LOGARITHMIC A-HYPERGEOMETRIC SERIES

MUTSUMI SAITO

Abstract. The method of Frobenius is a standard technique to construct series solutions of an ordinary linear differential equation around a regular singular point. In the classical case, when the roots of the indicial polynomial are separated by an integer, logarithmic solutions can be constructed by means of perturbation of a root.

The method for a regular A-hypergeometric system is a theme of the book by Saito, Sturmfels, and Takayama. Whereas they perturbed a parameter vector to obtain logarithmic A-hypergeometric series solutions, we adopt a different perturbation in this paper.
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1. Introduction

The method of Frobenius is a standard technique to construct series solutions of an ordinary linear differential equation around a regular singular point. In the classical case, when the roots of the indicial polynomial are separated by an integer, logarithmic solutions can be constructed by means of perturbation of a root; introducing a new parameter s in a series, differentiating it with respect to s several times, and taking the limit $s \to 0$ give logarithmic solutions ([4], see e.g. [3, pp. 243-258], [8, pp. 396-404]).

Let $A = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n) = (a_{ij})$ be a $d \times n$-matrix of rank d with coefficients in $\mathbb{Z}$. Throughout this paper, we assume the homogeneity of A, i.e., we assume that all $\boldsymbol{a}_j$ belong to one hyperplane off the origin in $\mathbb{Q}^d$. Let $\mathbb{N}$ be the set of nonnegative integers. Let I_A denote the toric ideal in the polynomial ring $\mathbb{C}[\partial] = \mathbb{C}[\partial_1, \ldots, \partial_n]$, i.e.,

$$I_A = \langle \partial^{\boldsymbol{u}} - \partial^{\boldsymbol{v}} \,:\, A\boldsymbol{u} = A\boldsymbol{v},\, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{N}^n \rangle \subseteq \mathbb{C}[\partial]. \tag{1}$$

Here and hereafter we use the multi-index notation; for example, $\partial^{\boldsymbol{u}}$ means $\partial^{u_1} \cdots \partial^{u_n}$ for $\boldsymbol{u} = (u_1, \ldots, u_n)^T$. Given a column vector $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_d)^T \in \mathbb{C}^d$, let $H_A(\boldsymbol{\beta})$ denote the left ideal of the Weyl algebra

$$D = \mathbb{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$$

generated by I_A and

$$\sum_{j=1}^{n} a_{ij}\theta_j - \beta_i \qquad (i = 1, \dots, d), \tag{2}$$

where $\theta_j = x_j\partial_j$. The quotient $M_A(\boldsymbol{\beta}) = D/H_A(\boldsymbol{\beta})$ is called the *A-hypergeometric system with parameter $\boldsymbol{\beta}$*, and a formal series annihilated by $H_A(\boldsymbol{\beta})$ an *A-hypergeometric series with parameter $\boldsymbol{\beta}$*. The homogeneity of A is known to be equivalent to the regularity of $M_A(\boldsymbol{\beta})$ by Hotta [7] and Schulze, Walther [11].

For a generic parameter $\boldsymbol{\beta}$, Gel'fand, Graev, Kapranov, and Zelevinsky [5], [6] constructed series solutions to $M_A(\boldsymbol{\beta})$. More generally, Saito, Sturmfels, and Takayama [10] constructed logarithm-free series solutions, which we will review in Section 2. Then they perturbed a parameter vector to construct logarithmic series solutions [10, §3.5]. This is reasonable, because this method perturbed the easier equations (2) keeping the difficult ones (1) unchanged. Then we can easily obtain perturbed solutions. However, we need to make a suitable linear combination of perturbed solutions before taking a limit, and it is not clear how to describe this linear combination (except the unimodular case [10, §3.6]).

Set

$$L := \mathrm{Ker}_{\mathbb{Z}}(A) = \{\boldsymbol{u} \in \mathbb{Z}^n \mid A\boldsymbol{u} = \boldsymbol{0}\}. \tag{3}$$

We know that the logarithmic coefficients of A-hypergeometric series solutions are polynomials of $\log x^{\boldsymbol{b}}$ ($\boldsymbol{b} \in L$) [9, Proposition 5.2]. In this paper, we adopt a perturbation by elements of L. At first glance, it does not seem a good idea, because we perturb a solution of the difficult equations (1) keeping the easier ones (2). But it turns out that we can evaluate the order of perturbation, and we can explicitly describe logarithmic series solutions (Theorems 5.4, 6.2 and Remarks 5.6, 6.3).

In [1], Adolphson and Sperber treated A-hypergeometric series solutions with logarithm mainly of degree 1 or 2, and considered an application to mirror symmetry.

This paper is organized as follows. In Sections 2 and 3, following [10], we recall some notions on A-hypergeometric series. In particular, we recall fake exponents and negative supports in Section 2, and then we recall that the fake exponents can be computed by the standard pairs of the initial ideal of I_A in Section 3.

In Section 4, for a generic weight $\boldsymbol{w}$ and a fake exponent $\boldsymbol{v}$, we define a set $\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})$ of negative supports, over which we consider a series. Then we introduce a Gale dual and a hyperplane arrangement to visualize negative supports.

In Sections 5 and 6, we state the main results (Theorems 5.4, 6.2 and Remarks 5.6, 6.3). We consider a perturbation by a single element and several elements of L in Sections 5 and 6, respectively. We make one section for the single element case, because it is much easier to consider. In both sections, we first consider orders of perturbations (Lemma 5.1, Corollaries 5.2, 5.3, and Lemma 6.1). Then we see that the perturbed solution operated by the difficult ones (1) has some positive orders, and we can prove the main results.

Throughout this paper we run three examples (Examples 3.1, 3.2, and 3.3) to illustrate the theory.

2. LOGARITHM-FREE CANONICAL A-HYPERGEOMETRIC SERIES

In this section, we recall logarithm-free canonical A-hypergeometric series. For details, see [10].

Fix a generic weight vector $\boldsymbol{w} = (w_1, \dots, w_n) \in \mathbb{R}^n$. The ideal of the polynomial ring $\mathbb{C}[\theta] = \mathbb{C}[\theta_1, \dots, \theta_n]$ defined by

$$\widetilde{\mathrm{fin}}_{\boldsymbol{w}}(H_A(\boldsymbol{\beta})) := D \cdot \mathrm{in}_{\boldsymbol{w}}(I_A) \cap \mathbb{C}[\theta] + \langle A\theta - \boldsymbol{\beta} \rangle \tag{4}$$

is called the *fake indicial ideal*, where $\mathrm{in}_{\boldsymbol{w}}(I_A)$ denotes the initial ideal of I_A with respect to $\boldsymbol{w}$, and $\langle A\theta - \boldsymbol{\beta} \rangle$ denotes the ideal generated by $\sum_{j=1}^n a_{ij}\theta_j - \beta_i$ $(i = 1, \dots, d)$. Each zero of $\widetilde{\mathrm{fin}}_{\boldsymbol{w}}(H_A(\boldsymbol{\beta}))$ is called a *fake exponent*.

Let $\boldsymbol{w} \cdot \boldsymbol{u}$ denote $w_1u_1 + \cdots + w_nu_n$ for $\boldsymbol{u} \in \mathbb{Q}^n$. An A-hypergeometric series

$$x^{\boldsymbol{v}} \cdot \sum_{\boldsymbol{u} \in L} g_{\boldsymbol{u}}(\log x) x^{\boldsymbol{u}} \qquad (g_{\boldsymbol{u}} \in \mathbb{C}[x] := \mathbb{C}[x_1, \dots, x_n]) \tag{5}$$

is said to be *in the direction of* $\boldsymbol{w}$ if there exists a basis $\boldsymbol{u}^{(1)}, \dots, \boldsymbol{u}^{(n)}$ of $\mathbb{Q}^n$ with $\boldsymbol{w} \cdot \boldsymbol{u}^{(j)} > 0$ $(j = 1, \dots, n)$ such that $g_{\boldsymbol{u}} = 0$ whenever $\boldsymbol{u} \notin \sum_{j=1}^n \mathbb{Q}_{\geq 0}\boldsymbol{u}^{(j)}$. A fake exponent $\boldsymbol{v}$ is called an *exponent* if there exists an A-hypergeometric series (5) in the direction of $\boldsymbol{w}$ with nonzero $g_{\boldsymbol{0}}$. Let $\prec$ be the lexicographic order on $\mathbb{N}^n$. Suppose that $\boldsymbol{u}^{(1)}, \dots, \boldsymbol{u}^{(n)}$ is a basis as above. Then a monomial like $x^{\boldsymbol{v}} \cdot \mathrm{in}_{\prec}(g_{\boldsymbol{0}})(\log x)$ in the A-hypergeometric series

$$x^{\boldsymbol{v}} \cdot \sum_{\boldsymbol{u} \in L \cap \sum_{j=1}^n \mathbb{Q}_{\geq 0}\boldsymbol{u}^{(j)}} g_{\boldsymbol{u}}(\log x) x^{\boldsymbol{u}} \qquad (g_{\boldsymbol{u}} \in \mathbb{C}[x]) \tag{6}$$

with nonzero $g_{\boldsymbol{0}}$ is called a *starting monomial*. The A-hypergeometric series (6) is said to be *canonical* with respect to $\boldsymbol{w}$ if no starting monomials other than $x^{\boldsymbol{v}} \cdot \mathrm{in}_{\prec}(g_0)(\log x)$ appear in the series.

Next we recall logarithm-free A-hypergeometric series $\phi_{\boldsymbol{v}}$. For $\boldsymbol{v} \in \mathbb{C}^n$, its *negative support* $\mathrm{nsupp}(\boldsymbol{v})$ is the set of indices i with $v_i \in \mathbb{Z}_{<0}$. When $\mathrm{nsupp}(\boldsymbol{v})$ is minimal with respect to inclusions among $\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})$ with $\boldsymbol{u} \in L$, $\boldsymbol{v}$ is said to have *minimal negative support.* For $\boldsymbol{v}$ satisfying $A\boldsymbol{v} = \boldsymbol{\beta}$ with minimal negative support, we define a formal series

$$\phi_{\boldsymbol{v}} = x^{\boldsymbol{v}} \cdot \sum_{\boldsymbol{u} \in N_{\boldsymbol{v}}} \frac{[\boldsymbol{v}]_{\boldsymbol{u}_-}}{[\boldsymbol{v}+\boldsymbol{u}]_{\boldsymbol{u}_+}} x^{\boldsymbol{u}}, \tag{7}$$

where

$$N_{\boldsymbol{v}} = \{\, \boldsymbol{u} \in L \mid \mathrm{nsupp}(\boldsymbol{v}) = \mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u}) \,\}, \tag{8}$$

and $\boldsymbol{u}_+, \boldsymbol{u}_- \in \mathbb{N}^n$ satisfy $\boldsymbol{u} = \boldsymbol{u}_+ - \boldsymbol{u}_-$ with disjoint supports, and $[\boldsymbol{v}]_{\boldsymbol{u}} = \prod_{j=1}^n [v_j]_{u_j} = \prod_{j=1}^n v_j(v_j-1)\cdots(v_j-u_j+1)$ for $\boldsymbol{u} \in \mathbb{N}^n$. Proposition 3.4.13 and Theorem 3.4.14 in [10] respectively state that the series $\phi_{\boldsymbol{v}}$ is A-hypergeometric, and that if $\boldsymbol{v}$ is a fake exponent of $M_A(\boldsymbol{\beta})$, then $\phi_{\boldsymbol{v}}$ is canonical, and $\boldsymbol{v}$ is an exponent.

3. Standard pairs and fake exponents

In this section, we review standard pairs and fake exponents following [10].

Let M be a monomial ideal in $\mathbb{C}[\partial] = \mathbb{C}[\partial_1, \dots \partial_n]$. A pair $(\boldsymbol{a}, \sigma)$ ($\boldsymbol{a} \in \mathbb{N}^n, \sigma \subseteq [1,n]$) is *standard* if it satisfies

(1) $a_i = 0$ for all $i \in \sigma$.
(2) For any $\boldsymbol{b} \in \mathbb{N}^\sigma$, $\partial^{\boldsymbol{a}}\partial^{\boldsymbol{b}} \notin M$.
(3) For any $l \notin \sigma$, there exists $\boldsymbol{b} \in \mathbb{N}^{\sigma\cup\{l\}}$ such that $\partial^{\boldsymbol{a}}\partial^{\boldsymbol{b}} \in M$.

Let $\mathcal{S}(M)$ denote the set of standard pairs of M. Then, by [10, Corollary 3.2.3], $\boldsymbol{v}$ is a fake exponent of $M_A(\boldsymbol{\beta})$ with respect to $\boldsymbol{w}$ if and only if $A\boldsymbol{v} = \boldsymbol{\beta}$ and there exists a standard pair $(\boldsymbol{a}, \sigma) \in \mathcal{S}(\mathrm{in}_{\boldsymbol{w}}(I_A))$ such that $v_j = a_j$ for all $j \notin \sigma$.

Example 3.1 (cf. Example 3.5.3 in [10]). Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$, and take $\boldsymbol{w}$ so that $\mathrm{in}_{\boldsymbol{w}}(I_A) = \langle \partial_1\partial_3 \rangle$. Hence

$$\mathcal{S}(\mathrm{in}_{\boldsymbol{w}}(I_A)) = \{(0,*,*),\ (*,*,0)\},$$

where, for a standard pair $(\boldsymbol{a}, \sigma)$, we put $*$ in the place of σ, a_j at $j \notin \sigma$.

Let $\boldsymbol{\beta} = \begin{pmatrix} 10 \\ 8 \end{pmatrix}$. Then the fake exponents are $(2,8,0)^T$ and $(0,12,-2)^T$. Since $(2,8,0)^T$ has minimal negative support, $\phi_{(2,8,0)^T}$ is a solution.

Example 3.2. Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$. Then

$$I_A = \langle\, \partial_1^2\partial_3 - \partial_2^3,\ \partial_2\partial_4^2 - \partial_3^3,\ \partial_1\partial_4 - \partial_2\partial_3,\ \partial_1\partial_3^2 - \partial_2^2\partial_4 \,\rangle.$$

Take $\boldsymbol{w}$ (e.g. $\boldsymbol{w} = (3,1,0,0)$) so that

$$\mathrm{in}_{\boldsymbol{w}} I_A = \langle\, \partial_1^2\partial_3,\ \partial_2\partial_4^2,\ \partial_1\partial_4,\ \partial_1\partial_3^2 \,\rangle.$$

Hence

$$\mathcal{S}(\mathrm{in}_{\boldsymbol{w}}(I_A)) = \{(0,0,*,*),\ (0,*,*,0),\ (0,*,*,1),\ (*,*,0,0),\ (1,*,1,0)\}.$$

Let $\boldsymbol{\beta} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$. Then the fake exponents (with their corresponding standard pairs) are

- $\boldsymbol{v}_3 := (0,0,-7,5)^T \quad \leftrightarrow \quad (0,0,*,*)$,
- $\boldsymbol{v}_\emptyset := (0,-5/2,1/2,0)^T \quad \leftrightarrow \quad (0,*,*,0)$,
- $\boldsymbol{v}_{2,3} := (0,-2,-1,1)^T \quad \leftrightarrow \quad (0,*,*,1)$,
- $\boldsymbol{v}_{1,2} := (-1,-1,0,0)^T \quad \leftrightarrow \quad (*,*,0,0)$,
- $\boldsymbol{v}_2 := (1,-4,1,0)^T \quad \leftrightarrow \quad (1,*,1,0)$.

Since $\boldsymbol{v}_\emptyset, \boldsymbol{v}_2, \boldsymbol{v}_3$ have minimal negative supports, $\phi_{\boldsymbol{v}_\emptyset}, \phi_{\boldsymbol{v}_2}, \phi_{\boldsymbol{v}_3}$ are solutions.

Example 3.3 (cf. Example 3.5.2 in [10])**.** Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 1 & 0 \end{pmatrix}$.

Then

$$I_A = \langle\, \partial_1\partial_3 - \partial_5^2,\ \partial_2\partial_4 - \partial_5^2 \,\rangle.$$

Take $\boldsymbol{w}$ so that $\mathrm{in}_{\boldsymbol{w}}(I_A) = \langle\, \partial_1\partial_3,\ \partial_2\partial_4 \,\rangle$. Hence

$$\mathcal{S}(\mathrm{in}_{\boldsymbol{w}}(I_A)) = \{(0,0,*,*,*),\ (*,0,0,*,*),\ (*,*,0,0,*),\ (0,*,*,0,*)\}.$$

Let $\boldsymbol{\beta} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Then $(0,0,0,0,1)^T$ is a unique fake exponent. Hence $\phi_{(0,0,0,0,1)^T} = x_5$ is a solution.

4. Negative supports and hyperplane arrangements

In this section, first we see that the sum (5) is taken over a set of negative supports. Then we define a set $\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})$ of negative supports, and we introduce a Gale dual and a hyperplane arrangement to visualize negative supports.

For an A-hypergeometric series ϕ (5), set

$$\mathrm{supp}(\phi) := \{\boldsymbol{u} \mid g_{\boldsymbol{u}} \neq 0\}.$$

Proposition 4.1. *Let ϕ be an A-hypergeometric series* (5)*. Suppose that $\boldsymbol{u} \in \operatorname{supp}(\phi)$ and $\operatorname{nsupp}(\boldsymbol{v}+\boldsymbol{u}) = \operatorname{nsupp}(\boldsymbol{v}+\boldsymbol{u}')$ for $\boldsymbol{u}, \boldsymbol{u}' \in L$.*

Then $\boldsymbol{u}' \in \operatorname{supp}(\phi)$. Furthermore, the terms of the highest total degree of $g_{\boldsymbol{u}}$ and those of $g_{\boldsymbol{u}'}$ are the same up to nonzero scalar multiplication.

Proof. Suppose that $p(\log x)$ is the terms of the highest total degree of $g_{\boldsymbol{u}}$. Consider

$$(\boldsymbol{u}'-\boldsymbol{u}) = (\boldsymbol{u}'-\boldsymbol{u})_+ - (\boldsymbol{u}'-\boldsymbol{u})_-$$

and

$$\partial^{(\boldsymbol{u}'-\boldsymbol{u})_-} x^{\boldsymbol{v}+\boldsymbol{u}} p(\log x).$$

Suppose that $u'_j - u_j < 0$.

- If $j \in \operatorname{nsupp}(\boldsymbol{v}+\boldsymbol{u}) = \operatorname{nsupp}(\boldsymbol{v}+\boldsymbol{u}')$, then $v_j + u'_j = v_j + u_j + (u'_j - u_j) < v_j + u_j < 0$. Hence $[v_j + u_j]_{u_j - u'_j} \neq 0$.
- Suppose that $j \notin \operatorname{nsupp}(\boldsymbol{v}+\boldsymbol{u}) = \operatorname{nsupp}(\boldsymbol{v}+\boldsymbol{u}')$. If $v_j \notin \mathbb{Z}$, then clearly $[v_j + u_j]_{u_j - u'_j} \neq 0$. If $v_j \in \mathbb{Z}$, then $0 \leq v_j + u'_j = v_j + u_j + (u'_j - u_j) < v_j + u_j$. Hence $[v_j + u_j]_{u_j - u'_j} \neq 0$.

Hence the highest log-term of $\partial^{(\boldsymbol{u}'-\boldsymbol{u})_-}(x^{\boldsymbol{v}+\boldsymbol{u}} p(\log x))$ is equal to

$$(\partial^{(\boldsymbol{u}'-\boldsymbol{u})_-} x^{\boldsymbol{v}+\boldsymbol{u}}) p(\log x),$$

which is not zero. Hence $\boldsymbol{u}' \in \operatorname{supp}(\phi)$. Do the same argument exchanging $\boldsymbol{u}$ and $\boldsymbol{u}'$, and find that the terms of the highest total tegree of $g_{\boldsymbol{u}}$ and those of $g_{\boldsymbol{u}'}$ are the same up to nonzero scalar multiplication. □

Let $\boldsymbol{w}$ be a generic weight. Suppose that $\mathcal{G} := \{\partial^{\boldsymbol{u}_+^{(i)}} - \partial^{\boldsymbol{u}_-^{(i)}} \mid i = 1, 2, \ldots, r\}$ is a Gröbner basis of I_A with respect to $\boldsymbol{w}$, and that $\partial^{\boldsymbol{u}_+^{(i)}} \in \operatorname{in}_{\boldsymbol{w}}(I_A)$ for all i. Set

$$C(\boldsymbol{w}) := \sum_{i=1}^{r} \mathbb{N}\boldsymbol{u}^{(i)}.$$

Here recall that $\mathbb{N} = \{0, 1, 2, \cdots\}$.

Lemma 4.2 (cf. Theorem 6.12.14 in [12])**.** *Suppose that $\boldsymbol{u} \in L$ satisfies $\partial^{\boldsymbol{u}_+} \in \operatorname{in}_{\boldsymbol{w}}(I_A)$.*

Then $\boldsymbol{u} \in C(\boldsymbol{w})$.

Proof. Since $\mathcal{G}$ is a Gröbner basis, $\partial^{\boldsymbol{u}_+} - \partial^{\boldsymbol{u}_-}$ is reduced to 0 by $\mathcal{G}$. This means that $\boldsymbol{u}$ belongs to $\sum_{i=1}^{r} \mathbb{N}\boldsymbol{u}^{(i)}$. □

Corollary 4.3. *Let ϕ be an A-hypergeometric series with exponent $\boldsymbol{v}$ in the direction of $\boldsymbol{w}$. Suppose that $\boldsymbol{u} \in \operatorname{supp}(\phi)$. Then*

$$\{\boldsymbol{u}' \in L \mid \operatorname{nsupp}(\boldsymbol{v}+\boldsymbol{u}') = \operatorname{nsupp}(\boldsymbol{v}+\boldsymbol{u})\} \subseteq C(\boldsymbol{w}).$$

Proof. We may assume that $w_1, \dots, w_n$ are linearly independent over $\mathbb{Q}$.

By Proposition 4.1, for $\boldsymbol{u}' \in L$ with $\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u}') = \mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})$, we have $\boldsymbol{u}' \in \mathrm{supp}(\phi)$. If $\boldsymbol{u}' = \boldsymbol{0}$, then clearly $\boldsymbol{u}' \in C(\boldsymbol{w})$.

Let $\boldsymbol{u}' \neq \boldsymbol{0}$. Since ϕ is in the direction of $\boldsymbol{w}$, we have $\boldsymbol{w}\cdot\boldsymbol{u}' > 0$, or $\boldsymbol{w}\cdot\boldsymbol{u}'_+ > \boldsymbol{w}\cdot\boldsymbol{u}'_-$. By Lemma 4.2, $\boldsymbol{u}'$ belongs to $C(\boldsymbol{w})$. □

For a fake exponent $\boldsymbol{v}$, set

(9)
$$\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v}) := \left\{ I \;\middle|\; \begin{array}{l} I = \mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u}) \text{ for some } \boldsymbol{u} \in C(\boldsymbol{w}). \\ \text{If } \mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u}') = I \text{ for } \boldsymbol{u}' \in L, \text{ then } \boldsymbol{u}' \in C(\boldsymbol{w}). \end{array} \right\},$$

and
$$\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c := \{\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u}) \mid \boldsymbol{u} \in L\} \setminus \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v}).$$

Proposition 4.4. *Let $\boldsymbol{w}$ be a generic weight, and $\boldsymbol{v}$ a fake exponent. Then*

$$\{\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u}) \mid \mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u}) \subseteq \mathrm{nsupp}(\boldsymbol{v}),\ \boldsymbol{u} \in L\} \subseteq \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v}).$$

In particular, $\mathrm{nsupp}(\boldsymbol{v}) \in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})$.

Proof. We may assume that $w_1, \dots, w_n$ are linearly independent over $\mathbb{Q}$.

Let $\boldsymbol{u} \in L \setminus \{\boldsymbol{0}\}$. We show

$$\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u}) \subseteq \mathrm{nsupp}(\boldsymbol{v}) \Rightarrow \boldsymbol{w}\cdot\boldsymbol{u} > 0. \tag{10}$$

Suppose that $\boldsymbol{w}\cdot\boldsymbol{u} < 0$. Then $\partial^{\boldsymbol{u}_-} x^{\boldsymbol{v}} = 0$ since $\boldsymbol{v}$ is a fake exponent. Hence there exists j such that $u_j < 0$, $v_j \in \mathbb{N}$, and $v_j + u_j < 0$. Namely, $j \in \mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u}) \setminus \mathrm{nsupp}(\boldsymbol{v})$, and we have proved (10). Then by Lemma 4.2, we have $\boldsymbol{u} \in C(\boldsymbol{w})$ (this is also valid for $\boldsymbol{u} = \boldsymbol{0}$.). □

Corollary 4.5. *Let $\boldsymbol{v}$ have the smallest $\boldsymbol{w}$-weight among the set of fake exponents in $\boldsymbol{v}+L$.*

If $\boldsymbol{v}+\boldsymbol{u}_0$ is a fake exponent, then $\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u}_0) \in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})$.

Proof. We may assume that $w_1, \dots, w_n$ are linearly independent over $\mathbb{Q}$.

If $\boldsymbol{u}_0 = \boldsymbol{0}$, then the assertion is in Proposition 4.4. Suppose that $\boldsymbol{u}_0 \neq \boldsymbol{0}$. By the minimality of $\boldsymbol{v}$, we have $\boldsymbol{w}\cdot\boldsymbol{u}_0 > 0$. If $\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u}_0+\boldsymbol{u}) = \mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u}_0)$, then $\boldsymbol{w}\cdot\boldsymbol{u} \geq 0$ by (10). Hence $\boldsymbol{u}_0+\boldsymbol{u} \in C(\boldsymbol{w})$ by Lemma 4.2, and $\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u}_0) \in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})$. □

To visualize $\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})$, we introduce a Gale dual (cf. e.g. [13]). Let $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_{n-d}\}$ be a basis of L. Set

$$B := (\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_{n-d}) = (\boldsymbol{g}_1, \boldsymbol{g}_2, \dots, \boldsymbol{g}_n)^T.$$

For $\boldsymbol{v} \in \mathbb{C}^n$, define

$$\psi_{\boldsymbol{v}} : \mathbb{R}^{n-d} \simeq \boldsymbol{v} + L_{\mathbb{R}} \subseteq \mathbb{C}^n$$

by $\psi_{\boldsymbol{v}}(\boldsymbol{x}) = \boldsymbol{v} + B\boldsymbol{x}$, where $L_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} L$. Set $Z(\boldsymbol{v}) := \{i \in [1, n] \,|\, v_i \in \mathbb{Z}\}$. For a subset $I \subseteq Z(\boldsymbol{v})$, set

$$N_I(\boldsymbol{v}) := \{\boldsymbol{v} + \boldsymbol{u} \,|\, \boldsymbol{u} \in L, \ \mathrm{nsupp}(\boldsymbol{v} + \boldsymbol{u}) = I\}. \tag{11}$$

Then

$$\begin{aligned} N_I(\boldsymbol{v}) &= \{\boldsymbol{z} \in \boldsymbol{v} + L \,|\, z_i < 0 \, (i \in I); \ z_j \geq 0 \, (j \in Z(\boldsymbol{v}) \setminus I)\} \\ &= \{\boldsymbol{z} \in \boldsymbol{v} + L \,|\, \boldsymbol{e}_i^*(\boldsymbol{z}) < 0 \, (i \in I); \ \boldsymbol{e}_j^*(\boldsymbol{z}) \geq 0 \, (j \in Z(\boldsymbol{v}) \setminus I)\}, \end{aligned}$$

where $\{\boldsymbol{e}_i^* \,|\, 1 \leq i \leq n\}$ is the dual basis of the standard basis $\{\boldsymbol{e}_i \,|\, 1 \leq i \leq n\}$ of $\mathbb{C}^n$. Hence $N_I(\boldsymbol{v})$ is the set of lattice points in a union of faces of the hyperplane arrangement $\{\boldsymbol{e}_i^* = 0 \,|\, i \in Z(\boldsymbol{v})\}$ on $\boldsymbol{v} + L_{\mathbb{R}}$. Transfer this to the hyperplane arrangement on $\mathbb{R}^{n-d}$ by $\psi_{\boldsymbol{v}}$. Since

$$\begin{aligned} (\psi_{\boldsymbol{v}}^*(\boldsymbol{e}_i^*))(\boldsymbol{x}) &= (\boldsymbol{e}_i^*)(\psi_{\boldsymbol{v}}(\boldsymbol{x})) = (\boldsymbol{e}_i^*)(\boldsymbol{v} + B\boldsymbol{x}) \\ &= v_i + (B^T \boldsymbol{e}_i)(\boldsymbol{x}) = v_i + \boldsymbol{g}_i(\boldsymbol{x}), \end{aligned}$$

we can regard $N_I(\boldsymbol{v})$ as the set of lattice points in a union of faces of the hyperplane arrangement $\{H_i \,|\, i \in Z(\boldsymbol{v})\}$ on $\mathbb{R}^{n-d}$, where

$$H_i = \{\boldsymbol{x} \in \mathbb{R}^{n-d} \,|\, \boldsymbol{g}_i(\boldsymbol{x}) + v_i = 0\}.$$

Example 4.6 (Continuation of Example 3.1)**.** Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$, and take $\boldsymbol{w}$ as before. Then $C(\boldsymbol{w}) = \mathbb{N}(1, -2, 1)$. Let

$$B = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = (\boldsymbol{b}) = (g_1, g_2, g_3)^T.$$

Let $\boldsymbol{\beta} = \begin{pmatrix} 10 \\ 8 \end{pmatrix}$. Then the fake exponents are $\boldsymbol{v} := (0, 12, -2)^T$ and $\boldsymbol{v'} := (2, 8, 0)^T$. We have

$$\psi_{\boldsymbol{v}} : \mathbb{R} \ni x \mapsto \boldsymbol{v} + x\boldsymbol{b} = (x, -2x + 12, x - 2)^T \in \mathbb{R}^3,$$

and

$$\mathrm{nsupp}(\boldsymbol{v} + x\boldsymbol{b}) = \begin{cases} \{2\} & (x \geq 7) \\ \emptyset & (x = 2, 3, 4, 5, 6) \\ \{3\} & (x = 0, 1) \\ \{1, 3\} & (x \leq -1). \end{cases}$$

Since $C(\boldsymbol{w})$ corresponds to $\{x \in \mathbb{N}\}$, we have

$$\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v}) = \{\{2\}, \{3\}, \emptyset\}, \qquad \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c = \{\{1, 3\}\}.$$

In Figure 4.1, a small arrow indicates the positive side. Note that a hyperplane (a point in this example) itself belongs to its positive side.

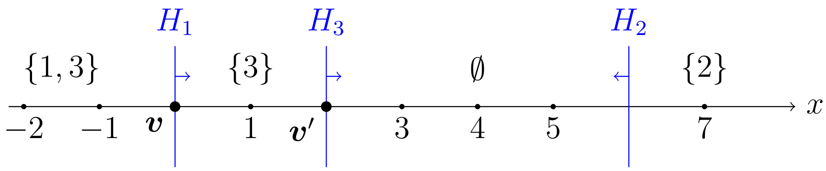

Figure 4.1

Example 4.7 (Continuation of Example 3.2)**.** Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$, and take $\boldsymbol{w}$ as before. Then

$$\begin{aligned} C(\boldsymbol{w}) &= \mathbb{N}(1,-1,-1,1)^T + \mathbb{N}(2,-3,1,0)^T + \mathbb{N}(0,1,-3,2)^T + \mathbb{N}(1,-2,2,-1)^T \\ &= \mathbb{N}(1,-2,2,-1)^T \oplus \mathbb{N}(0,1,-3,2)^T. \end{aligned}$$

Let

$$B := \begin{pmatrix} 1 & 0 \\ -2 & 1 \\ 2 & -3 \\ -1 & 2 \end{pmatrix} = (\boldsymbol{b}_1, \boldsymbol{b}_2) = (\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3, \boldsymbol{g}_4)^T.$$

Let $\boldsymbol{\beta} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$, and $\boldsymbol{v} := \boldsymbol{v}_{1,2} = (-1,-1,0,0)^T$. Then

$$\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v}) = \{\{2\},\{3\},\{2,3\},\{1,2\} = I_{\boldsymbol{0}}\},$$

$$\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c = \{\{1,3\},\{2,4\},\{1,4\},\{1,3,4\},\{1,2,4\}\}.$$

In Figure 4.2, we put I in the face where $\mathrm{nsupp}(\boldsymbol{v} + x_1\boldsymbol{b}_1 + x_2\boldsymbol{b}_2) = I$ for a lattice point $(x_1, x_2)^T$.

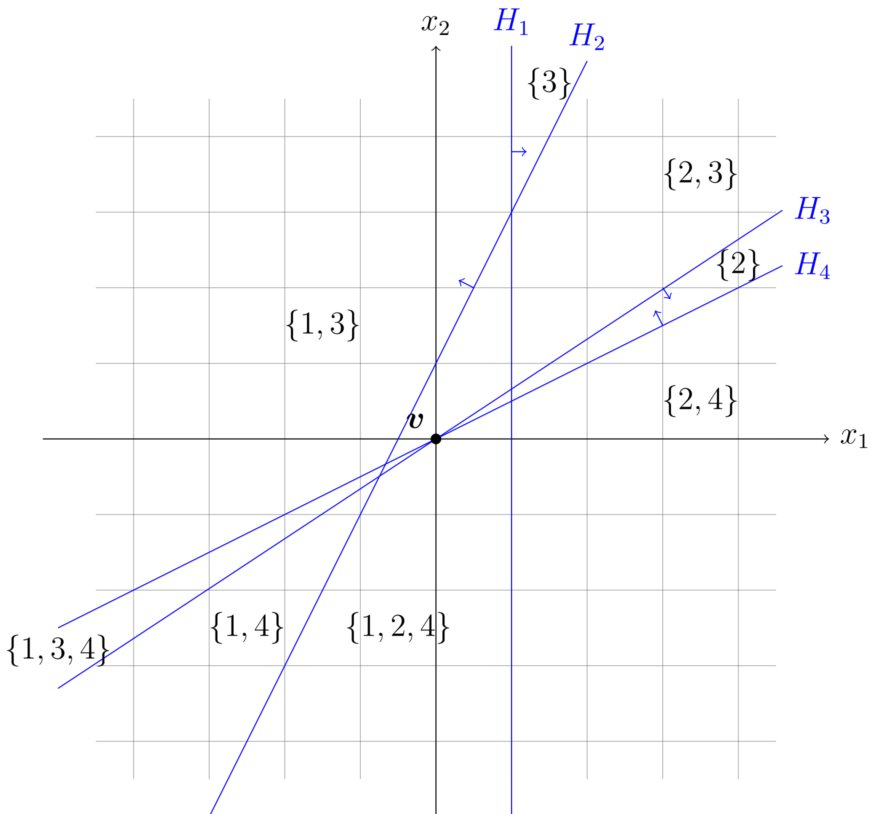

Figure 4.2

Example 4.8 (Continuation of Example 3.3)**.** Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 1 & 0 \end{pmatrix}$, and take $\boldsymbol{w}$ as before. Let

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ -2 & -2 \end{pmatrix} = (\boldsymbol{b}_1, \boldsymbol{b}_2) = (\boldsymbol{g}_1, \ldots, \boldsymbol{g}_5)^T.$$

Let $\boldsymbol{\beta} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Then $\boldsymbol{v} := (0,0,0,0,1)^T$ is the unique fake exponent.

$$\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v}) = \{\emptyset, \{5\}\},$$

$$\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c = \{\{1,3\}, \{2,4\}, \{1,3,5\}, \{2,4,5\}, \{1,2,3,4\}\}.$$

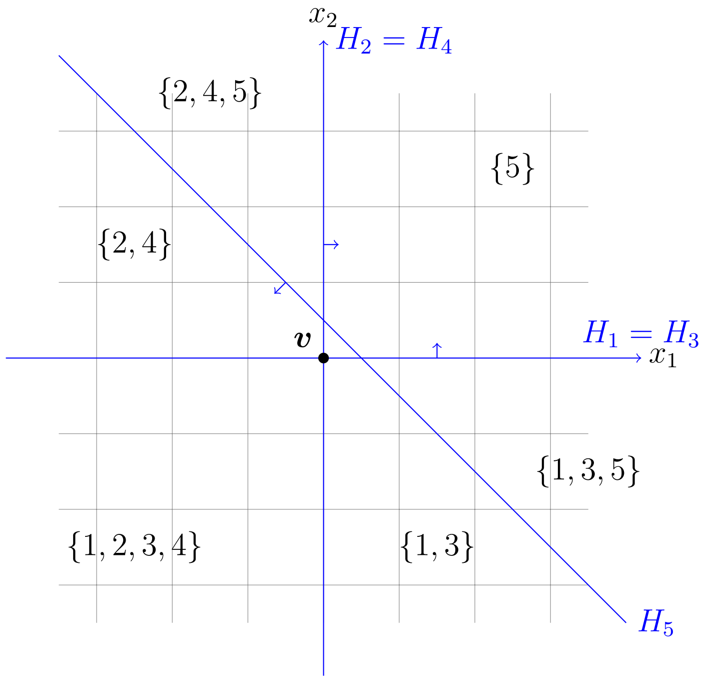

Figure 4.3

5. Method 1

In this section, we consider a Frobenius's method by perturbing an exponent with a single vector in L.

Lemma 5.1. *Let $\boldsymbol{b} \in L$, $\boldsymbol{u} \in \mathbb{N}^n$. Then*

$$\begin{aligned}[\boldsymbol{v}+s\boldsymbol{b}]_{\boldsymbol{u}} &= \Big(\prod_{i\in \mathrm{nsupp}(\boldsymbol{v}-\boldsymbol{u})\setminus \mathrm{nsupp}(\boldsymbol{v})} b_i\Big)\widehat{[\boldsymbol{v}]}_{\boldsymbol{u}} s^{|\mathrm{nsupp}(\boldsymbol{v}-\boldsymbol{u})|-|\mathrm{nsupp}(\boldsymbol{v})|} \\ &\qquad + o\big(s^{|\mathrm{nsupp}(\boldsymbol{v}-\boldsymbol{u})|-|\mathrm{nsupp}(\boldsymbol{v})|}\big),\end{aligned}$$

where $\widehat{[\boldsymbol{v}]}_{\boldsymbol{u}}$ is the product of nonzero factors of $[\boldsymbol{v}]_{\boldsymbol{u}}$;

$$\begin{aligned}\widehat{[\boldsymbol{v}]}_{\boldsymbol{u}} &= \Big(\prod_{i\notin \mathrm{nsupp}(\boldsymbol{v}-\boldsymbol{u})\setminus \mathrm{nsupp}(\boldsymbol{v})} [v_i]_{u_i}\Big) \\ &\qquad \times \Big(\prod_{i\in \mathrm{nsupp}(\boldsymbol{v}-\boldsymbol{u})\setminus \mathrm{nsupp}(\boldsymbol{v})} (v_i)!(-1)^{|v_i-u_i+1|}(|v_i-u_i+1|!)\Big).\end{aligned}$$

Proof. Note that

$$[\boldsymbol{v}+s\boldsymbol{b}]_{\boldsymbol{u}} = \prod_{i=1}^{n}(v_i+sb_i)(v_i-1+sb_i)\cdots(v_i-u_i+1+sb_i).$$

In $(v_i+sb_i)(v_i-1+sb_i)\cdots(v_i-u_i+1+sb_i)$, the factor s appears if and only if $v_i\in\mathbb{N}$ and $v_i-u_i\in\mathbb{Z}_{<0}$, and furthermore if that is the case, it appears only once and always with b_i.

Finally note that $\mathrm{nsupp}(\boldsymbol{v}-\boldsymbol{u})\supseteq\mathrm{nsupp}(\boldsymbol{v})$. Hence $|\mathrm{nsupp}(\boldsymbol{v}-\boldsymbol{u})\setminus\mathrm{nsupp}(\boldsymbol{v})|=|\mathrm{nsupp}(\boldsymbol{v}-\boldsymbol{u})|-|\mathrm{nsupp}(\boldsymbol{v})|$. □

Corollary 5.2. *Let $\boldsymbol{b},\boldsymbol{u}\in L$. Suppose that $b_j\neq 0$ for any j. Then*

(1) $\mathrm{ord}_s([\boldsymbol{v}+s\boldsymbol{b}]_{\boldsymbol{u}_-})=|\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})\setminus\mathrm{nsupp}(\boldsymbol{v})|$.
(2) $\mathrm{ord}_s([\boldsymbol{v}+s\boldsymbol{b}+\boldsymbol{u}]_{\boldsymbol{u}_+})=|\mathrm{nsupp}(\boldsymbol{v})\setminus\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})|$.

Proof. (1) By Lemma 5.1, $\mathrm{ord}_s([\boldsymbol{v}+s\boldsymbol{b}]_{\boldsymbol{u}_-})=|\mathrm{nsupp}(\boldsymbol{v}-\boldsymbol{u}_-)\setminus\mathrm{nsupp}(\boldsymbol{v})|$.

We have $\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})=\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u}_+-\boldsymbol{u}_-)\subseteq\mathrm{nsupp}(\boldsymbol{v}-\boldsymbol{u}_-)$. Hence $(\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})\setminus\mathrm{nsupp}(\boldsymbol{v}))\subseteq\mathrm{nsupp}(\boldsymbol{v}-\boldsymbol{u}_-)\setminus\mathrm{nsupp}(\boldsymbol{v})$.

Let $j\in\mathrm{nsupp}(\boldsymbol{v}-\boldsymbol{u}_-)\setminus\mathrm{nsupp}(\boldsymbol{v})$. Then $u_j<0$. Hence $(\boldsymbol{u}_+)_j=0$, and $j\in\mathrm{nsupp}(\boldsymbol{v}-\boldsymbol{u}_-+\boldsymbol{u}_+)\setminus\mathrm{nsupp}(\boldsymbol{v})$. Therefore $\mathrm{nsupp}(\boldsymbol{v}-\boldsymbol{u}_-)\setminus\mathrm{nsupp}(\boldsymbol{v})=\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})\setminus\mathrm{nsupp}(\boldsymbol{v})$.

(2) In (1), replace $\boldsymbol{v},\boldsymbol{u}$ by $\boldsymbol{v}+\boldsymbol{u},-\boldsymbol{u}$, respectively. □

Corollary 5.3. *Let $\boldsymbol{b},\boldsymbol{u}\in L$. Suppose that $b_j\neq 0$ for any j. Let*

$$a_{\boldsymbol{u}}(s):=\frac{[\boldsymbol{v}+s\boldsymbol{b}]_{\boldsymbol{u}_-}}{[\boldsymbol{v}+s\boldsymbol{b}+\boldsymbol{u}]_{\boldsymbol{u}_+}}.$$

Then

$$\mathrm{ord}_s(a_{\boldsymbol{u}}(s))=|\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})|-|\mathrm{nsupp}(\boldsymbol{v})|.$$

Indeed,

$$\begin{aligned} a_{\boldsymbol{u}}(s) &= \frac{\prod_{i\in\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})\setminus\mathrm{nsupp}(\boldsymbol{v})}b_i}{\prod_{j\in\mathrm{nsupp}(\boldsymbol{v})\setminus\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})}b_j}\frac{\widehat{[\boldsymbol{v}]}_{\boldsymbol{u}_-}}{\widehat{[\boldsymbol{v}+\boldsymbol{u}]}_{\boldsymbol{u}_+}}s^{|\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})|-|\mathrm{nsupp}(\boldsymbol{v})|} \\ &\qquad +o(s^{|\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})|-|\mathrm{nsupp}(\boldsymbol{v})|}). \end{aligned}$$

Proof. For finite sets X and Y,

$$\begin{aligned} |X\setminus Y|-|Y\setminus X| &= |X\setminus Y|+|X\cap Y|-(|Y\setminus X|+|X\cap Y|) \\ &= |X|-|Y|. \end{aligned}$$

Hence the statement follows from Lemma 5.1 and Corollary 5.2. □

Let $\boldsymbol{v}$ be a fake exponent. For $\boldsymbol{0}\neq\boldsymbol{b}\in L$ with $b_i\neq 0$ ($i\in\mathrm{nsupp}(\boldsymbol{v})$), set

$$F_{\boldsymbol{b}}(x,s):=\sum_{\boldsymbol{u}\in L,\ \mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})\in\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})}a_{\boldsymbol{u}}(s)x^{\boldsymbol{v}+s\boldsymbol{b}+\boldsymbol{u}},$$

where

$$a_{\boldsymbol{u}}(s)=\frac{[\boldsymbol{v}+s\boldsymbol{b}]_{\boldsymbol{u}_-}}{[\boldsymbol{v}+s\boldsymbol{b}+\boldsymbol{u}]_{\boldsymbol{u}_+}}.$$

The condition $b_i \neq 0$ ($i \in \mathrm{nsupp}(\boldsymbol{v})$) guarantees the denominator of $a_{\boldsymbol{u}}(s)$ not to be zero by Corollary 5.3.

Set $I_{\boldsymbol{u}} := \mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})$ for $\boldsymbol{u} \in L$.

Theorem 5.4. *Let $\boldsymbol{v}$ be a fake exponent. Put $m := \min_{I \in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})} |I|$ and $M := \min_{I \in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v}), J \in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c}(|I \cup J|)$. Since $I_{\boldsymbol{0}} \in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})$ (Proposition 4.4), we have $|I_{\boldsymbol{0}}| \geq m$. Let $\boldsymbol{b} \in L$ satisfy $\boldsymbol{b} \neq \boldsymbol{0}$ and $b_i \neq 0$ for $i \in \mathrm{nsupp}(\boldsymbol{v})$. Then*

(1) *$(\partial_s^j s^{|I_{\boldsymbol{0}}|-m} F_{\boldsymbol{b}}(x,s))_{|s=0}$ $(j = 0, 1, \ldots, M-m-1)$ are solutions to $M_A(\boldsymbol{\beta})$.*
If $M > |I_{\boldsymbol{0}}|$, then $\boldsymbol{v}$ is an exponent with multiplicity at least $\binom{n-d+M-|I_{\boldsymbol{0}}|-1}{M-|I_{\boldsymbol{0}}|-1}$.

(2) *If $\boldsymbol{b}^{(1)}, \boldsymbol{b}^{(2)}, \ldots \boldsymbol{b}^{(k)} \in L$ satisfy*

$$\sum_{i=1}^{k} \frac{b^{(i)}_{I \setminus I_{\boldsymbol{0}}} b^{(i)}_{J \setminus I}}{b^{(i)}_{I_{\boldsymbol{0}} \setminus I}} = 0$$

for all $I \in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})$, $J \in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c$ with $|I \cup J| = M$, then

$$(\partial_s^{M-m} s^{|I_{\boldsymbol{0}}|-m} \sum_{i=1}^{k} F_{\boldsymbol{b}^{(i)}}(x,s))_{|s=0}$$

is also a solution to $M_A(\boldsymbol{\beta})$, where $b_K = \prod_{k \in K} b_k$. If $M \geq |I_{\boldsymbol{0}}|$, then $\boldsymbol{v}$ is an exponent.

Proof. First of all, since $\boldsymbol{b} \in L$, we have

$$(\sum_{j=1}^{n} a_{ij}\theta_j - \beta_i) F_{\boldsymbol{b}}(x,s) = 0 \qquad (i = 1, \ldots, d).$$

Let $\boldsymbol{u}' \in L$.

Suppose that $I_{\boldsymbol{u}}, I_{\boldsymbol{u}+\boldsymbol{u}'} \in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})$. Then as in [10, (3.29)]

$$\partial^{\boldsymbol{u}'_-}(a_{\boldsymbol{u}}(s) x^{\boldsymbol{v}+s\boldsymbol{b}+\boldsymbol{u}}) = \partial^{\boldsymbol{u}'_+}(a_{\boldsymbol{u}+\boldsymbol{u}'}(s) x^{\boldsymbol{v}+s\boldsymbol{b}+\boldsymbol{u}+\boldsymbol{u}'}).$$

Hence

$$
\begin{aligned}
&(\partial^{\boldsymbol{u}'_+} - \partial^{\boldsymbol{u}'_-})F_{\boldsymbol{b}}(x,s)\\
= &\sum_{I_{\boldsymbol{u}}\in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v}),\, I_{\boldsymbol{u}-\boldsymbol{u}'}\in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c} \partial^{\boldsymbol{u}'_+}(a_{\boldsymbol{u}}(s)x^{\boldsymbol{v}+s\boldsymbol{b}+\boldsymbol{u}})\\
&- \sum_{I_{\boldsymbol{u}}\in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v}),\, I_{\boldsymbol{u}+\boldsymbol{u}'}\in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c} \partial^{\boldsymbol{u}'_-}(a_{\boldsymbol{u}}(s)x^{\boldsymbol{v}+s\boldsymbol{b}+\boldsymbol{u}})\\
= &\sum_{I_{\boldsymbol{u}}\in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v}),\, I_{\boldsymbol{u}-\boldsymbol{u}'}\in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c} \partial^{\boldsymbol{u}'_+}(a_{\boldsymbol{u}}(s)x^{\boldsymbol{v}+s\boldsymbol{b}+\boldsymbol{u}})\\
&- \sum_{I_{\boldsymbol{u}}\in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v}),\, I_{\boldsymbol{u}-(-\boldsymbol{u}')}\in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c} \partial^{(-\boldsymbol{u}')_+}(a_{\boldsymbol{u}}(s)x^{\boldsymbol{v}+s\boldsymbol{b}+\boldsymbol{u}}).
\end{aligned}
$$

Suppose that $I_{\boldsymbol{u}} \in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})$ and $J := I_{\boldsymbol{u}-\boldsymbol{u}'} \in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c$. Then $u'_j > 0\,(j \in J \setminus I_{\boldsymbol{u}})$, and

$$\partial^{\boldsymbol{u}'_+}(a_{\boldsymbol{u}}(s)x^{\boldsymbol{v}+s\boldsymbol{b}+\boldsymbol{u}}) = a_{\boldsymbol{u}}(s)[\boldsymbol{v}+s\boldsymbol{b}+\boldsymbol{u}]_{\boldsymbol{u}'_+}x^{\boldsymbol{v}+s\boldsymbol{b}+\boldsymbol{u}-\boldsymbol{u}'_+}.$$

By Corollary 5.3,

$$a_{\boldsymbol{u}}(s) = c\frac{b_{I_{\boldsymbol{u}}\setminus I_{\boldsymbol{0}}}}{b_{I_{\boldsymbol{0}}\setminus I_{\boldsymbol{u}}}}s^{|I_{\boldsymbol{u}}|-|I_{\boldsymbol{0}}|} + \text{higher terms},$$

where c is a nonzero constant unrelated to s and b_j's. Hence $s^{|I_{\boldsymbol{0}}|-m}a_{\boldsymbol{u}}(s)$ does not have a pole at $s = 0$.

By Lemma 5.1,

$$[\boldsymbol{v}+s\boldsymbol{b}+\boldsymbol{u}]_{\boldsymbol{u}'_+} = c' b_{J\setminus I_{\boldsymbol{u}}}s^{|J\setminus I_{\boldsymbol{u}}|} + \text{higher terms}.$$

Hence,

$$
\begin{aligned}
\partial^{\boldsymbol{u}'_+}(a_{\boldsymbol{u}}(s)x^{\boldsymbol{v}+s\boldsymbol{b}+\boldsymbol{u}}) &= c\frac{b_{I_{\boldsymbol{u}}\setminus I_{\boldsymbol{0}}}b_{J\setminus I_{\boldsymbol{u}}}}{b_{I_{\boldsymbol{0}}\setminus I_{\boldsymbol{u}}}}s^{|I_{\boldsymbol{u}}|-|I_{\boldsymbol{0}}|}s^{|J\setminus I_{\boldsymbol{u}}|} + \text{higher terms}\\
&= c\frac{b_{I_{\boldsymbol{u}}\setminus I_{\boldsymbol{0}}}b_{J\setminus I_{\boldsymbol{u}}}}{b_{I_{\boldsymbol{0}}\setminus I_{\boldsymbol{u}}}}s^{|I_{\boldsymbol{u}}\cup J|-|I_{\boldsymbol{0}}|} + \text{higher terms}. \qquad (12)
\end{aligned}
$$

Thus each coefficient of $\partial^{\boldsymbol{u}'_+}s^{|I_{\boldsymbol{0}}|-m}(a_{\boldsymbol{u}}(s)x^{\boldsymbol{v}+s\boldsymbol{b}+\boldsymbol{u}})$ has order at least $M-m$ in s, and we have proved the first part of (1). By looking at the coefficient of (12), we have (2).

Note that the starting part of $(\partial_s^{|I_{\boldsymbol{0}}|-m+k}s^{|I_{\boldsymbol{0}}|-m}F_{\boldsymbol{b}}(x,s))_{|s=0}$ is a nonzero multiple of $x^{\boldsymbol{v}}(\log x^{\boldsymbol{b}})^k$ $(k = 0, 1, \ldots, M-|I_{\boldsymbol{0}}|-1)$. Since $\operatorname{rank} L = n-d$, we have the second part of (1). □

Remark 5.5. Since the degrees of logarithmic polynomials in the coefficients of $(\partial_s^j s^{|I_{\boldsymbol{0}}|-m}F_{\boldsymbol{b}}(x,s))_{|s=0}$ are less than or equal to j,

$$(\partial_s^j s^{|I_{\boldsymbol{0}}|-m}F_{\boldsymbol{b}}(x,s))_{|s=0} \qquad (j = |I_{\boldsymbol{0}}|-m, \ldots, M-m-1)$$

in Theorem 5.4 (1) are basic Nilsson series solutions [2, Definition 2.6].

Remark 5.6. In Theorem 5.4, we may replace $\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})$ and $\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c$ by any $N \subseteq \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})$ with $N \ni \mathrm{nsupp}(\boldsymbol{v})$ and $N^c := (\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v}) \cup \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c) \setminus N$. Indeed, the proof of Theorem 5.4 is also valid for N and N^c.

Example 5.7. Let $\boldsymbol{v}$ be a fake exponent with minimal negative support, and $N := \{\mathrm{nsupp}(\boldsymbol{v})\}$.

Then for any $J \in N^c$ we have $I_{\boldsymbol{0}} \cup J \supsetneq I_{\boldsymbol{0}}$. Hence $M > |I_{\boldsymbol{0}}| = m$, and we see that $\phi_{\boldsymbol{v}}$ is a solution with exponent $\boldsymbol{v}$ by Theorem 5.4 for N.

Example 5.8 (Continuation of Example 4.6)**.** Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$, and take $\boldsymbol{w}$ as before.

Let $\boldsymbol{\beta} = \begin{pmatrix} 10 \\ 8 \end{pmatrix}$, and $\boldsymbol{v} := (0, 12, -2)^T$.

Then

$$\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v}) = \{\{2\}, \{3\}, \emptyset\}, \qquad \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c = \{\{1, 3\}\}.$$

Hence

$$M = 2, \quad m = 0, \quad I_0 = \{3\},$$

and by Theorem 5.4 $(\partial_s^j s F_{\boldsymbol{b}}(x, s))_{|s=0}$ $(j = 0, 1)$ are solutions. Here

$$(\partial_s^0 s F_{\boldsymbol{b}}(x, s))_{|s=0} = c\phi_{\boldsymbol{v}'},$$

where $\boldsymbol{b} = (1, -2, 1)^T$, $\boldsymbol{v'} = (2, 8, 0)^T = \boldsymbol{v} + 2\boldsymbol{b}$, and

$$c = (s a_{2\boldsymbol{b}}(s))_{|s=0} = (\frac{s[12 - 2s]_4}{[s+2]_2 [s]_2})_{|s=0} = -5940.$$

$$\begin{aligned} (\partial_s s F_{\boldsymbol{b}}(x, s))_{|s=0} &= \sum_{k \neq 2,3,\ldots,6} a_{k\boldsymbol{b}}(0) x^{\boldsymbol{v}+k\boldsymbol{b}} + \sum_{k=2}^{6} \partial_s (s a_{k\boldsymbol{b}}(s))_{|s=0} x^{\boldsymbol{v}+k\boldsymbol{b}} \\ &\quad + \sum_{k=2}^{6} (s a_{k\boldsymbol{b}}(s))_{|s=0} (\log x^{\boldsymbol{b}}) x^{\boldsymbol{v}+k\boldsymbol{b}}. \end{aligned}$$

Here note that $a_{k\boldsymbol{b}}(s)$ has a pole of order 1 at $s = 0$ for $k = 2, 3, 4, 5, 6$ by Corollary 5.3.

Note that the $\psi(0, x)$ in [10, Example 3.5.3] has a typo.

Example 5.9 (Continuation of Example 4.7)**.** Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$, and take $\boldsymbol{w}$ as before.

Let $\boldsymbol{\beta} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$, and $\boldsymbol{v} := \boldsymbol{v}_{1,2} = (-1,-1,0,0)^T$. Then

$$\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v}) = \{\{2\},\{3\},\{2,3\},\{1,2\} = I_{\boldsymbol{0}}\},$$

$$\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c = \{\{1,3\},\{2,4\},\{1,4\},\{1,3,4\},\{1,2,4\}\}.$$

Hence $M = 2, m = 1, I_{\boldsymbol{0}} = \{1,2\}$. By Theorem 5.4 (1), $(sF_{\boldsymbol{b}}(x,s))_{s=0}$ is a solution for any $\boldsymbol{b} \in L$ with $b_1, b_2 \neq 0$. We have $\boldsymbol{v}_2 = \boldsymbol{v} + (2,-3,1,0)^T$ and $\boldsymbol{v}_3 = \boldsymbol{v} + (1,1,-7,5)^T$. Hence by Corollary 5.3

$$\begin{aligned} (sF_{\boldsymbol{b}}(x,s))_{s=0} &= (sa_{(2,-3,1,0)^T}(s))_{s=0}\phi_{\boldsymbol{v}_2} + (sa_{(1,1,-7,5)^T}(s))_{s=0}\phi_{\boldsymbol{v}_3} \\ &= -\frac{6}{b_1}\phi_{\boldsymbol{v}_2} + \frac{6b_3}{b_1 b_2}\phi_{\boldsymbol{v}_3}. \end{aligned}$$

The (I,J)'s with the condition of Theorem 5.4 (2) are

$$(\{3\},\{1,3\}), \quad (\{2\},\{2,4\}).$$

Hence if $\boldsymbol{b}^{(1)}, \boldsymbol{b}^{(2)} \in L$ satisfy

$$\sum_{i=1}^{2} \frac{b_3^{(i)} b_1^{(i)}}{b_1^{(i)} b_2^{(i)}} = \sum_{i=1}^{2} \frac{b_3^{(i)}}{b_2^{(i)}} = 0, \quad \sum_{i=1}^{2} \frac{b_4^{(i)} b_1^{(i)}}{b_1^{(i)}} = \sum_{i=1}^{2} b_4^{(i)} = 0,$$

then $(\partial_s \sum_{i=1}^2 sF_{\boldsymbol{b}^{(i)}})_{|s=0}$ is a solution. We see that

$$\boldsymbol{b}^{(1)} = (1,-2,2,-1)^T, \quad \boldsymbol{b}^{(2)} = (1,-1,-1,1)^T$$

would do. Hence $\boldsymbol{v}_{1,2}$ is an exponent, and

$$\begin{aligned} &(\partial_s \sum_{i=1}^{2} sF_{\boldsymbol{b}^{(i)}}(x,s))_{|s=0} \\ = \ &(\partial_s(\sum_{i=1}^{2} \sum_{\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u}) \in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})} sa_{\boldsymbol{u}}^{(i)}(s) x^{\boldsymbol{v}+s\boldsymbol{b}^{(i)}+\boldsymbol{u}})_{|s=0} \\ = \ &\sum_{\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})=\{1,2\},\{2,3\}} (a_{\boldsymbol{u}}^{(1)}(0) + a_{\boldsymbol{u}}^{(2)}(0)) x^{\boldsymbol{v}+\boldsymbol{u}} \\ &+ \sum_{\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})=\{2\},\{3\}} (\partial_s(sa_{\boldsymbol{u}}^{(1)}(s) + sa_{\boldsymbol{u}}^{(2)}(s)))_{|s=0} x^{\boldsymbol{v}+\boldsymbol{u}} \\ &+ \sum_{\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})=\{2\},\{3\}} (sa_{\boldsymbol{u}}^{(1)}(s))_{|s=0} (\log x^{\boldsymbol{b}^{(1)}}) x^{\boldsymbol{v}+\boldsymbol{u}} \\ &+ \sum_{\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})=\{2\},\{3\}} (sa_{\boldsymbol{u}}^{(2)}(s))_{|s=0} (\log x^{\boldsymbol{b}^{(2)}}) x^{\boldsymbol{v}+\boldsymbol{u}}, \end{aligned}$$

where

$$a_{\boldsymbol{u}}^{(i)}(s) = \frac{[\boldsymbol{v} + s\boldsymbol{b}^{(i)}]_{\boldsymbol{u}_-}}{[\boldsymbol{v} + s\boldsymbol{b}^{(i)} + \boldsymbol{u}]_{\boldsymbol{u}_+}}.$$

6. Method 2

In this section, we consider a Frobenius's method by perturbing an exponent with several vectors in L.

Let $\boldsymbol{v}$ be a fake exponent, and let $\boldsymbol{b}^{(1)}, \boldsymbol{b}^{(2)}, \dots, \boldsymbol{b}^{(l)} \in L$ be non-zero. We suppose that for any $i \in \mathrm{nsupp}(\boldsymbol{v})$ there exists j such that $b_i^{(j)} \neq 0$.

For such $\boldsymbol{b}^{(1)}, \boldsymbol{b}^{(2)}, \dots, \boldsymbol{b}^{(l)} \in L$, set

$$F_{\boldsymbol{b}^{(1)},\boldsymbol{b}^{(2)},\dots,\boldsymbol{b}^{(l)}}(x, \boldsymbol{s}) := \sum_{\boldsymbol{u}\in L,\, \mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})\in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})} a_{\boldsymbol{u}}(\boldsymbol{s}) x^{\boldsymbol{v}+\boldsymbol{s}\boldsymbol{b}+\boldsymbol{u}},$$

where

$$\boldsymbol{s} = (s_1, s_2, \dots, s_l), \quad \boldsymbol{s}\boldsymbol{b} = s_1\boldsymbol{b}^{(1)} + s_2\boldsymbol{b}^{(2)} + \cdots + s_l\boldsymbol{b}^{(l)},$$

$$a_{\boldsymbol{u}}(\boldsymbol{s}) = \frac{[\boldsymbol{v} + \boldsymbol{s}\boldsymbol{b}]_{\boldsymbol{u}_-}}{[\boldsymbol{v} + \boldsymbol{s}\boldsymbol{b} + \boldsymbol{u}]_{\boldsymbol{u}_+}}.$$

Similarly to Lemma 5.1 and Corollary 5.3, we have the following lemma.

Lemma 6.1. (1)

$$a_{\boldsymbol{u}}(\boldsymbol{s}) = c\frac{\prod_{i\in \mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})\setminus \mathrm{nsupp}(\boldsymbol{v})}(s_1 b_i^{(1)} + s_2 b_i^{(2)} + \cdots + s_l b_i^{(l)}) + \textit{higher terms}}{\prod_{j\in \mathrm{nsupp}(\boldsymbol{v})\setminus \mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})}(s_1 b_j^{(1)} + s_2 b_j^{(2)} + \cdots + s_l b_j^{(l)}) + \textit{higher terms}}.$$

(2)

$$[\boldsymbol{v} + \boldsymbol{s}\boldsymbol{b} + \boldsymbol{u}]_{\boldsymbol{u}'_+} = c \prod_{i\in \mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u}-\boldsymbol{u}')\setminus \mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})} (s_1 b_i^{(1)} + s_2 b_i^{(2)} + \cdots + s_l b_i^{(l)}) + \textit{higher terms.}$$

Here c is a nonzero constant unrelated to $\boldsymbol{s}$ and $\boldsymbol{b}^{(k)}$'s.

Theorem 6.2. *Put $K := \cap_{I\in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})} I$. Set*

$$\widetilde{F}(x, \boldsymbol{s}) := \prod_{i\in I_{\boldsymbol{0}}\setminus K} (s_1 b_i^{(1)} + s_2 b_i^{(2)} + \cdots + s_l b_i^{(l)}) \cdot F_{\boldsymbol{b}^{(1)},\boldsymbol{b}^{(2)},\dots,\boldsymbol{b}^{(l)}}(x, \boldsymbol{s}).$$

Let M be the one in Theorem 5.4. Then

(1) $(\partial_{s_1}^{p_1} \cdots \partial_{s_l}^{p_l} \widetilde{F}(x, \boldsymbol{s}))_{|\boldsymbol{s}=\boldsymbol{0}}$ *are solutions to* $M_A(\boldsymbol{\beta})$ *for* $\sum_{k=1}^{l} p_k < M - |K|$.

(2) *Suppose that* $\sum_{k=1}^{l} p_k = M - |K|$.
If

$$\sum_{\coprod_{j=1}^{l} L_j = I\cup J\setminus K;\, |L_1|=p_1,\dots,|L_l|=p_l} \prod_{j=1}^{l} b_{L_j}^{(j)} = 0$$

for all $I \in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})$ *and* $J \in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c$ *with* $|I\cup J| = M$, *then* $(\partial_{s_1}^{p_1}\cdots\partial_{s_l}^{p_l}\widetilde{F}(x,\boldsymbol{s}))_{|\boldsymbol{s}=\boldsymbol{0}}$ *is also a solution to* $M_A(\boldsymbol{\beta})$.

Proof. By Lemma 6.1, we see that $\prod_{i\in I_{\boldsymbol{0}}\setminus K}(s_1b_i^{(1)} + s_2b_i^{(2)} + \cdots + s_lb_i^{(l)})\cdot a_{\boldsymbol{u}}(\boldsymbol{s})$ does not have a pole at $\boldsymbol{s} = \boldsymbol{0}$ for any $\boldsymbol{u}$ with $I_{\boldsymbol{u}} \in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})$.

Let $\boldsymbol{u'} \in L$, $\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u}) = I \in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})$, and $\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u}-\boldsymbol{u'}) = J \in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c$. By Lemma 6.1, the part of the lowest total degree in

$$\prod_{i\in I_{\boldsymbol{0}}\setminus K}(s_1b_i^{(1)} + s_2b_i^{(2)} + \cdots + s_lb_i^{(l)})\partial^{\boldsymbol{u'}_+}a_{\boldsymbol{u}}(\boldsymbol{s})x^{\boldsymbol{v}+\boldsymbol{s}\boldsymbol{b}+\boldsymbol{u}}$$

is a nonzero constant multiple of

$$\begin{aligned}(13)\qquad &\prod_{i\in(I_{\boldsymbol{0}}\setminus K)\setminus(I_{\boldsymbol{0}}\setminus I)}(s_1b_i^{(1)} + s_2b_i^{(2)} + \cdots + s_lb_i^{(l)})\\ &\times\prod_{i\in J\setminus I}(s_1b_i^{(1)} + s_2b_i^{(2)} + \cdots + s_lb_i^{(l)})\\ &\times\prod_{i\in I\setminus I_{\boldsymbol{0}}}(s_1b_i^{(1)} + s_2b_i^{(2)} + \cdots + s_lb_i^{(l)}).\end{aligned}$$

We have $(I_{\boldsymbol{0}}\setminus K)\setminus(I_{\boldsymbol{0}}\setminus I) = I_{\boldsymbol{0}}\cap I\setminus K$. Since the three sets $I_{\boldsymbol{0}}\cap I\setminus K$, $J\setminus I$, and $I\setminus I_{\boldsymbol{0}}$ are disjoint and their union equals $I\cup J\setminus K$, we see that (13) equals

$$\begin{aligned}&\prod_{i\in I\cup J\setminus K}(s_1b_i^{(1)} + s_2b_i^{(2)} + \cdots + s_lb_i^{(l)})\\ &= \sum_{I\cup J\setminus K=\coprod_{j=1}^{l} L_j}\prod_{j=1}^{l}\prod_{i\in L_j} s_jb_i^{(j)}\\ &= \sum_{\sum_{j=1}^{l} p_j=|I\cup J\setminus K|}\ \sum_{|L_1|=p_1,\dots,|L_l|=p_l}\prod_{j=1}^{l} b_{L_j}^{(j)} s_1^{p_1}s_2^{p_2}\cdots s_l^{p_l}.\end{aligned}$$

Hence, as in the proof of Theorem 5.4, we have (1).

Furthermore, if the assumption of (2) is satisfied, then the $s_1^{p_1}s_2^{p_2}\cdots s_l^{p_l}$ part of each $\partial^{\boldsymbol{u'}_+}\widetilde{F}(x,\boldsymbol{s})$ is zero. Hence we have (2). □

Remark 6.3. The proof of Theorem 6.2 is again valid for any $N \subseteq \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})$ with $N \ni \mathrm{nsupp}(\boldsymbol{v})$ and $N^c := (\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v}) \cup \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c) \setminus N$ in place of $\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})$ and $\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c$.

Example 6.4 (Continuation of Example 4.8)**.** Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 1 & 0 \end{pmatrix}$, and take $\boldsymbol{w}$ as before.

Let $\boldsymbol{\beta} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Then $\boldsymbol{v} := (0,0,0,0,1)^T$ is the unique fake exponent.

$$\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v}) = \{\emptyset, \{5\}\},$$

$$\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c = \{\{1,3\},\{2,4\},\{1,3,5\},\{2,4,5\},\{1,2,3,4\}\},$$

Hence $M = 2$, $m = 0$, $I_{\boldsymbol{0}} = \emptyset$, and

$$(\partial_s^j F_{\boldsymbol{b}}(x,s))_{|s=0} \quad (j = 0,1)$$

are solutions for any $\boldsymbol{b} \in L$ by Theorem 5.4.

Let $\boldsymbol{b}^{(1)}, \boldsymbol{b}^{(2)}$ be the column vectors of B in Example 4.8. We have $K = \emptyset$. The $(I \cup J \setminus K)$'s with $|I \cup J| = M$ are

$$\{1,3\}, \{2,4\}.$$

Let $p_1 = p_2 = 1$. Then

$$\begin{aligned} b_1^{(1)} b_3^{(2)} + b_3^{(1)} b_1^{(2)} &= 1 \cdot 0 + 1 \cdot 0 = 0 \quad (I \cup J \setminus K = \{1,3\}), \\ b_2^{(1)} b_4^{(2)} + b_4^{(1)} b_2^{(2)} &= 0 \cdot 1 + 0 \cdot 1 = 0 \quad (I \cup J \setminus K = \{2,4\}). \end{aligned}$$

Hence, by Theorem 6.2 (2), $(\partial_{s_1}\partial_{s_2} F_{\boldsymbol{b}_1,\boldsymbol{b}_2}(x,\boldsymbol{s}))_{|\boldsymbol{s}=\boldsymbol{0}}$ is also a solution.

Note that $\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u}) = \emptyset \Leftrightarrow \boldsymbol{u} = \boldsymbol{0}$, and $a_{\boldsymbol{0}}(\boldsymbol{s}) = 1$. We have

$$F_{\boldsymbol{b}_1,\boldsymbol{b}_2}(x,\boldsymbol{0}) = x_5,$$

$$\begin{aligned} (\partial_{s_1} F_{\boldsymbol{b}_1,\boldsymbol{b}_2}(x,\boldsymbol{s}))_{|\boldsymbol{s}=\boldsymbol{0}} &= (\partial_s F_{\boldsymbol{b}_1}(x,s))_{|s=0} \\ &= x_5(\log x^{\boldsymbol{b}_1}) + \sum_{\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})=\{5\}} (\partial_{s_1} a_{\boldsymbol{u}})(0) x^{\boldsymbol{v}+\boldsymbol{u}}, \end{aligned}$$

$$\begin{aligned} (\partial_{s_2} F_{\boldsymbol{b}_1,\boldsymbol{b}_2}(x,\boldsymbol{s}))_{|\boldsymbol{s}=\boldsymbol{0}} &= (\partial_s F_{\boldsymbol{b}_2}(x,s))_{|s=0} \\ &= x_5(\log x^{\boldsymbol{b}_2}) + \sum_{\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})=\{5\}} (\partial_{s_2} a_{\boldsymbol{u}})(0) x^{\boldsymbol{v}+\boldsymbol{u}}, \end{aligned}$$

$$
\begin{aligned}
(\partial_{s_1}\partial_{s_2}F_{\boldsymbol{b}_1,\boldsymbol{b}_2}(x,\boldsymbol{s}))_{|\boldsymbol{s}=\boldsymbol{0}} &= x_5(\log x^{\boldsymbol{b}_1})(\log x^{\boldsymbol{b}_2}) \\
&+ \sum_{\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})=\{5\}} (\partial_{s_1}a_{\boldsymbol{u}})(0)(\log x^{\boldsymbol{b}_2})x^{\boldsymbol{v}+\boldsymbol{u}} \\
&+ \sum_{\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})=\{5\}} (\partial_{s_2}a_{\boldsymbol{u}})(0)(\log x^{\boldsymbol{b}_1})x^{\boldsymbol{v}+\boldsymbol{u}} \\
&+ \sum_{\mathrm{nsupp}(\boldsymbol{v}+\boldsymbol{u})=\{5\}} (\partial_{s_1}\partial_{s_2}a_{\boldsymbol{u}})(0)x^{\boldsymbol{v}+\boldsymbol{u}}.
\end{aligned}
$$

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Department of Mathematics, Faculty of Science

Hokkaido University
Sapporo, 060-0810, Japan
e-mail: saito@math.sci.hokudai.ac.jp