k-1}+\varepsilon_k}^z (z-\zeta)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_\ell(z-\zeta)^\alpha)\left(\boldsymbol{B}\cdot\nabla v_k^{n-1}(\zeta,\,\cdot\,),\varphi_\ell\right)\mathrm{d}\zeta\right)\varphi_\ell,\quad z\in\overline{D_{k,\theta}}.\end{aligned}$$

Therefore, combining [41, Theorem 1.6] with the Lebesgue dominate convergence theorem, we see that

$$(v_k^{n+1}-v_k^n)(z,\,\cdot\,) = \int_{t_{2k-1}+\varepsilon_k}^z (z-\zeta)^{\alpha-1}\left(\sum_{\ell=1}^\infty E_{\alpha,\alpha}(-\lambda_\ell(z-\zeta)^\alpha)\left(\boldsymbol{B}\cdot\nabla\left(v_k^n-v_k^{n-1}\right)(\zeta,\,\cdot\,),\varphi_\ell\right)\varphi_\ell\right)\mathrm{d}\zeta.$$

Using this identity, we can obtain the following estimate of $\|v_k^{n+1}-v_k^n\|_{\mathcal{D}(\mathcal{A}^\gamma)}$ by an inductive argument.

Lemma 3.3. *For any $n\in\mathbb{N}$ and any compact subset K of $\overline{D_{k,\theta}}$, we have*

$$\left\|\left(v_k^{n+1}-v_k^n\right)(z,\,\cdot\,)\right\|_{H^{2\gamma}(\Omega)} \le C_K\frac{C^n\|\boldsymbol{B}\|_{(L^\infty(\Omega))^d}^n|z-t_{2k-1}-\varepsilon_k|^{\alpha n(1-\gamma)}}{\Gamma(\alpha n(1-\gamma)+1)},\quad z\in K. \tag{23}$$

For the sake of consistency, we provide the proof of Lemma 3.3 after finishing the proof of Proposition 3.2. In view of Lemma 3.3, it follows that the sequence $\{v_k^n\}_{n=0}^\infty$ converges uniformly on any compact set $K\subset\overline{D_{k,\theta}}$ to $\widetilde{v}_k\in C^1(\overline{D_{k,\theta}}; H^{2\gamma}(\Omega)) \cap C^\omega(D_{k,\theta}; H^{2\gamma}(\Omega))$. Moreover, in light of (21), we have

$$\widetilde{v}_k(z,\,\cdot\,) = \widetilde{u}_k(z,\,\cdot\,) + \widetilde{y}_k(z,\,\cdot\,) + \sum_{\ell=1}^\infty\left(\int_{t_{2k-1}+\varepsilon_k}^z (z-\zeta)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_\ell(z-\zeta)^\alpha)(\boldsymbol{B}\cdot\nabla\widetilde{v}_k(\zeta,\,\cdot\,),\varphi_\ell)\,\mathrm{d}\zeta\right)\varphi_\ell$$

for $z\in\overline{D_{k,\theta}}$. In particular, we find

$$\widetilde{v}_k(t,\,\cdot\,) = u_k(t,\,\cdot\,) + \widetilde{y}_k(t,\,\cdot\,) + \sum_{\ell=1}^\infty\left(\int_{t_{2k-1}+\varepsilon_k}^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_\ell(t-s)^\alpha)(\boldsymbol{B}\cdot\nabla\widetilde{v}_k(s,\,\cdot\,),\varphi_\ell)\,\mathrm{d}s\right)\varphi_\ell$$

for $t>t_{2k-1}+\varepsilon_k$. Combining this with the fact that

$$v_k(t,\,\cdot\,) = u_k(t,\,\cdot\,) + \widetilde{y}_k(t,\,\cdot\,) + \sum_{\ell=1}^\infty\left(\int_{t_{2k-1}+\varepsilon_k}^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_\ell(t-s)^\alpha)(\boldsymbol{B}\cdot\nabla v_k(s,\,\cdot\,),\varphi_\ell)\,\mathrm{d}s\right)\varphi_\ell$$

for $t > t_{2k-1} + \varepsilon_k$ and a solution of the integral equation

$$w(t,\,\cdot\,) = u_k(t,\,\cdot\,) + \widetilde{y}_k(t,\,\cdot\,) + \sum_{\ell=1}^{\infty} \left(\int_{t_{2k-1}+\varepsilon_k}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_\ell (t-s)^\alpha)(\boldsymbol{B}\cdot\nabla w(s,\,\cdot\,), \varphi_\ell)\,\mathrm{d}s \right) \varphi_\ell$$

for $t > t_{2k-1} + \varepsilon_k$ is unique, which can be deduced from arguments similar to [18, Proposition 1], we conclude that

$$\widetilde{v}_k(t,\,\cdot\,) = v_k(t,\,\cdot\,), \quad t > t_{2k-1} + \varepsilon_k.$$

This completes the proof of Proposition 3.2. □

Now we provide the proof of Lemma 3.3.

Proof of Lemma 3.3. Let $z \in D_{k,\theta}$. Using the fact that $z - t_{2k-1} - \varepsilon_k = |z - t_{2k-1} - \varepsilon_k|\,\mathrm{e}^{\mathrm{i}\beta} := r_k \mathrm{e}^{\mathrm{i}\beta}$ with $\beta \in (-\theta, \theta)$, we find

$$\begin{aligned}
&\left\| \left(v_k^{n+1} - v_k^n\right)(z,\,\cdot\,) \right\|_{\mathcal{D}(\mathcal{A}^\gamma)} \\
&\le \left| \int_{t_{2k-1}+\varepsilon_k}^{z} |z-\zeta|^{\alpha-1} \left(\sum_{\ell=1}^{\infty} \lambda_\ell^{2\gamma} \left| E_{\alpha,\alpha}(-\lambda_\ell (z-\zeta)^\alpha) \left(\boldsymbol{B}\cdot\nabla\left(v_k^n - v_k^{n-1}\right)(\zeta,\,\cdot\,), \varphi_\ell\right) \right|^2 \right)^{1/2} \mathrm{d}|\zeta| \right| \\
&\le C \int_0^{r_k} (r_k - \tau)^{\alpha-1} \\
&\quad \times \left(\sum_{\ell=1}^{\infty} \lambda_\ell^{2\gamma} \left| E_{\alpha,\alpha}(-\lambda_\ell (r_k-\tau)^\alpha \mathrm{e}^{\mathrm{i}\alpha\beta}) \left(\boldsymbol{B}\cdot\nabla\left(v_k^n - v_k^{n-1}\right)(t_{2k-1}+\varepsilon_k+\tau\,\mathrm{e}^{\mathrm{i}\beta},\,\cdot\,), \varphi_\ell\right) \right|^2 \right)^{1/2} \mathrm{d}\tau.
\end{aligned}$$

Therefore, applying [41, Theorem 1.6], we get

$$\begin{aligned}
\left\| \left(v_k^{n+1} - v_k^n\right)(z,\,\cdot\,) \right\|_{\mathcal{D}(\mathcal{A}^\gamma)} &\le C\|\boldsymbol{B}\|_{(L^\infty(\Omega))^d} \int_0^{r_k} \frac{(r_k-\tau)^{\alpha(1-\gamma)-1}}{\Gamma(\alpha(1-\gamma))} \\
&\qquad \times \left\| \left(v_k^n - v_k^{n-1}\right)(t_{2k-1}+\varepsilon_k+\tau\,\mathrm{e}^{\mathrm{i}\beta},\,\cdot\,) \right\|_{H^{2\gamma}(\Omega)} \mathrm{d}\tau \\
&\le C\|\boldsymbol{B}\|_{(L^\infty(\Omega))^d} \int_0^{|z-t_{2k-1}-\varepsilon_k|} \frac{(|z-t_{2k-1}-\varepsilon_k|-\tau)^{\alpha(1-\gamma)-1}}{\Gamma(\alpha(1-\gamma))} \\
&\qquad \times \left\| (v_k^n - v_k^{n-1})(t_{2k-1}+\varepsilon_k+\tau\,\mathrm{e}^{\mathrm{i}\beta},\,\cdot\,) \right\|_{H^{2\gamma}(\Omega)} \mathrm{d}\tau
\end{aligned}$$

for $z \in D_{k,\theta}$. Using this estimate and applying some arguments similarly to the proof of [18, Proposition 1], we can prove by iteration that

$$\begin{aligned}
\|v_k^{n+1}(z,\,\cdot\,) - v_k^n(z,\,\cdot\,)\|_{H^{2\gamma}(\Omega)} &\le C^n \|\boldsymbol{B}\|^n_{(L^\infty(\Omega))^d} \int_0^{|z-t_{2k-1}-\varepsilon_k|} \frac{(|z-t_{2k-1}-\varepsilon_k|-\tau)^{n\alpha(1-\gamma)-1}}{\Gamma(n\alpha(1-\gamma))} \\
&\qquad \times \left\| (v_k^1 - v_k^0)(t_{2k-1}+\varepsilon_k+\tau\,\mathrm{e}^{\mathrm{i}\beta},\,\cdot\,) \right\|_{H^{2\gamma}(\Omega)} \mathrm{d}\tau \\
&\le C^n \|\boldsymbol{B}\|^n_{(L^\infty(\Omega))^d} \int_0^{|z-t_{2k-1}-\varepsilon_k|} \frac{(|z-t_{2k-1}-\varepsilon_k|-\tau)^{n\alpha(1-\gamma)-1}}{\Gamma(n\alpha(1-\gamma))} \\
&\qquad \times \left\| (\widetilde{u}_k + \widetilde{y}_k)(t_{2k-1}+\varepsilon_k+\tau\,\mathrm{e}^{\mathrm{i}\beta},\,\cdot\,) \right\|_{H^{2\gamma}(\Omega)} \mathrm{d}\tau.
\end{aligned}$$

Combining this last estimate with the fact that $\widetilde{u}_k, \widetilde{y}_k \in C^1(\overline{D_{k,\theta}}\,; H^{2\gamma}(\Omega))$, we deduce (23). □

4 Proofs of the main results

Proof of Theorem 2.2. We divide the proof into four steps.

Step 1 We start by proving that (10) implies

$$a^1\partial_{\boldsymbol{\nu}} u_k^1 = a^2\partial_{\boldsymbol{\nu}} u_k^2 \quad \text{on } \mathbb{R}_+ \times \Gamma_{\mathrm{out}},\ k \in \mathbb{N}, \tag{24}$$

where u_k^j ($j = 1, 2$, $k \in \mathbb{N}$) is the solution of (18) with $(\rho, a, c) = (\rho^j, a^j, c^j)$ and $j = 1, 2$. We will prove (24) by induction. For $k = 1$, using the properties of the sequence $\{\psi_k\}_{k\in\mathbb{N}}$, we observe that

$$\psi_k = 0 \quad \text{in } (0, t_2),\ \forall\, k \geq 2.$$

Therefore, we have $u_1^j = u_j$ in $(0, t_2) \times \Omega$ ($j = 1, 2$), and the condition (10) implies

$$a^1\partial_{\boldsymbol{\nu}} u_1^1 = a^2\partial_{\boldsymbol{\nu}} u_1^2 \quad \text{on } (0, t_2) \times \Gamma_{\mathrm{out}}. \tag{25}$$

On the other hand, from Proposition 3.1 we know that $u_1^j \in C^\omega((t_1 + \varepsilon_1, \infty); H^2(\Omega))$, $j = 1, 2$. Thus, it follows from the trace theorem that $t \longmapsto \partial_{\boldsymbol{\nu}} u_1^j(t, \cdot)|_{\Gamma_{\mathrm{out}}} \in C^\omega((t_1 + \varepsilon_1, \infty); L^2(\Gamma_{\mathrm{out}}))$ for $j = 1, 2$, and the condition (25) implies (24) for $k = 1$. Now assume that for some $\ell \in \mathbb{N}$, the condition (24) is fulfilled for all $k = 1, 2, \ldots, \ell$. Since

$$\psi_k = 0 \quad \text{in } (0, t_{2\ell+2}),\ \forall\, k \geq \ell + 2,$$

we know that

$$\sum_{k=1}^{\ell+1} u_k^j = u^j \quad \text{in } (0, t_{2\ell+2}) \times \Omega.$$

Therefore, (10) implies

$$\sum_{k=1}^{\ell+1} a^1\partial_{\boldsymbol{\nu}} u_k^1 = \sum_{k=1}^{\ell+1} a^2\partial_{\boldsymbol{\nu}} u_k^2 \quad \text{in } (0, t_{2\ell+2}) \times \Gamma_{\mathrm{out}}.$$

Then, by the induction assumption, we deduce that

$$a^1\partial_{\boldsymbol{\nu}} u_{\ell+1}^1 = a^2\partial_{\boldsymbol{\nu}} u_{\ell+1}^2 \quad \text{in } (0, t_{2\ell+2}) \times \Gamma_{\mathrm{out}}.$$

Applying again Proposition 3.1, we deduce that $t \longmapsto \partial_{\boldsymbol{\nu}} u_{\ell+1}^j(t, \cdot)|_{\Gamma_{\mathrm{out}}} \in C^\omega((t_{2\ell+1} + \varepsilon_\ell, \infty); L^2(\Gamma_{\mathrm{out}}))$ for $j = 1, 2$, and we conclude (24) for $k = \ell + 1$. This proves that (24) holds for all $k \in \mathbb{N}$.

Step 2 For $j = 1, 2$ and $\xi > 0$, consider the boundary value problem

$$\begin{cases} -\mathrm{div}(a^j\nabla U^j(\xi)) + (\xi^\alpha \rho^j + c^j)U^j(\xi) = 0 & \text{in } \Omega, \\ U^j(\xi) = \chi\, h & \text{on } \partial\Omega, \end{cases} \qquad h \in H^{3/2}(\partial\Omega). \tag{26}$$

We associate this problem with the Dirichlet-to-Neumann map

$$\Lambda^j(\xi) : h \longmapsto a^j\partial_{\boldsymbol{\nu}} U^j(\xi)\big|_{\Gamma_{\mathrm{out}}}, \quad j = 1, 2,\ \xi > 0.$$

In this step, we show that (10) implies

$$\Lambda^1(\xi) = \Lambda^2(\xi), \quad j = 1, 2,\ \xi > 0. \tag{27}$$

For $j = 1, 2$, define the operator $\mathcal{A}^j$ by

$$\mathcal{A}^j w := \frac{-\mathrm{div}(a^j\nabla w) + c^j w}{\rho^j}, \quad w \in \mathcal{D}(\mathcal{A}^j)$$

with the domain $\mathcal{D}(\mathcal{A}^j) = \{w \in H_0^1(\Omega) \mid \mathrm{div}(a^j\nabla w) \in L^2(\Omega)\}$ acting on $L^2(\Omega; \rho^j\, \mathrm{d}\boldsymbol{x})$. Then we associate these operators with the eigensystems $\{(\lambda_\ell^j, \varphi_\ell^j)\}_{\ell\in\mathbb{N}}$. Similarly to the proof of Proposition 3.1, for all $k \in \mathbb{N}$, u_k^j, where u_k^j ($j = 1, 2$, $k \in \mathbb{N}$) is the solution of (18) with $(\rho, a, c) = (\rho^j, a^j, c^j)$, can be decomposed into $u_k^j = v_k^j + w_k^j$ with $v_k^j = -b_k\, \psi_k\, G_k^j$, where G_k^j and w_k^j solve

$$\begin{cases} -\mathrm{div}\left(a^j\nabla G_k^j\right) + c^j G_k^j = 0 & \text{in } \Omega, \\ G_k^j = -\chi\, \eta_k & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases}\rho^j \partial_t^\alpha w_k^j - \operatorname{div}\left(a^j \nabla w_k^j\right) + c^j w_k^j = b_k(\partial_t^\alpha \psi_k) G_k^j & \text{in } \mathbb{R}_+ \times \Omega,\\ \partial_t^m w_k^j = 0 \ (m = 0, \lceil \alpha \rceil - 1) & \text{in } \{0\} \times \Omega,\\ w_k^j = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega,\end{cases}$$

respectively. For $\gamma \in (3/4, 1)$, using [41, Theorem 1.6], we deduce that

$$\begin{aligned}\|w_k^j(t,\,\cdot\,)\|_{H^{2\gamma}(\Omega)}^2 &\le C \|w_k^j(t,\,\cdot\,)\|_{\mathcal{D}((\mathcal{A}^j)^\gamma)}^2\\ &\le C|b_k|^2 \sum_{\ell=1}^\infty \left| \int_0^t \left(\lambda_\ell^j\right)^\gamma (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_\ell^j (t-s)^\alpha\right) \partial_s^\alpha \psi_k(s)\,\mathrm{d}s \right|^2 \left| \left(G_k^j, \rho^j \varphi_\ell^j\right) \right|^2\\ &\le C_k \left(\int_0^t (t-s)^{\alpha(1-\gamma)-1} |\partial_s^\alpha \psi_k(s)|\,\mathrm{d}s \right)^2 \sum_{\ell=1}^\infty \left| \left(G_k^j, \rho^j \varphi_\ell^j\right) \right|^2\\ &\le C_k \|\psi_k\|_{W^{2,\infty}(\mathbb{R})}^2 t^{2(1-\gamma)} \|\rho^j\|_{L^\infty(\Omega)} \|G_k^j\|_{L^2(\Omega)}^2.\end{aligned}$$

This proves that

$$t \longmapsto \mathrm{e}^{-\xi t} w_k^j(t,\,\cdot\,) \in L^1(\mathbb{R}_+; H^{2\gamma}(\Omega)), \quad \xi > 0,$$

and we deduce that

$$t \longmapsto \mathrm{e}^{-\xi t} u_k^j(t,\,\cdot\,) \in L^1(\mathbb{R}_+; H^{2\gamma}(\Omega)), \quad \xi > 0.$$

Using the continuity of $H^{2\gamma}(\Omega) \ni w \longmapsto \partial_{\boldsymbol{\nu}} w|_{\Gamma_{\mathrm{out}}} \in L^2(\Gamma_{\mathrm{out}})$, we deduce that

$$t \longmapsto \mathrm{e}^{-\xi t} \partial_{\boldsymbol{\nu}} u_k^j(t,\,\cdot\,) \in L^1(\mathbb{R}_+; L^2(\Gamma_{\mathrm{out}})), \quad \xi > 0,\ k \in \mathbb{N}.$$

Therefore, applying the Laplace transform $\mathcal{L}$ in time on both sides of (24), we obtain

$$\Lambda^1(\xi) b_k (\mathcal{L}\psi_k)(\xi) \eta_k = \Lambda^2(\xi) b_k (\mathcal{L}\psi_k)(\xi) \eta_k, \quad \xi > 0,\ k \in \mathbb{N}.$$

Applying the fact that $\psi_k \ge 0, \not\equiv 0$, we deduce that $(\mathcal{L}\psi_k)(\xi) > 0$ for all $\xi > 0$. Then, using the fact $b_k > 0$ and the linearity of $\Lambda^j(\xi)$ $(j = 1, 2)$, we get

$$\Lambda^1(\xi) h = \Lambda^2(\xi) h, \quad \xi > 0,\ \forall\, h \in \operatorname{span}\{\eta_k\}_{k \in \mathbb{N}}.$$

Finally, the density of $\operatorname{span}\{\eta_k\}$ in $H^{3/2}(\partial\Omega)$ implies (27).

Step 3 In this step, by $\{\lambda_k^j\}_{k\in\mathbb{N}}$ and $m_k^j \in \mathbb{N}$ we denote the strictly increasing sequence of the eigenvalues of $\mathcal{A}^j$ and the multiplicity of λ_k^j, respectively. For each eigenvalue λ_k^j, we introduce a family $\{\varphi_{k,\ell}^j\}_{\ell=1}^{m_k^j}$ of eigenfunctions of $\mathcal{A}^j$, i.e.,

$$\mathcal{A}^j \varphi_{k,\ell}^j = \lambda_k^j \varphi_{k,\ell}^j, \quad \ell = 1, \ldots, m_k^j,$$

which forms an orthonormal basis in $L^2(\Omega; \rho^j\,\mathrm{d}\boldsymbol{x})$ of the eigenspace of $\mathcal{A}^j$ associated with λ_k^j. We fix also

$$\Theta_k^j(\boldsymbol{x}, \boldsymbol{x}') := a^j(\boldsymbol{x}) a^j(\boldsymbol{x}') \sum_{\ell=1}^{m_k^j} \partial_{\boldsymbol{\nu}} \varphi_{k,\ell}^j(\boldsymbol{x}) \partial_{\boldsymbol{\nu}} \varphi_{k,\ell}^j(\boldsymbol{x}'), \quad \boldsymbol{x}, \boldsymbol{x}' \in \partial\Omega,\ k \in \mathbb{N}.$$

In this step, we show that (10) implies

$$\begin{aligned}&\lambda_k^1 = \lambda_k^2, && k \in \mathbb{N},\\ &\Theta_k^1(\boldsymbol{x}, \boldsymbol{x}') = \Theta_k^2(\boldsymbol{x}, \boldsymbol{x}'), && \boldsymbol{x} \in \Gamma_{\mathrm{out}},\ \boldsymbol{x}' \in \Gamma_{\mathrm{in}}',\ k \in \mathbb{N}.\end{aligned} \tag{28}$$

Taking the scalar product of $\varphi_{k,\ell}^j$ with $U^j(\xi)$ solving (26), one can verify that

$$U^j(\xi) = -\sum_{k=1}^\infty \frac{\sum_{\ell=1}^{m_k^j} \left\langle \chi\, h, a^j \partial_{\boldsymbol{\nu}} \varphi_{k,\ell}^j \right\rangle \varphi_{k,\ell}^j}{\xi^\alpha + \lambda_k^j}.$$

Therefore, differentiating with respect to ξ on both side yields

$$\partial_\xi U^j(\xi)=\alpha\,\xi^{\alpha-1}\sum_{k=1}^\infty\frac{\sum_{\ell=1}^{m_k^j}\left\langle\chi\,h,a^j\partial_{\boldsymbol\nu}\varphi_{k,\ell}^j\right\rangle\varphi_{k,\ell}^j}{(\xi^\alpha+\lambda_k^j)^2}.$$

On the other hand, using the fact that

$$\sum_{k=1}^\infty\left|\frac{\sum_{\ell=1}^{m_k^j}\left\langle\chi\,h,a^j\partial_{\boldsymbol\nu}\varphi_{k,\ell}^j\right\rangle}{\xi^\alpha+\lambda_k^j}\right|^2=\|U^j(\xi)\|_{L^2(\Omega;\rho^j\,\mathrm{d}\boldsymbol{x})}^2<\infty,$$

we deduce that the sequence

$$\sum_{k=1}^\infty\frac{\sum_{\ell=1}^{m_k^j}\left\langle\chi\,h,a^j\partial_{\boldsymbol\nu}\varphi_{k,\ell}^j\right\rangle\varphi_{k,\ell}^j}{(\xi^\alpha+\lambda_k^j)^2}$$

converges in the sense of $\mathcal{D}(\mathcal{A}^j)$. Using the continuous embedding of $\mathcal{D}(\mathcal{A}^j)$ into $H^2(\Omega)$ and the continuity of $H^2(\Omega)\ni w\longmapsto\partial_{\boldsymbol\nu}w|_{\Gamma_{\mathrm{out}}}\in L^2(\Gamma_{\mathrm{out}})$, we deduce that

$$\partial_\xi\partial_{\boldsymbol\nu}U^j(\xi)\big|_{\Gamma_{\mathrm{out}}}=\partial_{\boldsymbol\nu}\partial_\xi U^j(\xi)\big|_{\Gamma_{\mathrm{out}}}=\alpha\,\xi^{\alpha-1}\sum_{k=1}^\infty\frac{\sum_{\ell=1}^{m_k^j}\left\langle\chi\,h,a^j\partial_{\boldsymbol\nu}\varphi_{k,\ell}^j\right\rangle\partial_{\boldsymbol\nu}\varphi_{k,\ell}^j\big|_{\Gamma_{\mathrm{out}}}}{(\xi^\alpha+\lambda_k^j)^2}. \tag{29}$$

On the other hand, in view of (27), we have

$$a^1\partial_{\boldsymbol\nu}U^1(\xi)\big|_{\Gamma_{\mathrm{out}}}=a^2\partial_{\boldsymbol\nu}U^2(\xi)\big|_{\Gamma_{\mathrm{out}}},\quad\xi>0,\ h\in H^{3/2}(\partial\Omega).$$

Differentiating the above identity with respect to ξ and applying (29), we deduce

$$\sum_{k=1}^\infty\frac{\sum_{\ell=1}^{m_k^1}\left\langle\chi\,h,a^1\partial_{\boldsymbol\nu}\varphi_{k,\ell}^1\right\rangle a^1\partial_{\boldsymbol\nu}\varphi_{k,\ell}^1\big|_{\Gamma_{\mathrm{out}}}}{(\xi^\alpha+\lambda_k^1)^2}=\sum_{k=1}^\infty\frac{\sum_{\ell=1}^{m_k^2}\left\langle\chi\,h,a^2\partial_{\boldsymbol\nu}\varphi_{k,\ell}^2\right\rangle a^2\partial_{\boldsymbol\nu}\varphi_{k,\ell}^2\big|_{\Gamma_{\mathrm{out}}}}{(\xi^\alpha+\lambda_k^2)^2} \tag{30}$$

for $h\in H^{3/2}(\partial\Omega)$ and $\xi>0$. Using this identity, we will complete the proof of (28) by applying some arguments borrowed from [26, Theorem 2.2]. Consider $D:=\mathbb{C}\setminus\{-\lambda_k^j\mid j=1,2,\ k\in\mathbb{N}\}$ which is clearly connected. In a similar way to [26, Lemma 4.1] (see also the estimate of [26, Lemma 2.3]), for any compact subset K contained in D, we can prove that the series

$$\sum_{k=1}^\infty\left(\frac{\int_{\partial\Omega}\Theta_k^j(\,\cdot\,,\boldsymbol{x}')\chi(\boldsymbol{x}')h(\boldsymbol{x}')\mathrm{d}\sigma(\boldsymbol{x}')}{(z+\lambda_k^j)^2}\right)=\sum_{k=1}^\infty\frac{\sum_{\ell=1}^{m_k^j}\left\langle\chi\,h,a^j\partial_{\boldsymbol\nu}\varphi_{k,\ell}^j\right\rangle a^j\partial_{\boldsymbol\nu}\varphi_{k,\ell}^j}{(z+\lambda_k^j)^2}$$

converges uniformly with respect to $z\in K$ as a function taking values in $L^2(\Gamma_{\mathrm{out}})$. Thus, since K is arbitrarily chosen in D, the function

$$z\longmapsto\sum_{k=1}^\infty\left(\frac{\int_{\partial\Omega}\Theta_k^j(\,\cdot\,,\boldsymbol{x}')\chi(\boldsymbol{x}')h(\boldsymbol{x}')\mathrm{d}\sigma(\boldsymbol{x}')}{(z+\lambda_k^j)^2}\right)$$

is holomorphic in $z\in D$ as a function taking values in $L^2(\Gamma_{\mathrm{out}})$. Combining this with (30), we obtain

$$\sum_{k=1}^\infty\left(\frac{\int_{\partial\Omega}\Theta_k^1(\boldsymbol{x},\boldsymbol{x}')\chi(\boldsymbol{x}')h(\boldsymbol{x}')\mathrm{d}\sigma(\boldsymbol{x}')}{(z+\lambda_k^j)^2}\right)=\sum_{k=1}^\infty\left(\frac{\int_{\partial\Omega}\Theta_k^2(\boldsymbol{x},\boldsymbol{x}')\chi(\boldsymbol{x}')h(\boldsymbol{x}')\mathrm{d}\sigma(\boldsymbol{x}')}{(z+\lambda_k^j)^2}\right),\quad z\in D,\ \text{a.e. }\boldsymbol{x}\in\Gamma_{\mathrm{out}} \tag{31}$$

by the unique continuation principle for analytic functions. Fixing $\lambda_1':=\min_{(j,k)\in\{1,2\}\times\mathbb{N}}\lambda_k^j$, multiplying both sides of (31) by $(z+\lambda_1')^2$ and letting z to $-\lambda_1'$, we get

$$\lambda_1^1=\lambda_1'=\lambda_1^2,\quad\int_{\partial\Omega}\Theta_1^1(\boldsymbol{x},\boldsymbol{x}')\chi(\boldsymbol{x}')h(\boldsymbol{x}')\mathrm{d}\sigma(\boldsymbol{x}')=\int_{\partial\Omega}\Theta_1^2(\boldsymbol{x},\boldsymbol{x}')\chi(\boldsymbol{x}')h(\boldsymbol{x}')\mathrm{d}\sigma(\boldsymbol{x}'),\quad\text{a.e. }\boldsymbol{x}\in\Gamma_{\mathrm{out}}.$$

By induction on $k \in \mathbb{N}$, we obtain

$$\lambda_k^1 = \lambda_k^2, \quad \int_{\partial\Omega} \Theta_k^1(\boldsymbol{x}, \boldsymbol{x}')\chi(\boldsymbol{x}')h(\boldsymbol{x}')\mathrm{d}\sigma(\boldsymbol{x}') = \int_{\partial\Omega} \Theta_k^2(\boldsymbol{x}, \boldsymbol{x}')\chi(\boldsymbol{x}')h(\boldsymbol{x}')\mathrm{d}\sigma(\boldsymbol{x}'), \quad \text{a.e. } \boldsymbol{x} \in \Gamma_{\mathrm{out}}$$

for any $h \in H^{3/2}(\partial\Omega)$ and any $k \in \mathbb{N}$. Since here $h \in H^{3/2}(\partial\Omega)$ is arbitrary chosen and $\chi = 1$ on Γ'_{in}, we deduce (28) from this last identity.

Step 4 We will complete the proof Theorem 2.2 in this step. For this purpose, we will use some arguments of [7, Theorem 1.1]. We use again the notations in Step 3. We recall that in the last step we have proved $\lambda_k^1 = \lambda_k^2 = \lambda_k$ for all $k \in \mathbb{N}$. Now our goal is to show that, for $j = 1, 2$ and $k \in \mathbb{N}$, there exists a family $\{\widetilde{\varphi}_{k,\ell}^j\}_{\ell=1}^{m_k^j}$ of eigenfunctions of $\mathcal{A}^j$ forming an orthonormal basis of $\ker(\lambda_k - \mathcal{A}^j)$ in $L^2(\Omega; \rho^j\,\mathrm{d}\boldsymbol{x})$ such that

$$m_k^1 = m_k^2, \quad a^1\partial_{\boldsymbol{\nu}}\widetilde{\varphi}_{k,\ell}^1 = a^2\partial_{\boldsymbol{\nu}}\widetilde{\varphi}_{k,\ell}^2 \text{ on } \partial\Omega, \quad k \in \mathbb{N},\ \ell = 1, \ldots, m_k^1. \tag{32}$$

To this end, using the fact that $\Gamma'_{\mathrm{in}} \cap \Gamma_{\mathrm{out}} \neq \emptyset$ and $\Gamma'_{\mathrm{in}} \cup \Gamma_{\mathrm{out}} = \partial\Omega$, we start by recalling an algebraic result in [7] which takes the following form in our context.

Lemma 4.1. *Let $M_1, M_2 \in \mathbb{N}$, $f_\ell \in C(\partial\Omega)$, $\ell = 1, \ldots, M_1$ and $g_\ell \in C(\partial\Omega)$, $\ell = 1, \ldots, M_2$. Assume that*

$$\sum_{\ell=1}^{M_1} f_\ell(\boldsymbol{x})f_\ell(\boldsymbol{x}') = \sum_{\ell=1}^{M_2} g_\ell(\boldsymbol{x})g_\ell(\boldsymbol{x}'), \quad \boldsymbol{x} \in \Gamma_{\mathrm{out}},\ \boldsymbol{x}' \in \Gamma'_{\mathrm{in}}, \tag{33}$$

and that the restrictions $\{f_\ell|_{\Gamma'_{\mathrm{in}}\cap\Gamma_{\mathrm{out}}}\}_{\ell=1}^{M_1}$ and $\{g_\ell|_{\Gamma'_{\mathrm{in}}\cap\Gamma_{\mathrm{out}}}\}_{\ell=1}^{M_2}$ are linearly independent respectively. Then there holds $M_1 = M_2$ and there exists an $M_1 \times M_1$ orthogonal matrix $\boldsymbol{O}$ such that

$$(f_1, \ldots, f_{M_1})^{\mathrm{T}}(\boldsymbol{x}) = \boldsymbol{O}(g_1, \ldots, g_{M_1})^{\mathrm{T}}(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega,$$

where $(\,\cdot\,)^{\mathrm{T}}$ stands for the transpose.

We refer to [7, Lemma 2.3] and Step 4 in the proof of [46, Theorem 3.7] for the proof of this lemma. Now we fix $k \in \mathbb{N}$ and set

$$M_1 = m_k^1, \quad M_2 = m_k^2, \quad f_\ell = a^1\partial_{\boldsymbol{\nu}}\varphi_{k,\ell}^1,\ \ell = 1, \ldots, M_1, \quad g_\ell = a^2\partial_{\boldsymbol{\nu}}\varphi_{k,\ell}^2,\ \ell = 1, \ldots, M_2$$

in the context of Lemma 4.1. According to [20, Lemma 2.4.2.2], since $\varphi_{k,\ell}^j$ solves the boundary value problem

$$\begin{cases} -\mathrm{div}\left(a^j\nabla\varphi_{k,\ell}^j\right) + \left(c^j - \lambda_k\rho^j\right)\varphi_{k,\ell}^j = 0 & \text{in } \Omega, \\ \varphi_{k,\ell}^j = 0 & \text{on } \partial\Omega, \end{cases} \quad j = 1, 2,\ \ell = 1, \ldots, m_k^j,$$

we deduce that $\varphi_{k,\ell}^j \in C^1(\overline{\Omega})$. Therefore, the functions f_ℓ, $\ell = 1, \ldots, M_1$ and g_ℓ, $\ell = 1, \ldots, M_2$ lie in $C(\partial\Omega)$. Moreover, according to (28), condition (33) is fulfilled. In addition, as a consequence of unique continuation results for elliptic equations (see e.g. [7, Lemma 2.1]), we deduce that $\{f_\ell|_{\Gamma'_{\mathrm{in}}\cap\Gamma_{\mathrm{out}}}\}_{\ell=1}^{M_1}$ and $\{g_\ell|_{\Gamma'_{\mathrm{in}}\cap\Gamma_{\mathrm{out}}}\}_{\ell=1}^{M_2}$ are linearly independent respectively. Therefore, Lemma 4.1 implies that $m_k^1 = m_k^2$ and there exists an $m_k^1 \times m_k^1$ orthogonal matrix $\boldsymbol{O}_k$ such that

$$\left(a^1\partial_{\boldsymbol{\nu}}\varphi_{k,1}^1, \ldots, a^1\partial_{\boldsymbol{\nu}}\varphi_{k,m_k^1}^1\right)^{\mathrm{T}}(\boldsymbol{x}) = \boldsymbol{O}_k\left(a^2\partial_{\boldsymbol{\nu}}\varphi_{k,1}^2, \ldots, a^2\partial_{\boldsymbol{\nu}}\varphi_{k,m_k^1}^2\right)^{\mathrm{T}}(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega.$$

Therefore, fixing $\widetilde{\varphi}_{k,\ell}^1 = \varphi_{k,\ell}^1$, $\ell = 1, \ldots, m_k^1$ and

$$\left(\widetilde{\varphi}_{k,1}^2, \ldots, \widetilde{\varphi}_{k,m_k^1}^2\right)^{\mathrm{T}} = \boldsymbol{O}_k\left(\varphi_{k,1}^2, \ldots, \varphi_{k,m_k^1}^2\right)^{\mathrm{T}},$$

we deduce that, for $j = 1, 2$, $\{\widetilde{\varphi}_{k,\ell}^j\}_{\ell=1}^{m_k^j}$ is a family of eigenfunctions of $\mathcal{A}^j$ forming an orthonormal basis of $\ker(\lambda_k - \mathcal{A}^j)$ in $L^2(\Omega; \rho^j\,\mathrm{d}\boldsymbol{x})$ such that (32) is fulfilled. This proves that (32) holds true for any $k \in \mathbb{N}$.

In order to complete the proof of Theorem 2.2, we recall the following inverse spectral result which follows from [8, Corollaries 1.5–1.7].

Lemma 4.2. *Under the conditions of Theorem 2.2, assume that either of the three assumptions (i), (ii) or (iii) holds. Then* (28) *and* (32) *imply* $(\rho^1, a^1, c^1) = (\rho^2, a^2, c^2)$.

Applying this result, we can complete the proof of Theorem 2.2. ☐

Proof of Theorem 2.3. Repeating the arguments used in the proof of Theorem 2.2 and utilizing Proposition 3.2, we deduce

$$\partial_{\boldsymbol{\nu}} v_k^1 = \partial_{\boldsymbol{\nu}} v_k^2 \quad \text{on } \mathbb{R}_+ \times \partial\Omega,\ k \in \mathbb{N}, \tag{34}$$

where v_k^j ($j = 1, 2$, $k \in \mathbb{N}$) solves (20) with $(\alpha, \rho, \boldsymbol{B}) = (\alpha^j, \rho^j, \boldsymbol{B}^j)$ ($j = 1, 2$). For $j = 1, 2$ and $\xi > 0$, consider the boundary value problem

$$\begin{cases} (-\triangle + \xi^{\alpha^j} \rho^j) V^j(\xi) + \boldsymbol{B}^j \cdot \nabla V^j(\xi) = 0 & \text{in } \Omega, \\ V^j(\xi) = \phi & \text{on } \partial\Omega, \end{cases} \quad \phi \in H^{3/2}(\partial\Omega).$$

According to [19, Theorem 8.3], this problem admits a unique solution $V^j(\xi) \in H^2(\Omega)$ and we can associate this problem with the Dirichlet-to-Neumann map

$$\mathcal{N}^j(\xi) : \phi \longmapsto \partial_{\boldsymbol{\nu}} V^j(\xi)\big|_{\partial\Omega}, \quad j = 1, 2,\ \xi > 0.$$

In a manner similar to the proof of Theorem 2.2, we can prove that (34) implies

$$\mathcal{N}^1(\xi) = \mathcal{N}^2(\xi), \quad \xi > 0. \tag{35}$$

Using the elliptic regularity of the operator $-\triangle + \boldsymbol{B}^j \cdot \nabla$, one can verify the continuity of $[0, \infty) \ni \xi \longmapsto \mathcal{N}^j(\xi) \in \mathcal{B}(H^{3/2}(\partial\Omega), H^{1/2}(\partial\Omega))$ ($j = 1, 2$). Therefore, making $\xi \to 0$ in condition (35), we obtain

$$\partial_{\boldsymbol{\nu}} V^1(0)\big|_{\partial\Omega} = \partial_{\boldsymbol{\nu}} V^2(0)\big|_{\partial\Omega} \tag{36}$$

for $V^j(0) \in H^2(\Omega)$ solving

$$\begin{cases} -\triangle V(0) + \boldsymbol{B}^j \cdot \nabla V(0) = 0 & \text{in } \Omega, \\ V(0) = \phi & \text{on } \partial\Omega, \end{cases} \quad \phi \in H^{3/2}(\partial\Omega).$$

Combining (36) with [40, Theorem 1.1] and using the density of $H^{3/2}(\partial\Omega)$ in $H^{1/2}(\partial\Omega)$, we deduce that $\boldsymbol{B}^1 = \boldsymbol{B}^2$. Now choosing $\xi = 1$ and applying [40, Proposition 2.1] (see also [30, 45] for equivalent results stated for magnetic Schrödinger operators), we deduce that $\rho^1 = \rho^2$. Finally, choosing $\xi = \mathrm{e}$ and applying [40, Proposition 2.1], we get $\rho^1 \exp(\alpha^1) = \rho^2 \exp(\alpha^2) = \rho^1 \exp(\alpha^2)$, indicating $\exp(\alpha^1) = \exp(\alpha^2)$ and thus $\alpha^1 = \alpha^2$. This completes the proof of Theorem 2.3. ☐

Proof of Corollary 2.4. For $j = 1, 2$, we define the operator $\mathcal{A}^j$ as

$$\mathcal{D}(\mathcal{A}^j) = H^2(\mathcal{M}^j) \cap H_0^1(\mathcal{M}^j), \quad \mathcal{A}^j := -\triangle_{g^j, \mu^j} + c^j.$$

Here we consider the weighted measure $\mu^j \, \mathrm{d}V_{g^j}$, where $\mathrm{d}V_{g^j}$ is the Riemannian volume measure associated with $(\mathcal{M}^j, g^j)$. Parallel to Step 3 in the proof of Theorem 2.2, we denote the eigensystem of $\mathcal{A}^j$ ($j = 1, 2$) by $\{(\lambda_k^j, \{\varphi_{k,\ell}^j\}_{\ell=1}^{m_k^j})\}_{k \in \mathbb{N}}$, where $\{\lambda_k^j\}$ is strictly increasing and m_k^j is the multiplicity of λ_k^j. Similarly, $\{\varphi_{k,\ell}^j \mid 1 \le \ell \le m_k^j,\ k \in \mathbb{N}\}$ forms an orthonormal basis in $L^2(\mathcal{M}^j)$ associated with λ_k^j. Fixing

$$\Theta_k^j(\boldsymbol{x}, \boldsymbol{x}') := \sum_{\ell=1}^{m_k^j} \partial_{\boldsymbol{\nu}} \varphi_{k,\ell}^j(\boldsymbol{x}) \partial_{\boldsymbol{\nu}} \varphi_{k,\ell}^j(\boldsymbol{x}'), \quad \boldsymbol{x}, \boldsymbol{x}' \in \partial\mathcal{M}^1,\ k \in \mathbb{N}$$

and repeating the arguments used in Step 3 of the proof of Theorem 2.2, we can prove that (11) implies the condition (28), especially, $\lambda_k^1 = \lambda_k^2 = \lambda_k$ for all $k \in \mathbb{N}$. Then, applying the arguments used in Step 4 of the proof of Theorem

2.2, we deduce that, for $j = 1, 2$, there exists a family $\{\widetilde{\varphi}^j_{k,\ell}\}_{\ell=1}^{m^j_k}$ of eigenfunctions of $\mathcal{A}^j$ forming an orthonormal basis of $\ker(\lambda_k - \mathcal{A}^j)$ in $L^2(\Omega; \rho^j\, \mathrm{d}\boldsymbol{x})$ such that

$$m^1_k = m^2_k, \quad \partial_{\boldsymbol{\nu}} \widetilde{\varphi}^1_{k,\ell} = \partial_{\boldsymbol{\nu}} \widetilde{\varphi}^2_{k,\ell} \text{ on } \Gamma_{\mathrm{out}}, \quad k \in \mathbb{N},\ \ell = 1, \ldots, m^1_k. \tag{37}$$

Here we apply again Lemma 4.1 with $\partial\Omega$ replaced by $\Gamma_{\mathrm{out}} = \Gamma'_{\mathrm{in}} \cup \Gamma_{\mathrm{out}}$ in its statement. Thus, in a way similar to [26, Sect. 5, pp. 1165–1166], we apply results in [23] to verify that (28) and (37) imply that $(\mathcal{M}^1, g^1)$ and $(\mathcal{M}^2, g^2)$ are isometric and conditions (12)–(13) are fulfilled. This completes the proof of Corollary 2.4. □

Proof of Corollary 2.6. Using the notation of the proof of Corollary 2.4 with $\mathcal{A}^j := -\triangle_{g^j}$ acting on $L^2(\mathcal{M}^j)$ and $\mathcal{D}(\mathcal{A}^j) := H^1_0(\mathcal{M}^j) \cap H^2(\mathcal{M}^j)$, we can prove again that (11) implies (28). For $j = 1, 2$, consider the initial-boundary value problem

$$\begin{cases} (\partial_t^2 - \triangle_{g^j}) u^j = 0 & \text{in } (0,\infty) \times \mathcal{M}^j, \\ u^j = \partial_t u^j = 0 & \text{in } \{0\} \times \mathcal{M}^j, \\ u^j = \Phi & \text{on } (0,\infty) \times \partial\mathcal{M}^j \end{cases}$$

and the associated hyperbolic partial Dirichlet-to-Neumann map

$$\mathcal{N}^j : C^\infty(\mathbb{R}_+ \times \Gamma'_{\mathrm{in}}) \ni \Phi \longmapsto \partial_{\boldsymbol{\nu}} u^j\big|_{(0,\infty)\times\Gamma_{\mathrm{out}}}.$$

Repeating arguments used in [26, Theorem 5.3], we can show that (28) implies $\mathcal{N}^1 = \mathcal{N}^2$. Thus, it follows from [31, Theorem 1] that $(\mathcal{M}^1, g^1)$ and $(\mathcal{M}^2, g^2)$ are isometric. This completes the proof of Corollary 2.6. □

Proof of Corollary 2.7. Using the notation of the proof of Corollary 2.4 with $\mathcal{A}^j := -\triangle_g - c^j$ acting on $L^2(\mathcal{M}^j)$ and $\mathcal{D}(\mathcal{A}^j) := H^1_0(\mathcal{M}^j) \cap H^2(\mathcal{M}^j)$, we can prove again that (11) implies (28). For $j = 1, 2$, consider the initial-boundary value problem

$$\begin{cases} (\partial_t^2 - \triangle_{g^j} + c^j) u^j = 0 & \text{in } (0,\infty) \times \mathcal{M}^j, \\ u^j = \partial_t u^j = 0 & \text{in } \{0\} \times \mathcal{M}^j, \\ u^j = \Phi & \text{on } (0,\infty) \times \partial\mathcal{M}^j \end{cases}$$

and the associated hyperbolic partial Dirichlet-to-Neumann map

$$\mathcal{N}^j : C^\infty(\mathbb{R}_+ \times \Gamma'_{\mathrm{in}}) \ni \Phi \longmapsto \partial_{\boldsymbol{\nu}} u^j\big|_{(0,\infty)\times\Gamma_{\mathrm{out}}}.$$

Similarly to the proof of Corollary 2.6, we deduce from (28) that $\mathcal{N}^1 = \mathcal{N}^2$. Thus, it follows from [25, Theorem 1] that there exists a neighborhood U of Γ_{out} in $\mathcal{M}$ such that $c^1 = c^2$ in U. This completes the proof of Corollary 2.7. □

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