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Accurate Results for Free Vibration of Doubly Curved Shallow Shells of Rectangular Planform (Part 1)

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Abstract

A method is presented for determining the free vibration frequencies of doubly curved, isotropic shallow shells under general edge conditions and is used to obtain accurate natural frequencies for wide range of geometric parameters. Based on the shallow shell theory applicable to thin thickness shells, a method of Ritz is extended to derive a frequency equation wherein the displacement functions are modified to accommodate arbitrary sets of edge conditions for both in-plane and out-of-plane motions. In numerical computation, convergence is tested against series terms and comparison study is made with existing results by other authors. Twenty one sets of frequency parameters are tabulated for a wide range of shell shape and curvature ratio to serve as data for future comparison and practical design purpose.

Keywords: Accuracy; free vibration; natural frequency; Ritz method; shallow shell

1. Introduction

Open shallow shell is one of practical shell shapes, and has been widely used as structural components in many mechanical, aeronautical, building and marine structures. These shallow shell components are often subjected to vibration environment and the vibration analysis of shallow shells constitutes important part of structural engineering.

Development in the field of general shell vibration is summarized in a famous monograph [1] compiled by Leissa, and shallow shell vibration is a part of this monograph. The mechanics of shells is described in textbooks, for example in [2], and a handbook on shell vibration was published [3] in 2003. Use of shallow shell structure in automobiles was introduced [4] in connection with composite material.

First notable work on this topic is one [5] by Leissa and Kadi that formulated the exact solution of shallow shells of rectangular planform supported along four edges by shear diaphrams. The shear diaphragm gives the edge condition similar to simple support in the flat plate theory, except that the in-plane displacement parallel to the edge is constrained but displacement perpendicular to the edge is free. For general boundary conditions other than the shear diaphragm, exact solution is not derivable.

By using approximate methods, vibration of cantilevered cylindrical [6] and doubly curved [7] shallow shells were studied in the first half of 1980's. Leissa and one of the present authors analyzed vibration of completely free [8] and corner point supported shallow shells [9]. In the 1990's, research on this topic became more active. Effects of edge constraints on natural frequencies were studied [10], and one of these authors published a review paper on shallow shell vibration [11]. Yu and his co-workers presented free vibration analysis of circular cylindrical shells [12]. Liew and Lim published analysis of shallow shells of curvilinear planform [13], doubly curved shell [14], and use of so-called pb-2 Ritz method on the topic [15]. They also presented a review paper on shallow shell vibration [16].

In the 2000's, Qatu once again summarized on development on shallow shell vibration [17] to follow up his previous review [11]. Design problem was studied [18] for maximizing natural frequencies of laminated shallow shell where more complicated stress-strain relations are included. Monterrubio [19] introduced penalty parameters in the analysis of shallow shell vibration. Qatu studied effect of in-plane edge constraints on frequencies of simply supported doubly curved shallow shells [20].

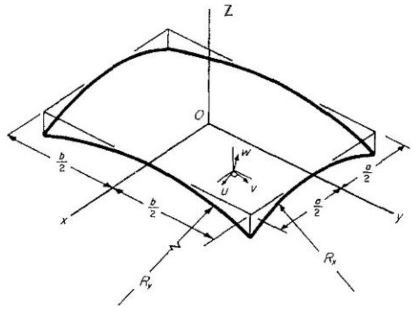

Figure 1. Shallow shell of rectangular planform

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More recently, Mochida et al. [21] and Qatu and Asadi [22] published lists of frequencies for general boundary conditions. Particularly in [21], they extended a method of superposition method to shallow shell vibration for the first time. In the authors' opinion, their numerical results seem most accurate in the literature published this far, but are limited to the case of very thin shell (edge length/ thickness=100). Considering the situations, it is obvious that thorough list of accurate frequency parameters should be tabulated for wide range of general edge conditions, which is the purpose of the present paper.

2. Analytical Method

2.1. Ritz method for general boundary condition

The quadratic mid-surface of a shallow shell (panel) may be expressed in a rectangular coordinate system as

$$\phi(x,y) = -\frac{1}{2}\left(\frac{x^2}{R_x} + 2\frac{xy}{R_{xy}} + \frac{y^2}{R_y}\right) \qquad (1)$$

where R_x and R_y are the radii of curvature in the *x* and *y* directions, respectively, and R_{xy} is the radius of twist but is not included in this study (i.e., $1/R_{xy}=0$). For a doubly curved shell, the orientation of the *x-y* coordinates may be chosen so that R_x and R_y are principal constant curvature radius as shown in Fig.1. The dimension of its planform is given by $a\times b$ and the thickness is *h*. The four sides are subjected to uniform in-plane (i.e., stretching) and out-of-plane (bending) boundary conditions. This shell takes geometric form of a cylindrical shell for "$1/R_x$=(finite) and $1/R_y=0$ ($R_y=\infty$)" or "$1/R_y$=(finite) and $1/R_x=0$ ($R_x=\infty$)". Similarly, it takes form of a spherical shell for $1/R_x=1/R_y$=(finite), and does form of a hyper paraboloidal shell for $1/R_x= -1/R_y$=(finite) where positive curvature exists in *x* direction and negative curvature in *y* direction.

Using the Kirchhoff hypothesis, the displacements $u^*(x,y,z,t)$, $v^*(x,y,z,t)$ and $w^*(x,y,z,t)$ of an arbitrary point in a shell are written as

$$u^* = u - z\frac{\partial w}{\partial x},\; v^* = v - z\frac{\partial w}{\partial y},\; w^* = w \qquad (2)$$

where *z* is the coordinate measured from the mid-surface in the direction of outer normal. The $u(x,y,t)$ and $v(x,y,t)$ are displacement components, tangent to the mid-surface and parallel to the *xz* and *yz* planes, respectively, and $w(x,y,t)$ is a displacement component normal to the mid-surface at a point on the mid-surface.

In the linear theory, the strain components at an arbitrary point (*x,y,z*) are

$$\varepsilon_x^* = \varepsilon_x + z\kappa_x,\; \varepsilon_y^* = \varepsilon_y + z\kappa_y,$$
$$\gamma_{xy}^* = \gamma_{xy} + z\kappa_{xy} \qquad (3)$$

assuming that *z* is negligible in comparison with *R* , where the membrane strains are given by

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{w}{R_x},\; \varepsilon_y = \frac{\partial v}{\partial y} + \frac{w}{R_y},\; \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \qquad (4)$$

and the curvature changes due to the vibratory displacements are

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2},\; \kappa_y = -\frac{\partial^2 w}{\partial y^2},\; \kappa_{xy} = -2\frac{\partial^2 w}{\partial x \partial y} \qquad (5)$$

For an isotropic shallow shell, the stress-strain equation is written as

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \begin{bmatrix}Q_{11} & Q_{12} & Q_{16}\\ Q_{12} & Q_{22} & Q_{26}\\ Q_{16} & Q_{26} & Q_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} \qquad (6)$$

where the coefficients are elastic constants

$$Q_{11} = Q_{22} = \frac{E}{1-\nu^2},\; Q_{12} = \nu Q_{11},\; Q_{66} = G \qquad (7)$$

where *E* is the moduli of elasticity, $G=E/2(1+\nu)$ is the shear modulus and ν is Poisson's ratios.

The force resultants and the moment resultants are obtained by integrating the stresses and the stresses multiplied by *z*, respectively, over the shell thickness *h*, and are written in matrix form as

$$\begin{Bmatrix}N\\ M\end{Bmatrix} = \begin{bmatrix}A & B\\ B & D\end{bmatrix}\begin{Bmatrix}\varepsilon\\ \kappa\end{Bmatrix} \qquad (8)$$

where $\{N\}$, $\{M\}$, $\{\varepsilon\}$ and $\{\kappa\}$ are the vectors of force resultants, moment resultants, mid-surface strains and curvatures, respectively, given by

$$\{N\} = \begin{Bmatrix}N_x\\ N_y\\ N_{xy}\end{Bmatrix},\; \{M\} = \begin{Bmatrix}M_x\\ M_y\\ M_{xy}\end{Bmatrix},$$
$$\{\varepsilon\} = \begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix},\; \{\kappa\} = \begin{Bmatrix}\kappa_x\\ \kappa_y\\ \kappa_{xy}\end{Bmatrix} \qquad \text{(9a,b,c,d)}$$

and $[A]$,$[B]$ and $[D]$ are the matrices of stiffness coefficients defined by

$$[A] = \begin{bmatrix}A_{11} & A_{12} & A_{16}\\ A_{12} & A_{22} & A_{26}\\ A_{16} & A_{26} & A_{66}\end{bmatrix} \qquad \text{(10a)}$$

$$[B] = \begin{bmatrix}B_{11} & B_{12} & B_{16}\\ B_{12} & B_{22} & B_{26}\\ B_{16} & B_{26} & B_{66}\end{bmatrix} \qquad \text{(10b)}$$

$$[D] = \begin{bmatrix}D_{11} & D_{12} & D_{16}\\ D_{12} & D_{22} & D_{26}\\ D_{16} & D_{26} & D_{66}\end{bmatrix} \qquad \text{(10c)}$$

The stiffness coefficients in Eqs. (10a,b,c) are values determined by

$$A_{ij} = hQ_{ij} \quad , \; B_{ij} = 0 \tag{11a,b}$$

$$D_{ij} = \left(\frac{h}{12}\right) Q_{ij} \tag{11c}$$

(i,j=1,2,6) for specific case of shallow shells composed of isotropic material. For more complicated shells including laminated composite materials, Eqs. (11) take more complicated form as shown in Ref. [18].

In the present study, the free vibration problem can be solved by means of the Ritz method. This requires the evaluation of energy functionals. The strain energy stored in a shell during elastic deformation is written in the classical (thin) shallow shell theory by

$$V = V_s + V_{bs} + V_b \tag{12}$$

where V_s , V_{bs} and V_b are the parts of the total strain energy due to stretching, bending- stretching coupling and bending, respectively:

$$V_s = \frac{1}{2}\iint \{\varepsilon\}^T [A]\{\varepsilon\} dArea \tag{13a}$$

$$V_{bs} = \frac{1}{2}\iint \left(\{\kappa\}^T [B]\{\varepsilon\} + \{\varepsilon\}^T [B]\{\kappa\}\right) dArea \tag{13b}$$

$$V_b = \frac{1}{2}\iint \{\kappa\}^T [D]\{\kappa\} dArea \tag{13c}$$

The kinetic energy of the panel due to translational motion only is given by

$$T = \frac{1}{2}\rho h \iint \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] dArea \tag{14}$$

where ρ is the mass density of the shell per unit volume.

For simplicity in the formulation, the following dimensionless quantities are introduced.

$$\xi = \frac{2x}{a}, \; \eta = \frac{2y}{b} \text{ (dimensionless coordinates)} \tag{15a}$$

$$\Omega = \omega a^2 \sqrt{\frac{\rho h}{D}} \text{ (dimensionless frequency parameter)} \tag{15b}$$

$$D = \frac{Eh^3}{12\left(1-\nu^2\right)} \text{ (reference plate stiffness)} \tag{15c}$$

In the Ritz method the displacements may be assumed in the form

$$u(\xi,\eta,t) = \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} P_{ij} X_i(\xi) Y_j(\eta) \sin\omega t \tag{16a}$$

$$v(\xi,\eta,t) = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} Q_{kl} X_k(\xi) Y_l(\eta) \sin\omega t \tag{16b}$$

$$w(\xi,\eta,t) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} R_{mn} X_m(\xi) Y_n(\eta) \sin\omega t \tag{16c}$$

where P_{ij} , Q_{kl} and R_{mn} are unknown coefficients and $X_i(\xi)$, $Y_j(\eta)$, .. and $Y_n(\eta)$ are the functions that satisfy at least the kinematical boundary conditions at the edges. The upper limit in each of the summations (16) is arbitrary but is unified here for simplicity in the convergence test.

After substituting Eqs.(16) into the functional L

$$L = T_{max} - V_{max} \tag{17}$$

composed of the maximum strain and kinetic energies obtained from Eqs.(12) and (14), the stationary value is obtained by

$$\frac{\partial L}{\partial P_{ij}} = 0, \; \frac{\partial L}{\partial Q_{kl}} = 0, \; \frac{\partial L}{\partial R_{mn}} = 0 \tag{18a,b,c}$$

$(i,k,m = 0,1,2,..,(M-1);\; j,l,n = 0,1,2,...,(N-1))$

The result of the minimization process (18) yields a set of homogeneous, linear simultaneous equations in the unknowns $\{P_{ij}, Q_{kl}, R_{mn}\}$. For non-trivial solutions, the determinant of the coefficient matrix is set to zero. The $(M\times N)\times 3$ eigenvalues may be extracted and the lowest several eigenvalues (natural frequencies) are important from a practical viewpoint.

The above procedure is a standard routine of the Ritz method, and is modified to incorporate arbitrary edge conditions. This approach introduces the following polynomials

$$X_i(\xi) = \xi^i (1+\xi)^{Bu1} (1-\xi)^{Bu3} \tag{19a}$$

$$Y_j(\eta) = \eta^j (1+\eta)^{Bu2} (1-\eta)^{Bu4} \tag{19b}$$

$$X_k(\xi) = \xi^k (1+\xi)^{Bv1} (1-\xi)^{Bv3} \tag{19c}$$

$$Y_l(\eta) = \eta^l (1+\eta)^{Bv2} (1-\eta)^{Bv4} \tag{19d}$$

$$X_m(\xi) = \xi^m (1+\xi)^{Bw1} (1-\xi)^{Bw3} \tag{19e}$$

$$Y_n(\eta) = \eta^n (1+\eta)^{Bw2} (1-\eta)^{Bw4} \tag{19f}$$

where B_{rs} $(r = u,v,w;\; s = 1,2,3,4)$ is the boundary index [18] which is used to satisfy the kinematic boundary conditions. The capital letter B stands for Boundary. The first subscript letter in B_{rs} indicates which displacement (u,v or w) is dealt with and the second subscript number indicates which edge, Edge (1),.. or Edge (4), is under consideration. The Edge (1),(2),(3) and (4) denote the boundary along x=-a/2, y=-b/2, x=a/2 and y=b/2, respectively (See Fig. 1).

For in-plane displacements u and v, B_{rs} =0 (r=u, v; s=1,2,3,4) denote that the specified displacement along the specified edge is free and B_{rs} =1 denote that the displacement is rigidly fixed. For out-of-plane displacement w, B_{rs} =0, 1 and 2 (r=w; s=1,2,3,4) denote that the specified displacement along the specified edge is free, simply supported and clamped, respectively. With such boundary indices, one can accommodate arbitrary sets of both in-plane and out-of-plane boundary conditions in the vibration analysis and computation.

The introduction of the boundary index makes it possible to deal with a tremendous number of edge conditions in the analysis. The number of combinations in the boundary condition is $(2\times2\times3)^4$ =20736, when one of the two conditions (free or fixed) in u and v and one of the three conditions (free, simply supported or clamped) in w are imposed along each of the four edges. This is significantly larger than the *plate* analysis where only out-of-plane displacement is concerned. The present vibration analysis can calculate natural frequencies for any of these combinations.

In the following numerical examples, however, the boundary conditions are limited to those similar to the standard plate boundary conditions, such as free edge, simply supported edge and clamped edge. For example, at the Edge (1) along x= -a/2

B_{u1}= B_{v1}= B_{w1}=0 for the free edge (no constraint), denoted by F (20a)

B_{u1}=0, B_{v1}=1, B_{w1}=1 for the simply supported edge denoted by S (20b)

B_{u1}= B_{v1}=1, B_{w1}=2 for the clamped edge denoted by C (20c)

The entire set of boundary conditions is denoted by four capital letter, such as CSFF, in the counterclockwise starting from Edge(1).

2.2. Exact solution for specific boundary condition

When a shallow shell of rectangular planform is supported along four edges by shear diaphragms, the exact solution is possible [1][3][5] by assuming

$$u(x,y,t)=P_{ij}\cos\frac{i\pi x}{a}\sin\frac{j\pi y}{b}\sin\omega t \qquad (21a)$$

$$v(x,y,t)=Q_{kl}\sin\frac{k\pi x}{a}\cos\frac{l\pi y}{b}\sin\omega t \qquad (21b)$$

$$w(x,y,t)=R_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\omega t \qquad (21c)$$

Equations (21) are substituted in the governing equation, and the exact solution is obtained in the form of 3×3 frequency matrix. This solution procedure is given in detail in Refs. [3][5].

3. Numerical Examples and Accuracy of Solution

3.1. Convergence and comparison of the solution

In numerical examples, square planform (a/b=1) and moderately thin thickness (a/h=20) are used. Poisson's ratio is kept as ν =0.3.

Table 1 Convergence and comparison of frequency parameters Ω of simply supported shallow shells (SSSS), a/b =1, a/h =20, ν =0.3.

	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
Spherical shell (Rx/Ry=1, a/Rx=0.2)						
8×8	23.715	51.052	51.052	80.021	99.563	99.563
10×10	23.715	51.052	51.052	80.021	99.543	99.543
12×12	23.715	51.052	51.052	80.021	99.542	99.542
Exact	23.715	51.052	51.052	80.021	99.542	99.542
Ref.[22]	23.70	51.04	51.04	80.02	—	—
Spherical shell (Rx/Ry=1, a/Rx=0.5)						
8×8	38.080	59.134	59.134	85.361	103.88	103.88
10×10	38.080	59.134	59.134	85.361	103.85	103.85
12×12	38.080	59.134	59.134	85.361	103.85	103.85
Exact	38.080	59.134	59.134	85.361	103.85	103.85
Ref.[22]	38.01	59.10	59.10	85.32	—	—
Cylindrical shell (Rx/Ry=0, a/Rx=0.2)						
8×8	20.786	49.391	50.451	79.203	98.700	99.420
10×10	20.786	49.391	50.451	79.203	98.680	99.400
12×12	20.786	49.391	50.451	79.203	98.680	99.400
Exact	20.786	49.391	50.451	79.203	98.680	99.400
Ref.[22]	20.78	49.39	50.44	79.19	—	—
Cylindrical shell (Rx/Ry=0, a/Rx=0.5)						
8×8	25.509	49.613	55.861	80.479	98.616	103.03
10×10	25.509	49.613	55.861	80.479	98.616	103.03
12×12	25.509	49.612	55.861	80.479	98.593	103.01
Exact	25.509	49.612	55.861	80.479	98.594	103.01
Ref.[22]	25.48	49.61	55.84	80.46	—	—
Hyperbolic paraboloidal shell (Rx/Ry=-1, a/Rx=0.2)						
8×8	19.659	49.924	49.924	78.873	99.244	99.244
10×10	19.659	49.924	49.924	78.873	99.224	99.224
12×12	19.659	49.924	49.924	78.873	99.224	99.224
Exact	19.659	49.924	49.924	78.873	99.224	99.224
Ref.[22]	19.66	49.92	49.92	78.85	—	—
Hyperbolic paraboloidal shell (Rx/Ry=-1, a/Rx=0.5)						
8×8	19.252	52.806	52.806	78.438	101.96	101.96
10×10	19.252	52.805	52.805	78.438	101.94	101.94
12×12	19.252	52.805	52.805	78.438	101.94	101.94
Exact	19.252	52.805	52.805	78.438	101.94	101.94
Ref.[22]	19.25	52.79	52.79	78.46	—	—

Table 1 presents convergence study of frequency parameters of spherical (R_x/R_y=1), cylindrical (R_x/R_y=0) and hyperbolic paraboloidal (R_x/R_y= -1) shells of square planform. These shells (SSSS) are supported by shear diaphram along four edges. For each shell configuration, two degrees of curvature a/R_x=0.2 and 0.5 are used. The present results are calculated for the number of terms 8×8, 10×10 and 12×12 for each of u, v and w that yield frequency matrix size of 192×192, 300×300 and 432×432, respectively.

The present parameters converge well within five significant figures, and show exact match with the exact results obtained from Eq. (21). They are also compared to the parameters in Ref. [22], and are generally in good agreement with slight differences. In the following tables, the present results are calculated by using the 12×12 solution and presented in five significant figures.

Table 2 Comparison of frequency parameters Ω of shallow shells, *a/Rx* =0.5, *a/b* =1, *a/h* =100, ν =0.3.

	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
CCSS						
Spherical shell (Rx/Ry=1)						
Present	171.60	180.27	186.99	195.32	204.04	210.09
Ref.[21]	171.6	180.3	187.0	195.3	204.1	210.1
Cylindrical shell (Rx/Ry=0)						
Present	72.375	105.48	127.24	133.02	147.49	168.62
Ref.[21]	72.37	105.5	127.2	133.0	147.5	168.6
Hyperbolic paraboloidal shell (Rx/Ry=−1)						
Present	94.181	122.38	136.98	149.24	173.01	173.62
Ref.[21]	94.18	122.4	137.0	149.2	173.0	173.6
CSCC						
Spherical shell (Rx/Ry=1)						
Present	179.90	187.88	192.50	202.47	208.02	228.73
Ref.[21]	179.9	187.9	192.5	202.5	208.0	228.7
Cylindrical shell (Rx/Ry=0)						
Present	95.798	116.43	145.24	150.67	170.30	187.31
Ref.[21]	95.81	116.4	145.2	150.6	170.3	187.3
Hyperbolic paraboloidal shell (Rx/Ry=−1)						
Present	131.20	139.90	154.35	158.42	186.02	189.83
Ref.[21]	131.2	139.9	154.4	158.4	186.0	189.8
CCCC						
Spherical shell (Rx/Ry=1)						
Present	191.99	191.99	196.93	209.96	216.19	242.22
Ref.[21]	192.0	192.0	169.9	210.0	216.2	242.2
Cylindrical shell (Rx/Ry=0)						
Present	99.263	119.00	151.13	156.35	172.52	192.43
Ref.[21]	99.26	119.0	151.1	156.3	172.5	192.4
Hyperbolic paraboloidal shell (Rx/Ry=−1)						
Present	157.35	157.35	157.41	166.52	204.03	208.69
Ref.[21]	157.3	157.3	157.4	166.5	204.0	208.7

Table 2 is another comparison study with values of Ref. [21] by Mochida and his co-workers. Note that only in Table 2, the thickness ratio is kept as *a/h*=100 (very thin) due to the need for comparison, while in all other tables, the ratio is *a/h*=20 (moderately thin) throughout. They [21] extended the method of superposition, commonly used in plate vibration analysis, to shallow shell vibration analysis. Three sets of boundary condition CCSS, CSCC and CCCC are considered. When the present values are rounded with four significant figures, most of the results are exactly identical with the referred values [21] for wide ranges of boundary conditions, shell configuration and degree of curvature. Thus, the accuracy of the present solution is well demonstrated

3.2. Comprehensive results of shallow shells

Table 3(a) presents the lowest six frequency parameters Ω of shallow spherical shells (R_x/R_y=1) of square planform (*a/b*=1) with moderate thickness (*a/h*=20) for 21 different sets of boundary conditions. The degree of curvature is taken as *a/R*=0.2. Table 3(b) is the same format as Table 3(a) except that the curvature is larger in *a/R*=0.5.

Table 3(a) Frequency parameters Ω of shallow spherical shells, R_x/R_y=1, $a/R_x=a/R_y$=0.2, *a/b* =1, *a/h* =20, ν =0.3.

B.C.	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
FFFF	13.460	19.563	25.991	34.838	34.838	61.770
SFFF	6.6294	15.343	25.400	27.084	49.196	51.604
CFFF	3.7518	8.4865	21.518	28.262	30.515	44.098
SSFF	3.3668	17.321	20.788	39.706	51.532	54.319
CSFF	5.5638	19.350	25.461	44.633	53.198	54.137
CCFF	7.8923	23.895	27.922	49.256	62.962	66.231
SFSF	10.066	16.105	38.895	39.592	46.867	71.588
CFSF	15.629	21.677	41.802	49.902	56.556	78.415
SSSF	12.221	30.508	41.852	60.149	63.216	64.384
CSSF	17.921	33.738	51.986	65.351	68.576	101.91
CCSF	18.464	38.646	52.338	72.133	75.507	106.00
CFCF	25.385	28.722	45.607	61.561	67.417	80.888
SCSF	12.951	35.762	42.269	64.111	64.384	73.590
CSCF	26.453	38.140	63.338	68.064	78.336	109.64
CCCF	26.907	42.737	63.637	77.905	81.563	117.42
SSSS	23.715	51.052	51.052	80.021	99.542	99.542
CSSS	27.338	53.342	60.177	87.149	101.11	113.97
CCSS	30.659	62.026	62.352	93.819	115.29	115.45
CSCS	32.577	56.377	70.634	95.546	103.05	128.77
CCCS	35.610	64.875	72.385	101.74	117.10	131.04
CCCC	40.422	74.713	74.713	109.14	132.24	133.06

Table3(b) Frequency parameters Ω of shallow spherical shells, R_x/R_y=1, $a/R_x=a/R_y$=0.5, *a/b* =1, *a/h* =20, ν =0.3.

B.C.	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
FFFF	13.414	19.392	32.522	34.952	34.952	64.993
SFFF	6.5501	16.251	25.460	31.034	52.846	57.290
CFFF	4.7227	8.3836	22.089	28.777	32.298	46.445
SSFF	3.3328	17.287	24.642	48.041	53.468	58.038
CSFF	6.3859	19.708	27.976	52.485	54.138	56.256
CCFF	11.139	23.818	32.726	57.426	63.892	69.438
SFSF	11.118	15.910	42.201	47.454	48.857	76.185
CFSF	17.086	26.233	50.914	52.202	57.953	83.723
SSSF	13.137	42.426	44.253	64.383	66.009	69.874
CSSF	21.832	45.242	54.358	71.892	74.084	105.86
CCSF	22.097	50.022	54.536	77.648	81.379	107.00
CFCF	37.260	38.449	55.161	63.247	68.605	86.273
SCSF	13.672	44.410	47.411	64.384	69.935	79.514
CSCF	37.802	49.827	65.270	74.501	83.417	113.60
CCCF	37.993	55.089	65.419	83.845	86.719	121.32
SSSS	38.080	59.134	59.134	85.361	103.85	103.85
CSSS	41.361	61.292	67.571	92.265	105.39	117.76
CCSS	44.688	69.242	69.928	98.789	119.07	119.22
CSCS	46.965	64.210	77.040	100.41	107.29	128.77
CCCS	50.623	72.403	78.805	106.52	120.86	134.53
CCCC	58.167	81.187	81.187	113.82	135.61	137.50

Table 4(a) Frequency parameters Ω of shallow cylindrical shells, R_x=(infinity), a/R_y=0.2, a/b =1, a/h =20, ν =0.3.

B.C.	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
FFFF	13.463	20.120	24.765	34.783	34.906	61.080
SFFF	6.6433	15.762	25.444	26.011	48.826	51.057
CFFF	3.8031	8.520	21.967	27.283	31.040	43.507
SSFF	3.3642	17.606	19.790	38.566	51.306	53.876
CSFF	5.6203	19.155	25.344	43.415	52.701	54.069
CCFF	7.2165	24.291	27.072	48.195	62.925	65.837
SFSF	10.293	16.180	36.802	39.703	46.889	70.965
CFSF	15.820	20.652	39.843	50.076	56.427	77.325
SSSF	12.306	28.033	41.887	59.537	61.866	64.383
CSSF	17.374	31.420	51.973	64.044	67.958	101.23
CCSF	18.147	36.578	52.369	71.598	74.409	106.18
CFCF	22.728	26.490	43.729	61.700	67.313	79.829
SCSF	13.345	33.608	42.359	63.602	64.383	72.471
CSCF	23.879	35.887	63.362	66.802	77.744	108.98
CCCF	24.466	40.558	63.695	76.806	81.039	116.81
SSSS	20.786	49.391	50.451	79.203	98.680	99.400
CSSS	24.612	51.746	59.574	86.374	100.26	113.84
CCSS	28.366	60.815	61.677	93.135	114.62	115.26
CSCS	29.790	54.843	70.118	94.807	102.22	128.77
CCCS	32.959	63.515	71.936	101.07	116.37	130.89
CCCC	37.829	73.541	74.364	108.53	131.81	132.63

Table 4(b) Frequency parameters Ω of shallow cylindrical shells, R_x=(infinity), a/R_y=0.5, a/b =1, a/h =20, ν =0.3.

B.C.	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
FFFF	13.439	21.283	28.119	34.685	35.450	61.010
SFFF	6.6395	19.427	25.800	26.100	49.939	54.147
CFFF	5.1616	8.5893	24.644	28.095	31.467	43.520
SSFF	3.3471	17.926	22.680	40.568	51.774	56.461
CSFF	6.7779	19.523	28.273	45.235	52.659	54.228
CCFF	8.4374	25.562	29.549	51.099	63.347	67.845
SFSF	13.103	16.415	37.219	43.261	47.666	72.235
CFSF	18.606	21.000	40.435	53.085	57.183	77.330
SSSF	14.789	29.570	44.943	61.893	62.205	64.383
CSSF	19.768	33.149	54.601	64.160	70.320	101.83
CCSF	20.504	39.505	54.881	74.396	74.918	108.09
CFCF	25.302	26.920	44.474	64.255	68.018	79.895
SCSF	15.790	36.420	45.243	64.384	66.824	72.867
CSCF	26.034	37.687	65.624	67.010	79.819	109.57
CCCF	26.625	43.567	65.872	77.322	83.608	117.65
SSSS	25.509	49.612	55.861	80.479	98.593	103.01
CSSS	29.072	52.119	64.209	87.612	100.21	116.97
CCSS	34.217	61.768	66.606	94.692	114.64	118.46
CSCS	33.797	55.363	74.117	95.957	102.21	128.77
CCCS	38.182	64.505	76.253	102.52	116.41	133.67
CCCC	46.241	74.300	79.239	110.14	132.35	135.51

Table 5(a) Frequency parameters Ω of shallow cylindrical shells, R_y=(infinity), a/R_x=0.2, a/b =1, a/h =20, ν =0.3.

B.C.	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
FFFF	13.463	20.120	24.765	34.783	34.906	61.080
SFFF	6.6292	14.925	25.340	26.889	48.649	50.799
CFFF	3.4675	8.4672	21.301	27.977	30.489	44.084
SSFF	3.3642	17.606	19.790	38.566	51.306	53.876
CSFF	5.3411	19.816	24.606	43.658	53.213	54.191
CCFF	7.2160	24.291	27.072	48.195	62.925	65.837
SFSF	9.6216	16.062	38.095	38.927	46.707	70.902
CFSF	15.125	21.586	41.173	49.350	56.387	78.225
SSSF	11.656	29.019	41.175	59.160	62.868	64.384
CSSF	17.291	32.612	51.402	65.052	67.756	101.44
CCSF	18.233	37.348	51.833	71.290	75.213	105.65
CFCF	24.782	28.631	45.173	61.099	67.253	80.746
SCSF	13.046	34.169	41.694	63.122	64.384	73.255
CSCF	25.874	37.417	62.847	67.846	77.615	109.24
CCCF	26.364	41.684	63.207	77.660	80.811	117.00
SSSS	20.786	49.391	50.451	79.203	98.680	99.400
CSSS	25.089	52.840	58.803	86.447	100.98	113.22
CCSS	28.366	60.815	61.677	93.135	114.62	115.26
CSCS	31.159	56.022	69.447	94.935	102.95	128.77
CCCS	33.866	64.448	71.213	101.12	117.01	130.41
CCCC	37.829	73.541	74.364	108.53	131.81	132.63

Addition of curvature causes frequencies to be increased. In Table 3(a), the average increase from flat plates for the fundamental frequencies in Ω_1 is 9 percent, including the highest increase 20 percent of SSSS shell. The deeper curvature a/R=0.5 in Table 3(b) shows the average increase of 39 percent in Ω_1 with the maximum 93 percent of SSSS shell.

Table 4(a) and (b) list up the lowest six frequency parameters of shallow cylindrical shells with R_x=(infinity) and a/R_y=0.2 and a/R_y=0.5, respectively. This represents straight edges of the shell along the x axis, and curvature is given only in y direction. When the increase of the fundamental frequencies due to the curvature increase is considered, the average percent increases are 4 percent and 20 percent in Table 4(a) and (b), respectively. Roughly speaking, this effect is a half of the spherical shells, and the effect of curvature increase in one direction is a half of curvatures in two directions of spherical shells.

Tables 5(a) and (b) also tabulate the lowest six frequency parameters of shallow cylindrical shells, but with R_y=(infinity) and a/R_x=0.2 and a/R_x=0.5, respectively. Straight edges of the shell exist along the y axis, and curvature is only in x direction. For cylindrical shells with FFFF, SSSS and CCCC, the results are the identical as in Table 4(a) and (b) due to uniform boundary condition along four edges, and also shells with SSFF, CCFF and CCSS give the identical results as in Table 4(a) and (b) since the 90 degree rotation (or flipping about a diagonal symmetric axis) of the shell gives essentially the same boundary conditions.

Table 5(b) Frequency parameters Ω of shallow cylindrical shells, R_y=(infinity), a/R_x=0.5, a/b=1, a/h=20, ν=0.3.

B.C.	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
FFFF	13.439	21.284	28.119	34.685	35.450	61.010
SFFF	6.5537	14.962	25.159	30.710	49.135	52.689
CFFF	3.4369	8.2686	21.129	28.647	31.417	46.386
SSFF	3.3474	17.927	22.681	40.569	51.774	56.461
CSFF	5.2853	22.217	24.740	46.690	53.805	56.634
CCFF	8.4374	25.562	29.549	51.099	63.347	67.845
SFSF	9.5542	15.696	38.833	44.561	46.547	71.739
CFSF	14.777	25.870	47.889	48.900	57.008	80.148
SSSF	11.506	34.831	41.059	59.650	64.384	67.893
CSSF	19.550	39.366	51.429	68.895	70.218	103.10
CCSF	21.313	43.499	51.948	72.439	79.696	105.04
CFCF	35.338	38.115	52.645	60.699	67.650	85.438
SCSF	14.725	39.385	41.653	63.678	64.384	77.594
CSCF	36.143	45.846	62.698	73.258	78.852	111.13
CCCF	36.473	49.547	63.134	82.043	82.448	118.79
SSSS	25.509	49.612	55.861	80.479	98.594	103.01
CSSS	31.345	58.522	59.672	88.079	104.65	113.21
CCSS	34.217	61.768	66.606	94.692	114.64	118.46
CSCS	40.744	62.278	70.066	96.739	106.73	128.77
CCCS	42.928	69.997	71.914	102.84	120.35	130.73
CCCC	46.241	74.300	79.239	110.14	132.35	135.51

Their identical results of six cases are underlined. As expected, the increase of the fundamental frequencies due to the curvature increase is basically same as in Table 4(a) and (b).

Finally, Table 6(a) and (b) list up the lowest six parameters of shallow hyperbolic paraboloidal shells with a/R_y=0.2 and a/R_y=0.5, respectively. The negative curvature ratio (R_x/R_y= -1) indicates that the shell geometry is convex in one direction and concave in another direction. This geometric feature gives rise unusual response in frequency. For shells of spherical and cylindrical curvature, addition of curvature causes more stiffness in the structure, and it results in the increase of natural frequencies. But for shell of hyperbolic paraboloidal shell, addition of negative curvature causes the decrease of frequencies, when shell has free edges. For example, the SFFF shell gives Ω_1=6.648 for a/R_x=0 (plate), Ω_1=6.628 for a/R_x=0.2 and Ω_1=6.548 for a/R_x=0.5. As the constrained is increased along the edges, this tendency disappears.

4. Conclusions

The purpose of this paper was to present lists of accurate natural frequencies for free vibration of doubly curved, isotropic shallow shells of rectangular (square) planform under different sets of boundary conditions. For this purpose, mathematical procedure was described and accuracy of the present numerical solutions was well established by convergence and comparison studies. Thus, with strong background in solution accuracy, all the present frequency parameters were given in five

Table 6(a) Frequency parameters Ω of shallow hyperbolic paraboloidal shells, R_x/R_y=-1, a/R_y=0.2, a/b=1, a/h=20, ν=0.3.

B.C.	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
FFFF	13.458	21.894	24.244	34.929	34.929	61.775
SFFF	6.6281	16.057	25.414	26.762	48.931	50.889
CFFF	3.8095	8.4752	22.356	27.892	30.617	44.083
SSFF	3.3591	18.781	19.323	38.159	51.803	54.022
CSFF	5.6190	20.369	25.086	43.345	53.237	54.290
CCFF	7.0680	25.562	27.018	48.207	63.436	65.918
SFSF	10.406	16.107	37.451	39.768	46.847	70.668
CFSF	15.839	21.623	40.753	50.048	56.513	78.037
SSSF	12.206	28.062	41.865	59.114	62.527	64.384
CSSF	17.735	32.074	51.964	64.796	67.770	101.19
CCSF	19.183	37.091	52.437	71.488	75.079	106.08
CFCF	25.159	28.691	44.999	61.683	67.361	80.627
SCSF	14.387	33.591	42.424	63.306	64.384	73.063
CSCF	26.181	37.301	63.325	67.705	77.629	109.05
CCCF	26.786	41.703	63.724	77.602	80.988	116.90
SSSS	19.659	49.924	49.924	78.873	99.224	99.224
CSSS	24.592	52.471	59.269	86.220	100.84	113.69
CCSS	28.542	61.369	61.547	93.043	115.07	115.19
CSCS	31.282	55.859	69.836	94.764	102.86	128.77
CCCS	34.232	64.379	71.753	101.07	116.93	130.85
CCCC	38.732	74.301	74.301	108.54	132.15	132.74

Table 6(b) Frequency parameters Ω of shallow hyperbolic paraboloidal shells, R_x/R_y=-1, a/R_y=0.5, a/b=1, a/h=20, ν=0.3.

B.C.	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
FFFF	13.406	24.102	31.229	35.444	35.444	63.315
SFFF	6.5483	19.020	25.562	31.448	49.885	54.027
CFFF	4.8347	8.3111	25.101	29.276	32.576	46.547
SSFF	3.3096	19.414	24.886	37.875	55.594	56.620
CSFF	6.3397	24.220	28.365	44.741	54.552	56.821
CCFF	7.5551	28.612	32.675	51.171	67.028	67.642
SFSF	12.909	15.945	41.288	43.783	47.396	70.296
CFSF	18.435	26.084	45.659	53.025	57.756	79.27
SSSF	14.054	29.794	45.098	59.410	64.383	65.895
CSSF	21.710	36.488	54.729	68.581	69.133	101.62
CCSF	25.820	41.998	55.400	73.706	78.889	107.56
CFCF	36.944	38.401	51.656	64.224	68.305	84.713
SCSF	20.858	36.118	45.769	64.383	64.882	76.477
CSCF	37.483	45.199	65.551	72.410	78.955	110.01
CCCF	38.249	49.610	66.142	82.098	83.171	118.21
SSSS	19.252	52.805	52.805	78.438	101.94	101.94
CSSS	28.811	56.335	62.406	86.712	103.76	116.05
CCSS	35.016	65.275	65.497	94.178	117.69	117.73
CSCS	41.267	61.320	72.397	95.686	106.13	128.77
CCCS	44.601	69.579	75.114	102.50	119.86	133.46
CCCC	50.691	78.836	78.836	110.19	135.10	135.53

significant figures, while other previous literature provide four significant figures.

It is hoped that the present comprehensive sets of frequency parameters will serve as good reference for future comparison. Due to the space limitation, the present results were given only for the case of relatively thick case (*a/h*=0.05) to complement Ref. [22]. In the next study, computation of frequency parameters will be done for very thin case (*a/h*=0.01) to match with Ref. [21], and the effect of the shell thickness will be clarified.

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