THE MOTION OF WEAKLY INTERACTING LOCALIZED PATTERNS FOR REACTION-DIFFUSION SYSTEMS WITH NONLOCAL EFFECT

SHIN-ICHIRO EI AND HIROSHI ISHII

Abstract. In this paper, we analyze the interaction of localized patterns such as traveling wave solutions for reaction-diffusion systems with nonlocal effect in one space dimension. We consider the case that a nonlocal effect is given by the convolution with a suitable integral kernel. At first, we deduce the equation describing the movement of interacting localized patterns in a mathematically rigorous way, assuming that there exists a linearly stable localized solution for general reaction-diffusion systems with nonlocal effect. When the distances between localized patterns are sufficiently large, the motion of localized patterns can be reduced to the equation for the distances between them. Finally, using this equation, we analyze the interaction of front solutions to some nonlocal scalar equation. Under some assumptions, we can show that the front solutions are interacting attractively for a large class of integral kernels.

1. Introduction

Pattern formation problem is one of the most interesting and attractive topics in natural science. There have been many mathematical models proposed for the theoretical understanding of the mechanisms. Among them, mathematical models in the type of reaction-diffusion systems have been proposed in order to describe spatio-temporal patterns in dissipative systems such as biology and chemistry [23, 25, 32]. In fact, reaction-diffusion type model equations are nicely fit to express Turing instability, which gives one of the most basic theoretical concepts as the mechanism of a spatial pattern formation in biology. Nowadays, reaction-diffusion type model equations are applied to so many kinds of phenomena arising in dissipative systems (e.g. [18, 20, 23, 24, 26]).

While reaction-diffusion systems can describe the structure with diffusive motions and local reactions, it has been known that there are nonlocal interactions e.g. by cell projections [27], contact inhibitions [7, 29] and so on. Such mechanisms with nonlocal interactions are described by the convolutions with suitable integral kernels.

On the other hand, as stated in [15, 22], some mechanisms which have been described by reaction-diffusion systems can be expressed by equations with nonlocal terms of convolution types. In [15], it was shown that activator-inhibitor systems written in reaction-diffusion systems which possess the mechanism of Turing instability are essentially reduced to the type of the equation

$$u_t = du_{xx} + f(u, K * u) \tag{1.1}$$

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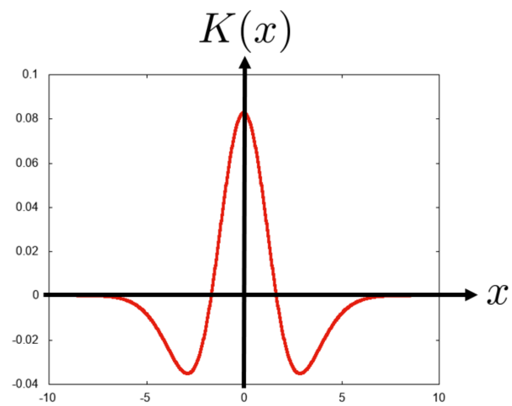

FIGURE 1. Integral kernel K with the Mexican hat profile on $\mathbb{R}$.

with an integral kernel $K(x)$ of the Mexican hat profile (Figure 1) and a function $f(u,v)$, where $K * u$ denotes the convolution defined by $(K * u)(t,x) := \int_{\mathbb{R}} K(x-y)u(t,y)dy$. This means (1.1) includes the mechanism of the Turing instability. Kondo [22] also showed that more complicated patterns which cannot be reproduced by two component reaction-diffusion systems can be easily done by the equations of the type of (1.1) with suitable kernels. In [15, 22, 28, 30], they pointed out that the mechanism of Turing instability related to the kernel with the Mexican hat profile. Furthermore, it is known that the diffusion itself can be expressed by the convolution [1, 5]. Thus, equations as (1.1) are the generalizations of reaction-diffusion equations in some sense.

In this paper, the equation in the type of

$$u_t = du_{xx} + K * u + f(u) \tag{1.2}$$

is mainly treated as one kind of equations (1.1), which appears in many fields such as neuro-science [25], dispersal motion of cells and organisms [19], optical illusion [31] and so on.

For the analysis of the equation (1.2), there have been many works [3, 4, 5, 6, 8, 9, 10, 11, 14, 28, 34]. In particular, the existence and the stability of pulses and fronts as localized solutions have been extensively investigated [3, 4, 5, 6, 8, 9, 10]. In their works, a single pulse and/or a single front solution were constructed together with the consideration of their stability.

From the pattern formation point of view, not only single localized patterns but also their interactions are important. For example, it is shown in Section 3 that multiple single front solutions of (1.2) are interacting attractively for a large class of integral kernels, which means the coarsening process of front localized patterns in time.

In this paper, we treat the following more general reaction-diffusion systems with non-local terms than (1.2):

$$\boldsymbol{u}_t = D\boldsymbol{u}_{xx} + \boldsymbol{K} * \boldsymbol{u} + F(\boldsymbol{u}),\ t > 0,\ x \in \mathbb{R}, \tag{1.3}$$

where $\boldsymbol{u} = \boldsymbol{u}(t,x) = {}^t(u_1(t,x), u_2(t,x), \cdots, u_n(t,x)) \in \mathbb{R}^n$, $D = diag(d_1, d_2, ..., d_n)$ $(d_j \geq 0)$, $F : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth nonlinear function, $\boldsymbol{K} = \boldsymbol{K}(x) \in \mathbb{R}^{n \times n}$,

$$
\begin{aligned}
(\boldsymbol{K} * \boldsymbol{u})(t,x) &:= \begin{pmatrix} K_{1,1} & K_{1,2} & \cdots & \cdots & K_{1,n} \\ K_{2,1} & K_{2,2} & \cdots & \cdots & K_{2,n} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ K_{n,1} & K_{n,2} & \cdots & \cdots & K_{n,n} \end{pmatrix} * \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_n \end{pmatrix} (t,x) \\
&= \begin{pmatrix} \displaystyle\sum_{k=1}^{n} (K_{1,k} * u_k)(t,x) \\ \displaystyle\sum_{k=1}^{n} (K_{2,k} * u_k)(t,x) \\ \vdots \\ \vdots \\ \displaystyle\sum_{k=1}^{n} (K_{n,k} * u_k)(t,x) \end{pmatrix},
\end{aligned}
$$

the $*$ denotes the convolution with respect to the spatial variable, in which the integral kernels $K_{j,k}$ are the functions satisfying

$$
\begin{cases} K_{j,k} \in C(\mathbb{R}) \cap L^1(\mathbb{R}), \ K_{j,k}(x) = K_{j,k}(-x) \ (x \in \mathbb{R}), \\ \forall \lambda \in \mathbb{R}, \quad \int_{\mathbb{R}} |K_{j,k}(y)| e^{\lambda y} dy < \infty. \end{cases} \tag{1.4}
$$

A typical example of $K_{j,k}$ is $K_{j,k}(x) = e^{-x^2}$. The purpose of this paper is to give a mathematical criteria for the interaction between multiple single pulse or single front solutions for (1.3).

We set

$$
A(\lambda) := \begin{pmatrix} \tilde{K}_{1,1}(\lambda) & \tilde{K}_{1,2}(\lambda) & \cdots & \cdots & \tilde{K}_{1,n}(\lambda) \\ \tilde{K}_{2,1}(\lambda) & \tilde{K}_{2,2}(\lambda) & \cdots & \cdots & \tilde{K}_{2,n}(\lambda) \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ \tilde{K}_{n,1}(\lambda) & \tilde{K}_{n,2}(\lambda) & \cdots & \cdots & \tilde{K}_{n,n}(\lambda) \end{pmatrix},
$$

where $\tilde{K}_{j,k}(\lambda) := \int_{\mathbb{R}} K_{j,k}(y) e^{\lambda y} dy$ for $j, \ k = 1, 2, \ldots, N$.

Hypothesis 1.1. *We suppose for* (1.3) *that:*

H1) [Existence of stable equilibria]
There exist linearly stable equilibria P^- and P^+ in the ODE

$$
\boldsymbol{u}_t = A(0)\boldsymbol{u} + F(\boldsymbol{u}).
$$

H2) [Existence of traveling wave solution]
There exist a constant θ, positive constants α, β and a function $P(z)$ satisfying the equation

$$\begin{cases} \mathbf{0} = DP_{zz} - \theta P_z + \boldsymbol{K} * P + F(P) \quad (z \in \mathbb{R}), \\ |P(z) - P^+| \leq O(e^{-\alpha z}) \ (z \to +\infty), \\ |P(z) - P^-| \leq O(e^{\beta z}) \ (z \to -\infty). \end{cases} \tag{1.5}$$

H3) [Linearized stability of traveling wave solution]
Let a differential operator L be

$$L\boldsymbol{v} = D\boldsymbol{v}_{zz} - \theta \boldsymbol{v}_z + \boldsymbol{K} * \boldsymbol{v} + F'(P(z))\boldsymbol{v},$$

for $\boldsymbol{v} \in \{H^2(\mathbb{R})\}^n$, where the domain $\mathcal{D}(L)$ is defined as

$$\mathcal{D}(L) = \{\boldsymbol{v} = {}^t(v_1, v_2, \cdots, v_n) \in \{L^2(\mathbb{R})\}^n \mid \theta \boldsymbol{v} \in \{H^1(\mathbb{R})\}^n, \ D\boldsymbol{v} \in \{H^2(\mathbb{R})\}^n\}.$$

This means

$$v_j \in \begin{cases} H^2(\mathbb{R}) & (if \ d_j > 0), \\ H^1(\mathbb{R}) & (if \ d_j = 0 \ and \ \theta \neq 0), \\ L^2(\mathbb{R}) & (if \ d_j = 0 \ and \ \theta = 0) \end{cases}$$

for any $j = 1, 2, \ldots, n$, when $\boldsymbol{v} \in \mathcal{D}(L)$. Then, the spectrum $\Sigma(L)$ of L is given by $\Sigma(L) = \Sigma_0 \cup \{0\}$, where 0 is a simple eigenvalue with a eigenfunction P_z and there exists a positive constant $\rho_0 > 0$ such that $\Sigma_0 \subset \{z \in \mathbb{C} \mid \Re(z) < -\rho_0\}$. Here, $\Re(z)$ denotes the real part of z.

We call $P(z)$ satisfying the Hypothesis 1.1 for a constant θ "(linearly) stable traveling wave solution with velocity θ". Many models of reaction-diffusion systems and nonlocal equations have linearly stable traveling wave solutions in this sense [4, 12, 17, 21, 33, 34].

Transforming (1.3) by $z := x + \theta t$, we have

$$\boldsymbol{u}_t = D\boldsymbol{u}_{zz} - \theta \boldsymbol{u}_z + \boldsymbol{K} * \boldsymbol{u} + F(\boldsymbol{u}) =: \mathcal{L}(\boldsymbol{u}). \tag{1.6}$$

We note that the stable traveling wave solution $P(z)$ is a stable stationary solution of (1.6). Throughout this paper, we call $P(z)$ "pulse solution" when $P^- = P^+$ and "front solution" when $P^- \neq P^+$, respectively.

The purpose of this paper is to give a general criterion for (1.3) to analyze their interaction together with applications under the above assumptions about the existence and the stability of a single traveling wave solution.

The organization of the paper as follows: in Section 2, we will state main results and its proofs for (1.3). An application of it to a nonlocal scalar equation (1.2) will be in Section 3, in which it is shown that front solutions interact attractively for fairly wide class of integral kernels. Finally, in Section 4, we will state future works related to the results of this paper.

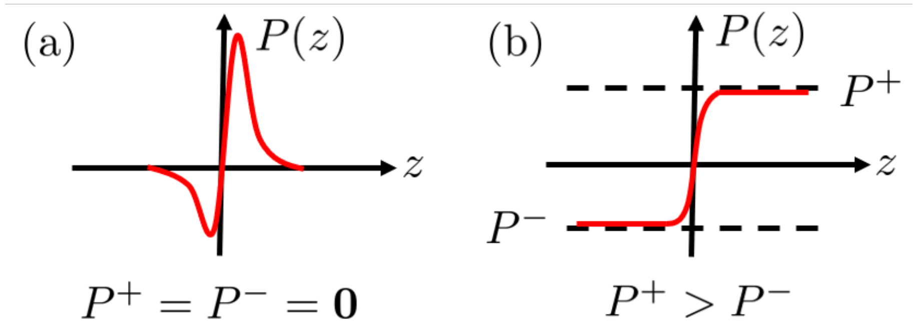

FIGURE 2. The image of localized pattern. (a) Pulse solution when $P^+ = P^- = \mathbf{0}$. (b) Front solution.

2. Main results

2.1. **Interaction of pulse solutions.** In this subsection, we consider the interaction of pulse solutions. Suppose that $P(z)$ is a stable pulse solution of (1.3) with velocity θ. Then, we can assume that $P^- = P^+ = \mathbf{0} = {}^t(0, 0, \cdots, 0) \in \mathbb{R}^n$ without loss of generality. Fixing an arbitrarily natural number N?諧e consider the interaction of $N+1$ pulse solutions. We define

$$P(z; \boldsymbol{h}) := P(z) + P(z - z_1) + \cdots + P(z - z_N),$$

where $\boldsymbol{h} = (h_1, h_2, ..., h_N)$ for $h_j > 0$, $z_0 = 0$ and

$$z_j = z_j(\boldsymbol{h}) = z_{j-1} + h_j \quad (j = 1, 2, ..., N).$$

Define the set

$$\mathcal{M}(h^*) = \{\Xi(l)P(z; \boldsymbol{h}) \mid l \in \mathbb{R},\ \min \boldsymbol{h} > h^*\},$$

where $\Xi(l)$ is translation operator defined as $(\Xi(l)\boldsymbol{v})(z) = \boldsymbol{v}(z - l)$ for $\boldsymbol{v} \in \{L^2(\mathbb{R})\}^n$.

Moreover, we set the quantity

$$\delta(\boldsymbol{h}) = \sup_{z \in \mathbb{R}} |\mathcal{L}(P(z; \boldsymbol{h}))|.$$

We note that $\delta(\boldsymbol{h})$ is sufficiently small as long as $\min \boldsymbol{h}$ is large enough. In fact, $\delta(\boldsymbol{h})$ satisfies $\delta(\boldsymbol{h}) \to 0$ as $\min \boldsymbol{h} \to +\infty$, since $\mathcal{L}(P(z - z_j)) = \mathbf{0}$ and $\mathcal{L}(\mathbf{0}) = \mathbf{0}$ for $j = 0, 1, \ldots, N$.

Furthermore, define functions

$$H_j(\boldsymbol{h}) = \langle\, \mathcal{L}(P(\cdot + z_j; \boldsymbol{h})), \Phi^*(\cdot)\, \rangle_{L^2}$$

for $j = 0, 1, ..., N$, where Φ^* is an eigenfunction corresponding to 0 eigenvalue of the adjoint operator L^* of L and normalized by $\langle\, P_z, \Phi^*\, \rangle_{L^2} = 1$. We note that the domain $\mathcal{D}(L^*)$ is equal to $\mathcal{D}(L)$ and Φ^* satisfies

$$L^*\Phi^* := D\Phi^*_{zz} + \theta\Phi^*_z + {}^t\boldsymbol{K} * \Phi^* + {}^tF'(P(z))\Phi^* = 0. \tag{2.1}$$

By applying the same line of argument in [13] based on the theory of infinite dimensional dynamical systems, we can obtain the following results.

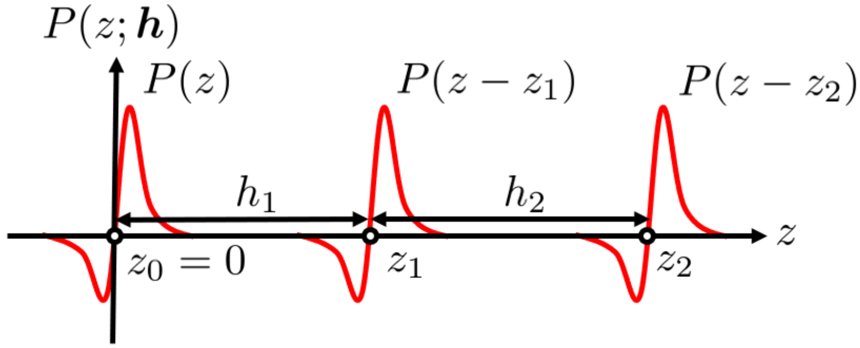

FIGURE 3. The image of $P(z;\boldsymbol{h})$ when $N=2$.

Theorem 2.1. [13]
There exist positive constants h^*, C_0 and a neighborhood $U=U(h^*)$ of $M(h^*)$ in $\{H^2(\mathbb{R})\}^n$ such that if $\boldsymbol{u}(0,\cdot)\in U$, then there exist functions $l(t)\in\mathbb{R}$ and $\boldsymbol{h}(t)\in\mathbb{R}^N$ such that

$$\|\boldsymbol{u}(t,\cdot)-\Xi(l(t))P(\cdot;\boldsymbol{h}(t))\|_\infty\le C_0\delta(\boldsymbol{h}(t)) \tag{2.2}$$

holds as long as $\min\boldsymbol{h}(t)>h^*$, where $\boldsymbol{u}(t,z)$ is a solution of (1.6) and $\|\cdot\|_\infty$ is the sup-norm on $\mathbb{R}$. Functions $l(t)\in\mathbb{R}$ and $\boldsymbol{h}(t)\in\mathbb{R}^N$ satisfy

$$\dot{\boldsymbol{h}} = \boldsymbol{H}(\boldsymbol{h})+O(\delta^2), \tag{2.3}$$
$$\dot{l} = -H_0(\boldsymbol{h})+O(\delta^2), \tag{2.4}$$

where $\delta=\delta(\boldsymbol{h}(t))$ and $\boldsymbol{H}=(H_0-H_1,H_1-H_2,\cdots,H_{N-1}-H_N)$.

Theorem 2.2. [13]
Suppose all of the elements d_j of D are positive. Then, there exist positive constants C_0, C_1 and h^* such that if

$$\dot{\boldsymbol{h}}=\boldsymbol{H}(\boldsymbol{h}) \tag{2.5}$$

has an equilblium $\overline{\boldsymbol{h}}$ satisfying $\min\overline{\boldsymbol{h}}>h^*$ and the set of eigenvalues $\Sigma(\boldsymbol{H}'(\overline{\boldsymbol{h}}))\subset\{z\in\mathbb{C}\mid\Re(z)<-C_0\delta(\overline{\boldsymbol{h}})\}$, there exists a stable traveling wave solution $\overline{P}(z+\overline{\theta}t)$ of (1.3) such that

$$\|\overline{P}(\cdot)-P(\cdot;\overline{\boldsymbol{h}})\|_\infty\le C_1\delta(\overline{\boldsymbol{h}})$$

and $\overline{\theta}=H_0(\overline{\boldsymbol{h}})+O(\delta^2(\overline{\boldsymbol{h}}))$. Here, $\boldsymbol{H}'(\overline{\boldsymbol{h}})$ denotes the linearized matrix of $\boldsymbol{H}$ with respect to $\overline{\boldsymbol{h}}$.

If (2.5) has an equilblium $\underline{\boldsymbol{h}}$ such that $\min\underline{\boldsymbol{h}}>h^*$ and the set of eigenvalues $\Sigma(H'(\underline{\boldsymbol{h}}))\subset\{z\in\mathbb{C}\mid\Re(z)<-C_0\delta(\underline{\boldsymbol{h}})\}\cup\{z\in\mathbb{C}\mid\Re(z)>C_0\delta(\underline{\boldsymbol{h}})\}$ and at least one eigenvalue of $H'(\underline{\boldsymbol{h}})$ is in $\{z\in\mathbb{C}\mid\Re(z)>C_0\delta(\underline{\boldsymbol{h}})\}$, there exists an unstable traveling wave solution $\underline{P}(z+\underline{\theta}t)$ of (1.3) such that

$$\|\underline{P}(\cdot)-P(\cdot;\underline{\boldsymbol{h}})\|_\infty\le C_1\delta(\underline{\boldsymbol{h}})$$

and $\underline{\theta}=H_0(\underline{\boldsymbol{h}})+O(\delta^2(\underline{\boldsymbol{h}}))$.

In [13], he constructed an attractive local invariant manifold giving the dynamics of interacting localized patterns in the case of Reaction-diffusion systems. In its proof, he used integral manifold theory. The proof of [13] can be also applied to reaction diffusion systems with perturbations given by bounded operators of $\{L^2(\mathbb{R})\}^n$. Now, the nonlocal term $\boldsymbol{K} * \boldsymbol{u}$ is bounded operator in $\{L^2(\mathbb{R})\}^n$. Therefore, we can extend theorems in [13].

From Theorem 2.1, when the distances between localized patterns are sufficiently large, the motion of localized patterns can be reduced to the equation (2.3) for the distances between them. However, it is difficult to analyze $H_j(\boldsymbol{h})$ directly. When the pulse solution $P(z)$ converges $\mathbf{0}$ in an exponentially monotone way, $H_j(h)$ can be represented by the explicit form approximately.

Theorem 2.3. *Suppose $P(z)$ converges $\mathbf{0}$ satisfying*

$$\begin{aligned} P(z) &= e^{-\alpha z}(\boldsymbol{a}^+ + O(e^{-\gamma z})) \quad (z \to +\infty), \\ P(z) &= e^{\beta z}(\boldsymbol{a}^- + O(e^{\gamma z})) \quad (z \to -\infty) \end{aligned}$$

for positive constants α, β and γ and non-zero constant vectors $\boldsymbol{a}^\pm \in \mathbb{R}^n$, and suppose Φ^* also converges $\mathbf{0}$ in an exponentially monotone way such that

$$\begin{aligned} \Phi^*(z) &= e^{-\beta z}(\boldsymbol{b}^+ + O(e^{-\gamma z})) \quad (z \to +\infty), \\ \Phi^*(z) &= e^{\alpha z}(\boldsymbol{b}^- + O(e^{\gamma z})) \quad (z \to -\infty) \end{aligned}$$

for non-zero constant vectors $\boldsymbol{b}^\pm \in \mathbb{R}^n$. Then, functions $H_j(h)$ are represented by

$$H_j(h) = (M_\beta e^{-\beta h_{j+1}} + M_\alpha e^{-\alpha h_j})(1 + O(e^{-\gamma' \min \boldsymbol{h}})) \ (j = 1, 2, \cdots, N-1), \tag{2.6}$$

$$H_0(h) = M_\beta e^{-\beta h_1}(1 + O(e^{-\gamma' \min \boldsymbol{h}})), \tag{2.7}$$

$$H_N(h) = M_\alpha e^{-\alpha h_N}(1 + O(e^{-\gamma' \min \boldsymbol{h}})), \tag{2.8}$$

for a constant $\gamma' > 0$ and the constants M_α, M_β are given by

$$M_\alpha = \langle (2\alpha D + \theta I + A'(\alpha))\boldsymbol{a}^+, \boldsymbol{b}^- \rangle, \tag{2.9}$$

$$M_\beta = \langle (2\beta D - \theta I + A'(\beta))\boldsymbol{a}^-, \boldsymbol{b}^+ \rangle, \tag{2.10}$$

where $\langle \cdot, \cdot \rangle$ stands for the inner product in $\mathbb{R}^n$, $I \in \mathbb{R}^{n\times n}$ is identity matrix and $A'(\lambda) \in \mathbb{R}^{n\times n}$ is the function with respect to λ defined by

$$A'(\lambda) := \begin{pmatrix} \tilde{K}'_{1,1}(\lambda) & \tilde{K}'_{1,2}(\lambda) & \cdots & \cdots & \tilde{K}'_{1,n}(\lambda) \\ \tilde{K}'_{2,1}(\lambda) & \tilde{K}'_{2,2}(\lambda) & \cdots & \cdots & \tilde{K}'_{2,n}(\lambda) \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ \tilde{K}'_{n,1}(\lambda) & \tilde{K}'_{n,2}(\lambda) & \cdots & \cdots & \tilde{K}'_{n,n}(\lambda) \end{pmatrix}$$

in which

$$\tilde{K}'_{j,k}(\lambda) := \int_{\mathbb{R}} y K_{j,k}(y) e^{\lambda y} dy, \ (j, k = 1, 2, \ldots n).$$

Remark 2.4. *Given a function $G(z) : \mathbb{R} \to \mathbb{R}^n$, we write*

$$G(z) = e^{-\alpha z}(\boldsymbol{a} + O(e^{-\gamma z}) \quad (z \to +\infty)$$

for some positive constants α, γ and a nonzero constant vector $\boldsymbol{a} \in \mathbb{R}^n$ if there exists a positive real number C_0 and a real number C_1 such that

$$|e^{\alpha z}G(z) - \boldsymbol{a}| \leq C_0 e^{-\gamma z} \quad (\forall z \geq C_1).$$

We also write

$$G(z) \;=\; e^{\alpha z}(\boldsymbol{a} + O(e^{\gamma z})) \quad (z \to -\infty)$$

if there exists a positive real number C_0 and a real number C_1 such that

$$|e^{-\alpha z}P(z) - \boldsymbol{a}| \leq C_0 e^{\gamma z} \quad (\forall z \leq C_1).$$

proof of Theorem 2.3. From the same calculation in [13], we can gain (2.6), (2.7) and (2.8), where

$$\begin{aligned} M_\alpha &= \int_{\mathbb{R}} e^{-\alpha z}\langle \{F'(P(z)) - F'(\mathbf{0})\}\boldsymbol{a}^+, \Phi^*(z)\rangle dz, \\ M_\beta &= \int_{\mathbb{R}} e^{\beta z}\langle \{F'(P(z)) - F'(\mathbf{0})\}\boldsymbol{a}^-, \Phi^*(z)\rangle dz. \end{aligned}$$

First, we consider M_α. Since positive constants α, β and non-zero vectors $\boldsymbol{a}^\pm \in \mathbb{R}^n$ satisfy

$$\begin{cases} \mathbf{0} = \alpha^2 D\boldsymbol{a}^+ + \alpha\theta\boldsymbol{a}^+ + A(\alpha)\boldsymbol{a}^+ + F'(\mathbf{0})\boldsymbol{a}^+, \\ \mathbf{0} = \beta^2 D\boldsymbol{a}^- - \beta\theta\boldsymbol{a}^- + A(\beta)\boldsymbol{a}^- + F'(\mathbf{0})\boldsymbol{a}^-, \end{cases}$$

we obtain

$$\begin{aligned} \langle F'(\mathbf{0})\boldsymbol{a}^+, \Phi^*(z)\rangle &= -\langle \{\alpha^2 D + \alpha\theta I + A(\alpha)\}\boldsymbol{a}^+, \Phi^*(z)\rangle \\ &= -\langle \boldsymbol{a}^+, \{\alpha^2 D + \alpha\theta I + {}^t A(\alpha)\}\Phi^*(z)\rangle. \end{aligned}$$

From the equation (2.1), we have

$$\begin{aligned} \langle F'(P(z))\boldsymbol{a}^+, \Phi^*(z)\rangle &= \langle \boldsymbol{a}^+, {}^t F'(P(z))\Phi^*(z)\rangle \\ &= -\langle \boldsymbol{a}^+, D\Phi^*_{zz} + \theta\Phi^*_z + {}^t\boldsymbol{K} * \Phi^*\rangle. \end{aligned}$$

Therefore, M_α is represented as

$$\begin{aligned} M_\alpha &= \int_{\mathbb{R}} e^{-\alpha z}\langle \boldsymbol{a}^+, D\{\alpha^2\Phi^*(z) - \Phi^*_{zz}(z)\}\rangle dz + \int_{\mathbb{R}} e^{-\alpha z}\langle \boldsymbol{a}^+, \theta\{\alpha\Phi^*(z) - \Phi^*_z(z)\}\rangle dz \\ &\quad + \int_{\mathbb{R}} e^{-\alpha z}\langle \boldsymbol{a}^+, \{{}^t A(\alpha)\Phi^*(z) - {}^t\boldsymbol{K} * \Phi^*\}\rangle dz =: I_1 + I_2 + I_3. \end{aligned}$$

Since we have

$$\lim_{z\to+\infty} e^{\beta z} D\Phi^*_z(z) = \beta D\boldsymbol{b}^+, \quad \lim_{z\to-\infty} e^{-\alpha z} D\Phi^*_z(z) = \alpha D\boldsymbol{b}^-$$

from the lemma in Appendix A, we have

$$\begin{aligned} I_1 &= \int_{\mathbb{R}} e^{-\alpha z}\langle \boldsymbol{a}^+, D\{\alpha^2\Phi^*(z) - \Phi^*_{zz}(z)\}\rangle dz \\ &= -\int_{\mathbb{R}} \frac{d}{dz}\left[e^{-\alpha z}\langle \boldsymbol{a}^+, D\Phi^*_z(z) + \alpha D\Phi^*(z)\rangle\right] dz \\ &= 2\alpha\langle \boldsymbol{a}^+, D\boldsymbol{b}^-\rangle = 2\alpha\langle D\boldsymbol{a}^+, \boldsymbol{b}^-\rangle. \end{aligned}$$

Similary, we obtain

$$\begin{aligned} I_2 &= \int_{\mathbb{R}} e^{-\alpha z}\langle \boldsymbol{a}^+, \theta\{\alpha\Phi^*(z) - \Phi^*_z(z)\}\rangle dz \\ &= -\theta \int_{\mathbb{R}} \frac{d}{dz}\left[e^{-\alpha z}\langle \boldsymbol{a}^+, \Phi^*(z)\rangle\right] dz \\ &= \theta\langle \boldsymbol{a}^+, \boldsymbol{b}^-\rangle. \end{aligned}$$

Finally, to compute I_3, we consider $\int_{\mathbb{R}} e^{-\alpha z}\{\tilde{K}_{k,j}(\alpha)\varphi^*_k(z) - (K_{k,j} * \varphi^*_k)(z)\}dz$ for $j, k = 1, 2, \cdots, n$, where $\Phi^* = {}^t(\varphi^*_1, \varphi^*_2, \ldots, \varphi^*_n)$. Since $\tilde{K}_{k,j}$ is the even function, the integrand can be rewritten as

$$\begin{aligned} &e^{-\alpha z}\{\tilde{K}_{k,j}(\alpha)\varphi^*_k(z) - (K_{k,j} * \varphi^*_k)(z)\} \\ &\quad = e^{-\alpha z}\{\tilde{K}_{k,j}(-\alpha)\varphi^*_k(z) - (K_{k,j} * \varphi^*_k)(z)\} \\ &\quad = \int_{\mathbb{R}} K_{k,j}(y)\left\{e^{-\alpha(z+y)}\varphi^*_k(z) - e^{-\alpha z}\varphi^*_k(z-y)\right\} dy \\ &\quad = \int_{\mathbb{R}} K_{k,j}(y) \int_0^y \frac{d}{ds}\left[e^{-\alpha(z+s)}\varphi^*_k(z-y+s)\right] ds dy. \end{aligned}$$

Notice that

$$\frac{d}{ds}\left[e^{-\alpha(z+s)}\varphi^*_k(z-y+s)\right] = \frac{d}{dz}\left[e^{-\alpha(z+s)}\varphi^*_k(z-y+s)\right],$$

so we obtain

$$\begin{aligned} &\int_{\mathbb{R}} e^{-\alpha z}\{\tilde{K}_{k,j}(\alpha)\varphi^*_k(z) - (K_{k,j} * \varphi^*_k)(z)\}dz \\ &\quad = \int_{\mathbb{R}} K_{k,j}(y)\int_0^y \int_{\mathbb{R}} \frac{d}{dz}\left[e^{-\alpha(z+s)}\varphi^*_k(z-y+s)\right] dz ds dy \\ &\quad = -b^-_k \int_{\mathbb{R}} yK_{k,j}(y)e^{-\alpha y}dy = b^-_k \tilde{K}'_{k,j}(\alpha), \end{aligned}$$

where $\boldsymbol{b}^- = {}^t(b^-_1, b^-_2, \ldots, b^-_n)$. Therefore,

$$I_3 = \int_{\mathbb{R}} e^{-\alpha z}\left\langle \boldsymbol{a}^+, \{{}^tA(\alpha)\Phi^*(z) - {}^t\boldsymbol{K} * \Phi^*\}\right\rangle dz = \langle \boldsymbol{a}^+, {}^tA'(\alpha)\boldsymbol{b}^-\rangle = \langle A'(\alpha)\boldsymbol{a}^+, \boldsymbol{b}^-\rangle.$$

From above calculation, we can gain (2.9). We can also obtain (2.10) by same argument. □

2.2. **Interaction of fronts.** In this subsection, let us consider the interaction of front solutions. We can consider only the case of the velocity $\theta = 0$. We use x as the space variable instead of z because $x = z$ in this case. Basically, we use the same notations as in the previous subsection with $\theta = 0$.

Suppose that $P(x)$ is a stable front solution of (1.3) with $\theta = 0$. We note that $P(-x)$ is also a stable front solution of (1.3) connecting from P^+ to P^-. We define the number of front solutions as $N + 1 = N^+ + N^-$, where N^+ and N^- are the numbers of front

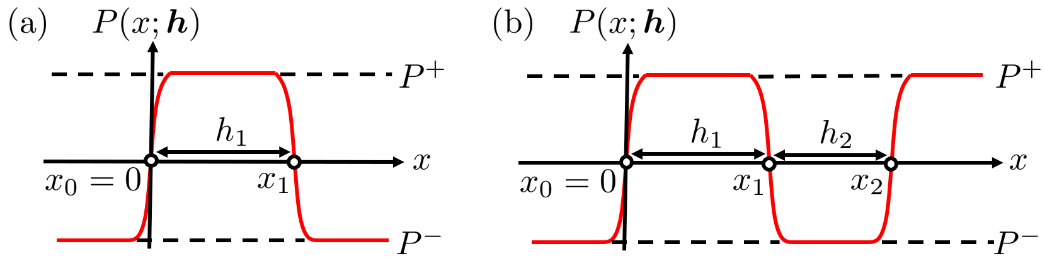

FIGURE 4. The image of $P(z;\boldsymbol{h})$. (a) $(N^+,N^-)=(1,1)$. (b) $(N^+,N^-)=(2,1)$.

solutions of the shapes $P(x)$ and $P(-x)$, respectively. We note that either $N^+=N^-$ or $N^+-1=N^-$ holds. Then, $N+1$ front solutions $P(x;\boldsymbol{h})$ are defined as

$$\begin{aligned} P(x;\boldsymbol{h}) &= P(x)+P(-(x-x_1))+P(x-x_2)+\cdots \\ &\quad +P((-1)^N(x-x_N))-\{N^+P^+ +(N^--1)P^-\} \end{aligned}$$

if $N^+=N^-$,

$$\begin{aligned} P(x;\boldsymbol{h}) &= P(x)+P(-(x-x_1))+P(x-x_2)+\cdots \\ &\quad +P((-1)^N(x-x_N))-\{(N^+-1)P^+ +N^-P^-\} \end{aligned}$$

if $N^+-1=N^-$, where $\boldsymbol{h}=(h_1,h_2,\cdots,h_N)\in\mathbb{R}^N$, $x_j=\sum_{k=1}^{k=j}h_k$ for $j=1,2,\cdots,N$ and $x_0=0$. Moreover, we define functions $H_j(\boldsymbol{h})$ $(j=0,1,\cdots,N)$ by

$$H_j(\boldsymbol{h})=\langle\mathcal{L}(P(x+x_j;\boldsymbol{h})),\Phi^*((-1)^jx)\rangle_{L^2}.$$

By applying the same line of argument in [13], we can also obtain the following result.

Theorem 2.5. [13] *Theorems 2.1 and 2.2 hold in the same statements but*

$$\dot{h}_j = (-1)^{j+1}(H_{j-1}(\boldsymbol{h})+H_j(\boldsymbol{h}))+O(\delta^2)\ (j=1,2,\cdots,N), \tag{2.11}$$

$$\dot{l} = -H_0(\boldsymbol{h})+O(\delta^2), \tag{2.12}$$

and

$$\boldsymbol{H}=(H_0+H_1,-(H_1+H_2),\cdots,(-1)^{N+1}(H_{N-1}+H_N)).$$

Just like Theorem 2.3, $H_j(h)$ can be represented by the explicit form approximately, when the front solution $P(z)$ converges $P^\pm$ in an exponentially monotone way.

Theorem 2.6. *Suppose $P(x)$ converges $P^\pm$ as*

$$\begin{aligned} P(x)-P^+ &= e^{-\alpha x}(\boldsymbol{a}^+ +O(e^{-\gamma x})) \quad (x\to+\infty), \\ P(x)-P^- &= e^{\beta x}(\boldsymbol{a}^- +O(e^{\gamma x})) \quad (x\to-\infty) \end{aligned}$$

for positive constants α, β and γ and non-zero constant vectors $\boldsymbol{a}^{\pm} \in \mathbb{R}^n$, and suppose Φ^* converges $\mathbf{0}$ in an exponentially monotone way such that

$$\begin{aligned}\Phi^*(x) &= e^{-\alpha x}(\boldsymbol{b}^+ + O(e^{-\gamma x})) \quad (x \to +\infty),\\ \Phi^*(x) &= e^{\beta x}(\boldsymbol{b}^- + O(e^{\gamma x})) \quad (x \to \infty)\end{aligned}$$

for non-zero constant vectors $\boldsymbol{b}^{\pm} \in \mathbb{R}^n$. Then, functions $H_j(h)$ are represented by

$$H_{2j-1}(\boldsymbol{h}) = (M^+ e^{-\alpha h_{2j-1}} - M^- e^{-\beta h_{2j}})(1 + O(e^{-\gamma' \min \boldsymbol{h}})) \quad (j = 1, 2, \cdots, N^+), \tag{2.13}$$

$$H_{2j}(\boldsymbol{h}) = (M^+ e^{-\alpha h_{2j+1}} - M^- e^{-\beta h_{2j}})(1 + O(e^{-\gamma' \min \boldsymbol{h}})) \quad (j = 1, 2, \cdots, N^-), \tag{2.14}$$

$$H_0(\boldsymbol{h}) = M^+ e^{-\alpha h_1}(1 + O(e^{-\gamma' \min \boldsymbol{h}})), \tag{2.15}$$

$$H_N(\boldsymbol{h}) = \begin{cases} M^+ e^{-\alpha h_N}(1 + O(e^{-\gamma' \min \boldsymbol{h}})) & (if\ N^+ = N^-),\\ M^- e^{-\beta h_N}(1 + O(e^{-\gamma' \min \boldsymbol{h}})) & (if\ N^+ - 1 = N^-)\end{cases} \tag{2.16}$$

for a constant $\gamma' > 0$ and the constants $M^{\pm}$ are given by

$$M^+ = \langle (2\alpha D + A'(\alpha))\boldsymbol{a}^+, \boldsymbol{b}^+ \rangle, \tag{2.17}$$

$$M^- = \langle (2\beta D + A'(\beta))\boldsymbol{a}^-, \boldsymbol{b}^- \rangle. \tag{2.18}$$

Proof. From the same calculation in [13], we can gain (2.13), (2.14), (2.15) and (2.16), where

$$\begin{aligned}M^+ &= \int_{\mathbb{R}} e^{\alpha x} \langle \{F'(P(x)) - F'(P^+)\}\boldsymbol{a}^+, \Phi^*(x)\rangle dx,\\ M^- &= \int_{\mathbb{R}} e^{-\beta x} \langle \{F'(P(x)) - F'(P^-)\}\boldsymbol{a}^-, \Phi^*(x)\rangle dx.\end{aligned}$$

By the argument similar to Theorem 2.3, we can obtain (2.17) and (2.18). □

3. Applications

In this section, we consider the interaction of two standing front solutions to the nonlocal scalar equation (1.2), where $d > 0$, $K \in C^1(\mathbb{R})$, $K' \in L^1(\mathbb{R})$, $\kappa := \int_{\mathbb{R}} K(y)dy$ and $g(u) := f(u) + \kappa u$ satisfies

$$\begin{cases} g \in C^3(\mathbb{R}),\ g(\pm 1) = g(a) = 0, g'(\pm 1) < 0 < g'(a),\ \int_{-1}^1 g(u)du = 0,\\ g < 0 \text{ in } (-1, a) \cup (1, \infty),\ g > 0 \text{ in } (-\infty, 0) \cup (a, 1),\\ g' \geq 0 \text{ in } [r_1, r_2],\ g' \leq 0 \text{ in } [-1, 1] \backslash [r_1, r_2]\end{cases}$$

for some constants $a,\ r_1,\ r_2 \in (-1, 1)$ with $r_1 < r_2$. A typical example of g is $g(u) = u(1 - u^2)$.

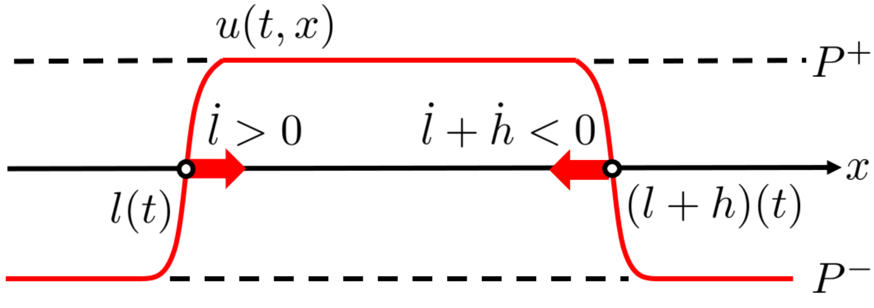

FIGURE 5. The image of the interaction of two standing fronts, when the kernel is non-negative function.

3.1. **Case of non-negative integral kernel.** In this subsection, we consider the interaction of two standing front solutions when $K(x) \geq 0$. In this case, (1.2) admits a strictly increasing stable standing front solution satisfying $P(\pm\infty) = \pm 1$ [4, 9, 34]. Furthermore, when K satisfies

$$\forall \lambda > 0, \quad A(\lambda) = \tilde{K}(\lambda) = \int_{\mathbb{R}} K(y) e^{\lambda y} dy < \infty,$$

$P(x)$ converges ± 1 in an exponentially monotone way [34]. Thus, suppose $P(x)$ converges 1 as

$$P(x) - 1 = e^{-\alpha x}(a^+ + O(e^{-\gamma x}))\ (x \to +\infty) \tag{3.1}$$

for some positive constant α and a non-zero constant a^+. Then α is a positive solution of

$$G(\lambda) := d\lambda^2 + A(\lambda) + g'(1) = 0.$$

It is easy to see that $G(\lambda)$ is strictly monotone increasing function for $\lambda > 0$, since $K \geq 0$. Therefore, we obtain $G'(\lambda) = 2d\lambda + A'(\lambda) > 0$ for $\lambda > 0$. Now, Φ^* is represented as

$$\Phi^*(x) \;=\; \frac{1}{\|P_x\|_{L^2}^2} P_x(x) \to -\frac{\alpha a^+}{\|P_x\|_{L^2}^2} e^{-\alpha x}\ (x \to +\infty)$$

and we have

$$M^+ = \frac{-\alpha (a^+)^2}{\|P_x\|_{L^2}^2} G'(\alpha) < 0. \tag{3.2}$$

Therefore, the equation for l and the front distance h is

$$\begin{cases} \dot{l} = -H_0(h) + O(\delta^2) \sim -M^+ e^{-\alpha h} > 0, \\ \dot{h} = H_0(h) + H_1(h) + O(\delta^2) \sim 2M^+ e^{-\alpha h} < 0. \end{cases}$$

This means the attractivity of two front solutions (Figure 5).

3.2. **Interaction of very slow front solutions.** In this subsection, we consider the interaction of two front solutions with very slow wave speed when $K(x) \geq 0$. We consider the equation (1.2) with small perturbation like

$$u_t = du_{xx} + K * u + f(u) + \epsilon f_1(u, K_1 * u), \tag{3.3}$$

where ϵ is a sufficiently small constant, $K_1 : \mathbb{R} \to \mathbb{R}$ belongs to $C(\mathbb{R}) \cap L^1(\mathbb{R})$, and $f_1(u,v) : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function satisfying

$$f(\pm 1, \pm \int_{\mathbb{R}} K_1(y)dy) = 0.$$

Let us consider the solution (3.3) with the initial function $u(0,x)$ close to $P(x - l(0), h(0))$ for sufficiently large $h(0)$. By a quite similar argument in [13, 16], if ϵ is sufficiently small, then we can show that the solution $u(t,x)$ is close to $P(x - l(t), h(t))$ and

$$\begin{cases} \dot{l} = -M^+ e^{-\alpha h} - \epsilon C_f + O(\delta^2 + \epsilon^2), \\ \dot{h} = 2M^+ e^{-\alpha h} + 2\epsilon C_f + O(\delta^2 + \epsilon^2) \end{cases} \tag{3.4}$$

holds as long as $h(t)$ is sufficiently large, where M^+ is the constant given by (3.2) and $C_f = \langle\, f_1(P, K_1 * P), P_x \,\rangle_{L^2}$. When $\epsilon C_f > 0$, we can understand the attractivity of two front solutions. If $\epsilon C_f < 0$, then (3.4) has a unstable equilibrium. Therefore, in this case, we can find a unstable stationary solution by a quite similar way to the proof of theorem 2.5.

For example, we consider the case that $f_1(u,v) = v$ and $K_1(x)$ is a odd function satisfying

$$K_1 < 0 \text{ in } (0, \infty).$$

Then, $C_f = \langle\, K_1 * P, P_x \,\rangle_{L^2}$. Since $P(x)$ is monotone increasing function, we obtain

$$K_1 * P(x) = \int_{\mathbb{R}} K_1(y) P(x-y) dy = \int_0^\infty K_1(y)(P(x-y) - P(x+y))dy > 0.$$

Thus, $C_f > 0$. From this, we can show that the existence of a unstable stationary solution for (3.3) if ϵ is a sufficiently small negative constant.

3.3. **Case of sign changing integral kernel.** When the integral kernel has negative parts, few have been known about the results of front solutions to (1.2). In the case that $d = 0$, the existence of standing front solutions to (1.2) with sign changing integral kernel was proved in [5] using variational method under the assumptions that K satisfies $\kappa > 0$, $\hat{K}(\xi) = \int_{\mathbb{R}} K(x) e^{-ix\xi} dx \leq \hat{K}(0) = \kappa$ for all $\xi \in \mathbb{R}$ and some conditions. However, there have been no results about the linearized stability of the front solutions to the best of our knowledge.

On the other hand, in numerical simulations, J. Siebert and E. Schöll [30] reported that the front solution of (1.2) has oscillatory tails when the integral kernel is Mexican hat profile as in Figure 1. From the result, it is natural to expect that there are cases when the front solutions of (1.2) with sign changing integral kernels have oscillatory tails while we were unfortunately unable to reveal the stability of front solutions in the case of sign changing integral kernels. We leave it as an open problem for a future day.

In the rest of this subsection, we assume that the existence of a stable single standing front solution for (1.2) with a sign changing integral kernel and consider the interaction of two single standing front solutions.

At first, we consider the interaction of two standing front solutions with exponentially decaying oscillatory tails. Suppose that there exists a stable standing front solution of (1.2) with an oscillatory tail such that

$$P(x) - 1 \to \Re(e^{\lambda^+ x} a^+ (1 + O(e^{-\gamma x}))) \ (x \to +\infty),$$

where $a^+ \in \mathbb{C}\backslash\{0\}$ and $\lambda^+ = -\alpha + i\nu^+$ for constants $\alpha > 0$ and $\nu^+ \neq 0$. Then, the equation for the distance h between front solutions is given by

$$\dot{h} = H_0 + H_1$$

as in Theorem 2.5. From the definition of $H_j(h)$ $(j = 0, 1)$, we can show $H_0(h) = H_1(h)$. Since $\Phi^* = \dfrac{1}{\|P_x\|^2} P_x$ and $P(\pm x)$ are front solutions to (1.2), we can calculate as

$$\begin{aligned} H_0(h) &= \frac{1}{\|P_x\|^2} \langle\, \mathcal{L}(P(\cdot; h)), P_x \,\rangle_{L^2} \\ &= \frac{1}{\|P_x\|^2} \langle\, f(P(\cdot; h)) - f(P(\cdot)) - f(P(-(\cdot - h))), P_x \,\rangle_{L^2} \end{aligned}$$

By a quite similar way to Subsection 4.5 in [13], we can show

$$H_0(h) = H_1(h) = \Re(M^+ e^{\lambda^+ h}(1 + O(e^{-\gamma' h})))$$

for a constant $\gamma' > 0$ as long as h is sufficiently large, where

$$M^+ = \frac{a^+}{\|P_x\|^2} \int_{-\infty}^{\infty} e^{-\lambda^+ x} P_x(x) \{f'(P(x)) - f'(1)\} dx.$$

The constant M^+ is well-defined because the integral is given as the Fourier transformation because of the form of λ^+. Let $M^+ = A^+ + iB^+$. Then, we have

$$H_0(h) = H_1(h) \sim e^{-\alpha h}(A^+ \cos(\nu^+ h) + B^+ \sin(\nu^+ h)).$$

Therefore, the equation on h is

$$\dot{h} = H_0 + H_1 + O(\delta^2(h)) \sim 2e^{-\alpha h}(A^+ \cos(\nu^+ h) + B^+ \sin(\nu^+ h)) \tag{3.5}$$

for sufficiently large h. From (3.5), we easily find that stable and unstable equilibria appear alternatively in (3.5). Thus, if there exists a stable standing front with oscillatory tails satisfying Hypothesis 1.1, we can easily give the proof on the existence and the stability for multiple front solutions from Theorem 2.5.

Secondly, we consider the interaction of two standing front solutions with exponentially monotone decaying tails. Suppose that there exists a stable standing front solution of (1.2) satisfying (3.1). By following the same line of arguments in Subsection 3.1, the equation of h is

$$\dot{h} \sim 2M^+ e^{-\alpha h},$$

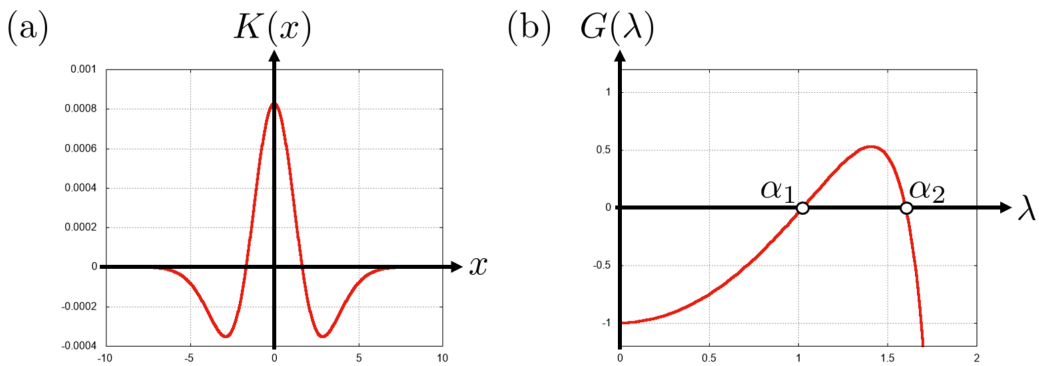

FIGURE 6. (a) The graph of (3.6) when $\epsilon = 0.01,\ q_1 = 1.0,\ q_2 = 2.0$. (b) The graph of $G(\lambda)$ when $d = 1.0$, $g'(1) = -1$ and the integral kernel is same as Figure 6 (a).

where

$$M^+ = \frac{-\alpha(a^+)^2}{\|P_x\|_{L^2}^2} G'(\alpha).$$

To reveal the sign of M^+, we consider $G'(\alpha)$. We note that α satisfies $G(\alpha) = 0$. In the case of a sign changing integral kernel, $G(\lambda)$ is not always monotone increasing. For example, when we consider the case that

$$K(x) = \frac{\epsilon}{\sqrt{4\pi}} \left\{ \frac{1}{\sqrt{q_1}} e^{-\frac{x^2}{4q_1}} - \frac{1}{\sqrt{q_2}} e^{-\frac{x^2}{4q_2}} \right\} \ (\epsilon, q_1, q_2 > 0), \tag{3.6}$$

then $A(\lambda)$ is represented as

$$A(\lambda) = \epsilon \left\{ e^{q_1 \lambda^2} - e^{q_2 \lambda^2} \right\}.$$

Therefore, we have

$$G(\lambda) = d\lambda^2 + \epsilon \left\{ e^{q_1 \lambda^2} - e^{q_2 \lambda^2} \right\} + g'(1).$$

When $d = 1.0,\ \epsilon = 0.01,\ q_1 = 1.0,\ q_2 = 2.0,\ g'(1) = -1$, it is observed that there exist two positive solutions α_1 and α_2 of $G(\lambda) = 0$ (Figure 6 (b)), where α_1 and α_2 denote the first and second positive root of $G(\lambda) = 0$, respectively. When $\alpha = \alpha_1$, we see

$$M^+ = \frac{-\alpha(a^+)^2}{\|P_x\|_{L^2}^2} G'(\alpha) < 0$$

by $G'(\alpha_1) > 0$, which means the attractivity of two front solutions. When $\alpha = \alpha_2$, we see

$$M^+ = \frac{-\alpha(a^+)^2}{\|P_x\|_{L^2}^2} G'(\alpha) > 0$$

by $G'(\alpha_2) < 0$. This means the repulsiveness of two front solutions.

In numerical calculations, α_1, α_2 can be computed approximately as $\alpha_1 = 1.0264\ldots$ and $\alpha_2 = 1.6022\ldots$. When we solve (1.2) numerically on the interval $(0, 40)$ in the same parameters as Figure 6, we can observe the stable standing front solution (Figure 7 (a))

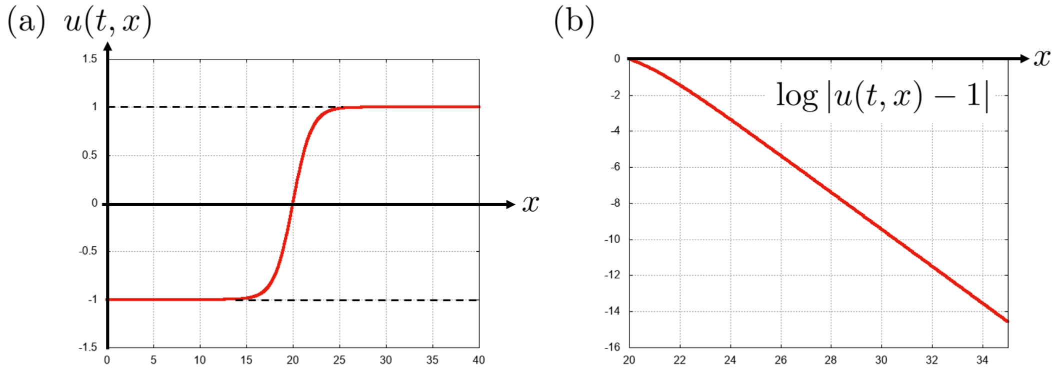

FIGURE 7. (a) The numerical solution of (1.2) on the interval $(0, 40)$ when $t = 100.0$, where $g(u) = \frac{1}{2}u(1-u^2)$ and the other parameters are same as that in Figure 6. (b) The graph of $\log|u(t,x)-1|$ on the interval $(20, 35)$ when $t = 100.0$, where $u(t,x)$ is the numerical solution of (1.2).

with the exponent α_1 as the exponential decay rate. In fact, Figure 7 (b) is a graph of $\log|u(t,x) - 1.0|$ at the place where $u(t,x)$ is close to 1, which shows that the numerical solution converges to 1 in an exponentially monotone way with the decay exponent $\alpha = 1.0260\ldots$ at $x = 32.0$. The value α is calculated as follows: Since $\log|u(t,x) - 1.0|$ looks like linear, the decay exponent at $x = a$ is calculated as

$$\alpha \sim -\frac{\partial}{\partial x}(\log|u(t,x) - 1.0|)\Big|_{x=a} \sim -\frac{\log|u(t,a+\eta) - 1.0| - \log|u(t,a) - 1.0|}{\eta},$$

in which η is a sufficiently small constant. Therefore, we expect that the front solution with the exponential decay rate α_1 is a stable one. Hence, we think that two stable front solutions are interacting attractively in the case of this example.

In general, if there exists a stable front solution of (1.2) satisfying (3.1), we expect that the decay rate α is generically given by $\alpha = \min\{\lambda > 0 \mid G(\lambda) = 0\}$. Then, $G'(\alpha) \geq 0$ always holds by the property of $G(0) = g'(1) < 0$. Thus, we find that the attractive motion will generically appear and suspect that the repulsive motion will not in most case.

4. DISCUSSION

In this section, we will state two future works related to the results of this paper.

First, we assume that an integral kernel decays faster than any exponential functions throughout this paper. An integral kernel satisfying this assumption is often appeared in the field of pattern formation problems, since many papers only concern effects of the shape of an integral kernel [15, 22, 30]. Thus, our results include the important case from the pattern formation point of view. Of course, we think that the condition (1.4) is technical and it might be replace this condition by some more general conditions. However, our results rely heavily on the condition (1.4). It would be interesting either to remove the condition (1.4) or to replace it by some more general conditions. We try to study as a future work.

Second, we only give the example of the nonlocal scalar equation in Section 3. To analyze the movement of traveling wave solutions, we need the property of the eigenfunction of L^* corresponding to eigenvalue 0. However, in general, it is difficult to analyze to the eigenfunction of L^* when L is not self-adjoint. Basically, L is not self-adjoint in the case of the n-component system with $n \geq 2$. Therefore, the example of this case is our future works.

Appendix A

In this appendix, we prove the following lemma:

Lemma A.1. *Let $\Phi^*(z)$ be an eigenfunction corresponding to* 0 *eigenvalue of the adjoint operator L^* and normalized by $\langle P_z, \Phi^* \rangle_{L^2} = 1$. If Φ^* converges $\mathbf{0}$ in an exponentially monotone way such that*

$$\begin{aligned} \Phi^*(z) &= e^{-\beta z}(\boldsymbol{b}^+ + O(e^{-\gamma z})) \quad (z \to +\infty), \\ \Phi^*(z) &= e^{\alpha z}(\boldsymbol{b}^- + O(e^{\gamma z})) \quad (z \to -\infty) \end{aligned}$$

for positive constants α, β and γ and non-zero constant vectors $\boldsymbol{b}^\pm \in \mathbb{R}^n$, then

$$\lim_{z \to +\infty} e^{\beta z} D\Phi_z^*(z) = \beta D\boldsymbol{b}^+, \tag{A1}$$

$$\lim_{z \to -\infty} e^{-\alpha z} D\Phi_z^*(z) = \alpha D\boldsymbol{b}^- \tag{A2}$$

holds.

We will only show the proof of (A2). When $D = \mathbf{0} \in \mathbb{R}^{n\times n}$, (A2) is trivial. We consider the case that $D \neq \mathbf{0} \in \mathbb{R}^{n\times n}$. We multiply (2.1) by $e^{-\alpha z}$, then we get

$$e^{-\alpha z}(D\Phi_{zz}^* + \theta\Phi_z^* + {}^t\boldsymbol{K} * \Phi^* + {}^tF'(P(z))\Phi^*) = 0.$$

Since we have

$$\lim_{z \to -\infty} e^{-\alpha z}({}^t\boldsymbol{K} * \Phi^*) = {}^tA(\alpha)\boldsymbol{b}^-$$

by Lebesgue dominated convergence, we obtain

$$\lim_{z \to -\infty} e^{-\alpha z}(D\Phi_{zz}^* + \theta\Phi_z^*) + \tilde{\boldsymbol{b}} = 0, \tag{A3}$$

where $\tilde{\boldsymbol{b}} := ({}^tA(\alpha) + {}^tF'(\mathbf{0}))\boldsymbol{b}^-$.

Lemma A.2. $\lim_{z \to -\infty} e^{-\alpha z} D\Phi_z^*(z)$ *exists.*

Proof. We fix $j \in \mathbb{N}$ satisfying $1 \leq j \leq n$ and $d_j > 0$. We write $\tilde{\boldsymbol{b}} = {}^t(\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n)$ and $\Phi^* = {}^t(\varphi_1^*, \varphi_2^*, \ldots, \varphi_n^*)$, then φ_j^* satisfies

$$\lim_{z \to -\infty} e^{-\alpha z}\{d_j(\varphi_j^*)_{zz}(z) + \theta(\varphi_j^*)_z(z)\} + \tilde{b}_j = 0$$

from (A3). Thus, for any $\epsilon > 0$, there exists $C_0 \in \mathbb{R}$ such that

$$-(\epsilon + \tilde{b}_j)e^{\alpha z} \leq d_j(\varphi_j^*)_{zz}(z) + \theta(\varphi_j^*)_z(z) \leq (\epsilon - \tilde{b}_j)e^{\alpha z} \tag{A4}$$

for all $z \leq C_0$.

Notice that $\varphi_j^* \in H^2(\mathbb{R})$ from the definition of $\mathcal{D}(L^*)$, φ_j^* is uniform continuos function from Morrey's inequality. This implies

$$\lim_{z\to-\infty} \varphi_j^*(z) = 0.$$

Integrating (A4) from $-\infty$ to $z < C_0$, we obtain

$$-\frac{\epsilon + \tilde{b}_j}{\alpha} e^{\alpha z} \le d_j(\varphi_j^*)_z(z) + \theta\varphi_j^*(z) \le \frac{\epsilon - \tilde{b}_j}{\alpha} e^{\alpha z}$$

for all $z \le C_0$. We multiply this inequality by $\alpha e^{\alpha z}$ and then take lower limit and upper limit as $z \to -\infty$, we can deduce

$$-\epsilon - \tilde{b}_j \le \alpha d_j \left(\liminf_{z\to-\infty} e^{-\alpha z}(\varphi_j^*)_z(z) \right) + \alpha\theta b_j^- \le \alpha d_j \left(\liminf_{z\to-\infty} e^{-\alpha z}(\varphi_j^*)_z(z) \right) + \alpha\theta b_j^- \le \epsilon - \tilde{b}_j$$

where $\boldsymbol{b}^- = {}^t(b_1^-, b_2^-, \dots, b_n^-)$. Since ϵ is an arbitrary positive constant, we can show that

$$\alpha d_j \left(\lim_{z\to-\infty} e^{-\alpha z}(\varphi_j^*)_z(z) \right) + \alpha\theta b_j^- + \tilde{b}_j = 0.$$

This implies

$$\alpha \lim_{z\to-\infty} e^{-\alpha z} D\Phi_z^*(z) + (\theta\boldsymbol{b}^- + {}^tA(\alpha)\boldsymbol{b}^- + {}^tF'(\boldsymbol{0})\boldsymbol{b}^-) = 0.$$

Therefore, we obtain the existence of $\lim_{z\to-\infty} e^{-\alpha z} D\Phi^*(z)$. □

From above lemma, we know that $\lim_{z\to-\infty} e^{-\alpha z} D\Phi^*(z)$ and $\lim_{z\to-\infty} e^{-\alpha z} D\Phi_z^*(z)$ exist. On the other hand, since

$$\frac{d}{dz}(e^{-\alpha z} D\Phi^*(z)) = e^{-\alpha z} D(\Phi_z(z) - \alpha\Phi(z)), \tag{A5}$$

$\lim_{z\to-\infty} \frac{d}{dz}(e^{-\alpha z} D\Phi^*(z))$ exists. Furthermore,

$$\lim_{z\to-\infty} \frac{d}{dz}(e^{-\alpha z} D\Phi^*(z)) = 0,$$

from the existence of $\lim_{z\to-\infty} e^{-\alpha z} D\Phi^*(z)$. Taking a limit of (A5) as $z \to -\infty$, we obtain (A2).

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