Settlement Fund Circulation Problem ☆

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Abstract

In the economic activities, the central bank has an important role to cover payments of banks, when they are short of funds to clear their debts. For this purpose, the central bank timely puts funds so that the economic activities go smooth. Since payments in this mechanism are processed sequentially, the total amount of funds put by the central bank critically depends on the order of the payments. Then an interest goes to the amount to prepare if the order of the payments can be controlled by the central bank, or if it is determined under the worst case scenario. This motivates us to introduce a brand-new problem, which we call the settlement fund circulation problem. The problems are formulated as follows: Let $G = (V, A)$ be a directed multigraph with a vertex set V and an arc set A. Each arc $a \in A$ is endowed debt $d(a) \geq 0$, and the debts are settled sequentially under a sequence π of arcs. Each vertex $v \in V$ is put fund in the amount of $p_\pi(v) \geq 0$ under the sequence. The minimum/maximum settlement fund circulation problem (Min-SFC/Max-SFC) in a given graph G with debts $d : A \to \mathbb{R}_+ \cup \{0\}$ asks to find a bijection $\pi : A \to \{1, 2, \ldots, |A|\}$ that minimizes/maximizes the total funds $\sum_{v \in V} p_\pi(v)$. In this paper, we show that both Min-SFC and Max-SFC are NP-hard; in particular, Min-SFC is (I) strongly NP-hard even if G is (i) a multigraph

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with $|V| = 2$ or (ii) a simple graph with treewidth at most two, and is (II) (not necessarily strongly) NP-hard for simple trees of diameter four, while it is solvable in polynomial time for stars. Also, we identify several polynomial time solvable cases for both problems.

Keywords: Fund settlement, Algorithm, Digraph, Scheduling

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1. Introduction

Background. In the economic activities, when a company borrows money, it owes a debt and the debt is not cleared until the debtor pays its amount. If the debtor fails to prepare cash for the payment until the deadline, it will go bankrupt. Such bankruptcy should be avoided when it could cause significant damage to the economy, and it is particularly true for the case of banks since their debts are highly interconnected each other and bankruptcy of a bank may cause chain reaction of bankruptcy. It is one of the reasons that debts among banks are cleared in a special system, called *interbank settlement system*, in which the central bank supports cash management of the banks.

In the system, cash held by the central bank is used as the fund for the payments. When a bank does not have enough funds for clearing its debts, the central bank will lend the necessary amount. Suppose, for example, that there are three banks, say A, B, and C, and they form debts such that A owes 50 to B, and B owes 30 to C, and A and B currently have 10 each on its own. Now if A pays for its debt, then A is short of 40. Therefore, the central bank is requested to put 40 in order to fill the shortage. Once 40 is put on A, it can clear its debt 50 to B, and then B can also clear its debt 30 by using its own funds 10 and a part of the received funds 50. Note that we assume each debt has to be cleared independently and "sequentially", that is, it is not allowed to cancel out payments; A pays 30 directly to C, and the rest 20 to B, for example.[1]

Objective. Now, suppose that B pays before A does. Then, the central bank has to put 20 to B, and in addition, 40 to A. This illustrates, in general, that the total amount of funds put to clear all debts depends on the order of the

[1]Sequential clearing is standard in the modern interbank settlement systems, as World Bank documents that 116 of 139 surveyed countries have adopted sequential clearing based systems up to 2010 [23].

payments. Since funds in an interbank settlement system is scarce resource in the public interest, the efficient usage is socially desirable. Accordingly, one of the important roles of the central bank is to minimize the total funds put to clear the debts. Then we can consider a problem that finds the minimum total funds put to clear all debts by deciding a sequence of the payments, which we formulate as MIN-SFC.

In a different perspective, another role of the central bank is to prepare for the worst case scenario such that it could hardly control the sequence of the payments. It is typical at the time of financial disruption and is crucially important. These observations again motivate us to define a corresponding maximization version of the problem, that is, to estimate the maximum funds that have to be put to clear all given debts, which we formulate as MAX-SFC. It is quite significant to obtain insights concerning the desirable sequence of the payments in order to argue relevant policies.

Technically, both problems are formulated as optimization problems on networks. However, the nature of our problems is essentially different from the classic flow problems in the sense that the amount of each "debt" (flow) cannot be split at the time of the payment. On the contrary, such unsplittable flows come to have a feature that once some debt is cleared, then the transferred funds are accumulated in the bank's "account" and they can be split arbitrarily for the subsequent payments.

History and Perspective in Economics. Historically, we can find a primitive concern of fund circulation in the renowned Quesnay's "Economic Table" [18]. Only recently, Rotemberg explicitly discusses the amount of required funds in the context of interbank settlements [21], though he does not give its general formulation. A general formulation to derive each of the minimum and maximum amount of required funds is then given by Hayakawa [10] for the purpose of economic analysis.[2] This paper now gives, from the computational aspect, detailed mathematical formulations for these problems as MIN-SFC and MAX-SFC, and presents a series of algorithmic or complexity results based on solid observations for the first time.

In the wake of the recent world-wide financial crisis, analyzing "dominos" of default comes to have critical importance. Seminal studies in the literature effectively assume "simultaneous" clearing that makes payments cancel

[2] The relevant chapter of the paper [10] is reorganized as an independent article [11] with additional results.

out whenever possible, not only bilaterally but also multilaterally, though "sequential" clearing, which we assume, is standard in the modern interbank settlement systems. The assumption of simultaneous clearing lets the relevant analyses be highly tractable [1, 4], [3] however, it could considerably underestimate the amount of funds required to prevent "dominos" of default. In the light of these, we believe that the study in this paper serves as fundamental tools of the estimation and suggests a new methodology in the analyses that is applicable to complex economic situations in reality.

Related work. In a context of logistics or scheduling, a *reallocation problem* is investigated [15, 16, 17]. The setting of the reallocation problem is as follows: There are blocks and boxes where blocks can be stored. Each block has a certain volume, and each box has a certain capacity. Blocks can be stored in a box if the sum of the volumes of the blocks is less than or equal to the box capacity. In the initial state, each block is stored in a box. Each block has a target (or destination) box where it must be eventually stored. There are also buffer (or bypass, evacuation) boxes where a block can be temporarily put aside. In each move, only one block can be moved from one box to another. The problem is to determine whether all the blocks can be moved from their initial boxes into their target boxes within a given number of moves.

This problem is very similar to our SFC if no buffer box is prepared. A box corresponds to a vertex of a graph of SFC. A block at u with target v corresponds to arc (u, v) and the volume corresponds to a debt. The initial room of a box corresponds to a *residual*, which will be defined in Section 2.2, though the initial residuals of all the vertices are set to 0 in SFC. In fact, the problems without buffer boxes are also considered (e.g., [16, Theorem 5], [17, Theorem 3.1]). Although the setting is not exactly the same, some hardness result of our MIN-SFC, Theorem 1, can be deduced from the results of the reallocation problem.

This paper is organized as follows. In Section 2, after giving several terminologies, we formalize our problem of interests and show some examples. Sections 3 and 4 discuss the minimization version of the problem, and show tractable and intractable cases, respectively. Section 5 deals with the maximization version. Finally in Section 6, we discuss some open problems and

[3] The assumption of simultaneous clearing is maintained, either implicitly or explicitly, in the subsequent studies [6, 2, 3, 19, 5, 9].

conclude the paper.

2. Preliminaries

2.1. Definitions and Terminology

For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$. For a finite set V, a family $\mathcal{X}$ of subsets in V is a *partition of* V if $\bigcup_{X \in \mathcal{X}} X = V$ holds and every two distinct sets in $\mathcal{X}$ are disjoint.

A directed graph (digraph) D is an ordered pair of its vertex set $V(D)$ and arc set $A(D)$ and is denoted by $D = (V(D), A(D))$, or simply $D = (V, A)$. An arc, an element of $A(D)$, is an ordered pair of vertices, and is denoted by $a = (u, v)$; this is distinct from (v, u). For an arc $a = (u, v)$, u is its *start vertex* and v is *end vertex*; they are sometimes denoted by $s(a)$ and $t(a)$, respectively. A digraph D is *multiple* when $A(D)$ is a multiple set; otherwise it is *simple*.

The *underlying graph* of a digraph D is an undirected graph G_D whose vertex set is $V(D)$ and edge set $E(G_D)$ has an edge $\{u, v\}$ as its element if and only if $(u, v) \in A(D)$ or $(v, u) \in A(D)$. A digraph D is *weakly connected* if its underlying graph G_D is connected. We assume throughout the paper that all digraphs are weakly connected. We usually use n and m to denote the number of vertices and arcs (edges), respectively, of a graph.

In a digraph D, a vertex $u \in V(D)$ is an *out-neighbor* (resp., *in-neighbor*) of a vertex $v \in V(D)$ if $(v, u) \in A(D)$ (resp., $(u, v) \in A(D)$). Let $N^+_D(v)$ (resp., $N^-_D(v)$) denote the set of out-neighbors (resp., in-neighbors) of $v \in V(D)$ in D, and $N_D(v) = N^+_D(v) \cup N^-_D(v)$. In a digraph D, the *out-degree* (*in-degree*) of a vertex v, denoted by $\deg^+_D(v)$ (resp., $\deg^-_D(v)$), is $|\{(v, w) \in A(D) \mid w \in V(D)\}|$ (resp., $|\{(u, v) \in A(D) \mid u \in V(D)\}|$), and the *degree* of v, denoted by $\deg_D(v)$, equals $\deg^+_D(v) + \deg^-_D(v)$. We use $\Delta(D)$ to denote the maximum degree of a digraph D.

Note that for a simple digraph D, we have $\deg^+_D(v) = |N^+_D(v)|$ and $\deg^-_D(v) = |N^-_D(v)|$. Let $D[V']$ (resp., $D[A']$) denote the subgraph of D induced by a subset $V' \subseteq V(D)$ of vertices (resp., a subset $A' \subseteq A(D)$ of arcs). For a digraph D and a subset $A' \subseteq A(D)$ of arcs, we denote by $D \setminus A'$ the subgraph of D obtained from D by deleting A'.

2.2. Models and Problem Description

In the paper, we describe our problem by a digraph whose nodes are banks and arcs are loan relationship from one bank to another together with debts as arc weights.

Given a digraph $D = (V, A)$, *debt* of arcs is a function $d : A \to \mathbb{R}_+ \cup \{0\}$. For convenience, we sometimes introduce a (virtual) arc $a = (u, v)$ with $d(a) = 0$ if $(v, u) \in A$ and $(u, v) \notin A$. A debt function d is *uniform* if $d(a) = c$ (constant) for all $a \in A$, otherwise *non-uniform*; it is *unit* if d is uniform and $c = 1$. Debts on a vertex v are *balanced* if $\sum_{(u,v)\in A} d(u, v) = \sum_{(v,w)\in A} d(v, w)$, and debts on a pair of vertices u and v is *symmetric* if $d(u, v) = d(v, u)$.

In our model, debts on arcs are settled individually in a single installment and sequentially. We say that a debt on an arc is *cleared* when it is settled (or, simply *clear* the arc), and we *put* funds on vertices to clear debts on their out-going arcs. When an arc is cleared, the amount for it is accumulated on the end vertex of the arc and can be reused for subsequent settlements. The amount of funds existing on a vertex is called its *residual*. A sequence of arcs, which corresponds to the order of selecting arcs to be cleared, can be represented as a permutation $\pi : A \to \{1, 2, \ldots, |A|\}$. We sometimes refer to this permutation as a *sequence* of A. We denote by $p_\pi(u, i)$ the fund put on u and by $r_\pi(u, i)$ the residual of u, immediately before putting fund $p_\pi(u, i)$ to clear arc $\pi^{-1}(i)$ for all $u \in V$ and for $i = 1, \ldots, |A|$. Then we can clear the debt on arc (u, v) if $r_\pi(u, \pi(u, v)) + p_\pi(u, \pi(u, v)) \geq d(u, v)$. We assume that we always put the minimum amount of funds to clear an arc, that is, $p_\pi(s(a), \pi(a)) = \max\{0, d(a) - r_\pi(s(a), \pi(a))\}$.

Now we define the *minimum settlement fund circulation problem* (Min-SFC) and the corresponding maximization problem (Max-SFC), which are introduced in [10] in the context of the interbank fund settlement systems, as follows.

Min-SFC (Max-SFC)
Instance: a digraph $D = (V, A)$ and debt $d : A \to \mathbb{R}_+ \cup \{0\}$.
Question: minimize (maximize) $\sum_{v\in V} p_\pi(v)$ ($\triangleq p_\pi(V)$)
subject to permutation $\pi : A \to \{1, \ldots, |A|\}$
and $p_\pi(s(a), \pi(a)) + r_\pi(s(a), \pi(a)) \geq d(a)$ for all $a \in A$, where

$$p_\pi(u, 0) = 0, r_\pi(u, 0) = 0 \text{ for all } u \in V,$$

$$r_\pi(u, i) = \begin{cases} r_\pi(u, i-1), & \pi^{-1}(i-1) \text{ is not incident on } u, \\ r_\pi(u, i-1) + d(\pi^{-1}(i-1)), & \pi^{-1}(i-1) \text{ is incident to } u, \\ \max\{0, r_\pi(u, i-1) \\ \qquad -d(\pi^{-1}(i-1))\}, & \pi^{-1}(i-1) \text{ is incident from } u. \end{cases}$$

Here, we define $p_\pi(v) = \sum_{i=1}^{|A|} p(v,i)$, and for notational convenience, we often let $p_\pi(X) = \sum_{v\in X} p_\pi(v)$ for a subset X of $V(D)$ in a digraph D and a sequence π of $A(D)$.

We show examples of Min-SFC and Max-SFC in Figure 1, and see how debts are cleared in detail. In the sequence π_1, 20 funds are put on v_5 to clear the debt $d(v_5,v_6) = 20$ for the first arc $\pi_1^{-1}(1) = (v_5,v_6)$; $p_{\pi_1}(v_5,1) = 20$. The 20 funds are transferred to and accumulated in v_6; $r_{\pi_1}(v_6,2) = 20$. The second arc $\pi_1^{-1}(2) = (v_3,v_6)$ is cleared by putting 10 funds on v_3; $p_{\pi_1}(v_3,2) = 10$. The 10 funds are transferred to v_6, and it turns out that the residual on v_6 becomes $20 + 10 = 30$; $r_{\pi_1}(v_6,3) = 30$. Now, the residual are used for clearing the third arc $\pi_1^{-1}(3) = (v_6,v_1)$ and no additional fund needs to be put on v_6; $r_{\pi_1}(v_1,4) = 30$. We remark here again that a debt can only be cleared by a single installment. Also a residual can be split. Next, therefore, a part 20 of the residual 30 of v_1 is used for clearing $\pi_1^{-1}(4) = (v_1,v_2)$, where $d(v_1,v_2) = 20$, and so on. In the sequence π_2, 20 funds are put on v_1 to clear the debt $d(v_1,v_2) = 20$ for the first arc $\pi_2^{-1}(1) = (v_1,v_2)$, and transferred to and accumulated in v_2; $p_{\pi_2}(v_1,1) = 20$ and $r_{\pi_2}(v_2,2) = 20$. In π_2, no residual is used for clearing arcs except (v_2,v_3) and (v_4,v_5); $p_{\pi_2}(s(a),\pi_2(a)) = d(a)$ for each $a \in A \setminus \{(v_2,v_3),(v_4,v_5)\}$.

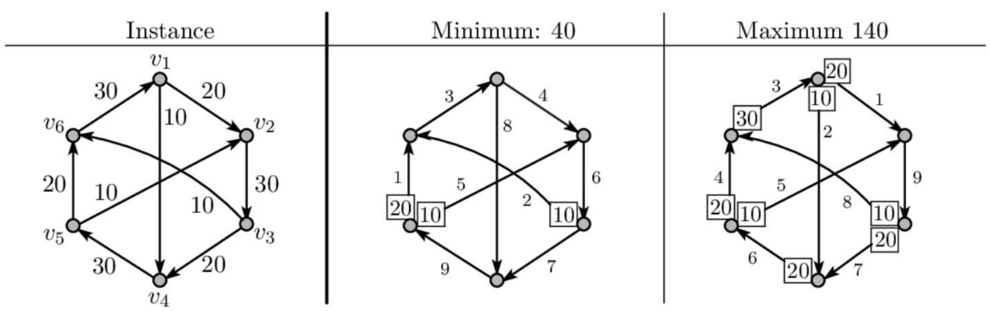

Figure 1: **[Left]** An instance of both Min-SFC and Max-SFC; a digraph $D = (V,A)$ with the debts $d(a)$ beside each arc a. **[Middle, Right]** Sequences π_1 and π_2 of A, respectively. The number beside each arc $a \in A$ indicates $\pi_i(a)$ and the number in the square attached to $s(a)$ indicates the amount of funds put on $s(a)$ for clearing the debt of a in π_i, i.e., $p_{\pi_i}(s(a),\pi_i(a))$ $(i = 1,2)$. In fact, π_1 and π_2 are optimal solutions of Min-SFC and Max-SFC for D, with $p_{\pi_1}(V) = 40$ and $p_{\pi_2}(V) = 140$, respectively.

2.3. Summary of the Results

The results of this paper are summarized in Table 1. To explain the table and for the use throughout the paper, we introduce some additional definitions. For a digraph D, if the underlying graph G_D of D belongs to some class $\mathcal{C}$ of graphs, then we may simply say that D belongs to $\mathcal{C}$ if no confusion occurs. A digraph is called *balanced* if debts on each vertex is balanced. A digraph D is called *symmetric* if debts on each pair of vertices u and v with $\{u, v\} \in E(G_D)$ is symmetric. A digraph is called *uniform* if its debt function is uniform.

We emphasize here that all the results are new. Especially, we can see that those for general and simple graphs show sharp border with respect to the complexity in the sense that it is tractable for stars, but is intractable for trees.

Table 1: Summary of our results in this paper together with their corresponding theorem/lemma numbers; Linear and P stand for linear and polynomial time solvable, respectively, and T, C and L in brackets stand for Theorem, Corollary and Lemma, respectively.

<table>
<tr><th colspan="2">arcs</th><th colspan="5">graphs</th></tr>
<tr><th>debt</th><th>multiplicity</th><th>dag</th><th>path</th><th>star</th><th>tree</th><th>larger classes</th></tr>
<tr><td colspan="7">MIN-SFC</td></tr>
<tr><td>uniform</td><td>multiple</td><td colspan="5">Linear [T5]</td></tr>
<tr><td>symmetric</td><td>simple</td><td>Linear [L4]</td><td colspan="4">P [T7]</td></tr>
<tr><td rowspan="2">balanced</td><td>multiple</td><td>Linear [L4]</td><td colspan="4">strongly NP-hard for two vertices [T1]</td></tr>
<tr><td>simple</td><td>Linear [L4]</td><td>P [C8]</td><td>P [C8]</td><td>P [C8]</td><td>strongly NP-hard for bipartite or tw$\leq$ 2 [T2]</td></tr>
<tr><td rowspan="2">general</td><td>multiple</td><td>Linear [L4]</td><td colspan="4">strongly NP-hard for two vertices [T1]</td></tr>
<tr><td>simple</td><td>Linear [L4]</td><td>P [C18]</td><td>P [T15]</td><td colspan="2">NP-hard [T3]
FPT wrt. Δ [T17]</td></tr>
<tr><td colspan="7">MAX-SFC</td></tr>
<tr><td>uniform</td><td></td><td>Linear [L22]</td><td colspan="3"></td><td>NP-hard for bipartite [T23]</td></tr>
<tr><td>general</td><td>multiple</td><td>Linear [L22]</td><td colspan="4"></td></tr>
</table>

3. Min-SFC: Intractable Cases

In the subsequent two sections (Sections 3 and 4), we discuss about MIN-SFC, which is our main interest in the context of analyzing settlements of debts. We first observe in this section that the problem is hard in general, but later in Section 4 we will see that it is tractable in some practical cases.

Throughout these two sections, for an instance (D, d) of Min-SFC, we denote by $opt(D, d)$ the minimum amount of funds put on $V(D)$ for clearing all arcs in D, i.e., $opt(D, d) = \min\{p_\pi(V(D)) \mid \pi \text{ is a sequence of } A(D)\}$.

Now let $D = (V, A)$ be a multiple digraph. We show that even if $|V| = 2$ and D is balanced, Min-Sfc with D is strongly NP-hard by a reduction from 3-Partition, which is known to be strongly NP-hard [8, p.224].

3-Partition

Instance: $(\{x_1, x_2, \ldots, x_{3m}\}, B)$: A set of $3m$ positive integers $x_1, x_2, \ldots, x_{3m}$ and an integer B such that $\sum_{i\in[3m]} x_i = mB$ and $B/4 < x_i < B/2$ for each $i \in [3m]$.

Question: Is there a partition $\{X_1, X_2, \ldots, X_m\}$ of $[3m]$ such that $\sum_{i\in X_j} x_i = B$ for each $j \in [m]$?

Theorem 1. *For a multiple digraph D,* Min-SFC *is strongly NP-hard even if $|V(D)| = 2$ and D is balanced.*

Proof. Before showing the proof, we note that Miwa and Ito independently showed essentially the same result [16, Theorem 5] as our Theorem 1 as mentioned in Related Work paragraph of Section 1. Since the proof is also similar, we could omit it, but we do not here, because this proof can be extended to Theorem 2.

Take an instance $I_{\text{3PART}} = (\{x_1, x_2, \ldots, x_{3m}\}, B)$ of 3-Partition. From the I_{3PART}, we construct an instance $I_{\text{SFC}} = (D = (V, A), d)$ of Min-SFC as follows. Let $V = \{u, v\}$ and A be the set of arcs consisting of $3m$ multiple arcs from u to v and m multiple arcs from v to u; denote an arc from u to v by a_i, $i \in [3m]$, and an arc from v to u by b_j, $j \in [m]$. Let $d(a_i) = x_i$ for $i \in [3m]$ and $d(b_j) = B$ for $j \in [m]$. Note that D is balanced.

We claim that the instance I_{3PART} is a yes-instance of 3-Partition if and only if there exists a sequence π of A with $p_\pi(V) \le B$. Notice that since I_{SFC} can be constructed from I_{3PART} in polynomial time, this claim proves the theorem.

First, we show "only-if" part. Assume that I_{3PART} is a yes-instance of 3-Partition; there exists a partition $\{X_1, X_2, \ldots, X_m\}$ of $[3m]$ such that $\sum_{i\in X_j} x_i = B$ for each $j \in [m]$. Without loss of generality, let $X_j = \{3j-2, 3j-1, 3j\}$ for $j \in [m]$ (note that $|X_j| = 3$ holds since $B/4 < x_i < B/2$ for each $i \in [3m]$). Then the sequence π of A defined as (b_1, a_1, a_2, a_3, b_2,

$a_4,\ a_5,\ a_6,\ b_3, \ldots,\ b_m, a_{3m-2}, a_{3m-1}, a_{3m})$ satisfies $p_\pi(V) = B$. Notice that $p_\pi(v,1) = B$, $p_\pi(v,\ell) = 0$ for all $\ell \geq 2$, and $p_\pi(u) = 0$.

Next we show "if" part. Assume that there exists a sequence π of A with $p_\pi(V) \leq B$. Since $d(b_j) = B$, we have $opt(D,d) \geq B$ and hence $p_\pi(V) = B$. Without loss of generality, assume that $\pi(b_1) < \pi(b_2) < \cdots < \pi(b_m)$. For $j \in [m-1]$, let X_j be the set of indices $i \in [3m]$ such that $\pi(b_j) < \pi(a_i) < \pi(b_{j+1})$. Since we need funds with amount B for clearing b_1 and $p_\pi(V) = B$ holds, no additional fund is put on V when any arc $a' \in A$ with $\pi(a') > \pi(b_1)$ is cleared. Hence, the total debts of arcs cleared between b_j and b_{j+1} is exactly B, i.e., $\sum_{i \in X_j} x_i = B$ for each $j \in [m-1]$. Furthermore, since $B/4 < x_i < B/2$ for $i \in [3m]$, we have $|X_j| = 3$ for each $j \in [m-1]$. Let $X_m = [3m] \setminus (\bigcup_{i=1}^{m-1} X_j)$. Note that $|X_m| = 3m - \sum_{j=1}^{m-1} |X_j| = 3$ and $\sum_{i \in X_m} x_i = mB - \sum_{j=1}^{m-1} \sum_{i \in X_j} x_i = B$. Thus, the partition $\{X_1, X_2, \ldots, X_m\}$ of $[3m]$ shows that I_{3PART} is a yes-instance of 3-Partition. □

Let D_1 be the graph obtained from the graph $D = (V, A)$ of I_{SFC} in the proof of Theorem 1 by introducing new vertices w_i, $i \in [3m]$, and w'_j, $j \in [m]$, replacing each arc a_i with two arcs (u, w_i) and (w_i, v) with $d(u, w_i) = d(w_i, v) = x_i$, and replacing each arc b_j with two arcs (v, w'_j) and (w'_j, u) with $d(v, w'_j) = d(w'_j, u) = B$. Notice that D_1 is a simple and balanced graph, and the underlying graph G_{D_1} of D_1 is bipartite and series-parallel. Also we can prove that I_{3PART} is a yes-instance of 3-Partition if and only if there exists a sequence π of $A(D_1)$ with $p_\pi(V(D_1)) \leq B$, in a similar way to the proof of Theorem 1. Hence, we have the following theorem.

Theorem 2. *For a simple digraph D,* Min-Sfc *is strongly NP-hard even if D is balanced, G_D is bipartite, and series-parallel* (*i.e., the treewidth of G_D is at most two*).

Furthermore, we can show that the problem Min-SFC is NP-hard even in the case of trees, while it is open whether it is strongly NP-hard or not. The proof is given later in Section 4.5, since it uses an optimality criteria (Lemma 21), which are based on positive results in the case of stars (Section 4.4).

Theorem 3. *For a simple digraph D,* Min-Sfc *is NP-hard even if G_D is a tree of diameter at most four.*

4. Min-SFC: Tractable Cases

In this section, we show that in some practical cases the problem Min-SFC becomes tractable. We assume in this section that D is a simple digraph, unless otherwise mentioned.

4.1. Acyclic Digraphs

Let $D = (V, A)$ be a digraph. For a vertex $v \in V$, we need to put on v funds with amount at least $\max\{0, \sum_{a \in A_D^+(v)} d(a) - \sum_{a \in A_D^-(v)} d(a)\}$ for clearing all arcs incident from v, where $A_D^+(v)$ (resp., $A_D^-(v)$) denotes the set of all arcs incident from v (resp., to v) in D. Hence, we have $opt(D, d) \geq \sum_{v \in V} \max\{0, \sum_{a \in A_D^+(v)} d(a) - \sum_{a \in A_D^-(v)} d(a)\}$.

If D is an acyclic digraph, a sequence π of A with $p_\pi(V) = \sum_{v \in V} \max\{0, \sum_{a \in A_D^+(v)} d(a) - \sum_{a \in A_D^-(v)} d(a)\}$ can be found in linear time, i.e., Min-SFC is solvable in linear time, as shown in the following lemma.

Lemma 4. *For an acyclic digraph $D = (V, A)$, there exists a sequence π of A such that $p_\pi(V) = \sum_{v \in V} \max\{0, \sum_{a \in A_D^+(v)} d(a) - \sum_{a \in A_D^-(v)} d(a)\}$. Also, such a sequence π can be found in linear time.*

Proof. Any acyclic graph has at least one vertex with in-degree zero. Also note that any subgraph of an acyclic graph is acyclic. Based on these observations, we can obtain a sequence π of A in the following manner:

Step 1: Let $D' := D$ and $i := 1$.
Step 2: While D' has an arc, repeat the following procedures (i) and (ii).
(i) Choose an arc $a = (u, v)$ with $\deg^-_{D'}(u) = 0$ and let $\pi(a) := i$.
(ii) Let $D' := D' \setminus \{a\}$ and $i := i + 1$.

Clearly, π can be computed in linear time. Furthermore, it is not difficult to see that $p_\pi(v) = \max\{0, \sum_{a \in A_D^+(v)} d(a) - \sum_{a \in A_D^-(v)} d(a)\}$ holds for each vertex $v \in V$, since $\pi(a_1) < \pi(a_2)$ holds for every pair of an arc a_1 incident to v and an arc a_2 incident from v. □

4.2. Uniform Digraphs

In the case of uniform debt, Min-SFC is equivalent to the problem which asks to partition a given graph into a minimum number of directed paths, which is known to be solvable in linear time (e.g., see [7, Lemma 2]).

Theorem 5. *If each debt is uniform,* Min-SFC *can be solved in linear time.*

Proof. Let $D = (V, A)$ be a digraph such that all arcs $a \in A$ satisfy $d(a) = c$ for $c \in \mathbb{R}_+$. We first show that a sequence of A for clearing A corresponds to a family of pairwise arc-disjoint directed paths in D whose union is D, where two paths P_1 and P_2 are *arc-disjoint* if $A(P_1) \cap A(P_2) = \emptyset$.

Let π be an arbitrary sequence of A for clearing A. Observe that the residual on a vertex is always a multiple of c during the sequence, since each debt on an arc is c. Hence, for an arc $(u, v) \in A$, if $r_\pi(u, \pi(u, v)) > 0$, then $r_\pi(u, \pi(u, v)) \geq c$ holds and we need not put any additional fund for clearing (u, v); $p_\pi(u, \pi(u, v)) = 0$. Also, note that if $r_\pi(u, \pi(u, v)) = 0$, then $p_\pi(u, \pi(u, v)) = c$. Hence, $p_\pi(V) = c|A^*|$ holds, where $A^* = \{a \in A \mid p_\pi(s(a), \pi(a)) > 0\}$. Furthermore, we can consider that for each arc $a \in A \setminus A^*$, funds used for clearing a (whose amount is c) have been put for clearing some arc in A^* and circulated to $s(a)$. Based on this observation, we can partition $A \setminus A^*$ into $\{A^*(a) \mid a \in A^*\}$, where $A^*(a)$ denotes the set of arcs in $A \setminus A^*$ whose debts are cleared by funds circulated from $a \in A^*$. Note that for each $a \in A^*$, $\{a\} \cup A^*(a)$ composes a directed path starting at a. Thus, from π, we can obtain a family $\mathcal{P}$ of pairwise arc-disjoint directed paths with $|\mathcal{P}| = p_\pi(V)/c$ and $\bigcup_{P \in \mathcal{P}} A(P) = A$.

Conversely, given a family $\mathcal{P}$ of pairwise arc-disjoint directed paths, it is not difficult to see that all arcs in $A(P)$ can be cleared by putting funds with amount c for each path $P \in \mathcal{P}$, since P is a directed path and $d(a) = c$ holds for all $a \in A(P)$. It follows that we can obtain a sequence π of A with $p_\pi(V) = c|\mathcal{P}|$ if $\bigcup_{P \in \mathcal{P}} A(P) = A$.

Thus, an optimal sequence for Min-SFC for D can be obtained from a minimum family $\mathcal{P}^*$ of pairwise arc-disjoint paths in D with $\bigcup_{P \in \mathcal{P}^*} A(P) = A$. The theorem follows since $\mathcal{P}^*$ can be found in linear time (e.g., see [7, Lemma 2]). $\square$

4.3. Symmetric Digraphs

Let $D = (V, A)$ be a digraph. We call a subgraph D' of D a *component* of D if D' is weakly connected and any supergraph of D' is not weakly connected. Let $\mathit{Comp}(D)$ be the family of components in D and $\mathit{comp}(D) = |\mathit{Comp}(D)|$ be the number of components in D. Let A_δ denote the set of arcs a in A with $d(a) \leq \delta$. We denote $\{d(a) \mid a \in A\}$ by $\{\delta_1, \delta_2, \dots, \delta_q\}$ with $\delta_1 < \delta_2 < \cdots < \delta_q$; note that $q \leq |A|$. Then, we have the following lemma about lower bounds on $\mathit{opt}(D, d)$.

Lemma 6. (i) *Let* $D = (V, A)$ *be a digraph and* $D_1 = D[A \setminus A_{\delta_1}]$. *Then, we have* $opt(D, d) \geq \delta_1 + opt(D_1, d_1)$, *where* $d_1(a) = d(a) - \delta_1$ *for all* $a \in A(D_1)$ $(= A \setminus A_{\delta_1})$.
(ii) *For a digraph* $D = (V, A)$, *we have* $opt(D, d) \geq \sum_{i=1}^{q} comp(D[A \setminus A_{\delta_{i-1}}])(\delta_i - \delta_{i-1})$, *where we let* $\delta_0 = 0$.

Proof. (i) Since each arc $a \in A$ satisfies $d(a) \geq \delta_1$, we need funds with amount at least δ_1. Furthermore, even if the funds with amount δ_1 are circulated to be used for clearing all arcs, we need additional funds for clearing debts $d(a) - \delta_1 = d_1(a)$ for each $a \in A \setminus A_{\delta_1}$. Hence, we see that $opt(D, d) \geq \delta_1 + opt(D_1, d_1)$.

(ii) By (i), we have $opt(D, d) \geq \delta_1 + opt(D_1, d_1)$. Note that we have $opt(D_1, d_1) = \sum_{D' \in \mathcal{C}omp(D_1)} opt(D', d_1)$. By applying (i) to each component $D' \in \mathcal{C}omp(D_1)$ of D_1, we have $opt(D', d_1) \geq opt(D'', d_2) + (\delta_2 - \delta_1)$, where $D'' = D[A(D') \setminus A_{\delta_2}]$ and $d_2(a) = d(a) - \delta_2$ for all $a \in A(D'')$. It follows that

$$\begin{aligned} opt(D, d) &\geq \textstyle\sum_{D' \in \mathcal{C}omp(D_1)} \{opt(D[A(D') \setminus A_{\delta_2}], d_2) + (\delta_2 - \delta_1)\} + \delta_1 \\ &= \delta_1 + comp(D_1)(\delta_2 - \delta_1) + \textstyle\sum_{D' \in \mathcal{C}omp(D_1)} opt(D[A(D') \setminus A_{\delta_2}], d_2) \\ &= \delta_1 + comp(D_1)(\delta_2 - \delta_1) + \textstyle\sum_{D' \in \mathcal{C}omp(D_2)} opt(D', d_2), \end{aligned}$$

where $D_2 = D[A \setminus A_{\delta_2}]$. By repeating these observations, the statement (ii) is proved. □

Below, we assume that D is a symmetric digraph, i.e., $d(u, v) = d(v, u)$ holds for all $\{u, v\} \in E(G_D)$. Then, we can show that MIN-SFC is solvable in polynomial time.

Theorem 7. *For a symmetric digraph* D, MIN-SFC *can be solved in* $O(m^2)$ *time.*

Let $D_{\delta_i} = D[A_{\delta_i} \setminus A_{\delta_{i-1}}]$; that is, D_{δ_i} is the subgraph of D induced by the set of arcs with debt δ_i. Let $low(D, d) = \sum_{i=1}^{q} comp(D[A \setminus A_{\delta_{i-1}}])(\delta_i - \delta_{i-1})$. Below, we show that the following algorithm, named MINSYMMETRIC(D, d), returns a sequence π of $A(D)$ with $p_\pi(V(D)) = low(D, d)$; by Lemma 6, such a sequence π is an optimal solution for MIN-SFC. We here say that two sets A_1 and A_2 of arcs are *adjacent* if some two arcs $a_1 \in A_1$ and $a_2 \in A_2$ have an end-vertex in common, i.e., $\{s(a_1), t(a_1)\} \cap \{s(a_2), t(a_2)\} \neq \emptyset$.

Algorithm MINSYMMETRIC(D, d) finds a sequence π' of arcs A' which composes an Eulerian cycle in each component in $\mathcal{C}omp(D[A_{\delta_{q-i+1}}])$ in the i-th iteration, and connects π' and the sequence of A'' whenever A' is adjacent to some set A'' of arcs which has been found in the j-th iteration for some $j < i$. Thus, as a result of the i-th iteration, the algorithm obtains a sequence of arcs which composes an Eulerian cycle in each component in $\mathcal{C}omp(D[A \setminus A_{\delta_{q-i}}])$. A more precise description of the algorithm is given below.

Algorithm MinSymmetric(D, d)
Input: A symmetric digraph $D = (V, A)$.
Output: A sequence π of A such that $p_\pi(V) = low(D, d)$.
Step 1: Let $\mathcal{P} := \emptyset$.
Step 2: For $i = q, q-1, \ldots, 2,1$, execute the following procedures (i) and (ii) for each component $D_0 \in \mathcal{C}omp(D_{\delta_i})$ of D_{δ_i}.

(i) Compute an Eulerian cycle C of D_0, and let π_0 be a sequence of $A(D_0)$ obtained from traversing C, $\mathcal{P} := \mathcal{P} \cup \{\pi_0\}$, and $A_0 := A(D_0)$.

(ii) For each sequence $\pi' \in \mathcal{P}$ of a set A' of arcs such that $A(D_0) \cap A' = \emptyset$ holds and $A(D_0)$ are adjacent to A' (say, for some $a_0 \in A(D_0)$ and $a_1 \in A'$, $t(a_0) = s(a_1)$ holds), execute the following procedure (ii-1) and (ii-2).

(ii-1) Construct a sequence π_1 of $A' \cup A_0$ by inserting π' into π_0 at $t(a_0)(= s(a_1))$ as

$\pi_1(a) := \pi_0(a)$ for all $a \in A_0$ with $\pi_0(a) \leq \pi_0(a_0)$,
$\pi_1(a) := \pi_0(a_0) + 1 + \pi'(a) - \pi'(a_1)$ for all $a \in A'$ with $\pi'(a_1) \leq \pi'(a) \leq |A'|$,
$\pi_1(a) := \pi_0(a_0) + 1 + |A'| - \pi'(a_1) + \pi'(a)$ for all $a \in A'$ with $\pi'(a) < \pi'(a_1)$, and
$\pi_1(a) := \pi_0(a) + |A'|$ for all $a \in A_0$ with $\pi_0(a) > \pi_0(a_0)$.

(ii-2) Let $A_0 := A_0 \cup A'$, $\pi_0 := \pi_1$, and $\mathcal{P} := \mathcal{P} \setminus \{\pi'\}$.

Proof of Theorem 7. Let $D = (V, A)$ be a symmetric digraph. We prove the theorem by showing that Algorithm MinSymmetric(D, d) finds an optimal sequence of A in $O(m^2)$ time.

We first prove that Algorithm MinSymmetric(D, d) finds a sequence π of A with $p_\pi(V) = low(D, d)$. For this, we show by induction that immediately after executing the procedures (i) and (ii) for the ℓ-th selected component of D_{δ_i} during the $(q-i+1)$-th iteration of Step 2, for each sequence π' of a set A' of arcs in the current $\mathcal{P}$,

(I) $D' = D[A']$ satisfies $p_{\pi'}(V[D']) = low(D', d)$,
(II) π' composes an Eulerian cycle of D',

(III) $\pi' + k$ satisfies $p_{\pi'+k}(V[D']) = low(D', d)$ for each $0 \leq k \leq |A'| - 1$, and

(IV) A' is not adjacent to A'' for any sequence of a set A'' of arcs in $\mathcal{P} \setminus \{A'\}$,

where for a sequence π' of an set A' of arcs and an integer k with $0 \leq k \leq |A'| - 1$, let $\pi' + k$ denotes the sequence π'' satisfying $\pi''(a) = (\pi(a) - 1 + k) \pmod{|A'|} + 1$. Note that $\pi' + k$ is the sequence of A' obtained from π' by shifting elements in π' by k positions, and the condition (III) implies that we can start from any arc in A' to clear all arcs in A' with funds with amount $p_{\pi'}(V[D'])$, if the order of A' defined by π' is preserved. Also note that when the algorithm halts, the resulting family $\mathcal{P}''$ of sequences satisfies $|\mathcal{P}''| = 1$ by the connectedness of D, and if the sequence π'' with $\mathcal{P}'' = \{\pi''\}$ satisfies (I), then it is a sequence of A with $p_{\pi''}(V) = low(D, d)$.

Let D_0 be the digraph in $\mathcal{C}omp(D_{\delta_q})$ first selected in the first iteration of Step 2. In the procedure (i), we compute a sequence π_0 of $A(D_0)$ based on an Eulerian cycle of D_0, and insert π_0 into $\mathcal{P}$; π_0 satisfies the condition (II). We here remark that since D is symmetric, each vertex v in D_{δ_i} satisfies $\deg^+_{D_{\delta_i}}(v) = \deg^-_{D_{\delta_i}}(v)$ and each component of D_{δ_i} has an Eulerian cycle for $i \in [q]$. Since $d(a) = \delta_q$ for each $a \in A(D_0)$, it follows that we have $p_{\pi_0}(V(D_0)) = \delta_q$ and $low(D_0, d) = \delta_q$. Hence, it is not difficult to see that the conditions (I) and (III) are satisfied. Clearly, Step 2(ii) is not executed and π_0 satisfies the condition (IV). Thus, at the end of the procedures for D_0, we have $\mathcal{P} = \{\pi_0\}$, and π_0 satisfies (I)–(IV).

Next consider the situation immediately before we select $D_0 \in \mathcal{C}omp(D_{\delta_i})$ of D_{δ_i} in the $(q - i + 1)$-th iteration of Step 2. Then, we assume that in the current $\mathcal{P}$ each sequence satisfies the conditions (I)–(IV), and we will show that the family $\mathcal{P}'$ resulting from the procedures (i) and (ii) for D_0 also satisfies all of these conditions.

In the procedure (i), similarly to the above case, we can observe that D_0 has an Eulerian cycle C_0, $d(a) = \delta_i$ for all $a \in A(D_0)$, and π_0 satisfies (I)–(III). If the procedure (ii) is not executed, then it follows that $A(D_0)$ and A' are not adjacent for any sequence π' of a set A' of arcs in $\mathcal{P} \setminus \{\pi_0\}$, and π_0 satisfies (IV) as well.

We consider the case where the procedure (ii) is executed. Let $\pi'_j \in \mathcal{P}$, $j = 1, 2, \ldots, k$ denote a sequence of a set A'_j of arcs such that $A(D_0) \cap A'_j = \emptyset$ holds and $A(D_0)$ and A'_j are adjacent. Denote by C_j the Eulerian cycle of $D[A'_j]$ corresponding to π'_j. For each $j = 1, 2, \ldots, k$, let $a_{0j} \in A(D_0)$ and

$a_j \in A'_j$ be two arcs with $v_j \equiv t(a_{0j}) = s(a_j)$, at which π'_j and the current π_0 are combined in the procedure (ii); such two arcs exist since the in-degree and out-degree of each vertex are both positive in each of D_0 and $D[A'_j]$. Let π'' be the sequence resulting from the procedure (ii). Observe that π'' is the sequence of $A'' \equiv A(D_0) \cup A'_1 \cup A'_2 \cup \cdots \cup A'_k$ which corresponds to an Eulerian cycle of $D[A'']$ obtained by combining C_0 and C_j at $v_j (= t(a_{0j}) = s(a_j))$; π'' satisfies the condition (II).

We here claim that

$$p_{\pi''}(V(D[A''])) = \delta_i + \sum_{j=1}^{k}(low(D[A'_j], d) - \delta_i). \tag{4.1}$$

Consider the amount of funds needed for clearing A'_j by π'', i.e., $\sum_{a \in A'_j} p_{\pi''}(s(a), \pi''(a))$. From construction, the sequence $\pi''[A'_j]$ restricted to A'_j in π'' corresponds to C_j. By this and the property that π'_j satisfies (I)–(III), we can observe that it is equal to $low(D[A'_j], d) - \delta_i$. It is not difficult to see that funds with amount $low(D[A'_j], d)$ suffice for clearing A'_j even in π'' since π'_j satisfies (I) and (III). Furthermore, since $\pi''^{-1}(1)$ (i.e., the arc first cleared in π'') belongs to A_0 from construction, we can use funds with amount δ_i which are circulated to v_j through an arc in $A(D_0)$. Thus, it follows that $\sum_{a \in A'_j} p_{\pi''}(s(a), \pi''(a)) = low(D[A'_j], d) - \delta_i$. Next we claim that the amount of funds needed for clearing A_0 by π'' is δ_i. Indeed, for each $j \in \{1, 2, \ldots, k\}$, $A'_j \subseteq A'' \setminus A_0$ is cleared between some two arcs $a'_{j1}, a'_{j2} \in A_0$ with $t(a'_{j1}) = s(a'_{j2}) = v_j$, and funds with amount at least δ_i is circulated to v_j by clearing the arc $a'_j \in A'_j$ with $\pi''(a'_j) = \pi''(a'_{j2}) - 1$, and hence it follows that $r(v_j, \pi''(a'_{j2})) \geq \delta_i$; funds with amount δ_i suffice for clearing A_0 even in π'' (note that each debt of an arc in A'_j is at least δ_i and funds with amount at least δ_i is circulated to v_j by clearing a'_j). Thus, (4.1) is proved.

Since $D[A'']$ is connected, each debt of an arc in $D[A'']$ is at least δ_i, and we have $V(D[A'_j]) \cap V(D[A'_{j'}]) = \emptyset$ for each pair j, j' with $j \neq j'$ (by induction hypothesis), it follows that $low(D[A''], d) = \delta_i + \sum_{j=1}^{k}(low(D[A'_j]) - \delta_i)$. Hence, we have $p_{\pi''}(V(D[A''])) = low(D[A''], d)$ by (4.1). Also, by the above arguments, it is not difficult to see that the amount of funds needed for clearing A'' does not depend on which arc in A'' is first cleared. It follows that π'' satisfies (I) and (III). Furthermore, we remark that for each sequence π' of a set A' of arcs in $\mathcal{P} \setminus \{\pi_0, \pi'_1, \pi'_2, \ldots, \pi'_k\}$, A' is adjacent to neither $A(D_0)$ nor A'_j by the conditions of the procedure (ii) and induction hypothesis. Thus, π''

satisfies (IV). Consequently, we can observe that each member in the family $\mathcal{P}'$ resulting from the procedures (i) and (ii) for D_0 also satisfies (I)–(IV).

Thus, we have proved that Algorithm MinSymmetric(D, d) returns a sequence π of A with $p_\pi(V) = low(D, d)$. Also, it can be implemented to run in $O(m \min\{m, q\})$ time, since each iteration of Step 2 can be computed in $O(m)$ time; the running time is $O(m^2)$ in general by $q \leq m$. □

For a tree D, if debts on each vertex is balanced, then debts on each pair of two vertices u and v with $\{u, v\} \in E(G_D)$ becomes symmetric. Therefore, as a corollary of Theorem 7, we can show that Min-SFC for a tree D is solvable in polynomial time if D is balanced.

Corollary 8. *For a balanced tree,* Min-Sfc *can be solved in $O(n^2)$ time.*

Proof. Let $D = (V, A)$ be a balanced tree. By Theorem 7, it suffices to show a claim that each pair of two vertices u and v with $\{u, v\} \in E(G_D)$ is symmetric. For a vertex $r \in V$ chosen arbitrarily, we regard D as a rooted tree with root r. We will show the claim for each arc from leaves to the root in a bottom-up way, where a vertex v is called a *leaf* if the degree of v is one in G_D.

For a leaf v, since debts on v is balanced, we have $d(u, v) = d(v, u)$, where u is the parent of v. Hence, debts on a pair of a leaf and its parent is symmetric.

Let v $(\neq r)$ be a non-leaf and u be the parent of v, and suppose that debts on a pair of v and w has been shown to be symmetric for each child w of v. Since debts on v is balanced, it follows that $d(u, v) = d(v, u)$ holds; debts on a pair of u and v is also symmetric.

Thus, we can conclude that debts on each pair of two vertices composing an arc in D is symmetric. □

Before closing this subsection, we remark that Algorithm MinSymmetric(D, d) can be applied to a wider class of graphs D such that each component in the graph D_{δ_i} has an Eulerian cycle. This follows since the symmetricity of D is used only for showing that each component of D_{δ_i} has an Eulerian cycle in the proof of Theorem 7. Note that each component of D_{δ_i} has an Eulerian cycle if and only if every vertex v in D_{δ_i} satisfies $\deg^-_{D_{\delta_i}}(v) = \deg^+_{D_{\delta_i}}(v)$, i.e., D_{δ_i} is balanced. Namely, we have the following theorem.

Theorem 9. *Let $D = (V, A)$ be a digraph with $\{d(a) \mid a \in A\} = \{\delta_1, \delta_2, \ldots, \delta_q\}$. If D_{δ_i} is balanced for each $i \in [q]$, then Algorithm* MinSymmetric(D, d) *finds an optimal sequence of A for* Min-Sfc *in $O(m^2)$ time.*

In applications of symmetric graphs, we can consider a situation where the maximum amount of funds which one can borrow from the other is established between two banks; it is called a bilateral credit line. When $d(u, v)$ and $d(v, u)$ denote bilateral credit lines between banks u and v for each pair of (u, v) and (v, u), it is naturally assumed that a graph D is symmetric. Assume that one of the banks for each pair faces temporary needs of finance (the assumption would be probable since banks taking the opposite risks are inclined to engage in mutual credit line contracts as insurance). Such a situation would be particularly relevant in times of financial distress, when banks cannot borrow in the market. We could ask what is an optimal sequence of borrowings and repayments through the bilateral credit lines. This problem is similar to but different from the settlement fund circulation problem, since repayment is essentially done after borrowing for each pair. However, the analyses or algorithms for symmetric graphs in this subsection may give some implications to this problem.

4.4. Stars with General Debts

We next consider the case where the underlying graph of $D = (V, A)$ is a star with arbitrary debts. We here remark that the interbank network system in Japan was a kind of star structures before 1997 [12]. We also remark that the discussion about the case of stars will be extended to deal with the case of trees. Throughout this subsection, we assume that for each pair of vertices v and v' in V with $\{v, v'\} \in E(G_D)$, both of (v, v') and (v', v) belong to A; otherwise (say, $(v, v') \notin A$), then we add an arc (v, v') with debt 0 to D and redenote the resulting graph by D (recall that D is simple). Note that the existence of arcs with debt 0 does not affect to the optimal value for Min-SFC.

We first show a series of auxiliary lemmas that hold for more general graphs or graphs that have leaf vertices.

Lemma 10. *Let $D = (V, A)$ be a digraph and π be a sequence of A. For a subset A' of A, let π' be a sequence of A obtained from π by changing positions (in π) of arcs in a subset A' of A. Then, if no arc in A' is incident on a vertex $v \in V$, then we have $p_{\pi'}(v) = p_\pi(v)$.*

Proof. Let A_v be the set of arcs incident on v, i.e., $A_v = A^+_D(v) \cup A^-_D(v)$. How much funds need to be put on v depend only on the arcs in A_v and their order to be cleared. Since $A' \cap A_v = \emptyset$, this order does not change in both of π and π'. Hence, $p_{\pi'}(v) = p_\pi(v)$ holds. $\square$

We call two arcs incident from/to a leaf *leaf arcs*. The following lemmas show some properties about the order of leaf arcs to be cleared.

Lemma 11. *Assume that a digraph D has a leaf v; denote the two arcs incident on v by (u, v) and (v, u). For an arbitrary sequence π of $A(D)$, if $\pi(v, u) < \pi(u, v)$, then $p_\pi(v) = d(v, u)$, and otherwise $p_\pi(v) = \max\{0, d(v, u)$ $-d(u, v)\}$.*

Proof. The lemma follows since if $\pi(v, u) > \pi(u, v)$, then the residual on v is $d(u, v)$ immediately before clearing (v, u) and additional funds with amount $\max\{0, d(v, u) - d(u, v)\}$ suffice for clearing (v, u). $\square$

Lemma 12. *Assume that a digraph $D = (V, A)$ has a leaf v and two leaf arcs (u, v) and (v, u), where $N_D(v) = \{u\}$. For a sequence π of A, the following properties* (i)–(iii) *hold.*
(i) *If $\pi(u, v) < \pi(v, u) - 1$, then we have $p_{\pi'}(V) \leq p_\pi(V)$ for the sequence π' of A obtained from π by moving (v, u) to the position immediately after (u, v).*
(ii) *If $\pi(v, u) < \pi(u, v)$ and $d(v, u) \geq d(u, v)$, then we have $p_{\pi'}(V) \leq p_\pi(V)$ for the sequence π' of A obtained from π by moving (u, v) to the position immediately before (v, u).*
(iii) *If $\pi(v, u) < \pi(u, v)$ and $d(v, u) < d(u, v)$, then we have $p_{\pi'}(V) \leq p_\pi(V)$ for the sequence π' of A obtained from π by moving (v, u) to the position immediately after (u, v).*

Proof. In each statement (i)–(iii), π is modified in a different way. Intuitively, the statement (i) indicates that as (v, u) is cleared sooner, the amount of funds put into u is smaller, since the possibility that funds circulated from v are reused is higher. The statements (ii) and (iii) indicate that the case where (u, v) is cleared before (v, u) is more efficient than the other case, since funds circulated to v can be reused. The difference between (ii) and (iii) depends on which debt of (u, v) and (v, u) is larger; the position of an arc with smaller debt is moved.

(i) Note that $\pi'(a) = \pi(a)$ for all arcs $a \in A$ with $\pi(a) \leq \pi(u, v)$ or $\pi(a) > \pi(v, u)$, $\pi'(a) = \pi(a)+1$ for all $a \in A$ with $\pi(u, v)+1 \leq \pi(a) < \pi(v, u)$, and $\pi'(v,u) = \pi(u,v)+1$. By Lemma 10, we have $p_{\pi'}(V) - p_\pi(V) = p_{\pi'}(\{u,v\}) - p_\pi(\{u,v\})$. Since (u,v) is cleared before (v,u), both in π and π', and no arc other than (v,u) is incident from v, Lemma 11 implies that $p_{\pi'}(v) = p_\pi(v) = \max\{0, d(v,u) - d(u,v)\}$. Also, by $\pi'(v,u) < \pi(v,u)$, it is not difficult to see that $r_{\pi'}(u,\pi'(a)) \geq r_\pi(u,\pi(a))$ and hence $p_{\pi'}(u,\pi'(a)) \leq p_\pi(u,\pi(a))$ holds for all arcs a incident from u. Hence, $p_{\pi'}(V) \leq p_\pi(V)$ holds.

(ii) Note that $\pi'(a) = \pi(a)$ for all arcs $a \in A$ with $\pi(a) < \pi(v,u)$ or $\pi(a) > \pi(u,v)$, $\pi'(a) = \pi(a)+1$ for all $a \in A$ with $\pi(v,u) \leq \pi(a) \leq \pi(u,v)-1$, and $\pi'(u,v) = \pi(v,u)$. By Lemma 10, we have $p_{\pi'}(V) - p_\pi(V) = p_{\pi'}(\{u,v\}) - p_\pi(\{u,v\})$. We can observe from Lemma 11 that $p_\pi(v,\pi(v,u)) = d(v,u)$ and $p_\pi(u,\pi(u,v)) = \max\{0, d(u,v) - r_\pi(u,\pi(u,v))\}$, while $p_{\pi'}(u,\pi'(u,v)) = \max\{0, d(u,v) - r_{\pi'}(u,\pi'(u,v))\}$ and $p_{\pi'}(v,\pi'(v,u)) = d(v,u) - d(u,v)$. Also, note that $p_\pi(v) = p_\pi(v,\pi(v,u))$ and $p_{\pi'}(v) = p_{\pi'}(v,\pi(v,u))$, since no other arc other than (v,u) is incident from v. The residual on u after clearing (v,u) in π' is less than that in π by $\min\{d(u,v), r_{\pi'}(u,\pi'(u,v))\}$ (note that $r_\pi(u,\pi(v,u)) = r_{\pi'}(u,\pi'(u,v))$). It follows that the total funds needed to be put on u for clearing arcs in $\{a \in A \setminus \{(u,v)\} \mid \pi(a) > \pi(v,u)\}$ in π' is more than that in π by at most $\min\{d(u,v), r_{\pi'}(u,\pi'(u,v))\}$. Hence, we have

$$\begin{aligned}
p_{\pi'}(V) - p_\pi(V) &\leq p_{\pi'}(u,\pi'(u,v)) + p_{\pi'}(v,\pi'(v,u)) - p_\pi(u,\pi(u,v)) \\
&\qquad -p_\pi(v,\pi(v,u)) + \min\{d(u,v), r_{\pi'}(u,\pi'(u,v))\} \\
&= -p_\pi(u,\, \pi(u,v)) \\
&\leq 0.
\end{aligned}$$

(iii) Note that $\pi'(a) = \pi(a)$ for all $a \in A$ with $\pi(a) < \pi(v,u)$ or $\pi(a) > \pi(u,v)$, $\pi'(a) = \pi(a) - 1$ for all $a \in A$ with $\pi(v,u)+1 \leq \pi(a) \leq \pi(u,v)$, and $\pi'(v,u) = \pi(u,v)$. By Lemma 10, we have $p_{\pi'}(V) - p_\pi(V) = p_{\pi'}(\{u,v\}) - p_\pi(\{u,v\})$. We can observe from Lemma 11 that $p_\pi(v) = p_\pi(v,\pi(v,u)) = d(v,u)$ and $p_\pi(u,\pi(u,v)) = \max\{0,\, d(u,v) - r_\pi(u,\pi(u,v))\}$ as above, while $p_{\pi'}(u,\pi'(u,v)) = \max\{0, d(u,v) - r_{\pi'}(u,\pi'(u,v))\}$ and $p_{\pi'}(v) = p_{\pi'}(v,\, \pi'(v,u)) = 0$ (note that $d(v,u) < d(u,v)$). Let a_1 be the arc cleared immediately after (v,u) in π ($a_1 = (u,v)$ may hold). The residual on u immediately before clearing a_1 in π' is less than that in π by $d(v,u)$. It follows that the total funds needed to be put on u for clearing arcs in $A_1 = \{a \in A \mid \pi(v,u) < \pi(a) \leq \pi(u,v)\}$ in π' is more than that in π by at most $d(v,u)$ (note that A_1 includes (u,v)). On the other hand, the residual on u after clearing (v,u) in π' is at least that in π, i.e., we have $r_{\pi'}(u,\pi'(a)) \geq r_\pi(u,\pi(a))$ for all $a \in A_2 = \{e \in A \mid \pi(a) > \pi(u,v)\}$ and the total fund needed on u for clearing A_2 in π' is at most that in π. Hence, we have $p_{\pi'}(V) - p_\pi(V) \leq p_{\pi'}(v, \pi'(v,u)) - p_\pi(v, \pi(v,u)) + d(v,u) = 0$. □

Lemma 13. *Assume that a digraph $D = (V, A)$ has four leaf arcs (u, v_1), (v_1, u), (u, v_2), and (v_2, u) for two leaves v_1 and v_2. Let π be a sequence of A such that $\pi(v_2, u) = \pi(u, v_2) + 1 = \pi(v_1, u) + 2 = \pi(u, v_1) + 3$, and π' be the sequence of A obtained from π by exchanging the position of the ordered pair $((u, v_1), (v_1, u))$ and that of $((u, v_2), (v_2, u))$. If one of the following conditions* (i)–(iii) *holds, then $p_{\pi'}(V) \leq p_\pi(V)$ holds.*
(i) *$d(u, v_1) > d(v_1, u)$ and $d(u, v_2) \leq d(v_2, u)$.*
(ii) *$d(u, v_1) \leq d(v_1, u)$, $d(u, v_2) \leq d(v_2, u)$, and $d(u, v_1) > d(u, v_2)$.*
(iii) *$d(u, v_1) > d(v_1, u)$, $d(u, v_2) > d(v_2, u)$, and $d(v_1, u) < d(v_2, u)$.*

Proof. Note that $\pi'(u, v_1) = \pi(u, v_2)$, $\pi'(v_1, u) = \pi(v_2, u)$, $\pi'(u, v_2) = \pi(u, v_1)$, $\pi'(v_2, u) = \pi(v_1, u)$, and $\pi'(a) = \pi(a)$ for all other arcs a. By Lemma 10, we have $p_{\pi'}(V) - p_\pi(V) = p_{\pi'}(\{u, v_1, v_2\}) - p_\pi(\{u, v_1, v_2\})$. For $i = 1, 2$, since $\pi(u, v_i) < \pi(v_i, u)$ and $\pi'(u, v_i) < \pi'(v_i, u)$ hold and no arc in A other than (v_i, u) is incident from v_i, Lemma 11 implies that $p_{\pi'}(v_i) = p_\pi(v_i) = \max\{0, d(v_i, u) - d(u, v_i)\}$. Thus, $p_{\pi'}(V) - p_\pi(V) = p_{\pi'}(u) - p_\pi(u)$ holds. Let $A_1 = \{(u, v_1), (v_1, u), (u, v_2), (v_2, u)\}$. We consider the amount q^1_π (resp., $q^1_{\pi'}$) of funds needed on u for clearing A_1 and the amount q^2_π (resp., $q^2_{\pi'}$) of funds remaining on u after clearing A_1 among the funds with amount q^1_π (resp., $q^1_{\pi'}$) in π (resp., π'). We here note that q^1_π is the summation of the amount of funds put on u and that of funds having been remaining on u by the sequence before clearing A_1, which are used for clearing A_1 in π. Also note that q^2_π is the amount of funds which returns to u through (v_1, u) and (v_2, u) during the sequence for clearing A_1 and still remains on u immediately after clearing A_1. Observe that $q^1_\pi = d(u, v_1) + \max\{0, d(u, v_2) - d(v_1, u)\}$, $q^1_{\pi'} = d(u, v_2) + \max\{0, d(u, v_1) - d(v_2, u)\}$, $q^2_\pi = \max\{0, d(v_1, u) - d(u, v_2)\} + d(v_2, u)$, and $q^2_{\pi'} = \max\{0, d(v_2, u) - d(u, v_1)\} + d(v_1, u)$.

Below, we give a proof of each statement (i)–(iii). Intuitively, the statement (i) indicates that a strategy in which a pair $\{(u, v), (v, u)\}$ with $d(u, v) \leq d(v, u)$ precedes a pair $\{(u, v'), (v', u)\}$ with $d(u, v') > d(v', u)$ is more efficient, since after clearing the former pair, more funds are circulated to u than those put on u for clearing the arc outgoing from u in the pair. The statement (ii) indicates that among pairs $\{(u, v), (v, u)\}$ with $d(u, v) \leq d(v, u)$, a sequence based on a nondecreasing order of $d(u, v)$ is efficient, since funds put on u are saved (note that more funds are circulated after clearing each pair than that put on u for clearing). The statement (iii) indicates that among pairs $\{(u, v), (v, u)\}$ with $d(u, v) > d(v, u)$, a sequence based on a nonincreasing order of $d(v, u)$ is efficient, since more funds circulated to u can be reused.

(i) We first consider the case of $d(u, v_2) \geq d(v_1, u)$; $d(v_2, u) \geq d(u, v_2) \geq d(v_1, u)$ holds by the assumption that $d(v_2, u) \geq d(u, v_2)$. Then, we have $q^1_\pi = d(u, v_1) + d(u, v_2) - d(v_1, u)$ and $q^2_\pi = d(v_2, u)$. There are two possible cases (I) $d(u, v_1) \geq d(v_2, u)$ and (II) $d(u, v_1) < d(v_2, u)$.

In the case of (I), we have $q^1_{\pi'} = d(u, v_2) + d(u, v_1) - d(v_2, u)$ and $q^2_{\pi'} = d(v_1, u)$. Hence, $q^1_{\pi'} - q^1_\pi = q^2_{\pi'} - q^2_\pi = -d(v_2, u) + d(v_1, u) \leq 0$ holds by $d(v_2, u) \geq d(u, v_2) \geq d(v_1, u)$. This means that the amount of funds needed on u for clearing A_1 in π' is less than that in π by $d(v_2, u) - d(v_1, u)$, while the residual on u after clearing A_1 in π' is less than that in π by at most $d(v_2, u) - d(v_1, u)$. Hence, for clearing all arcs processed after A_1 in π (i.e., $\{a \in A \mid \pi(a) > \pi(v_2, u)\}$), we may need more fund in π' than in π, while additional funds with amount at most $d(v_2, u) - d(v_1, u)$ suffice. It follows that $p_{\pi'}(u) \leq p_\pi(u)$.

In the case of (II), we have $q^1_{\pi'} = d(u, v_2)$ and $q^2_{\pi'} = d(v_2, u) - d(u, v_1) + d(v_1, u)$. Hence, $q^1_{\pi'} - q^1_\pi = q^2_{\pi'} - q^2_\pi = -d(u, v_1) + d(v_1, u) < 0$ holds by the assumption that $d(u, v_1) > d(v_1, u)$. Similarly to the previous case, it follows that $p_{\pi'}(u) \leq p_\pi(u)$.

We next consider the case of $d(u, v_2) < d(v_1, u)$; $d(u, v_1) > d(v_1, u) > d(u, v_2)$ holds by the assumption that $d(u, v_1) > d(v_1, u)$. Then, we have $q^1_\pi = d(u, v_1)$ and $q^2_\pi = d(v_1, u) - d(u, v_2) + d(v_2, u)$. If $d(u, v_1) \geq d(v_2, u)$, then we have $q^1_{\pi'} = d(u, v_2) + d(u, v_1) - d(v_2, u)$, $q^2_{\pi'} = d(v_1, u)$, and $q^1_{\pi'} - q^1_\pi = q^2_{\pi'} - q^2_\pi = d(u, v_2) - d(v_2, u) \leq 0$ (by the assumption that $d(u, v_2) \leq d(v_2, u)$). If $d(u, v_1) < d(v_2, u)$, then we have $q^1_{\pi'} = d(u, v_2)$, $q^2_{\pi'} = d(v_2, u) - d(u, v_1) + d(v_1, u)$, and $q^1_{\pi'} - q^1_\pi = q^2_{\pi'} - q^2_\pi = d(u, v_2) - d(u, v_1) < 0$ (by $d(u, v_1) > d(v_1, u) > d(u, v_2)$). Thus, $p_{\pi'}(u) \leq p_\pi(u)$ holds.

(ii) Observe that $q^1_\pi = d(u, v_1) + \max\{0, d(u, v_2) - d(v_1, u)\} = d(u, v_1)$ and $q^2_\pi = \max\{0, d(v_1, u) - d(u, v_2)\} + d(v_2, u) = d(v_1, u) - d(u, v_2) + d(v_2, u)$, since $d(v_1, u) \geq d(u, v_1) > d(u, v_2)$ holds by the assumption that $d(u, v_1) \leq d(v_1, u)$ and $d(u, v_1) > d(u, v_2)$. If $d(v_2, u) \geq d(u, v_1)$, then $q^1_{\pi'} - q^1_\pi = q^2_{\pi'} - q^2_\pi = d(u, v_2) - d(u, v_1) < 0$. If $d(v_2, u) < d(u, v_1)$, then $q^1_{\pi'} - q^1_\pi = q^2_{\pi'} - q^2_\pi = d(u, v_2) - d(v_2, u) \leq 0$. Hence, in both cases, we can see that $p_{\pi'}(u) \leq p_\pi(u)$, in a similar way to the above arguments.

(iii) Observe that $q^1_\pi = d(u, v_1) + \max\{0, d(u, v_2) - d(v_1, u)\} = d(u, v_1) + d(u, v_2) - d(v_1, u)$ and $q^2_\pi = \max\{0, d(v_1, u) - d(u, v_2)\} + d(v_2, u) = d(v_2, u)$, since $d(u, v_2) > d(v_2, u) > d(v_1, u)$ holds by the assumption that $d(u, v_2) >$

$d(v_2, u)$ and $d(v_2, u) > d(v_1, u)$. If $d(v_2, u) \geq d(u, v_1)$, then $q^1_{\pi'} - q^1_\pi = q^2_{\pi'} - q^2_\pi = d(v_1, u) - d(u, v_1)$ (< 0). If $d(v_2, u) < d(u, v_1)$, then $q^1_{\pi'} - q^1_\pi = q^2_{\pi'} - q^2_\pi = d(v_1, u) - d(v_2, u)$ (< 0). Hence, in both cases, we can see that $p_{\pi'}(u) \leq p_\pi(u)$ in a similar way to the above arguments. □

Now let $D = (V, A)$ be a star with center u. Then, $E(G_D) = \{\{u, v\} \mid v \in V \setminus \{u\}\}$ holds. Let $V^+ = \{v \in V \setminus \{u\} \mid d(v, u) \geq d(u, v)\}$ and $V^- = \{v \in V \setminus \{u\} \mid d(v, u) < d(u, v)\}$. From Lemmas 12 and 13, we have the following theorem about an optimal solution for MIN-SFC for a star.

Theorem 14. *Let $D = (V, A)$ be a star with center u. There exists an optimal sequence π of A for* MIN-SFC *satisfying the following* (a)–(d)*:*
(a) $\pi(u, v) = \pi(v, u) - 1$ *for all* $v \in V \setminus \{u\}$.
(b) $\pi(u, v) < \pi(u, v')$ *for all* $v \in V^+$ *and* $v' \in V^-$.
(c) $\pi(u, v) < \pi(u, v')$ *if and only if* $d(u, v) \leq d(u, v')$ *for all* $v, v' \in V^+$.
(d) $\pi(u, v) < \pi(u, v')$ *if and only if* $d(v, u) \geq d(v', u)$ *for all* $v, v' \in V^-$.

Proof. Let π^* be an optimal sequence of A for MIN-SFC. By applying Lemma 12 to each pair of (u, v) and (v, u) violating (a), we can obtain another optimal sequence π^*_1 of A satisfying (a).

Assume that π^*_1 does not satisfy (b). Then, by $V^+ \cup V^- = V \setminus \{u\}$, there exist two vertices $v_1 \in V^-$ and $v_2 \in V^+$ such that $\pi(v_2, u) = \pi(u, v_2) + 1 = \pi(v_1, u) + 2 = \pi(u, v_1) + 3$. By applying Lemma 13(i), we can obtain another optimal sequence π^*_2 of A by exchanging the position of the ordered pair $((u, v_1), (v_1, u))$ and that of $((u, v_2), (v_2, u))$. Clearly, π^*_2 also satisfies (a). Hence, by repeating applying Lemma 13(i), we can obtain an optimal sequence π^*_3 of A satisfying (a) and (b).

Similarly, by applying Lemma 13(ii), we can obtain from π^*_3 an optimal sequence π^*_4 of A satisfying (a)–(c). Also, by applying Lemma 13(iii), we can obtain from π^*_4 an optimal sequence π^*_5 of A satisfying (a)–(d). □

This theorem shows that we can obtain an optimal solution of MIN-SFC by the following algorithm MINSTAR(D, d).

Algorithm MINSTAR(D, d)
Input: A star $D = (V, A)$ with center u and a debt function d.
Output: A sequence π of A such that $p_\pi(V)$ is minimized.
Step 1: Order vertices of V^+ such that $d(u, v_1) \leq d(u, v_2) \leq \cdots \leq d(u, v_{|V^+|})$ and let $\pi(u, v_i) = 2i - 1$ and $\pi(v_i, u) = 2i$ for $i = 1, 2, \ldots, |V^+|$.

Step 2: Order vertices of V^- such that $d(v_{|V^+|+1}, u) \geq d(v_{|V^+|+2}, u) \geq \cdots \geq d(v_{|V^+|+|V^-|}, u)$ and let $\pi(u, v_i) = 2i - 1$ and $\pi(v_i, u) = 2i$ for $i = |V^+| + 1, |V^+| + 2, \ldots, |V^+| + |V^-|$.

It is fairly straightforward to see that the time complexity of this algorithm is $O(n \log n)$, since it is dominated by that of sorting $O(n)$ arcs.

Theorem 15. *For a star,* MIN-SFC *can be solved in $O(n \log n)$ time.*

Before closing this section, we show the following property about funds put on the center of a star.

Lemma 16. *Let $D = (V, A)$ be a star with center u, and π^* be an optimal sequence of A obtained by applying Algorithm* MINSTAR(D, d) *to D. Then, we have $p_{\pi^*}(u) = p_{\pi^*}(V) - \sum_{v \in V \setminus \{u\}} \max\{0,\, d(v, u) - d(u, v)\}$.*

Proof. By construction, we have $\pi^*(u, v) < \pi^*(v, u)$ for all $v \in V \setminus \{u\}$. It follows from Lemma 11 that $p_{\pi^*}(v) = \max\{0, d(v, u) - d(u, v)\}$ for all $v \in V \setminus \{u\}$. $\square$

4.5. Trees with General Debts

We consider the case where the underlying graph of D is a tree. As shown in Corollary 8, the case where a given tree is balanced can be solved in polynomial time. In this subsection, we will show that MIN-SFC is fixed-parameter tractable with respect to the maximum degree Δ for trees, and that MIN-SFC is NP-hard even for trees, which proves Theorem 3.

Throughout this subsection, we assume that for each pair of vertices v and v' in V with $\{v, v'\} \in E(G_D)$, both of (v, v') and (v', v) belong to A.

A Fixed-parameter Algorithm

We first show the following theorem.

Theorem 17. *For a tree D,* MIN-SFC *can be solved in $O(2^{\Delta(D)} n \log n)$ time.*

As a corollary of this theorem, we can see that MIN-SFC for paths is solvable in polynomial time.

Corollary 18. *For a path,* MIN-SFC *can be solved in $O(n \log n)$ time.*

Before proving Theorem 17, we prepare some auxiliary lemmas. For a leaf v in a digraph D, *splitting* v is to introduce a new vertex v' and to replace the arc $(u,v) \in A(D)$ incident to v with an arc (u,v') with debt $d(u,v)$. We denote the resulting digraph and its debt function by $D_{v,v'}$ and $d_{v,v'}$, respectively.

Lemma 19. *Assume that a digraph $D=(V,A)$ has a leaf v; denote the two arcs incident on v by (u,v) and (v,u). Let π be a sequence of $A(D_{v,v'})$. If $\pi(u,v') < \pi(v,u)$, then we have $p_{\pi'}(V(D_{v,v'})) \leq p_\pi(V(D_{v,v'}))$ for the sequence π' of $A(D_{v,v'})$ obtained from π by changing the position of (v,u) before (u,v').*

Proof. By Lemma 10, we have $p_{\pi'}(V(D_{v,v'})) - p_\pi(V(D_{v,v'})) = p_{\pi'}(\{u,v,v'\})$ $-p_\pi(\{u,v,v'\})$. Also, we have $p_{\pi'}(v) = p_\pi(v) = d(v,u)$ and $p_{\pi'}(v') = p_\pi(v') = 0$. Hence, $p_{\pi'}(V(D_{v,v'})) - p_\pi(V(D_{v,v'})) = p_{\pi'}(u) - p_\pi(u)$ holds. Also, by $\pi'(v,u) < \pi(v,u)$, it is not difficult to see that $r_{\pi'}(u,\pi'(a)) \geq r_\pi(u,\pi(a))$ and hence $p_{\pi'}(u,\pi'(a)) \leq p_\pi(u,\pi(a))$ hold for all arcs a incident from u. Hence, it follows that $p_{\pi'}(V(D_{v,v'})) \leq p_\pi(V(D_{v,v'}))$. □

Lemma 20. *Assume that a digraph $D=(V,A)$ has a leaf v; denote the two arcs incident on v by (u,v) and (v,u). Let π be a sequence of A with $\pi(u,v) > \pi(v,u)$ and π' be the sequence of $A(D_{v,v'})$ such that $\pi'(u,v') = \pi(u,v)$ and $\pi'(a) = \pi(a)$ for all other arcs $a \in A \setminus \{(u,v)\}$ $(= A(D_{v,v'}) \setminus \{(u,v')\})$. Then, π' is an optimal sequence of* Min-SFC *for $D_{v,v'}$ if and only if π is an optimal sequence for D under the assumption that (v,u) is cleared before (u,v).*

Proof. Note that $p_\pi(V) = p_{\pi'}(V(D_{v,v'}))$. First, we show "only-if" part. Assume for contradiction that π' is an optimal sequence of Min-SFC for $D_{v,v'}$, and that some sequence π_1 of A satisfies $p_{\pi_1}(V) < p_\pi(V)$ and $\pi_1(u,v) > \pi_1(v,u)$. Let π_1' be the sequence of $A(D_{v,v'})$ such that $\pi_1'(u,v') = \pi_1(u,v)$ and $\pi_1'(a) = \pi_1(a)$ for all other arcs $a \in A \setminus \{(u,v)\}$. Clearly, we have $p_{\pi_1'}(V(D_{v,v'})) = p_{\pi_1}(V) < p_\pi(V) = p_{\pi'}(V(D_{v,v'}))$, contradicting the minimality of $p_{\pi'}(V(D_{v,v'}))$.

We here show "if" part. Assume for contradiction that π is an optimal sequence of Min-SFC for D under the assumption that (v,u) is cleared before (u,v), and that some sequence π_1' of $A(D_{v,v'})$ satisfies $p_{\pi_1'}(V(D_{v,v'})) < p_{\pi'}(V(D_{v,v'}))$. Then, from Lemma 19, we can assume that $\pi_1'(u,v') > \pi_1'(v,u)$. Let π_1 be the sequence of A such that $\pi_1(u,v) = \pi_1'(u,v')$ and $\pi_1(a) = \pi_1'(a)$ for all other arcs $a \in A \setminus \{(u,v)\}$. Clearly, we have $p_{\pi_1}(V) = p_{\pi_1'}(V(D_{v,v'})) <$

$p_{\pi'}(V(D_{v,v'})) = p_\pi(V)$ and $\pi_1(u,v) > \pi_1(v,u)$, contradicting the minimality of $p_\pi(V)$. $\square$

For a vertex u in a tree D, let D_0 be the star induced by $\{u\} \cup N_D(u)$, and $D_1, D_2, \ldots, D_q$ be subtrees in the graph obtained from D by deleting u, where $q = |N_D(u)|$. We denote two arcs connecting u and D_i by (u, v_i) and (v_i, u), where $v_i \in V(D_i)$. The following lemma shows that if we know in advance whether (u, v_i) is cleared after (v_i, u) or not for each $v_i \in N_D(u)$, then the minimum amount $opt(D, d)$ of funds for clearing $A(D)$ follows from optimal solutions for the star D_0, and either trees $D_i + u$ or $(D_i + u)_{u,u'}$ obtained from $D_i + u$ by splitting u, where for a subgraph D' of D and a vertex $u \in V \setminus V(D')$, we denote $(V(D') \cup \{u\}, A(D') \cup \bigcup_{v \in N_D(u) \cap V(D')} \{(u,v),(v,u)\})$ by $D' + u$.

Lemma 21. *For a vertex u in a digraph $D = (V, A)$, let v_i, D_0, D_i, and $D_i + u$, $i = 1, 2, \ldots, q$ be defined as above. Let N_1 and N_2 be a partition of $N_D(u)$ (N_1 or N_2 may be empty). Let $opt(D, d, u, N_1, N_2)$ denote the minimum amount of funds put on V for clearing all arcs in A under the assumption that (v, u) is cleared before (u, v) for each $v \in N_1$ and (u, v) is cleared before (v, u) for each $v \in N_2$. Then,*

$$\begin{aligned} opt(D,d,u,N_1,N_2) \;=\;& opt((D_0)_{N_1,N_1'}, d_{N_1}) - \textstyle\sum_{v \in N_1} d(v,u) \\ & - \textstyle\sum_{v \in N_2} \max\{0, d(v,u) - d(u,v)\} \\ & + \textstyle\sum_{v_i \in N_1} (opt(D_i + u, d) - \max\{0, d(u,v_i) - d(v_i,u)\}) \\ & + \textstyle\sum_{v_i \in N_2} (opt((D_i + u)_{u,u'}, d_{u,u'}) - d(u,v_i)), \end{aligned}$$

where $(D_0)_{N_1,N_1'}$ denotes the star obtained from the star D_0 by splitting all vertices in N_1, N_1' denotes the set of vertices generated by these splitting operations, and d_{N_1} denotes the resulting debt function on $A((D_0)_{N_1,N_1'})$.

Proof. Let

$$\begin{aligned} f(D,d,u,N_1,N_2) \;=\;& opt((D_0)_{N_1,N_1'}, d_{N_1}) - \textstyle\sum_{v \in N_1} d(v,u) \\ & - \textstyle\sum_{v \in N_2} \max\{0, d(v,u) - d(u,v)\} \\ & + \textstyle\sum_{v_i \in N_1} (opt(D_i + u, d) - \max\{0, d(u,v_i) - d(v_i,u)\}) \\ & + \textstyle\sum_{v_i \in N_2} (opt((D_i + u)_{u,u'}, d_{u,u'}) - d(u,v_i)). \end{aligned}$$

Let π be an arbitrary sequence of A such that $\pi(v,u) < \pi(u,v)$ for each $v \in N_1$ and $\pi(u,v) < \pi(v,u)$ for each $v \in N_2$. First we show that $p_\pi(V) \geq f(D,d,u,N_1,N_2)$, from which $opt(D,d,u,N_1,N_2) \geq f(D,d,u,N_1,N_2)$. We will consider lower bounds L_1, L_2, and L_3 on $p_\pi(u)$, $\sum_{v_i \in N_1} p_\pi(V(D_i))$, and $\sum_{v_i \in N_2} p_\pi(V(D_i))$, respectively; we have $p_\pi(V) \geq L_1 + L_2 + L_3$.

Consider a lower bound on $p_\pi(u)$. Since how much funds need to be put on u depends only on debts of arcs incident from/to u, we consider the minimum amount p^* of funds for clearing all arcs in the star D_0 with center u. By the assumption that (v_i, u) is cleared before (u, v_i) for each $v_i \in N_1$ and (u, v_i) is cleared before (v_i, u) for each $v_i \in N_2$ and Lemma 20 for leaves $v_i \in N_1$ of D_0, we can see that $p^* = \mathit{opt}((D_0)_{N_1, N_1'}, d_{N_1})$. Also by the assumption and Lemma 11, any sequence π' of $A((D_0)_{N_1, N_1'})$ satisfies $p_{\pi'}(N_{(D_0)_{N_1, N_1'}}(u)) = \sum_{v \in N_1} d(v, u) + \sum_{v \in N_2} \max\{0, d(v, u) - d(u, v)\}$. Hence, the amount $p_\pi(u)$ of funds put on u is at least $\mathit{opt}((D_0)_{N_1, N_1'}, d_{N_1}) - (\sum_{v \in N_1} d(v, u) + \sum_{v \in N_2} \max\{0, d(v, u) - d(u, v)\})$.

Consider a lower bound on $p_\pi(V(D_i))$ for $v_i \in N_1$. Note that how much funds need to be put on $V(D_i)$ depends on debts of $A(D_i) \cup \{(u, v_i), (v_i, u)\}$. By taking into account the assumption that (v_i, u) is cleared before (u, v_i), we can observe that $p_\pi(V(D_i))$ is at least the minimum amount of funds put on $V(D_i)$ among any funds for clearing $A(D_i + u)$ in $D_i + u$. Here notice that the amount of funds put on u in $D_i + u$ is always $\max\{0, d(u, v_i) - d(v_i, u)\}$, by the assumption and Lemma 11 for the leaf u of $D_i + u$. It follows that $p_\pi(V(D_i)) \geq \mathit{opt}(D_i + u, d) - \max\{0, d(u, v_i) - d(v_i, u)\}$.

Consider a lower bound on $p_\pi(V(D_i))$ for $v_i \in N_2$. Note that how much funds need to be put on $V(D_i)$ depends on debts of $A(D_i) \cup \{(u, v_i), (v_i, u)\}$. By taking into account the assumption that (u, v_i) is cleared before (v_i, u) and Lemma 20 for the leaf u of $D_i + u$, we can observe that $p_\pi(V(D_i))$ is at least the minimum amount of funds put on $V(D_i)$ among any funds for clearing $A((D_i + u)_{u,u'})$ in $(D_i + u)_{u,u'}$. Note that the amount of funds put on u in $(D_i + u)_{u,u'}$ is always $d(u, v_i)$ by Lemma 11 for the leaf u of $(D_i + u)_{u,u'}$. It follows that $p_\pi(V(D_i)) \geq \mathit{opt}((D_i + u)_{u,u'}, d_{u,u'}) - d(u, v_i)$.

Thus, we can see that $p_\pi(V) \geq f(D, d, u, N_1, N_2)$. Finally, we show that some sequence π^* of A satisfies $p_{\pi^*}(V) = f(D, d, u, N_1, N_2)$; π^* is optimal and proves this lemma. Let π_0' be a sequence of $A((D_0)_{N_1, N_1'})$ obtained by applying Algorithm $\textsc{MinStar}((D_0)_{N_1, N_1'}, d_{N_1})$, and π_0 be the sequence of $A(D_0)$ obtained from π_0' by letting $\pi_0(u, v) = \pi_0'(u, v')$ for all $v \in N_1$ and $\pi_0(a) = \pi_0'(a)$ for all other arcs a incident on u. By Lemmas 16 and 20, we have $p_{\pi_0}(u) = p_{\pi_0'}(u) = \mathit{opt}((D_0)_{N_1, N_1'}, d_{N_1}) - (\sum_{v \in N_1} d(v, u) + \sum_{v \in N_2} \max\{0, d(v, u) - d(u, v)\})$. For a tree D_i with $v_i \in N_1$, let π_i be a sequence of $A(D_i + u)$ with $p_{\pi_i}(V(D_i + u)) = \mathit{opt}(D_i + u, d)$. Then, we can assume that $\pi_i(v_i, u) < \pi_i(u, v_i)$, since otherwise we can apply Lemma 12 to rechoose an optimal sequence π'_i of $A(D_i + u)$ with $\pi'_i(v_i, u) < \pi'_i(u, v_i)$. Let $\pi_{i,1}$ be the subsequence of π_i before (v_i, u), $\pi_{i,2}$ be the subsequence of π_i after (v_i, u) and before (u, v_i), and $\pi_{i,3}$ be the subsequence of π_i after (u, v_i). For a tree D_i with $v_i \in N_2$, let π_i be a sequence of $A((D_i + u)_{u,u'})$ with $p_{\pi_i}(V((D_i + u)_{u,u'})) = \mathit{opt}((D_i + u)_{u,u'}, d_{u,u'})$. Then, we can assume that $\pi_i(v_i, u') > \pi_i(u, v_i)$, since otherwise we can apply Lemma 19 to rechoose an optimal sequence π'_i of $A((D_i + u)_{u,u'})$ with $\pi'_i(v_i, u') > \pi'_i(u, v_i)$. Let $\pi_{i,1}$ be the subsequence of π_i before (u, v_i), $\pi_{i,2}$ be the subsequence of π_i after (u, v_i) and before (v_i, u'), and $\pi_{i,3}$ be the subsequence of π_i after (v_i, u'). Let π^* be a sequence of A obtained by the following procedure:

Step 1: Let $\pi^* := \pi_0$.

Step 2: For each tree D_i, execute the following procedure: If $v_i \in N_1$, then insert $\pi_{i,1}$, $\pi_{i,2}$, and $\pi_{i,3}$ into the position immediately before (v_i, u) in π^*, the position immediately after (v_i, u) in π^*, and the position immediately after (u, v_i) in π^*, respectively. If $v_i \in N_2$, then insert $\pi_{i,1}$, $\pi_{i,2}$, and $\pi_{i,3}$ into the position immediately before (u, v_i) in π^*, into the position immediately after (u, v_i) in π^*, and into the position immediately after (v_i, u) in π^*, respectively.

From the above discussion, it is not difficult to see that the resulting sequence π^* clears all arcs in A and $p_{\pi^*}(V) = f(D, d, u, N_1, N_2)$ holds. □

Let $D = (V, A)$ be a tree. Based on Lemma 21, we will give a dynamic programming algorithm for finding an optimal sequence of A in $O(2^{\Delta} n)$ time, which proves Theorem 17.

Here, for a vertex $r \in V$ chosen arbitrarily, we regard D as a rooted tree with root r. For a vertex u in D, let $\mathit{pa}(u)$ be the parent of u if it exists, $Ch(u)$ be the children of u, and $D(u)$ be the subtree of D rooted at u. For a partition $\{N_1, N_2\}$ of $Ch(u)$, we define $\mathit{opt}_+(u, N_1, N_2)$ (resp., $\mathit{opt}_-(u, N_1, N_2)$) as the minimum amount of funds clearing $A(D(u) + \mathit{pa}(u))$ under the assumption that (v, u) is cleared before (u, v) for each $v \in N_1$ (resp., $v \in N_1 \cup \{\mathit{pa}(u)\}$) and (u, v) is cleared before (v, u) for each $v \in N_2 \cup \{\mathit{pa}(u)\}$ (resp., $v \in N_2$) (note that the notation '+' indicates the assumption that $(u, \mathit{pa}(u))$ (i.e., the arc outgoing from u into $\mathit{pa}(u)$) is cleared before $(\mathit{pa}(u), u)$). Note that $\mathit{opt}_+(r, N_1, N_2) = \mathit{opt}_-(r, N_1, N_2)$. Let $\mathit{opt}^*_i(u) = \min\{\mathit{opt}_i(u, N_1, N_2) \mid N_1 \subseteq Ch(u)\}$ for $i \in \{+, -\}$. We here remark that $\mathit{opt}^*_+(u)$ (resp., $\mathit{opt}^*_-(u)$) is the minimum amount of funds for clearing $A(D(u) + \mathit{pa}(u))$ under the assumption that $(u, \mathit{pa}(u))$ (resp., $(\mathit{pa}(u), u)$) is cleared before $(\mathit{pa}(u), u)$ (resp., $(u, \mathit{pa}(u))$).

Our dynamic programming algorithm proceeds in a bottom-up manner in D, while computing these two values $opt^*_+(u)$ and $opt^*_-(u)$ for each vertex u in D. We remark that $opt^*_+(r) = opt^*_-(r)$ is the optimal value for Min-SFC. Lemma 21 indicates that $opt_+(u, N_1, N_2)$ and $opt_-(u, N_1, N_2)$ can be computed by using $opt^*_+(v)$ and $opt^*_-(v)$ for $v \in Ch(u)$. Namely, we have

$$\begin{aligned} opt_+(u, N_1, N_2) \;=\; & opt((D_0)_{N_1, N_1'}, d_{N_1}) - \textstyle\sum_{v \in N_1} d(v, u) \\ & - \textstyle\sum_{v \in N_2} \max\{0, d(v, u) - d(u, v)\} \\ & + \textstyle\sum_{v \in N_1} (opt^*_+(v) - \max\{0, d(u, v) - d(v, u)\}) \\ & + \textstyle\sum_{v \in N_2} (opt^*_-(v) - d(u, v)), \end{aligned}$$

and

$$\begin{aligned} opt_-(u, N_1, N_2) \;=\; & opt((D_0)_{N_1 \cup \{pa(u)\}, N_1' \cup \{pa'\}}, d_{N_1 \cup \{pa(u)\}}) - \textstyle\sum_{v \in N_1} d(v, u) \\ & - \textstyle\sum_{v \in N_2} \max\{0, d(v, u) - d(u, v)\} \\ & + \textstyle\sum_{v \in N_1} (opt^*_+(v) - \max\{0, d(u, v) - d(v, u)\}) \\ & + \textstyle\sum_{v \in N_2} (opt^*_-(v) - d(u, v)), \end{aligned}$$

where $D_0 = D[\{u, pa(u)\} \cup Ch(u)]$, pa' denotes the vertex generated by splitting $pa(u)$ in $(D_0)_{N_1, N_1'}$, and $d_{N_1 \cup \{pa(u)\}}$ denotes the debt function on $A((D_0)_{N_1 \cup \{pa(u)\}, N_1' \cup \{pa'\}})$. Here we note that in these two equations, $opt^*_+(v) = opt(D(v) + u, d)$ and $opt^*_-(v) = opt((D(v) + u)_{u,u'}, d_{u,u'})$, by the assumption on N_1 and N_2. For stars $(D_0)_{N_1, N_1'}$ and $(D_0)_{N_1 \cup \{pa(u)\}, N_1' \cup \{pa'\}}$, we can compute $opt((D_0)_{N_1, N_1'}, d_{N_1})$ and $opt((D_0)_{N_1 \cup \{pa(u)\}, N_1' \cup \{pa'\}}, d_{N_1 \cup \{pa(u)\}})$ in $O(|N_D(u)| \log |N_D(u)|)$ time by Theorem 15. Hence, if we know $opt^*_+(v)$ and $opt^*_-(v)$ for all $v \in Ch(u)$, then we can compute $opt^*_+(u)$ and $opt^*_-(u)$ in $O(2^{|N_D(u)|} |N_D(u)| \log |N_D(u)|)$ time by computing $opt_+(u, N_1, N_2)$ and $opt_-(u, N_1, N_2)$ for all possible N_1 and N_2. Thus, we can compute $opt(D, d) = opt^*_+(r) = opt^*_-(r)$ in $O(2^\Delta n \log n)$ time. Also, notice that an optimal sequence of A can be obtained in the same time complexity, according to the algorithm mentioned in the last paragraph of the proof of Lemma 21.

Summarizing the arguments given so far, we have shown Theorem 17.

NP-hardness

Next, we give a proof of Theorem 3; we show the NP-hardness of Min-SFC for a tree. We will reduce from Partition, which is known to be NP-hard [13].

Partition

Instance: $\{x_1, x_2, \ldots, x_n\}$: A set of n positive integers $x_1, x_2, \ldots, x_n$.

Question: Is there a partition $\{X_1, X_2\}$ of $[n]$ such that $\sum_{i \in X_1} x_i = \sum_{i \in X_2} x_i$?

Take an instance $I_{\text{PART}} = \{x_1, x_2, \ldots, x_n\}$ of Partition. From the I_{PART}, we construct an instance $I_{\text{SFC}} = (D = (V, A), d)$ of Min-SFC as follows. Let $V = \{r, u\} \cup \bigcup_{i=1}^{n} \{v_i, w_i\}$ and $E(G_D) = \{\{r, u\}\} \cup \bigcup_{i=1}^{n} \{\{u, v_i\}, \{v_i, w_i\}\}$. Let $x^* = \sum_{i \in [n]} x_i$, $d(r, u) = d(u, r) = x^*/2$, $d(u, v_i) = d(v_i, u) = d(v_i, w_i) = x_i$, and $d(w_i, v_i) = x_i/2$ for $i \in [n]$. Note that the diameter of D is four.

We here claim that there exists a partition $\{X_1, X_2\}$ of $[n]$ such that $\sum_{i \in X_1} x_i = \sum_{i \in X_2} x_i$ if and only if there exists a sequence π of A with $p_\pi(V) \leq 3x^*/4$. Notice that since I_{SFC} can be constructed from I_{PART} in polynomial time, this claim proves Theorem 3.

Claim 1. *There exists a partition $\{X_1, X_2\}$ of $[n]$ such that $\sum_{i \in X_1} x_i = \sum_{i \in X_2} x_i$ if and only if there exists a sequence π of A with $p_\pi(V) \leq 3x^*/4$.*

Proof. Consider a sequence $\pi(X_1)$ of A such that $p_{\pi(X_1)}(V)$ is minimized, under the assumption that (v_i, u) is cleared before (u, v_i) for all $i \in X_1$ and (u, v_i) is cleared before (v_i, u) for all $i \in [n] \setminus X_1$, and that (u, r) is cleared before (r, u). Let D_0 be the star with center u induced by $\{u, r, v_1, \ldots, v_n\}$ in D, D_i be the star with center v_i induced by $\{u, v_i, w_i\}$ in D for $i \in [n]$, and $D_r = D[\{u, r\}]$. Let $(D_0)_{X_1}$ be the star obtained from D_0 by splitting all vertices in v_i with $i \in X_1$ and d_{X_1} be the debt function on $A((D_0)_{X_1})$. Lemma 21 implies that

$$\begin{aligned}
p_{\pi(X_1)}(V) &= opt((D_0)_{X_1}, d_{X_1}) - \textstyle\sum_{i \in X_1} d(v_i, u) \\
&\quad - \textstyle\sum_{i \in [n] \setminus X_1} \max\{0, d(v_i, u) - d(u, v_i)\} \\
&\quad - \max\{0, d(r, u) - d(u, r)\} \\
&\quad + \textstyle\sum_{i \in X_1} (opt(D_i, d) - \max\{0, d(u, v_i) - d(v_i, u)\}) \\
&\quad + \textstyle\sum_{i \in [n] \setminus X_1} (opt((D_i)_{u,u'}, d_{u,u'}) - d(u, v_i)) \\
&\quad + opt((D_r)_{u,u'}, d_{u,u'}) - d(u, r) \\
&= opt((D_0)_{X_1}, d_{X_1}) - \textstyle\sum_{i \in X_1} x_i + \sum_{i \in X_1} opt(D_i, d) \\
&\quad + \textstyle\sum_{i \in [n] \setminus X_1} (opt((D_i)_{u,u'}, d_{u,u'}) - x_i)
\end{aligned}$$

by $d(u, v_i) = d(v_i, u) = x_i$ and $d(u, r) = d(r, u) = x^*/2$. By applying Algorithm MinStar, we can see that

$$\begin{aligned}
opt((D_0)_{X_1}, d_{X_1}) &= \textstyle\sum_{i \in X_1} d(v_i, u) + \max\{0, x^*/2 - \sum_{i \in X_1} d(v_i, u)\} \\
&= \textstyle\sum_{i \in X_1} x_i + \max\{0, x^*/2 - \sum_{i \in X_1} x_i\},
\end{aligned}$$

$opt(D_i, d) = d(u, v_i) = x_i$, and $opt((D_i)_{u,u'}, d_{u,u'}) = d(u, v_i)+d(v_i, u)-d(w_i, v_i)$ $= 3x_i/2$. If follows that

$$\begin{aligned} p_{\pi(X_1)}(V) &= \max\{0, x^*/2 - \textstyle\sum_{i\in X_1} x_i\} + \textstyle\sum_{i\in X_1} x_i + \textstyle\sum_{i\in[n]\setminus X_1} x_i/2 \\ &= \max\{0, x^*/2 - \textstyle\sum_{i\in X_1} x_i\} + \textstyle\sum_{i\in X_1} x_i/2 + x^*/2. \end{aligned}$$

If $\sum_{i\in X_1} x_i - \sum_{i\in[n]\setminus X_1} x_i = \epsilon \geq 0$, then $\sum_{i\in X_1} x_i \geq x^*/2$ and $p_{\pi(X_1)}(V) = 3x^*/4 + \epsilon/4$. If $\sum_{i\in[n]\setminus X_1} x_i - \sum_{i\in X_1} x_i = \epsilon \geq 0$, then $\max\{0, x^*/2 - \sum_{i\in X_1} x_i\} = \epsilon/2$ and $p_{\pi(X_1)}(V) = 3x^*/4 + \epsilon/4$. This implies that $\sum_{i\in X_1} x_i = \sum_{i\in[n]\setminus X_1} x_i$ if and only if $p_{\pi(X_1)}(V) \leq 3x^*/4$.

Next consider a sequence π' of A under the assumption that (r, u) is cleared before (u, r). By this assumption, we have $p_{\pi'}(r) = x^*/2$. As observed in the proof of Lemma 21, we need to put on $V(D_i)$ funds with amount at least $\min\{opt(D_i, d) - \max\{0, d(u, v_i) - d(v_i, u)\}, opt((D_i)_{u,u'}, d_{u,u'}) - d(u, v_i)\} = x_i/2$ for each $i \in [n]$. Thus, we have $p_{\pi'}(V) \geq x^*/2 + \sum_{i\in[n]} x_i/2 = x^*$.

Consequently, we can observe that there exists a partition $\{X_1, [n] \setminus X_1\}$ of $[n]$ such that $\sum_{i\in X_1} x_i = \sum_{i\in[n]\setminus X_1} x_i$ if and only if there exists a sequence π of A with $p_\pi(V) \leq 3x^*/4$. $\square$

5. Max-SFC

In this section, we study Max-SFC which analyzes the worst case scenario in the settlement fund circulation.

5.1. Tractable Case

The following lemma shows that the case of acyclic digraphs is solvable in linear time, since the summation of all debts on arcs is clearly an upper bound on the optimal value.

Lemma 22. *For an acyclic digraph $D = (V, A)$, there exists a sequence π of A such that $p_\pi(V) = \sum_{a\in A} d(a)$. Also, such a sequence π can be found in linear time.*

Proof. Any acyclic graph has at least one vertex with out-degree zero. Also note that any subgraph of an acyclic graph is acyclic. Based on these observations, we can obtain a sequence π of A in the following manner:

Step 1: Let $D' := D$ and $i := 1$.
Step 2: While D' has an arc, repeat the following procedures (i) and (ii).

(i) Choose an arc $a = (u, v)$ with $\deg^+_{D'}(v) = 0$ and let $\pi(a) := i$.
(ii) Let $D' := D' \setminus \{a\}$ and $i := i + 1$.

Clearly, π can be computed in linear time. We here claim that $p_\pi(s(a), \pi(a)) = d(a)$ holds for each $a \in A$; this claim implies that $p_\pi(V) = \sum_{a \in A} d(a)$, proving the lemma. For proving this claim, we will show by induction on order of arcs in π that immediately before clearing an arc $a_i \in A$ with $\pi(a_i) = i$, the residual on $s(a)$ is zero for each $a \in A$ with $\pi(a) \geq \pi(a_i)$ (i.e., each $a \in A$ which has not been cleared). Note that if the induction statement holds, then we need to put funds with amount $d(a_i)$ on $s(a_i)$ for clearing a_i, i.e., $p_\pi(s(a_i), i) = d(a_i)$, since $r_\pi(s(a_i), i) = 0$.

Before clearing the first arc in π, every vertex clearly has no residual. Suppose that immediately before clearing an arc $a_i \in A$ with $\pi(a_i) = i \geq 1$, the residual on $s(a)$ is zero for each $a \in A$ with $\pi(a) \geq \pi(a_i)$. Since the residual on $s(a_i)$ is zero, we need to put funds with amount $d(a_i)$ on $s(a_i)$; $p_\pi(s(a_i), i) = d(a_i)$. The funds are circulated to $t(a_i)$ after clearing a_i, and it turns out that the residual on $s(a_i)$ remains zero. Now by the construction of π, each arc incident from $t(a_i)$ has been cleared if it exists. Thus, we can see that the induction statement holds immediately before clearing the $(i + 1)$-th arc in π. □

5.2. NP-hardness

Below, we show that Max-SFC is NP-hard, even in the case where each debt is unit and a given graph is bipartite.

Theorem 23. *For a digraph D,* Max-SFC *is NP-hard even if each debt of an arc in $A(D)$ is unit and D is bipartite.*

We prove this theorem by reducing from Vertex Cover, which is known to be NP-hard [13]. For an undirected graph $G = (V, E)$, a set $V' \subseteq V$ of vertices is called a *vertex cover* if every edge $e = \{u, v\} \in E$ satisfies $\{u, v\} \cap V' \neq \emptyset$.

Vertex Cover

Instance: An undirected graph $G = (V, E)$ and an integer k, that is, $(G = (V, E), k)$.

Question: Is there a vertex cover X with $|X| \leq k$ in G?

Take an instance $I_{\text{VC}} = (G = (V, E), k)$ of VERTEX COVER. From the I_{VC}, we construct an instance $I_{\text{SFC}} = (D = (V', A), d)$ of MAX-SFC as follows. For each vertex $v_i \in V$, we introduce two copies v_i^1 and v_i^2 of v_i and an arc (v_i^1, v_i^2), and let $V' = \bigcup_{v_i \in V}\{v_i^1, v_i^2\}$ and $A_1 = \bigcup_{v_i \in V}(v_i^1, v_i^2)$. For each edge $\{v_i, v_j\} \in E$, we introduce two arcs (v_i^2, v_j^1) and (v_j^2, v_i^1), and let $A_2 = \bigcup_{\{v_i,v_j\}\in E}\{(v_i^2, v_j^1), (v_j^2, v_i^1)\}$. Let $A = A_1 \cup A_2$ and $d(u, v) = 1$ for all $(u, v) \in A$. Note that D is bipartite.

The following lemma completes the proof of Theorem 23.

Lemma 24. *G has a vertex cover with cardinality at most k if and only if there exists a sequence π of A such that $p_\pi(V') \geq |A| - k$ in D.*

Proof. First, we show "only-if" part. Assume that G has a vertex cover S with $|S| \leq k$. Let A_S be the set of arcs in D corresponding to S, i.e., $A_S = \{(v_i^1, v_i^2) \mid v_i \in S\}$.

We here claim that every cycle in D contains an arc in A_S. Let C be an arbitrary cycle in D. From the construction of D, arcs in A_1 and A_2 appear alternately on C and we have $|C| \geq 4$. For three consecutive arcs a_1, a_2, and a_3 on C with $\{a_1, a_3\} \subseteq A_1$ and $a_2 \in A_2$, let v_i be the vertex in V corresponding to a_i for $i \in \{1, 3\}$ and e_2 be the edge in E corresponding to a_2. Also by the construction of D, v_1 and v_3 are two end-vertices of e_2. Since S is a vertex cover in G, it follows that $v_1 \in S$ or $v_3 \in S$ hold and hence $a_1 \in A_S$ or $a_3 \in A_S$. Thus, C contains an arc in A_S.

By this claim, $D \setminus A_S$ is acyclic. It follows from Lemma 22 and the property that $d(a) = 1$ for each $a \in A$ that there exists a sequence π of A such that $p_\pi(V') \geq |A \setminus A_S| \geq |A| - k$.

Next, we show "if" part. Assume that there exists a sequence π of A such that $p_\pi(V') \geq |A| - k$. Since $d(a) = 1$ for all $a \in A$, we can observe that immediately before clearing an arc $a \in A$ according to π, if the residual $r_\pi(s(a), \pi(a))$ on $s(a)$ is positive, then $r_\pi(s(a), \pi(a))$ is a natural number and no additional funds is needed for clearing a (i.e., $p_\pi(s(a), \pi(a)) = 0$), and otherwise (i.e., $r_\pi(s(a), \pi(a)) = 0$) then $p_\pi(s(a), \pi(a)) = 1$ holds. In π, we call an arc a for which no additional funds is needed (i.e., $p_\pi(s(a), \pi(a)) = 0$) *free* and denote the set of free arcs by $A_F(\pi)$. Note that $A_F(\pi)$ is uniquely determined by π and that $p_\pi(V') \geq |A| - k$ if and only if $|A_F(\pi)| \leq k$.

Let π^* be a sequence of A such that $|A_F(\pi^*)| \leq k$ and $|A_F(\pi^*) \cap A_1|$ is maximized. We then claim that for each edge $\{v_i, v_j\} \in E$, we have $\{(v_i^1, v_i^2), (v_j^1, v_j^2)\} \cap A_F(\pi^*) \neq \emptyset$. This claim implies that G has a vertex cover with cardinality at most k, since the set $\{v_i \in V \mid (v_i^1, v_i^2) \in A_F(\pi^*)\}$ of vertices is a vertex cover in G. Below, we prove this claim by assuming that for some edge $\{v_i, v_j\} \in E$, we have $\{(v_i^1, v_i^2), (v_j^1, v_j^2)\} \cap A_F(\pi^*) = \emptyset$ and deriving a contradiction.

Assume that $\{(v_i^1, v_i^2), (v_j^1, v_j^2)\} \cap A_F(\pi^*) = \emptyset$ holds for an edge $\{v_i, v_j\} \in E$. Observe that $\pi^*(v_i^1, v_i^2) < \pi^*(v_j^2, v_i^1)$. This follows since if $\pi^*(v_i^1, v_i^2) > \pi^*(v_j^2, v_i^1)$ holds, then funds circulated through (v_j^2, v_i^1) is used for clearing (v_i^1, v_i^2) and (v_i^1, v_i^2) would belong to $A_F(\pi^*)$ (note that there is no arc incident from v_i^1 other than (v_i^1, v_i^2)). Similarly, $\pi^*(v_j^1, v_j^2) < \pi^*(v_i^2, v_j^1)$. Without loss of generality, assume that $\pi^*(v_j^2, v_i^1) > \pi^*(v_i^2, v_j^1)$; (v_j^2, v_i^1) is the last arc cleared among four arcs (v_i^1, v_i^2), (v_j^1, v_j^2), (v_i^2, v_j^1), and (v_j^2, v_i^1) in the sequence π^*. Note that there is no arc incident to v_j^2 other than (v_j^1, v_j^2) and $d(v_j^1, v_j^2) = 1$ holds. Hence, exactly one arc a' incident from v_j^2 belongs to $A_F(\pi^*)$, since there exists at least one arc a'' (say, (v_j^2, v_i^1)) incident from v_j^2 with $\pi^*(v_j^1, v_j^2) < \pi^*(a'')$. Let π_1 be the sequence of A obtained from π^* by changing the position of (v_j^1, v_j^2) to the last one; $\pi_1(a) = \pi^*(a)$ for each arc $a \in A$ with $\pi^*(a) < \pi^*(v_j^1, v_j^2)$, $\pi_1(a) = \pi^*(a) - 1$ for each arc $a \in A$ with $\pi^*(a) > \pi^*(v_j^1, v_j^2)$, and $\pi_1(v_j^1, v_j^2) = |A|$. Then we have $(v_j^1, v_j^2) \in A_F(\pi_1)$ since there is no arc incident from v_j^1 other than (v_j^1, v_j^2) and funds are circulated through (v_i^2, v_j^1). Furthermore, $A_F(\pi_1) \setminus A_F(\pi^*) = \{(v_j^1, v_j^2)\}$ holds since no arc is cleared after (v_j^1, v_j^2) in π_1. Also, we can observe that $A_F(\pi^*) \setminus A_F(\pi_1) \supseteq \{a'\}$ since there is no arc incident to v_j^2 other than (v_j^1, v_j^2). Thus, $A_F(\pi_1) \subseteq (A_F(\pi^*) \setminus \{a'\}) \cup \{(v_j^1, v_j^2)\}$. It follows that $|A_F(\pi_1)| \le |A_F(\pi^*)| \le k$ and $|A_F(\pi_1) \cap A_1| > |A_F(\pi^*) \cap A_1|$, which contradicts the maximality of $|A_F(\pi^*) \cap A_1|$. $\square$

6. Conclusion

This paper studied the problem of fund settlement among banks which practically arises in the daily economic activities, and introduced a precise mathematical model based on graphs/networks for the first time. We defined two related optimization problems, which we call Min-SFC and Max-SFC, to minimize/maximize the total amount of settlement fund put for clearing all the debts among banks. Systematic and exhaustive analyses from the algorithmic and computational viewpoints revealed that both Min-SFC and Max-SFC are computationally intractable in general even for highly structured instances, but also showed that it becomes polynomial-time solvable by putting some conditions on debts or by restricting the graph classes more strictly.

One of the most important future work is to deal with more appropriate graphs classes that reflect well the debts relationship among banks in our real economic activities. As we mentioned in Section 4.4, it is known that the interbank network system in Japan was a kind of star structures before 1997 [12]. On the contrary, Imakubo and Soejima [12] also showed that in the year of 2005 it had changed and turned to be a core-periphery structure, which is a certain kind of classic hub-authority biclique model [14] and thus one of the so-called complex networks. In the model, banks are classified into either one of the two categories, *core banks* or *periphery* banks, such that payments among the core banks are more densely connected among them compared to those among the periphery banks. Recent research observed similar facts in some other countries, e.g., in the US in 2004 [22], in the Netherlands in 2006 [20], and so on. In view of these recent observations, it would be extremely important to consider our problem on this realistic model and develop efficient algorithms for it.

In practical situations of putting fund for clearing debts, it does not necessarily seem important to carry out a plan of putting exact minimum amount obtained by time consuming computation, but rather preferable to get its rough estimate more quickly. This implies that it would be more practical and meaningful to get a plan that requires fund amount within some desired error bound. In this regard, we think that designing approximation algorithms with bounded approximation ratio on real network models is quite important, even if it is intractable to obtain exact algorithms.

We think that our new models and results based on solid observations and analyses in this paper can give fundamental insights to this classic but important problem in economics. We hope that this will contribute to open new directions and to ignite subsequent research on this rich topic.

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