Abstract of Doctoral Dissertation

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Title of Doctoral Dissertation

Long time solution of some types of nonlinear wave equations
（種々の非線形波動方程式の長時間解について）

In this paper, we study the long-time behavior of some semilinear wave equations with small initial data. It is well known that the *Strauss* conjecture is derived from the discussion of the equation $\Box u := (\partial_t^2 - \Delta)u = |u|^p, p > 1$. Here, we have reached similar relation between the nonlinear power p, the smallness of initial data (ε order), and the longest existence time T_ε on more related issues.

Firstly we consider a semilinear wave system $\Box_g u_1 = |u_2|^{p_1}$, $\Box_g u_2 = |u_1|^{p_2}$ on 3+1-dimensional asymptotically flat space-time, where $\Box_g$ is the *d'Alembertian* operator of this space-time. The difficulty of such a problem is that most known results break down in non-flat space-time, so that we can not use Fourier transform, fundamental solution, and so on. In this work, we proved a kind of weighted *Strichartz* estimate, which robust enough for this kind of equations. Adopting this estimate to our problem, we show that for $p_2 \geq p_1 \geq 2$ (similar for $p_1 \geq p_2$), there exists a constant c such that

$$T_\varepsilon \geq \begin{cases} c\varepsilon^{-\frac{1}{\sigma}}, & if \ \sigma > 0, \\ \exp(c\varepsilon^{-2(p_1-1)}), & if \ \sigma = 0, p_1 > 2, \\ \exp(c\varepsilon^{-2+\delta}), & if \ \sigma = 0, p_1 = 2, \\ \infty & if \ \sigma < 0, \end{cases} \qquad \sigma := \frac{p_2 + 2 + p_1^{-1}}{p_1 p_2 - 1} - 1.$$

This result generalizes the former existence result for $p_1, p_2 > 2$, $\sigma < 0$ area to the result for all of $p_1, p_2 \geq 2$ area, and covers all the known existence results in general asymptotically flat spacetime. Comparing with the results in flat space, it can be found that our results are sharp except for the critical case. Also, when reducing to the single solution case, our results make some improvements to the known results.

Next, we consider a class of semilinear wave equations with damping and potential in flat space-time, which has the form

$$(\partial_t^2 + Dr^{-1}\partial_t - \Delta + Vr^{-2})u = |u|^p.$$

This problem is complicated because both damping and potential are in the same scaling as the wave operator, which means that they provide a comparable effect to the development of the solution. Meanwhile, the extra singularity at the origin also needs to be taken care of. As a result, we introduce $A := 2 + \sqrt{(n-2)^2 + 4V}$, $p_d := \frac{2}{A-D}$, p_F be the root of $h_F := \frac{n+A-2}{2}p - \frac{n+A+2}{2} = 0$, and p_S be the positive root of $h_S := (n + D - 1)p^2 - (n + D + 1)p - 2 = 0$. Then we prove that there exists a constant C, when $(D + 3 - A)(A + n + 2) < 8$, where $p_d < p_F < p_S$, we have

$$T_\varepsilon \leq \begin{cases} C\varepsilon^{\frac{p-1}{h_F}}, & p \leq p_d, \\ C\varepsilon^{\frac{2p(p-1)}{h_S}}, & p_d < p < p_S, \\ \exp(C\varepsilon^{-p(p-1)}), & p = p_S. \end{cases}$$

When $(D + 3 - A)(A + n + 2) = 8$, where $p_d = p_F = p_S$, we have

$$T_\varepsilon \leq \begin{cases} C\varepsilon^{\frac{p-1}{h_F}}, & p \leq p_F, \\ \exp(C\varepsilon^{-(p-1)}), & p = p_F. \end{cases}$$

When $(D+3-A)(A+n+2)>8$, where $p_d > p_F > p_S$, we have

$$T_\varepsilon \le \begin{cases} C\varepsilon^{\frac{p-1}{h_F}}, & p \le p_F, \\ \exp\left(C\varepsilon^{-(p-1)}\right), & p = p_F. \end{cases}$$

On the other hand, for $D=0$, under some technical assumptions we show that there exists a constant c. When $(3-A)(A+n+2)<8$, where $p_d < p_F < p_S$, we have

$$T_\varepsilon \ge \begin{cases} c\varepsilon^{\frac{p-1}{h_F}}, & p \le p_d, \\ c\varepsilon^{\frac{p-1}{h_F}}|\ln\varepsilon|^{\frac{1}{h_F}}, & p = p_d, \\ c\varepsilon^{\frac{2p(p-1)}{h_S}}, & p_d < p < p_S, \\ \exp\left(c\varepsilon^{-p(p-1)}\right), & p = p_S, \\ \infty, & p > p_S. \end{cases}$$

When $(3-A)(A+n+2)=8$, where $p_d = p_F = p_S$, we have

$$T_\varepsilon \ge \begin{cases} c\varepsilon^{\frac{p-1}{h_F}}, & p \le p_F, \\ \exp\left(C\varepsilon^{-\frac{p-1}{2}}\right), & p = p_F, \\ \infty, & p > p_F. \end{cases}$$

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Comparing the upper and lower bounds, we reached the critical curve of this problem for $D=0$, and gave the optimal bound of lifespan except for the critical case. Such results are innovative. Moreover, through the study of the lifespan, we find that there are two effects determining the lifespan. Normally, the critical exponent and the lifespan are obtained by the stronger effect. When the two effects are approximately equivalent, they will promote each other and show a different lifespan.

Then, we study the equation $\Box u = |u|^{p_S} + |\partial u|^q$ with small initial data in 2 and 3 dimensions, where p_S is the critical exponent of *Strauss* exponent and $q \ge p_S$. We already know some results of this equation by adopting the energy method. However, to get a better bound of lifespan, we use different methods for different nonlinear terms. When $n=3$, we improve the known lower bound of T_ε from $\exp\left(c\varepsilon^{-(p_S-1)}\right)$ to the sharp one $\exp\left(c\varepsilon^{-p_S(p_S-1)}\right)$, with some constant c. When $n=2$, we also improve the lower bound from $\exp\left(c\varepsilon^{-(p_S-1)}\right)$ to $\exp\left(c\varepsilon^{-\frac{(p_S-1)^2}{2}}\right)$, which is the optimal bound among all of that deduced by weighted Strichartz type estimate we have known.

At last, we study the system $\Box u = |v|^q$, $\Box v = |\partial_t u|^p$ in 3 dimensions. It is strange that the energy method can not adapt well to this equation. Instead, we analyze the fundamental solution directly. Finally, we find a curve

$$\frac{p+2}{pq-1} - \frac{1}{pq} - 1, \quad q < 2,$$

in (p,q) plane. We prove that when (p,q) is above the curve and $q<2$, the solution is global while the initial data is small, spherically symmetric, and regular enough. On the other hand, we also prove that the solution must blow up in finite time, provided that (p,q) is below the curve, and initial data is positive in some meanings. These means the curve we get is the critical curve of this problem for $q<2$. Adding the known $q>2$ branch of the critical curve, we reach the completely critical curve of this system.

Keywords: Wave equation; *Strauss* problem; long time existence