Continuity of derivatives of a convex solution to a perturbed one-Laplace equation by p-Laplacian

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Abstract

We consider a one-Laplace equation perturbed by p-Laplacian with $1 < p < \infty$. We prove that a weak solution is continuously differentiable (C^1) if it is convex. Note that similar result fails to hold for the unperturbed one-Laplace equation. The main difficulty is to show C^1-regularity of the solution at the boundary of a facet where the gradient of the solution vanishes. For this purpose we blow-up the solution and prove that its limit is a constant function by establishing a Liouville-type result, which is proved by showing a strong maximum principle. Our argument is rather elementary since we assume that the solution is convex. A few generalization is also discussed.

Keywords C^1-regularity, one-Laplace equation, strong maximum principle

1 Introduction

We consider a one-Laplace equation perturbed by p-Laplacian of the form

$$L_{b,p}u = f \quad \text{in} \quad \Omega \tag{1.1}$$

with

$$L_{b,p}u := -b\Delta_1 u - \Delta_p u,$$

where

$$\Delta_1 u := \operatorname{div}\left(\nabla u/|\nabla u|\right), \quad \Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$$

in a domain Ω in $\mathbb{R}^n$, $\nabla u = (\partial_{x_1}u, \ldots, \partial_{x_n}u)$ with $\partial_{x_j}u = \partial u/\partial x_j$ for a function $u = u(x_1, \ldots, x_n)$, and $\operatorname{div} X = \sum_{i=1}^{n} \partial_{x_i} X_i$ for a vector field $X = (X_1, \ldots, X_n)$. The constants $b > 0$ and $p \in (1, \infty)$ are given and fixed. It has been a long-standing open problem whether its weak solution is C^1 up to a facet, the place where the gradient ∇u vanishes, even if f is smooth. This is a non-trivial question since a weak solution to the unperturbed one-Laplace equation, i.e., $-\Delta_1 u = f$ may not be C^1. This is because the ellipticity degenerates in the direction of ∇u for $\Delta_1 u$. Our goal in this paper is to solve this open problem under the assumption that a solution is convex.

1.1 Main theorems and our strategy

Throughout the paper, we assume $f \in L^q_{\mathrm{loc}}(\Omega)$ $(n < q \leq \infty)$, i.e., $|f|^q$ is locally integrable in Ω. Our main result is

Theorem 1 (C^1-regularity theorem)**.** *Let u be a convex weak solution to (1.1) with $f \in L^q_{\mathrm{loc}}(\Omega)$ $(n < q \leq \infty)$. Then u is in $C^1(\Omega)$.*

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Difficulty on proving regularity on gradients of solutions to (1.1) can be explained from a viewpoint of ellipticity ratio. We set a convex function $E\colon \mathbb{R}^n \to [0,\infty)$ by

$$E(z) := b|z| + \frac{|z|^p}{p} \quad \text{for} \quad z \in \mathbb{R}^n.$$

We rewrite (1.1) by

$$-\operatorname{div}(\nabla_z E(\nabla u)) = f \quad \text{in } \Omega. \tag{1.2}$$

By differentiating (1.2) by x_i ($i \in \{1, \dots, n\}$), we get

$$-\operatorname{div}\left(\nabla_z^2 E(\nabla u) \nabla \partial_{x_i} u\right) = \partial_{x_i} f. \tag{1.3}$$

By elementary calculations, ellipticity ratio of the Hessian $\nabla_z^2 E$ at $z_0 \in \mathbb{R}^n \setminus \{0\}$ is given by

$$\begin{aligned}\left(\text{ellipticity ratio of } \nabla_z^2 E(z_0)\right) &:= \frac{(\text{the largest eigenvalue of } \nabla_z^2 E(z_0))}{(\text{the lowest eigenvalue of } \nabla_z^2 E(z_0))} \\ &= \frac{\max(p-1, 1) + b|z_0|^{1-p}}{\min(p-1, 1)}.\end{aligned}$$

Since the exponent $1-p$ is negative, the ellipticity ratio of $\nabla_z^2 E(z_0)$ blows up as $z_0 \to 0$. From this we can observe that the equation (1.2) becomes non-uniformly elliptic near the facet. It should be noted that our problem is substantially different from the (p, q)-growth problem, since for (p, q)-growth equations, non-uniform ellipticity appears as a norm of a gradient blows up [26, Section 6.2]. Although regularity of minimizers of double phase functionals, including

$$\mathcal{H}(u) := \int E_p(\nabla u)\,dx + \int a(x) E_q(\nabla u)\,dx \quad \text{with} \quad 1 < p \le q < \infty, a(x) \ge 0$$

were discussed in scalar and even in vectorial cases by Colombo and Mingione [6, 7], their results do not recover our C^1-regularity results. This is basically derived from the fact that, unlike $\nabla_z^2 E_p$ with $1 < p < \infty$, the Hessian matrix $\nabla_z^2 E_1(z_0)$ ($z_0 \neq 0$) always takes 0 as its eigenvalue. In other words, ellipticity of the operator $\Delta_1 u$ degenerates in the direction of ∇u, which seems to be difficult to handle analytically.

On the other hand, the ellipticity ratio of $\nabla_z^2 E(z_0)$ is uniformly bounded over $|z_0| > \delta$ for each fixed $\delta > 0$. In this sense we may regard the equation (1.3) as locally uniformly elliptic outside the facet. To show Lipschitz bound, we do not need to study over the facet. In fact, local Lipschitz continuity of solutions to (1.1) are already established in [32]; see also [33] for a weaker result. To study continuity of derivatives, we have to study regularity up to the facet. Thus, it seems to be impossible to apply standard arguments based on De Giorgi–Nash–Moser theory. In this paper, we would like to show continuity of derivatives of convex solutions by elementary arguments based on convex analysis.

Let us give a basic strategy to prove Theorem 1. Since the problem is local, we may assume that Ω is convex, or even a ball. By C^1-regularity criterion for a convex function, to show u is C^1 at $x \in \Omega$ it suffices to prove that

$$\text{the subdifferential } \partial u(x) \text{ at } x \in \Omega \text{ is a singleton;} \tag{1.4}$$

see [1, Appendix D], [30, §25] and Remark 1 for more detail. Here the subdifferential of u at $x_0 \in \Omega$ is defined by

$$\partial u(x_0) := \{z \in \mathbb{R}^n \mid u(x) \ge u(x_0) + \langle z \mid x - x_0 \rangle \text{ for all } x \in \Omega\}.$$

Here $\langle \cdot \mid \cdot \rangle$ stands for the standard inner product in $\mathbb{R}^n$. For a convex function $u\colon \Omega \to \mathbb{R}$, we can simply express the facet of u as

$$F := \{x \in \Omega \mid \partial u(x) \ni 0\} = \{x \in \Omega \mid u(x) \le u(y) \text{ for all } y \in \Omega\}.$$

By definition it is clear that the facet F is non-empty if and only if a minimum of u in Ω exists. By convexity of u, we can easily check that $F \subset \Omega$ is a relatively closed convex set in Ω. We also define an open set

$$D := \Omega \setminus F = \{x \in \Omega \mid u(y) < u(x) \text{ for some } y \in \Omega\}.$$

Our strategy to show (1.4) depends on whether x is inside F or not.

Remark 1. [Some properties on differentiability of convex functions] Let v a real-valued convex function in a convex domain $\Omega \subset \mathbb{R}^n$, then following property holds.

1. v is locally Lipschitz continuous in Ω, and therefore v is a.e. differentiable in Ω by Rademacher's theorem ([1, Theorem 1.19], see also [9, Theorem 3.1 and 3.2] and [30, Theorem 25.5]).

2. For $x \in \Omega$, v is differentiable at x if and only if the subdifferential set $\partial v(x)$ is a singleton. Moreover, if $x \in \Omega$ satisfies either of these equivalent conditions, then we have $\partial v(x) = \{\nabla v(x)\}$ ([1, Proposition D.5], see also [30, Theorem 25.1]). In particular, Rademacher's theorem implies that $\partial v(x) = \{\nabla v(x)\}$ for a.e. $x \in \Omega$.

3. $v \in C^1(\Omega)$ if and only if ∂v is single-valued ([1, Remark D.3 (iii)], see also [30, Theorem 25.5]).

Throughout the paper we use these well-known results without proofs.

We first discuss the case $x \in D$. Our goal is to show directly that u is $C^{1,\alpha}$ near a neighborhood of x and therefore $\partial u(x) = \{\nabla u(x)\} \neq \{0\}$ for all $x \in D$. This strategy roughly consists of three steps. Among them the first step, a kind of separation of $x \in D$ from the facet F, plays an important role. Precisely speaking, we first find a neighborhood $B_r(x) \subset D$, an open ball centered at x with its radius $r > 0$, such that

$$\partial_\nu u \geq \mu > 0 \quad \text{a.e. in} \quad B_r(x) \tag{1.5}$$

for some direction ν and some constant $\mu > 0$. In order to justify (1.5), we fully make use of convexity of u (Lemma 8 in Section A), not elliptic regularity theory. Then with the aid of local Lipschitz continuity of u, the inclusion $B_r(x) \subset \{0 < \mu \leq \partial_\nu u \leq |\nabla u| \leq M\}$ holds for some finite positive constant M. Secondly, this inclusion allows us to check that u admits local $W^{2,2}$-regularity in $B_r(x)$ by the standard difference quotient method. Therefore we are able to obtain the equation (1.3) in the distributional sense. Finally, we appeal to the classical De Giorgi–Nash–Moser theory to obtain local $C^{1,\alpha}$-regularity at $x \in D$, since the equation (1.3) is uniformly elliptic in $B_r(x)$. Here the constant $\alpha \in (0,1)$ we have obtained may depend on the location of $x \in D$ through ellipticity, so α may tend to zero as x tends to the facet.

It takes much efforts to prove that $\partial u(x) = \{0\}$ for all $x \in F$. Our strategy for justifying this roughly consists of three parts; a blow-argument for solutions, a strong maximum principle, and a Liouville-type theorem. Here we describe each individual step.

We first make a blow-argument. Precisely speaking, for a given convex solution $u\colon \Omega \to \mathbb{R}$ and a point $x_0 \in \Omega$, we set a sequence of rescaled functions $\{u_a\}_{a>0}$ defined by

$$u_a(x) := \frac{u(a(x-x_0)+x_0) - u(x_0)}{a}.$$

We show that u_a locally uniformly converges to some convex function $u_0\colon \mathbb{R}^n \to \mathbb{R}$, which satisfies $\partial u(x_0) \subset \partial u_0(x_0)$ by construction. Moreover, we prove that u_0 satisfies $L_{b,p}u_0 = 0$ in $\mathbb{R}^n$ in the distributional sense. There we will face to justify a.e. convergence of gradients, and this is elementarily shown by regarding gradients in the classical sense as subgradients (Lemma 9 in the appendices).

Next we prove that if $x_0 \in F$, then the convex weak solution u_0 constructed as above satisfies $\partial u_0(x_0) = \{0\}$. Moreover, we are going to prove that u_0 is constant (a Liouville-type theorem). For this purpose we establish the maximum principle.

Theorem 2 (Strong maximum principle)**.** *Let u be a convex weak solution to $L_{b,p}u = 0$ in a convex domain $\Omega \subset \mathbb{R}^n$ and $F \subset \Omega$ be the facet of u. Then u is affine in each connected component of the open set $D := \Omega \setminus F$. In particular, if $F = \emptyset$, then u is affine in Ω.*

It should be noted that this result is a kind of strong maximum principle in the sense that

$$u \geq a \text{ in } D_0 \text{ and } u(x_0) = a(x_0) \text{ for } x_0 \in D_0 \quad \text{imply that} \quad u \equiv a \text{ in } D_0, \tag{1.6}$$

where $a(x) := u(x_0) + \langle \nabla u(x_0) \mid x - x_0 \rangle$ and D_0 is a connected component of D. The affine function a clearly satisfies $L_{b,p}a = 0$ in the classical sense.

In order to justify (1.6), we will face three problems. The first is a justification of the comparison principle, the second is regularity of u, and the third is a construction of suitable barrier subsolutions, all of which are essentially needed in the classical proof of E. Hopf's strong maximum principle [20]. In order to overcome these obstacles, we appeal to both classical and distributional approaches, and restrict our analysis only over regular points. For details, see Section 1.2.

Even though our strong maximum principle is somewhat weakened in the sense that this holds only on each connected component of $D \subset \Omega$, we are able to show the following Liouville-type theorem.

Theorem 3 (Liouville-type theorem)**.** *Let u be a convex weak solution to $L_{b,\,p}u = 0$ in $\mathbb{R}^n$. Then $F \subset \mathbb{R}^n$, the facet of u, satisfies either $F = \emptyset$ or $F = \mathbb{R}^n$. In particular, u satisfies either of the followings.*

1. *If u attains its minimum in $\mathbb{R}^n$, then u is constant.*
2. *If u does not attains its minimum in $\mathbb{R}^n$, then u is a non-constant affine function in $\mathbb{R}^n$.*

In the proof of the Liouville-type theorem, our strong maximum principle plays an important role. Precisely speaking, if a convex solution in the total space does not satisfy $\emptyset \subsetneq F \subsetneq \mathbb{R}^n$, then Theorem 2 and the supporting hyperplane theorem from convex analysis help us to determine the shape of convex solutions. In particular, the convex solution can be classified into three types of piecewise-linear functions of one-variable. These non-smooth piecewise-linear functions are, however, no longer weak solutions, which we will prove by some explicit calculations.

By applying the Liouville-type theorem and our blow-argument, we are able to show that subgradients at points of the facet are always 0, i.e., $\partial u(x) = \{0\}$ for all $x \in F$, and we complete the proof of the C^1-regularity theorem. Note that the statements in Theorem 2 and 3 should not hold for unperturbed one-Laplace equation $-\Delta_1 u = f$, since any absolutely continuous non-decreasing function of one variable $u = u(x_1)$ satisfies $-\Delta_1 u = 0$.

Finally we mention that we are able to refine our strategy, and obtain C^1-regularity of convex solutions to more general equations. We replace the one-Laplacian Δ_1 by another operator which is derived from a general convex functional of degree 1. This generalization requires us to modify some of our arguments, including a blow-up argument and the Liouville-type theorem. For further details, see Section 1.4 and Section 6.2.

1.2 Literature overview on maximum principles

We briefly introduce maximum principles related to the paper. We also describe our strategy to establish the strong maximum principle.

Maximum principles, including comparison principles and strong maximum principles, have been discussed by many mathematicians in various settings. In the classical settings, E. Hopf proved a variety of maximum principles on elliptic partial differential equations of second order, by elementary arguments based on constructions of auxiliary functions. E. Hopf's strong maximum principle is one of the well-known results on maximum principles. In Hopf's proof of the strong maximum principle [20], he defined an auxiliary function

$$h(x) \coloneqq e^{-\alpha|x-x_*|^2} - e^{-\alpha R^2} \quad \text{for } x \in \mathbb{R}^n, \tag{1.7}$$

which becomes a classical subsolution in a fixed open annulus $E_R = E_R(x_*) \coloneqq B_R(x_*) \setminus \overline{B_{R/2}(x_*)}$ for sufficiently large $\alpha > 0$. An alternative function

$$h(x) \coloneqq |x-x_*|^{-\alpha} - R^{-\alpha} \quad \text{for } x \in \mathbb{R}^n \setminus \{x_*\} \tag{1.8}$$

is given in [29, Chapter 2.8]. E. Hopf's classical results on maximum principles are extensively contained in [17, Chapter 3], [28, Chapter 2] and [29, Chapter 2].

The materials [17, Chapter 8–9] and [29, Chapter 3–6] provide proofs of maximum principles, including strong maximum principles, even for distributional solutions. Among them, [29, Theorem 5.4.1] deals with a justification of the strong maximum principle for distributional supersolutons to certain quasilinear elliptic equations with divergence structures,

$$\text{i.e., } -\operatorname{div}(A(x, \nabla u(x))) = 0,$$

which covers the p-Laplace equation with $1 < p < \infty$. Even in the distributional schemes, the proof of the maximum principle [29, Theorem 5.4.1] is partially similar to E. Hopf's classical one, in the sense that it is completed by calculating directional derivatives of auxiliary functions. The significant difference is, however, the construction of spherically symmetric subsolutions of C^1 class, which is given in [29, Chapter 4], is based on Leray–Schauder's fixed point theorem [17, Theorem 11.6]. Also it should be noted that the proofs of comparison principles [29, Theorem 2.4.1 and 3.4.1] are just based on strict monotonicity of the mapping $A(x, \cdot)\colon \mathbb{R}^n \to \mathbb{R}^n$, whereas Hopf's proof appeals to direct constructions of auxiliary functions.

With our literature overview in mind, we describe our strategy for showing (1.6). A justification of comparison principles is easily obtained in the distributional schemes (see [29, Chapter 3] as a related material). However, the remaining two obstacles, the differentiability of u and the construction of subsolutions, cannot be resolved affirmatively by just imitating arguments given in [29, Chapter 4–5]. In the first place, it should be mentioned that convex weak solutions we treat in this paper are assumed to have only local Lipschitz regularity, whereas supersolutions treated in [29, Chapter 5] are required to be in C^1. We recall that C^1-regularity of convex weak solutions can be guaranteed in $D \subset \Omega$ (the outside of the facet) by the classical De Giorgi–Nash–Moser theory, and this result enables us to overcome the problem whether u is differentiable at certain points. This is the reason why Theorem 2 need to restrict on D. Although the construction of distributional subsolutions is generally discussed in [29, Chapter 4], we do not appeal to this. Instead, we directly construct a function $v = \beta h + a$ in $\mathbb{R}^n \setminus \{x_*\}$, where $\beta > 0$ is a constant and h is defined as in (1.7) or (1.8). We will determine the constants $\alpha, \beta > 0$ so precisely that v satisfies $L_{b,\,p} v \le 0$ in the classical sense over a fixed open annulus $E_R = E_R(x_*)$. We also make $|\nabla v|$ very close to $|\nabla a| \equiv |\nabla u(x_0)| > 0$ over E_R, so that ∇v no longer degenerates there. By direct calculation of $L_{b,\,p} v$, we explicitly construct classical subsolutions to $L_{b,\,p} u = 0$ in E_R. Finally we are able to deduce (1.6) by an indirect proof.

Another type of definitions of subsolutions and supersolutions to (1.1) in the distributional schemes can be found in F. Krügel's thesis in 2013 [25]. The significant difference is that Krügel did not regard the term $\nabla u / |\nabla u|$ as a subgradient vector field. Since monotonicity of $\partial |\cdot|$ is not used at all, it seems that Krügel's proof of comparison principle [25, Theorem 4.8] needs further explanation. For details, see Remark 3.

1.3 Mathematical models and previous researches

Our problem is derived from a minimizing problem of a certain energy functional, which involves the total variation energy. The equation (1.1) is deduced from the following Euler–Lagrange equation;

$$f = \frac{\delta G}{\delta u}, \quad \text{where} \quad G(u) := b \int_\Omega |\nabla u|\, dx + \frac{1}{p} \int_\Omega |\nabla u|^p\, dx.$$

The energy functional G often appears in fields of materials science and fluid mechanics.

In [31], Spohn modeled the relaxation dynamics of a crystal surface below the roughening temperature. On h describing the height of the crystal for a two-dimensional domain Ω is modeled as

$$h_t + \operatorname{div} j = 0$$

with $j = -\nabla \mu$, where μ is a chemical potential. In [31], its evolution is given as

$$\mu = \frac{\delta \Phi}{\delta h} \quad \text{with} \quad \Phi(h) = \int_\Omega |\nabla h|\, dx + \kappa \int_\Omega |\nabla h|^3\, dx$$

with $\kappa > 0$. This Φ is essentially the same as G with $p = 3$. Then, the resulting evolution equation for h is of the form

$$b h_t = \Delta L_{b,3} h \quad \text{with} \quad b = \frac{1}{3\kappa}.$$

This equation can be defined as a limit of step motion, which is microscopic in the direction of height [23]; see also [27]. The initial value problem of this equation can be solved based on the theory of maximal monotone operators [12] under the periodic boundary condition. Subdifferentials describing the evolution are characterized by Kashima [21], [22]. Its evolution speed is calculated by [21] for one dimensional setting and by [22] for radial setting. It is known that the solution stops in finite time [13], [14]. In [27], numerical calculation based on step motion is calculated. If one considers a stationary solution, h must satisfies

$$\Delta L_{b,3} h = 0.$$

If $L_{b,3} h$ is a constant, our Theorem 1 implies that the height function h is C^1 provided that h is convex.

For a second order problem,

$$\text{i.e., } b h_t = L_{b,\,p} h,$$

its analytic formulation goes back to [4], [8, Chapter VI] for $p = 2$, and its numerical analysis is given in [19]. For the fourth order problem, its numerical study is more recent. The reader is referred to papers by [15], [16], [24].

Another important mathematical model for the equation (1.1) is found in fluid mechanics. Especially for $p = 2$ and $n = 2$, the energy functional G appears when modeling stationary laminar incompressible flows of a material called Bingham fluid, which is a typical non Newtonian fluid. Bingham fluid reflects the effect of plasticity corresponding to $\Delta_1 u$ as well as that of viscosity corresponding $\Delta_2 u = \Delta u$ in (1.1). Let us consider a parallel stationary flow with velocity $U = (0, 0, u(x_1, x_2))$ in a cylinder $\Omega \times \mathbb{R}$. Of course, this is incompressible flow, i.e., $\operatorname{div} U = 0$. If this flow is the classical Newtonian fluid, then the Navier–Stokes equations become (1.1) in Ω with $b = 0$ and $f = -\partial_{x_3} \pi$, where π denotes the pressure. In the case that plasticity effects appears, one obtains (1.1), following [8, Chapter VI, Section 1]. There it is also mentioned that since the velocity is assumed to be uni-directional, the external force term in (1.1) is considered as constant in this laminar flow model. The significant difference is that motion of the Bingham fluid is blocked if the stress of the Bingham fluid exceeds a certain threshold. This physical phenomenon is essentially explained by the nonlinear term $b\Delta_1 u$, which reflects rigidity of the Bingham fluid. For more details, see [8, Chapter VI] and the references therein.

On continuity of derivatives for solutions, less is known even for the second order elliptic case. Although Krügel gave an observation that solutions can be continuously differentiable [25, Theorem 1.2] on the boundary of a facet, mathematical justifications of C^1-regularity have not been well-understood. Our main result (Theorem 1) mathematically establishes continuity of gradient for convex solutions.

1.4 Organization of the paper

We outline the contents of the paper.

Section 2 establishes $C^{1,\alpha}$-regularity at regular points of convex weak solutions (Lemma 1). In order to apply De Giorgi–Nash–Moser theory, we will need to justify local $W^{2,2}$-regularity by the difference quotient method. The key lemma, which is proved by convex analysis, is contained in the appendices (Lemma 8).

Section 3 provides a blow-up argument for convex weak solutions. The aim of Section 3 is to prove that $u_0 \colon \mathbb{R}^n \to \mathbb{R}$, a limit of rescaled solutions, satisfies $L_{b,p} u_0 = 0$ in the weak sense over the whole space $\mathbb{R}^n$ (Proposition 1). To assure this, we will make use of an elementary result on a.e. convergence of gradients, which is given in the appendices (Lemma 9).

Section 4 is devoted to justifications of maximum principles for the equation $L_{b,p} u = 0$. We first give definitions of sub- and supersolution in the weak sense. Section 4.1 provides a justification of the comparison principle (Proposition 2). Section 4.2 establishes an existence result of classical barrier subsolutions in an open annulus (Lemma 2). Applying these results in Section 4.1–4.2, we prove the strong maximum principle outside the facet (Theorem 2).

In Section 5, we will show the Liouville-type theorem (Theorem 3) by making use of Theorem 2, and complete the proof of our main theorem (Theorem 1).

Finally in Section 6, we discuss a few generalization of the operators Δ_1 and Δ_p. Since the general strategy for the proof is the same, we only indicate modification of our arguments. Among them, we especially treat with a Liouville-type theorem and a blow-up argument, since these proofs require basic facts of a general convex functional which is positively homogeneous of degree 1. These well-known facts are contained in the appendices for completeness.

2 Regularity outside the facet

In Section 2, we would like to show that u is C^1 at any $x \in D$, and therefore (1.4) holds for all $x \in D$. This result will be used in the proof of the strong maximum principle (Theorem 2).

We first give a precise definition of weak solutions to $L_{b,p} u = f$ in a convex domain $\Omega \subset \mathbb{R}^n$, which is not necessarily bounded.

Definition 1. Let $\Omega \subset \mathbb{R}^n$ be a domain, which is not necessarily bounded, and $f \in L^q_{\mathrm{loc}}(\Omega)$ $(n < q \leq \infty)$. We say that a function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a *weak* solution to (1.1), when for any bounded Lipschitz domain $\omega \Subset \Omega$, there exists a vector field $Z \in L^\infty(\omega, \mathbb{R}^n)$ such that the pair $(u, Z) \in W^{1,p}(\omega) \times L^\infty(\omega, \mathbb{R}^n)$ satisfies

$$b \int_\omega \langle Z \mid \nabla \phi \rangle \, dx + \int_\omega \left\langle |\nabla u|^{p-2} \nabla u \;\middle|\; \nabla \phi \right\rangle dx = \int_\omega f \phi \, dx \tag{2.1}$$

for all $\phi \in W_0^{1,p}(\omega)$, and

$$Z(x) \in \partial|\cdot|(\nabla u(x)) \tag{2.2}$$

for a.e. $x \in \omega$. For such pair (u, Z), we say that (u, Z) satisfies $L_{b,p}u = f$ in $W^{-1,p'}(\omega)$ or simply say that u satisfies $L_{b,p}u = f$ in $W^{-1,p'}(\omega)$. Here $p' \in (1, \infty)$ denotes the Hölder conjugate exponent of $p \in (1, \infty)$.

The aim of Section 2 is to show Lemma 1 below.

Lemma 1. *Let u be a convex weak solution to (1.1) in a convex domain $\Omega \subset \mathbb{R}^n$, and $f \in L^q_{\mathrm{loc}}(\Omega)$ $(n < q \le \infty)$. If $x_0 \in D$, then we can take a small radius $r_0 > 0$, a unit vector $v_0 \in \mathbb{R}^n$, and a small number $\mu > 0$ such that*

$$\overline{B_{r_0}(x_0)} \subset D \text{ and } \langle \nabla u(x) \mid v_0 \rangle \ge \mu \quad \text{for a.e. } x \in B_{r_0}(x_0), \tag{2.3}$$

and there exists a small number $\alpha = \alpha(\mu) \in (0, 1)$ such that $u \in C^{1,\alpha}(B_{r_0/2}(x_0))$. In particular, u is C^1 in D, and $\partial u(x) = \{\nabla u(x)\} \ne \{0\}$ for all $x \in D$.

Before proving Lemma 1, we introduce difference quotients. For given $g\colon \Omega \to \mathbb{R}^m$ $(m \in \mathbb{N})$, $j \in \{1, \dots, n\}$, $h \in \mathbb{R} \setminus \{0\}$, we define

$$\Delta_{j,h} g(x) := \frac{g(x + he_j) - g(x)}{h} \in \mathbb{R}^m \quad \text{for } x \in \Omega \text{ with } x + he_j \in \Omega,$$

where $e_j \in \mathbb{R}^n$ denotes the unit vector in the direction of the x_j-axis.

In the proof of Lemma 1, we will use Lemma 7–8 without proofs. For precise proofs, see Section A.

Proof. For each fixed $x_0 \in D$, we may take and fix $x_1 \in \Omega$ such that $u(x_0) > u(x_1)$. We set $3\delta_0 := u(x_0) - u(x_1) > 0$, $d_0 := |x_0 - x_1| > 0$ and $v_0 := d_0^{-1}(x_0 - x_1)$. By $u \in C(\Omega)$, we may take a sufficiently small $r_0 > 0$ such that

$$u(y_0) - u(y_1) \ge \delta_0 > 0 \quad \text{for all} \quad y_0 \in B_{r_0}(x_0),\, y_1 \in B_{r_0}(x_1). \tag{2.4}$$

From (2.4), the inclusion $\overline{B_{r_0}(x_0)} \subset D$ clearly holds. (2.4) also allows us to check that for all $y_0 \in B_{r_0}(x_0)$, $z_0 \in \partial u(y_0)$,

$$\langle z_0 \mid v_0 \rangle \ge \frac{u(y_0) - u(y_0 - dv_0)}{d_0} \ge \frac{\delta_0}{d_0} =: \mu_0 > 0. \tag{2.5}$$

For the first inequality in (2.5), we have used Lemma 8, which is basically derived from convexity of u. Recall that $\partial u(x) = \{\nabla u(x)\}$ for a.e. $x \in \Omega$, and hence we are able to recover (2.3) from (2.5).

In order to obtain C^1-regularity in D, we will appeal to the classical De Giorgi–Nash–Moser theory. For preliminaries, we check that the operator $L_{b,p}u$ assures uniform ellipticity in $B_{r_0}(x_0)$. Local Lipschitz continuity of u implies that there exists a sufficiently large number $M_0 \in (0, \infty)$ such that

$$\operatorname*{ess\,sup}_{B_{r_0}(x_0)} |\nabla u| \le M_0 \text{ and } |u(x) - u(y)| \le M_0 |x - y| \quad \text{for all } x, y \in B_{r_0}(x_0). \tag{2.6}$$

For notational simplicity, we write subdomains by

$$U_1 := B_{r_0}(x_0) \Supset U_2 := B_{15r_0/16}(x_0) \Supset U_3 := B_{7r_0/8}(x_0) \Supset U_4 := B_{3r_0/4}(x_0) \Supset U_5 := B_{r_0/2}(x_0).$$

It should be noted that $E(z) := b|z| + |z|^p/p$ $(z \in \mathbb{R}^n)$ satisfies $E \in C^\infty(\mathbb{R}^n \setminus \{0\})$, and there exists two constants $0 < \lambda(p, \mu_0, M_0) \le \Lambda(b, p, \mu_0, M_0) < \infty$ such that

$$\lambda|\zeta|^2 \le \left\langle \nabla_z^2 E(z_0)\zeta \,\middle|\, \zeta \right\rangle \tag{2.7}$$

$$\left\langle \nabla_z^2 E(z_0)\zeta \,\middle|\, \omega \right\rangle \le \Lambda|\zeta||\omega| \tag{2.8}$$

for all $z_0, \zeta, \omega \in \mathbb{R}^n$ with $\mu_0 \le |z_0| \le M_0$. We can explicitly determine $0 < \lambda \le \Lambda < \infty$ by

$$\begin{cases} \lambda(p, \mu_0, M_0) & := & \min_{\mu_0 \le t \le M_0}\left(\min\{1, p-1\} t^{p-2}\right), \\ \Lambda(b, p, \mu_0, M_0) & := & \max_{\mu_0 \le t \le M_0}\left(bt^{-1} + \max\{1, p-1\} t^{p-2}\right) \end{cases}$$

Now we check that $u \in W^{2,2}(U_4)$ by the difference quotient method. We refer the reader to [18, Theorem 8.1] as a related result. By [18, Lemma 8.2], it suffices to check that

$$\sup\left\{ \int_{U_4} |\nabla(\Delta_{j,h} u)|^2 \, dx \,\middle|\, h \in \mathbb{R},\, 0 < |h| < \frac{r_0}{16} \right\} < \infty \quad \text{for each } j \in \{1, \dots, n\}. \tag{2.9}$$

Since $u \in W^{1,p}(U_1)$ satisfies $L_{b,p}u = f$ in $W^{-1,p'}(U_1)$, we obtain

$$\int_{U_1} \langle \nabla_z E(\nabla u) \mid \nabla\phi \rangle \, dx = \int_{U_1} f\phi \, dx \tag{2.10}$$

for all $\phi \in W_0^{1,p}(U_1)$. Here we note that ∇u no longer degenerates in U_1 by (2.3). We fix a cutoff function $\eta \in C_c^1(U_3)$ such that

$$0 \le \eta \le 1 \text{ in } U_3, \eta \equiv 1 \text{ in } U_4, |\nabla\eta| \le \frac{c}{r_0} \tag{2.11}$$

for some constant $c > 0$. For each fixed $j \in \{1, \dots, n\}$, $h \in \mathbb{R}$ with $0 < |h| < r_0/16$, we test $\phi := \Delta_{j,-h}(\eta^2 \Delta_{j,h} u)$ into (2.10). We note that $\phi \in W^{1,\infty}(U_1) \subset W^{1,p}(U_1)$ by (2.8), and this is compactly supported in U_2. Hence $\phi \in W_0^{1,p}(U_2)$ is an admissible test function. By testing ϕ, we have

$$\begin{aligned}
0 &= \int_{U_2} \left\langle \Delta_{j,h}(\nabla_z E(\nabla u(x))) \,\middle|\, \eta^2 \nabla(\Delta_{j,h}u) + 2\eta\Delta_{j,h}u\nabla\eta \right\rangle - \int_{U_2} f\Delta_{-j,h}(\eta^2\Delta_{j,h}u)\, dx \\
&= \int_{U_2} \eta^2 \left\langle A_h(x, \nabla u(x))\nabla(\Delta_{j,h}u) \,\middle|\, \nabla(\Delta_{j,h}u) \right\rangle dx \\
&\quad + 2\int_{U_2} \eta\Delta_{j,h}u \left\langle A_h(x, \nabla u(x))\nabla(\Delta_{j,h}u) \,\middle|\, \nabla\eta \right\rangle dx \\
&\quad - \int_{U_2} f\Delta_{-j,h}(\eta^2\Delta_{j,h}u)\, dx \\
&=: I_1 + I_2 - I_3.
\end{aligned} \tag{2.12}$$

Here $A_h = A_h(x, \nabla u(x))$ denotes a matrix-valued function in U_2 given by

$$A_h(x, \nabla u(x)) := \int_0^1 \nabla_z^2 E((1-t)\nabla u(x) + t\nabla u(x + he_j))\, dt.$$

We note that with the aid of (2.3)–(2.6), we obtain

$$\mu_0 \le |(1-t)\nabla u(x) + t\nabla u(x+he_j)| \le M_0$$

for a.e. $x \in U_2$ and for all $0 \le t \le 1$. Combining this result with (2.7)–(2.8), we conclude that A_h satisfies

$$\lambda|\zeta|^2 \le \langle A_h(x, \nabla u(x))\zeta \mid \zeta \rangle \tag{2.13}$$

$$\langle A_h(x, \nabla u(x))\zeta \mid \omega \rangle \le \Lambda|\zeta||\omega| \tag{2.14}$$

for all $\zeta, \omega \in \mathbb{R}^n$ and for a.e. $x \in U_2$. We set an integral

$$J := \int_{U_2} \eta^2 |\nabla(\Delta_{j,h}u)|^2 \, dx.$$

By (2.13), it is clear that $I_1 \ge \lambda J$. By Young's inequality and applying a Poincaré-type inequality (Lemma 7) to $\eta^2\Delta_{j,h}u \in W_0^{1,2}(U_2)$, we obtain for any $\varepsilon > 0$,

$$\begin{aligned}
|I_3| &\le \frac{1}{4\varepsilon}\|f\|_{L^2(U_2)}^2 + \varepsilon\int_{U_2} |\nabla(\eta^2\Delta_{j,h}u)|^2\, dx \\
&\le \frac{1}{4\varepsilon}\|f\|_{L^2(U_2)}^2 + 4\varepsilon\int_{U_2} |\Delta_{j,h}u|^2|\nabla\eta|^2\, dx + 2\varepsilon\int_{U_2} \eta^2|\nabla(\Delta_{j,h}u)|^2\, dx.
\end{aligned}$$

Here we have invoked the property $0 \le \eta \le 1$ in U_2. We fix $\varepsilon := \lambda/6 > 0$. By (2.14) and Young's inequality, we have

$$\begin{aligned}
|I_2| &\le 2\Lambda\int_{U_2} \eta|\nabla(\Delta_{j,h}u)| \cdot |\Delta_{j,h}u||\nabla\eta|\, dx \\
&\le \frac{\lambda}{3}J + \frac{3\Lambda^2}{\lambda}\int_{U_2} |\Delta_{j,h}u|^2|\nabla\eta|^2\, dx.
\end{aligned}$$

It follows from (2.6) that $\|\Delta_{j,h}u\|_{L^\infty(U_2)} \le M_0$. Therefore we obtain from (2.12),

$$\int_{U_4} |\nabla(\Delta_{j,h}u)|^2\,dx \le J = \int_{U_2} \eta^2 |\nabla(\Delta_{j,h}u)|^2\,dx \le C(\lambda,\Lambda)\Big(M_0^2\|\nabla\eta\|_{L^2(U_2)}^2 + \|f\|_{L^2(U_2)}^2\Big).$$

The estimate (2.9) follows from this, and therefore $u \in W^{2,2}(U_4)$.

For each $\psi \in C_c^\infty(U_4)$, we test $\partial_{x_j}\psi \in C_c^\infty(U_4)$ into (2.10). Integrating by parts, we obtain

$$\int_{U_4} \big\langle \nabla_z^2 E(\nabla u)\nabla\partial_{x_j}u \;\big|\; \nabla\psi \big\rangle\,dx = -\int_{U_4} f\partial_{x_j}\psi\,dx \tag{2.15}$$

for all $\psi \in C_c^\infty(U_4)$. Noting that $f \in L^q(U_4) \subset L^2(U_4)$, $\partial_{x_j}u \in W^{1,2}(U_4)$, and (2.7)–(2.8), we may extend $\psi \in W_0^{1,2}(U_4)$ by a density argument. The conditions (2.7)–(2.8) imply that $\nabla_z^2 E(\nabla u)$ is uniformly elliptic over U_1. Hence by [17, Theorem 8.22], there exists $\alpha = \alpha(\lambda,\Lambda,n,q) \in (0,1)$ such that $\partial_{x_j}u \in C^\alpha(U_5)$ for each $j \in \{1,\dots,n\}$. This regularity result implies $\partial u(x) = \{\nabla u(x)\} \neq \{0\}$ for all $x \in D$. □

3 A blow-up argument

In order to show that (1.4) holds true even for $x \in F$, we first make a blow-argument and construct a convex weak solution in the whole space $\mathbb{R}^n$, in the sense of Definition 1.

Proposition 1. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain, and $f \in L^q_{\mathrm{loc}}(\Omega)$ $(n < q \le \infty)$. Assume that u is a convex weak solution to (1.1), and $x_0 \in \Omega$. Then there exists a convex function $u_0\colon \mathbb{R}^n \to \mathbb{R}$ such that*

1. *u_0 is a weak solution to $L_{b,p}u_0 = 0$ in $\mathbb{R}^n$.*
2. *The inclusion $\partial u(x_0) \subset \partial u_0(x_0)$ holds. That is, if $c \in \partial u(x_0)$, then we have*

$$u_0(x) \ge u_0(x_0) + \langle c \mid x - x_0\rangle \quad \text{for all } x \in \mathbb{R}^n.$$

In particular, if $x_0 \in F$, then the facet of u_0 is non-empty.

Proof. Without loss of generality, we may assume that $x_0 = 0$ and $u(x_0) = 0$. First we fix a closed ball $\overline{B_R(0)} = \overline{B_R} \subset \Omega$. We note that $u \in \mathrm{Lip}(\overline{B_R})$ since u is convex. Hence there exists a sufficiently large number $M \in (0,\infty)$ such that

$$\operatorname*{ess\,sup}_{B_R} |\nabla u| \le M \text{ and } |u(x) - u(y)| \le M|x-y| \quad \text{for all } x, y \in B_R.$$

We take and fix a vector field $Z \in L^\infty(B_R,\mathbb{R}^n)$ such that the pair $(u,Z) \in W^{1,p}(B_R) \times L^\infty(B_R,\mathbb{R}^n)$ satisfies $L_{b,p}u = f$ in $W^{-1,p'}(B_R)$. For each $a > 0$, we define a rescaled convex function $u_a\colon B_{R/a} \to \mathbb{R}$ and a dilated vector field $Z_a \in L^\infty(B_{R/a},\mathbb{R}^n)$ by

$$u_a(x) \coloneqq \frac{u(ax)}{a}, \quad Z_a(x) \coloneqq Z(ax) \quad \text{for } x \in B_{R/a}.$$

We also set $f_a \in L^q(B_{R/a})$ by

$$f_a(x) \coloneqq af(ax) \quad \text{for } x \in B_{R/a}.$$

Then it is easy to check that the pair $(u_a, Z_a) \in W^{1,\infty}(B_{R/a}) \times L^\infty(B_{R/a},\mathbb{R}^n)$ satisfies $L_{b,p}u_a = f_a$ in $W^{-1,p'}(B_{R/a})$. By definition of u_a, we clearly have

$$\sup_{B_{R/a}} |u_a| \le M < \infty, \quad \|\nabla u_a\|_{L^\infty(B_{R/a})} \le M < \infty \quad \text{for all } a > 0. \tag{3.1}$$

Hence by the Arzelà–Ascoli theorem and a diagonal argument, we can take a decreasing sequence $\{a_N\}_{N=1}^\infty \subset (0,\infty)$, such that $a_N \to 0$ as $N \to \infty$, and

$$u_{a_N} \to u_0 \quad \text{locally uniformly in } \mathbb{R}^n. \tag{3.2}$$

for some function $u_0\colon \mathbb{R}^n \to \mathbb{R}$. Clearly u_0 is convex in $\mathbb{R}^n$, and the inclusion $\partial u(x_0) \subset \partial u_0(x_0)$ holds true by the construction of rescaled functions u_a. If $x_0 \in F$, then we have $\{0\} \subset \partial u(x_0) \subset \partial u_0(x_0)$ and therefore x_0 lies in the facet of u_0. We are left to show that u_0 is a weak solution to $L_{b,p}u_0 = 0$ in $\mathbb{R}^n$. Before proving this, we note that from (3.1)–(3.2) and Lemma 9, it follows that

$$\nabla u_{a_N}(x) \to \nabla u_0(x) \quad \text{and} \quad |\nabla u_0(x)| \le M \quad \text{for a.e. } x \in \mathbb{R}^n \tag{3.3}$$

as $N \to \infty$. We arbitrarily fix an open ball $B_r \subset \mathbb{R}^n$. Note that the inclusion $B_r \subset B_{R/a}$ holds for all $0 < a < R/r$. Hence we easily realize that a family of pairs $\{(u_a, Z_a)\}_{0<a<R/r} \subset W^{1,\infty}(B_r) \times L^\infty(B_r, \mathbb{R}^n)$ satisfies

$$Z_a(x) \in \partial|\cdot|(\nabla u_a(x)) \quad \text{for a.e. } x \in B_r, \tag{3.4}$$

$$b\int_{B_r} \langle Z_a \mid \nabla\phi\rangle\,dx + \int_{B_r} \left\langle |\nabla u_a|^{p-2}\nabla u_a \,\middle|\, \nabla\phi\right\rangle dx = \int_{B_r} f_a\phi\,dx \quad \text{for all } \phi \in W_0^{1,p}(B_r). \tag{3.5}$$

By definition of f_a, we get $\|f_a\|_{L^q(B_r)} = a^{1-n/q}\|f\|_{L^q(B_{ar})} \le a^{1-n/q}\|f\|_{L^q(B_R)}$ for all $0 < a < R/r$. Hence by the continuous embedding $L^q(B_r) \hookrightarrow W^{-1,p'}(B_r)$, we obtain

$$f_{a_N} \to 0 \quad \text{in } W^{-1,p'}(B_r) \quad \text{as } N \to \infty. \tag{3.6}$$

By (3.1) and (3.3), we can apply Lebesgue's dominated convergence theorem and get

$$|\nabla u_{a_N}|^{p-2}\nabla u_{a_N} \to |\nabla u_0|^{p-2}\nabla u_0 \quad \text{in } L^{p'}(B_r, \mathbb{R}^n) \quad \text{as } N \to \infty. \tag{3.7}$$

It is clear that $\|Z_a\|_{L^\infty(B_r,\mathbb{R}^n)} \le 1$ for all $0 < a < R/r$. Hence by [5, Corollary 3.30], up to a subsequence, we may assume that

$$Z_{a_N} \overset{*}{\rightharpoonup} Z_{0,r} \quad \text{in } L^\infty(B_r, \mathbb{R}^n) \quad \text{as } N \to \infty \tag{3.8}$$

for some $Z_{0,r} \in L^\infty(B_r, \mathbb{R}^n)$. By lower-semicontinuity of the norm with respect to the weak* topology and (3.3)–(3.4), we get

$$\|Z_{0,r}\|_{L^\infty(B_r,\mathbb{R}^n)} \le 1, \quad Z_{0,r}(x) = \frac{\nabla u_0(x)}{|\nabla u_0(x)|} \quad \text{for a.e. } x \in B_r \text{ with } \nabla u_0(x) \ne 0,$$

which implies that

$$Z_{0,r}(x) \in \partial|\cdot|(\nabla u_0(x)) \quad \text{for a.e. } x \in B_r. \tag{3.9}$$

Letting $a = a_N$ in (3.5) and $N \to \infty$, we obtain

$$b\int_{B_r} \langle Z_{0,r} \mid \nabla\phi\rangle\,dx + \int_{B_r} \left\langle |\nabla u_0|^{p-2}\nabla u_0 \,\middle|\, \nabla\phi\right\rangle dx = 0 \quad \text{for all } \phi \in W_0^{1,p}(B_r) \tag{3.10}$$

by (3.5)–(3.8). Since $B_r \subset \mathbb{R}^n$ is arbitrary, (3.9)–(3.10) means that u_0 is a weak solution to $L_{b,p}u_0 = 0$ in $\mathbb{R}^n$, in the sense of Definition 1. □

4 Maximum principles

In Section 4, we justify maximum principles for the equation $L_{b,p}u = 0$.

We first define subsolutions and supersolutions in the weak sense.

Definition 2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. A pair $(u, Z) \in W^{1,p}(\Omega) \times L^\infty(\Omega, \mathbb{R}^n)$ is called a *weak* subsolution to $L_{b,p}u = 0$ in Ω, if it satisfies

$$b\int_\Omega \langle Z \mid \nabla\phi\rangle\,dx + \int_\Omega \left\langle |\nabla u|^{p-2}\nabla u \,\middle|\, \nabla\phi\right\rangle dx \le 0 \tag{4.1}$$

for all $0 \le \phi \in C_c^\infty(\Omega)$, and

$$Z(x) \in \partial|\cdot|(\nabla u(x)) \quad \text{for a.e. } x \in \Omega. \tag{4.2}$$

Similarly we call a pair $(u, Z) \in W^{1,p}(\Omega)\times L^{\infty}(\Omega,\mathbb{R}^n)$ a *weak* supersolution $L_{b,p}u=0$ in Ω, if it satisfies (4.2) and

$$b\int_{\Omega}\langle Z \mid \nabla\phi\rangle\,dx+\int_{\Omega}\left\langle|\nabla u|^{p-2}\nabla u \,\middle|\, \nabla\phi\right\rangle dx \ge 0 \tag{4.3}$$

for all $0\le\phi\in C_c^{\infty}(\Omega)$. For $u\in W^{1,p}(\Omega)$, we simply say that u is respectively a subsolution and a supersolution to $L_{b,p}u=0$ in the *weak* sense if there is $Z\in L^{\infty}(\Omega,\mathbb{R}^n)$ such that the pair (u, Z) is a weak subsolution and a weak supersolution to $L_{b,p}u=0$ in Ω.

Remark 2. We describe some remarks on our definitions of weak solutions, subsolutions and supersolutions.

1. By an approximation argument, we may extend the test function class of (4.1) to

   $$D_+(\Omega) := \{\phi\in W^{1,p}(\Omega) \mid \phi\ge 0 \text{ a.e. in } \Omega,\ \operatorname{supp}\phi\subset\Omega\}.$$

   Indeed, for $\phi\in D_+(\Omega)$ and $0<\varepsilon<\operatorname{dist}(\operatorname{supp}\phi,\partial\Omega)$, the function,

   $$\phi_{\varepsilon}(x)=\int_{\Omega}\phi(x-y)\rho_{\varepsilon}(y)\,dy \quad \text{for } x\in\Omega$$

   satisfies $0\le\phi_{\varepsilon}\in C_c^{\infty}(\Omega)$. Here for $0<\varepsilon<\infty$, $0\le\rho_{\varepsilon}\in C_c^{\infty}(B_{\varepsilon}(0))$ denotes a standard mollifier so that

   $$0\le\rho\in C_c^{\infty}(B_1),\quad \|\rho\|_{L^1(\mathbb{R}^n)}=1,\quad \rho_{\varepsilon}(x):=\varepsilon^{-n}\rho(x/\varepsilon) \text{ for } x\in\mathbb{R}^n.$$

   By testing ϕ_{ε} into (4.1) for sufficiently small $\varepsilon>0$ and letting $\varepsilon\to 0$, we conclude that if the pair (u, Z) satisfies (4.1) for all $0\le\phi\in C_c^{\infty}(\Omega)$, then (4.1) holds for all $\phi\in D_+(\Omega)$. A similar result is also valid for (4.3).

2. By Definition 1–2, if a pair $(u, Z)\in W^{1,p}(\Omega)\times L^{\infty}(\Omega,\mathbb{R}^n)$ satisfies $L_{b,p}u=0$ in $W^{-1,p'}(\Omega)$, then u is clearly both a subsolution and a supersolution to $L_{b,p}u=0$ in Ω in the weak sense. Conversely, if a pair $(u, Z)\in W^{1,p}(\Omega)\times L^{\infty}(\Omega,\mathbb{R}^n)$ is both a weak subsolution and a weak supersolution to $L_{b,p}u=0$ in Ω, then the pair (u, Z) satisfies $L_{b,p}u=0$ in $W^{-1,p'}(\Omega)$. Indeed, by the previous remark we have already known that the pair (u, Z) satisfies (4.1) and (4.3) for all $\phi\in D_+(\Omega)$, which clearly yields

   $$b\int_{\Omega}\langle Z \mid \nabla\phi\rangle\,dx+\int_{\Omega}\left\langle|\nabla u|^{p-2}\nabla u \,\middle|\, \nabla\phi\right\rangle dx = 0 \tag{4.4}$$

   for all $\phi\in D_+(\Omega)$. We decompose arbitrary $\phi\in C_c^{\infty}(\Omega)$ by $\phi=\phi_+-\phi_-$, where $\phi_+:=\max\{\phi,0\}$, $\phi_-:=\max\{-\phi,0\}\in D_+(\Omega)$. By testing $\phi_+,\phi_-\in D_+(\Omega)$ into (4.4), we conclude that (4.4) holds for all $\phi\in C_c^{\infty}(\Omega)$. By density of $C_c^{\infty}(\Omega)\subset W_0^{1,p}(\Omega)$, it is clear that (4.4) is valid for all $\phi\in W_0^{1,p}(\Omega)$.

3. For a bounded domain $\Omega\subset\mathbb{R}^n$, let $u\in C^2(\overline{\Omega})$ satisfy the following two conditions (4.5)–(4.6);

   $$\nabla u(x)\neq 0 \quad \text{for all } x\in\Omega, \tag{4.5}$$

   $$(L_{b,p}u)(x)=-(b\Delta_1 u+\Delta_p u)(x)\le 0 \quad \text{for all } x\in\Omega. \tag{4.6}$$

   Then for any fixed $0\le\phi\in C_c^{\infty}(\Omega)$, we have

   $$0\ge\int_{\Omega}(L_{b,p}u)\phi\,dx=b\int_{\Omega}\left\langle\frac{\nabla u}{|\nabla u|}\,\middle|\,\nabla\phi\right\rangle dx+\int_{\Omega}\left\langle|\nabla u|^{p-2}\nabla u\,\middle|\,\nabla\phi\right\rangle dx,$$

   with the aid of integration by parts and (4.6). We also note that

   $$\partial|\cdot|(\nabla u(x))=\left\{\frac{\nabla u(x)}{|\nabla u(x)|}\right\} \quad \text{for all } x\in\Omega$$

   by (4.5). Therefore the pair $(u,\nabla u/|\nabla u|)\in W^{1,p}(\Omega)\times L^{\infty}(\Omega,\mathbb{R}^n)$ satisfies (4.1)–(4.2). For such u, we simply say that u satisfies $L_{b,p}u\le 0$ in Ω in the *classical* sense.

4.1 Comparison principle

We justify the comparison principle, i.e., for any subsolution u^- and supersolution u^+,

$$u^- \leq u^+ \quad \text{on } \partial\Omega \quad \text{implies that} \quad u^- \leq u^+ \quad \text{in } \Omega,$$

under the condition that u^+ and u^- admits continuity properties in $\overline{\Omega}$.

Proposition 2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Assume that $u^+, u^- \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ is a subsolution and a supersolution to $L_{b,p}u = 0$ in the weak sense respectively. If u^+, u^- satisfies*

$$u^-(x) \leq u^+(x) \quad \text{for all } x \in \partial\Omega, \tag{4.7}$$

then $u^- \leq u^+$ in $\overline{\Omega}$.

Before proving Proposition 2, we recall that the mapping $A\colon \mathbb{R}^n \ni z \mapsto |z|^{p-2}z \in \mathbb{R}^n$ satisfies strict monotonicity,

$$\text{i.e., } \langle A(z_2) - A(z_1) \mid z_2 - z_1 \rangle > 0 \quad \text{for all} \quad z_1, z_2 \in \mathbb{R}^n \quad \text{with} \quad z_1 \neq z_2. \tag{4.8}$$

Proof. We take arbitrary $\delta > 0$. By $u^+, u^- \in C(\overline{\Omega})$ and (4.7), we can take a subdomain $\Omega' \Subset \Omega$ such that $u^- \leq u^+ + \delta$ in $\Omega \setminus \Omega'$. This implies that the support of the truncated non-negative function $w_\delta := (u^+ - u^- + \delta)_- \in W^{1,p}(\Omega)$ is contained in $\overline{\Omega'} \subsetneq \Omega$ and therefore $w_\delta \in D_+(\Omega)$. Let $Z^+, Z^- \in L^\infty(\Omega, \mathbb{R}^n)$ be vector fields such that (u^+, Z^+), (u^-, Z^-) satisfies (4.1)–(4.2), (4.2)–(4.3) respectively. As in Remark 2, we may test w_δ in (4.1) and (4.3). Note that $\nabla w_\delta = -\chi_\delta \nabla(u^+ - u^-)$, where χ_δ denotes the characteristic function of $A_\delta := \{x \in \Omega \mid u^+ + \delta \leq u^-\}$. Hence, we have

$$\begin{aligned} 0 &\leq -b \int_{A_\delta} \left\langle Z_+ - Z_- \;\middle|\; \nabla u^+ - \nabla u^- \right\rangle dx - \int_{A_\delta} \left\langle |\nabla u^+|^{p-2}\nabla u^+ - |\nabla u^-|^{p-2}\nabla u^- \;\middle|\; \nabla u^+ - \nabla u^- \right\rangle dx \\ &\leq -\int_{A_\delta} \left\langle |\nabla u^+|^{p-2}\nabla u^+ - |\nabla u^-|^{p-2}\nabla u^- \;\middle|\; \nabla u^+ - \nabla u^- \right\rangle dx. \end{aligned}$$

Here we have invoked (4.2) and monotonicity of the subdifferential operator $\partial|\cdot|$. From (4.8) we can easily check that $\nabla u^+ = \nabla u^-$ in A_δ, and therefore $w_\delta = 0$ in $W_0^{1,p}(\Omega)$. This means that $u^- \leq u^+ + \delta$ a.e. in Ω. By regularity assumptions $u^+, u^- \in C(\overline{\Omega})$, we conclude that $u^- \leq u^+ + \delta$ in $\overline{\Omega}$. Since $\delta > 0$ is arbitrary, this completes the proof. □

Remark 3. In 2013, Krügel gave another type of definitions of weak subsolutions and weak supersolutions to $L_{b,p} = a$, where $a \in \mathbb{R}$ is a constant. In Krügel's definition [25, Definition 4.6], a function $u^- \in W^{1,p}(\Omega)$ is called a *subsolution* to $L_{b,p} = a$ if u^- satisfies

$$\int_{D^-} \left\langle \frac{\nabla u^-}{|\nabla u^-|} \;\middle|\; \nabla\phi \right\rangle dx + \int_{F^-} |\nabla\phi|\, dx + \left\langle |\nabla u^-|^{p-2}\nabla u^- \;\middle|\; \nabla\phi \right\rangle dx \leq \int_\Omega a\phi\, dx \tag{4.9}$$

for all $\phi \in D_+(\Omega)$. Here $F^- := \{x \in \Omega \mid \nabla u^-(x) = 0\}$, $D^- := \Omega \setminus F^-$. Similarly a function $u^+ \in W^{1,p}(\Omega)$ is called a *supersolution* to $L_{b,p} = a$ if u^+ satisfies

$$\int_{D^+} \left\langle \frac{\nabla u^+}{|\nabla u^+|} \;\middle|\; \nabla\phi \right\rangle dx + \int_{F^+} |\nabla\phi|\, dx + \left\langle |\nabla u^+|^{p-2}\nabla u^+ \;\middle|\; \nabla\phi \right\rangle dx \geq \int_\Omega a\phi\, dx \tag{4.10}$$

for all $\phi \in D_+(\Omega)$. Here $F^+ := \{x \in \Omega \mid \nabla u^+(x) = 0\}$, $D^+ := \Omega \setminus F^+$.

The comparison principle discussed by Krügel [25, Theorem 4.8] states that

$$(u^- - u^+)_+ \in D_+(\Omega) \text{ implies } u^- \leq u^+ \quad \text{a.e. in } \Omega. \tag{4.11}$$

By testing $(u^- - u^+)_+ \in D_+(\Omega)$ into (4.9)(4.10) and substracting the two inequalities, Krügel claims that $\nabla u^- = \nabla u^+$ over $\Omega' := \{x \in \Omega \mid u^-(x) \geq u^+(x)\}$ and hence $u^- = u^+$ a.e. in Ω'. Despite Krügel's comment that integrals over F^- and F^+ cancel out, however, it seems unclear whether

$$\int_{F^-} |\nabla(u^- - u^+)_+|\, dx = \int_{F^+} |\nabla(u^- - u^+)_+|\, dx \tag{4.12}$$

is valid. This problem is essentially due to the fact that Krügel did not appeal to monotonicity of the subdifferential operator $\partial|\cdot|$ and did not regard the term $\nabla u/|\nabla u|$ as an L^∞-vector field satisfying the property (4.2). In our proof of the comparison principle (Proposition 2), we make use of monotonicity of the operator $\partial|\cdot|$. Compared to our argument based on monotonicity, the equality (4.12) itself seems to be too strong to hold true.

4.2 Construction of classical subsolutions

In Section 4.2, we construct a classical subsolution to $L_{b,p}u = 0$ in an open annulus.

Lemma 2. *Let $c \in \mathbb{R}^n \setminus \{0\}, m > 0$. Then for each fixed open ball $B_R(x_*) \subset \mathbb{R}^n$, there exists a function $h \in C^\infty(\mathbb{R}^n \setminus \{x_*\})$ such that*

$$h = 0 \quad \text{on } \partial B_R(x_*), \quad 0 \le h \le m \quad \text{on } \overline{E_R(x_*)}, \tag{4.13}$$

$$\partial_\nu h < 0 \quad \text{on } \partial B_R(x_*), \tag{4.14}$$

$$|\nabla h| \le \frac{|c|}{2} \quad \text{in } E_R(x_*), \tag{4.15}$$

$$v(x) \coloneqq h(x) + \langle c \mid x \rangle \text{ satisfies} \quad L_{b,p}v \le 0 \quad \text{in } E_R(x_*), \text{ in the classical sense.} \tag{4.16}$$

Here $E_R(x_*) \coloneqq B_R(x_*) \setminus \overline{B_{R/2}(x_*)}$ is an open annulus, and ν in (4.14) denotes the exterior unit vector normal to $B_R(x_*)$.

Before proving Lemma 2, we fix some notations on matrices. For a given $n \times n$ matrix A, we write $\mathrm{tr}(A)$ as the trace of A. We denote $\mathbf{1}_n$ by the $n \times n$ unit matrix. For column vectors $x = (x_i)_i$, $y = (y_i)_i \in \mathbb{R}^n$, we define a tensor $x \otimes y$, which is regarded as a real-valued $n \times n$ matrix

$$x \otimes y \coloneqq (x_i y_j)_{i,j} = \begin{pmatrix} x_1 y_1 & \cdots & x_1 y_n \\ \vdots & \ddots & \vdots \\ x_n y_1 & \cdots & x_n y_n \end{pmatrix}.$$

Assume that h satisfies (4.15). Then the triangle inequality implies that

$$0 < \frac{1}{2}|c| \le |\nabla v| \le \frac{3}{2}|c| \quad \text{in} \quad E_R(x_*). \tag{4.17}$$

The estimate (4.17) allows us to calculate $L_{b,p}v$ in the classical sense over $E_R(x_*)$. By direct calculations we have

$$-L_{b,p}v = +\mathrm{div}(\nabla_z E(\nabla v)) = \sum_{i,j=1}^{n} \partial_{z_i z_j} E(\nabla v) \partial_{x_i x_j} v = \mathrm{tr}\left(\nabla_z^2 E(\nabla v) \nabla^2 h\right) \quad \text{in } E_R(x_*).$$

We note that $\nabla^2 v = \nabla^2 h$ by definition. Here we recall a well-known result on Pucci's extremal operators. For given constants $0 < \lambda \le \Lambda < \infty$ and a fixed $n \times n$ symmetric matrix M, we define

$$\mathcal{M}^-(M, \lambda, \Lambda) \coloneqq \lambda \sum_{\lambda_i > 0} \lambda_i + \Lambda \sum_{\lambda_i < 0} \lambda_i,$$

where $\lambda_i \in \mathbb{R}$ are the eigenvalues of M. The following formula is a well-known result [1, Remark 5.36] ;

$$\mathcal{M}^-(M, \lambda, \Lambda) = \inf\left\{\mathrm{tr}(AM) \,\middle|\, A \in \mathcal{A}_{\lambda,\Lambda}\right\},$$

where $\mathcal{A}_{\lambda,\Lambda}$ denotes the set of all symmetric matrices whose eigenvalues all belong to the closed interval $[\lambda, \Lambda]$. By (4.17) $L_{b,p}v$ is an uniformly elliptic operator in $E_R(x_*)$. This enables us to find constants $0 < \lambda \le \Lambda < \infty$, depending on $0 < b < \infty$, $1 < p < \infty$, $|c| > 0$, such that $\nabla_z^2 E(\nabla v) \in [\lambda, \Lambda]$ in $E_R(x_*)$. Combining these results, it suffices to show that

$$\mathcal{M}^-\left(\nabla^2 h(x), \lambda, \Lambda\right) = \lambda \sum_{\lambda_i > 0} \lambda_i(x) + \Lambda \sum_{\lambda_i < 0} \lambda_i(x) > 0 \quad \text{for all } x \in E_R(x_*), \tag{4.18}$$

where $\lambda_i(x) \in \mathbb{R}$ denotes the eigenvalues of $\nabla^2 h(x)$.

Now we construct classical subsolutions. Our first construction is a modification of that by E. Hopf [20].

Proof. Without loss of generality we may assume $x_* = 0$. We define

$$h(x) := e^{-\alpha|x|^2} - e^{-\alpha R^2} \quad \text{for } x \in \mathbb{R}^n. \tag{4.19}$$

Here $\alpha = \alpha(b, n, p, |c|, R) > 0$ is a sufficiently large constant to be chosen later. It is clear that $0 \le h(x) \le e^{-\alpha R^2/4} - e^{-\alpha R^2}$ in $\overline{E_R(0)}$. We first let $\alpha > 0$ be so large that

$$me^{\alpha R^2} \ge e^{3\alpha R^2/4} - 1. \tag{4.20}$$

From (4.20), we can easily check (4.13). By direct calculation we get

$$\nabla h(x) = -2\alpha e^{-\alpha|x|^2} x, \text{ and } \nabla^2 h(x) = -2\alpha e^{-\alpha|x|^2} \mathbf{1}_n + 4\alpha^2 e^{-\alpha|x|^2} x \otimes x \quad \text{for each } x \in \mathbb{R}^n.$$

From this result, (4.14) is clear. Also, we have

$$|\nabla h(x)| \le 2\alpha R e^{-\alpha R^2/4} \quad \text{for all } x \in E_R(0).$$

Let $\alpha > 0$ be so large that

$$\alpha e^{-\alpha R^2/4} \le \frac{|c|}{4R}, \tag{4.21}$$

then we can check that h satisfies (4.15). Now we prove (4.16) to complete the proof. For $x \ne 0$, the eigenvalues of $\nabla^2 h(x)$ are given by

$$\begin{cases} \lambda_{\parallel}(x) & := & 4\alpha^2|x|^2 e^{-\alpha|x|^2} - 2\alpha e^{-\alpha|x|^2}, \\ \lambda_{\perp}(x) & := & -2\alpha e^{-\alpha|x|^2}, \end{cases} \quad \text{and the geometric multiplicities are } \begin{cases} 1, \\ n-1. \end{cases}$$

Assume that α satisfies

$$\alpha > \frac{2}{R^2}, \tag{4.22}$$

so that $\lambda_{\parallel} > 0 > \lambda_{\perp}$ in $E_R(0)$. Therefore we get

$$\begin{aligned} \mathcal{M}^-\left(\nabla^2 h(x), \lambda, \Lambda\right) &= \lambda\lambda_{\parallel}(x) + (n-1)\Lambda\lambda_{\perp}(x) = 2\alpha e^{-\alpha|x|^2}\left[\lambda(2\alpha|x|^2 - 1) - (n-1)\Lambda\right] \\ &\ge 2\alpha e^{-\alpha|x|^2}\left[\lambda\left(\frac{R^2}{2}\alpha - 1\right) - (n-1)\Lambda\right]. \end{aligned}$$

We can take sufficiently large $\alpha = \alpha(|c|, m, n, R, \lambda, \Lambda) > 0$ so that α satisfies (4.18) and (4.20)–(4.22). For such constant $\alpha > 0$, the function v defined as in (4.19) satisfies (4.13)–(4.16). □

It is possible to construct an alternative function $h \in C^\infty(\mathbb{R}^n \setminus \{x_0\})$ which satisfies (4.13)–(4.16). We give another proof of Lemma 2, which is derived from [29, Chapter 2.8].

Proof. Without loss of generality we may assume $x_* = 0$. We define

$$h(x) := \beta[|x|^{-\alpha} - R^{-\alpha}] \quad \text{for } x \in \mathbb{R}^n \setminus \{0\}. \tag{4.23}$$

We will later determine positive constants $\alpha, \beta > 0$, depending on $b, m, n, p, |c|, R$. It is clear that $0 \le h(x) \le \beta R^{-\alpha}(2^\alpha - 1)$ in $\overline{E_R(0)}$. We first let $\alpha, \beta > 0$ satisfy

$$\beta \le \frac{mR^\alpha}{2^\alpha - 1}. \tag{4.24}$$

Then h satisfies (4.13). By direct calculation we get

$$\nabla h(x) = -\frac{\alpha\beta x}{|x|^{\alpha+2}}, \text{ and } \nabla^2 h(x) = \frac{\alpha\beta}{|x|^{\alpha+2}}\left[(\alpha+2)\frac{x \otimes x}{|x|^2} - \mathbf{1}_n\right]$$

for each $x \in \mathbb{R}^n \setminus \{0\}$. The estimate (4.14) is clear by this result. Also, we have

$$|\nabla h(x)| \leq \frac{\alpha\beta}{(R/2)^{\alpha+1}} \quad \text{for all } x \in E_R(0).$$

Let $\alpha, \beta > 0$ satisfy

$$\beta \leq \frac{|c|(R/2)^{\alpha+1}}{2\alpha}, \tag{4.25}$$

then we can check that h satisfies (4.15). Now we prove (4.16) to complete the proof. For $x \neq 0$, the eigenvalues of $\nabla^2 h(x)$ are given by

$$\begin{cases} \lambda_{\|}(x) & := & (\alpha+1)\alpha\beta|x|^{-\alpha-2}, \\ \lambda_{\perp}(x) & := & -\alpha\beta|x|^{-\alpha-2}, \end{cases} \quad \text{and the geometric multiplicities are} \begin{cases} 1, \\ n-1. \end{cases}$$

It is clear that $\lambda_{\|} > 0 > \lambda_{\perp}$ in $\mathbb{R}^n \setminus \{0\}$, and therefore

$$\mathcal{M}^-\left(\nabla^2 h(x), \lambda, \Lambda\right) = \alpha\beta|x|^{-\alpha-2}\left[(\alpha+1)\lambda - (n-1)\Lambda\right]$$

for all $x \in E_R(0)$. We take and fix sufficiently large $\alpha = \alpha(n, \lambda, \Lambda) > 0$ so that α satisfies (4.18). For such $\alpha > 0$, we choose sufficiently small $\beta = \beta(|c|, R, \alpha) > 0$ so that β satisfies (4.24)–(4.25). Then the function h defined as in (4.23) satisfies (4.13)–(4.16). □

4.3 Strong maximum principle

We prove the strong maximum principle (Theorem 2).

Proof. Let $D_0 \subset D$ be a connected component of the open set D, and $x_0 \in D_0$. Without loss of generality we may assume that $x_0 = 0$ and $u(0) = 0$. By Lemma 1, it is clear that $\partial u(0) = \{\nabla u(0)\} \neq \{0\}$. We set a vector $c := \nabla u(0) \in \mathbb{R}^n \setminus \{0\}$ and a relatively closed set

$$\Sigma := \{x \in D_0 \mid u(x) = \langle c \mid x \rangle\}.$$

and we will prove that $\Sigma = D_0$. It is also clear that $0 \in \Sigma$ and hence $\Sigma \neq \emptyset$. Suppose for contradiction that $\Sigma \subsetneq D_0$. Then it follows that $\partial\Sigma \cap D_0 \neq \emptyset$, since D_0 is connected. We may take and fix a point $x_* \in D_0 \setminus \Sigma$ such that $\mathrm{dist}(x_*, \Sigma) < \mathrm{dist}(x_*, \partial D_0)$. By extending a closed ball centered at x_* until it hits Σ, we can take a point $y_* \in D_0$ and a closed ball $\overline{B_R(x_*)} \subset D_0$ such that $y_* \in \partial B_R(x_*) \cap \Sigma$ and $u(x) > \langle c \mid x \rangle$ for all $x \in B_R(x_*)$. We note that

$$\begin{cases} 0 & = & \min\limits_{x \in \partial B_R(x_*)} (u(x) - \langle c \mid x \rangle), & \text{achieved at } y_* \in \partial B_R(x_*), \\ m & := & \min\limits_{x \in \partial B_{R/2}(x_*)} (u(x) - \langle c \mid x \rangle) > 0, & \end{cases} \tag{4.26}$$

by construction of $B_R(x_*)$. Let $h \in C^\infty(\mathbb{R}^n \setminus \{x_*\})$ be an auxiliary function as in Lemma 2. Then from (4.26) it is easy to check that $v := h + \langle c \mid x \rangle$ satisfies $v \leq u$ on $\partial E_R(x_*)$, in the sense of (4.7). By Proposition 2, we have $v \leq u$ on $\overline{E_R(x_*)}$. Hence $0 \leq u - \langle c \mid x \rangle - h$ in $\overline{E_R(x_*)}$. This inequality becomes equality at $y_* \in \partial E_R(x_*)$ by (4.13) and (4.26). Therefore the function $u(x) - \langle c \mid x \rangle - h(x)\ (x \in \overline{E_R(x_*)})$ takes its minimum at $y_* \in \partial B_R(x_*)$. Also by $y_* \in \Sigma$ and the subgradient inequality

$$u(x) \geq \langle c \mid x \rangle \quad \text{for all } x \in \Omega,$$

it is clear that the function $w(x) := u(x) - \langle c \mid x \rangle\ (x \in D_0)$ takes its minimum 0 at $y_* \in D_0$. We note that $w, w - h \in C^1(D_0)$ by Lemma 1. By calculating classical partial derivatives at y_* in the direction $\nu_0 := (y_* - x_*)/R$, we obtain

$$0 \geq \partial_{\nu_0}(w - h)(y_*) = -\partial_{\nu_0} h(y_*) > 0.$$

This is a contradiction, and therefore $\Sigma = D_0$. □

5 Proofs of main theorems

In Section 5, we give proofs of the Liouville-type theorem (Theorem 3) and the C^1-regularity theorem (Thorem 1).

5.1 Liouville-type theorem

For a preparation, we prove Lemma 3 below.

Lemma 3. *Let u be a real-valued convex function in $\mathbb{R}^n$. Assume that u satisfies the following,*

1. *The facet of u, $F \subset \mathbb{R}^n$, satisfies $\emptyset \subsetneq F \subsetneq \mathbb{R}^n$.*
2. *u attains its minimum 0.*
3. *u is affine in each connected component of $D := \mathbb{R}^n \setminus F$.*

Then up to a rotation and a shift translation, u can be expressed as either of the following three types of piecewise-linear functions.

$$u(x) = \max\{\, t_1 x_1, 0 \,\} \quad \text{for all } x \in \mathbb{R}^n, \tag{5.1}$$

$$u(x) = \max\{\, t_1 x_1, -t_2 x_1 \,\} \quad \text{for all } x \in \mathbb{R}^n, \tag{5.2}$$

$$u(x) = \max\{\, t_1 x_1, 0, -t_2(x_1 + l_0) \,\} \quad \text{for all } x \in \mathbb{R}^n. \tag{5.3}$$

Here $t_1, t_2, l > 0$ are constants.

Before starting the proof of Lemma 3, we introduce notations on affine hyperplanes. For $c \in \mathbb{R}^n \setminus \{0\}$ and $x_0 \in \mathbb{R}^n$, we define

$$\begin{cases} H_{c,x_0} & := & \{x \in \mathbb{R}^n \mid \langle c \mid x - x_0 \rangle = 0\}, \\ H^-_{c,x_0} & := & \{x \in \mathbb{R}^n \mid \langle c \mid x - x_0 \rangle < 0\}, \\ H^+_{c,x_0} & := & \{x \in \mathbb{R}^n \mid \langle c \mid x - x_0 \rangle > 0\}. \end{cases}$$

In order to prove the Liouville-type theorem, we will make use of the supporting hyperplane theorem, which states that for any non-empty closed convex set $C \subset \mathbb{R}^n$ and $x_0 \in \partial C$, there exists $c \in \mathbb{R}^n \setminus \{0\}$ such that

$$\sup_{x \in C} \langle c \mid x \rangle \le \langle c \mid x_0 \rangle, \quad \text{and in particular} \quad H^+_{c,x_0} \subset \mathbb{R}^n \setminus C.$$

For such $c \in \mathbb{R}^n \setminus \{0\}$, a hyperplane H_{c,x_0} is often called a supporting hyperplane for C at the boundary point x_0. For the proof of the supporting hyperplane theorem, see [3, Proposition 1.5.1].

Proof. Since $\mathbb{R}^n$ is connected and $F \subset \mathbb{R}^n$ is a closed convex set, it follows that $\partial F \neq \emptyset$. Without loss of generality we may assume that $0 \in \partial F$ and $u(0) = 0$.

By the supporting hyperplane theorem, we can take and fix a supporting hyperplane for F at the boundary point 0, which we write $H_{c,0} \subset \mathbb{R}^n$. By rotation, we may assume that $c = e_1$. Let D_1 be the connected component of D which contains $H^+_{e_1,0} \subset \mathbb{R}^n \setminus F = D$. By the assumption 3 and $u(0) = 0$, it follows that there exists $c \in \mathbb{R}^n \setminus \{0\}$ such that $u(x) = \langle c \mid x \rangle$ for all $x \in D_1$. We should note that $H_{c,0} = H_{e_1,0}$ and hence $c = t_1 e_1$ for some $t_1 \in (0, \infty)$, since otherwise it follows that $H^+_{e_1,0} \cap H^-_{c,0} \neq \emptyset$ and $0 \le u(x_0) = \langle c \mid x_0 \rangle < 0$ for any $x_0 \in H^+_{e_1,0} \cap H^-_{c,0}$. The result $H_{c,0} = H_{e_1,0}$ also implies that $H_{e_1,0} \subset \partial F \subset F \subset \{x \in \mathbb{R}^n \mid x_1 \le 0\} = H^-_{e_1,0} \cup H_{e_1,0}$. Now we will deduce three possible representations of u.

If $\partial F = H_{e_1,0}$, then we have either $F = H^-_{e_1,0} \cup H_{e_1,0}$ or $F = H_{e_1,0}$, since the open set $H^-_{e_1,0} = \{x \in \mathbb{R}^n \mid x_1 < 0\}$ is connected. For the first case, u is clearly expressed by (5.1). For the second case, it is clear that D consists of two connected components $D_1 = H^+_{e_1,0}$ and $D_2 = H^-_{e_1,0}$. Again by the condition 3 and similar arguments to the above, we can determine $u|_{D_2}$ as $u(x) = \langle -t_2 e_1 \mid x \rangle$ for all $x \in D_2$. Here $t_2 \in (0, \infty)$ is a constant. Hence we obtain (5.2). For the case $H_{e_1,0} \subsetneq \partial F$, we take and fix $z_0 \in \partial F \setminus H_{e_1,0}$ and a supporting hyperplane for F at z_0, which we write by H_{c',z_0}. Let D_2 be the connected component of D which contains $H^+_{c',z_0} \subset D$. By the assumption 3 and $u(z_0) = 0$, it follows that there exists $c'' \in \mathbb{R}^n \setminus \{0\}$ such that $u(x) = \langle c'' \mid x - z_0 \rangle$ for all $x \in D_2$. Completely similarly to the arguments above for showing that $H_{c,0} = H_{e_1,0}$, we can easily notice that $H_{c'',z_0} = H_{c',z_0}$ and hence $c'' = t'_1 c'$ for some constant $t'_1 \in (0, \infty)$. Moreover, we also realize that $c' = t_* e_1$ for some $t_* \in \mathbb{R} \setminus \{0\}$. Otherwise it follows that the two hyperplanes $H_{e_1,0}$ and H_{c',z_0} cross, and hence we get $D_1 = D_2$ and $H^+_{e_1,0} \cap H^-_{c',z_0} \neq \emptyset$, which implies that there exists a point $x_0 \in D$ such that $u(x_0) < 0$. This is clearly a contradiction. This result and convexity of u imply that D consists of two connected components $D_1 = H^+_{e_1,0}$ and $D_2 = H^+_{-e_1,z_0}$, and that $F = \{x \in \mathbb{R}^n \mid -l_0 \le x_1 \le 0\}$. Here $l_0 := \mathrm{dist}(H_{e_1,0}, H_{-e_1,z_0}) > 0$. Finally we obtain the last possible expression (5.3). u can be expressed by either of (5.1)–(5.3). $\square$

Now we give the proof of Theorem 3.

Proof. Assume by contradiction that F, the facet of u, would satisfy $\emptyset \subsetneq F \subsetneq \mathbb{R}^n$. Without loss of generality, we may assume that u attains its minimum 0. By the strong maximum principle (Theorem 2), the convex weak solution u is affine in each connected component of $D := \mathbb{R}^n \setminus F$. Therefore we are able to apply Lemma 3. By rotation and translation, u can be expressed as (5.1)–(5.3). Now we prove that u is no longer a weak solution to $L_{b,\,p}u = 0$ in $\mathbb{R}^n$. We set open cubes $Q' := (-1, 1)^{n-1} \subset \mathbb{R}^{n-1}$ and $Q := (-d, d) \times Q' \subset \mathbb{R}^n$, where $d > 0$ is to be chosen later. We claim that u does not satisfy $L_{b,\,p}u = 0$ in $W^{-1,\,p'}(Q)$. Assume by contradiction that there exists a vector field $Z \in L^\infty(Q, \mathbb{R}^n)$ such that the pair $(u, Z) \in W^{1,\,p}(Q) \times L^\infty(Q, \mathbb{R}^n)$ satisfies $L_{b,\,p}u = 0$ in $W^{-1,\,p'}(Q)$.

For the first case (5.1), we have

$$|Z(x)| \le 1 \quad \text{for a.e. } x \in Q, \quad \text{and } Z(x) = e_1 \quad \text{for a.e. } x \in Q_r := (0, d) \times Q' \subset \mathbb{R}^n. \tag{5.4}$$

by definition of Z. We also set another open cube $Q_l := (-d, 0) \times Q' \subset \mathbb{R}^n$. We take and fix non-negative functions $\phi_1 \in C_c^1((-d, d))$, $\phi_2 \in C_c^1(Q')$ such that

$$\phi_1' \ge 0 \text{ in } (-d, 0),\ \max_{(-d,\,d)} \phi_1 = \phi_1(0) > 0, \text{ and } \phi_2 \not\equiv 0. \tag{5.5}$$

We define an admissible test function $\phi \in C_c^1(Q)$ by $\phi(x_1, x') := \phi_1(x_1)\phi_2(x')$ for $(x_1, x') \in (-d, d) \times Q' = Q$. Test $\phi \in C_c^1(Q)$ into $L_{b,\,p}u = 0$ in $W^{-1,\,p'}(Q)$, and divide the integration over Q into that over Q_l and Q_r. Then (5.4) implies that

$$\begin{aligned}
0 &= b\int_{Q_l} \langle Z + |0|^{p-2}0 \mid \nabla(\phi_1\phi_2)\rangle\, dx + \int_{Q_r} \left\langle (b + t_1^{p-1})e_1 \,\middle|\, \nabla(\phi_1\phi_2)\right\rangle dx \\
&\le b\int_{Q_l} \phi_1'\phi_2\, dx + b\int_{Q_l} \phi_1 |\nabla\phi_2|\, dx \\
&\quad + b\phi_1(0)\int_{Q'} \phi_2(x')\langle e_1 \mid -e_1\rangle\, dx' + t_1^{p-1}\phi_1(0)\int_{Q'} \phi_2(x')\langle e_1 \mid -e_1\rangle\, dx' \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Here we have applied the Gauss–Green theorem to the integration over Q_r, and the Cauchy–Schwarz inequality to the integration over Q_l. For the integrations I_1 and I_2, we make use of Fubini's theorem and (5.5). Then we have

$$I_1 = \int_{Q'} \left(\int_{-d}^0 \phi_1'(x_1)dx_1\right)\phi_2(x')dx' = b\phi_1(0)\int_{Q'} \phi_2(x')dx' = b\phi_1(0)\|\phi_2\|_{L^1(Q')} = -I_3,$$

$$I_2 \le b\phi_1(0)\int_{-d}^0 dx_1 \int_{Q'} |\nabla\phi_2(x')|\, dx' = bd\phi_1(0)\|\nabla\phi_2\|_{L^1(Q')}.$$

Finally we obtain

$$0 \le I_1 + I_2 + I_3 + I_4 \le I_2 + I_4 \le \phi_1(0)\left(bd\|\nabla\phi_2\|_{L^1(Q')} - t_1^{p-1}\|\phi_2\|_{L^1(Q')}\right). \tag{5.6}$$

From (5.6), we can easily deduce a contradiction by choosing sufficiently small $d = d(b, p, t_1, \phi_2) > 0$. Similarly we can prove that u defined as in (5.3) does not satisfy $L_{b,\,p}u = 0$ in $W^{-1,\,p'}(Q)$, since it suffices to restrict $d < l_0$. We consider the remaining case (5.2). We have

$$Z(x) = \begin{cases} e_1 & \text{for a.e. } x \in Q_r, \\ -e_1 & \text{for a.e. } x \in Q_l. \end{cases}$$

by definition of Z. We test the same function $\phi \in C_c^1(Q)$ in $L_{b,\,p}u = 0$, then it follows that

$$\begin{aligned}
0 &= \int_{Q_l} \left\langle -(b + t_2^{p-1})e_1 \,\middle|\, \nabla(\phi_1\phi_2)\right\rangle dx + \int_{Q_r} \left\langle (b + t_1^{p-1})e_1 \,\middle|\, \nabla(\phi_1\phi_2)\right\rangle dx \\
&= -(b + t_2^{p-1})\int_{Q'} \phi_1(0)\phi_2(x')\langle e_1 \mid e_1\rangle\, dx' + (b + t_1^{p-1})\int_{Q'} \phi_1(0)\phi_2(x')\langle e_1 \mid -e_1\rangle\, dx' \\
&= -\phi_1(0)\left(2b + t_1^{p-1} + |t_2|^{p-1}\right)\int_{Q'} \phi_2(x')\, dx' < 0,
\end{aligned}$$

which is a contradiction. This completes the proof. $\square$

Remark 4. The estimate (5.6) breaks for $p = 1$, since the equation $|0|^{p-2}0 = 0$ is no longer valid for $p = 1$. This means that we have implicitly used differentiability of the function $|z|^p/p$ at $0 \in \mathbb{R}^n$. Also it should be noted that for the one-variable case, functions as in (5.1), which are in general not in C^1, are one-harmonic in $\mathbb{R}$.

5.2 C^1-regularity theorem

We give the proof of Theorem 1.

Proof. We may assume that Ω is convex. By [30, Theorem 25.1 and 25.5] and Lemma 1, it suffices to show that $\partial u(x_0) = \{0\}$ for all $x_0 \in F$. Let $x_0 \in F$. We get a convex function $u_0 \colon \mathbb{R}^n \to \mathbb{R}$ as a blow-up limit as in Proposition 1. We note that the facet of u_0 is non-empty by Proposition 1. Hence by the Liouville-type theorem (Theorem 3), u_0 is constant and we obtain $\partial u_0(x_0) = \{0\}$. Combining these results, we have $\{0\} \subset \partial u(x_0) \subset \partial u_0(x_0) = \{0\}$ and therefore $\partial u(x_0) = \{0\}$. This completes the proof. □

6 Generalization

In Section 6, we would like to discuss C^1-regularity of convex weak solutions to

$$Lu \coloneqq -\operatorname{div}(\nabla_z \Psi(\nabla u)) - \operatorname{div}(\nabla_z W(\nabla u)) = f \quad \text{in} \quad \Omega \subset \mathbb{R}^n, \tag{6.1}$$

which covers (1.1). Precisely speaking, throughout Section 6, we make these following assumptions for Ψ and W on regularity and ellipticity. For regularity, we only require

$$\Psi \in C(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{0\}),\ W \in C^1(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{0\}). \tag{6.2}$$

For W, we assume that for each fixed $0 < \mu \le M < \infty$, there exist constants $0 < \gamma < \Gamma < \infty$ such that W satisfies

$$\gamma|\zeta|^2 \le \left\langle \nabla_z^2 W(z_0)\zeta \,\middle|\, \zeta \right\rangle, \tag{6.3}$$

$$\left|\left\langle \nabla_z^2 W(z_0)\zeta \,\middle|\, \omega \right\rangle\right| \le \Gamma|\zeta||\omega| \tag{6.4}$$

for all $z_0, \zeta, \omega \in \mathbb{R}^n$ with $\mu \le |z_0| \le M$. Also, there is no loss of generality in assuming that

$$\nabla_z W(0) = 0. \tag{6.5}$$

Finally, we assume that Ψ is positively homogeneous of degree 1. In other words, Ψ satisfies

$$\Psi(\lambda z_0) = \lambda \Psi(z_0) \tag{6.6}$$

holds for all $z_0 \in \mathbb{R}^n$ and $\lambda > 0$. This clearly yields $\Psi(0) = 0$.

By modifying some of our arguments, we are able to show that

Theorem 4 (C^1-regularity theorem for general equations)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain. Assume that $f \in L^q_{\mathrm{loc}}(\Omega)$ ($n < q \le \infty$) and the functionals Ψ and W satisfy (6.2)–(6.5). If u is a convex weak solution to (6.1), then u is in $C^1(\Omega)$.*

If we set

$$\Psi(z) \coloneqq b|z|, \quad W(z) \coloneqq \frac{|z|^p}{p}, \quad \text{where } 1 < p < \infty,$$

then the equation (6.1) becomes (1.1). Therefore Theorem 4 generalizes Theorem 1.

6.1 Preliminaries

In Section 6.1, we mention some basic properties of Ψ and W, which are derived from the assumptions (6.2)–(6.5).

For W, by (6.2)–(6.3) and (6.5) it is easy to check that the continuous mapping $A \colon \mathbb{R}^n \ni z \mapsto \nabla W(z) \in \mathbb{R}^n$ satisfies strict monotonicity (4.8). In particular, by (6.5) we have

$$\langle A(z) \mid z \rangle > 0 \quad \text{for all } z \in \mathbb{R}^n \setminus \{0\}. \tag{6.7}$$

For the proof, see Lemma 10 in the appendices.

For Ψ, we first note that Ψ satisfies the triangle inequality

$$\Psi(z_1+z_2) \le \Psi(z_1)+\Psi(z_2) \quad \text{for all } z_1, z_2 \in \mathbb{R}^n. \tag{6.8}$$

We define a function $\tilde{\Psi}\colon \mathbb{R}^n \to [0,\infty]$ by

$$\tilde{\Psi}(\zeta) := \sup\{\langle \zeta \mid z\rangle \mid z \in \mathbb{R}^n, \Psi(z) \le 1\}.$$

$\tilde{\Psi}$ is the support function for the closed convex set $C_\Psi := \{z \in \mathbb{R}^n \mid \Psi(z) \le 1\}$. By definition it is easy to check that $\tilde{\Psi}$ is convex and lower semicontinuous. Also, if $\zeta \in \mathbb{R}^n$ satisfies $\tilde{\Psi}(\zeta) < \infty$, then the following Cauchy–Schwarz-type inequality holds;

$$\langle z \mid \zeta\rangle \le \Psi(z)\tilde{\Psi}(\zeta) \quad \text{for all} \quad z \in \mathbb{R}^n. \tag{6.9}$$

If a convex function Ψ is positively homogeneous of degree 1, then the subdifferential operator $\partial\Psi$ is explicitly given by

$$\partial\Psi(z) = \left\{\zeta \in \mathbb{R}^n \,\middle|\, \tilde{\Psi}(\zeta) \le 1, \Psi(z) = \langle z \mid \zeta\rangle\right\} \tag{6.10}$$

for all $z \in \mathbb{R}^n$. In particular, we have the following formula

$$\langle \nabla_z \Psi(z_0) \mid z_0\rangle = \Psi(z_0) \quad \text{for all } z_0 \in \mathbb{R}^n \setminus \{0\}, \tag{6.11}$$

which is often called Euler's identity. Also, assumptions (6.2) and (6.6) imply that

$$\nabla\Psi(\lambda z_0) = \nabla\Psi(z_0), \quad \nabla^2\Psi(\lambda z_0) = \lambda^{-1}\nabla^2\Psi(z_0) \tag{6.12}$$

for all $\lambda > 0$ and $z_0 \in \mathbb{R}^n \setminus \{0\}$. Proofs of (6.8)–(6.10) are given in Lemma 11 of the appendices for the reader's convenience.

Remark 5. The results (6.11)–(6.12) give us the following basic property for Ψ.

1. We set a constant
$$K := \sup\{|\nabla_z \Psi(z_0)| \mid z_0 \in \mathbb{R}^n, |z_0| = 1\},$$
which is finite. Then we have $\partial\Psi(z_0) \subset \overline{B_K(0)}$ for all $z_0 \in \mathbb{R}^n$. For the case $z_0 \neq 0$, this inclusion is clear by (6.12) and $\partial\Psi(z_0) = \{\nabla_z\Psi(z_0)\}$. For $z_0 = 0$, we take arbitrary $w \in \partial\Psi(0) \setminus \{0\}$. Then by the subgradient inequality, Euler's identity (6.11) and the Cauchy–Schwarz inequality, we have
$$\begin{aligned} |w|^2 &= \langle w \mid w-0\rangle + \Psi(0) \\ &\le \Psi(w) = \langle \nabla_z\Psi(w) \mid w\rangle \le K|w|. \end{aligned}$$
This estimate yields the inclusion $\partial\Psi(0) \subset \overline{B_K(0)}$.

2. For $z_0 \in \mathbb{R}^n \setminus \{0\}$, the Hessian matrix $\nabla_z^2\Psi(z_0)$ satisfies
$$0 \le \left\langle \nabla_z^2\Psi(z_0)\zeta \,\middle|\, \zeta\right\rangle, \tag{6.13}$$
$$\left|\left\langle \nabla_z^2\Psi(z_0)\zeta \,\middle|\, \omega\right\rangle\right| \le \frac{C}{|z_0|}|\zeta||\omega| \tag{6.14}$$
for all $\zeta, \omega \in \mathbb{R}^n$. Here the finite constant C is explicitly given by
$$C := \sup\left\{\left|\left\langle \nabla_z^2\Psi(w)\zeta \,\middle|\, \omega\right\rangle\right| \,\middle|\, z, \zeta, \omega \in \mathbb{R}^n, |w| = |\eta| = |\omega| = 1\right\}.$$

Lemma 4 states lower semicontinuity of a functional in the weak* topology of an L^∞-space. This result is used in the justification of a blow-up argument for the equation (6.1).

Lemma 4. *Let $\Omega \subset \mathbb{R}^m$ be a Lebesgue measurable set, and let $\Psi\colon \mathbb{R}^n \to [0,\infty)$ be a convex function which satisfies (6.6). Assume that a vector field $Z \in L^\infty(\Omega, \mathbb{R}^n)$ and a sequence $\{Z_N\}_N \subset L^\infty(\Omega,\mathbb{R}^n)$ satisfy $Z_N \overset{*}{\rightharpoonup} Z$ in $L^\infty(\Omega,\mathbb{R}^n)$. Then we have*

$$\operatorname*{ess\,sup}_{x\in\Omega} \tilde{\Psi}(Z(x)) \le \liminf_{N\to\infty} \operatorname*{ess\,sup}_{x\in\Omega} \tilde{\Psi}(Z_N(x)), \tag{6.15}$$

where $\tilde{\Psi}$ denotes the support function of the closed convex set $C_\Psi := \{z \in \mathbb{R}^n \mid \Psi(z) \le 1\}$.

We give an elementary proof of Lemma 4, which is based on a definition of $\tilde{\Psi}$.

Proof. We consider the case $C_\infty \coloneqq \liminf_{N\to\infty} \left\|\tilde{\Psi}(Z_N)\right\|_{L^\infty(\Omega)} < \infty$, since otherwise (6.15) is clear. Fix arbitrary $\varepsilon > 0$. Then we may take a subsequence $\{Z_{N_j}\}_{j=1}^\infty$ such that

$$\operatorname*{ess\,sup}_{x\in\Omega} \tilde{\Psi}(Z_{N_j}(x)) \le C_\infty + \varepsilon < \infty. \tag{6.16}$$

Take arbitrary $0 \le \phi \in L^1(\Omega)$ and $w \in C_\Psi$. Then with the aid of (6.9), we have

$$\langle Z_{N_j}(x) \mid w\rangle \le C_\infty + \varepsilon$$

for all $j \in \mathbb{N}$ and for a.e. $x \in \Omega$, which yields

$$\int_\Omega \left[C_\infty + \varepsilon - \langle Z_{N_j}(x) \mid w\rangle\right]\phi(x)\,dx \ge 0 \tag{6.17}$$

for all $j \in \mathbb{N}$. Letting $j \to \infty$, we have

$$\int_\Omega [C_\infty + \varepsilon - \langle Z(x) \mid w\rangle]\phi(x)\,dx \ge 0$$

by $Z_{N_j} \overset{*}{\rightharpoonup} Z$ in $L^\infty(\Omega, \mathbb{R}^n)$. Since $0 \le \phi \in L^1(\Omega)$ is arbitrary, for each $w \in C_\Psi$, there exists an $\mathcal{L}^n$-measurable set $U_w \subset \Omega$, such that $\mathcal{L}^n(U_w) = 0$ and

$$\langle Z(x) \mid w\rangle \le C_\infty + \varepsilon \quad \text{for all } x \in \Omega \setminus U_w.$$

Here we denote $\mathcal{L}^n$ by the n-dimensional Lebesgue measure. Since $C_\Psi \subset \mathbb{R}^n$ is separable, we may take a countable and dense set $D \subset C_\psi$. We set an $\mathcal{L}^n$-measurable set

$$U \coloneqq \bigcup_{w\in D} U_w \subset \Omega,$$

which clearly satisfies $\mathcal{L}^n(U) = 0$. Then we conclude that

$$\langle Z(x) \mid w\rangle \le C_\infty + \varepsilon \quad \text{for all } x \in \Omega \setminus U,\, w \in C_\Psi$$

from density of $D \subset C_\Psi$. Hence by definition of $\tilde{\Psi}$, it is clear that

$$\tilde{\Psi}(Z(x)) \le C_\infty + \varepsilon \quad \text{for a.e. } x \in \Omega.$$

Since $\varepsilon > 0$ is arbitrary, this completes the proof of (6.15). □

6.2 Sketches of the proofs

We first give definitions of weak solutions to (6.1). We also define weak subsolutions, and supersolutions to an equation $Lu = 0$ in a bounded domain.

Definition 3. Let $\Omega \subset \mathbb{R}^n$ be a domain.

1. Let $f \in L^q_{\mathrm{loc}}(\Omega)\,(n < q \le \infty)$. We say that a function $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ is a weak solution to (6.1), when for any bounded Lipschitz domain $\omega \Subset \Omega$, there exists a vector field $Z \in L^\infty(\omega, \mathbb{R}^n)$ such that the pair $(u, Z) \in W^{1,\infty}(\omega) \times L^\infty(\omega, \mathbb{R}^n)$ satisfies

$$\int_\omega \langle Z \mid \nabla\phi\rangle\,dx + \int_\omega \langle A(\nabla u) \mid \nabla\phi\rangle\,dx = \int_\omega f\phi\,dx \tag{6.18}$$

   for all $\phi \in W^{1,1}_0(\omega)$, and

$$Z(x) \in \partial\Psi(\nabla u(x)) \tag{6.19}$$

   for a.e. $x \in \omega$. Here A denotes the continuous mapping $A\colon \mathbb{R}^n \ni x \mapsto \nabla_z W(x) \in \mathbb{R}^n$. For such pair (u, Z), we say that (u, Z) satisfies $Lu = f$ in $W^{-1,\infty}(\omega)$ or simply say that u satisfies $Lu = f$ in $W^{-1,\infty}(\omega)$.

2. Assume that Ω is bounded. A pair $(u, Z) \in W^{1,\infty}(\Omega) \times L^{\infty}(\Omega, \mathbb{R}^n)$ is called a *weak* subsolution to $Lu = 0$ in Ω, if it satisfies

$$\int_{\Omega} \langle Z \mid \nabla\phi \rangle \, dx + \int_{\Omega} \langle A(\nabla u) \mid \nabla\phi \rangle \, dx \le 0 \tag{6.20}$$

for all $0 \le \phi \in C_c^{\infty}(\Omega)$, and

$$Z(x) \in \partial\Psi(\nabla u(x)) \quad \text{for a.e. } x \in \Omega. \tag{6.21}$$

Similarly we call a pair $(u, Z) \in W^{1,p}(\Omega) \times L^{\infty}(\Omega, \mathbb{R}^n)$ a *weak* supersolution $L_{b,p}u = 0$ in Ω, if it satisfies (6.21) and

$$\int_{\Omega} \langle Z \mid \nabla\phi \rangle \, dx + \int_{\Omega} \langle A(\nabla u) \mid \nabla\phi \rangle \, dx \ge 0$$

for all $0 \le \phi \in C_c^{\infty}(\Omega)$. For $u \in W^{1,p}(\Omega)$, we simply say that u is respectively a subsolution and a supersolution to $Lu = 0$ in the *weak* sense if there is $Z \in L^{\infty}(\Omega, \mathbb{R}^n)$ such that the pair (u, Z) is a weak subsolution and a weak supersolution to $Lu = 0$ in Ω.

Remark 6. We describe some remarks on Definition 3.

1. In this paper we treat a convex solution, which clearly satisfies local Lipschitz regularity. Hence it is not restrictive to assume local or global $W^{1,\infty}$-regularity for solutions in Definition 3. Also it should be noted that if a vector field Z satisfies (6.19), then Z is in L^{∞} by Remark 5. Hence our regularity assumptions of the pair (u, Z) involve no loss of generality.

2. Integrals in (6.18) make sense by $Z, \nabla u \in L^{\infty}(\omega, \mathbb{R}^n)$, $A \in C(\mathbb{R}^n, \mathbb{R}^n)$, and the continuous embedding $W_0^{1,1}(\omega) \hookrightarrow L^{q'}(\omega)$.

3. For a bounded domain $\Omega \subset \mathbb{R}^n$, let $u \in C^2(\overline{\Omega})$ satisfy

$$\nabla u(x) \neq 0 \quad \text{for all } x \in \Omega, \text{ and}$$

$$Lu(x) \le 0 \quad \text{for all } x \in \Omega.$$

Then the pair $(u, \nabla_z \Psi(\nabla u)) \in W^{1,p}(\Omega) \times L^{\infty}(\Omega, \mathbb{R}^n)$ satisfies (6.20)–(6.21). For such u, we simply say that u satisfies $Lu \le 0$ in Ω in the *classical* sense.

To prove Theorem 4, we may assume that Ω is a bounded convex domain, since our argument is local. As described in Section 1.1, we would like to prove that a convex solution u to (6.1) satisfies (1.4) for all $x \in \Omega$.

For the case $x \in D$, we can show (1.4) by De Giorgi–Nash–Moser theory. This is basically due to the fact that the functional

$$E(z) \coloneqq \Psi(z) + W(z) \quad \text{for } z \in \mathbb{R}^n$$

satisfy the following property. For each fixed constants $0 < \mu \le M < \infty$, there exists constants $0 < \lambda \le \Lambda < \infty$ such that the estimates (2.7)–(2.8) hold for all $z_0, \zeta, \omega \in \mathbb{R}^n$ with $\mu \le |z_0| \le M$. In other words, the operator L is *locally uniformly elliptic outside a facet*, in the sense that for a function v the operator Lv becomes uniformly elliptic in a place where $0 < \mu \le |\nabla v| \le M < \infty$ holds. This ellipticity is an easy consequence of (6.3)–(6.4) and (6.13)–(6.14). Appealing to local uniform ellipticity of the operator L outside the facet and De Giorgi–Nash–Moser theory, we are able to show that a convex solution to $Lu = f$ is $C^{1,\alpha}$ near a neighborhood of each fixed point $x \in D$, similarly to the proof of Lemma 1.

For the case $x \in F$, we first make a blow-argument to construct a convex function $u_0 \colon \mathbb{R}^n \to \mathbb{R}$ satisfying $\partial u(x) \subset \partial u_0(x)$, and $Lu_0 = 0$ in $\mathbb{R}^n$ in the sense of Definition 3. Next we justify a maximum principle, which is described as in (1.6), holds on each connected component of D. This result enables us to apply Lemma 3, and thus similarly in Section 5.1, we are able to prove a Liouville-type theorem. Hence it follows that a convex solution u_0, which is constructed by the previous blow-argument, should be constant. Finally the inclusions $\{0\} \subset \partial u(x) \subset \partial u_0(x) \subset \{0\}$ hold, and this completes the proof of (1.4), i.e., $\partial u(x) = \{0\}$.

For maximum principles on the equation $Lu = 0$, the proofs are almost similar to those in Section 4. Indeed, we first recall that the operator $A \colon \mathbb{R}^n \ni z_0 \mapsto \nabla_z W(z_0) \in \mathbb{R}^n$ satisfies strict monotonicity (4.8). Combining with monotonicity of the subdifferential operator $\partial\Psi$, we can easily prove a comparison principle as in Proposition 2. Also, similarly to Lemma 2, we can construct classical barrier subsolutions to $Lu = 0$ in an open annulus, since the operator L is locally uniformly elliptic outside a facet. These results enable us to prove a maximum principle outside a facet.

We are left to justify the remaining two problems, a blow-argument and the Liouville-type theorem. To show them, we have to make use of some basic facts on a convex functional which is homogeneous of degree 1. These fundamental results are contained in Section A.3.

For a blow-up argument as in Section 3, we similarly define rescaled solutions. Existence of a limit of these rescaled functions are guaranteed by the Arzelà–Ascoli theorem and a diagonal argument. By proving Lemma 5 below, we are able to demonstrate that u_0, a limit of rescaled solutions, is a weak solution to $Lu = 0$ in $\mathbb{R}^n$, and this finishes our blow-up argument.

Lemma 5. *Let $U \subset \mathbb{R}^n$ be a bounded domain. Assume that sequences of functions $\{u_N\}_{N=1}^{\infty} \subset W^{1,\infty}(U)$ and $\{f_N\}_{N=1}^{\infty} \subset L^q(U)$ ($n < q \le \infty$) satisfy all of the following.*

1. *For each $N \in \mathbb{N}$, u_N satisfies $Lu_N = f_N$ in $W^{-1,\infty}(U)$.*
2. *There exists a constant $M > 0$, independent of $N \in \mathbb{N}$, such that*
$$|\nabla u_N(x)| \le M \quad \text{for a.e. } x \in U. \tag{6.22}$$
3. *There exists a function $u \in W^{1,\infty}(U)$ such that*
$$\nabla u_N(x) \to \nabla u(x) \quad \text{for a.e. } x \in U. \tag{6.23}$$
4. *f_N strongly converges to 0 in $L^q(U)$.*

Then u satisfies $Lu = 0$ in $W^{-1,\infty}(U)$.

Proof. For each $N \in \mathbb{N}$, there exists a vector field $Z_N \in L^\infty(U, \mathbb{R}^n)$ such that

$$Z_N(x) \in \partial\Psi(\nabla u(x)) \quad \text{for a.e. } x \in U, \tag{6.24}$$

$$\int_U \langle Z_N \mid \nabla\phi\rangle\, dx + \int_U \langle A(\nabla u_N) \mid \nabla\phi\rangle\, dx = \int_U f_N \phi\, dx \quad \text{for all } \phi \in W_0^{1,1}(U). \tag{6.25}$$

Combining the assumption $f_N \to f$ in $L^q(U)$ with the continuous embedding $L^q(U) \hookrightarrow W^{-1,\infty}(U)$, we get

$$f_N \to 0 \quad \text{in } W^{-1,\infty}(U). \tag{6.26}$$

By $A \in C(\mathbb{R}^n, \mathbb{R}^n)$ and (6.22), the vector fields $\{A(\nabla u_N)\}_{N=1}^{\infty}$ satisfy

$$A(\nabla u_N(x)) \to A(\nabla u(x)) \quad \text{for a.e. } x \in U,$$

$$|A_N(\nabla u_N(x)) - A(\nabla u(x))| \le C \quad \text{for a.e. } x \in U,$$

where C is independent of $N \in \mathbb{N}$. From these and Lebesgue's dominated convergence theorem, it follows that

$$A(\nabla u_N) \overset{*}{\rightharpoonup} A(\nabla u) \quad \text{in } L^\infty(U, \mathbb{R}^n). \tag{6.27}$$

As mentioned in Remark 5–6, the $\{Z_N\}_{N=1}^{\infty} \subset L^\infty(U, \mathbb{R}^n)$ is bounded. Hence by [5, Corollary 3.30], we may take a subsequence $\{Z_{N_j}\}_{j=1}^{\infty}$ so that

$$Z_{N_j} \overset{*}{\rightharpoonup} Z \quad \text{in } L^\infty(U, \mathbb{R}^n) \tag{6.28}$$

for some $Z \in L^\infty(U, \mathbb{R}^n)$. By (6.25)–(6.28) we obtain

$$\int_U \langle Z \mid \phi\rangle\, dx + \int_U \langle A(\nabla u) \mid \nabla\phi\rangle\, dx = 0 \quad \text{for all } \phi \in W_0^{1,1}(U).$$

Now we are left to prove that

$$Z(x) \in \partial\Psi(\nabla u(x)) \quad \text{for a.e. } x \in U.$$

By (6.10), it suffices to show that Z satisfies

$$\tilde{\Psi}(Z(x)) \le 1, \tag{6.29}$$

$$\Psi(\nabla u(x)) = \langle Z \mid \nabla u(x)\rangle \tag{6.30}$$

for a.e. $x \in U$. Similarly, it follows that for each $N \in \mathbb{N}$, the vector field Z_N satisfies

$$\begin{cases} \tilde{\Psi}(Z_N(x)) & \le & 1, \\ \Psi(\nabla u_N(x)) & = & \langle Z_N \mid \nabla u(x)\rangle, \end{cases} \quad \text{for a.e. } x \in U.$$

Hence (6.29) is an easy consequence of Lemma 4. We recall (6.2), and thus $\partial\Psi(z_0) = \{\nabla_z \Psi(z_0)\}$ holds for all $z_0 \in \mathbb{R}^n \setminus \{0\}$. Combining (6.23), we can check that $Z_N(x) \to Z(x)$ for a.e. $x \in D := \{x \in U \mid \nabla u(x) \neq 0\}$. Hence (6.30) holds for a.e. $x \in D$. Note that (6.30) is clear for $x \in U \setminus D$, and this completes the proof. □

We prove a Liouville-type theorem as in Theorem 3. In other words, for a convex solution to $Lu = 0$ in $\mathbb{R}^n$, we show that F, the facet of u, would satisfy either $F = \emptyset$ or $F = \mathbb{R}^n$. Assume by contradiction that F satisfies $\emptyset \subsetneq F \subsetneq \mathbb{R}^n$. Then by Lemma 3, we may write a convex solution u by either of (5.1)–(5.3). However, Lemma 6 below states that u is no longer a weak solution, and this completes our proof.

Lemma 6. *Let u be a piecewise-linear function defined as in either of (5.1)–(5.3). Then u is not a weak solution to $Lu = 0$ in $\mathbb{R}^n$.*

Proof. As in the proof of Theorem 3, we introduce a constant $d > 0$, and set open cubes $Q' \subset \mathbb{R}^{n-1}$ and $Q, Q_l, Q_r \subset \mathbb{R}^n$. By choosing sufficiently small $d > 0$, we show that u does not satisfy $Lu = 0$ in $W^{-1,\infty}(Q)$. Assume by contradiction that there exists a vector field $Z \in L^\infty(Q, \mathbb{R}^n)$ such that the pair (u, Z) satisfies $Lu = 0$ in $W^{-1,\infty}(Q)$.

We first show that a function u defined as in (5.1) is not a weak solution. For this case, (6.12) implies that Z satisfies $Z(x) = \nabla_z \Psi(e_1)$ for a.e. $x \in Q_r$. We take and fix non-negative functions $\phi_1 \in C_c^1((-d, d))$, $\phi_2 \in C_c^1(Q')$ such that (5.5) holds, and define $\phi \in C_c^1(Q)$ by $\phi(x_1, x') := \phi_1(x_1)\phi_2(x')$ for $(x_1, x') \in (-d, d) \times Q' = Q$. Testing ϕ into $Lu = 0$ in $W^{-1,\infty}(Q)$, we have

$$\begin{aligned} 0 &= \int_{Q_l} \langle Z + A(0) \mid \nabla(\phi_1\phi_2)\rangle\, dx + \int_{Q_r} \langle \nabla_z \Psi(e_1) + A(t_1 e_1) \mid \nabla(\phi_1\phi_2)\rangle\, dx \\ &\le \int_{Q_l} \Psi(\nabla(\phi_1\phi_2))\tilde{\Psi}(Z(x))\, dx \\ &\quad + \phi_1(0)\int_{Q'} \phi_2(x')\langle \nabla_z \Psi(e_1) \mid -e_1\rangle\, dx' + \phi_1(0)\int_{Q'} \phi_2(x')\langle A(t_1 e_1) \mid -e_1\rangle\, dx' \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

Here we have used the Cauchy–Schwarz-type inequality (6.5) for the integral over Q_l, and applied the Gauss–Green theorem to the integration over Q_r. For I_1, we make use of (6.9)–(6.8), Fubini's theorem and (5.5). Then we have

$$\begin{aligned} I_1 &\le \int_{Q_l} \phi_1(x_1)\Psi(0, \nabla_{x'}\phi_2(x'))\, dx + \int_{Q_l} \phi_1'(x_1)\phi_2(x')\Psi(e_1)\, dx \\ &\le \phi_1(0)\Big(d \cdot \|\Psi(0, \nabla_{x'}\phi_2)\|_{L^1(Q')} + \Psi(e_1)\|\phi_2\|_{L^1(Q')}\Big), \end{aligned}$$

where $\nabla_{x'}\phi_2 := (\partial_{x_2}\phi_2, \dots, \partial_{x_n}\phi_2)$. For I_2, recalling Euler's identity (6.11), we get $I_2 = -\phi_1(0)\Psi(e_1)\|\phi_2\|_{L^1(Q')}$. We set a constant $\mu := \langle A(t_1 e_1) \mid e_1\rangle$, which is positive by (6.7). Then we obtain

$$I_1 + I_2 + I_3 \le \phi_1(0)\Big(d \cdot \|\Psi(0, \nabla_{x'}\phi_2)\|_{L^1(Q')} - \mu\|\phi_2\|_{L^1(Q')}\Big).$$

Choosing $d = d(\mu, \Psi, \phi_2) > 0$ sufficiently small, we have $0 \le I_1 + I_2 + I_3 < 0$, which is a contradiction. Similarly we can deduce that u defined as in (5.3) does not satisfy $Lu = 0$ in $W^{-1,\infty}(Q)$, since it suffices to restrict $d < l_0$. For the remaining case (5.2), we have already known that

$$Z(x) = \begin{cases} \nabla_z \Psi(e_1) & \text{for a.e. } x \in Q_r, \\ \nabla_z \Psi(-e_1) & \text{for a.e. } x \in Q_l \end{cases}$$

by definition of Z and (6.12). We set two constants $\mu_1 := \langle A(t_1e_1) \mid e_1\rangle$, $\mu_2 := \langle A(-t_2e_1) \mid -e_1\rangle$, both of which are positive by (6.7). Testing the same function $\phi \in C_c^1(Q)$ into $Lu = 0$ in $W^{-1,\infty}(Q)$, we obtain

$$\begin{aligned} 0 &= \int_{Q_l} \langle \nabla_z \Psi(-e_1) + A(-t_2e_1) \mid \nabla(\phi_1\phi_2)\rangle\, dx + \int_{Q_r} \langle \nabla_z \Psi(e_1) + A(t_1e_1) \mid \nabla(\phi_1\phi_2)\rangle\, dx \\ &= \int_{Q'} \phi_1(0)\phi_2(x')\langle \nabla_z \Psi(-e_1) + A(-t_2e_1) \mid e_1\rangle\, dx' + \int_{Q'} \phi_1(0)\phi_2(x')\langle \nabla_z \Psi(e_1) + A(t_1e_1) \mid -e_1\rangle\, dx' \\ &= -\phi_1(0)(\Psi(e_1) + \Psi(-e_1) + \mu_1 + \mu_2) \int_{Q'} \phi_2(x')\, dx' < 0, \end{aligned}$$

which is a contradiction. Here we have used the Gauss–Green theorem and Euler's identity (6.11). This completes the proof. □

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A Proofs for a few basic facts

In this section, we give proofs for a few basic facts used in this paper for completeness.

A.1 A Poincaré-type inequality

We give a precise proof of Lemma 7, a Poincaré-type inequality for difference quotients of functions in $W_0^{1,p}$ ($1 \le p < \infty$). This result is used in the proof of Lemma 1. The proof of Lemma 7 is essentially a modification of that of the Poincaré inequality for the Sobolev space $W_0^{1,p}$ [10, Proposition 3.10].

Lemma 7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $1 \le p < \infty$. For all $u \in W_0^{1,p}(\Omega)$, $j \in \{1, \dots, n\}$, $h \in \mathbb{R} \setminus \{0\}$, we have*

$$\|\Delta_{j,h}u\|_{L^p(\Omega)} \le \|\nabla u\|_{L^p(\Omega)}. \tag{A.1}$$

Here $\Delta_{j,h}u$ is defined by

$$\Delta_{j,h}u(x) := \frac{\overline{u}(x + he_j) - u(x)}{h} \quad \textit{for } x \in \Omega.$$

Before the proof of Lemma 7, we note that $\Delta_{j,h}u(x)$ makes sense for a.e. $x \in \Omega$ by the zero extension of $u \in W_0^{1,p}(U)$. That is, for a given $u \in W_0^{1,p}(U)$, we set $\overline{u} \in W^{1,p}(\mathbb{R}^n)$ by

$$\overline{u}(x) := \begin{cases} u(x) & x \in U, \\ 0 & x \in \mathbb{R}^n \setminus U. \end{cases} \tag{A.2}$$

Proof. We fix $j \in \{1, \dots, n\}$, $h \in \mathbb{R} \setminus \{0\}$. We first note that the operator $\Delta_{j,h} \colon W_0^{1,p}(U) \to L^p(U)$ is bounded, since for all $u \in W_0^{1,p}(U)$ we have

$$\begin{aligned} \|\Delta_{j,h}u\|_{L^p(U)} &\le \frac{1}{|h|}\left[\left(\int_U |\overline{u}(x+h)|^p\, dx\right)^{1/p} + \left(\int_U |u(x)|^p\, dx\right)^{1/p}\right] \\ &\le \frac{2}{|h|}\|u\|_{L^p(U)} \le \frac{C(p,U)}{|h|}\|\nabla u\|_{L^p(U)} \end{aligned}$$

by the Minkowski inequality and the Poincaré inequality. Here $\overline{u} \in W^{1,p}(\mathbb{R}^n)$ is defined as in (A.2). Hence by a density argument, it suffices to check that (A.1) holds true for all $u \in C_c^\infty(U)$. Let $u \in C_c^\infty(U)$. Then for all $x \in U$, we have

$$
\begin{aligned}
|\overline{u}(x+he_j)-u(x)| &= \left|\int_0^1 \left\langle \nabla\overline{u}(x+the_j) \;\middle|\; he_j \right\rangle dt\right| \\
&\leq |h|\int_0^1 |\nabla\overline{u}(x+the_j)|\,dt \leq |h|\left(\int_0^1 |\nabla\overline{u}(x+the_j)|^p\,dt\right)^{1/p}
\end{aligned}
$$

by the Cauchy-Schwarz inequality and Hölder's inequality. From this estimate we get

$$
\begin{aligned}
\|\Delta_{j,h}u\|_{L^p(U)}^p &\leq \int_\Omega\int_0^1 |\nabla\overline{u}(x+the_j)|^p\,dt\,dx \\
&= \int_0^1 \underbrace{\int_U |\nabla\overline{u}(x+the_j)|^p\,dx}_{\leq \|\nabla u\|_{L^p(U)}^p}\,dt \quad \text{(by Fubini's theorem)} \\
&\leq \|\nabla u\|_{L^p(U)}^p.
\end{aligned}
$$

Hence we obtain (A.1) for all $u \in C_c^\infty(U)$, and this completes the proof. □

A.2 Convex analysis

Lemma 8 is used in the proof of Lemma 1 for a justification of local $W^{2,2}$-regularity of a convex weak solution outside of the facet.

Lemma 8. *Let u be a real-valued convex function in a convex domain $\Omega \subset \mathbb{R}^n$. Assume that $x_1, x_2 \in \Omega$ satisfy $x_1 \neq x_2$, and set $d := |x_2 - x_1| > 0$, $\nu := d^{-1}(x_2 - x_1)$. Then for all $z_2 \in \partial u(x_2)$, we have*

$$
\langle z_2 \mid \nu \rangle \geq \frac{u(x_2)-u(x_1)}{d}. \tag{A.3}
$$

Proof. By $z_2 \in \partial u(x_2)$, we have a subgradient inequality

$$
u(x) \geq u(x_2) + \langle z_2 \mid x - x_2 \rangle
$$

for all $x \in \Omega$. Substituting $x := x_1 = x_2 - d\nu \in \Omega$, we obtain

$$
u(x_1) \geq u(x_2) - d\langle z_2 \mid \nu \rangle,
$$

which yields (A.3). □

Remark 7. Instead of subgradient inequalities, we are able to show (A.3) by monotonicity of ∂u. For each fixed $x_1, x_2 \in \Omega$ with $x_1 \neq x_2$, we may take and fix $x_3 := x_1 + t(x_2 - x_1)$ for some $0 < t < 1$ and $z_3 \in \partial u(x_3)$ such that

$$
u(x_2) - u(x_1) = \langle z_3 \mid x_2 - x_1 \rangle, \tag{A.4}
$$

with the aid of the mean value theorem for non-smooth convex functions [1, Theorem D.6]. $x_2 - x_1 = d\nu$ is clear by definitions of d, ν. Noting $x_2 - x_3 = (1-t)d\nu$, we can check that

$$
\langle z_2 - z_3 \mid \nu \rangle = \frac{1}{(1-t)d}\langle z_2 - z_3 \mid x_2 - x_3 \rangle \geq 0
$$

by monotonicity of ∂u. Combining these results with (A.4), we obtain

$$
u(x_2) - u(x_1) = d\langle z_3 \mid \nu \rangle \leq d\langle z_2 \mid \nu \rangle,
$$

which yields (A.3).

The following lemma is used in the proof of Proposition 1.

Lemma 9. *Let $U \subset \mathbb{R}^n$ be a convex open set, and let $\{u_N\}_{N=1}^{\infty}$ be a sequence of real-valued convex functions in U. Assume that this sequence is uniformly Lipschitz. In other words, there is a constant $L > 0$ independent of $N \in \mathbb{N}$ such that*

$$|u_N(x) - u_N(y)| \leq L|x - y| \quad \text{for all } x, y \in U. \tag{A.5}$$

If there exists a function $u_\infty \colon U \to \mathbb{R}$ such that

$$u_N(x) \to u_\infty(x) \quad \text{for all } x \in U, \tag{A.6}$$

then we have $\nabla u_N(x) \to \nabla u_\infty(x)$ for a.e. $x \in U$.

Remark 8. From (A.5)–(A.6), it is easy to show that u_∞ is also convex, $u_N \to u_\infty$ uniformly in U, and

$$|u_\infty(x) - u_\infty(y)| \leq L|x - y| \quad \text{for all } x, y \in U.$$

Our proof of Lemma 9 is inspired by [11, Lemma A.3].

Proof. We define $\mathcal{L}^n$-measurable sets

$$P_N \coloneqq \{x \in U \mid u_N \text{ is not differentiable at } x\} \text{ for } N \in \mathbb{N} \cup \{\infty\}.$$

Clearly P_N ($N \in \mathbb{N} \cup \{\infty\}$) satisfies $\mathcal{L}^n(P_N) = 0$ by Lipschitz continuity of u_N, and therefore the $\mathcal{L}^n$-measurable set

$$P \coloneqq \bigcup_{N \in \mathbb{N} \cup \{\infty\}} P_N \subset U$$

also satisfies $\mathcal{L}^n(P) = 0$. We claim that

$$\nabla u_N(x_0) \to \nabla u_\infty(x_0) \quad \text{for all } x_0 \in U \setminus P. \tag{A.7}$$

We take and fix arbitrary $x_0 \in U \setminus P$. We note that $\nabla u_N(x_0)$ exists for each $N \in \mathbb{N}$ since $x_0 \notin P_N$, and we obtain

$$\sup_{N \in \mathbb{N}} |\nabla u_N(x_0)| \leq L$$

with the aid of (A.5). Hence it suffices to check that, if a subsequence $\{u_{N_k}\}_k \subset \{u_N\}_N$ satisfies

$$\nabla u_{N_k}(x_0) \to v \; (k \to \infty) \quad \text{for some } v \in \mathbb{R}^N, \tag{A.8}$$

then $v = \nabla u_\infty(x_0)$. Since $x_0 \notin P_{N_k}$ and therefore $\partial u_{N_k}(x_0) = \{\nabla u_{N_k}(x_0)\}$ for each $k \in \mathbb{N}$, we easily get

$$u_{N_k}(x) \geq u_{N_k}(x_0) + \langle \nabla u_{N_k}(x_0) \mid x - x_0 \rangle \quad \text{for all } x \in U,\ k \in \mathbb{N}.$$

Letting $k \to \infty$, we have

$$u_\infty(x) \geq u_\infty(x_0) + \langle v \mid x - x_0 \rangle \quad \text{for all } x \in U$$

by (A.6) and (A.8). This means that $v \in \partial u_\infty(x_0)$. Note again that $x_0 \notin P_\infty$ and therefore $\partial u_\infty(x_0) = \{\nabla u_\infty(x_0)\}$, which yields $v = \nabla u_\infty(x_0)$. This completes the proof of (A.7). □

A.3 Convex functionals

We prove some basic property of convex functionals Ψ and W in Section 6.

Lemma 10. *Let W be a convex function which satisfies (6.2)-(6.3) and (6.5). Then the mapping $A \colon \mathbb{R}^n \ni z \mapsto \nabla W(z) \in \mathbb{R}^n$ satisfies strict monotonicity (4.8).*

Proof. We take arbitrary $z_1, z_2 \in \mathbb{R}^n$ with $z_1 \neq z_2$ and define a line segment $L \coloneqq \{z_1 + t(z_2 - z_1) \in \mathbb{R}^n \mid 0 \leq t \leq 1\}$.

We first consider the case $0 \notin L$. Then there exist constants $0 < \mu \leq M < \infty$ such that $\mu \leq |z_0| \leq M$ holds for all $z_0 \in L$. Here we can take a constant $\gamma > 0$ such that (6.3) holds for all $z_0 \in L$. Then by $W \in C^2(\mathbb{R}^n \setminus \{0\})$, we have

$$\langle A(z_1) - A(z_2) \mid z_2 - z_1 \rangle = \int_0^1 \left\langle \nabla_z^2 W(z_1 + t(z_2 - z_1))(z_2 - z_1) \;\middle|\; z_2 - z_1 \right\rangle dt \geq \gamma |z_2 - z_1|^2 > 0.$$

To consider the remaining case $0 \in L$, it suffices to show (6.7). Indeed, the assumption $0 \in L$ allows us to write $z_1 = -l_1 \nu$, $z_2 = l_2 \nu$ for some unit vector ν and some constants $l_1, l_2 \geq 0$. Under this notation, we obtain

$$\langle A(z_2) - A(z_1) \mid z_2 - z_1 \rangle = \langle A(l_2\nu) \mid (l_1 + l_2)\nu \rangle + \langle A(-l_1\nu) \mid -(l_1 + l_2)\nu \rangle > 0$$

by (6.7). Here we note that at least one of l_1, l_2 is positive since $l_1 + l_2 = |z_2 - z_1| > 0$.

We prove (6.7) to complete the proof. Let $z \in \mathbb{R}^n \setminus \{0\}$. Then we obtain

$$d_N \coloneqq \left\langle A(z/2^{N-1}) - A(z/2^N) \,\middle|\, z \right\rangle > 0$$

for each $N \in \mathbb{N}$, since we have already shown (4.8) for the case $0 \notin L$. By definition of d_j ($j \in \mathbb{N}$), it is clear that

$$\left\langle A(z) - A(z/2^N) \,\middle|\, z \right\rangle = d_1 + \cdots + d_N \geq d_1.$$

Letting $N \to \infty$, we obtain $\langle A(z) \mid z \rangle \geq d_1 > 0$ by $A \in C(\mathbb{R}^n, \mathbb{R}^n)$. □

We precisely prove (6.8)–(6.10) in Lemma 11. See also [2, Section 1.3] and [30, §13] as related items.

Lemma 11. *Let* $\Psi\colon \mathbb{R}^n \to [0, \infty)$ *be a convex function which is positively homogeneous of degree* 1.

1. Ψ *satisfies the triangle inequality (6.8).*
2. *Assume that* $\zeta \in \mathbb{R}^n$ *satisfies* $\tilde{\Psi}(\zeta) < \infty$. *Then the Cauchy–Schwarz-type inequality (6.9) holds.*
3. *The subdifferential operator* $\partial\Psi$ *is given by (6.10).*

Proof. By convexity of Ψ and (6.6), Ψ satisfies

$$\frac{\Psi(z_1 + z_2)}{2} = \Psi\Big(\frac{z_1 + z_2}{2}\Big) \leq \frac{\Psi(z_1) + \Psi(z_2)}{2} \quad \text{for all } z_1, z_2 \in \mathbb{R}^n,$$

which yields (6.8).

We next show the Cauchy–Schwarz inequality (6.9). Let $z \in \mathbb{R}^n$. If $\Psi(z) > 0$, then we have

$$\langle z \mid \zeta \rangle = \Psi(z) \left\langle \frac{z}{\Psi(z)} \,\middle|\, \zeta \right\rangle \leq \Psi(z)\tilde{\Psi}(\zeta)$$

by $z/\Psi(z) \in C_\Psi$. For the case $\Psi(z) = 0$, we note that $\lambda z \in C_\Psi$ for all $\lambda > 0$. Hence it follows that

$$\langle z \mid \zeta \rangle = \frac{\langle \lambda z \mid \zeta \rangle}{\lambda} \leq \frac{\tilde{\Psi}(w)}{\lambda}$$

for all $\lambda > 0$. By $\tilde{\Psi}(\zeta) < \infty$, we obtain $\langle z \mid \zeta \rangle \leq 0 = \Psi(z)\tilde{\Psi}(\zeta)$. This completes the proof of (6.9).

Finally we prove (6.10). Let $z_0 \in \mathbb{R}^n$ be arbitrarily fixed. Assume that $\zeta \in \mathbb{R}^n$ satisfies $\tilde{\Psi}(\zeta) \leq 1$ and $\Psi(z_0) = \langle z_0 \mid \zeta \rangle$. Then by combining these assumptions with (6.9), we have

$$\begin{aligned} \Psi(z) &\geq \Psi(z)\tilde{\Psi}(\zeta) \\ &\geq \langle z \mid \zeta \rangle = \langle z_0 \mid \zeta \rangle + \langle z - z_0 \mid \zeta \rangle \\ &= \Psi(z_0) + \langle \zeta \mid z - z_0 \rangle \end{aligned}$$

for all $z \in \mathbb{R}^n$. Hence $\zeta \in \partial\Psi(z_0)$. Conversely, if $\zeta \in \partial\Psi(z_0)$, then we have the subgradient inequality

$$\Psi(z) \geq \Psi(z_0) + \langle \zeta \mid z - z_0 \rangle \quad \text{for all } z \in \mathbb{R}^n. \tag{A.9}$$

By testing kz_0 into (A.9), where $k \in [0, \infty)$ is arbitrary, we have

$$(k-1)\Psi(z_0) = \Psi(kz_0) - \Psi(z_0) \geq \langle \zeta \mid (k-1)z_0 \rangle = (k-1)\langle \zeta \mid z_0 \rangle. \tag{A.10}$$

If we let $0 \leq k < 1$ so that $k - 1 < 0$, then we have $\Psi(z_0) \leq \langle \zeta \mid z_0 \rangle$. Similarly, letting $1 < k < \infty$, we have $\Psi(z_0) \geq \langle \zeta \mid z_0 \rangle$. Hence we obtain $\Psi(z_0) = \langle \zeta \mid z_0 \rangle$. Combining with (A.9), we have

$$\langle z \mid \zeta \rangle \leq \Psi(z) \quad \text{for all } z \in \mathbb{R}^n,$$

which yields $\tilde{\Psi}(\zeta) \leq 1$ by definition of $\tilde{\Psi}$. This completes the proof of (6.10). □

References

[1] L. Ambrosio, A. Carlotto, and A. Massaccesi. *Lectures on elliptic partial differential equations*, volume 18 of *Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]*. Edizioni della Normale, Pisa, 2018.

[2] F. Andreu-Vaillo, V. Caselles, and J. M. Mazón. *Parabolic quasilinear equations minimizing linear growth functionals*, volume 223 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2004.

[3] D. P. Bertsekas. *Convex optimization theory*. Athena Scientific, Nashua, NH, 2009.

[4] H. Brézis. Monotonicity methods in Hilbert spaces and some applications to nonlinear partial differential equations. In *Contributions to nonlinear functional analysis (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1971)*, pages 101–156, 1971.

[5] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.

[6] M. Colombo and G. Mingione. Bounded minimisers of double phase variational integrals. *Arch. Ration. Mech. Anal.*, 218(1):219–273, 2015.

[7] M. Colombo and G. Mingione. Regularity for double phase variational problems. *Arch. Ration. Mech. Anal.*, 215(2):443–496, 2015.

[8] G. Duvaut and J.-L. Lions. *Inequalities in mechanics and physics*, volume 219 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin-New York, 1976. Translated from the French by C. W. John.

[9] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition, 2015.

[10] M. Giaquinta and L. Martinazzi. *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*, volume 11 of *Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]*. Edizioni della Normale, Pisa, second edition, 2012.

[11] M.-H. Giga and Y. Giga. Stability for evolving graphs by nonlocal weighted curvature. *Comm. Partial Differential Equations*, 24(1-2):109–184, 1999.

[12] M.-H. Giga and Y. Giga. Very singular diffusion equations: second and fourth order problems. *Jpn. J. Ind. Appl. Math.*, 27(3):323–345, 2010.

[13] Y. Giga and R. V. Kohn. Scale-invariant extinction time estimates for some singular diffusion equations. *Discrete Contin. Dyn. Syst.*, 30(2):509–535, 2011.

[14] Y. Giga, H. Kuroda, and H. Matsuoka. Fourth-order total variation flow with Dirichlet condition: characterization of evolution and extinction time estimates. *Adv. Math. Sci. Appl.*, 24(2):499–534, 2014.

[15] Y. Giga, M. Muszkieta, and P. Rybka. A duality based approach to the minimizing total variation flow in the space H^{-s}. *Jpn. J. Ind. Appl. Math.*, 36(1):261–286, 2019.

[16] Y. Giga and Y. Ueda. Numerical computations of split Bregman method for fourth order total variation flow. *J. Comput. Phys.*, 405:109114, 24, 2020.

[17] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[18] E. Giusti. *Direct methods in the calculus of variations*. World Scientific Publishing Co., Inc., River Edge, NJ, 2003.

[19] R. Glowinski, J.-L. Lions, and R. Trémolières. *Numerical analysis of variational inequalities*, volume 8 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam-New York, 1981. Translated from the French.

[20] E. Hopf. A remark on linear elliptic differential equations of second order. *Proc. Amer. Math. Soc.*, 3:791–793, 1952.

[21] Y. Kashima. A subdifferential formulation of fourth order singular diffusion equations. *Adv. Math. Sci. Appl.*, 14(1):49–74, 2004.

[22] Y. Kashima. Characterization of subdifferentials of a singular convex functional in Sobolev spaces of order minus one. *J. Funct. Anal.*, 262(6):2833–2860, 2012.

[23] R. V. Kohn. Surface relaxation below the roughening temperature: some recent progress and open questions. In *Nonlinear partial differential equations*, volume 7 of *Abel Symp.*, pages 207–221. Springer, Heidelberg, 2012.

[24] R. V. Kohn and H. M. Versieux. Numerical analysis of a steepest-descent PDE model for surface relaxation below the roughening temperature. *SIAM J. Numer. Anal.*, 48(5):1781–1800, 2010.

[25] F. Krügel. *A variational problem leading to a singular elliptic equation involving the 1-Laplacian*. Berlin: Mensch und Buch Verlag, 2013.

[26] G. Mingione. Regularity of minima: an invitation to the dark side of the calculus of variations. *Appl. Math.*, 51(4):355–426, 2006.

[27] I. V. Odisharia. *Simulation and analysis of the relaxation of a crystalline surface*. New York University, 2006.

[28] M. H. Protter and H. F. Weinberger. *Maximum principles in differential equations*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1967.

[29] P. Pucci and J. Serrin. *The maximum principle*, volume 73 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Verlag, Basel, 2007.

[30] R. T. Rockafellar. *Convex analysis*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1997. Reprint of the 1970 original, Princeton Paperbacks.

[31] H. Spohn. Surface dynamics below the roughening transition. *Journal de Physique I*, 3(1):69–81, 1993.

[32] S. Tsubouchi. Local Lipschitz bounds for solutions to certain singular elliptic equations involving the one-Laplacian. *Calc. Var. Partial Differential Equations*, 60(1):Paper No. 33, 35, 2021.

[33] X. Xu. Mathematical validation of a continuum model for relaxation of interacting steps in crystal surfaces in 2 space dimensions. *Calc. Var. Partial Differential Equations*, 59(5):158, 2020.