Weak comparison principles for fully nonlinear degenerate parabolic equations with discontinuous source terms

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Abstract

We study the initial value problem for a fully nonlinear degenerate parabolic equation with discontinuous source terms, to which a usual type of comparison principles does not apply. Examples include singular equations appearing in surface evolution problems such as the level-set mean curvature flow equation with a driving force term and a discontinuous source term. By a suitable scaling, we establish weak comparison principles for a viscosity sub- and supersolution to the equation. We also present uniqueness and existence results of possibly discontinuous viscosity solutions.

Key words: Weak comparison principle; Viscosity solution; Fully nonlinear equation; Discontinuous source term; Level-set mean curvature flow equation
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1 Introduction

Equation and purpose. We study a fully nonlinear parabolic partial differential equation of the form

$$u_t(x,t) + H(x,t,\nabla u(x,t)) + F(\nabla u(x,t), \nabla^2 u(x,t)) = f(x) \quad \text{in } \mathbf{R}^n \times (0,\infty) \tag{1.1}$$

under the initial condition

$$u(x,0) = u_0(x) \quad \text{in } \mathbf{R}^n. \tag{1.2}$$

Here $u : \mathbf{R}^n \times [0,\infty) \to \mathbf{R}$ is the unknown function and u_t, $\nabla u = (u_{x_i})_{i=1}^n$ and $\nabla^2 u = (u_{x_i x_j})_{i,j=1}^n$ stand for the time derivative, the spatial gradient and the Hessian matrix of u, respectively. Moreover, we assume the following conditions throughout this paper:

- $F : (\mathbf{R}^n \setminus \{0\}) \times \mathbf{S}^n \to \mathbf{R}$ is a continuous function, where $\mathbf{S}^n$ denotes the set of $n \times n$ real symmetric matrices with the usual ordering,

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- $H : \mathbf{R}^n \times [0,\infty) \times \mathbf{R}^n \to \mathbf{R}$ is a continuous function called a Hamiltonian,
- $f : \mathbf{R}^n \to \mathbf{R}$ is a locally bounded, possibly discontinuous source term,
- $u_0 : \mathbf{R}^n \to \mathbf{R}$ is a continuous initial datum such that $\lim_{|x|\to\infty} u_0(x) = 0$.

Further assumptions on F, H and f will be given later. We note that $F = F(p, X)$ is allowed to be singular at $p = 0$.

The goal of this paper is to establish new comparison principles for a viscosity sub- and supersolution of (1.1)–(1.2). A difficulty lies in the discontinuity of the source term f, and due to this, classical comparison results do not apply to (1.1). Under a condition on discontinuity of f, we prove a weak version of comparison principles for (1.1). Moreover, we derive uniqueness of solutions to (1.1)–(1.2) in suitable classes. Our results guarantee uniqueness of solutions which are possibly discontinuous. We also investigate existence of solutions.

Typical equation and physical background. Our assumptions on F are very mild; indeed, we only need (2.1)–(2.3) in Section 2. Examples of F include typical second order operators such as the Laplacian $-\Delta u(x,t)$, the Pucci extremal operators $\mathcal{P}^{\pm}(\nabla^2 u(x,t))$ which are fully nonlinear ([5]) and Bellman–Isaacs operators arising in stochastic control problems ([11]).

Among many other second order equations, a typical one in our mind is the level-set mean curvature flow equation with a driving force term and a source term. Namely,

$$u_t(x,t) - \nu(x,t)|\nabla u(x,t)| - \Delta_1 u(x,t) = f(x) \quad \text{in } \mathbf{R}^n \times (0,\infty), \tag{1.3}$$

where $|\cdot|$ stands for the standard Euclidean norm. In this case, the function F in (1.1) is given by

$$F(\nabla u(x,t), \nabla^2 u(x,t)) = -\Delta_1 u(x,t) := -\frac{1}{n-1}|\nabla u(x,t)| \mathrm{div}\left(\frac{\nabla u(x,t)}{|\nabla u(x,t)|}\right),$$

or equivalently

$$F(p,X) = -\frac{1}{n-1}\mathrm{tr}\left(\left(I - \frac{p\otimes p}{|p|^2}\right)X\right) \quad ((p,X) \in (\mathbf{R}^n \setminus \{0\}) \times \mathbf{S}^n). \tag{1.4}$$

Here $p \otimes p = (p_i p_j)_{i,j=1}^n$ for a vector $p = (p_1, \dots, p_n) \in \mathbf{R}^n$. Also, the Hamiltonian is

$$H(x,t,p) = -\nu(x,t)|p|,$$

where a continuous function $\nu : \mathbf{R}^n \times [0,\infty) \to \mathbf{R}$ stands for the driving force. A typical source term f is a characteristic function

$$f(x) = c\chi_\Omega(x) \quad (c > 0,\ \Omega \subset \mathbf{R}^n), \tag{1.5}$$

where $\chi_\Omega(x) = 1$ if $x \in \Omega$ and $\chi_\Omega(x) = 0$ if $x \notin \Omega$.

The equation of the form (1.3) appears in a crystal growth phenomenon called *two dimensional nucleation* ([4, 28, 30]). In this phenomenon, crystals grow by catching molecules on some area of the crystal surface. Let us briefly explain the derivation of the equation describing this growth. Let $u(x,t)$ be the height of the crystal surface at a position $x \in \mathbf{R}^n$ and a time $t \in [0,\infty)$. See Figure 1. We assume that the crystal growth in the horizontal direction and the vertical direction are respectively governed by the following laws (A) and (B):

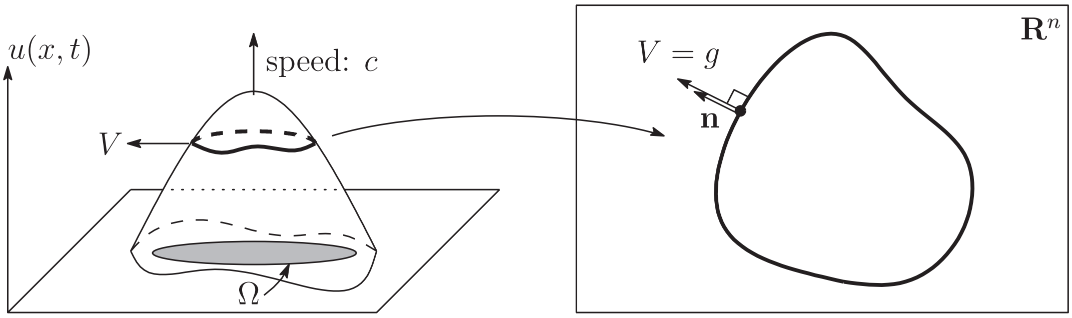

Figure 1: Growth laws in the case of (1.5).

(A) For each $l \in \mathbf{R}$, the level-set $\Gamma_l(t) = \{x \in \mathbf{R}^n \mid u(x,t) = l\}$ of u evolves in the horizontal direction according to a surface evolution equation of the form

$$V = g(x, t, \mathbf{n}, \nabla \mathbf{n}) \quad \text{on } \Gamma_l(t). \tag{1.6}$$

(B) The height u changes at a rate of $f(x)$ due to nucleation.

Here g is a given function, $\mathbf{n} = \mathbf{n}(x,t) \in \mathbf{R}^n$ is the unit normal vector to $\Gamma_l(t)$ at x from $\{x \in \mathbf{R}^n \mid u(x,t) > l\}$ to $\{x \in \mathbf{R}^n \mid u(x,t) < l\}$, $V = V(x,t)$ is the normal velocity of $\Gamma_l(t)$ at x in the direction of $\mathbf{n}$, and $-\nabla \mathbf{n}$ is the second fundamental form in the direction of $\mathbf{n}$. If $\nabla u(x,t) \neq 0$ and u is smooth near (x,t), we have the following representation:

$$V = \frac{u_t(x,t)}{|\nabla u(x,t)|}, \quad \mathbf{n} = -\frac{\nabla u(x,t)}{|\nabla u(x,t)|}, \quad \nabla \mathbf{n} = -\frac{1}{|\nabla u(x,t)|} Q_{\nabla u(x,t)}(\nabla^2 u(x,t)),$$

where

$$Q_p(X) = R_p X R_p \quad \left(R_p := I - \frac{p \otimes p}{|p|^2} \right).$$

See [13, Chapter 1] for derivations of these representations. Substituting these for (1.6), we obtain

$$u_t(x,t) + G(x, t, \nabla u(x,t), \nabla^2 u(x,t)) = 0 \quad \text{in } \mathbf{R}^n \times (0, \infty) \tag{1.7}$$

with

$$G(x,t,p,X) = -|p| g \left(x, t, -\frac{p}{|p|}, -\frac{1}{|p|} Q_p(X) \right), \tag{1.8}$$

which is a possibly singular function at $p = 0$. The equation (1.7) is often called a *level-set equation*. See [6, 10, 13] for rigorous analysis of such level-set equations. We turn to the condition (B). Since the growth speed in the vertical direction is given by $u_t(x,t)$, the condition requires that

$$u_t(x,t) = f(x) \quad \text{in } \mathbf{R}^n \times (0, \infty). \tag{1.9}$$

As both (A) and (B) occur in the two dimensional nucleation, the equation describing this phenomenon is the mixed one of (1.7) and (1.9), that is,

$$u_t(x,t) + G(x, t, \nabla u(x,t), \nabla^2 u(x,t)) = f(x) \quad \text{in } \mathbf{R}^n \times (0, \infty). \tag{1.10}$$

We thus get (1.1) provided that the second order term of G is separable.

In [15] derivation of (1.10) is presented by use of Trotter–Kato product formula.

One of typical surface evolution equations is the mean curvature flow equation with a driving force, which is of the form

$$V = \kappa + \nu \quad \text{on } \Gamma_l(t).$$

Here $\kappa = -(\operatorname{div}_{\Gamma_l(t)} \mathbf{n})(x)/(n-1)$ is the mean curvature of $\Gamma_l(t)$ at x, and $\nu = \nu(x,t)$ is a driving force. The equation (1.10) in this case is given by (1.3).

In this paper we often focus on (1.3) with a constant driving force $\nu \in \mathbf{R}$, that is,

$$u_t(x,t) - \nu|\nabla u(x,t)| - \Delta_1 u(x,t) = f(x) \quad \text{in } \mathbf{R}^n \times (0,\infty), \tag{1.11}$$

where the Hamiltonian is

$$H(x,t,p) = H(p) = -\nu|p|. \tag{1.12}$$

Results. A usual comparison principle for viscosity solutions to initial value problems asserts that, if u and v are respectively a viscosity subsolution and a viscosity supersolution and if $u^*(\cdot,0) \leqq v_*(\cdot,0)$ in $\mathbf{R}^n$, then

$$u^* \leqq v_* \quad \text{in } \mathbf{R}^n \times (0,\infty). \tag{1.13}$$

Here the asterisks $*$ stand for the semicontinuous envelopes; see Section 2 for the definitions. This comparison result guarantees that viscosity solutions are unique and that the unique solution is continuous. Also, this type of comparison principle is established under a suitable continuity of equations ([7]).

When the equation is discontinuous, it is possible that viscosity solutions are not unique and that discontinuous solutions exist. See Section 6 and [14, 15]. Thus, we cannot expect the inequality (1.13) as a conclusion of our comparison principle for (1.1). For this reason, we establish a weaker version of the comparison principle. We prove

$$(u^*)_* \leqq v_*, \quad u^* \leqq (v_*)^* \quad \text{in } \mathbf{R}^n \times (0,\infty). \tag{1.14}$$

Since $(u^*)_* \leqq u^*$ and $v_* \leqq (v_*)^*$, these estimates are actually weaker than (1.13). In this sense, we call our result (1.14) a *weak comparison principle*.

We establish two kinds of weak comparison principles in Section 3 under different assumptions. In both the proofs, we change the scale of either a subsolution u or a supersolution v. Such an idea can be found in [23] for elliptic equations. For the first comparison principle (Theorem 3.1), we assume that either u or v is Lipschitz continuous with respect to the space variable x; see (2.7). This Lipschitz regularity guarantees that derivatives of a test function are bounded uniformly in parameters, and it enables us to extract a subsequence of the derivatives and take a limit for viscosity inequalities.

The condition (2.7) requires Lipschitz continuity in x which is locally uniform in $t \in (0,\infty)$. In particular, the initial time is excluded in the condition (2.7). Thanks to this, there is a chance to build a unique solution even if the initial datum is not Lipschitz continuous, provided that the Lipschitz regularizing effect occurs for (1.1). See Remark 5.2 (3) for more comments on this.

Our second comparison principle (Theorem 3.2) does not need the Lipschitz regularity of one of the solutions. Instead, we assume that the Hamiltonian H satisfies an additional condition (2.8), which covers the case (1.11) with a nonpositive driving force $\nu \leqq 0$.

In Section 4 we derive some uniqueness results of solutions. The results are obtained as a consequence of the weak comparison principles. Among other things, we prove that semicontinuous solutions are unique. We also discuss existence of solutions in Section 5 via approximation of the source term f by continuous ones. For (1.11) with a negative driving force $\nu < 0$, with the aid of Perron's method, we prove that solutions have compact supports which are uniform in $t \geqq 0$.

Literature overview. In [14] Hamilton-Jacobi equations with a discontinuous source term

$$u_t(x,t) + H(x, \nabla u(x,t)) = f(x) \quad \text{in } \mathbf{R}^n \times (0,\infty) \tag{1.15}$$

are studied. This is the case where $F = 0$ in (1.1). Introducing a new notion of solutions, the authors of [14] prove uniqueness and existence of solutions when H is coercive. The large time behavior of the solution is investigated in [19].

The level-set mean curvature flow equation (1.11) is studied in [15] when the driving force ν is a positive constant. The asymptotic speed of the maximal solution is investigated. We remark that, in [15, Proposition 2.1] a comparison result named a "weak comparison principle" is presented, but its assertion is different from ours. It asserts that (1.13) holds if v is a viscosity supersolution of (1.11) with $g(x)$ on the right-hand side such that $f^* \leqq g_*$ in $\mathbf{R}^n$. Our comparison results differ from [15, Proposition 2.1] in that ours apply to equations with the common source term.

When the source term f is continuous, some further results are obtained in [16, 17]. The asymptotic shape is studied in [16] for a radially symmetric source term. In [17] the asymptotic speed is investigated for equations with a general degenerate elliptic operator F.

Our second weak comparison principle (Theorem 3.2) is applied in the forthcoming paper [21], where we consider (1.11) with a negative driving force $\nu < 0$. The asymptotic shape of solutions is investigated. In [21] we also provide a game-theoretic interpretation for the equation and apply the result to study the asymptotic speed of solutions.

In [12] some weak versions of comparison principles are established for first order equations

$$u_t(x,t) + H(u(x,t), \nabla u(x,t)) = 0 \quad \text{in } \mathbf{R}^n \times (0,\infty),$$

whose solutions may develop jump discontinuity (shock). We also refer the reader to the paper [3], which studies connection between nonempty interior condition on evolving surfaces and uniqueness of semicontinuous solutions to geometric equations without source terms. See also [2] for uniqueness results of semicontinuous solutions to first order Hamilton–Jacobi equations whose Hamiltonian is convex.

Organization. This paper is organized as follows: Section 2 is devoted to preliminaries. In Section 3 we establish two kinds of weak comparison principles as main results of this paper. In Sections 4 and 5 we study uniqueness and existence of solutions, respectively. Some examples are given in Section 6.

A part of the results in this paper is announced in [20].

2 Preliminaries

2.1 Assumptions

Let us denote by $B_r(x)$ the open ball centered at x with a radius $r > 0$. We first recall a notion of semicontinuous envelopes. For a subset $K \subset \mathbf{R}^N$ and a function

$h : K \to \mathbf{R}$, we define the *upper semicontinuous envelope* $h^* : \overline{K} \to \mathbf{R} \cup \{\infty\}$ and the *lower semicontinuous envelope* $h_* : \overline{K} \to \mathbf{R} \cup \{-\infty\}$ by

$$h^*(x) = \lim_{r \to +0} \sup\{h(y) \mid y \in B_r(x) \cap K\}, \quad h_*(x) = \lim_{r \to +0} \inf\{h(y) \mid y \in B_r(x) \cap K\}$$

for $x \in \overline{K}$.

In addition to the continuity (Section 1) we assume the following conditions on F:

$$F(p, X) \leqq F(p, Y) \text{ for all } p \in \mathbf{R}^n \setminus \{0\} \text{ and } X, Y \in \mathbf{S}^n \text{ such that } X \geqq Y, \tag{2.1}$$

$$-\infty < F_*(0, O) = F^*(0, O) < \infty, \tag{2.2}$$

$$F(rp, X) = F(p, X) \text{ for all } (p, X) \in (\mathbf{R}^n \setminus \{0\}) \times \mathbf{S}^n \text{ and } r > 0. \tag{2.3}$$

Remark 2.1. It is not difficult to see that the level-set mean curvature flow operator (1.4) satisfies all of (2.1)–(2.3) above.

A key assumption on the source term f is

$$\begin{cases} \text{For any discontinuous point } x \in \mathbf{R}^n \text{ of } f \text{ and any sequence } \{x_\lambda\}_{\lambda > 1} \subset \mathbf{R}^n \\ \text{such that } \lim\limits_{\lambda \to 1+0} x_\lambda = x, \text{ we have } \limsup\limits_{\lambda \to 1+0} \{f^*(\lambda x_\lambda) - f_*(x_\lambda)\} \leqq 0. \end{cases} \tag{2.4}$$

Remark 2.2. It is clear that (2.4) is fulfilled if

$$f^*(\lambda x) \leqq f_*(x) \text{ for all } x \in \mathbf{R}^n \text{ and } \lambda > 1. \tag{2.5}$$

When f is a characteristic function (1.5), the condition (2.5) holds if and only if Ω is star-shaped with respect to the origin, that is,

$$\overline{\Omega} \subset \lambda \Omega^\circ \text{ for all } \lambda > 1, \tag{2.6}$$

where Ω° is the interior of Ω and $\lambda \Omega^\circ = \{\lambda x \mid x \in \Omega^\circ\}$.

We present two weak comparison principles in Section 3. For the first comparison principle, we assume that either a subsolution u or a supersolution v is Lipschitz continuous with respect to the space variable. More precisely, the following condition is imposed on $w = u$ or $w = v$:

$$\begin{cases} \text{For every } \gamma > 0 \text{ and } T > \gamma \text{ there is } L > 0 \text{ such that} \\ |w(x, t) - w(y, t)| \leqq L|x - y| \text{ for all } x, y \in \mathbf{R}^n \text{ and } t \in [\gamma, T]. \end{cases} \tag{2.7}$$

We note that the Lipschitz continuity is not required at the initial time $t = 0$.

Our second comparison principle does not need (2.7); instead we assume that the Hamiltonian H satisfies

$$H \text{ is independent of } (x, t), \text{ and } H(p) \leqq H(\lambda p) \text{ for all } p \in \mathbf{R}^n \text{ and } \lambda > 1. \tag{2.8}$$

Remark 2.3. When H is of the form (1.12) with a nonpositive $\nu \leqq 0$, the condition (2.8) is fulfilled. More generally, if

$$H(x, t, p) = H(p) = -\nu |p|^a \quad (\nu \leqq 0,\ a \geqq 0 \text{ are constants}),$$

then H satisfies (2.8). Indeed, for $p \in \mathbf{R}^n$ and $\lambda > 1$, we have

$$H(\lambda p) - H(p) = -\nu \lambda^a |p|^a + \nu |p|^a = -\nu(\lambda^a - 1)|p|^a \geqq 0.$$

2.2 Viscosity solution

We next introduce a notion of viscosity solutions. The reader is referred to [7, 13] for the basic theory of viscosity solutions. Let $C^{2,1}(\mathbf{R}^n \times (0,\infty))$ denote the set of functions $\phi = \phi(x,t)$ that are of class C^2 in x and C^1 in t.

Definition 2.4 (Viscosity solution)**.** (1) Let $u : \mathbf{R}^n \times (0,\infty) \to \mathbf{R}$. We say that u is a *viscosity subsolution* (resp. a *viscosity supersolution*) *of* (1.1) if the following (i)–(ii) hold:

(i) $u^* < \infty$ (resp. $u_* > -\infty$) in $\mathbf{R}^n \times (0,\infty)$,

(ii) Whenever $u^* - \phi$ (resp. $u_* - \phi$) attains a local maximum (resp. a local minimum) at $(x_0,t_0) \in \mathbf{R}^n \times (0,\infty)$ for $\phi \in C^{2,1}(\mathbf{R}^n \times (0,\infty))$, we have

$$\phi_t(x_0,t_0) + H(x_0,t_0,\nabla\phi(x_0,t_0)) + F_*(\nabla\phi(x_0,t_0),\nabla^2\phi(x_0,t_0)) \leqq f^*(x_0)$$
$$(\text{resp. } \phi_t(x_0,t_0) + H(x_0,t_0,\nabla\phi(x_0,t_0)) + F^*(\nabla\phi(x_0,t_0),\nabla^2\phi(x_0,t_0)) \geqq f_*(x_0)).$$

(2) Let $u : \mathbf{R}^n \times [0,\infty) \to \mathbf{R}$. We say that u is a *viscosity subsolution* (resp. a *viscosity supersolution*) *of* (1.1)–(1.2) if u is a viscosity subsolution of (1.1) and $u^*(\cdot,0) \leqq u_0$ (resp. $u_*(\cdot,0) \geqq u_0$) in $\mathbf{R}^n$.

(3) A function u is called a *viscosity solution* if u is both a viscosity subsolution and a viscosity supersolution.

Remark 2.5. When u is a viscosity solution of (1.1)–(1.2), it is continuous on $\mathbf{R}^n \times \{0\}$.

Remark 2.6. The definition of viscosity solutions can be rephrased by using parabolic semijets $\mathcal{P}^{2,\pm}u(x,t)$ and the extended ones $\overline{\mathcal{P}}^{2,\pm}u(x,t)$; for their definitions, see [7, 13]. In fact, the condition (ii) in Definition 2.4 can be replaced by the following one:

(ii)′ Whenever $(x_0,t_0) \in \mathbf{R}^n \times (0,\infty)$ and $(p,\tau,X) \in \mathcal{P}^{2,+}u^*(x_0,t_0)$ (resp. $(p,\tau,X) \in \mathcal{P}^{2,-}u_*(x_0,t_0)$), we have

$$\tau + H(x_0,t_0,p) + F_*(p,X) \leqq f^*(x_0) \quad (\text{resp. } \tau + H(x_0,t_0,p) + F^*(p,X) \geqq f_*(x_0)).$$

Moreover, one may replace "$\mathcal{P}^{2,\pm}u(x,t)$" in (ii)′ by "$\overline{\mathcal{P}}^{2,\pm}u(x,t)$".

In our comparison principles we assume decay conditions on a subsolution u and a supersolution v as follows:

$$\text{For every } \delta, T > 0 \text{ there exists some } R > 0 \text{ such that } u(x,t) \leqq \delta \text{ in } B_R(0)^c \times [0,T], \tag{2.9}$$

$$\text{For every } \delta, T > 0 \text{ there exists some } R > 0 \text{ such that } v(x,t) \geqq -\delta \text{ in } B_R(0)^c \times [0,T]. \tag{2.10}$$

For later use we prepare notations of classes of viscosity sub- and supersolutions.

Definition 2.7.

$$\mathrm{SUB} := \{u \mid u \text{ is a viscosity subsolution of (1.1)–(1.2) and satisfies (2.9)}\},$$
$$\mathrm{SUP} := \{u \mid u \text{ is a viscosity supersolution of (1.1)–(1.2) and satisfies (2.10)}\}$$

and $\mathrm{SOL} := \mathrm{SUB} \cap \mathrm{SUP}$. Moreover,

$$\mathrm{SUB}_{\mathrm{Lip}} := \{u \in \mathrm{SUB} \mid u \text{ is continuous in } \mathbf{R}^n \times [0,\infty) \text{ and satisfies (2.7)}\},$$
$$\mathrm{SUP}_{\mathrm{Lip}} := \{u \in \mathrm{SUP} \mid u \text{ is continuous in } \mathbf{R}^n \times [0,\infty) \text{ and satisfies (2.7)}\}$$

and $\mathrm{SOL}_{\mathrm{Lip}} := \mathrm{SUB}_{\mathrm{Lip}} \cap \mathrm{SUP}_{\mathrm{Lip}}$.

3 Weak comparison principles

We establish two kinds of weak comparison principles for a viscosity subsolution and a viscosity supersolution of (1.1).

3.1 Comparison under Lipschitz continuity of solutions

We prove a weak comparison principle under the assumption that either a subsolution or a supersolution satisfies the Lipschitz condition (2.7).

Theorem 3.1 (Weak comparison principle 1). *Assume* (2.1)–(2.4). *Let* $u : \mathbf{R}^n \times [0,\infty) \to \mathbf{R}$ *be a viscosity subsolution of* (1.1) *satisfying* (2.9)*, and let* $v : \mathbf{R}^n \times [0,\infty) \to \mathbf{R}$ *be a viscosity supersolution of* (1.1) *satisfying* (2.10)*. Assume that either* u *or* v *satisfies* (2.7)*. If* $u^*(\cdot,0) \leqq v_*(\cdot,0)$ *in* $\mathbf{R}^n$*, then* $(u^*)_* \leqq v_*$ *and* $u^* \leqq (v_*)^*$ *in* $\mathbf{R}^n \times [0,\infty)$*.*

Proof. For simplicity let us write u and v for u^* and v_*, respectively. We only give the proof of $u_* \leqq v$ in $\mathbf{R}^n \times (0,\infty)$ since the other one is derived in a parallel way.

1. Let $\lambda > 1$. We rescale the subsolution u by

$$u_\lambda(x,t) = \frac{1}{\lambda^2} u(\lambda x, \lambda^2 t).$$

A direct calculation shows that u_λ is a viscosity subsolution of

$$u_t(x,t) + H(\lambda x, \lambda^2 t, \lambda \nabla u(x,t)) + F(\lambda \nabla u(x,t), \nabla^2 u(x,t)) = f(\lambda x) \quad \text{in } \mathbf{R}^n \times (0,\infty),$$

and, by (2.3), it is also a viscosity subsolution of

$$u_t(x,t) + H(\lambda x, \lambda^2 t, \lambda \nabla u(x,t)) + F(\nabla u(x,t), \nabla^2 u(x,t)) = f(\lambda x) \quad \text{in } \mathbf{R}^n \times (0,\infty). \tag{3.1}$$

Note that $u_* \leqq \liminf_{\lambda\to 1+0} u_\lambda$ in $\mathbf{R}^n \times [0,\infty)$. Thus, in order to derive $u_* \leqq v$ in $\mathbf{R}^n \times (0,\infty)$, it suffices to prove that

$$\liminf_{\lambda\to 1+0} u_\lambda \leqq v \quad \text{in } \mathbf{R}^n \times (0,T) \tag{3.2}$$

for every $T > 0$. Suppose by contradiction that

$$\theta := \sup_{\mathbf{R}^n \times (0,T)} \left(\liminf_{\lambda\to 1+0} u_\lambda - v \right) > 0.$$

We choose a point $(x_0,t_0) \in \mathbf{R}^n \times (0,T)$ such that

$$\liminf_{\lambda\to 1+0} u_\lambda(x_0,t_0) - v(x_0,t_0) \geqq \frac{4\theta}{5},$$

and then there exists some $\lambda_0 > 1$ such that

$$u_\lambda(x_0,t_0) - v(x_0,t_0) \geqq \frac{3\theta}{5} \quad \text{for all } \lambda \in (1,\lambda_0). \tag{3.3}$$

We fix $\lambda \in (1,\lambda_0)$ and define a function $\Psi_\lambda : \mathbf{R}^n \times [0,T] \times \mathbf{R}^n \times [0,T] =: \mathcal{O} \to \mathbf{R} \cup \{-\infty\}$ by

$$\Psi_\lambda(x,t,y,s) := u_\lambda(x,t) - v(y,s) - \phi(x,t,y,s)$$

with

$$\phi(x,t,y,s) = \frac{|x-y|^4}{\varepsilon} + \frac{|t-s|^2}{\varepsilon} + \frac{\sigma}{T-t}.$$

Here $\varepsilon \in (0,1]$ and

$$\sigma = \frac{\theta(T-t_0)}{5} > 0. \tag{3.4}$$

We interpret $\sigma/(T-t) = \infty$ when $t = T$. Note that σ is independent of λ. From (3.3) and (3.4) it follows that

$$\Psi_\lambda(x_0,t_0,x_0,t_0) \geqq \frac{2\theta}{5}. \tag{3.5}$$

2. For later use we prepare some constants.

- Since u and v satisfy (2.9) and (2.10) respectively, there exists a constant $R > 0$ independent of $\lambda \in (1,\lambda_0)$ such that

$$u_\lambda \leqq \frac{\theta}{5}, \quad v \geqq -\frac{\theta}{5} \quad \text{in } B_R(0)^c \times [0,T]. \tag{3.6}$$

- By (3.6) and the semicontinuity of u and v, there exists a constant $M > 0$ independent of $\lambda \in (1,\lambda_0)$ such that

$$u_\lambda \leqq M, \quad v \geqq -M \quad \text{in } \mathbf{R}^n \times [0,T]. \tag{3.7}$$

- Hereafter we take $\varepsilon \in (0,1]$ so small that $2\varepsilon M \leqq R^4$.

3. Define

$$K := \left\{ (x,t,y,s) \in \mathcal{O} \,\middle|\, \Psi_\lambda(x,t,y,s) \geqq \frac{2\theta}{5} \right\}, \tag{3.8}$$

which is nonempty by (3.5) and is closed by the upper semicontinuity of Ψ_λ. Let us prove that

$$K \subset B_{2R}(0) \times [0,T] \times B_{2R}(0) \times [0,T] =: K_*.$$

Since K_* is bounded, the above inclusion guarantees that K is a nonempty compact set. Take $(x,t,y,s) \in K$ arbitrarily. Then, by (3.7)

$$\frac{|x-y|^4}{\varepsilon} + \frac{|t-s|^2}{\varepsilon} + \frac{\sigma}{T-t} \leqq u_\lambda(x,t) - v(y,s) - \frac{2\theta}{5} \leqq 2M - \frac{2\theta}{5} < 2M,$$

and especially

$$|x-y|^4 \leqq 2\varepsilon M, \quad |t-s|^2 \leqq 2\varepsilon M. \tag{3.9}$$

The former inequality implies that $|x-y| \leqq R$ due to the smallness of ε. Suppose that $(x,t,y,s) \notin K_*$, i.e., $x \notin B_{2R}(0)$ or $y \notin B_{2R}(0)$. Since $|x-y| \leqq R$, we then have $x \notin B_R(0)$ and $y \notin B_R(0)$. It follows from (3.6) that

$$\Psi_\lambda(x,t,y,s) \leqq \frac{\theta}{5} + \frac{\theta}{5} - \frac{|x-y|^4}{\varepsilon} - \frac{|t-s|^2}{\varepsilon} - \frac{\sigma}{T-t} < \frac{2\theta}{5},$$

which is contrary to the assumption $(x,t,y,s) \in K$. We thus conclude that $K \subset K_*$.

4. Since K is a nonempty compact set, Ψ_λ attains a maximum over $\mathcal{O}$ at some $Z_{\lambda,\varepsilon} = (x_{\lambda,\varepsilon}, t_{\lambda,\varepsilon}, y_{\lambda,\varepsilon}, s_{\lambda,\varepsilon}) \in K \subset K_*$. In particular, letting $\phi_1(x,t) := \phi(x,t,y_{\lambda,\varepsilon}, s_{\lambda,\varepsilon})$ and $\phi_2(y,s) := -\phi(x_{\lambda,\varepsilon}, t_{\lambda,\varepsilon}, y, s)$, we have

$$\begin{cases} u_\lambda - \phi_1 \text{ attains a maximum at } (x_{\lambda,\varepsilon}, t_{\lambda,\varepsilon}), \\ v - \phi_2 \text{ attains a minimum at } (y_{\lambda,\varepsilon}, s_{\lambda,\varepsilon}). \end{cases} \tag{3.10}$$

Moreover, the definition of K_* implies that the family of maximizers $\{Z_{\lambda,\varepsilon}\}$ is bounded uniformly in λ and ε. Thus, for every $\lambda \in (1, \lambda_0)$, we may assume that there exists some $(\bar{x}_\lambda, \bar{t}_\lambda, \bar{y}_\lambda, \bar{s}_\lambda) \in \overline{K_*}$ such that

$$\lim_{\varepsilon \to +0} Z_{\lambda,\varepsilon} = \lim_{\varepsilon \to +0} (x_{\lambda,\varepsilon}, t_{\lambda,\varepsilon}, y_{\lambda,\varepsilon}, s_{\lambda,\varepsilon}) = (\bar{x}_\lambda, \bar{t}_\lambda, \bar{y}_\lambda, \bar{s}_\lambda).$$

By (3.9) we have $\bar{x}_\lambda = \bar{y}_\lambda \in \overline{B_{2R}(0)}$ and $\bar{t}_\lambda = \bar{s}_\lambda \in [0, T]$. Since $\{\bar{x}_\lambda\}$ and $\{\bar{t}_\lambda\}$ are bounded uniformly in λ, we may again assume that they are convergent. Namely,

$$\lim_{\lambda \to 1+0} (\bar{x}_\lambda, \bar{t}_\lambda) = (\bar{x}, \bar{t})$$

for some $(\bar{x}, \bar{t}) \in \overline{B_{2R}(0)} \times [0, T]$.

Let us show that $\bar{t} \in (0, T)$. Set

$$\Theta_\lambda := \sup_{(x,t) \in \mathbf{R}^n \times (0,T)} \Psi_\lambda(x,t,x,t) = \sup_{(x,t) \in \mathbf{R}^n \times (0,T)} \left\{ u_\lambda(x,t) - v(x,t) - \frac{\sigma}{T-t} \right\}.$$

From (3.5) it follows that $\Theta_\lambda \geqq 2\theta/5 > 0$. Moreover, since $\Psi_\lambda(Z_{\lambda,\varepsilon}) \geqq \Psi_\lambda(x,t,x,t)$ for any $(x,t) \in \mathbf{R}^n \times (0,T)$, we deduce that $\Psi_\lambda(Z_{\lambda,\varepsilon}) \geqq \Theta_\lambda$. This inequality implies that

$$u_\lambda(x_{\lambda,\varepsilon}, t_{\lambda,\varepsilon}) - v(y_{\lambda,\varepsilon}, s_{\lambda,\varepsilon}) - \frac{\sigma}{T - t_{\lambda,\varepsilon}} \geqq \frac{|x_{\lambda,\varepsilon} - y_{\lambda,\varepsilon}|^4}{\varepsilon} + \frac{|t_{\lambda,\varepsilon} - s_{\lambda,\varepsilon}|^2}{\varepsilon} + \Theta_\lambda \geqq \frac{2\theta}{5}.$$

Taking $\limsup_{\varepsilon \to +0}$, we obtain

$$u_\lambda(\bar{x}_\lambda, \bar{t}_\lambda) - v(\bar{x}_\lambda, \bar{t}_\lambda) - \frac{\sigma}{T - \bar{t}_\lambda} \geqq \frac{2\theta}{5},$$

and then sending $\limsup_{\lambda \to 1+0}$ yields

$$u(\bar{x}, \bar{t}) - v(\bar{x}, \bar{t}) - \frac{\sigma}{T - \bar{t}} \geqq \frac{2\theta}{5}.$$

By this inequality we see that $\bar{t} \neq T$. Moreover, we have $\bar{t} \neq 0$; otherwise the initial conditions on u and v would imply that

$$u(\bar{x}, 0) - v(\bar{x}, 0) - \frac{\sigma}{T} \leqq 0 - \frac{\sigma}{T} < 0,$$

which is a contradiction. Therefore $\bar{t} \in (0, T)$.

When $\lambda \in (1, \lambda_0)$ is sufficiently close to 1, we have $\bar{t}_\lambda \in (\bar{t}/2, T)$. In addition, for every such λ, we have $t_{\lambda,\varepsilon}, s_{\lambda,\varepsilon} \in (\bar{t}/2, T)$ if $\varepsilon \in (0, 1]$ is sufficiently small.

5. Let us define

$$p_{\lambda,\varepsilon} := \nabla_x \phi(Z_{\lambda,\varepsilon}) = -\nabla_y \phi(Z_{\lambda,\varepsilon}) = \frac{4}{\varepsilon}|x_{\lambda,\varepsilon} - y_{\lambda,\varepsilon}|^2(x_{\lambda,\varepsilon} - y_{\lambda,\varepsilon}),$$
$$\tau_{\lambda,\varepsilon} := \phi_t(Z_{\lambda,\varepsilon}) - \frac{\sigma}{(t_{\lambda,\varepsilon} - T)^2} = -\phi_s(Z_{\lambda,\varepsilon}) = \frac{2}{\varepsilon}(t_{\lambda,\varepsilon} - s_{\lambda,\varepsilon}).$$

For $\gamma := \bar{t}/2$ we apply the assumption that either u or v satisfies the Lipschitz condition (2.7). In either case, we see by (3.10) that $\{p_{\lambda,\varepsilon}\}$ is bounded uniformly in λ and ε. Thus, for a fixed $\lambda \in (1, \lambda_0)$, extracting a subsequence if necessary, we deduce that

$$\lim_{\varepsilon \to +0} p_{\lambda,\varepsilon} = \bar{p}_\lambda$$

for some $\bar{p}_\lambda \in \mathbf{R}^n$. Furthermore, we may assume that there is $\bar{p} \in \mathbf{R}^n$ such that

$$\lim_{\lambda \to 1+0} \bar{p}_\lambda = \bar{p}.$$

Fix $\lambda \in (1, \lambda_0)$, and let us divide the situation into two cases.

Case 1: We study the case where there exists a sequence $\{\varepsilon_k\}_{k=1}^\infty \subset (0, 1]$ such that $\lim_{k\to\infty} \varepsilon_k = 0$ and $p_{\lambda,\varepsilon_k} \neq 0$ for all $k \in \mathbf{N}$. To simplify notation we omit the index k below. We apply Crandall-Ishii's lemma (see, e.g., [7, Theorems 3.2 and 8.3]) for Ψ_λ at $Z_{\lambda,\varepsilon}$. Then there exist $X_{\lambda,\varepsilon}, Y_{\lambda,\varepsilon} \in \mathbf{S}^n$ such that $X_{\lambda,\varepsilon} + Y_{\lambda,\varepsilon} \leqq O$ and

$$\left(p_{\lambda,\varepsilon}, \tau_{\lambda,\varepsilon} + \frac{\sigma}{(t_{\lambda,\varepsilon} - T)^2}, X_{\lambda,\varepsilon}\right) \in \overline{\mathcal{P}}^{2,+} u_\lambda(x_{\lambda,\varepsilon}, t_{\lambda,\varepsilon}), \quad (p_{\lambda,\varepsilon}, \tau_{\lambda,\varepsilon}, -Y_{\lambda,\varepsilon}) \in \overline{\mathcal{P}}^{2,-} v(y_{\lambda,\varepsilon}, s_{\lambda,\varepsilon}).$$

Recall that u_λ is a subsolution of (3.1). By Remark 2.6 we then have

$$\tau_{\lambda,\varepsilon} + \frac{\sigma}{(t_{\lambda,\varepsilon} - T)^2} + H(\lambda x_{\lambda,\varepsilon}, \lambda^2 t_{\lambda,\varepsilon}, \lambda p_{\lambda,\varepsilon}) + F(p_{\lambda,\varepsilon}, X_{\lambda,\varepsilon}) \leqq f^*(\lambda x_{\lambda,\varepsilon}),$$
$$\tau_{\lambda,\varepsilon} + H(y_{\lambda,\varepsilon}, s_{\lambda,\varepsilon}, p_{\lambda,\varepsilon}) + F(p_{\lambda,\varepsilon}, -Y_{\lambda,\varepsilon}) \geqq f_*(y_{\lambda,\varepsilon}).$$

Here, since $p_{\lambda,\varepsilon} \neq 0$, we have used the fact that $F_*(p_{\lambda,\varepsilon}, X_{\lambda,\varepsilon}) = F(p_{\lambda,\varepsilon}, X_{\lambda,\varepsilon})$ and $F^*(p_{\lambda,\varepsilon}, -Y_{\lambda,\varepsilon}) = F(p_{\lambda,\varepsilon}, -Y_{\lambda,\varepsilon})$. Also, note that $F(p_{\lambda,\varepsilon}, X_{\lambda,\varepsilon}) \geqq F(p_{\lambda,\varepsilon}, -Y_{\lambda,\varepsilon})$ by (2.1). Hence, subtracting the two inequalities above, we obtain

$$\frac{\sigma}{T^2} + H(\lambda x_{\lambda,\varepsilon}, \lambda^2 t_{\lambda,\varepsilon}, \lambda p_{\lambda,\varepsilon}) - H(y_{\lambda,\varepsilon}, s_{\lambda,\varepsilon}, p_{\lambda,\varepsilon}) \leqq f^*(\lambda x_{\lambda,\varepsilon}) - f_*(y_{\lambda,\varepsilon}). \tag{3.11}$$

Sending $\limsup_{\varepsilon\to+0}$ gives

$$\frac{\sigma}{T^2} + H(\lambda \bar{x}_\lambda, \lambda^2 \bar{t}_\lambda, \lambda \bar{p}_\lambda) - H(\bar{x}_\lambda, \bar{t}_\lambda, \bar{p}_\lambda) \leqq f^*(\lambda \bar{x}_\lambda) - f_*(\bar{x}_\lambda). \tag{3.12}$$

Case 2: We next consider the case where $p_{\lambda,\varepsilon} = 0$ for all $\varepsilon > 0$ small enough. Let us recall the fact (3.10). For the functions ϕ_1 and ϕ_2 in (3.10), we have

$$\nabla\phi_1(x_{\lambda,\varepsilon}, t_{\lambda,\varepsilon}) = \nabla\phi_2(y_{\lambda,\varepsilon}, s_{\lambda,\varepsilon}) = 0, \quad \nabla^2\phi_1(x_{\lambda,\varepsilon}, t_{\lambda,\varepsilon}) = \nabla^2\phi_2(y_{\lambda,\varepsilon}, s_{\lambda,\varepsilon}) = O.$$

Thus, the definition of viscosity solutions imply

$$\tau_{\lambda,\varepsilon} + \frac{\sigma}{(t_{\lambda,\varepsilon} - T)^2} + H(\lambda x_{\lambda,\varepsilon}, \lambda^2 t_{\lambda,\varepsilon}, 0) + F_*(0, O) \leqq f^*(\lambda x_{\lambda,\varepsilon}),$$
$$\tau_{\lambda,\varepsilon} + H(y_{\lambda,\varepsilon}, s_{\lambda,\varepsilon}, 0) + F^*(0, O) \geqq f_*(y_{\lambda,\varepsilon}).$$

By (2.2), subtracting these two inequalities and letting ε go to 0, we are again led to (3.12) with $\bar{p}_\lambda = 0$.

Recall that σ does not depend on λ. We take $\limsup_{\lambda\to 1+0}$ in (3.12) and apply (2.4) to obtain

$$\frac{\sigma}{T^2} + H(\bar{x}, \bar{t}, \bar{p}) - H(\bar{x}, \bar{t}, \bar{p}) \leqq 0,$$

which is a contradiction since $\sigma/T^2 > 0$. $\square$

3.2 Comparison for special Hamiltonians

We establish the other version of a weak comparison principle which is valid for H satisfying (2.8). In this case, we do not need the Lipschitz continuity (2.7).

Theorem 3.2 (Weak comparison principle 2)**.** *Assume* (2.1)–(2.4) *and* (2.8)*. Let* $u : \mathbf{R}^n \times [0, \infty) \to \mathbf{R}$ *be a viscosity subsolution of* (1.1) *satisfying* (2.9)*, and let* $v : \mathbf{R}^n \times [0, \infty) \to \mathbf{R}$ *be a viscosity supersolution of* (1.1) *satisfying* (2.10)*. If* $u^*(\cdot, 0) \leqq v_*(\cdot, 0)$ *in* $\mathbf{R}^n$*, then* $(u^*)_* \leqq v_*$ *and* $u^* \leqq (v_*)^*$ *in* $\mathbf{R}^n \times [0, \infty)$*.*

Proof. We only state the difference from the proof of Theorem 3.1. Due to a lack of the Lipschitz continuity (2.7) of u or v, $\{p_{\lambda,\varepsilon}\}$ may not have a convergent subsequence. However, (3.11) and (2.8) give

$$\frac{\sigma}{T^2} \leqq f^*(\lambda x_{\lambda,\varepsilon}) - f_*(y_{\lambda,\varepsilon}).$$

Sending $\varepsilon \to +0$, we deduce that $\sigma/T^2 \leqq f^*(\lambda\bar{x}_\lambda) - f_*(\bar{x}_\lambda)$. Thus, taking $\limsup_{\lambda\to 1+0}$, we reach a contradiction by (2.4). $\square$

4 Uniqueness of solutions

From Theorems 3.1 and 3.2, we derive uniqueness results of solutions to (1.1)–(1.2).

4.1 Uniqueness under Theorem 3.1

As an immediate consequence of Theorem 3.1, we see that that Lipschitz continuous solutions are unique in the following sense:

Theorem 4.1 (Uniqueness of Lipschitz continuous solutions)**.** *Assume* (2.1)–(2.4)*. Let* $u \in \mathrm{SOL}_{\mathrm{Lip}}$*. If* $v \in \mathrm{SOL}$*, then* $u = v$ *in* $\mathbf{R}^n \times [0, \infty)$*.*

Proof. Note that u is continuous in $\mathbf{R}^n \times [0, \infty)$. We apply Theorem 3.1 for $u \in \mathrm{SOL}_{\mathrm{Lip}} \subset \mathrm{SUB}_{\mathrm{Lip}}$ and $v \in \mathrm{SOL} \subset \mathrm{SUP}$ to obtain $u = (u^*)_* \leqq v_*$ in $\mathbf{R}^n \times [0, \infty)$. Next, changing the role of u and v, we deduce that $v^* \leqq (u_*)^* = u$ in $\mathbf{R}^n \times [0, \infty)$. Combining the two inequalities implies that $u = v$ in $\mathbf{R}^n \times [0, \infty)$. $\square$

We next show that solutions given as an envelope of Lipschitz continuous solutions are unique. For this purpose, we introduce

Definition 4.2 (Envelope solution)**.**

$$\mathrm{SOL}^+ = \{u \in \mathrm{SOL} \mid \text{there exists a family } \mathcal{G} \subset \mathrm{SUP}_{\mathrm{Lip}} \text{ such that } u = \inf\nolimits_{w\in\mathcal{G}} w\},$$
$$\mathrm{SOL}^- = \{u \in \mathrm{SOL} \mid \text{there exists a family } \mathcal{G} \subset \mathrm{SUB}_{\mathrm{Lip}} \text{ such that } u = \sup\nolimits_{w\in\mathcal{G}} w\}.$$

We call $u \in \mathrm{SOL}^+$ an *upper envelope solution* and $u \in \mathrm{SOL}^-$ a *lower envelope solution.*

Remark 4.3. We do not require that Lipschitz constants of $w \in \mathcal{G}$ are uniform in the definitions above, and hence upper- and lower envelope solutions may not satisfy (2.7). However, since $\mathrm{SUP}_{\mathrm{Lip}}$ and $\mathrm{SUB}_{\mathrm{Lip}}$ consist of continuous functions in $\mathbf{R}^n \times [0, \infty)$, we see that every upper envelope solution is upper semicontinuous in $\mathbf{R}^n \times [0, \infty)$ and every lower envelope solution is lower semicontinuous in $\mathbf{R}^n \times [0, \infty)$.

The following comparison result immediately follows from Theorem 3.1.

Corollary 4.4. *Assume* (2.1)–(2.4)*.*

(1) *If* $u \in \mathrm{SUB}$ *and* $v \in \mathrm{SOL}^+$*, then* $u^* \leqq v$ *in* $\mathbf{R}^n \times [0, \infty)$*.*

(2) *If* $u \in \mathrm{SOL}^-$ *and* $v \in \mathrm{SUP}$*, then* $u \leqq v_*$ *in* $\mathbf{R}^n \times [0, \infty)$*.*

(3) *If* $u \in \mathrm{SOL}^-$ *and* $v \in \mathrm{SOL}^+$*, then* $u^* \leqq v$ *and* $u \leqq v_*$ *in* $\mathbf{R}^n \times [0, \infty)$*.*

Proof. (1) Take $\mathcal{G} \subset \mathrm{SUP}_{\mathrm{Lip}}$ such that $v = \inf_{w \in \mathcal{G}} w$. For any $w \in \mathcal{G} \subset \mathrm{SUP}_{\mathrm{Lip}}$, Theorem 3.1 implies that $u^* \leqq (w_*)^* = w$ since w is continuous. Thus, taking the infimum, we conclude that $u^* \leqq \inf_{w \in \mathcal{G}} w = v$ in $\mathbf{R}^n \times [0, \infty)$. The proof of (2) is parallel, and (3) is a consequence of (1) and (2). □

Let us prove that upper envelope solutions and lower envelope solutions are unique.

Theorem 4.5 (Uniqueness of envelope solutions)**.** *Assume* (2.1)–(2.4)*. Let* $u^+ \in \mathrm{SOL}^+$ *and* $u^- \in \mathrm{SOL}^-$*.*

(1) *If* $v \in \mathrm{SOL}$*, then* $u^- \leqq v_* \leqq v \leqq v^* \leqq u^+$ *in* $\mathbf{R}^n \times [0, \infty)$*.*

(2) *If* $v \in \mathrm{SOL}^+$*, then* $u^+ = v$ *in* $\mathbf{R}^n \times [0, \infty)$*.*

(3) *If* $v \in \mathrm{SOL}^-$*, then* $u^- = v$ *in* $\mathbf{R}^n \times [0, \infty)$*.*

Proof. (1) Since $v \in \mathrm{SOL} \subset \mathrm{SUB}$ and $u^+ \in \mathrm{SOL}^+$, Corollary 4.4 (1) implies that $v^* \leqq u^+$ in $\mathbf{R}^n \times [0, \infty)$. Similarly, since $u^- \in \mathrm{SOL}^-$ and $v \in \mathrm{SOL} \subset \mathrm{SUP}$, we deduce from Corollary 4.4 (2) that $u^- \leqq v_*$ in $\mathbf{R}^n \times [0, \infty)$.

(2) We have $u^+ \in \mathrm{SOL}^+ \subset \mathrm{SOL} \subset \mathrm{SUB}$ and $v \in \mathrm{SOL}^+$. Thus, noting that u^+ is upper semicontinuous (Remark 4.3), we see that $u^+ = (u^+)^* \leqq v$ in $\mathbf{R}^n \times [0, \infty)$ by Corollary 4.4 (1). The same argument shows that $v \leqq u^+$ in $\mathbf{R}^n \times [0, \infty)$. One can prove (3) in a similar manner. □

Remark 4.6. Let $u^+ \in \mathrm{SOL}^+$ and $u^- \in \mathrm{SOL}^-$. Theorem 4.5 (1) asserts that u^+ and u^- are respectively a *maximal solution* and a *minimal solution*. In this sense, envelope solutions can characterize maximal solutions and minimal solutions. In [15, Section 2] uniqueness and existence of maximal solutions are established for the equation (1.11) with a positive $\nu > 0$.

4.2 Uniqueness under Theorem 3.2

We begin with a result similar to Theorem 4.1 as a consequence of Theorem 3.2. We omit the proof since it is almost the same as before.

Theorem 4.7 (Uniqueness of continuous solutions)**.** *Assume* (2.1)–(2.4) *and* (2.8)*. Let* $u \in \mathrm{SOL} \cap C(\mathbf{R}^n \times [0,\infty))$*. If* $v \in \mathrm{SOL}$*, then* $u = v$ *in* $\mathbf{R}^n \times [0,\infty)$*.*

Theorem 3.2 also guarantees that semicontinuous solutions are unique.

Theorem 4.8 (Uniqueness of semicontinuous solutions)**.** *Assume* (2.1)–(2.4) *and* (2.8)*. Let* $u \in \mathrm{SOL}$*.*

(1) $(u_*)^* = u^*$ *and* $(u^*)_* = u_*$ *in* $\mathbf{R}^n \times [0,\infty)$*. In particular,* $u^*, u_* \in \mathrm{SOL}$*.*

(2) *If* $v \in \mathrm{SOL}$*, then* $u^* = v^*$ *and* $u_* = v_*$ *in* $\mathbf{R}^n \times [0,\infty)$*. In particular, if both* $u, v \in \mathrm{SOL}$ *are upper semicontinuous or lower semicontinuous in* $\mathbf{R}^n \times [0,\infty)$*, then* $u = v$ *in* $\mathbf{R}^n \times [0,\infty)$*.*

Proof. Let $v \in \mathrm{SOL}$. Then Theorem 3.2 yields

$$(u^*)_* \leqq v_*, \quad u^* \leqq (v_*)^*, \quad (v^*)_* \leqq u_*, \quad v^* \leqq (u_*)^* \quad \text{in } \mathbf{R}^n \times [0,\infty). \tag{4.1}$$

By the first and third inequalities, we have

$$(u^*)_* \leqq v_* \leqq (v^*)_* \leqq u_* \leqq (u^*)_* \quad \text{in } \mathbf{R}^n \times [0,\infty),$$

and hence $(u^*)_* = u_* = v_*$ in $\mathbf{R}^n \times [0,\infty)$. Similarly, the second and fourth inequalities in (4.1) imply that $(u_*)^* = u^* = v^*$ in $\mathbf{R}^n \times [0,\infty)$. The former assertions of (1) and (2) are thus proved. The proofs of the latter ones are immediate. □

Remark 4.9. Let $u \in \mathrm{SOL}$. Then the unique upper semicontinuous solution and the unique lower semicontinuous solution are given by u^* and u_*, respectively. Moreover, as a consequence of Theorem 4.8, we see that u^* is a maximal solution and u_* is a minimal solution. Therefore they give another characterization of the maximal and minimal solution (Remark 4.6).

Let us recall that every upper envelope solution $u^+ \in \mathrm{SOL}^+$ is upper semicontinuous (Remark 4.3). Accordingly, we have $u^* = u^+$ since upper semicontinuous solutions are unique. Similaly, if $u^- \in \mathrm{SOL}^-$, then $u_* = u^-$.

5 Existence of solutions

We turn to the issue of existence of solutions.

5.1 Envelope solutions

We discuss construction of upper- and lower envelope solutions, which are unique under the assumptions in Theorem 4.5. To do this, we approximate the source term f by continuous functions f^ε and f_ε such that $f_\varepsilon \leqq f \leqq f^\varepsilon$, and we solve

$$u_t(x,t) + H(x,t,\nabla u(x,t)) + F(\nabla u(x,t), \nabla^2 u(x,t)) = f^\varepsilon(x) \quad \text{in } \mathbf{R}^n \times (0,\infty), \tag{5.1}$$
$$u_t(x,t) + H(x,t,\nabla u(x,t)) + F(\nabla u(x,t), \nabla^2 u(x,t)) = f_\varepsilon(x) \quad \text{in } \mathbf{R}^n \times (0,\infty). \tag{5.2}$$

We define $\mathrm{SOL}^\varepsilon_{\mathrm{Lip}}$ as the set of viscosity solutions u of (5.1)–(1.2) satisfying (2.7), (2.9) and (2.10). In a similar manner, we define $(\mathrm{SOL}_\varepsilon)_{\mathrm{Lip}}$ by replacing "(5.1)" by "(5.2)" above.

Proposition 5.1. *Assume* (2.1)–(2.4)*. Let* $\{f^\varepsilon\}_{\varepsilon>0}, \{f_\varepsilon\}_{\varepsilon>0} \subset C(\mathbf{R}^n)$ *be sequences such that*

$$f_\varepsilon \leqq f_\delta \leqq f_* \leqq f^* \leqq f^\delta \leqq f^\varepsilon \quad \text{in } \mathbf{R}^n \text{ for } 0<\delta<\varepsilon, \tag{5.3}$$

$$f^* = \inf_{\varepsilon>0} f^\varepsilon,\ \ f_* = \sup_{\varepsilon>0} f_\varepsilon \quad \text{in } \mathbf{R}^n. \tag{5.4}$$

Assume that $u^\varepsilon \in \mathrm{SOL}^\varepsilon_{\mathrm{Lip}}$, $u_\varepsilon \in (\mathrm{SOL}_\varepsilon)_{\mathrm{Lip}}$ *for* $\varepsilon>0$ *and that*

$$u_\varepsilon \leqq u_\delta \leqq u^\delta \leqq u^\varepsilon \quad \text{in } \mathbf{R}^n\times[0,\infty) \text{ for } 0<\delta<\varepsilon. \tag{5.5}$$

Define $u^+ := \inf_{\varepsilon>0} u^\varepsilon$ *and* $u^- := \sup_{\varepsilon>0} u_\varepsilon$*. Then*

(1) $u^+ \in \mathrm{SOL}^+$ *and* $u^- \in \mathrm{SOL}^-$.

(2) *If either* u^+ *or* u^- *satisfies* (2.7)*, then* $u^+ = u^-$ *in* $\mathbf{R}^n\times[0,\infty)$ *and* $u^\pm = v$ *in* $\mathbf{R}^n\times[0,\infty)$ *for any* $v\in\mathrm{SOL}$.

Proof. (1) We first note that (5.3) implies that $u^\varepsilon \in \mathrm{SUP}_{\mathrm{Lip}}$ and $u_\varepsilon \in \mathrm{SUB}_{\mathrm{Lip}}$ for $\varepsilon>0$.

Next, by (5.5) we have $u_\varepsilon \leqq u^- \leqq u^+ \leqq u^\varepsilon$ for every $\varepsilon>0$. This shows that $u^\pm$ satisfy the initial condition (1.2) and the decay conditions (2.9) and (2.10).

Let us prove that u^+ is a viscosity solution of (1.1). To do this, we apply stability results for viscosity solutions ([7, Sections 4 and 6]).

- Since u^ε is a viscosity supersolution of (1.1), the infimum $u^+ = \inf_{\varepsilon>0} u^\varepsilon$ is also a viscosity supersolution of (1.1).
- We next apply stability under the relaxed half limits. From the monotonicity (5.5) and (5.4) it follows that

  $$\limsup_{\varepsilon\to+0}{}^* u^\varepsilon = \inf_{\varepsilon>0} u^\varepsilon = u^+, \quad \limsup_{\varepsilon\to+0}{}^* f^\varepsilon = \inf_{\varepsilon>0} f^\varepsilon = f^*.$$

  Since u^ε is a viscosity subsolution of (5.1), the limit u^+ is a viscosity subsolution of (1.1).

We thus conclude that $u^+\in\mathrm{SOL}^+$. One can prove that $u^-\in\mathrm{SOL}^-$ in the same way.

(2) This follows from Theorem 4.1. □

A technique similar to the above proof can be found in [14, Proposition 3.7] and [15, Theorem 2.2].

Remark 5.2. We comment on the assumptions in Proposition 5.1.

(1) When the usual comparison principle (in the sense of (1.13)) holds for (5.1) and (5.2), it implies the monotonicity (5.5) of solutions.

(2) When the initial datum u_0 is Lipschitz continuous or more regular, there is a chance that the unique solutions of (5.1)–(1.2) and (5.2)–(1.2) preserve the Lipschitz continuity, i.e., $u^\varepsilon \in \mathrm{SOL}^\varepsilon_{\mathrm{Lip}}$ and $u_\varepsilon \in (\mathrm{SOL}_\varepsilon)_{\mathrm{Lip}}$. See, e.g., [27] for linear and quasi-linear equations, [26, Lemma 7.28] for viscous Hamilton–Jacobi equations and [1, Theorems 8.1 and 8.2] for first order equations. For the equation (1.11) with $\nu > 0$, Lipschitz continuity of solutions is shown in [17, Section 4]. See also [22], [13, Chapter 3.5] and [18, Section 5] for related results.

(3) Let us recall that (2.7) does not require the Lipschitz regularity at the initial time. This implies that, even though the initial datum u_0 is not Lipschitz continuous, Proposition 5.1 can be applied if Lipschitz regularizing effect occurs for (5.1) and (5.2). Here, by Lipschitz regularizing, we mean that the solution $u(x,t)$ immediately gets Lipschitz regularity in x after the initial time. Such Lipschitz regularizing effect occurs for some uniformly parabolic equations and Hamilton–Jacobi equations. See, e.g., [8, 29, 9] for second order equations and [24, 25] for first order equations.

5.2 Semicontinuous solutions

To build semicontinuous solutions, whose uniqueness are guaranteed in Theorem 4.8, we only have to find some solution $u \in \mathrm{SOL}$. Indeed, the semicontinuous envelopes u^* and u_* then give the unique upper semicontinuous solution and the unique lower semicontinuous solution, respectively.

To find a viscosity solution $u \in \mathrm{SOL}$, *Perron's method* ([7, Section 4], [13, Chapter 2.4]) is a well-known and powerful tool; the method can give a solution without approximating f by continuous ones. For Perron's method we need so-called barrier functions. Namely, we need $h^- \in \mathrm{SUB}$ and $h^+ \in \mathrm{SUP}$ such that

$$\begin{cases} (h^-)^* \leqq (h^+)_* \text{ in } \mathbf{R}^n \times [0,\infty), \\ h^\pm(\cdot,0) = u_0 \text{ in } \mathbf{R}^n,\ h^\pm \text{ are continuous on } \mathbf{R}^n \times \{0\}. \end{cases} \tag{5.6}$$

If we define

$$u_P(x,t) := \sup\{w(x,t) \mid w \in \mathrm{SUB} \text{ and } (h^-)^* \leqq w \leqq (h^+)_* \text{ in } \mathbf{R}^n \times [0,\infty)\}, \tag{5.7}$$

then $u_P \in \mathrm{SOL}$.

The remaining problem is the existence of barrier functions. In this paper we do not pursue this issue too far; the reader is refereed to [13, Chapter 4.3] and so on.

We state a simple sufficient condition for existence of the barriers for (1.1).

Proposition 5.3. *Assume that $u_0 \in C^2(\mathbf{R}^n)$ and that both ∇u_0 and $\nabla^2 u_0$ are bounded in $\mathbf{R}^n$. Assume that F is locally bounded in $\mathbf{R}^n \times \mathbf{S}^n$, H is bounded in $\mathbf{R}^n \times [0,\infty) \times B_R(0)$ for every $R > 0$ and that f is bounded in $\mathbf{R}^n$. For $M > 0$ we define*

$$h^-(x,t) = -Mt + u_0(x), \quad h^+(x,t) = Mt + u_0(x).$$

If $M > 0$ is large enough, then $h^- \in \mathrm{SUB}$, $h^+ \in \mathrm{SUP}$ and $h^\pm$ satisfy (5.6).

Proof. We define

$$M := \sup_{(x,t)\in\mathbf{R}^n\times[0,\infty)} |H(x,t,\nabla u_0(x)) + F(\nabla u_0(x), \nabla^2 u_0(x)) - f(x)|,$$

which is finite by assumptions. It is then easily seen that h^- and h^+ are respectively a classical subsolution and a classical supersolution of (1.1). Furthermore, h^- satisfies (2.9) and h^+ satisfies (2.10) since $\lim_{|x|\to\infty} u_0(x) = 0$. We thus have $h^- \in \mathrm{SUB}$ and $h^+ \in \mathrm{SUP}$. The condition (5.6) is obvious. □

Remark 5.4. Let $\nu \in \mathbf{R}$ and consider a geometric equation

$$u_t(x,t) - \nu|\nabla u(x,t)| - \Delta_1 u(x,t) = 0 \quad \text{in } \mathbf{R}^n \times (0,\infty). \tag{5.8}$$

Then, there exist barrier functions $h_0^- \in \mathrm{SUB}$ and $h_0^+ \in \mathrm{SUP}$ satisfying (5.6). See [13, Chapter 4.3]. Furthermore, if the support

$$\operatorname{supp} u_0 = \overline{\{x \in \mathbf{R}^n \mid u_0(x) \neq 0\}}$$

of the initial datum u_0 is bounded, then $h_0^\pm$ can be chosen so that $\operatorname{supp} h_0^\pm(\cdot,t)$ are bounded locally uniformly in $t \geqq 0$.

Modifying $h_0^\pm$, one easily obtains barrier functions for (1.11) provided that f is bounded in $\mathbf{R}^n$. In fact, if

$$m_1 := \inf_{\mathbf{R}^n} f > -\infty, \quad m_2 := \sup_{\mathbf{R}^n} f < \infty,$$

then it is easily seen that the functions

$$h^-(x,t) = \min\{m_1,\, 0\}t + h_0^-(x,t), \quad h^+(x,t) = \max\{m_2,\, 0\}t + h_0^+(x,t)$$

are barriers for (1.11).

We next restrict ourselves to (1.11) with a negative driving force $\nu < 0$ and build barrier functions $h^\pm$ whose supports $\operatorname{supp} h^\pm(\cdot,t)$ are bounded uniformly in $t \geqq 0$. As a consequence, we see that the support $\operatorname{supp} u(\cdot,t)$ of any solution $u \in \mathrm{SOL}$ is also bounded uniformly in $t \geqq 0$.

For this purpose, we prepare solutions to the elliptic problem associated to

$$u_t(x,t) - \nu|\nabla u(x,t)| - \Delta_1 u(x,t) = c\chi_{B_R(0)}(x) \quad \text{in } \mathbf{R}^n \times (0,\infty), \tag{5.9}$$

where $c, R > 0$. We solve the elliptic problem in $B_R(0)$, so that discontinuity of the source term disappears.

Example 5.5. Let $c, R > 0$ and $\nu < 0$. We consider

$$-\nu|\nabla U(x)| - \Delta_1 U(x) = c \quad \text{in } B_R(0) \tag{5.10}$$

under the Dirichlet boundary condition:

$$U(x) = 0 \quad \text{on } \partial B_R(0). \tag{5.11}$$

Here $U : \overline{B_R(0)} \to \mathbf{R}$ is unknown. We now suppose that there is a smooth solution $U(x)$ and that it is radially symmetric $U(x) = \psi(|x|)$. By direct calculations we have

$$\nabla U(x) = \psi'(|x|)\frac{x}{|x|}, \quad \nabla^2 U(x) = \psi''(|x|)\frac{x \otimes x}{|x|^2} + \psi'(|x|)\frac{1}{|x|}\left(I - \frac{x \otimes x}{|x|^2}\right) \quad (x \neq 0).$$

Substituting these for (5.10), we find that

$$-\nu|\psi'(r)| - \frac{1}{r}\psi'(r) = c \quad \text{in } (0, R). \tag{5.12}$$

Also, by (5.11) we have

$$\psi(R) = 0.$$

We now assume that $\psi' \leqq 0$. Then, the equation (5.12) gives $\psi'(r) = cr/(\nu r - 1)$ in $(0, R)$, and thus

$$\psi(r) = \int_R^r \frac{cs}{\nu s - 1}\, ds = -\frac{c}{\nu}(R - r) + \frac{c}{\nu^2}\log\frac{-\nu r + 1}{-\nu R + 1} \quad (0 \leqq r \leqq R).$$

Therefore, we conclude that

$$U_{R,c}(x) := U(x) = \psi(|x|) = -\frac{c}{\nu}(R - |x|) + \frac{c}{\nu^2}\log\frac{-\nu|x| + 1}{-\nu R + 1} \quad (|x| \leqq R). \tag{5.13}$$

See Figure 2 for the graph. One can check that the function U above is a viscosity solution of (5.10)–(5.11). In fact, $U \in C^2(\overline{B_R(0)})$ and U solves (5.10) in the classical sense in $B_R(0) \setminus \{0\}$. At the origin $x = 0$, we have

$$\nabla U(0) = 0, \quad \nabla^2 U(0) = -cI.$$

These facts show that $F_*(0, -cI) = F^*(0, -cI) = -c$, where F is the operator given by (1.4). This implies that U solves (5.10) in the viscosity sense at $x = 0$.

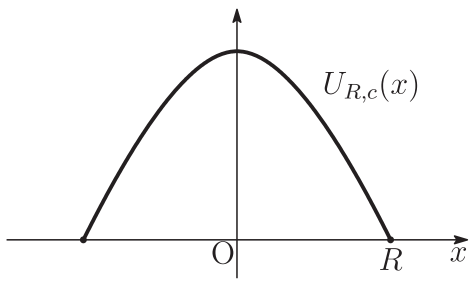

Figure 2: The graph of $U_{R,c}(x)$.

In the following proposition we use notations SOL etc. for the problem (1.11)–(1.2).

Proposition 5.6. *Let $\nu < 0$. Assume that both f and u_0 are nonnegative and their supports* $\operatorname{supp} f$ *and* $\operatorname{supp} u_0$ *are bounded in $\mathbf{R}^n$. Let $h_0^\pm$ are barrier functions for* (5.8) *given in Remark 5.4. For $c, R > 0$ we define*

$$\tilde{U}_{R,c}(x) := \begin{cases} U_{R,c}(x) & \text{if } |x| \leqq R, \\ 0 & \text{if } |x| > R, \end{cases}$$

where $U_{R,c}$ is the function defined in (5.13). *We further define*

$$h^-(x,t) = \max\{(h_0^-)^*(x,t),\, 0\}, \quad h^+(x,t) = \min\{(h_0^+)_*(x,t) + ct,\, \tilde{U}_{R,c}(x)\}.$$

If c, R are chosen so that $f \leqq c\chi_{B_R(0)}$ in $\mathbf{R}^n$ and $u_0 \leqq \tilde{U}_{R,c}$ in $\mathbf{R}^n$, then $h^- \in$ SUB, *$h^+ \in$* SUP, *$h^\pm$ satisfy* (5.6) *and* $\operatorname{supp} h^\pm(\cdot, t) \subset \overline{B_R(0)}$ *for all $t \geqq 0$. In particular,* $\operatorname{supp} u(\cdot, t) \subset \overline{B_R(0)}$ *for all $u \in$* SOL *and $t \geqq 0$.*

Proof. **1.** We prove that $\tilde{U}_{R,c}$ is a viscosity supersolution of (1.11). First, recall that $U_{R,c}$ is a solution to (5.10). Since $f \leqq c$ in $B_R(0)$ by assumption, we see that $\tilde{U}_{R,c}$ is a viscosity supersolution of (1.11) in $B_R(0) \times (0,\infty)$. Next, it is easily seen that a constant function 0 is a supersolution of (1.11) in $\overline{B_R(0)}^c \times (0,\infty)$.

It remains to prove the viscosity property of $\tilde{U}_{R,c}$ on $\partial B_R(0) \times (0,\infty)$. Assume that $\tilde{U}_{R,c} - \phi$ attains a local minimum at $(x_0,t_0) \in \partial B_R(0) \times (0,\infty)$ for $\phi \in C^{2,1}(\mathbf{R}^n \times (0,\infty))$. Since $\tilde{U}_{R,c}$ is independent of t, we have $\phi_t(x_0,t_0) = 0$. Also, by [15, Lemma A.1 (i)] we see that there is $s \leqq 0$ such that

$$F^*(\nabla\phi(x_0,t_0), \nabla^2\phi(x_0,t_0)) \geqq -\frac{s}{R},$$

where F is the operator defined in (1.4). Accordingly, we have

$$\phi_t(x_0,t_0) - \nu|\nabla\phi(x_0,t_0)| + F^*(\nabla\phi(x_0,t_0), \nabla^2\phi(x_0,t_0)) \geqq 0 = c\chi_{B_R(0)}(x_0) = f(x_0).$$

We thus conclude that $\tilde{U}_{R,c}$ is a supersolution of (1.11).

2. By the previous step and the stability result for viscosity solutions ([7, Lemma 4.2]), we see that $h^+ \in \mathrm{SUP}$. Similarly, we have $h^- \in \mathrm{SUB}$ since the constant 0 is a subsolution of (1.11). Moreover, since $h_0^\pm$ satisfy (5.6) and $0 \leqq u_0 \leqq \tilde{U}_{R,c}$ in $\mathbf{R}^n$, we deduce from Theorem 3.2 that $(h^-)^* \leqq (h^+)_*$ in $\mathbf{R}^n \times [0,\infty)$. The remaining conditions in (5.6) also hold since $(h_0^-)^* \leqq (h^-)^* \leqq (h^+)_* \leqq (h_0^+)_*$ in $\mathbf{R}^n \times [0,\infty)$ and $h_0^\pm$ satisfy (5.6).

We now have

$$0 \leqq h^- \leqq h^+ \leqq \tilde{U}_{R,c} \quad \text{in } \mathbf{R}^n \times [0,\infty)$$

and $\operatorname{supp} \tilde{U}_{R,c} = \overline{B_R(0)}$. This shows that $\operatorname{supp} h^\pm(\cdot,t) \subset \overline{B_R(0)}$ for all $t \geqq 0$.

3. Let $u_P \in \mathrm{SOL}$ be the solution given by (5.7). By definition we have $\operatorname{supp} u_P(\cdot,t) \subset \overline{B_R(0)}$ for all $t \geqq 0$. Take any $u \in \mathrm{SOL}$. Then Theorem 4.8 guarantees that $(u_P)_* \leqq u \leqq (u_P)^*$ in $\mathbf{R}^n \times [0,\infty)$, which implies that $\operatorname{supp} u(\cdot,t) \subset \overline{B_R(0)}$ for all $t \geqq 0$. The proof is complete. □

6 Examples

Let us give some examples of solutions to illustrate our results. Throughout this section we let $\nu, c > 0$. We consider the level-set mean curvature flow equation

$$u_t(x,t) - \nu|\nabla u(x,t)| - \Delta_1 u(x,t) = c\chi_\Omega(x) \quad \text{in } \mathbf{R}^n \times (0,\infty) \tag{6.1}$$

or the Hamilton–Jacobi equation

$$u_t(x,t) - \nu|\nabla u(x,t)| = c\chi_\Omega(x) \quad \text{in } \mathbf{R}^n \times (0,\infty) \tag{6.2}$$

with the source term $f = c\chi_\Omega$ given as (1.5). We solve them under the initial condition

$$u(x,0) = 0 \quad \text{in } \mathbf{R}^n. \tag{6.3}$$

We study solutions for several Ω. Solutions of (6.1) and (6.2) are respectively investigated in [15] and [14] in the case of $\nu = 1$. We utilize the results, but we present them for a general $\nu > 0$.

6.1 Discontinuous solutions

Example 6.1. Let
$$\Omega = \overline{B_R(0)} \quad (R > 0),$$
and let us study (6.1) with this Ω. Clearly, (2.6) is fulfilled and hence Theorem 3.1 is applicable to (6.1). In [15, Section 4] the maximal solution of (6.1)–(6.3) is investigated for $\nu = 1$. For a general $\nu > 0$ the unique maximal solution u_R of (6.1)–(6.3) is given as follows: Set
$$\Psi_R(r) := \frac{c}{\nu}\left(r - R + \frac{1}{\nu}\log\frac{-\nu r + 1}{-\nu R + 1}\right).$$

- If $R < 1/\nu$, then
$$u_R(x,t) = \begin{cases} \min\{ct,\ \Psi_R(|x|)\} & \text{if } |x| \leqq R, \\ 0 & \text{if } |x| > R. \end{cases}$$

- If $R = 1/\nu$, then $u_{1/\nu}(x,t) = ct\chi_{\overline{B_{1/\nu}(0)}}(x)$. This is a discontinuous solution.

- If $R > 1/\nu$, then
$$u_R(x,t) = \begin{cases} ct & \text{if } |x| \leqq R, \\ \max\{ct - \Psi_R(|x|),\ 0\} & \text{if } |x| > R. \end{cases}$$

We discuss in what sense u_R is the unique solution. Let $R \neq 1/\nu$. Then it is easily seen that u_R satisfies the Lipschitz continuity (2.7), i.e., $u_R \in \mathrm{SOL_{Lip}}$, where we use notations $\mathrm{SOL_{Lip}}$ etc. for the problem (6.1)–(6.3). Thus, u_R is the unique solution in the sense of Theorem 4.1. We next let $R = 1/\nu$. Observe that $u_R \in \mathrm{SUP_{Lip}}$ if $R > 1/\nu$, $u_R \in \mathrm{SUB_{Lip}}$ if $0 < R < 1/\nu$, and that
$$u_{1/\nu} = \inf_{R > 1/\nu} u_R, \quad (u_{1/\nu})_* = ct\chi_{B_{1/\nu}(0)} = \sup_{0<R<1/\nu} u_R.$$
We therefore have $u_{1/\nu} \in \mathrm{SOL}^+$ and $(u_{1/\nu})_* \in \mathrm{SOL}^-$. From Theorem 4.5 it follows that $u_{1/\nu}$ and $(u_{1/\nu})_*$ are respectively the unique upper envelope solution and the unique lower envelope solution of (6.1)–(6.3).

6.2 Counter-examples

We give counter-examples to our weak comparison principles when f does not satisfy (2.4).

Example 6.2. Let $n = 2$. Let $\Omega \subset \mathbf{R}^2$ be two touching disks given by
$$\Omega = \overline{B_{1/\nu}((-1/\nu, 0))} \cup \overline{B_{1/\nu}((1/\nu, 0))}.$$
We study (6.1) with this Ω.

We check that (2.4) does not hold for $f = c\chi_\Omega$. Note that $f^* = c\chi_\Omega$ and $f_* = c\chi_{\Omega^\circ}$. By this we see that f is discontinuous at the origin $x = 0 \in \mathbf{R}^2$. Then, letting $x_\lambda = 0$ for all $\lambda > 1$, we have $\lim_{\lambda\to 1+0} x_\lambda = x = 0$ and
$$\limsup_{\lambda\to 1+0}\{f^*(\lambda x_\lambda) - f_*(x_\lambda)\} = \limsup_{\lambda\to 1+0}\{f^*(0) - f_*(0)\} = \limsup_{\lambda\to 1+0}(c - 0) = c > 0.$$

Thus, (2.4) does not hold.

Solutions to (6.1)–(6.3) is studied in [15, Appendix B], where the authors prove that there are at least two different viscosity solutions $u, v \in \mathrm{SOL}$. One solution u is a trivial one given as

$$u(x,t) = ct\chi_\Omega(x).$$

On the other hand, it is shown in [15] that the maximal solution v satisfies

$$\liminf_{t\to\infty} \frac{v(x,t)}{t} \geqq \alpha \quad \text{locally uniformly in } x \in \mathbf{R}^2$$

for some $\alpha > 0$. This implies that neither $(v^*)_* \leqq u_*$ nor $v^* \leqq (u_*)^*$ holds in $\mathbf{R}^n \times (0,\infty)$.

In [21] we study the behavior of the maximal solution v in more detail by applying the game theoretic interpretation of (6.1) ([23]).

Example 6.3. We study the first order equation (6.2). Let

$$\Omega = \{0\}.$$

Since $f^*(0) = c$ and $f_*(0) = 0$, for the same reason as the previous example, the condition (2.4) is not satisfied at $x = 0$. For $\alpha \in \mathbf{R}$ let us set

$$u^\alpha(x,t) = \max\left\{\alpha\left(t - \frac{|x|}{\nu}\right), 0\right\},$$

which belongs to $\mathrm{SOL}_{\mathrm{Lip}}$. Then, as in [14, Examples 2.3, 5.5 and 5.6], u^α is a viscosity solution of (6.2)–(6.3) for every $\alpha \in [0,c]$. In other words, there are infinitely many Lipschitz continuous solutions, and thus Theorem 4.1 fails.

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References

[1] Y. Achdou, G. Barles, H. Ishii, G. L. Litvinov, *Hamilton–Jacobi equations: approximations, numerical analysis and applications*, Lecture Notes in Mathematics **2074**, Springer, Heidelberg; Fondazione C.I.M.E., Florence, 2011.

[2] G. Barles, *Discontinuous viscosity solutions of first-order Hamilton–Jacobi equations: a guided visit*, Nonlinear Anal. **20** (1993), 1123–1134.

[3] G. Barles, H. M. Soner, P. E. Souganidis, *Front propagation and phase field theory*, SIAM J. Control Optim. **31** (1993), 439–469.

[4] W. K. Burton, N. Cabrera, F. C. Frank, *The growth of crystals and the equilibrium structure of their surfaces*, Philos. Trans. Roy. Soc. London. Ser. A. **243** (1951), 299–358.

[5] L. A. Caffarelli, X. Cabré, *Fully nonlinear elliptic equations*, Amer. Math. Soc. Colloq. Publ. **43**, American Mathematical Society, Providence, RI, 1995.

[6] Y.-G. Chen, Y. Giga, S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Differential Geom. **33** (1991), 749–786.

[7] M. G. Crandall, H. Ishii, P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.) **27** (1992), 1–67.

[8] M. G. Crandall, P.-L. Lions, P. E. Souganidis, *Maximal solutions and universal bounds for some partial differential equations of evolution*, Arch. Rational Mech. Anal. **105** (1989), 163–190.

[9] N. Dirr, V. D. Nguyen, *Some new results on Lipschitz regularization for parabolic equations*, J. Evol. Equ. **19** (2019), 1149–1166.

[10] L. C. Evans, J. Spruck, *Motion of level sets by mean curvature. I*, J. Differential Geom. **33** (1991), 635–681.

[11] W. H. Fleming, H. M. Soner, *Controlled Markov processes and viscosity solutions*, Second edition, Stochastic Modelling and Applied Probability **25**, Springer, New York, 2006.

[12] Y. Giga, *Viscosity solutions with shocks*, Comm. Pure Appl. Math. **55** (2002), 431–480.

[13] Y. Giga, *Surface evolution equations: A level set approach*, Monographs in Mathematics **99**, Birkhäuser Verlag, Basel, 2006.

[14] Y. Giga, N. Hamamuki, *Hamilton-Jacobi equations with discontinuous source terms*, Comm. Partial Differential Equations **38** (2013), 199–243.

[15] Y. Giga, H. Mitake, H. V. Tran, *On asymptotic speed of solutions to level-set mean curvature flow equations with driving and source terms*, SIAM J. Math. Anal. **48** (2016), 3515–3546.

[16] Y. Giga, H. Mitake, H. V. Tran, *Remarks on large time behavior of level-set mean curvature flow equations with driving and source terms*, Discrete Contin. Dyn. Syst. Ser. B **25** (2020), 3983–3999.

[17] Y. Giga, H. Mitake, T. Ohtsuka, H. V. Tran, *Existence of asymptotic speed of solutions to birth-and-spread type nonlinear partial differential equations*, Indiana Univ. Math. J. **70** (2021), 121–156.

[18] Y. Giga, N. Požár, *Viscosity solutions for the crystalline mean curvature flow with a nonuniform driving force term*, Partial Differ. Equ. Appl. **1** (2020), Paper No. 39.

[19] N. Hamamuki, *On large time behavior of Hamilton-Jacobi equations with discontinuous source terms*, GAKUTO Internat. Ser. Math. Sci. Appl. **36** (2013), Nonlinear Analysis in Interdisciplinary Sciences, 83–112.

[20] N. Hamamuki, K. Misu, *A weak comparison principle and asymptotic behavior of viscosity solutions to the mean curvature flow equation with discontinuous source terms*, to appear in RIMS Kôkyûroku (Japanese).

[21] N. Hamamuki, K. Misu, *Asymptotic shape of solutions to the mean curvature flow equation with discontinuous source terms*, in preparation.

[22] B. Kawohl, N. Kutev, *Comparison principle and Lipschitz regulality for viscosity solutions of some classes of nonlinear partial differencial equations*, Funkcialaj Ekvacioj **43** (2000), 241–253.

[23] R. V. Kohn, S. Serfaty, *A deterministic-control-based approach to motion by curvature*, Comm. Pure Appl. Math. **59** (2006), 344–407.

[24] P.-L. Lions, *Generalized solutions of Hamilton–Jacobi equations*, Research Notes in Mathematics **69**, Pitman (Advanced Publishing Program), Boston, Mass.-London, 1982.

[25] P.-L. Lions, *Regularizing effects for first-order Hamilton–Jacobi equations*, Applicable Anal. **20** (1985), 283–307.

[26] N. Q. Le, H. Mitake, H. V. Tran, *Dynamical and geometric aspects of Hamilton–Jacobi and linearized Monge-Ampère equations*, Lecture Notes in Mathematics **2183**, Springer, Cham, 2017.

[27] O. A. Ladyženskaya, V. A. Solonnikov, N. N. Ural′ceva, *Linear and quasilinear equations of parabolic type*, Translations of Mathematical Monographs **23**, American Mathematical Society, Providence, RI, 1968.

[28] M. Ohara, R. C. Reid, *Modeling Crystal Growth Rates from Solution*, Prentice-Hall, Englewood Cliffs, NJ, 1973.

[29] A. Porretta, E. Priola, *Global Lipschitz regularizing effects for linear and nonlinear parabolic equations*, J. Math. Pures Appl. **100** (2013), 633–686.

[30] G. Sazaki, S. Zepeda, S. Nakatsubo, E. Yokoyama, Y. Furukawa, *Elementary steps at the surface of ice crystals visualized by advanced optical microscopy*, Proc. Nat. Acad. Sci. USA. **107** (2010), 19702–19707.