G-TUTTE POLYNOMIALS AND ABELIAN LIE GROUP ARRANGEMENTS

YE LIU, TAN NHAT TRAN, AND MASAHIKO YOSHINAGA

ABSTRACT. We introduce and study the notion of the G-Tutte polynomial for a list $\mathcal{A}$ of elements in a finitely generated abelian group Γ and an abelian group G, which is defined by counting the number of homomorphisms from associated finite abelian groups to G.

The G-Tutte polynomial is a common generalization of the (arithmetic) Tutte polynomial for realizable (arithmetic) matroids, the characteristic quasi-polynomial for integral arrangements, Brändén-Moci's arithmetic version of the partition function of an abelian group-valued Potts model, and the modified Tutte-Krushkal-Renhardy polynomial for a finite CW-complex.

As in the classical case, G-Tutte polynomials carry topological and enumerative information (e.g., the Euler characteristic, point counting and the Poincaré polynomial) of abelian Lie group arrangements.

We also discuss differences between the arithmetic Tutte and the G-Tutte polynomials related to the axioms for arithmetic matroids and the (non-)positivity of coefficients.

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Date: December 2, 2018.

Key words and phrases. Tutte polynomial, characteristic quasi-polynomial, Poincare polynomial, arithmetic matroids.

1. Introduction

The Tutte polynomial is one of the most important invariants of a graph. The significance of the Tutte polynomial is that it has several important specializations, including chromatic polynomials, partition functions of Potts models ([32]), and Jones polynomials for alternating links ([34]). Another noteworthy aspect of the Tutte polynomial is that it depends only on the (graphical) matroid structure, and thus one can define the Tutte polynomial for a matroid. Matroids and (specializations of) Tutte polynomials play a role in several diverse areas of mathematics ([31, 35]).

Matroids and Tutte polynomials are particularly important in the study of hyperplane arrangements ([30]), because the Tutte polynomial and one of its specializations, the characteristic polynomial, carry enumerative and topological information about the arrangement. For instance, the number of points over a finite field, the number of chambers for a real arrangement and the Betti numbers for a complex arrangement are all obtained from the characteristic polynomial.

It is natural to consider arrangements of subsets of other types. Recently, arrangements of subtori in a torus, or so-called toric arrangements, have received considerable attention ([15]), which has origin in the study of the moduli space of curves ([26]) and regular semisimple elements in an algebraic group ([25]).

The notions of arithmetic Tutte polynomials and arithmetic matroids invented by Moci and collaborators ([28, 14, 10, 19]) are particularly useful for studying toric arrangements. As in the case of hyperplane arrangements, arithmetic Tutte polynomials carry enumerative and topological information about toric arrangements. It is generally difficult to explicitly compute the arithmetic Tutte polynomial. Arithmetic Tutte polynomials for classical root systems were computed by Ardila, Castillo and Henley ([1]).

Another (quasi-)polynomial invariant for a hyperplane arrangement defined over integers, the characteristic quasi-polynomial introduced by Kamiya, Takemura and Terao [22], is a refinement of the characteristic polynomial of an arrangement. The notion of the characteristic quasi-polynomial is closely related to Ehrhart theory on counting lattice points, and has increased in combinatorial importance recently. The characteristic quasi-polynomial for root systems was essentially computed by Suter [33] (see also [23]). By comparing the computations of Suter with those of Ardila, Castillo and Henley, it has been observed that the most degenerate constituent of the characteristic quasi-polynomial is a specialization of the arithmetic Tutte polynomial.

The purpose of this paper is to introduce and study a new class of polynomial invariant that forms a common generalization of the Tutte, arithmetic Tutte and characteristic quasi-polynomials, among others.

The key observation to unify the above "Tutte-like polynomials" is that they are all defined by means of counting homomorphisms between certain abelian groups (this formulation appeared in [10, §7]). This observation has prompted us to introduce the notion of the *G-Tutte polynomial* $T_{\mathcal{A}}^G(x, y)$ for a list of elements $\mathcal{A}$ in a finitely generated abelian group Γ and an abelian group G with a certain finiteness assumption on the torsion elements (see §4 for details). We mainly consider abelian Lie groups G of the form

$$G = F \times (S^1)^p \times \mathbb{R}^q,$$

where F is a finite abelian group and $p, q \geq 0$. Typical examples are $\mathbb{C} \simeq \mathbb{R}^2$ and $\mathbb{C}^\times \simeq S^1 \times \mathbb{R}$.

When the group G is $\mathbb{C}$, $\mathbb{C}^\times$, or the finite cyclic group $\mathbb{Z}/k\mathbb{Z}$, the G-Tutte polynomial is precisely the Tutte polynomial, the arithmetic Tutte polynomial, or a constituent of the characteristic quasi-polynomial, respectively. We will see that many known properties (deletion-contraction formula, Euler characteristic of the complement, point counting, Poincaré polynomial, convolution formula) for (arithmetic) Tutte polynomials are shared by G-Tutte polynomials. (See [17] for another attempt to generalize arithmetic Tutte polynomials.)

The organization of this paper is as follows.

§2 gives a summary of background material. We recall definitions of the Tutte polynomial $T_{\mathcal{A}}(x, y)$, arithmetic Tutte polynomial $T_{\mathcal{A}}^{\mathrm{arith}}(x, y)$ and the characteristic quasi-polynomial $\chi_{\mathcal{A}}^{\mathrm{quasi}}(q)$ for a given list of elements $\mathcal{A}$ in $\Gamma = \mathbb{Z}^\ell$.

As pointed out by D'Adderio-Moci [14], it is more convenient to consider a list $\mathcal{A}$ in a finitely generated abelian group Γ. Following their ideas, in §3 we define arrangements $\mathcal{A}(G)$ of subgroups in $\mathrm{Hom}(\Gamma, G)$ and its complements $\mathcal{M}(\mathcal{A}; \Gamma, G)$ for arbitrary abelian group G. We see that the set-theoretic deletion-contraction formula holds.

In §4, the G-Tutte polynomial $T_{\mathcal{A}}^G(x, y)$ is defined using the number of homomorphisms of certain finite abelian groups to G (the G-multiplicities). We also define the multivariate version $Z_{\mathcal{A}}^G(q, \boldsymbol{v})$ and the G-characteristic polynomial $\chi_{\mathcal{A}}^G(t)$. We then show that the G-Tutte polynomial satisfies the deletion-contraction formula . We also see that the G-Tutte polynomial has several specializations.

In §5, we show that the Euler characteristic $e(\mathcal{M}(\mathcal{A}; \Gamma, G))$ of the complement can be computed as a special value of the G-Tutte polynomial (or G-characteristic polynomial) when G is an abelian Lie group with finitely many components. As a special case, when G is finite, we obtain a formula that counts the cardinality $\#\mathcal{M}(\mathcal{A}; \Gamma, G)$. The equality between the arithmetic characteristic polynomial and the most degenerate constituent of the characteristic quasi-polynomial is also proved.

In §6 we compute the Poincaré polynomial for toric arrangements associated with root systems (considering positive roots to be a list in the root lattice). Applying recent results on characteristic quasi-polynomials, we show that the Poincaré polynomial satisfies a certain self-duality when the root system differs from E_7, E_8. We also recover Moci's results on Euler characteristics [28, Corollary 7.3].

In §7, we prove a formula that expresses the Poincaré polynomial of $\mathcal{M}(\mathcal{A}; \Gamma, G)$ in terms of G-characteristic polynomials under the assumption that G is a non-compact abelian Lie group with finitely many connected components. This formula covers several classical results, including hyperplane arrangements (Orlik-Solomon [29] and Zaslavsky [38]), certain subspace arrangements (Goresky-MacPherson [20], Björner [8]) and toric arrangements (De Concini-Procesi [15], Moci [28]).

If $G = S^1$ or $\mathbb{C}^\times$, then the G-multiplicities satisfy the five axioms of arithmetic matroids given in [14]. A natural question to ask is whether the G-multiplicities satisfy these axioms for general groups G. In §8, we show that four of the five axioms are satisfied by the G-multiplicities. However, one of the axioms is not necessarily satisfied. We also present a counterexample that does not satisfy this axiom. However, we prove that the G-multiplicity function satisfies another important formula, the so-called convolution formula, which has been a formula of interest recently [3, 16]. Unlike the cases of arithmetic Tutte polynomials, the coefficients of G-Tutte polynomials are not necessarily positive. We show this with an example.

For the purpose of giving a combinatorial framework that describes intersection patterns of arrangements over an abelian Lie group G, an interesting problem would be to axiomatize G-multiplicities, which is left for future research.

Conventions: In this paper, the term **list** is synonymous with multiset. We follow the convention in [14, §2.1]. For example, the list $\mathcal{A} = \{\alpha, \alpha\}$ has 4 distinct sublists: $\mathcal{S}_1 = \emptyset, \mathcal{S}_2 = \{\alpha\}, \mathcal{S}_3 = \{\alpha\}, \mathcal{S}_4 = \{\alpha, \alpha\} = \mathcal{A}$. We distinguish $\mathcal{S}_2$ and $\mathcal{S}_3$, and hence $\mathcal{A} \smallsetminus \mathcal{S}_2 = \mathcal{S}_3$. If $\mathcal{A}$ is a list, then $\mathcal{S} \subset \mathcal{A}$ indicates that $\mathcal{S}$ is a sublist of $\mathcal{A}$.

(In §4.5 and §8) A dot under a letter indicates the parameter in the summation. For instance, $\sum_{\mathcal{S} \subset \underset{\bullet}{\mathcal{B}} \subset \mathcal{T}}$ indicates that $\mathcal{S}$ and $\mathcal{T}$ are fixed, and $\mathcal{B}$ is running between them.

2. Background

In this section, we recall the definitions of Tutte polynomials, arithmetic Tutte polynomials, characteristic quasi-polynomials and related results.

2.1. **(Arithmetic) Tutte polynomials.** Let $\mathcal{A} = \{\alpha_1, \dots, \alpha_n\} \subset \mathbb{Z}^\ell$ be a list of integer vectors, let $\alpha_i = (a_{i1}, \dots, a_{i\ell})$. We may consider α_i to be a linear form defined by

$$\alpha_i(x_1, \dots, x_\ell) = a_{i1}x_1 + \cdots + a_{i\ell}x_\ell.$$

A sublist $\mathcal{S} \subset \mathcal{A}$ determines a homomorphism $\alpha_{\mathcal{S}} : \mathbb{Z}^\ell \longrightarrow \mathbb{Z}^{\#\mathcal{S}}$.

Let G be an abelian group. Define the subgroup $H_{\alpha_i,G}$ of G^ℓ by

$$H_{\alpha_i,G} = \operatorname{Ker}(\alpha_i \otimes G : G^\ell \longrightarrow G).$$

The list $\mathcal{A}$ determines an arrangement $\mathcal{A}(G) = \{H_{\alpha,G} \mid \alpha \in \mathcal{A}\}$ of subgroups in G^ℓ. Denote their complement by

$$\mathcal{M}(\mathcal{A}; \mathbb{Z}^\ell, G) := G^\ell \smallsetminus \bigcup_{\alpha_i \in \mathcal{A}} H_{\alpha_i,G}.$$

The arrangement $\mathcal{A}(G)$ of subgroups and its complement $\mathcal{M}(\mathcal{A}; \mathbb{Z}^\ell, G)$ are important objects of study in many contexts. We list some of these below.

(i) When G is the additive group of a field (e.g., $G = \mathbb{C}, \mathbb{R}, \mathbb{F}_q$), $\mathcal{A}(G)$ is the associated hyperplane arrangement ([30]).
(ii) When $G = \mathbb{R}^c$ with $c > 0$, $\mathcal{A}(G)$ is called the c-plexification of $\mathcal{A}$ (see [8, §5.2]).
(iii) When G is $\mathbb{C}^\times$ or S^1, $\mathcal{A}(G)$ is called a toric arrangement.
(iv) When $G = S^1 \times S^1$ (viewed as an elliptic curve), $\mathcal{A}(G)$ is called an elliptic (or abelian) arrangement. ([7]).
(v) When G is a finite cyclic group $\mathbb{Z}/q\mathbb{Z}$, $\mathcal{A}(G)$ is related to the characteristic quasi-polynomial studied in [22, 23] (see 2.2). There is also an important connection with Ehrhart theory and enumerative problems ([9, 36, 37]).

To define the arithmetic Tutte polynomial, we need further notation. The linear map $\alpha_{\mathcal{S}}$ is expressed by the matrix $M_{\mathcal{S}} = (a_{ij})_{i\in\mathcal{S},1\le j\le\ell}$ of size $\#\mathcal{S}\times\ell$. Denote by $r_{\mathcal{S}}$ the rank of $M_{\mathcal{S}}$. Suppose that the Smith normal form of $M_{\mathcal{S}}$ is

$$\begin{pmatrix} d_{\mathcal{S},1} & 0 & \cdots & 0 & \cdots & \cdots & 0 \\ 0 & d_{\mathcal{S},2} & & \vdots & & & \vdots \\ \vdots & & \ddots & 0 & & & \\ 0 & \cdots & 0 & d_{\mathcal{S},r_{\mathcal{S}}} & & & \\ \vdots & & & & 0 & & \\ \vdots & & & & & \ddots & \vdots \\ 0 & \cdots & & & & \cdots & 0 \end{pmatrix},$$

where $1 \le d_{\mathcal{S},i}$ is a positive integer and $d_{\mathcal{S},i}$ divides $d_{\mathcal{S},i+1}$.

The Tutte polynomial $T_{\mathcal{A}}(x,y)$ and the arithmetic Tutte polynomial $T^{\mathrm{arith}}_{\mathcal{A}}(x,y)$ of $\mathcal{A}$ are defined as follows ([28, 10]).

$$T_{\mathcal{A}}(x,y) = \sum_{\mathcal{S}\subset\mathcal{A}} (x-1)^{r_{\mathcal{A}}-r_{\mathcal{S}}}(y-1)^{\#\mathcal{S}-r_{\mathcal{S}}},$$

$$T^{\mathrm{arith}}_{\mathcal{A}}(x,y) = \sum_{\mathcal{S}\subset\mathcal{A}} m(\mathcal{S})(x-1)^{r_{\mathcal{A}}-r_{\mathcal{S}}}(y-1)^{\#\mathcal{S}-r_{\mathcal{S}}},$$

where $m(\mathcal{S}) = \prod_{i=1}^{r_{\mathcal{S}}} d_{\mathcal{S},i}$. ($m(\mathcal{S})$ can also be defined as the cardinality of torsion subgroup of $\mathbb{Z}^\ell$ quotient by the subgroup generated by the row vectors of $M_{\mathcal{S}}$. See also §4.2). It should be noted that the arithmetic Tutte polynomial $T^{\mathrm{arith}}_{\mathcal{A}}(x,y)$ is defined for more general objects, arithmetic matroids [14] and matroids over $\mathbb{Z}$ [19].

These polynomials encode combinatorial and topological information about the arrangements. For instance, the characteristic polynomial of the ranked poset of flats of the hyperplane arrangement is $\chi_{\mathcal{A}}(t) = (-1)^{r_{\mathcal{A}}}t^{\ell-r_{\mathcal{A}}}T_{\mathcal{A}}(1-t,0)$, and the Poincaré polynomial of $\mathcal{M}(\mathcal{A};\mathbb{Z}^\ell,\mathbb{R}^c)$ is ([20, 8])

$$P_{\mathcal{M}(\mathcal{A};\mathbb{Z}^\ell,\mathbb{R}^c)}(t) = t^{r_{\mathcal{A}}\cdot(c-1)}\cdot T_{\mathcal{A}}\left(\frac{1+t}{t^{c-1}},0\right). \tag{2.1}$$

Note that the special cases $c = 1$ and $c = 2$ reduce to the famous formulas given by Zaslavsky [38] and Orlik-Solomon [29], respectively. Similarly, as proved by De Concini-Procesi [15] and Moci [28], the characteristic polynomial of the layers (connected components of intersections) of the corresponding toric arrangement is $\chi^{\mathrm{arith}}_{\mathcal{A}}(t) = (-1)^{r_{\mathcal{A}}}t^{\ell-r_{\mathcal{A}}}T^{\mathrm{arith}}_{\mathcal{A}}(1-t,0)$, and the Poincaré polynomial of $\mathcal{M}(\mathcal{A};\mathbb{Z}^\ell,\mathbb{C}^\times)$ is

$$P_{\mathcal{M}(\mathcal{A};\mathbb{Z}^\ell,\mathbb{C}^\times)}(t) = (1+t)^{\ell-r_{\mathcal{A}}}\cdot t^{r_{\mathcal{A}}}\cdot T^{\mathrm{arith}}_{\mathcal{A}}\left(\frac{1+2t}{t},0\right). \tag{2.2}$$

The cohomology ring structure of $\mathcal{M}(\mathcal{A}; \mathbb{Z}^\ell, \mathbb{C}^\times)$ was recently described in [11].

Contrary to the above cases, Bibby [7, Remark 4.4] pointed out that when $G = S^1 \times S^1$, a similar formula for the Poincaré polynomial does not hold. We will see that the non-compactness of G plays an important role in Poincaré polynomial formulas (Theorem 7.7 and Remark 7.9).

2.2. **Characteristic quasi-polynomials.** In [22], Kamiya, Takemura and Terao proved that $\#\mathcal{M}(\mathcal{A}; \mathbb{Z}^\ell, \mathbb{Z}/q\mathbb{Z})$ is a quasi-polynomial in q ($q \in \mathbb{Z}_{>0}$), denoted by $\chi_{\mathcal{A}}^{\mathrm{quasi}}(q)$, with period

$$\rho_{\mathcal{A}} := \mathrm{lcm}(d_{\mathcal{S}, r_{\mathcal{S}}} \mid \mathcal{S} \subset \mathcal{A}).$$

More precisely, there exist polynomials $f_1(t), f_2(t), \cdots, f_{\rho_{\mathcal{A}}}(t) \in \mathbb{Z}[t]$ such that for any positive integer q,

$$\chi_{\mathcal{A}}^{\mathrm{quasi}}(q) := \#\mathcal{M}(\mathcal{A}; \mathbb{Z}^\ell, \mathbb{Z}/q\mathbb{Z}) = f_k(q),$$

where $k \equiv q \mod \rho_{\mathcal{A}}$. The polynomial $f_k(t)$ is called the k-constituent. They also proved that $f_k(t) = f_m(t)$ if $\gcd(k, \rho_{\mathcal{A}}) = \gcd(m, \rho_{\mathcal{A}})$. Furthermore, the 1-constituent $f_1(t)$ (and more generally, $f_k(t)$ with $\gcd(k, \rho_{\mathcal{A}}) = 1$) is known to be equal to the characteristic polynomial $\chi_{\mathcal{A}}(t)$ ([2]).

We will show that the most degenerate constituent $f_{\rho_{\mathcal{A}}}(t)$ is obtained as a specialization of the arithmetic Tutte polynomial, and that the other constituents can also be described in terms of the G-Tutte polynomials introduced later (Theorem 5.5, Corollary 5.6).

3. Arrangements over abelian groups and deletion-contraction

Let Γ be a finitely generated abelian group, $\mathcal{A} = \{\alpha_1, \dots, \alpha_n\} \subset \Gamma$ a list (multiset) of finitely many elements, and G an arbitrary abelian group with the unit $e \in G$. In this section, we define arrangements over G, and prove the set-theoretic deletion-contraction formula by generalizing an idea of D'Adderio-Moci [14, §3.2].

3.1. G**-plexification.** Let us denote the subgroup of torsion elements of Γ by $\Gamma_{\mathrm{tor}} \subset \Gamma$, and the rank of Γ by r_Γ. More generally, for a list $\mathcal{S} \subset \Gamma$, denote the rank

of the subgroup $\langle \mathcal{S} \rangle \subset \Gamma$ generated by $\mathcal{S}$ by

$$r_{\mathcal{S}} = \mathrm{rank}\langle \mathcal{S} \rangle.$$

We now define the "arrangement" associated with a list $\mathcal{A}$ over an arbitrary abelian group G. The total space is the abelian group

$$\mathrm{Hom}(\Gamma, G) = \{\varphi : \Gamma \longrightarrow G \mid \varphi \text{ is a homomorphism}\}$$

of all homomorphisms from Γ to G. The element $\alpha \in \Gamma$ naturally determines a homomorphism

$$\alpha : \mathrm{Hom}(\Gamma, G) \longrightarrow G, \varphi \longmapsto \varphi(\alpha),$$

with the kernel

$$H_{\alpha,G} := \{\varphi \in \mathrm{Hom}(\Gamma, G) \mid \varphi(\alpha) = e\}.$$

The collection of subgroups $\mathcal{A}(G) = \{H_{\alpha,G} \mid \alpha \in \mathcal{A}\}$ is called the G-plexification of $\mathcal{A}$. Denote the complement of $\mathcal{A}(G)$ by

$$\mathcal{M}(\mathcal{A}; \Gamma, G) := \mathrm{Hom}(\Gamma, G) \smallsetminus \bigcup_{\alpha \in \mathcal{A}} H_{\alpha,G}.$$

Example 3.1. (1) Suppose that $\Gamma = \mathbb{Z}^\ell$. Then $\mathrm{Hom}(\Gamma, G) \simeq G^\ell$.

(2) Suppose that $\Gamma = \mathbb{Z}/d\mathbb{Z}$. Then

$$\mathrm{Hom}(\Gamma, G) \simeq G[d] := \{x \in G \mid d \cdot x = e\}$$

is the subgroup of d-torsion points.

Example 3.2. Let $\Gamma = \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and $G = \mathbb{C}^\times$. Then $\mathrm{Hom}(\Gamma, G) \simeq \mathbb{C}^\times \times \{\pm 1, \pm i\}$, which is a (real 2-dimensional) Lie group with 4 connected components. If $\alpha_1 = (2, 2) \in \Gamma$, then $H_{\alpha_1,G} = \{(\pm 1, \pm 1), (\pm i, \pm i)\}$ consists of 8 points. If $\alpha_2 = (0, 2) \in \Gamma$, then $H_{\alpha_2,G} = \mathbb{C}^\times \times \{\pm 1\}$ is a union of two copies of $\mathbb{C}^\times$.

Remark 3.3. G-plexification can be considered as a generalization of complexification, c-plexification, toric arrangements and $\mathbb{Z}/q\mathbb{Z}$ reduction (see §2).

3.2. **Set-theoretic deletion-contraction formula.** D'Adderio-Moci [14] defined two other lists for a fixed $\alpha \in \mathcal{A}$, the deletion $\mathcal{A}'$ and the contraction $\mathcal{A}'' = \mathcal{A}/\alpha$. The deletion is just $\mathcal{A}' = \mathcal{A} \smallsetminus \{\alpha\}$, a list of elements in the same group $\Gamma' := \Gamma$. To define $\mathcal{A}''$, let $\Gamma'' := \Gamma/\langle\alpha\rangle$, and $\mathcal{A}'' := \{\overline{\alpha'} \mid \alpha' \in \mathcal{A}'\} \subset \Gamma''$. By the exact sequence

$$0 \longrightarrow \mathrm{Hom}(\Gamma'', G) \longrightarrow \mathrm{Hom}(\Gamma, G) \longrightarrow \mathrm{Hom}(\langle\alpha\rangle, G),$$

the group $\mathrm{Hom}(\Gamma'', G)$ can be identified with

$$H_{\alpha,G} = \{\varphi \in \mathrm{Hom}(\Gamma, G) \mid \varphi(\alpha) = e\}.$$

Therefore, we can consider both $\mathcal{M}(\mathcal{A}; \Gamma, G)$ and $\mathcal{M}(\mathcal{A}''; \Gamma'', G)$ as subsets of $\mathrm{Hom}(\Gamma, G)$ (actually, the subsets of $\mathcal{M}(\mathcal{A}'; \Gamma', G)$). These three sets are related by the following deletion-contraction formula.

Proposition 3.4. *Using the above identification, we have the following decomposition:*

$$\mathcal{M}(\mathcal{A}'; \Gamma', G) = \mathcal{M}(\mathcal{A}''; \Gamma'', G) \sqcup \mathcal{M}(\mathcal{A}; \Gamma, G).$$

Proof. The set $\mathcal{M}(\mathcal{A}';\Gamma',G)$ can be decomposed as $\{\varphi \in \mathcal{M}(\mathcal{A}';\Gamma',G) \mid \varphi(\alpha) = e\} \sqcup \{\varphi \in \mathcal{M}(\mathcal{A}';\Gamma',G) \mid \varphi(\alpha) \neq e\}$. The first term on the right-hand side can be identified with $\{\varphi \in \mathcal{M}(\mathcal{A}';\Gamma',G) \mid \varphi(\alpha) = e\} \simeq \mathcal{M}(\mathcal{A}'';\Gamma'',G)$, and the second term is equal to $\mathcal{M}(\mathcal{A};\Gamma,G)$. □

More generally, for any sublist $\mathcal{S} \subset \mathcal{A}$, we can define the contraction $\mathcal{A}/\mathcal{S}$ as the list of cosets $\{\overline{\alpha} \mid \alpha \in \mathcal{A} \smallsetminus \mathcal{S}\}$ in the group $\Gamma/\langle\mathcal{S}\rangle$. Because $\operatorname{Hom}(\Gamma/\langle\mathcal{S}\rangle, G)$ can be naturally identified with the set $\{\varphi \in \operatorname{Hom}(\Gamma,G) \mid \varphi(\alpha) = e, \forall\alpha \in \mathcal{S}\}$, it can be seen as a subset of $\operatorname{Hom}(\Gamma,G)$. Under this identification, we can describe $\mathcal{M}(\mathcal{A}/\mathcal{S};\Gamma/\langle\mathcal{S}\rangle,G)$ as

$$\mathcal{M}(\mathcal{A}/\mathcal{S};\Gamma/\langle\mathcal{S}\rangle,G) = \left\{\varphi \in \operatorname{Hom}(\Gamma,G) \;\middle|\; \begin{array}{ll} \varphi(\alpha) = e, & \text{for } \alpha \in \mathcal{S} \\ \varphi(\alpha) \neq e, & \text{for } \alpha \in \mathcal{A} \smallsetminus \mathcal{S} \end{array} \right\}.$$

It is easily seen that for any $\varphi \in \operatorname{Hom}(\Gamma,G)$, $\mathcal{S} = \mathcal{A}_\varphi := \{\alpha \in \mathcal{A} \mid \varphi(\alpha) = e\}$ is the unique sublist $\mathcal{S} \subset \mathcal{A}$ that satisfies $\varphi \in \mathcal{M}(\mathcal{A}/\mathcal{S};\Gamma/\langle\mathcal{S}\rangle,G)$. This yields the following.

Proposition 3.5. *Let Γ be a finitely generated abelian group, $\mathcal{A}$ be a finite list of elements in Γ, and G be an abelian group. Then*

$$\operatorname{Hom}(\Gamma,G) = \bigsqcup_{\mathcal{S}\subset\mathcal{A}} \mathcal{M}(\mathcal{A}/\mathcal{S};\Gamma/\langle\mathcal{S}\rangle,G).$$

The structure of the intersection of the $H_{\alpha,G}$ is described by the next proposition.

Proposition 3.6. *Let $\mathcal{A}$ be a finite list of elements in a finitely generated abelian group Γ. Then*

$$\begin{aligned} \bigcap_{\alpha\in\mathcal{A}} H_{\alpha,G} &\simeq \operatorname{Hom}(\Gamma/\langle\mathcal{A}\rangle, G) \\ &\simeq \operatorname{Hom}((\Gamma/\langle\mathcal{A}\rangle)_{\mathrm{tor}}, G) \times G^{r_\Gamma - r_\mathcal{A}}. \end{aligned}$$

Proof. Recall that $H_{\alpha,G} \simeq \operatorname{Hom}(\Gamma/\langle\alpha\rangle, G)$. Hence $\bigcap_{\alpha\in\mathcal{A}} H_{\alpha,G} \simeq \operatorname{Hom}(\Gamma/\langle\mathcal{A}\rangle, G)$. From the structure theorem for finitely generated abelian groups, we may assume that $\Gamma/\langle\mathcal{A}\rangle \simeq (\Gamma/\langle\mathcal{A}\rangle)_{\mathrm{tor}} \oplus \mathbb{Z}^{r_\Gamma - r_\mathcal{A}}$. The result follows from this isomorphism. □

4. G-TUTTE POLYNOMIALS

In this section, we define the (multivariate) G-Tutte polynomial and the G-characteristic polynomial for a finite list $\mathcal{A} \subset \Gamma$ and an abelian group G. We present the deletion-contraction formulas for these polynomials. We also give several specializations of G-Tutte polynomials.

4.1. **Torsion-wise finite abelian groups.**

Definition 4.1. An abelian group G is called *torsion-wise finite* if the subgroup of d-torsion points $G[d]$ is finite for all $d > 0$.

Example 4.2. The following are examples of torsion-wise finite abelian groups.

- Every torsion-free abelian group (e.g., $\{0\}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$) is torsion-wise finite.
- Every finitely generated abelian group is torsion-wise finite.
- Every subgroup of the multiplicative group $K^\times$ for any field K is torsion-wise finite (e.g., $(S^1, \times)$ and $(\mathbb{C}^\times, \times)$).

Example 4.3. $(\mathbb{Z}/2\mathbb{Z})^\infty$ is not a torsion-wise finite group.

The class of torsion-wise finite groups is closed under taking subgroups and finite direct products. We mainly study torsion-wise finite groups of the form

$$G \simeq F \times (S^1)^p \times \mathbb{R}^q,$$

where F is a finite abelian group and $p, q \geq 0$.

Proposition 4.4. *Let G be a torsion-wise finite abelian group. Let F be a finite abelian group. Then $\operatorname{Hom}(F, G)$ is finite.*

Proof. By the structure theorem, we may assume that $F \simeq \mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_k\mathbb{Z}$. Then

$$\begin{aligned}\operatorname{Hom}(F, G) &= \operatorname{Hom}(\mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_k\mathbb{Z}, G)\\ &= G[d_1] \times \cdots \times G[d_k]\end{aligned}$$

is finite by definition. $\square$

The next proposition is useful later (we omit the proofs).

Proposition 4.5. *(1)* $\operatorname{Hom}(\Gamma, G_1 \times G_2) \simeq \operatorname{Hom}(\Gamma, G_1) \times \operatorname{Hom}(\Gamma, G_2)$. *In particular, if* $\operatorname{Hom}(\Gamma, G_1 \times G_2)$ *is finite, then* $\# \operatorname{Hom}(\Gamma, G_1 \times G_2) = \# \operatorname{Hom}(\Gamma, G_1) \times \# \operatorname{Hom}(\Gamma, G_2)$.

(2) Let d_1, d_2 be positive integers. Then

$$\operatorname{Hom}(\mathbb{Z}/d_1\mathbb{Z}, \mathbb{Z}/d_2\mathbb{Z}) \simeq \operatorname{Hom}(\mathbb{Z}/d_2\mathbb{Z}, \mathbb{Z}/d_1\mathbb{Z}) \simeq \mathbb{Z}/\gcd(d_1, d_2)\mathbb{Z}.$$

In particular, $\# \operatorname{Hom}(\mathbb{Z}/d_1\mathbb{Z}, \mathbb{Z}/d_2\mathbb{Z}) = \gcd(d_1, d_2)$.

4.2. G**-Tutte polynomials for torsion-wise finite abelian groups.**

Definition 4.6. Let Γ be a finitely generated abelian group, $\mathcal{A}$ a list of elements in Γ, and G a torsion-wise finite abelian group. Define the G-multiplicity $m(\mathcal{A}; G) \in \mathbb{Z}$ by

$$m(\mathcal{A}; G) := \# \operatorname{Hom}\left((\Gamma/\langle\mathcal{A}\rangle)_{\mathrm{tor}}, G\right).$$

(Recall that $(\Gamma/\langle\mathcal{A}\rangle)_{\mathrm{tor}}$ is the torsion part of the group $\Gamma/\langle\mathcal{A}\rangle$.)

Let us describe $m(\mathcal{A};G)$ more explicitly. Because $\Gamma/\langle\mathcal{A}\rangle$ is a finitely generated abelian group, it is isomorphic to a group of the form $\mathbb{Z}/d_1\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}/d_r\mathbb{Z}\oplus\mathbb{Z}^{r_\Gamma-r_{\mathcal{A}}}$. Thus $(\Gamma/\langle\mathcal{A}\rangle)_{\mathrm{tor}}\simeq\bigoplus_{i=1}^r\mathbb{Z}/d_i\mathbb{Z}$, and

$$\mathrm{Hom}\left((\Gamma/\langle\mathcal{A}\rangle)_{\mathrm{tor}},G\right)\simeq\bigoplus_{i=1}^r G[d_i].$$

Therefore, $m(\mathcal{A};G)=\prod_{i=1}^r\#G[d_i]$.

Remark 4.7. It is also easily seen that $\mathrm{Hom}\left((\Gamma/\langle\mathcal{A}\rangle)_{\mathrm{tor}},G\right)$ is (non-canonically) isomorphic to $\mathrm{Tor}_1^{\mathbb{Z}}(\Gamma/\langle\mathcal{A}\rangle,G)$. Hence we may define $m(\mathcal{A};G):=\#\,\mathrm{Tor}_1^{\mathbb{Z}}(\Gamma/\langle\mathcal{A}\rangle,G)$.

Definition 4.8. Let $\mathcal{A}=\{\alpha_1,\ldots,\alpha_n\}\subset\Gamma$ be a finite list of elements in a finitely generated abelian group Γ, and G a torsion-wise finite group. Recall that r_Γ and $r_{\mathcal{S}}$ denote the rank of Γ and $\langle\mathcal{S}\rangle$, respectively (§3.1).

(1) Define the *multivariate G-Tutte polynomial* of $\mathcal{A}$ and G by

$$Z_{\mathcal{A}}^G(q,v_1,\ldots,v_n):=\sum_{\mathcal{S}\subset\mathcal{A}}m(\mathcal{S};G)q^{-r_{\mathcal{S}}}\prod_{\alpha_i\in\mathcal{S}}v_i.$$

(2) Define the *G-Tutte polynomial* of $\mathcal{A}$ and G by

$$T_{\mathcal{A}}^G(x,y):=\sum_{\mathcal{S}\subset\mathcal{A}}m(\mathcal{S};G)(x-1)^{r_{\mathcal{A}}-r_{\mathcal{S}}}(y-1)^{\#\mathcal{S}-r_{\mathcal{S}}}.$$

(3) Define the *G-characteristic polynomial* of $\mathcal{A}$ and G by

$$\chi_{\mathcal{A}}^G(t):=\sum_{\mathcal{S}\subset\mathcal{A}}(-1)^{\#\mathcal{S}}m(\mathcal{S};G)\cdot t^{r_\Gamma-r_{\mathcal{S}}}.$$

These three polynomials are related by the following formulas, as in the cases of the Tutte and the arithmetic Tutte polynomials ([10, 28, 32]).

$$\begin{aligned}T_{\mathcal{A}}^G(x,y)&=(x-1)^{r_{\mathcal{A}}}\cdot Z_{\mathcal{A}}^G((x-1)(y-1),y-1,\ldots,y-1),\\ \chi_{\mathcal{A}}^G(t)&=(-1)^{r_{\mathcal{A}}}\cdot t^{r_\Gamma-r_{\mathcal{A}}}\cdot T_{\mathcal{A}}^G(1-t,0).\end{aligned}$$

Recall that $\alpha\in\mathcal{A}$ is called a loop (resp. coloop) if $\alpha\in\Gamma_{\mathrm{tor}}$ (resp. $r_{\mathcal{A}}=r_{\mathcal{A}\smallsetminus\{\alpha\}}+1$). An element α that is neither a loop nor a coloop is called proper ([14, §4.4]).

Lemma 4.9. *Let $(\mathcal{A},\mathcal{A}',\mathcal{A}'')$ be the triple associated with $\alpha_i\in\mathcal{A}$. Then*

$$Z_{\mathcal{A}}^G(q,\boldsymbol{v})=\begin{cases}Z_{\mathcal{A}'}^G(q,\boldsymbol{v})+v_i\cdot Z_{\mathcal{A}''}^G(q,\boldsymbol{v}), & \text{if } \alpha_i \text{ is a loop,}\\ Z_{\mathcal{A}'}^G(q,\boldsymbol{v})+v_i\cdot q^{-1}\cdot Z_{\mathcal{A}''}^G(q,\boldsymbol{v}), & \text{otherwise.}\end{cases}$$

Proof. Similar to [10, Lemma 3.2]. □

Corollary 4.10. *The G-Tutte polynomials satisfy*

$$T^G_{\mathcal{A}}(x,y) = \begin{cases} T^G_{\mathcal{A}'}(x,y) + (y-1)T^G_{\mathcal{A}''}(x,y), & \text{if } \alpha_i \text{ is a loop,} \\ (x-1)T^G_{\mathcal{A}'}(x,y) + T^G_{\mathcal{A}''}(x,y), & \text{if } \alpha_i \text{ is a coloop,} \\ T^G_{\mathcal{A}'}(x,y) + T^G_{\mathcal{A}''}(x,y), & \text{if } \alpha_i \text{ is proper.} \end{cases}$$

Corollary 4.11. *The G-characteristic polynomials satisfy*

$$\chi^G_{\mathcal{A}}(t) = \chi^G_{\mathcal{A}'}(t) - \chi^G_{\mathcal{A}''}(t).$$

4.3. **Specializations.** The G-Tutte polynomial has several specializations.

Proposition 4.12. *Let $\mathcal{A}$ be a list in the free abelian group $\Gamma = \mathbb{Z}^\ell$. Suppose that G is a torsion-free abelian group. Then $T^G_{\mathcal{A}}(x,y) = T_{\mathcal{A}}(x,y)$ and $\chi^G_{\mathcal{A}}(t) = \chi_{\mathcal{A}}(t)$.*

Proof. This follows from $\#\operatorname{Hom}((\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{tor}}, G) = 1$ for $\mathcal{S} \subset \mathcal{A}$. □

Proposition 4.13. *Let $\mathcal{A}$ be a list in the finitely generated abelian group Γ, and let $G = S^1$ or $\mathbb{C}^\times$. Then $T^G_{\mathcal{A}}(x,y) = T^{\mathrm{arith}}_{\mathcal{A}}(x,y)$.*

Proof. Note that $\#\operatorname{Hom}((\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{tor}}, G) = \#(\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{tor}}$, which is equal to the multiplicity $m(\mathcal{S})$ in the definition of the arithmetic Tutte polynomial. □

The arithmetic Tutte polynomial can also be obtained as another specialization. Suppose that $(\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{tor}} \simeq \bigoplus_{i=1}^{k_{\mathcal{S}}} \mathbb{Z}/d_{\mathcal{S},i}\mathbb{Z}$, where $k_{\mathcal{S}} \geq 0$ and $d_{\mathcal{S},i} \mid d_{\mathcal{S},i+1}$. Define $\rho_{\mathcal{A}}$ by

$$\rho_{\mathcal{A}} := \operatorname{lcm}(d_{\mathcal{S},k_{\mathcal{S}}} \mid \mathcal{S} \subset \mathcal{A}).$$

Proposition 4.14. $T^{\mathbb{Z}/\rho_{\mathcal{A}}\mathbb{Z}}_{\mathcal{A}}(x,y) = T^{\mathrm{arith}}_{\mathcal{A}}(x,y)$.

Proof. Let $G = \mathbb{Z}/\rho_{\mathcal{A}}\mathbb{Z}$. By Proposition 4.5, because $d_{\mathcal{S},i} \mid \rho_{\mathcal{A}}$, we have $\#\operatorname{Hom}(\mathbb{Z}/d_{\mathcal{S},i}\mathbb{Z}, G) = d_{\mathcal{S},i}$ for all $\mathcal{S} \subset \mathcal{A}$ and $1 \leq i \leq k_{\mathcal{S}}$. Furthermore, $m(\mathcal{S}; G) = m(\mathcal{S})$ for all $\mathcal{S}$. □

Example 4.15. In [12, 32], partition functions of abelian group valued Potts models were studied, which were generalized to the arithmetic matroid setting by Brändén-Moci [10, Theorem 7.4]. Brändén-Moci's polynomial $Z_{\mathcal{L}}(\Gamma, H, \boldsymbol{v})$ is, in our terminology, equal to $(\#H)^{r_\Gamma} \cdot Z^H_{\mathcal{L}}(\#H, v_1, \ldots, v_n)$, where $\mathcal{L}$ is a list of n elements in Γ, and H is a finite abelian group. In the same paper, Brändén-Moci also defined the Tutte quasi-polynomial $Q_{\mathcal{L}}(x,y)$, which is equal to $T^{\mathbb{Z}/(x-1)(y-1)\mathbb{Z}}_{\mathcal{L}}(x,y)$ for any fixed integers x and y.

Example 4.16. Both the modified Tutte-Krushkal-Renhardy polynomial for a finite CW-complex (see [4, §3], [16, §4] for details) and Bibby's Tutte polynomial for an elliptic arrangement ([7, Remark 4.4]) can be expressed using $T^{S^1\times S^1}_{\mathcal{A}}(x,y)$.

We give a representation of the constituents of characteristic quasi-polynomials in terms of G-characteristic polynomials in §5.2.

4.4. **Changing the group Γ.** Let $\sigma : \Gamma_1 \longrightarrow \Gamma_2$ be a homomorphism between finitely generated abelian groups. The map σ induces a homomorphism

$$\sigma^* : \mathrm{Hom}(\Gamma_2, G) \longrightarrow \mathrm{Hom}(\Gamma_1, G). \tag{4.1}$$

Let $\alpha \in \Gamma_1$. It is easily seen that $(\sigma^*)^{-1}(H_{\alpha,G}) = H_{\sigma(\alpha),G}$. Hence (4.1) induces a map between the complements

$$\sigma^*|_{\mathcal{M}(\sigma(\mathcal{A});\Gamma_2,G)} : \mathcal{M}(\sigma(\mathcal{A});\Gamma_2,G) \longrightarrow \mathcal{M}(\mathcal{A};\Gamma_1,G).$$

A natural question is to compare $T^G_{\mathcal{A}}(x,y)$ and $T^G_{\sigma(\mathcal{A})}(x,y)$. This comparison is in general difficult. However, in the case where $\Gamma_1 = \Gamma_2 = \mathbb{Z}^\ell$ and G is a connected Lie group, the constant terms of the G-characteristic polynomials can be controlled by $\det(\sigma)$.

Proposition 4.17. *Let $\Gamma = \mathbb{Z}^\ell$, $\sigma : \Gamma \longrightarrow \Gamma$ be a homomorphism, $\mathcal{A}$ be a finite list of elements in Γ, and $G = (S^1)^p \times \mathbb{R}^q$ with $p > 0$. Then*

$$\chi^G_{\sigma(\mathcal{A})}(0) = |\det(\sigma)|^p \cdot \chi^G_{\mathcal{A}}(0).$$

Proof. By the definition (Definition 4.8) of the G-characteristic polynomial, $\chi^G_{\mathcal{A}}(t)$ is divisible by $t^{r_\Gamma - r_{\mathcal{A}}}$. If $\det(\sigma) = 0$, then $r_{\sigma(\mathcal{A})} < \ell = r_\Gamma$, and $\chi^G_{\sigma(\mathcal{A})}(t)$ is divisible by t. Therefore the left-hand side vanishes, and the assertion holds trivially.

We assume instead that $\det(\sigma) \neq 0$. Note that for a sublattice $L \subset \Gamma$ of rank ℓ, we have $(\Gamma : \sigma(L)) = |\det(\sigma)| \cdot (\Gamma : L)$. Second, if $r_{\mathcal{S}} = \ell$, then $(\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{tor}} = \Gamma/\langle\mathcal{S}\rangle$, and we have $m(\mathcal{S};G) = m(\mathcal{S};S^1)^p = \#(\Gamma/\langle\mathcal{S}\rangle)^p$. Third, because $\sigma : \Gamma \longrightarrow \Gamma$ is injective, $r_{\sigma(\mathcal{S})} = r_{\mathcal{S}}$ and $\#\sigma(\mathcal{S}) = \#\mathcal{S}$ for every sublist $\mathcal{S} \subset \mathcal{A}$. Therefore,

$$\begin{aligned}
\chi^G_{\sigma(\mathcal{A})}(0) &= \sum_{\substack{\sigma(\mathcal{S}) \subset \sigma(\mathcal{A}) \\ r_{\sigma(\mathcal{S})} = \ell}} (-1)^{\#\sigma(\mathcal{S})} m(\sigma(\mathcal{S});G) \\
&= \sum_{\substack{\mathcal{S} \subset \mathcal{A} \\ r_{\mathcal{S}} = \ell}} (-1)^{\#\mathcal{S}} m(\sigma(\mathcal{S});G) \\
&= \sum_{\substack{\mathcal{S} \subset \mathcal{A} \\ r_{\mathcal{S}} = \ell}} (-1)^{\#\mathcal{S}} |\det(\sigma)|^p m(\mathcal{S};G) \\
&= |\det(\sigma)|^p \cdot \chi^G_{\mathcal{A}}(0).
\end{aligned}$$

□

4.5. **The case Γ is finite.** If the group Γ is finite (or equivalently $r_\Gamma = 0$), then $r_{\mathcal{A}} = r_{\mathcal{S}} = 0$. Hence $T_{\mathcal{A}}^G(x,y)$ is a polynomial in y by definition. In this case, the coefficients of the G-Tutte polynomial can be explicitly expressed. More generally, we can prove the following.

Theorem 4.18. *Let $\mathcal{A}$ be a finite list of elements in a finitely generated abelian group Γ, and let G be a torsion-wise finite abelian group. Suppose that $\mathcal{A}$ is contained in Γ_{tor}. Then*

$$T_{\mathcal{A}}^G(x,y) = \sum_{k=0}^{\#\mathcal{A}} \left(\sum_{\substack{\mathcal{S}\subset\mathcal{A}\\ \#\mathcal{S}=k}} \#\mathcal{M}(\mathcal{A}/\mathcal{S}; \Gamma_{\mathrm{tor}}/\langle\mathcal{S}\rangle, G) \right) y^k.$$

In particular, $T_{\mathcal{A}}^G(x,y)$ is a polynomial in y with positive coefficients.

Proof. By assumption, $r_{\mathcal{A}} = r_{\mathcal{S}} = 0$ and $(\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{tor}} = \Gamma_{\mathrm{tor}}/\langle\mathcal{S}\rangle$ for every $\mathcal{S} \subset \mathcal{A}$. Using Proposition 3.5, we have

$$\begin{aligned}
T_{\mathcal{A}}^G(x,y) &= \sum_{\mathcal{S}\subset\mathcal{A}} \#\operatorname{Hom}(\Gamma_{\mathrm{tor}}/\langle\mathcal{S}\rangle, G)\cdot(y-1)^{\#\mathcal{S}} \\
&= \sum_{\mathcal{S}\subset\mathcal{A}} \#\operatorname{Hom}(\Gamma_{\mathrm{tor}}/\langle\mathcal{S}\rangle, G)\cdot\sum_{k=0}^{\#\mathcal{S}} y^k\cdot(-1)^{\#\mathcal{S}-k}\cdot\binom{\#\mathcal{S}}{k} \\
&= \sum_{k=0}^{\#\mathcal{A}} y^k\cdot\sum_{\substack{\mathcal{S}\subset\mathcal{A}\\ \#\mathcal{S}\geq k}}(-1)^{\#\mathcal{S}-k}\binom{\#\mathcal{S}}{k}\sum_{\mathcal{S}\subset\mathcal{T}\subset\mathcal{A}}\#\mathcal{M}(\mathcal{A}/\mathcal{T};\Gamma_{\mathrm{tor}}/\langle\mathcal{T}\rangle, G) \\
&= \sum_{k=0}^{\#\mathcal{A}} y^k\cdot\sum_{\substack{\mathcal{T}\subset\mathcal{A}\\ \#\mathcal{T}\geq k}}\#\mathcal{M}(\mathcal{A}/\mathcal{T};\Gamma_{\mathrm{tor}}/\langle\mathcal{T}\rangle, G)\cdot\sum_{\substack{\mathcal{S}\subset\mathcal{T}\\ \#\mathcal{S}\geq k}}(-1)^{\#\mathcal{S}-k}\binom{\#\mathcal{S}}{k} \\
&= \sum_{k=0}^{\#\mathcal{A}} y^k\cdot\sum_{\substack{\mathcal{T}\subset\mathcal{A}\\ \#\mathcal{T}\geq k}}\#\mathcal{M}(\mathcal{A}/\mathcal{T};\Gamma_{\mathrm{tor}}/\langle\mathcal{T}\rangle, G)\cdot\sum_{k\leq m\leq\#\mathcal{T}}(-1)^{m-k}\binom{m}{k}\cdot\binom{\#\mathcal{T}}{m} \\
&= \sum_{k=0}^{\#\mathcal{A}} y^k\cdot\sum_{\substack{\mathcal{T}\subset\mathcal{A}\\ \#\mathcal{T}\geq k}}\#\mathcal{M}(\mathcal{A}/\mathcal{T};\Gamma_{\mathrm{tor}}/\langle\mathcal{T}\rangle, G)\cdot\binom{\#\mathcal{T}}{k}\sum_{k\leq m\leq\#\mathcal{T}}(-1)^{m-k}\binom{\#\mathcal{T}-k}{m-k} \\
&= \sum_{k=0}^{\#\mathcal{A}} y^k\cdot\sum_{\substack{\mathcal{T}\subset\mathcal{A}\\ \#\mathcal{T}= k}}\#\mathcal{M}(\mathcal{A}/\mathcal{T};\Gamma_{\mathrm{tor}}/\langle\mathcal{T}\rangle, G).
\end{aligned}$$

$\square$

5. EULER CHARACTERISTIC AND POINT COUNTING

In this section, we prove formulas that express the Euler characteristic (in the case that G is an abelian Lie group with finitely many components) and the cardinality (in the case that G is finite) of the complement as a special value of the G-characteristic polynomial. We then describe the constituents of characteristic quasi-polynomials in terms of G-characteristic polynomials.

5.1. **Euler characteristic of the complement.** We first recall the notion of Euler characteristic for semialgebraic sets (see [13, 5] for further details). Every semialgebraic set X has a decomposition $X = \bigsqcup_{i=1}^N X_i$ such that each X_i is a semialgebraic subset that is semialgebraically homeomorphic to the open simplex $\sigma_{d_i} = \{(x_1, \ldots, x_{d_i}) \in \mathbb{R}^{d_i} \mid x_i > 0, \sum x_i < 1\}$ for some $d_i = \dim X_i$. The semialgebraic Euler characteristic of X is defined by

$$e_{\text{semi}}(X) = \sum_{i=1}^{N} (-1)^{d_i}.$$

The Euler characteristic $e_{\text{semi}}(X)$ is independent of the choice of decomposition. Furthermore, it satisfies the following additivity and multiplicativity properties:

- Let X be a semialgebraic set. Let $Y \subset X$ be a semialgebraic subset. Then $e_{\text{semi}}(X) = e_{\text{semi}}(X \smallsetminus Y) + e_{\text{semi}}(Y)$.
- Let X and Y be semialgebraic sets. Then $e_{\text{semi}}(X \times Y) = e_{\text{semi}}(X) \times e_{\text{semi}}(Y)$.

Remark 5.1. Unlike the topological Euler characteristic $e_{\text{top}}(X) = \sum(-1)^i b_i(X)$, the semialgebraic Euler characteristic $e_{\text{semi}}(X)$ is not homotopy invariant. (Even contractible semialgebraic sets have different values: e.g., $e_{\text{semi}}([0,1]) = 1$, $e_{\text{semi}}(\mathbb{R}_{\geq 0}) = 0$, $e_{\text{semi}}(\mathbb{R}) = -1$.) However, if X is a manifold (without boundary), then $e_{\text{semi}}(X)$ and $e_{\text{top}}(X)$ are related by the following formula:

$$e_{\text{semi}}(X) = (-1)^{\dim X} \cdot e_{\text{top}}(X).$$

Here we assume that G is an abelian Lie group with finitely many connected components. Then G is of the form $G = (S^1)^p \times \mathbb{R}^q \times F$, where F is a finite abelian group. Such a group G can be realized as a semialgebraic set, with the group operations defined by C^∞ semialgebraic maps. Hence subsets defined by using group operations are always semialgebraic sets.

The Euler characteristics of G are easily computed as

$$e_{\text{semi}}(G) = \begin{cases} 0, & \text{if } p > 0, \\ (-1)^{p+q} \cdot \#F, & \text{if } p = 0, \end{cases}$$
$$e_{\text{top}}(G) = \begin{cases} 0, & \text{if } p > 0, \\ \#F, & \text{if } p = 0. \end{cases}$$

Let $\mathcal{A}$ be a finite list of elements in a finitely generated abelian group Γ. The space $\mathcal{M}(\mathcal{A}; \Gamma, G)$ is a semialgebraic set, and, if it is not empty, it is also a manifold (without boundary) of $\dim \mathcal{M}(\mathcal{A}; \Gamma, G) = r_\Gamma \cdot \dim G$.

The G-Tutte polynomial can be used to compute the Euler characteristic of $\mathcal{M}(\mathcal{A}; \Gamma, G)$.

Theorem 5.2. *Let G be an abelian Lie group with finitely many connected components, and let $g = \dim G$. Then,*

$$e_{\text{semi}}(\mathcal{M}(\mathcal{A}; \Gamma, G)) = \chi^G_{\mathcal{A}}(e_{\text{semi}}(G)),$$

or equivalently,

$$e_{\mathrm{top}}(\mathcal{M}(\mathcal{A};\Gamma,G)) = (-1)^{g\cdot r_\Gamma} \cdot \chi_{\mathcal{A}}^{G}\left((-1)^g \cdot e_{\mathrm{top}}(G)\right).$$

Proof. By the additivity of $e_{\mathrm{semi}}(-)$, we can compute $e_{\mathrm{semi}}(\mathcal{M}(\mathcal{A};\Gamma,G))$ by using the principle of inclusion-exclusion together with Proposition 3.6 as follows:

$$\begin{aligned} e_{\mathrm{semi}}(\mathcal{M}(\mathcal{A};\Gamma,G)) &= \sum_{\mathcal{S}\subset\mathcal{A}} (-1)^{\#\mathcal{S}} \cdot e_{\mathrm{semi}}\left(\bigcap_{\alpha\in\mathcal{S}} H_{\alpha,G}\right) \\ &= \sum_{\mathcal{S}\subset\mathcal{A}} (-1)^{\#\mathcal{S}} \cdot m(\mathcal{S};G) \cdot e_{\mathrm{semi}}(G)^{r_\Gamma - r_{\mathcal{S}}} \\ &= \chi_{\mathcal{A}}^{G}(e_{\mathrm{semi}}(G)). \end{aligned}$$

□

Remark 5.3. We can also prove Theorem 5.2 by using deletion-contraction formula. Note that if $\mathcal{A} = \emptyset$, then $\chi_{\mathcal{A}}^{G}(t) = \#\operatorname{Hom}(\Gamma_{\mathrm{tor}}, G) \cdot t^{r_\Gamma}$. Hence $\chi_{\mathcal{A}}^{G}(e_{\mathrm{semi}}(G)) = \#\operatorname{Hom}(\Gamma_{\mathrm{tor}}, G) \cdot e_{\mathrm{semi}}(G)^{r_\Gamma} = e_{\mathrm{semi}}(\operatorname{Hom}(\Gamma, G))$. Theorem 5.2 then follows easily from Proposition 3.4 and Corollary 4.11 by induction on $\#\mathcal{A}$.

5.2. **Point counting in complements.** In the case that G is finite, the complement $\mathcal{M}(\mathcal{A};\Gamma,G)$ is also a finite set. Every finite set can be considered as a 0-dimensional semialgebraic set whose Euler characteristic is equal to its cardinality. The following theorem immediately follows from Theorem 5.2.

Theorem 5.4. *Let $\mathcal{A}$ be a finite list of elements in a finitely generated abelian group Γ, and let G be a finite abelian group. Then*

$$\#\mathcal{M}(\mathcal{A};\Gamma,G) = \chi_{\mathcal{A}}^{G}(\#G).$$

We can now describe the constituents of characteristic quasi-polynomials as G-characteristic polynomials (see §2.2).

Theorem 5.5. *(See §2.2 for notation.) Let $\mathcal{A}$ be a finite list of elements in $\Gamma = \mathbb{Z}^\ell$, and let k be a divisor of $\rho_{\mathcal{A}}$. The k-constituent $f_k(t)$ of the characteristic quasi-polynomial $\chi_{\mathcal{A}}^{\mathrm{quasi}}(q)$ is equal to*

$$f_k(t) = \chi_{\mathcal{A}}^{\mathbb{Z}/k\mathbb{Z}}(t).$$

Proof. Let $q \in \mathbb{Z}_{>0}$ be a positive integer, and suppose that $\gcd(q, \rho_{\mathcal{A}}) = k$. Because $d_{\mathcal{S},i} \mid \rho_{\mathcal{A}}$, we have

$$\gcd(q, d_{\mathcal{S},i}) = \gcd(k, d_{\mathcal{S},i})$$

for any $\mathcal{S} \subset \mathcal{A}$ and $1 \le i \le r_{\mathcal{S}}$. It follows from Proposition 4.5 that $\#(\mathbb{Z}/q\mathbb{Z})[d_{\mathcal{S},i}] = \#(\mathbb{Z}/k\mathbb{Z})[d_{\mathcal{S},i}]$, and hence

$$m(\mathcal{S}; \mathbb{Z}/q\mathbb{Z}) = m(\mathcal{S}; \mathbb{Z}/k\mathbb{Z}).$$

Using Theorem 5.4,

$$\begin{aligned} f_k(q) &= \#\mathcal{M}(\mathcal{A}; \Gamma, \mathbb{Z}/q\mathbb{Z}) = \chi_{\mathcal{A}}^{\mathbb{Z}/q\mathbb{Z}}(q) \\ &= \sum_{\mathcal{S}\subset\mathcal{A}} (-1)^{\#\mathcal{S}} m(\mathcal{S}; \mathbb{Z}/q\mathbb{Z}) \cdot q^{r_\Gamma - r_{\mathcal{S}}} \\ &= \sum_{\mathcal{S}\subset\mathcal{A}} (-1)^{\#\mathcal{S}} m(\mathcal{S}; \mathbb{Z}/k\mathbb{Z}) \cdot q^{r_\Gamma - r_{\mathcal{S}}} \\ &= \chi_{\mathcal{A}}^{\mathbb{Z}/k\mathbb{Z}}(q). \end{aligned}$$

Because $f_k(t)$ and $\chi_{\mathcal{A}}^{\mathbb{Z}/k\mathbb{Z}}(t)$ are polynomials in t that have common values for infinitely many $q > 0$, $f_k(t) = \chi_{\mathcal{A}}^{\mathbb{Z}/k\mathbb{Z}}(t)$. □

Corollary 5.6. *The most degenerate constituent $f_{\rho_{\mathcal{A}}}(t)$ of the characteristic quasi-polynomial $\chi_{\mathcal{A}}^{\mathrm{quasi}}(q)$ is equal to both $\chi_{\mathcal{A}}^{\mathbb{C}^\times}(t)$ and $\chi_{\mathcal{A}}^{\mathrm{arith}}(t)$.*

Proof. By Proposition 4.13, Proposition 4.14 (and its specialization to the characteristic polynomial) and Theorem 5.5, we have

$$\chi_{\mathcal{A}}^{\mathrm{arith}}(t) = \chi_{\mathcal{A}}^{\mathbb{C}^\times}(t) = \chi_{\mathcal{A}}^{\mathbb{Z}/\rho_{\mathcal{A}}\mathbb{Z}}(t) = f_{\rho_{\mathcal{A}}}(t).$$

□

Remark 5.7. Corollary 5.6 enables us to compute the Poincaré polynomial $P_{\mathcal{M}(\mathcal{A};\Gamma,\mathbb{C}^\times)}(t)$ of the associated toric arrangement $\mathcal{A}(\mathbb{C}^\times)$ via modulo q counting, where $\rho_{\mathcal{A}} \mid q$.

In particular, the Poincaré polynomial of a toric arrangement can be computed if the characteristic quasi-polynomial is known. We will compute the Poincaré polynomial of the toric arrangement for exceptional root systems in §6.

6. Examples: root systems

As we saw in §5.2 (Remark 5.7), the Poincaré polynomial of a toric arrangement can be computed from the characteristic quasi-polynomial. By applying recent results on characteristic quasi-polynomials of root systems, we prove that Poincaré polynomials satisfy a certain functional equation. We also prove a formula that expresses the Euler characteristic.

Let Φ be an irreducible root system of rank ℓ, and let $\Gamma = \mathbb{Z}\cdot\Phi$ be the root lattice of Φ. Consider the list $\mathcal{A}_\Phi := \Phi^+ \subset \Gamma$ of positive roots. The characteristic quasi-polynomial $\chi_{\mathcal{A}_\Phi}^{\mathrm{quasi}}(q)$ was computed by Suter [33] and

Kamiya-Takemura-Terao [23]. The most degenerate constituents $f_{\rho_{\mathcal{A}_\Phi}}(t)$ are shown in Table 1.

Φ	$\#W$	h	f	$\rho_{\mathcal{A}_\Phi}$	$f_{\rho_{\mathcal{A}}}(t) = \chi_{\mathcal{A}}^{\mathbb{C}^\times}(t)$
A_ℓ	$(\ell+1)!$	$\ell+1$	$\ell+1$	1	$\prod_{k=1}^{\ell}(t-k)$
B_ℓ, C_ℓ	$2^\ell \cdot \ell!$	2ℓ	2	2	$(t-\ell)\prod_{k=1}^{\ell-1}(t-2k)$
D_ℓ	$2^{\ell-1}\cdot \ell!$	$2\ell-2$	4	2	$(t^2-2(\ell-1)t+\frac{\ell(\ell-1)}{2})\prod_{k=1}^{\ell-2}(t-2k)$
E_6	$2^7\cdot 3^4\cdot 5$	12	3	6	$(t-6)^2(t^4-24t^3+186t^2-504t+480)$
E_7	$2^{10}\cdot 3^4\cdot 5\cdot 7$	18	2	12	$(t-12)(t^6-51t^5+1005t^4-9675t^3$ $+47784t^2-116064t+120960)$
E_8	$2^{14}\cdot 3^5\cdot 5^2\cdot 7$	30	1	60	$t^8-120t^7+6020t^6-163800t^5+2626008t^4$ $-25260480t^3+142577280t^2-445824000t$ $+696729600$
F_4	$2^7\cdot 3^2$	12	1	12	$t^4-24t^3+208t^2-768t+1152$
G_2	$2^2\cdot 3$	6	1	6	$t^2-6t+12$

TABLE 1. Table of root systems. (Notation: W is the Weyl group, h is the Coxeter number, f is the index of connection, and $\rho_{\mathcal{A}_\Phi}$ is the minimal period of the characteristic quasi-polynomial. See [21] for details.)

Using formula (2.2), or Theorem 7.7, the Poincaré polynomial for the corresponding toric arrangement is $P_{\mathcal{M}(\mathcal{A}_\Phi;\Gamma,\mathbb{C}^\times)}(t) = (-t)^\ell \chi_{\mathcal{A}_\Phi}^{\mathbb{C}^\times}\left(-\frac{1+t}{t}\right)$. We only show exceptional cases. (See [6] for more detailed information about the cohomology groups including the Weyl group action (except for E_8).)

$$\begin{aligned}
P_{\mathcal{M}(\mathcal{A}_{E_6};\Gamma,\mathbb{C}^\times)}(t) =& 1+42t+705t^2+6020t^3+27459t^4+63378t^5+58555t^6,\\
P_{\mathcal{M}(\mathcal{A}_{E_7};\Gamma,\mathbb{C}^\times)}(t) =& 1+70t+2016t^2+30800t^3+268289t^4+1328670t^5\\
&+3479734t^6+3842020t^7,\\
P_{\mathcal{M}(\mathcal{A}_{E_8};\Gamma,\mathbb{C}^\times)}(t) =& 1+128t+6888t^2+202496t^3+3539578t^4+37527168t^5\\
&+235845616t^6+818120000t^7+1313187309t^8,\\
P_{\mathcal{M}(\mathcal{A}_{F_4};\Gamma,\mathbb{C}^\times)}(t) =& 1+28t+286t^2+1260t^3+2153t^4,\\
P_{\mathcal{M}(\mathcal{A}_{G_2};\Gamma,\mathbb{C}^\times)}(t) =& 1+8t+19t^2.
\end{aligned}$$

It was proved in [36, Corollary 3.8] that the characteristic quasi-polynomial of a root system Φ satisfies the functional equation

$$\chi_{\mathcal{A}_\Phi}^{\text{quasi}}(h-q) = (-1)^\ell \chi_{\mathcal{A}_\Phi}^{\text{quasi}}(q),$$

where h is the Coxeter number. This equation holds even at the level of k-constituents.

$$f_k(h-q) = (-1)^\ell f_k(q), \tag{6.1}$$

for some k. (More precisely, (6.1) holds for admissible divisors k in the sense of [37, §5.3].) In particular, equation (6.1) holds for $k = \rho_{\mathcal{A}_\Phi}$ if and only if the period $\rho_{\mathcal{A}_\Phi}$ divides the Coxeter number h, which is equivalent to $\Phi \neq E_7, E_8$ (see [37, §5.3] for details). Thus we have the following.

Proposition 6.1. *Let Φ be an irreducible root system. Assume that $\Phi \neq E_7, E_8$. Then the characteristic polynomial of the associated toric arrangement satisfies*

$$\chi^{\mathbb{C}^\times}_{\mathcal{A}_\Phi}(h-t) = (-1)^\ell \chi^{\mathbb{C}^\times}_{\mathcal{A}_\Phi}(t),$$

or, equivalently, the Poincaré polynomial satisfies the following relation:

$$P_{\mathcal{M}(\mathcal{A}_\Phi;\Gamma,\mathbb{C}^\times)}(t) = ((h+2)t+1)^\ell \cdot P_{\mathcal{M}(\mathcal{A}_\Phi;\Gamma,\mathbb{C}^\times)}\left(\frac{-t}{(h+2)t+1}\right).$$

We next describe the Euler characteristic of $\mathcal{M}(\mathcal{A}_\Phi;\Gamma,\mathbb{C}^\times)$.

Proposition 6.2. *Let W be the Weyl group of Φ, and let f be the index of connection. The constant term of the characteristic polynomial of the associated toric arrangement can be computed as follows:*

$$\chi^{\mathbb{C}^\times}_{\mathcal{A}_\Phi}(0) = \frac{(-1)^\ell \# W}{f}.$$

Proof. By Corollary 5.6,

$$\chi^{\mathbb{C}^\times}_{\mathcal{A}_\Phi}(0) = f_{\rho_{\mathcal{A}_\Phi}}(0) = \frac{(-1)^\ell \# W}{f} L_{\overline{A^\circ}}(0) = \frac{(-1)^\ell \# W}{f},$$

where $L_{\overline{A^\circ}}(q)$ is the Ehrhart quasi-polynomial of the closed fundamental alcove $\overline{A^\circ}$ (see [21, 36] for the definition of $\overline{A^\circ}$). The second equality is obtained from [36, Proposition 3.7], and the last equality follows from $L_{\overline{A^\circ}}(0) = 1$. □

Remark 6.3. The Cartan matrix of Φ whose determinant is the index of connection f expresses the change of basis between the root lattice and the weight lattice. It follows from Propositions 4.17 and 6.2 that the constant term of the characteristic polynomial of the toric arrangement with respect to the weight lattice equals $(-1)^\ell \# W$. This gives a new proof for [28, Corollary 7.4].

Using the notation in Section 5, we can compute the Euler characteristic of $\mathcal{M}(\mathcal{A}_\Phi;\Gamma,\mathbb{C}^\times)$ as follows, noting that $e_{\mathrm{top}}(\mathbb{C}^\times) = e_{\mathrm{semi}}(\mathbb{C}^\times) = 0$:

Corollary 6.4. *(*[27, 28]*)*

$$e_{\mathrm{semi}}(\mathcal{M}(\mathcal{A}_\Phi;\Gamma,\mathbb{C}^\times)) = e_{\mathrm{top}}(\mathcal{M}(\mathcal{A}_\Phi;\Gamma,\mathbb{C}^\times)) = \frac{(-1)^\ell \# W}{f}.$$

Proof. This follows directly from Theorem 5.2 and Proposition 6.2. □

7. Poincaré polynomials for non-compact groups

In this section, we prove a formula that expresses the Poincaré polynomial in terms of the G-characteristic polynomial.

7.1. **Torus cycles.** We introduce a special class of homology cycles called torus cycles in $H_*(\mathcal{M}(\mathcal{A};\Gamma,G),\mathbb{Z})$, which are lifts of cycles in a compact torus.

Let $G = F \times (S^1)^p \times \mathbb{R}^q$, where F a finite abelian group. Write $G_{\mathrm{c}} = F \times (S^1)^p$ (compact part) and $V = \mathbb{R}^q$ (non-compact part).

Let Γ be a finitely generated abelian group. Fix a decomposition $\Gamma = \Gamma_{\mathrm{tor}} \oplus \Gamma_{\mathrm{free}}$, where $\Gamma_{\mathrm{free}} \simeq \mathbb{Z}^{r_\Gamma}$. Then

$$\mathrm{Hom}(\Gamma,G) \simeq \mathrm{Hom}(\Gamma,G_{\mathrm{c}}) \times \mathrm{Hom}(\Gamma_{\mathrm{free}},V). \tag{7.1}$$

(Note that $\mathrm{Hom}(\Gamma_{\mathrm{tor}},V)=0$.) We can decompose this further as follows:

$$\mathrm{Hom}(\Gamma,G) \simeq \mathrm{Hom}(\Gamma_{\mathrm{tor}},G_{\mathrm{c}}) \times \mathrm{Hom}(\Gamma_{\mathrm{free}},G_{\mathrm{c}}) \times \mathrm{Hom}(\Gamma_{\mathrm{free}},V). \tag{7.2}$$

The first component $\mathrm{Hom}(\Gamma_{\mathrm{tor}},G_{\mathrm{c}})$ of (7.2) is a finite abelian group, the second component $\mathrm{Hom}(\Gamma_{\mathrm{free}},G_{\mathrm{c}})$ is a compact abelian Lie group (not necessarily connected), and the third component is $\mathrm{Hom}(\Gamma_{\mathrm{free}},V) \simeq V^{r_\Gamma} \simeq \mathbb{R}^{q\cdot r_\Gamma}$.

Let $\alpha = (\beta,\eta) \in \Gamma_{\mathrm{tor}} \oplus \Gamma_{\mathrm{free}}$. According to decomposition (7.1), the subgroup $H_{\alpha,G} \subset \mathrm{Hom}(\Gamma,G)$ can be expressed as

$$H_{\alpha,G} = H_{\alpha,G_{\mathrm{c}}} \times H_{\eta,V},$$

where $H_{\alpha,G_{\mathrm{c}}} \subset \mathrm{Hom}(\Gamma,G_{\mathrm{c}})$ and $H_{\eta,V} \subset \mathrm{Hom}(\Gamma_{\mathrm{free}},V)$.

If $\alpha \in \Gamma_{\mathrm{tor}}$, or equivalently $\alpha = (\beta,0)$, then using (7.2) gives

$$H_{\alpha,G} = H_{\beta,G_{\mathrm{c}}} \times \mathrm{Hom}(\Gamma_{\mathrm{free}},G_{\mathrm{c}}) \times \mathrm{Hom}(\Gamma_{\mathrm{free}},V),$$

where $H_{\beta,G_{\mathrm{c}}}$ is a subgroup of the finite abelian group $\mathrm{Hom}(\Gamma_{\mathrm{tor}},G_{\mathrm{c}})$. In this case, $H_{\alpha,G}$ is a collection of connected components of $\mathrm{Hom}(\Gamma,G)$ (note that $H_{\beta,G_{\mathrm{c}}}$ is a finite subset of the finite abelian group $\mathrm{Hom}(\Gamma_{\mathrm{tor}},G_{\mathrm{c}})$). Similarly, the complement can be expressed as

$$\begin{aligned}\mathcal{M}(\{\alpha\};\Gamma,G) &= \mathrm{Hom}(\Gamma,G) \smallsetminus H_{\alpha,G}\\ &= (\mathrm{Hom}(\Gamma_{\mathrm{tor}},G_{\mathrm{c}}) \smallsetminus H_{\beta,G_{\mathrm{c}}}) \times \mathrm{Hom}(\Gamma_{\mathrm{free}},G_{\mathrm{c}}) \times \mathrm{Hom}(\Gamma_{\mathrm{free}},V).\end{aligned}$$

More generally, if $\mathcal{A} \subset \Gamma_{\mathrm{tor}} \subset \Gamma$, then

(7.3)
$$\begin{aligned}\mathcal{M}(\mathcal{A};\Gamma,G) &= \left(\mathrm{Hom}(\Gamma_{\mathrm{tor}},G_{\mathrm{c}}) \smallsetminus \bigcup_{\alpha\in\mathcal{A}} H_{\alpha,G_{\mathrm{c}}}\right) \times \mathrm{Hom}(\Gamma_{\mathrm{free}},G_{\mathrm{c}}) \times \mathrm{Hom}(\Gamma_{\mathrm{free}},V)\\ &= \mathcal{M}(\mathcal{A};\Gamma_{\mathrm{tor}},G_{\mathrm{c}}) \times \mathrm{Hom}(\Gamma_{\mathrm{free}},G_{\mathrm{c}}) \times \mathrm{Hom}(\Gamma_{\mathrm{free}},V).\end{aligned}$$

Therefore, $\mathcal{M}(\mathcal{A};\Gamma,G)$ is a collection of some of connected components of $\mathrm{Hom}(\Gamma,G)$.

Let $\mathcal{A} \subset \Gamma$ be a list of elements. Define $\mathcal{A}_{\mathrm{tor}} := \mathcal{A} \cap \Gamma_{\mathrm{tor}}$. As mentioned above, $\mathcal{M}(\mathcal{A}_{\mathrm{tor}};\Gamma,G)$ is a collection of components of $\mathrm{Hom}(\Gamma,G)$.

Consider the following diagram:

(7.4)
$$\begin{tikzcd}
\mathcal{M}(\mathcal{A};\Gamma,G) \arrow[r, "\subset"] & \mathcal{M}(\mathcal{A}_{\mathrm{tor}};\Gamma,G) \arrow[r, "\subset"] \arrow[d] & \mathrm{Hom}(\Gamma,G) \ni (f,t,v) \arrow[d, "\pi"] \\
 & \mathcal{M}(\mathcal{A}_{\mathrm{tor}};\Gamma,G_{\mathrm{c}}) \arrow[r, "\subset"] & \mathrm{Hom}(\Gamma,G_{\mathrm{c}}) \ni (f,t),
\end{tikzcd}$$

where $\pi : \mathrm{Hom}(\Gamma,G) \longrightarrow \mathrm{Hom}(\Gamma,G_{\mathrm{c}})$ is the projection defined by $\pi(f,t,v) = (f,t)$ for $(f,t,v) \in \mathrm{Hom}(\Gamma_{\mathrm{tor}},G_{\mathrm{c}}) \times \mathrm{Hom}(\Gamma_{\mathrm{free}},G_{\mathrm{c}}) \times \mathrm{Hom}(\Gamma_{\mathrm{free}},V) \simeq \mathrm{Hom}(\Gamma,G)$.

Now assume that $q > 0$. The fiber of the projection π is isomorphic to $\mathrm{Hom}(\Gamma,V) \simeq V^{r_\Gamma} \simeq \mathbb{R}^{q\cdot r_\Gamma}$. Then

$$\mathcal{M}(\mathcal{A}\smallsetminus\mathcal{A}_{\mathrm{tor}};\Gamma,V) = \mathrm{Hom}(\Gamma,V) \smallsetminus \bigcup_{\alpha\in\mathcal{A}\smallsetminus\mathcal{A}_{\mathrm{tor}}} H_{\alpha,V}$$

is a complement of proper subspaces. Hence it is non-empty. Fix an element $v_0 \in \mathcal{M}(\mathcal{A}\smallsetminus\mathcal{A}_{\mathrm{tor}};\Gamma,V)$. For a given $(f,t) \in \mathrm{Hom}(\Gamma,G_{\mathrm{c}})$, define $i_{v_0}(f,t) := (f,t,v_0)$. This induces a map

$$i_{v_0} : \mathcal{M}(\mathcal{A}_{\mathrm{tor}};\Gamma,G_{\mathrm{c}}) \longrightarrow \mathcal{M}(\mathcal{A};\Gamma,G),$$

which is a section of the projection $\pi|_{\mathcal{M}(\mathcal{A};\Gamma,G)} : \mathcal{M}(\mathcal{A};\Gamma,G) \longrightarrow \mathcal{M}(\mathcal{A}_{\mathrm{tor}};\Gamma,G_{\mathrm{c}})$ in (7.4).

Definition 7.1. Assume that $q > 0$. A cycle $\gamma \in H_*(\mathcal{M}(\mathcal{A};\Gamma,G),\mathbb{Z})$ is said to be a *torus cycle* if there exist a connected component $T \subset \mathcal{M}(\mathcal{A}_{\mathrm{tor}};\Gamma,G_{\mathrm{c}})$, a cycle $\widetilde{\gamma} \in H_*(T,\mathbb{Z}) \subset H_*(\mathcal{M}(\mathcal{A}_{\mathrm{tor}};\Gamma,G_{\mathrm{c}}),\mathbb{Z})$ and $v_0 \in \mathcal{M}(\mathcal{A}\smallsetminus\mathcal{A}_{\mathrm{tor}};\Gamma,V)$ such that

$$\gamma = (i_{v_0})_*(\widetilde{\gamma}).$$

The subgroup of $H_*(\mathcal{M}(\mathcal{A};\Gamma,G),\mathbb{Z})$ generated by torus cycles is denoted by $H_*^{\mathrm{torus}}(\mathcal{A}(G))$.

Remark 7.2. If $q > 1$, then the homology class $(i_{v_0})_*(\widetilde{\gamma})$ is independent of the choice of $v_0 \in \mathcal{M}(\mathcal{A}\smallsetminus\mathcal{A}_{\mathrm{tor}};\Gamma,V)$, because $\mathcal{M}(\mathcal{A}\smallsetminus\mathcal{A}_{\mathrm{tor}};\Gamma,V)$ is connected. On the other hand, if $q = 1$, then the subspace $H_{\alpha,V}$ is a real hyperplane in $\mathrm{Hom}(\Gamma, V) \simeq V^{r_\Gamma}$. Hence the homology class $(i_{v_0})_*(\widetilde{\gamma})$ may depend on the chamber containing v_0.

Lemma 7.3. *Assume that $q > 0$. Let $\alpha \in \mathcal{A} \smallsetminus \mathcal{A}_{\mathrm{tor}}$, and $\mathcal{A}' = \mathcal{A} \smallsetminus \{\alpha\}$. Then the map $\iota : H_*^{\mathrm{torus}}(\mathcal{A}(G)) \longrightarrow H_*^{\mathrm{torus}}(\mathcal{A}'(G))$ induced by the inclusion $\mathcal{M}(\mathcal{A}; \Gamma, G) \hookrightarrow \mathcal{M}(\mathcal{A}'; \Gamma, G)$ is surjective.*

Proof. Let $(i_{v_0})_*(\widetilde{\gamma}) \in H_*(\mathcal{M}(\mathcal{A}'; \Gamma, G), \mathbb{Z})$ be a torus cycle. If $v_0 \notin H_{\alpha,V}$, then $(i_{v_0})_*(\widetilde{\gamma})$ is clearly contained in the image of the map ι. If $v_0 \in H_{\alpha,V}$, since $\mathcal{M}(\mathcal{A} \smallsetminus \mathcal{A}_{\mathrm{tor}}; \Gamma, V)$ is nonempty, there exists a small perturbation v_0' of v_0 such that $v_0' \in \mathcal{M}(\mathcal{A} \smallsetminus \mathcal{A}_{\mathrm{tor}}; \Gamma, V)$ (see Remark 7.2). Then $H_*^{\mathrm{torus}}(\mathcal{A}(G)) \ni (i_{v_0'})_*(\widetilde{\gamma}) \longmapsto (i_{v_0})_*(\widetilde{\gamma}) \in H_*^{\mathrm{torus}}(\mathcal{A}'(G))$. □

7.2. **Meridian cycles.** The torus cycles introduced in the previous section are not enough to generate the homology group $H_*(\mathcal{M}(\mathcal{A}; \Gamma, G), \mathbb{Z})$. We also need to consider meridians of $H_{\alpha,G}$ to generate $H_*(\mathcal{M}(\mathcal{A}; \Gamma, G), \mathbb{Z})$.

Let us first recall the notion of layers. A layer of $\mathcal{A}(G)$ is a connected component of a non-empty intersection of elements of $\mathcal{A}(G)$. Let $\mathcal{S} \subset \mathcal{A}$. By Proposition 3.6, every connected component of $H_{\mathcal{S},G} := \bigcap_{\alpha \in \mathcal{S}} H_{\alpha,G}$ is isomorphic to

$$\left((S^1)^p \times \mathbb{R}^q\right)^{r_\Gamma - r_\mathcal{S}}.$$

We sometimes call the number $r_\mathcal{S}$ the rank of the layer. Since $H_{\emptyset,G} = \mathrm{Hom}(\Gamma, G)$, a connected component of $\mathrm{Hom}(\Gamma, G)$ is a layer of rank 0. Similarly, a connected component of $H_{\alpha,G}$ for $\alpha \in \mathcal{A} \smallsetminus \mathcal{A}_{\mathrm{tor}}$ is a layer of rank 1.

Let L be a layer. Denote the set of α such that $H_{\alpha,G}$ contains L by $\mathcal{A}_L := \{\alpha \in \mathcal{A} \mid L \subset H_{\alpha,G}\}$, and the contraction by $\mathcal{A}^L := \mathcal{A}/\mathcal{A}_L$. Note that L can be considered to be a rank 0 layer of $\mathcal{A}^L(G)$. Define

$$\begin{aligned}\mathcal{M}^L(\mathcal{A}) :=& L \smallsetminus \bigcup_{H_{\alpha,G} \not\supset L} H_{\alpha,G} \\ =& L \cap \mathcal{M}(\mathcal{A}^L; \Gamma/\langle \mathcal{A}_L \rangle, G).\end{aligned}$$

(We consider $\mathcal{M}(\mathcal{A}^L; \Gamma/\langle \mathcal{A}_L \rangle, G)$ as a subset of $\mathrm{Hom}(\Gamma, G)$ as in Proposition 3.5.)

Let $L_1 \subset \mathrm{Hom}(\Gamma, G)$ be a rank 1 layer of $\mathcal{A}(G)$, and let L_0 be the rank 0 layer that contains L_1. We wish to define the meridian homomorphism

$$\mu^\varepsilon_{L_0/L_1} : H_*(\mathcal{M}^{L_1}(\mathcal{A}), \mathbb{Z}) \longrightarrow H_{*+\varepsilon\cdot(g-1)}(\mathcal{M}^{L_0}(\mathcal{A}), \mathbb{Z}),$$

where $g = \dim G = p + q > 0$ and $\varepsilon \in \{0, 1\}$.

Since the normal bundle of L_1 in L_0 is trivial, there is a tubular neighborhood U of $\mathcal{M}^{L_1}(\mathcal{A})$ in L_0 such that $U \simeq \mathcal{M}^{L_1}(\mathcal{A}) \times D^g$ with the identification $\mathcal{M}^{L_1}(\mathcal{A}) = \mathcal{M}^{L_1}(\mathcal{A}) \times \{0\}$, where D^g is the g-dimensional disk. Then $U \cap \mathcal{M}^{L_0}(\mathcal{A}) \simeq \mathcal{M}^{L_1}(\mathcal{A}) \times D^{g*}$, where $D^{g*} = D^g \smallsetminus \{0\}$. We denote the corresponding inclusion by $i : \mathcal{M}^{L_1}(\mathcal{A}) \times D^{g*} \hookrightarrow M^{L_0}(\mathcal{A})$. For a given $\gamma \in H_*(\mathcal{M}^{L_1}(\mathcal{A}), \mathbb{Z})$, define the element $\mu^{\varepsilon}_{L_0/L_1}(\gamma) \in H_{*+\varepsilon\cdot(g-1)}(\mathcal{M}^{L_0}(\mathcal{A}), \mathbb{Z})$ as follows.

(0) For $\varepsilon = 0$, let $p_0 \in D^{g*}$. Then $\gamma \times [p_0] \in H_*(\mathcal{M}^{L_1}(\mathcal{A})) \otimes H_0(D^{g*}) \subset H_*(\mathcal{M}^{L_1}(\mathcal{A}) \times D^{g*})$, and $\mu^0_{L_0/L_1}(\gamma) := i_*(\gamma \times [p_0])$.

(1) For $\varepsilon = 1$, let $S^{g-1} \subset D^{g*}$ be a sphere of small radius. Then $\gamma \times [S^{g-1}] \in H_*(\mathcal{M}^{L_1}(\mathcal{A})) \otimes H_{g-1}(D^{g*}) \subset H_{*+g-1}(\mathcal{M}^{L_1}(\mathcal{A}) \times D^{g*})$ (this part is essentially the Gysin homomorphism). Now define $\mu^1_{L_0/L_1}(\gamma) := i_*(\gamma \times [S^{g-1}])$.

Similarly, we can define the meridian map

$$\mu^{\varepsilon}_{L_j/L_{j+1}} : H_*(\mathcal{M}^{L_{j+1}}(\mathcal{A}), \mathbb{Z}) \longrightarrow H_{*+\varepsilon\cdot(g-1)}(\mathcal{M}^{L_j}(\mathcal{A}), \mathbb{Z})$$

between layers $L_j \supset L_{j+1}$ with consecutive ranks.

Definition 7.4. A cycle $\gamma \in H_d(\mathcal{M}(\mathcal{A}; \Gamma, G), \mathbb{Z})$ is called a *meridian cycle* if there exists some $k \geq 0$ and

(a) a flag $L_0 \supset L_1 \supset \cdots \supset L_k$ of layers with $\operatorname{rank} L_j = j$, such that $L_0 \cap \mathcal{M}(\mathcal{A}; \Gamma, G) \neq \emptyset$ (or equivalently, $L_0 \subset \mathcal{M}(\mathcal{A}_{\mathrm{tor}}; \Gamma, G)$),

(b) a sequence $\varepsilon_1, \dots, \varepsilon_k \in \{0, 1\}$, and

(c) a torus cycle $\tau \in H_{d-(g-1)\sum_{i=1}^{k} \varepsilon_i}(\mathcal{M}^{L_k}(\mathcal{A}), \mathbb{Z})$,

such that

$$\gamma = \mu^{\varepsilon_1}_{L_0/L_1} \circ \mu^{\varepsilon_2}_{L_1/L_2} \circ \cdots \circ \mu^{\varepsilon_k}_{L_{k-1}/L_k}(\tau).$$

We call the minimum such k the depth of γ.

By definition, a meridian cycle of depth 0 is a torus cycle of a connected component L_0. Furthermore, a cycle $\gamma \in H_*(\mathcal{M}(\mathcal{A}; \Gamma, G), \mathbb{Z})$ is a meridian cycle of depth $k > 0$ if and only if there exist layers $L_0 \supset L_1$ of rank 0 and 1 respectively, with $L_0 \cap \mathcal{M}(\mathcal{A}; \Gamma, G) \neq \emptyset$, $\varepsilon \in \{0, 1\}$ and a meridian cycle $\gamma' \in H_{*-\varepsilon\cdot(g-1)}(\mathcal{M}^{L_1}(\mathcal{A}), \mathbb{Z})$ of depth $(k-1)$ such that $\gamma = \mu^{\varepsilon}_{L_0/L_1}(\gamma')$.

Note that in Definition 7.4, $\mathcal{M}^{L_0}(\mathcal{A})$ is a non-empty open subset of $\mathcal{M}(\mathcal{A}; \Gamma, G)$. Hence we have the induced injection $H_*(\mathcal{M}^{L_0}(\mathcal{A}), \mathbb{Z}) \hookrightarrow H_*(\mathcal{M}(\mathcal{A}; \Gamma, G), \mathbb{Z})$. We denote by $H^{\mathrm{merid}}_*(\mathcal{A}(G))$ the submodule of $H_*(\mathcal{M}(\mathcal{A}; \Gamma, G), \mathbb{Z})$ generated by the images of meridian cycles. It is clear that

$$H^{\mathrm{torus}}_*(\mathcal{A}(G)) \subset H^{\mathrm{merid}}_*(\mathcal{A}(G)) \subset H_*(\mathcal{M}(\mathcal{A}; \Gamma, G), \mathbb{Z}).$$

Lemma 7.5. *Assume that $q > 0$. Let $\alpha \in \mathcal{A} \smallsetminus \mathcal{A}_{\mathrm{tor}}$, and let $\mathcal{A}' := \mathcal{A} \smallsetminus \{\alpha\}$. Then*

$$H^{\mathrm{merid}}_*(\mathcal{A}(G)) \longrightarrow H^{\mathrm{merid}}_*(\mathcal{A}'(G)) \tag{7.5}$$

is surjective.

Proof. We prove this by induction on $\#(\mathcal{A} \smallsetminus \mathcal{A}_{\rm tor})$ and the depth k of the meridian cycle γ. If $\#(\mathcal{A} \smallsetminus \mathcal{A}_{\rm tor}) = 1$, then $\mathcal{A}' = \mathcal{A}_{\rm tor}$. In this case, the meridian cycles of $\mathcal{M}(\mathcal{A}'; \Gamma, G)$ are torus cycles, and the result follows from Lemma 7.3. Now assume that $\#(\mathcal{A} \smallsetminus \mathcal{A}_{\rm tor}) > 1$. Let $\gamma \in H_*(\mathcal{M}(\mathcal{A}'; \Gamma, G), \mathbb{Z})$ be a meridian cycle of $\mathcal{A}'$. Suppose that γ can be expressed as $\gamma = \mu^{\varepsilon_1}_{L_0/L_1} \circ \cdots \circ \mu^{\varepsilon_k}_{L_{k-1}/L_k}(\tau)$, as in Definition 7.4. If $k = 0$, then $\gamma = \tau$ is a torus cycle. Hence, again by Lemma 7.3, γ is contained in the image of the map (7.5). We may therefore assume that $k > 0$. Let $\gamma' = \mu^{\varepsilon_2}_{L_1/L_2} \circ \cdots \circ \mu^{\varepsilon_k}_{L_{k-1}/L_k}(\tau)$. Then $\gamma' \in H^{\rm merid}_{*-\varepsilon_1\cdot(g-1)}((\mathcal{A}')^{L_1}(G))$ is a meridian cycle of depth $(k-1)$.

We separate the proof into two cases depending on whether $\overline{\alpha}$ is a loop in $\mathcal{A}^{L_1}$.

Suppose that $\overline{\alpha}$ is a loop in $\mathcal{A}^{L_1}$. Then $H_{\alpha,G}$ either contains L_1 or does not intersect L_1. In either case, $\mathcal{M}^{L_1}(\mathcal{A}) = \mathcal{M}^{L_1}(\mathcal{A}')$. Hence the tubular neighborhood U of $\mathcal{M}^{L_1}(\mathcal{A}')$ satisfies $U \cap \mathcal{M}^{L_0}(\mathcal{A}') = U \cap \mathcal{M}^{L_0}(\mathcal{A}) = U \smallsetminus L_1$, and the meridian cycle $\gamma = \mu^{\varepsilon}_{L_0/L_1}(\gamma')$ can be constructed in $\mathcal{M}^{L_0}(\mathcal{A}) \subset \mathcal{M}^{L_0}(\mathcal{A}')$. Consequently, γ is contained in the image $H^{\rm merid}_*(\mathcal{A}(G)) \longrightarrow H^{\rm merid}_*(\mathcal{A}'(G))$.

Suppose that $\overline{\alpha}$ is not a loop in $\mathcal{A}^{L_1}$. Then, by the induction hypothesis, there exists a meridian cycle $\widetilde{\gamma}' \in H^{\rm merid}_{*-\varepsilon_1\cdot(g-1)}(\mathcal{A}^{L_1}(G))$ that is sent to γ' by the induced map

$$H^{\rm merid}_{*-\varepsilon_1\cdot(g-1)}(\mathcal{A}^{L_1}(G)) \longrightarrow H^{\rm merid}_{*-\varepsilon_1\cdot(g-1)}((\mathcal{A}')^{L_1}(G)), \widetilde{\gamma}' \longmapsto \gamma'.$$

Using the following commutative diagram, we can conclude that γ is also contained in the image:

$$\begin{array}{ccc} \widetilde{\gamma}' \in H^{\rm merid}_{*-\varepsilon_1\cdot(g-1)}(\mathcal{A}^{L_1}(G)) & \longrightarrow & H^{\rm merid}_{*-\varepsilon_1\cdot(g-1)}((\mathcal{A}')^{L_1}(G)) \ni \gamma' \\ {\scriptstyle \mu^{\varepsilon_1}_{L_0/L_1}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mu^{\varepsilon_1}_{L_0/L_1}} \\ H^{\rm merid}_*(\mathcal{A}(G)) & \longrightarrow & H^{\rm merid}_*(\mathcal{A}'(G)) \ni \gamma. \end{array}$$

$\square$

7.3. **Mayer-Vietoris sequences and Poincaré polynomials.** For simplicity, in this section, we will set $\mathcal{M}(\mathcal{A}) := \mathcal{M}(\mathcal{A}; \Gamma, G)$, $\mathcal{M}(\mathcal{A}') := \mathcal{M}(\mathcal{A}'; \Gamma, G)$, and $\mathcal{M}(\mathcal{A}'') := \mathcal{M}(\mathcal{A}''; \Gamma'', G)$.

Theorem 7.6. *Let $\mathcal{A}$ be a finite list of elements in a finitely generated abelian group Γ, and let $G = (S^1)^p \times \mathbb{R}^q \times F$, where F is a finite abelian group. Assume that $q > 0$, and set $g = \dim G = p + q$. Then the following hold.*

(i) *$H_*(\mathcal{M}(\mathcal{A}), \mathbb{Z})$ is generated by meridian cycles. That is $H_*(\mathcal{M}(\mathcal{A}), \mathbb{Z}) = H^{\rm merid}_*(\mathcal{A}(G))$, and furthermore it is torsion free.*

(ii) *If α is not a loop, then $H_*(\mathcal{M}(\mathcal{A}),\mathbb{Z}) \longrightarrow H_*(\mathcal{M}(\mathcal{A}'),\mathbb{Z})$ is surjective.*

(iii) *Let $\alpha \in \mathcal{A}$. Then*

$$P_{\mathcal{M}(\mathcal{A})}(t) = \begin{cases} P_{\mathcal{M}(\mathcal{A}')}(t) - P_{\mathcal{M}(\mathcal{A}'')}(t), & \textit{if } \alpha \textit{ is a loop,} \\ P_{\mathcal{M}(\mathcal{A}')}(t) + t^{g-1} \cdot P_{\mathcal{M}(\mathcal{A}'')}(t), & \textit{if } \alpha \textit{ is not a loop.} \end{cases}$$

Proof. We first note that when α is a loop, $\mathcal{M}(\mathcal{A}') = \mathcal{M}(\mathcal{A}) \sqcup \mathcal{M}(\mathcal{A}'')$ is a decomposition into disjoint open subsets. Thus (iii) is obvious when α is a loop.

We prove the other results by induction on $\#(\mathcal{A} \smallsetminus \mathcal{A}_{\mathrm{tor}})$. If $\mathcal{A} = \mathcal{A}_{\mathrm{tor}}$, then (i) follows from

$$H_*(\mathcal{M}(\mathcal{A}),\mathbb{Z}) = H_*^{\mathrm{merid}}(\mathcal{A}(G)) = H_*^{\mathrm{torus}}(\mathcal{A}(G))$$

(see (7.3) in §7.1), and there is nothing to prove for (ii) and (iii).

Assume that $\mathcal{A} \smallsetminus \mathcal{A}_{\mathrm{tor}} \neq \emptyset$, and suppose that $\alpha \in \mathcal{A} \smallsetminus \mathcal{A}_{\mathrm{tor}}$. Let U be a tubular neighborhood of $\mathcal{M}(\mathcal{A}'')$ in $\mathcal{M}(\mathcal{A}')$, as in §7.2. Set $U^* := U \cap \mathcal{M}(\mathcal{A}) \simeq \mathcal{M}(\mathcal{A}'') \times D^{g*}$. Consider the Mayer-Vietoris sequence associated with the covering $\mathcal{M}(\mathcal{A}') = U \cup \mathcal{M}(\mathcal{A})$. We have the following diagram:

$$\begin{tikzcd}
{} \arrow[r] & H_k(U^*) \arrow[r, "f_k"] & H_k(U) \oplus H_k(\mathcal{M}(\mathcal{A})) \arrow[r, "g_k"] & H_k(\mathcal{M}(\mathcal{A}')) \arrow[r] & {} \\
& H_k^{\mathrm{merid}}(U^*) \arrow[u, "h_1"] \arrow[r, "f_k'"] & H_k^{\mathrm{merid}}(U) \oplus H_k^{\mathrm{merid}}(\mathcal{A}(G)) \arrow[u, "h_2"] \arrow[r, "g_k'"] & H_k^{\mathrm{merid}}(\mathcal{A}'(G)), \arrow[u, "h_3"] &
\end{tikzcd}$$

where $H_*^{\mathrm{merid}}(U^*) = H_*^{\mathrm{merid}}(\mathcal{A}''(G)) \otimes H_*(D^{g*})$ and $H_k^{\mathrm{merid}}(U) \simeq H_k^{\mathrm{merid}}(\mathcal{A}''(G))$. The first line is a part of the Mayer-Vietoris long exact sequence. The vertical arrows h_1, h_2 and h_3 are the inclusion of the subgroup generated by meridian cycles. By the induction hypothesis, h_1 and h_3 are isomorphic. Lemma 7.5 implies that g_k' is surjective. Hence, g_k is also surjective. The surjectivity of g_{k+1} implies that f_k is injective. Therefore, the long exact sequence breaks into short exact sequences. The torsion freeness follows immediately. Thus

$$\operatorname{rank} H_k(U) + \operatorname{rank} H_k(\mathcal{M}(\mathcal{A})) = \operatorname{rank} H_k(U^*) + \operatorname{rank} H_k(\mathcal{M}(\mathcal{A}')),$$

which implies the inductive formula (iii). A diagram chase shows that h_2 is also surjective. Hence $H_*(\mathcal{M}(\mathcal{A}),\mathbb{Z}) = H_*^{\mathrm{merid}}(\mathcal{A}(G))$. □

If $G = (S^1)^p \times \mathbb{R}^q \times F$ as in the previous theorem, the Poincaré polynomial of G is

$$P_G(t) = (1+t)^p \times \#F.$$

We can compute the Poincaré polynomial of the complement $\mathcal{M}(\mathcal{A})$ using $P_G(t)$ and the G-Tutte polynomial $T_{\mathcal{A}}^G(x,y)$.

Theorem 7.7. *Let G be a non-compact abelian Lie group with finitely many connected components. Set $g = \dim G$. Then*

$$\begin{aligned} P_{\mathcal{M}(\mathcal{A})}(t) &= P_G(t)^{r_\Gamma - r_\mathcal{A}} \cdot t^{r_\mathcal{A}(g-1)} \cdot T^G_\mathcal{A}\left(\frac{P_G(t)}{t^{g-1}} + 1, 0\right) \\ &= (-t^{g-1})^{r_\Gamma} \cdot \chi^G_\mathcal{A}\left(-\frac{P_G(t)}{t^{g-1}}\right). \end{aligned} \tag{7.6}$$

Proof. We prove the result by induction on $\#\mathcal{A}$. Suppose that $\mathcal{A} = \emptyset$. Then $\mathcal{M}(\mathcal{A}) = \operatorname{Hom}(\Gamma, G) \simeq \operatorname{Hom}(\Gamma_{\mathrm{tor}}, G) \times G^{r_\Gamma}$, and $\chi^G_\mathcal{A}(t) = \# \operatorname{Hom}(\Gamma_{\mathrm{tor}}, G) \times t^{r_\Gamma}$. The formula (7.6) follows immediately.

Suppose $\mathcal{A} \neq \emptyset$. Then, using Corollary 4.11 and Theorem 7.6 (iii), Formula (7.6) can be proved by induction. □

Remark 7.8. Theorem 7.7 recovers the known formulas (2.1) and (2.2).

Remark 7.9. If G is a compact group, then formula (7.6) does not hold unless $\mathcal{A} = \emptyset$. There are several steps that fail for compact groups. For example the surjectivity of torus cycles (Lemma 7.3) fails, so the proof of the surjectivity of meridian cycles (Lemma 7.5) does not work. Furthermore, the existence of the fundamental class is an obstruction for breaking the Mayer-Vietoris sequence into short exact sequences.

8. Relationship with arithmetic matroids

In this section, we discuss the relationship between G-multiplicities and arithmetic matroid structures.

8.1. **Properties of G-multiplicities.** We summarize the construction of the dual of a representable arithmetic matroid. Let $\mathcal{A} = \{\alpha_1, \dots, \alpha_n\}$ be a finite list of elements in a finitely generated abelian group Γ. In [14], D'Adderio and Moci constructed another finitely generated abelian group $\Gamma^\dagger$ and a list $\mathcal{A}^\dagger = \{\alpha_1^\dagger, \dots, \alpha_n^\dagger\}$ of elements in $\Gamma^\dagger$ labelled by the same index set $[n] = \{1, \dots, n\}$ (see [14, §3.4] for details). Let us recall the construction briefly. Assume that Γ can be expressed as $\Gamma = \mathbb{Z}^m / \langle \boldsymbol{v}_1, \dots, \boldsymbol{v}_h \rangle$. Choose representatives $\widetilde{\alpha}_i \in \mathbb{Z}^m$ of $\alpha_i \in \Gamma$. Define

$$\Gamma^\dagger := \mathbb{Z}^{n+h} / \langle {}^t(\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_n, \boldsymbol{v}_1, \dots, \boldsymbol{v}_h) \rangle,$$

where the denominator is the subgroup generated by m columns of the $(n+h) \times m$ matrix ${}^t(\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_n, \boldsymbol{v}_1, \dots, \boldsymbol{v}_h)$. Let $\boldsymbol{e}_i$ be the standard basis of $\mathbb{Z}^{n+h}$. Set $\alpha_i^\dagger := \overline{\boldsymbol{e}_i} \in \Gamma^\dagger$ for $i = 1, \dots, n$. Now we have the list $\mathcal{A}^\dagger = \{\alpha_1^\dagger, \dots, \alpha_n^\dagger\}$. For a subset $S \subset [n]$, we have ([14, §3.4])

$$\begin{aligned} r_S^\dagger &= \#S - r_{[n]} + r_{S^c}, \\ (\Gamma^\dagger / \langle \alpha_i^\dagger \mid i \in S \rangle)_{\mathrm{tor}} &\simeq (\Gamma / \langle \alpha_i \mid i \in S^c \rangle)_{\mathrm{tor}}, \end{aligned} \tag{8.1}$$

where $S^c = [n] \smallsetminus S$, $r_S = \operatorname{rank}\langle \alpha_i \mid i \in S\rangle$ and $r^\dagger_S = \operatorname{rank}\langle \alpha^\dagger_i \mid i \in S\rangle$ (the second relation in (8.1) is not a canonical isomorphism). Note that $\mathcal{A}^\dagger$ has rank $r_{\mathcal{A}^\dagger} = \#\mathcal{A} - r_{\mathcal{A}}$.

Let G be a torsion-wise finite abelian group. Recall from Definition 4.6 that $m(\mathcal{S}; G) := \#\operatorname{Hom}\left((\Gamma/\langle \mathcal{S}\rangle)_{\mathrm{tor}}, G\right)$ for any $\mathcal{S} \subset \mathcal{A}$.

Denote the G-multiplicity of $(\Gamma^\dagger, \mathcal{A}^\dagger)$ by

$$m^\dagger(S; G) := \#\operatorname{Hom}\left((\Gamma^\dagger/\langle \alpha^\dagger_i \mid i \in S\rangle)_{\mathrm{tor}}, G\right).$$

The second relation in (8.1) implies that

$$m^\dagger(S; G) = m(S^c; G).$$

The operation $(-)^\dagger$ is reflexive in the sense that

$$r_S = \#S - r^\dagger_{[n]} + r^\dagger_{S^c},$$
$$m(S; G) = m^\dagger(S^c; G),$$

and G-Tutte polynomials satisfy

$$T^G_{\mathcal{A}^\dagger}(x, y) = T^G_{\mathcal{A}}(y, x). \tag{8.2}$$

Theorem 8.1. *The G-multiplicities satisfy the following four properties (we borrow the numbering from* [14, §2.3]*).*

(1) *If $\mathcal{S} \subset \mathcal{A}$ and $\alpha \in \mathcal{A}$ satisfy $r_{\mathcal{S}\cup\{\alpha\}} = r_{\mathcal{S}}$, then $m(\mathcal{S} \cup \{\alpha\}; G)$ divides $m(\mathcal{S}; G)$.*
(2) *If $\mathcal{S} \subset \mathcal{A}$ and $\alpha \in \mathcal{A}$ satisfy $r_{\mathcal{S}\cup\{\alpha\}} = r_{\mathcal{S}} + 1$, then $m(\mathcal{S}; G)$ divides $m(\mathcal{S} \cup \{\alpha\}; G)$.*
(4) *If $\mathcal{S} \subset \mathcal{T} \subset \mathcal{A}$ and $r_{\mathcal{S}} = r_{\mathcal{T}}$, then*

$$\rho_{\mathcal{T}}(\mathcal{S}; G) := \sum_{\mathcal{S}\subset \underset{\bullet}{\mathcal{B}} \subset\mathcal{T}} (-1)^{\#\mathcal{B}-\#\mathcal{S}} m(\mathcal{B}; G) \geq 0.$$

(5) *If $\mathcal{S} \subset \mathcal{T} \subset \mathcal{A}$ and $r_{\mathcal{T}} = r_{\mathcal{S}} + \#(\mathcal{T} \smallsetminus \mathcal{S})$, then*

$$\rho^*_{\mathcal{T}}(\mathcal{S}; G) := \sum_{\mathcal{S}\subset \underset{\bullet}{\mathcal{B}} \subset\mathcal{T}} (-1)^{\#\mathcal{T}-\#\mathcal{B}} m(\mathcal{B}; G) \geq 0.$$

Additionally, if G is a (torsion-wise finite) divisible abelian group, that is, the multiplication-by-k map $k : G \longrightarrow G$ is surjective for any positive integer k, then the G-multiplicities satisfy the following.

(3) *If $\mathcal{S} \subset \mathcal{T} \subset \mathcal{A}$ and $\mathcal{T}$ is a disjoint union $\mathcal{T} = \mathcal{S} \sqcup \mathcal{B} \sqcup \mathcal{C}$ such that for all $\mathcal{S} \subset \mathcal{R} \subset \mathcal{T}$, we have $r_{\mathcal{R}} = r_{\mathcal{S}} + \#(\mathcal{R} \cap \mathcal{B})$, then*

$$m(\mathcal{S}; G) \cdot m(\mathcal{T}; G) = m(\mathcal{S} \sqcup \mathcal{B}; G) \cdot m(\mathcal{S} \sqcup \mathcal{C}; G).$$

Proof. Property (1) follows from the fact that there exists a group epimorphism $(\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{tor}} \longrightarrow (\Gamma/\langle\mathcal{S}\cup\{\alpha\}\rangle)_{\mathrm{tor}}$ ([10, Lemma 5.2]), and by applying the functor $\mathrm{Hom}(-, G)$ to this epimorphism.

By the above construction, $(r^{\dagger}, m^{\dagger})$ satisfies (1), which is equivalent to property (2) for (r, m).

We prove (4) by showing that $\rho_{\mathcal{T}}(\mathcal{S}; G)$ is the cardinality of a certain finite set. Property (4) is clearly true if $\mathcal{S} = \mathcal{T}$, so assume that $\mathcal{S} \subsetneq \mathcal{T}$. Let us define Γ' by

$$\Gamma' := \{g \in \Gamma \mid \exists n > 0 \text{ such that } n \cdot g \in \langle\mathcal{S}\rangle\}.$$

It is also characterized by $(\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{tor}} = \Gamma'/\langle\mathcal{S}\rangle$. By the assumption $r_{\mathcal{S}} = r_{\mathcal{T}}$, we have $\mathcal{S} \subset \mathcal{T} \subset \Gamma'$. If $\mathcal{S} \subset \mathcal{B} \subset \mathcal{T}$, we also have $(\Gamma/\langle\mathcal{B}\rangle)_{\mathrm{tor}} = \Gamma'/\langle\mathcal{B}\rangle$. Therefore, $\mathrm{Hom}((\Gamma/\langle\mathcal{B}\rangle)_{\mathrm{tor}}, G) = \mathrm{Hom}(\Gamma'/\langle\mathcal{B}\rangle, G)$ can be considered as a subset of $\mathrm{Hom}((\Gamma/\langle\mathcal{S}\rangle)_{\mathrm{tor}}, G) = \mathrm{Hom}(\Gamma'/\langle\mathcal{S}\rangle, G)$. By the principle of inclusion-exclusion and Proposition 3.6, we have

$$\begin{aligned}
\rho_{\mathcal{T}}(\mathcal{S}; G) &= \sum_{\mathcal{S}\subset \underset{\bullet}{\mathcal{B}} \subset\mathcal{T}} (-1)^{\#\mathcal{B}-\#\mathcal{S}} \cdot m(\mathcal{B}; G) \\
&= \sum_{\mathcal{S}\subset \underset{\bullet}{\mathcal{B}} \subset\mathcal{T}} (-1)^{\#\mathcal{B}-\#\mathcal{S}} \cdot \#\,\mathrm{Hom}(\Gamma'/\langle\mathcal{B}\rangle, G) \\
&= \#\mathcal{M}(\mathcal{T}/\mathcal{S}; \Gamma'/\langle\mathcal{S}\rangle, G),
\end{aligned}$$

which is clearly non-negative.

We can prove (5) by an argument similar to that for (2) by using duality.

Finally, to prove property (3) we generalize the argument used in [14, Lemma 2.6]. We consider the following diagram composing of two short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \left(\frac{\Gamma}{\langle\mathcal{S}\sqcup\mathcal{C}\rangle}\right)_{\mathrm{tor}} \arrow[r] & \left(\frac{\Gamma}{\langle\mathcal{T}\rangle}\right)_{\mathrm{tor}} \arrow[r] & \left(\frac{\Gamma}{\langle\mathcal{T}\rangle}\right)_{\mathrm{tor}} \Big/ \left(\frac{\Gamma}{\langle\mathcal{S}\sqcup\mathcal{C}\rangle}\right)_{\mathrm{tor}} \arrow[r] & 0 \\
0 \arrow[r] & \left(\frac{\Gamma}{\langle\mathcal{S}\rangle}\right)_{\mathrm{tor}} \arrow[r] & \left(\frac{\Gamma}{\langle\mathcal{S}\sqcup\mathcal{B}\rangle}\right)_{\mathrm{tor}} \arrow[r] & \left(\frac{\Gamma}{\langle\mathcal{S}\sqcup\mathcal{B}\rangle}\right)_{\mathrm{tor}} \Big/ \left(\frac{\Gamma}{\langle\mathcal{S}\rangle}\right)_{\mathrm{tor}} \arrow[r] \arrow[u, "\simeq"'] & 0.
\end{tikzcd}$$

(The isomorphism indicated by the vertical arrow is proved in [10, Lemma 5.3].) Since G is divisible, G is an injective $\mathbb{Z}$-module and the functor $\mathrm{Hom}(-, G)$ is exact. Applying the functor $\mathrm{Hom}(-, G)$ to the diagram we obtain property (3). $\square$

Remark 8.2. When G is a connected abelian Lie group, that is, $G = (S^1)^p \times \mathbb{R}^q$, G is a torsion-wise finite and divisible group. It is easily seen that property (3) fails in many cases. For example, let $\Gamma := \mathbb{Z}^2, \mathcal{S} := \{(0, 2)\}, \mathcal{B} := \{(2, 1)\}, \mathcal{C} := \{(0, 1)\}$ and $G := \mathbb{Z}/2\mathbb{Z}$. Then $(\Gamma/\langle S\rangle)_{\mathrm{tor}} \simeq \mathbb{Z}/2\mathbb{Z}$, $(\Gamma/\langle S \cup$

$B\rangle)_{\mathrm{tor}} \simeq \mathbb{Z}/4\mathbb{Z}$, $(\Gamma/\langle S \cup C\rangle)_{\mathrm{tor}} \simeq \{0\}$, $(\Gamma/\langle T\rangle)_{\mathrm{tor}} \simeq \mathbb{Z}/2\mathbb{Z}$, and $m(\mathcal{S}; G) \cdot m(\mathcal{T}; G) = 4 \neq 2 = m(\mathcal{S} \sqcup \mathcal{B}; G) \cdot m(\mathcal{S} \sqcup \mathcal{C}; G)$.

8.2. **(Non-)positivity of coefficients.** As was proved in [28, Theorem 3.5], the arithmetic Tutte polynomial $T^{\mathrm{arith}}_{\mathcal{A}}(x, y)$ is a polynomial with positive coefficients. In this section, we show that the G-Tutte polynomial has positive coefficients for some special cases. However, for a general group G, we show that the G-Tutte polynomial can have negative coefficients by exhibiting an explicit example.

Theorem 8.3. *Let G be a torsion-wise finite divisible abelian group. Then the coefficients of the G-Tutte polynomial $T^G_{\mathcal{A}}(x, y)$ are positive integers.*

Proof. When G is a torsion-wise finite divisible group, the pair $(\Gamma, \mathcal{A})$ together with the G-multiplicities form an arithmetic matroid. It is proved in [10, Theorem 4.5] that the coefficients of the arithmetic Tutte polynomial of a pseudo-arithmetic matroid (and hence of an arithmetic matroid) are positive integers. □

Proposition 8.4. *Let Γ be a finitely generated abelian group, and let G be a torsion-wise finite group.*

(i) *If $\mathcal{A} \subset \Gamma$ consists of loops (i.e., $\mathcal{A} \subset \Gamma_{\mathrm{tor}}$), then $T^G_{\mathcal{A}}(x, y)$ has positive coefficients.*
(ii) *If $\mathcal{A} \subset \Gamma$ consists of coloops (i.e., $r_{\mathcal{A}} = \#\mathcal{A}$), then $T^G_{\mathcal{A}}(x, y)$ has positive coefficients.*

Proof. (i) follows immediately from Theorem 4.18. (ii) follows immediately from (i) and (8.2) (note that if $\mathcal{A}$ consists of coloops, then $\mathcal{A}^{\dagger}$ consists of loops). □

In general, the G-Tutte polynomial can have negative coefficients as in the next example.

Example 8.5. Let $\Gamma = \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, let $\mathcal{A} = \{\alpha, \beta\}$ with $\alpha = (2, \overline{1})$ and $\beta = (0, \overline{2}) \in \Gamma$, and let $G = \mathbb{Z}/4\mathbb{Z}$. Then by direct computation, we have

$$T^G_{\mathcal{A}}(x, y) = 2xy + 2x + 2y - 2.$$

This also produces a counter-example to axiom (P) of Brändén-Moci [10, §2]. With notation in [10, §2], $[\emptyset, \mathcal{A}]$ is a molecule, and

$$\rho(\emptyset, \mathcal{A}; G) = (-1) \cdot \sum_{\mathcal{B} \subset \mathcal{A}} (-1)^{2-\#\mathcal{B}} m(\mathcal{B}; G) = -2 < 0.$$

Therefore, in this case, the multiplicity $m(\mathcal{B}; G)$ ($\mathcal{B} \subset \mathcal{A}$) does not form a pseudo-arithmetic matroid in the sense of Brändén-Moci [10, §2].

8.3. **Convolution formula.** The following result is the G-Tutte polynomial version of the so-called convolution formula [18, 24].

Theorem 8.6. *Let $\mathcal{A} \subset \Gamma$ be a list in a finitely generated group Γ, and let G_1 and G_2 be torsion-wise finite groups. Then*

$$T_{\mathcal{A}}^{G_1\times G_2}(x,y) = \sum_{\underset{\bullet}{\mathcal{B}}\subset\mathcal{A}} T_{\mathcal{B}}^{G_1}(0,y)\cdot T_{\mathcal{A}/\mathcal{B}}^{G_2}(x,0). \tag{8.3}$$

Proof. The right-hand side of the formula is equal to

$$\begin{aligned}
&\sum_{\underset{\bullet}{\mathcal{B}}\subset\mathcal{A}} \left\{ \sum_{\underset{\bullet}{\mathcal{S}}\subset\mathcal{B}} m(\mathcal{S};G_1)(-1)^{r_{\mathcal{B}}-r_{\mathcal{S}}}(y-1)^{\#\mathcal{S}-r_{\mathcal{S}}} \right\}\\
&\times \left\{ \sum_{\mathcal{B}\subset\underset{\bullet}{\mathcal{T}}\subset\mathcal{A}} m(\mathcal{T};G_2)(x-1)^{r_{\mathcal{A}}-r_{\mathcal{B}}-(r_{\mathcal{T}}-r_{\mathcal{B}})}(-1)^{\#\mathcal{T}-\#\mathcal{B}-(r_{\mathcal{T}}-r_{\mathcal{B}})} \right\}\\
=& \sum_{\underset{\bullet}{\mathcal{S}}\subset\underset{\bullet}{\mathcal{B}}\subset\underset{\bullet}{\mathcal{T}}\subset\mathcal{A}} m(\mathcal{S};G_1)m(\mathcal{T};G_2)(x-1)^{r_{\mathcal{A}}-r_{\mathcal{T}}}(y-1)^{\#\mathcal{S}-r_{\mathcal{S}}}(-1)^{\#\mathcal{T}-\#\mathcal{B}-r_{\mathcal{T}}-r_{\mathcal{S}}}\\
=& \sum_{\underset{\bullet}{\mathcal{S}}=\underset{\bullet}{\mathcal{B}}=\underset{\bullet}{\mathcal{T}}\subset\mathcal{A}} m(\mathcal{S};G_1)m(\mathcal{S};G_2)(x-1)^{r_{\mathcal{A}}-r_{\mathcal{S}}}(y-1)^{\#\mathcal{S}-r_{\mathcal{S}}}\\
&+ \sum_{\underset{\bullet}{\mathcal{S}}\subsetneq\underset{\bullet}{\mathcal{T}}\subset\mathcal{A}} \left\{ m(\mathcal{S};G_1)m(\mathcal{T};G_2)(x-1)^{r_{\mathcal{A}}-r_{\mathcal{T}}}(y-1)^{\#\mathcal{S}-r_{\mathcal{S}}} \sum_{\mathcal{S}\subset\underset{\bullet}{\mathcal{B}}\subset\mathcal{T}} (-1)^{\#\mathcal{T}-\#\mathcal{B}-r_{\mathcal{T}}-r_{\mathcal{S}}} \right\}.
\end{aligned}$$

The first term is equal to $T_{\mathcal{A}}^{G_1\times G_2}(x,y)$ from the multiplicativity $m(\mathcal{S};G_1\times G_2) = m(\mathcal{S};G_1)m(\mathcal{S};G_2)$ (see Proposition 4.5). The second term vanishes because, when $\mathcal{S}\subsetneq\mathcal{T}$, we have $\sum_{\mathcal{S}\subset\underset{\bullet}{\mathcal{B}}\subset\mathcal{T}}(-1)^{\#\mathcal{B}} = 0$. $\square$

The classical convolution formula [18, 24] for matroids representable over $\mathbb{Q}$ is obtained from Theorem 8.6 by replacing G_1 and G_2 by $\{0\}$. Theorem 8.6 can also be specialized to the Backman-Lenz [3] convolution formula when $G_1\times G_2 = S^1\times\{0\}$ or $\{0\}\times S^1$.

Acknowledgements: YL is partially supported by JSPS KAKENHI Grant Number 16J00125. TNT gratefully acknowledges the support of the scholarship program of the Japanese Ministry of Education, Culture, Sports, Science, and Technology (MEXT) under grant number 142506. MY is partially supported by JSPS KAKENHI Grant Number JP15KK0144, JP16K13741, JP18H01115 and Humboldt Foundation. The authors thank Emanuele Delucchi for sharing observations on arithmetic Tutte and characteristic quasi-polynomials, Luca Moci for suggestions concerning the convolution formula, and Roberto Pagaria for pointed out some mistakes in the previous version. The authors also thank ALTA group in Bremen University for their hospitality where part of this work was done.

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YE LIU, DEPARTMENT OF MATHEMATICS, HOKKAIDO UNIVERSITY, KITA 10, NISHI 8, KITA-KU, SAPPORO 060-0810, JAPAN.
E-mail address: liu@math.sci.hokudai.ac.jp

TAN NHAT TRAN, DEPARTMENT OF MATHEMATICS, HOKKAIDO UNIVERSITY, KITA 10, NISHI 8, KITA-KU, SAPPORO 060-0810, JAPAN.
E-mail address: trannhattan@math.sci.hokudai.ac.jp

MASAHIKO YOSHINAGA, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, HOKKAIDO UNIVERSITY, KITA 10, NISHI 8, KITA-KU, SAPPORO 060-0810, JAPAN.
E-mail address: yoshinaga@math.sci.hokudai.ac.jp