Title	CRYSTALLINE SURFACE DIFFUSION FLOW FOR GRAPH-LIKE CURVES
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Citation	Hokkaido University Preprint Series in Mathematics, 1142, 1-35
Issue Date	2022-01-13
DOI	10.14943/100842
Doc URL	http://hdl.handle.net/2115/83816
Type	bulletin (article)
File Information	CrySurDiff.pdf

CRYSTALLINE SURFACE DIFFUSION FLOW FOR GRAPH-LIKE CURVES

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Abstract. This paper studies a fourth-order crystalline curvature flow for a curve represented by the graph of a spatially periodic function. This is a special example of general crystalline surface diffusion flow. We consider a special class of piecewise linear functions and calculate its speed. We introduce notion of firmness and prove that the solution stays firm if initially it is firm at least for a short time. We also give an example that a facet (flat part) may split if the initial profile is not firm. Moreover, an example of facet-merging is given as well as several estimates for the speed of each facet.

1. **Introduction.** A surface diffusion flow equation was introduced by W. W. Mullins [23] to model motion of a phase boundary in a relaxation dynamics; see also [4] for a review. When two phases are bounded by a closed curve Γ_t depending on time t in the plane $\mathbf{R}^2$, its typical form is

$$V = -\kappa_{ss} \quad \text{on} \quad \Gamma_t.$$

Here, V denotes the normal velocity of Γ_t in the direction of unit normal $\mathbf{n}$ of Γ_t and κ denotes the curvature in the direction of $\mathbf{n}$. The subscript s denotes the derivative with respect to arclength so that $\kappa_{ss} = \Delta_{\Gamma_t}\kappa$, where Δ_Γ denotes the Laplace-Beltrami operator on a curve Γ. (Its higher dimensional version is $V = -\Delta_\Gamma \kappa$, where κ denotes the mean curvature of an evolving hypersurface Γ_t in the direction of $\mathbf{n}$ in $\mathbf{R}^d$.) This equation is considered as a gradient flow of the

2020 *Mathematics Subject Classification.* Primary: 35K67; Secondary: 34A12, 35K25.
Key words and phrases. Crystalline surface diffusion flow, firm, Cahn-Hoffman field, facet-splitting.
The work of the second author was partly supported by the Japan Society for the Promotion of Science (JSPS) through the grants KAKENHI No. 19H00639, No. 18H05323, No. 17H01091 and by Arithmer Inc. and Daikin Industries, Ltd. through collaborative grant.
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length of Γ_t, which is similar to the curve shortening equation $V = \kappa$. However, for the curve shortening equation the gradient is taken in $L^2(\Gamma_t)$-metric while for the surface diffusion equation it is taken in $H^{-1}(\Gamma_t)$-metric.

In materials science, one has to take care of anisotropic structure of the phase boundary. One typical anisotropy appears in interface energy which equals the length of the interface Γ in isotropic setting. A typical form of an anisotropic interface energy is of the form

$$I(\Gamma) = \int_\Gamma \gamma(\mathbf{n}) ds.$$

Here $\gamma : \mathbf{R}^2 \to [0, \infty)$ is a given convex function, which is positively one-homogeneous; we always assume that $\gamma > 0$ outside the origin. Such an anistropic interface energy density γ is often called just an anisotropy. The gradient of I with respect to $L^2(\Gamma)$-metric is denoted by κ_γ called anisotropic curvature. In the case when γ loses C^1 regularity, the problem is substantially different even for $V = \kappa_\gamma$, because the speed is determined by nonlocal quantity. A typical example is the case when γ is piecewise linear. In this case, I is called a crystalline interfacial energy and κ_γ is a crystalline curvature. Note that κ_γ is determined by a nonlocal quantity of the interface. The equation $V = \kappa_\gamma$ with crystalline curvature κ_γ is called a crystalline flow equation.

A crystalline flow equation was introduced in mathematical community by Taylor [27] and independently by Angenent and Gurtin [1] around 1990. They restrict evolving curves into a special class of polygons called admissible and reduce the original flow to a system of ordinary differential equations. Even if the initial shape is a general polygon, one is able to solve this system as suggested by [28] and proved by [14]. Moreover, it turns out this approach is consistent with the approach by the theory of maximal monotone operators [7] and by the level set approach [9], [10]; in these approaches, a non-polygonal initial data is allowed. The theory of crystalline flow equations corresponding to the second-order problems has been well established including higher dimensional problems based on comparison principle; see a review article [17].

For a surface diffusion, it is natural to study its anisotropic version

$$V = -(\kappa_\gamma)_{ss}.$$

When γ is crystalline, the equation is called a crystalline surface diffusion equation. Indeed, such a problem is introduced by [5] and numerical calculation is carried out for this equation by restricting a class of polygon called admissible. Apparently, in their numerics they reduce the problem to a certain system of ordinary differential equations (ODEs), but its property is not analysed.

The goal of this paper is to study a singular fourth-order problem, which can be regarded as a kind of crystalline surface diffusion equation when Γ_t is given as the graph of an admissible piecewise linear periodic function. To formulate the problem, let W be a piecewise linear convex function in $\mathbf{R}$. For later convenience, we assume that W is coercive in the sense that

$$\lim_{|p|\to\infty} W(p)/|p| = \infty.$$

Let P denote the set of jump discontinuity of W' in $\mathbf{R}$. We say that a piecewise linear periodic function v is an *admissible* function if all slopes of v belong to and the slope on adjacent interval where v is affine is adjacent in P. This definition is the same as the second-order problem. We call a non-differentiable point of v a *nod* and its totality is denoted by $S(v)$. We consider

$$u_t = -\partial_x^2 \partial_x \left(W'(u_x) \right). \tag{1}$$

This is not exactly the same as the crystalline surface diffusion equation but it has a similar nature. Fortunately, the abstract theory of maximal monotone operators developed by Y. Kōmura [21] and H. Brézis [2] provides a unique solution for arbitrary H^{-1} data. There, we regard (1) as a gradient flow

$$u_t \in -\partial_{H^{-1}} \Phi(u)$$

with $\Phi(u) = \int_{\mathbf{T}} W(u_x)dx$, where $\mathbf{T} = \mathbf{R}/\omega\mathbf{Z}$ with some $\omega > 0$, which is the period. Here $\partial_{H^{-1}}$ denotes the subdifferential in H^{-1} metric; a rigorous formation will be given in Section 2.

We are interested in behavior of a solution when the initial function u_0 is admissible. Our main concern is to give a class that the solution stays admissible and no nod is disappearing nor created during evolution. We also derive the system of ODEs to describle evolution when the solution stays admissible.

One key issue is a characterization of subdifferential for an admissible function v. As in [12], [11], it is easily reduced to a second-order problem and it is of the form

$$\partial_{H^{-1}}\Phi(v) = \left\{\xi_{xxx} \;\middle|\; \xi \in CH_1(v)\right\},$$
$$CH_1(v) = \left\{\xi \in H^1(\mathbf{T}) \;\middle|\; \xi(x) \in \partial W\left(v_x(x)\right) \text{ for } x \in \mathbf{T}\backslash S(v)\right\}.$$

An element of $CH_1(v)$ is called a *Cahn-Hoffman field*. The right derivative d^+u/dt (which is the speed) must be equal to the minimal section of $\partial_{H^{-1}}\Phi\left(u(t)\right)$. It must be

$$\partial^0_{H^{-1}}\Phi(v) = \eta_{xxx},$$
$$\eta := \operatorname{argmin}\left\{\|\xi_{xx}\|_{L^2(\mathbf{T})} \mid \xi \in CH_1(v)\right\}.$$

This η is called a *minimal Cahn-Hoffman field*. If v is admissible, $CH_1(v)$ is nonempty so that $\partial_{H^{-1}}\Phi(v)$ is not empty. Here is a key observation.

Lemma 1.1. *A minimal Cahn-Hoffman field η is in C^2. Moreover, η is cubic outside the union of $S(v)$ and the set of all x where $\eta(x)$ touches the boundary of $\partial W\left(v_x(x)\right)$ for $x \in \mathbf{T}\backslash S(v)$. A minimal Cahn-Hoffman field is uniquely determined by v when there are at least two nods in $\mathbf{T}$.*

We consider a subclass of admissible functions. We say that an admissible function v is *firm* if the minimal Cahn-Hoffman field η satisfies $\eta(x) \in \operatorname{int}\partial W\left(v_x(x)\right)$ for $x \in \mathbf{T}\backslash S(v)$ and $\eta_x(x) \neq 0$ for $x \in S(v)$. Our main result is summerized as follows.

Theorem 1.2. *Assume that the initial data u_0 is firm. Then there is $t_0 > 0$ such that the solution u of (1) stays firm for $t \in [0, t_0)$. Moreover, motion of nod is smooth (actually, analytic) during evolution in $[0, t_0)$. This motion is governed by a system of ODEs.*

Instead of motion of nods, we are interested in the behavior of a faceted region of a solution $u = u(x,t)$. For an admissible function v, we number its nods as

$$S(v) = \{x_1 < x_2 < \cdots < x_n\} \subset [0, \omega).$$

We say that (x_i, x_{i+1}) is a *faceted region* of v with understanding that $x_{n+1} = x_1$. We assume $n \geq 2$. Let ℓ_i denote its length, i.e., $\ell_i = x_{i+1} - x_i$. We define a value d_i by

$$\{d_i\} = \partial W\left(v_x(x_i - 0)\right) \cap \partial W\left(v_x(x_i + 0)\right) \quad \text{for} \quad i = 1, \ldots, n.$$

To describe the minimal Cahn-Hoffman field for v when it is firm, we note the unique existence of $\zeta \in C^2(\mathbf{T})$ which is cubic in all faceted regions and $\zeta(x_i) = d_i$, $i = 1, \cdot\cdot n$. For this purpose, we set $\zeta''(x_i) = b_i$ for $i = 1, \dots, n$. If $\vec{b} = (b_i)_{i=1}^n$ is given, then it must satisfy

$$M(\vec{\ell})\vec{b} = \vec{f}(\vec{\ell}, \vec{d}),$$

where $M(\vec{\ell})$ is a symmetric matrix and $\vec{f}$ is an n-vector. The dependence of $\vec{f}$ and M on $\vec{\ell} = (\ell_i)_{i=1}^n$, $\vec{d} = (d_i)_{i=1}^n$ is analytic provided that all ℓ_i's are positive. Fortunately, $M(\vec{\ell})$ is positive definite so it has an inverse (Lemma 5.1). Thus, we are able to determine $\vec{b}$.

Based on above preparation, we are able to derive evolution equations for $\ell_i(t)$'s, where ℓ_i is the length of a faceted region $(x_i(t), x_{i+1}(t))$ of $u(x,t)$. By a transport equation (see e.g. [1] or [27]), we observe that

$$\dot{\ell}_i(t) = c_{i-1}V_{i-1} + c_iV_i + c_{i+1}V_{i+1},$$

where $V_i = u_t(x)$ for $x \in (x_i, x_{i+1})$ and c_i is a constant idependent of time. Note that (1) determines the velocity

$$V_i = -\eta_{xxx} \quad \text{on} \quad (x_i, x_{i+1})$$

with the minimal Cahn-Hoffman field η of v. If $u(\cdot, t)$ is firm, then $\eta \in C^2(\mathbf{T})$ should be the above piecewise cubic funsion ζ so that

$$V_i = -\frac{b_{i+1} - b_i}{\ell_i}.$$

Since $\vec{b} = \vec{b}(\vec{\ell})$ is uniquely determined by $\vec{\ell} = \vec{\ell}(t)$, we end up with a system of ODEs for $\vec{\ell}$. This system has a unique local (analytic) solution for a given initial data $\vec{\ell}(0)$.

For a given firm initial data u_0, we construct a local solution $\vec{\ell}(t)$, which gives an admissible function $u(x,t)$ and $\zeta(x,t)$ that satisfies

$$u_t = -\zeta_{xxx} \quad x \in \mathbf{T} \backslash S(v).$$

Since the notion of a firmness is an open property, this evolving $\zeta(x,t)$ (which depends on t smoothly) still stays in $\operatorname{int} \partial W(u_x(x,t))$ for $x \in \mathbf{T} \backslash S(v)$ and $\zeta_x(x_j, t) \neq 0$ for $x_j \in S(v)$ at least for a short time. By the uniqueness of the minimal Cahn-Hoffman field, this ζ must be the minimal Cahn-Hoffman fields. By the uniqueness of the solution in abstract theory, our constructed u must agree with the solution of (1).

If the initial data is not firm but admissible, some facets may split and new nods may appear. We give a class of examples of initial data resulting splitting. Note that in our examples, the solution of (1) instantaneously becomes firm for $t > 0$. A numerical simulation by [5] suggests that facet-splitting phenomenon may occur during evolution even if initial data is firm but we do not have an explicit example. We also give an example that some facet may disappear during evolution for some firm initial data. This phenomenon is called facet-merging. Different from the second order problem, the speed $-\zeta_{xxx}$ is not determined just by its length. What we have is an estimate like $|\zeta_{xxx}| < 3 \cdot 2^5 |d_{i+1} - d_i| / \ell_i^3$ on (x_i, x_{i+1}) when the function is firm and $d_{i+1} \neq d_i$; see Theorem 7.3.

There are few articles for crystalline surface diffusion flow although there are a few articles on 4th-order total variation flow, i.e., in the case $W(p) = |p|$. In this case, the energy Φ is not coercive so the solution may form jump discontinuity instantaneously for Lipschitz initial data as proved in [11]. This type of equation with $W = W_q(p) = |p| + \alpha|p|^q$, $\alpha > 0$, $q > 1$ is first analysed by Y. Kashima [19] (See also [20] for improvement) where the subdifferential was characterized even for multi-dimensional problems. Note that in this case, W is coercive so the solution is always spatial continuous. The pure fourth-order total variation flow ($W(p) = |p|$) was proposed for image denoising and restoring by S. Osher, A. Solé and L. Vese [24], while the model with W_q was introduced to model relaxation phenomena in crystal growth by H. Spohn [26]. For such an equation, it is known that the solution stopps in finite time [12], [13] when the spatial dimension is less than or equal to 4. Such a phenomenon is well-known for second-order singular diffusion equations [29, Section 11.5]. This stopping phenomenon is also observed in our example of facet merging (Section 7.2). Note that other than the theory of maximal monotone operators, there is a way to construct a solution based on the Galerkin method [6]. There are by now several ways to calculate numerically such 4th-order singular diffusion equations. We just list a few of them including [22], [15] and [18].

This paper is organized as follows. In Section 2, we give a rigorous formulation of (1). In Section 3, we calculate sundifferentials even at a function not necessarily admissible. In Section 4, we calculate the minimal section. In Section 5, we give an explicit form of the minimal Cahn-Hoffman field by constructing a spline curve. In Section 6, we reduce our (1) to a system of ODEs when u is firm. In Section 7, we study facet-splitting (Section 7.1) and facet-merging phenomenon (Section 7.2). We also give the estimate of the speed (Section 7.3). In Section 8, we give an elementary proof for a few well-known facts used in this paper for the reader's convenience.

The main results of this paper has been announced in [8].

2. **Rigorous formulation.** We consider (1) with the periodic boundary condition. To give a rigorous formulation, let $H^1(\mathbf{T})$ denote the Sobolev space of all real-valued periodic functions with period ω equipped with the inner product :

$$(f,g)_{H^1} = ((f,g))_1 + \int_{\mathbf{T}} f \cdot g \; dx, \quad f,g \in H^1(\mathbf{T}),$$

with $\mathbf{T} = \mathbf{R}/\omega\mathbf{Z}$, where

$$((f,g))_1 = \int_{\mathbf{T}} f_x \cdot g_x \; dx.$$

We consider a subspace so that $((f,g))_1$ becomes an equivalent inner product; let $H^1_{\mathrm{av}}(\mathbf{T})$ be the codimension one subspace of the form

$$H^1_{\mathrm{av}}(\mathbf{T}) = \left\{ f \in H^1(\mathbf{T}) \;\middle|\; \int_{\mathbf{T}} f \; dx = 0 \right\}.$$

By the Poincaré or Wirtinger inequality $((f,g))_1$ is an equivalent inner product in $H^1_{\mathrm{av}}(\mathbf{T})$.

Let $H^{-1}_{\mathrm{av}}(\mathbf{T})$ be the dual space of $H^1_{\mathrm{av}}(\mathbf{T})$. Let $\langle \;\cdot\; , \;\cdot\; \rangle$ denotes the canonical pairing of H^1 and H^{-1}, i.e., for $g \in H^{-1}(\mathbf{T})$, $g(f) = \langle f, g \rangle$ for $f \in H^1(\mathbf{T})$, which is formally written as $\int_{\mathbf{T}} fg \, dx$. Then

$$H^{-1}_{\mathrm{av}}(\mathbf{T}) = \left\{ f \in H^{-1}(\mathbf{T}) \;\middle|\; \langle 1, f \rangle = \int_{\mathbf{T}} f \; dx = 0 \right\},$$

which is the Hilbert space equipped with an inner product

$$((f,g))_{-1} = \left\langle (-\Delta)^{-1} f, g \right\rangle = \int_{\mathbf{T}} \left((-\Delta)^{-1} f \right) g \, dx,$$

where $-\Delta$ denotes an isometry from $H^1_{\rm av}(\mathbf{T})$ to $H^{-1}_{\rm av}(\mathbf{T})$ defined by $f \mapsto ((f,\cdot))_1$.
(A) Assume that interfacial energy W is a piecewise linear convex function with coercivity: $\lim_{|p|\to\infty} W(p)/p = +\infty$.

In $H^{-1}_{\rm av}(\mathbf{T})$ consider a functional $\Phi : H^{-1}_{\rm av} \to (-\infty,\infty]$ of the form

$$\Phi(v) = \begin{cases} \int_{\mathbf{T}} W(v_x(x))dx, & v \in W^{1,1}(\mathbf{T}) \cap H^{-1}_{\rm av}(\mathbf{T}), \\ \infty, & \text{otherwise.} \end{cases}$$

We set $D_{H^{-1}_{\mathbf{av}}}(\Phi) = \{v \in H^{-1}_{\rm av}(\mathbf{T}) \mid \Phi(v) < \infty\}$. We consider an $H^{-1}_{\rm av}$ gradient flow equation:

$$\begin{aligned} &\frac{du}{dt}(t) \in -\partial_{H^{-1}}\Phi\,(u(t)) \quad \text{a.e. } t>0, \\ &u|_{t=0} = u_0 \in H^{-1}_{\rm av}(\mathbf{T}). \end{aligned} \tag{2}$$

Here $\partial_{H^{-1}}\Phi$ denotes the subdifferential of Φ in $H^{-1}_{\rm av}(\mathbf{T})$, i.e.,

$$\begin{aligned} &\partial_{H^{-1}}\Phi(v) \\ &= \{f \in H^{-1}_{\rm av}(\mathbf{T}) | \Phi(v+h) - \Phi(v) \ge ((h,f))_{-1} \\ &\qquad\qquad \text{for all } h \in H^{-1}_{\rm av}(\mathbf{T})\}. \end{aligned}$$

Proposition 1. *Assume (A). Then, the functional Φ is proper (i.e., $\Phi \not\equiv \infty$), lower semicontinuous and convex on $H^{-1}_{\rm av}(\mathbf{T})$.*

The proof is standard. We give it in the Appendix as Proposition 7. An abstract theory (initiated by Kōmura [21]; see [2]) yields

Proposition 2 (Unique existence from abstract theory)**.** *Assume (A). Then there exists a unique solution $u \in C([0,\infty), H^{-1}_{\rm av}(\mathbf{T}))$ of (2) which is absolutely continuous in any compact interval in $(0,\infty)$ with values in $H^{-1}_{\rm av}(\mathbf{T})$. Moreover, $t \mapsto u(t)$ is right differentiable for all $t>0$, $\partial_{H^{-1}}\Phi(u(t)) \neq \emptyset$ for all $t>0$ and*

$$\begin{aligned} &\frac{d^+u}{dt}(t) = -\partial^0_{H^{-1}}\Phi\,(u(t)) \quad \text{for } t>0, \\ &u|_{t=0} = u_0 \in H^{-1}_{\rm av}(\mathbf{T}). \end{aligned} \tag{3}$$

Here $\partial^0_{H^{-1}}\Phi$ is the minimal section (or canonical restriction), i.e.,

$$\partial^0_{H^{-1}}\Phi(v) = \text{argmin}\ \{||f||_{H^{-1}_{\text{av}}(\mathbf{T})} | f \in \partial_{H^{-1}}\Phi(v)\},$$

which is the unique element of the closed convex set $\partial_{H^{-1}}\Phi(v)$ closest to the origin of $H^{-1}_{\text{av}}(\mathbf{T})$.

3. **Characterization of the subdifferentials.** To characterize the subdifferential $\partial_{H^{-1}_{\text{av}}}\Phi$, we recall the subdifferential $\partial_{L^2}\tilde{\Phi}$ of $\tilde{\Phi} := \Phi|_{L^2_{\text{av}}}$ in $L^2_{\text{av}}(\mathbf{T})$, i.e.,

$$\partial_{L^2_{\text{av}}}\tilde{\Phi}(u) := \left\{ g \in L^2_{\text{av}}(\mathbf{T}) \;\middle|\; \tilde{\Phi}(u+k) - \tilde{\Phi}(u) \geq \int_{\mathbf{T}} gk\, dx \quad \text{for all} \quad k \in L^2_{\text{av}}(\mathbf{T}) \right\}$$

for $u \in D_{L^2_{\text{av}}}(\tilde{\Phi}) = \left\{ u \in L^2_{\text{av}}(\mathbf{T}) \;\middle|\; \tilde{\Phi}(u) < \infty \right\}$, where

$$L^2_{\text{av}}(\mathbf{T}) = \left\{ g \in L^1(\mathbf{T}) \;\middle|\; \int_{\mathbf{T}} g\, dx = 0 \right\}.$$

Proposition 3. *Assume (A). For $u \in L^2_{\text{av}}(\mathbf{T})$, a function $g \in L^2_{\text{av}}(\mathbf{T})$ belongs to the subdifferential $\partial_{L^2_{\text{av}}}\tilde{\Phi}(u)$ if and only if there is $\eta \in L^2(\mathbf{T})$ such that $\eta(x) \in \partial W\,(u_x(x))$ a.e. $x \in \mathbf{T}$ and $g = -\eta_x$. In other words,*

$$\partial_{L^2_{\text{av}}}\tilde{\Phi}(u) = \left\{ -\eta_x \in L^2_{\text{av}}(\mathbf{T}) \;\middle|\; \eta \in CH_0(u) \right\}$$

with

$$CH_0(u) = \left\{ \eta \in L^1(\mathbf{T}) \;\middle|\; \eta_x \in L^2_{\text{av}}(\mathbf{T}),\ \eta(x) \in \partial W\,(u_x(x))\,,\ \text{a.e. } x \in \mathbf{T} \right\},$$

which we call L^2_{av}-Cahn-Hoffman (vector) field.

This can be proved as in [7] based on [3]. We give an elementary proof when u is an admissible piecewise linear function in the Appendix for L^2 function (Proposition 8); it is easy to replace L^2 by L^2_{av}.

The characterization of L^2_{av} subdifferential of Φ enables us to characterize H^{-1}_{av} subdifferential.

Theorem 3.1. *Assume (A). For $v \in D_{H^{-1}_{\text{av}}}(\Phi)$, a function f belongs to the subdifferential $\partial_{H^{-1}_{\text{av}}}\Phi(v)$ if and only if there is $\eta \in L^1(\mathbf{T})$ such that $\eta_x \in H^1_{\text{av}}(\mathbf{T})$ with $\eta(x) \in \partial W\,(v_x(x))$ a.e. $x \in \mathbf{T}$ and $f = \eta_{xxx}$. In other words,*

$$\partial_{H^{-1}}\Phi(v) \left(= \partial_{H^{-1}_{\text{av}}}\Phi(v)\right) = \left\{ \eta_{xxx} \;\middle|\; \eta \in CH_1(v) \right\}$$

with

$$CH_1(v) := \left\{ \eta \in L^1(\mathbf{T}) \;\middle|\; \eta_x \in H^1_{\text{av}}(\mathbf{T}),\ \eta(x) \in \partial W\left(v_x(x)\right) \ \text{a.e. } x \in \mathbf{T} \right\},$$

which we call H^1_{av}-Cahn-Hoffman (vector) field.

Proof. We reduce the problem in L^2-setting as in [12]. We give the full proof for completeness. We observe that $v \in D_{H^{-1}_{\text{av}}}(\Phi)$ implies $v \in W^{1,1}(\mathbf{T})$. In particular, $v \in D_{L^2_{\text{av}}}(\tilde{\Phi})$. If $f \in \partial\Phi_{H^{-1}_{\text{av}}}(\Phi)$, then

$$\Phi(v+k) - \Phi(v) \geq ((f,k))_{-1} = \int_{\mathbf{T}} \left((-\Delta)^{-1} f\right) k \, dx \quad \text{for all} \quad k \in H^{-1}_{\text{av}}(\mathbf{T}).$$

Since $H^{-1}_{\text{av}}(\mathbf{T}) \supset L^2_{\text{av}}(\mathbf{T})$, this implies $(-\Delta)^{-1} f \in \partial_{L^2_{\text{av}}} \tilde{\Phi}(v)$. By Proposition 3, we see that $f = \eta_{xxx}$, $\eta \in CH_1(v)$. Conversely, for $f = \eta_{xxx}$, $\eta \in CH_1(v)$,

$$\Phi(v+k) - \Phi(v) \geq \int_{\mathbf{T}} \left((-\Delta)^{-1} f\right) k \, dx$$

for $k \in L^2_{\text{av}}(\mathbf{T})$. Since $L^2_{\text{av}}(\mathbf{T})$ is dense in $H^{-1}_{\text{av}}(\mathbf{T})$, one is able to extend this inequality for $k \in H^{-1}_{\text{av}}(\mathbf{T})$. The proof is now complete. □

Remark 1. A similar characterization is given when $W_q(p) = |p| + \alpha|p|^q$, $\alpha > 0$, $q > 1$ by Y. Kashima [19] for dimension up to 4 and [20] for general dimension. Relation between L^2 subdifferential and H^{-1}_{av} differential is discussed for a periodic function [12]. For a generalization to a domain with boundary, see [13].

4. **Minimal Cahn-Hoffman field.** We shall characterize the minimal section of a subdifferential $\partial_{H^{-1}}\Phi(v)$ when v is an admissible function. From our characterization of the subdifferential (Theorem 3.1), we are able to describe its minimal section by a Cahn-Hoffman field.

Proposition 4. *Assume (A). Let v is in $D_{H^{-1}_{\mathbf{av}}}(\Phi)$. For $\zeta \in CH_1(v)$,*

$$\zeta_{xxx} = \partial^0_{H^{-1}} \Phi(v)$$

if and only if ζ is a minimizer of $\|\eta_{xx}\|^2_{L^2_{av}(\mathbf{T})}$. In other words,

$$\zeta = \arg\min \left\{ \|\eta_{xx}\|^2_{L^2_{av}(\mathbf{T})} \;\middle|\; \eta \in CH_1(v) \right\}.$$

Proof. This follows from Theorem 3.1 and the definition of the minimal section if we note

$$\|\eta_{xxx}\|_{H^{-1}_{av}} = \|\eta_{xx}\|_{L^2}.$$

□

Definition 4.1. The field ζ is called the minimal Cahn-Hoffman field. A general theory says that ζ_{xxx} is uniquely determined. (In one-dimensional setting, it turns out that ζ itself is uniquely determined as we see later.)

If the constraint $\eta \in \partial W(v_x)$ is missing, it is clear that the minimizer must satisfy $\eta_{xxxx} = 0$, i.e., a cubic polynominal. Because of the constraint, the situation is more complicated but we are able to characterize a minimal Cahn-Hoffman field.

Lemma 4.2. *Let v be an admissible function having at least two nods in $\mathbf{T}$. Let $\zeta \in CH_1(v)$ be a minimal Cahn-Hoffman field for v. Then $\zeta \in C^2(\mathbf{T})$ and ζ is a piecewise polynomial of degree 3. Moreover, the minimal Cahn-Hoffman field ζ is uniquely determined and fulfills following properties.*

(i) *If $\zeta_x(\overline{x}) = 0$ at nod of v, i.e., $\overline{x} \in S(v)$, then $\zeta_{xx}(\overline{x}) = 0$.*

(ii-sup) *For $\hat{x} \notin S(v)$, if*

$$\zeta(\hat{x}) = \sup \partial W\left(v_x(\hat{x})\right), \tag{4}$$

then

$$\zeta_{xxx}(\hat{x}-0) \geq \zeta_{xxx}(\hat{x}+0).$$

(ii-inf) *For $\check{x} \notin S(v)$, if*

$$\zeta(\check{x}) = \inf \partial W\left(v_x(\check{x})\right), \tag{5}$$

then

$$\zeta_{xxx}(\check{x}-0) \leq \zeta_{xxx}(\check{x}+0).$$

(iii) *In each faceted region $I = (a, b)$, i.e., the maximal open interval where v is affine, the set K_+ of all $\hat{x} \in I$ satisfying (4) is at most a single (relatively) closed interval. (It can be a singleton or an empty set.) Similarly, the set K_- of all $\check{x}$ satisfying (5) is at most a single closed interval. If the length of K_+ (resp. K_-) is positive, then K_- (resp. K_+) must be empty and $\zeta(a) = \zeta(b)$. If*

$$\zeta(a) = \zeta(b) = \inf_{x\in I} \partial W\left(v_x(x)\right) \ \textit{(resp. } \sup_{x\in I} \partial W\left(v_x(x)\right) \textit{)},$$

then K_- (resp. K_+) must be empty.

(iv) *Outside $K_\pm$, ζ must be a cubic polynomial in each faceted region.*

Proof. Let ζ be a minimizer of $\|\eta_{xx}\|_{L^2}$ for $\eta \in CH_1(v)$. Let I be a faceted region and $x_0 \in I$ which is not a nodal point. We consider the case that

$$m \leq \zeta(\overline{x}) < M, \quad M = \sup \partial W\left(v_x(\overline{x})\right), \quad m = \inf \partial W\left(v_x(\overline{x})\right).$$

Since $\zeta \in CH_1(v)$ is continuous, there is an open interval $U \subset I$ containing x_0 such that

$$\sup_U \zeta < M.$$

We take non-negative $\varphi \in C_c^\infty(U)$ such that $\zeta^\varepsilon := \zeta + \varepsilon\varphi \in CH_1(v)$ at least for small $\varepsilon > 0$. Here, $C_c^\infty(U)$ denotes the space of all smooth functions with compact support in U. Since ζ is a minimizer of $\|\eta_{xx}\|_{L^2}^2$, we see that

$$0 \leq \frac{1}{2} \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \|\zeta_{xx}^\varepsilon\|_{L^2}^2 = \int_U \zeta_{xx}\varphi_{xx}\, dx.$$

We recall a distributional characterization of a convex function [25].

Lemma 4.3. *Let U be an open interval. A locally integrable function f in U is convex if and only if*

$$\int_U f\varphi_{xx}\, dx \geq 0$$

for all non-negative $\varphi \in C_c^\infty(U)$. (The statement is still valid even if f is a distribution.)

We only discuss "if part" only since the other part is easier. By the Riesz representation of non-negative measure, our assumption implies that the distributional second derivative is a measure. This implies that distributional first derivative is a non-decreasing function, which implies that the original function f is convex.

We apply Lemma 4.3 to observe that ζ_{xx} is convex in U. Thus, ζ_{xxx} is non-decreasing near K_-, which implies (ii-inf). A symmetric argument yields the concavity of ζ_{xx} near K_+. This implies (ii-sup). We now conclude ζ is C^2 in I since ζ_{xx} is continuous. Outside K_+ and K_-, ζ_{xx} must be convex and concave so it must be linear. This prove (iv).

We shall prove (iii). Assume that $\alpha = \sup K_- < b$. By geometry, we see that $\zeta_{xx}(\alpha) \geq 0$. Assume that $\zeta_{xx}(\alpha) > 0$. If the slope of ζ_{xx} for $x < \alpha$ near α is non-positive, then ζ_{xx} must be linear up to $x = a$ since ζ is strictly decreasing in (a, α). This implies that K_- is a singleton. Moreover,

$$K_+ \cap \{x \in I \mid x < \alpha\} = \emptyset$$

since $\zeta_{xx} \leq 0$ on K_+ by geometry. For $x > \alpha$, there may exist a point in K_+. We set $\beta = \inf K_+$. Since $\zeta_{xx}(\alpha) > 0$ and $\zeta_{xx}(\beta) \leq 0$, the slope of ζ_{xx} for $x > \beta$ is negative by the concavity of ξ_{xx}. Thus for $x > \beta$, we see $\zeta_x \leq 0$ so there is no chance to have a point of K_+ for $x > \beta$.

If the slope of ζ_{xx} for $x < \alpha$ near α is positive, it must be positive for $x > \alpha$. This implies there is no point of K_+ for $x > \alpha$. We argue in the same way and observe that for $x < \alpha$, $K_+ \cap \{x \in I \mid x < \alpha\}$ is at most singleton.

If $\zeta_{xx}(\alpha) = 0$, then ζ must be $a_1(x-\alpha)^3$ for $x > \alpha$ with some $a_1 > 0$. If the slope of ζ_{xx} is not zero for $x < \alpha$ near $x = \alpha$, then ζ must be $-a_2(x-\alpha)^3$ with some $a_2 > 0$. If the slope ζ_{xx} is zero for $x < \alpha$ near $x = \alpha$. $\zeta(x) = \inf \partial W(v_x(\alpha))$ near $x < \alpha$. In this case, K_- is a relatively closed interval and if $\gamma = \inf K_- > a$, then ζ must be $-a_3(x-\gamma)^3$ for $x < \alpha$ with some $a_3 > 0$. In this case $\zeta(a) = \zeta(b) = \sup \partial W(v_x(\gamma))$. A symmetric argument yields the desired result for K_+.

We now conclude that K_+ and K_- are at most closed intervals in I. If K_- (resp. K_+) is not a singleton, K_+ (resp. K_-) must be empty by geometry. If $\zeta(a) = \zeta(b)$, then one of K_+ and K_- cannot exist. We have thus proved (iii).

It remains to prove (i). We only discuss the case

$$\inf \partial W\left(v_x(\overline{x} - 0)\right) \leq \sup \partial W\left(v_x(\overline{x} + 0)\right),$$

since a symmetric argument applies the other case. By (ii) and (iii), we may assume that $f = \zeta_{xx}$ is linear for $x > \overline{x}$ and for $x < \overline{x}$. We set $\overline{x} = 0$. Since ζ is a minimizer, for a small $\delta > 0$,

$$\int_0^\delta f \varphi_{xx}\, dx \geq 0$$

for all $\varphi \in C_c^\infty[0,\delta)$, $\varphi \geq 0$ with $\varphi(0) = 0$. If $\zeta_x(0) = 0$, then $\varphi_x(0)$ should be non-negative. Integrating by parts yields

$$0 \leq \int_0^\delta f\varphi_{xx}\,dx = -f(+0)\varphi_x(0) - \int_0^\delta f_x \cdot \varphi_x\,dx$$
$$= -f(+0)\varphi_x(0) + f_x(+0)\varphi(0)$$
$$= -f(+0)\varphi_x(0).$$

Since $\varphi_x(0) \geq 0$ is taken arbitrary, we conclude that $f(+0) \leq 0$. The geometry for ζ yields $\zeta_{xx}(+0) = f(+0) \geq 0$. Thus, $\zeta_{xx}(+0) = 0$ if $\zeta_x(0) = 0$. Similar argument implies $\zeta_{xx}(-0) = 0$. In the case $\zeta_x(0) \neq 0$, we argue in a similar way to conclude that $\zeta_{xx}(+0) = \zeta_{xx}(-0)$.

We thus conclude that $\zeta \in C^2(\mathbf{T})$. Moreover, since ζ_{xxx} is unique, (i)–(iii) implies that ζ itself is uniquely determined. See Figure 1 for profiles of ζ on I when $K_\pm$ is a singleton and $\zeta_x \neq 0$ on ∂I. □

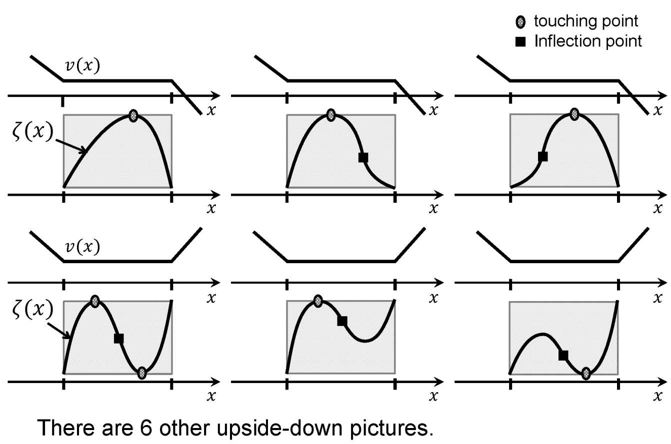

FIGURE 1. 12 possible patterns when $K_\pm$ is a singleton

5. **An explicit formula of the minimal Cahn-Hoffman field.** We give an explicit formula of the minimal Cahn-Hoffman field when v is firm. We begin with basic properties of a piecewise cubic polynomial ξ in $C^2(\mathbf{T})$.

Proposition 5. *Let $\xi \in C^2(\mathbf{T})$ be cubic on each sub interval (z_i, z_{i+1}) with $i = 1, \dots, m$ $(m \geq 2)$, where $0 \leq z_1 < z_2 < \dots < z_m < \omega$ with interpretation $z_{m+1} = z_1$. Then we have*

(i) $\xi_{xxx}(x) = \dfrac{b_{i+1} - b_i}{\ell_i}$ *with* $\xi_{xx}(x_i) = b_i$ *for* $x \in (z_i, z_{i+1})$*, where* $\ell_i = z_{i+1} - z_i$*,*

(ii) $\dfrac{1}{6}\{\ell_{i-1}b_{i-1} + 2(\ell_{i-1} + \ell_i)b_i + \ell_i b_{i+1}\} = -\dfrac{d_i - d_{i-1}}{\ell_{i-1}} + \dfrac{d_{i+1} - d_i}{\ell_i}$ *for* $i = 1, \ldots, m$*, where* $d_i = \xi(z_i)$*.*

Proof. On $[z_i, z_{i+1}]$, ξ is of the form

$$\xi(x) = \xi_i(x) := \frac{a_i}{6}(x - z_i)^3 + \frac{b_i}{2}(x - z_i)^2 + c_i(x - z_i) + d_i$$

with some a_i and c_i. Since ξ is C^2, we have

$$b_{i+1} = \partial_x^2 \xi_{i+1}(z_{i+1}) = \partial_x^2 \xi_i(z_{i+1}) = a_i \ell_i + b_i$$

which yields (i).

By (i), we see that

$$\begin{aligned} d_{i+1} = \xi_{i+1}(z_{i+1}) = \xi_i(z_{i+1}) &= \frac{a_i}{6}\ell_i^3 + \frac{b_i}{2}\ell_i^2 + c_i\ell_i + d_i \\ &= \frac{1}{6}\frac{b_{i+1} - b_i}{\ell_i}\ell_i^3 + \frac{b_i}{2}\ell_i^2 + c_i\ell_i + d_i \\ &= \frac{1}{6}(b_{i+1} + 2b_i)\ell_i^2 + c_i\ell_i + d_i. \end{aligned}$$

This implies

$$c_i = \frac{d_{i+1} - d_i}{\ell_i} - \frac{b_{i+1} + 2b_i}{6}\ell_i. \tag{6}$$

Similarly, by (i), we have

$$\begin{aligned} c_i = \partial_x \xi_i(z_i) = \partial_x \xi_{i-1}(z_i) &= \frac{a_{i-1}}{2}\ell_{i-1}^2 + b_{i-1}\ell_{i-1} + c_{i-1} \\ &= \frac{b_i - b_{i-1}}{2\ell_{i-1}}\ell_{i-1}^2 + b_{i-1}\ell_{i-1} + c_{i-1} \\ &= \frac{b_i + b_{i+1}}{2}\ell_{i-1} + c_{i-1}. \end{aligned}$$

Thus

$$c_i - c_{i-1} = \frac{b_i + b_{i-1}}{2}\ell_{i-1}.$$

By (6), we have

$$\frac{d_{i+1} - d_i}{\ell_i} - \frac{b_{i+1} + 2b_i}{6}\ell_i - \frac{d_i - d_{i-1}}{\ell_{i-1}} + \frac{b_i + 2b_{i-1}}{6}\ell_{i-1} = \frac{b_i + b_{i-1}}{2}\ell_{i-1}.$$

Rearranging this identity yields (ii). □

Remark 2. Since a_i and c_i is determined by d_i,

$$\xi_i(x) = \frac{b_{i+1} - b_i}{6\ell_i}(x - z_i)^3 + \frac{b_i}{2}(x - z_i)^2 + \left\{ \frac{d_{i+1} - d_i}{\ell_i} - \frac{b_{i+1} + 2b_i}{6}\ell_i \right\}(x - z_i) + d_i$$

for $x \in [z_i, z_{i+1}]$, $i = 1, \ldots, m$.

It is convenient to write a relation stated in Proposition 5 (iii) in a concise form. For a vector $\vec{\ell} = (\ell_i)_{i=1}^m$, $\vec{d} = (d_i)_{i=1}^m$, $\vec{b} = (b_i)_{i=1}^m$, we set a vector $\vec{f} = (f_i)_{i=1}^m$ by

$$f_i = -\frac{d_i - d_{i-1}}{\ell_{i-1}} + \frac{d_{i+1} - d_i}{\ell_i} \quad \text{for} \quad i = 1, \ldots m$$

and $m \times m$ matrix $M(\vec{\ell}) = (M_{jk})$ by

$$\begin{aligned} 6M_{jj} &= 2(\ell_{j-1} + \ell_j) \\ 6M_{jj-1} &= \ell_{j-1}, \quad 6M_{jj+1} = \ell_j \\ M_{jk} &= 0 \quad \text{if} \quad |j - k| \geq 2. \end{aligned}$$

Here, we regard $\vec{b}$ and $\vec{f}$ as column vectors. We always interpret the indices modulo m. For example, $b_{m+1} = b_1$, $b_0 = b_m$. Using this notation, the relation in Proposition 5 (ii) can be written of the form

$$M(\vec{\ell})\vec{b} = \vec{f}(\vec{\ell}, \vec{d}). \tag{7}$$

The matrix M is symmetric. Moreover, it is positive definite as shown below. Thus M is invertible, i.e., it is regular.

Lemma 5.1. *Set* $\lambda_i = \ell_i + \ell_{i-1}$. *Then*

$$6M(\vec{\ell}) \geq \min_{1 \leq i \leq m} \lambda_i I,$$

where I *denotes the* $m \times m$ *identity matrix. In particular,* M *is a regular matrix.*

Proof. We consider the quadratic form $6\vec{b}^T M\vec{b}$. By definition,

$$6\vec{b}^T M\vec{b} = \sum_{i=1}^{m} b_i \left\{\ell_{i-1}b_{i-1} + 2(\ell_{i-1}+\ell_i)b_i + \ell_i b_{i+1}\right\}$$
$$= \sum_{i=1}^{m} \left\{\ell_{i-1}b_{i-1}b_i + 2(\ell_{i-1}+\ell_i)b_i^2 + \ell_i b_i b_{i+1}\right\}.$$

Since indices are modulo m, we see that

$$\sum_{i=1}^{m} \ell_{i-1}b_{i-1}b_i = \sum_{i=1}^{m} \ell_i b_i b_{i+1}.$$

Thus

$$6\vec{b}^T M\vec{b} = \sum_{i=1}^{m} \left\{2(\ell_{i-1}+\ell_i)b_i^2 + 2\ell_i b_i b_{i+1}\right\}$$
$$\geq \sum_{i=1}^{m} \left\{2(\ell_{i-1}+\ell_i)b_i^2 - \ell_i(b_i^2 + b_{i+1}^2)\right\}$$

by the Cauchy-Schwarz inequality. Since

$$\sum_{i=1}^{m} \ell_i b_{i+1}^2 = \sum_{i=1}^{m} \ell_{i-1} b_i^2,$$

we see that

$$6\vec{b}^T M\vec{b} \geq \sum_{i=1}^{m} \lambda_i b_i^2 \geq \min_{1\leq j\leq m} \lambda_j |\vec{b}|^2.$$

□

By Lemma 5.1, $\vec{b}$ is uniquely determined by $\vec{b} = M(\vec{\ell})^{-1}\vec{f}(\vec{\ell},\vec{d})$. Thus Remark 2 implies

Theorem 5.2. *For a give division $0 \leq z_1 < z_2 < \cdots z_m < \omega$, a piecewise cubic function $\xi \in C^2(\mathbf{T})$ which is cubic in each (z_i, z_{i+1}) uniquely exists if the value of $\xi(z_i)$ is fixed for $i = 1, \ldots, m$, where $z_{m+1} = z_1$.*

We now consider the minimal Cahn-Hoffman vector field where v is firm. If v is firm, then the minimal Cahn-Hoffman field ζ must be cubic on each faceted interval and $\zeta \in C^2(\mathbf{T})$. Let (x_i, x_{i+1}) $(i = 1, \ldots, n)$ be a faceted region of v so that $S(v) = \{x_1, \ldots, x_n\} \subset [0, \omega)$ with interpretation that $x_{n+1} = x_1$. Let ℓ_i denote its length, i.e., $\ell_i = x_{i+1} - x_i$. We define the value d_i by

$$\{d_i\} = \partial W\left(v_x(x_i - 0)\right) \cap \partial W\left(v_x(x_i - 0)\right).$$

By definition, $\zeta(x_i) = d_i$, so applying Proposition 5 and its Remark 2 with $\xi = \zeta$, $z_i = x_i$, $m = n$ implies that ζ can be written explicitly by using $\vec{\ell}$ and $\vec{d}$ only since $\vec{b} = \vec{M}(\vec{\ell})^{-1}\vec{f}(\vec{\ell}, \vec{d})$ by Lemma 5.1.

6. **Reduction to a system of ordinary differential equations.** In this section, we reduce the evolution equation $u_t = -\zeta_{xxx}$ assuming that u is firm. By elementary geometry, we see

$$\dot{x}_i = \frac{V_i - V_{i-1}}{(u_x)_{i-1} - (u_x)_i}, \tag{8}$$

where $V_i = u_t(x)$ for $x \in (x_i, x_{i+1})$ and $(u_x)_i$ denotes the slope of u in (x_i, x_{i+1}), where $\dot{x}_i$ denotes the time derivative dx_i/dt. This yields a transport equation

$$\begin{aligned} \dot{\ell}_i(t) &= \dot{x}_{i+1}(t) - \dot{x}_i(t) \\ &= -\frac{V_{i+1} - V_i}{(u_x)_{i+1} - (u_x)_i} + \frac{V_i - V_{i-1}}{(u_x)_i - (u_x)_{i-1}}. \end{aligned} \tag{9}$$

We thus obtain

$$\dot{\ell}_i(t) = c_{i-1}V_{i-1} + c_iV_i + c_{i+1}V_{i+1}$$

with c_i independent of t. By Proposition 5 (i), we see $V_i = -(b_{i+1} - b_i)/\ell_i$. Thus

$$\frac{d\vec{\ell}}{dt} = \vec{g}(\vec{\ell}, \vec{b})$$

where $\vec{g}$ depends on $\vec{b}$ and $\vec{\ell}$ analytically provided all ℓ_i's of $\vec{\ell}$ is positive. By Lemma 5.1, $\vec{b}$ can be explicitly written by $\vec{\ell}$. More precisely,

$$\vec{b} = M(\vec{\ell})^{-1}\vec{f}(\vec{\ell}, \vec{d}).$$

We thus obtain an system of ordinally differential eqiations ODEs of the form

$$\frac{d\vec{\ell}}{dt} = \vec{G}(\vec{\ell}) \tag{10}$$

with $\vec{G}(\vec{\ell}) = \vec{g}\left(\vec{\ell}, M(\vec{\ell})^{-1}\vec{f}(\vec{\ell}, \vec{d})\right)$. Since $\vec{G}$ is analytic in $\vec{\ell} = (\ell_i)_{i=1}^n$ for $\ell_i > 0$, a general theory of ODEs yields

Proposition 6. *The system of ordinary differential equation* (10) *admits a unique local-in-time solution provided that all component* ℓ_{0i} *of the initial data* $\vec{\ell}_0$ *is positive.*

Remark 3. The minimal Cahn-Hoffman field ζ is uniquely determined by $\vec{\ell}$. This mapping

$$Z : (0, \infty)^n \longrightarrow C^2(\mathbf{T}), \quad \vec{\ell} \longmapsto Z(\vec{\ell})$$

is continuous by Remark 2. Note that $\zeta(x, t) = Z\left(\vec{\ell}(t)\right)(x)$.

Proof of Theorem 1.2. We may assume that $n \geq 2$. If initial data u_0 is firm, we have unique Cahn-Hoffman vector ζ_0 in $C^2(\mathbf{T})$ and it is cubic in each faceted region. Moreover, $\zeta_{0x}(x) \neq 0$ for $x \in S(u_0)$ and $\zeta_0(x) \in \operatorname{int} \partial W(u_{0x}(x))$ for $x \notin S(u_0)$. By Proposition 6, there is a unique local-in-time solution $\vec{\ell}$ of (10), say in the time interval $[0, t_0)$. By Remark 2 and invertibility of M (Lemma 5.1), we observe that $(\zeta_i)_{i=1}^n$ is determined from $(\ell_i)_{i=1}^n$. Since the property firmness is an open propoerty, our constructed $(\zeta_i)_{i=1}^n$ for $t \in [0, t_0)$ is still in $\partial W(u_x(x, t))$ on $(x_i(t), x_{i+1}(t))$ provided that t_0 is taken smaller. Although we know the dynamics of ℓ_i's, this is not enough to determine the motion of x_i's unless one of x_i's is determined.

Since V_i is completely determined, (8) determines motion of x_i's, which determines evolution of firm u for a short time. This u solves (2) which is continuous up to $t = 0$. Since the solution of (2) is unique, the proof of Theorem 1.2 is now complete. □

7. **Facet splitting and merging.** In this section, we give a few examples of facet-splitting and merging.

7.1. **Starting from non-firm admissible functions.** We consider an admissible function u_0 with the set $S(u_0)$ of nods. We assume the number n of elements of $S(v)$ is at least 2, i.e., $n \geq 2$.

Theorem 7.1 (Facet splitting)**.** *Let* u_0 *be an admissible function with* $u_0 \not\equiv 0$. *Assume that there is a faceted region* I *such that the minimal Cahn-Hoffman field* $\eta \in \operatorname{int} \partial W(u_0(x))$ *for* $x \in \mathbf{T} \setminus \left(\overline{I} \cup S(u_0)\right)$ *and* $\eta_x(x) \neq 0$ *for* $x \in S(u_0)$. *Assume that there is a unique point* $\alpha^0 \in I$ *such that* $\eta(\alpha^0)$ *is a boundary point of the interval* $\partial W(u_{0x}(x))$ *for* $x \in I$. *Moreover, assume that* $\eta_{xx}(\alpha^0) \neq 0$ *and* $\eta_{xxx}(\alpha^0 - 0) \neq \eta_{xxx}(\alpha^0 + 0)$. *Then, the unique solution of* (1) *starting from* u_0 *instantaneously* *becomes firm with $n+2$ facets. (The faceted region I will be broken into three faceted region for $t>0$.)*

Proof. We may assume that $S(u_0)=\{0<x_1^0<\cdots<x_n^0<\omega\}$ with $n\geq 2$. Let I be (x_i^0,x_{i+1}^0) and $\alpha^0\in I$. We may assume that $\eta(\alpha)=\inf\partial W\left(u_{0x}(\alpha^0)\right)$ by considering $-u_0$ if necessary.

We shall construct a solution of (10) with (8) satisfying

$$
\begin{aligned}
\ell_j(t) &:= x_{j+1}(t)-x_j(t), \quad j\leq i-1 \text{ or } j\geq i+1\\
\ell_-(t) &:= \alpha_-(t)-x_i(t),\\
\ell(t) &:= \alpha_+(t)-\alpha_-(t),\\
\ell_+(t) &= x_{i+1}(t)-\alpha_+(t)
\end{aligned}
$$

with initial data

$$
\begin{aligned}
x_j(0) &= x_j^0 \quad \text{for} \quad j=1,2,\ldots,n,\\
\alpha_+(0) &= \alpha_-(0)=\alpha^0,
\end{aligned}
$$

so that $\ell_j(0)>0$ for $j\neq i$ and $\ell_-(0),\ell_+(0)>0$, $\ell(0)=0$. We consider the equation (10) with $\vec{\ell}=(\ell_1,\ldots,\ell_{i-1},\ell_-,\ell,\ell_+,\ell_{i+1},\ldots,\ell_n)$ with

$$
S\left(u(x,t)\right)=\{x_1(t)<\ldots<x_i(t)<\alpha_-(t)<\alpha_+(t)<x_{i+1}(t)<\cdots<x_n(t)\} \quad \text{for} \quad t>0.
$$

By Lemma 5.1, $M(\vec{\ell})$ is invertible provided that all components except one component (which is non-negative) of $\vec{\ell}$ are positive. We then set $d_i=\eta(x_i^0), i=1,\ldots n$

and $d = \eta(\alpha^0)$ so that $\vec{f} = (f_1, \dots, f_{i-1}, f_-, f, f_+, f_{i+1}, \dots, f_n)$ with

$$\begin{aligned} f_j &:= -\frac{d_j - d_{j-1}}{\ell_{j-1}} + \frac{d_{j+1} - d_j}{\ell_j} \quad \text{for} \quad j \le i-1,\ j \ge i+2, \\ f_- &:= -\frac{d_i - d_{i-1}}{\ell_{i-1}} + \frac{d - d_i}{\ell_-}, \\ f &:= -\frac{d - d_i}{\ell_-}, \\ f_+ &:= \frac{d_{i+1} - d}{\ell_+}, \\ f_{i+1} &:= -\frac{d_{i+1} - d}{\ell_+} + \frac{d_{i+2} - d_{i+1}}{\ell_{i+1}}. \end{aligned}$$

These quantities appear in Proposition 5, if the cubic polynomial ξ satisfies $d = \xi(\alpha_+) = \xi(\alpha_-)$, where ξ is cubic in $[\alpha_-, \alpha_+]$. Note that $\vec{f}$ has no singularity at $\ell = 0$. We thus observe that

$$\vec{b} = M(\vec{\ell})^{-1} \vec{f}(\vec{\ell}, \vec{d}) \tag{11}$$

is analytic in $\vec{\ell}$ including the place where $\ell = 0$.

We shall study how singular the ODE (10) at $t = 0$. We consider the evolution of the length ℓ of (α_-, α_+) by (9):

$$\dot{\ell}(t) = -\frac{V_+ - V}{(u_x)_+ - (u_x)} + \frac{V - V_-}{(u_x) - (u_x)_-},$$

where V_+, V, V_- and $(u_x)_+, (u_x), (u_x)_-$ denote the speed and the slope in faceted region (x_i, α_-), (α_-, α_+) and (α_+, x_{i+1}), respectively. By geometry, we see $(u_x)_+ = (u_x)_-$, which implies

$$\dot{\ell} = (V_- - V_+) \,/\, ((u_x)_+ - (u_x)) \,. \tag{12}$$

By Lemma 4.2 (ii), $\eta_{xxx}(\alpha^0 + 0) > \eta_{xxx}(\alpha^0 - 0)$. Since $V_+(0) = -\eta_{xxx}(\alpha^0 + 0)$, $V_-(0) = -\eta_{xxx}(\alpha^0 - 0)$ and since $(u_x)_+ > (u_x)$, we end up with

$$\dot{\ell}(0) > 0. \tag{13}$$

Since

$$V_- = -\frac{b_- - b_i}{\ell_-}, \quad V = -\frac{b_+ - b_-}{\ell}, \quad V_+ = -\frac{b_{i+1} - b_+}{\ell_+},$$

with $\vec{b} = (b_1, \ldots, b_i, b_-, b_+, b_{i+1}, \ .. \ b_n)$, the formula (12) with (11) says that its right-hand side depends analytically in $\vec{\ell}$ including the place where $\ell = 0$. The derivatives of $\ell_\pm$ includes V so it looks singular at $\ell = 0$. However, by (11) we see that

$$b_\pm(\ell) - b_\pm(0) = 0(\ell)$$

as $\ell \to 0$. Thus, V depends on $\vec{\ell}$ analytically including the place where $\ell = 0$. We thus observe that (10) is solvable at least for a short time. Moreover, $\dot{\ell} > 0$ by (13) so a new faceted region (α_-, α_+) appears for $t > 0$. Thus, we are able to construct a $\zeta \in C^2(\mathbf{T})$ (smoothly depending on time $t > 0$) which is cubic on

$$\begin{aligned} &(x_j(t), x_{j+1}(t)) \quad (j \leq i-1, \ j \geq i+1) \\ &(x_i(t), \alpha_-(t))\,, \ (\alpha_-(t), \alpha_+(t))\,, \ (\alpha_+(t), x_{i+1}(t))\,. \end{aligned}$$

We shall prove that our ζ belongs to $\partial W\left(u_x(x)\right)$ near $[\alpha_-, \alpha_+]$ for small $t > 0$. It suffices to prove that $\zeta_x(\alpha_-) < 0$ and $\zeta_x(\alpha_+) > 0$ for a short time. Since

$$\zeta_x(\alpha_-) = \frac{d-d}{\ell} - \frac{b_+ + 2b_-}{6}\ell$$

by (6), $\zeta_x(\alpha_-) < 0$ provided that b_+, b_- are positive. Since we have assumed that $\eta_{xx}(\alpha^0) \neq 0$ so that $\eta_{xx}(\alpha^0) > 0$, this $b_\pm$ is positive at time zero. Thus $\zeta_x(\alpha_-) < 0$. We replace x by $-x$ and argue in the same way to get $\zeta_x(\alpha_+) > 0$. We thus conclude that ζ belongs to $\partial W\left(u_x(x)\right)$ for all x since we have assumed that $\eta_x \neq 0$ on $S(u_0)$ and η does not touch $\partial W\left(u_x(x)\right)$ on $\mathbf{T} \setminus \left(S(v) \cup \bar{I}\right)$. Thus, we are able to construct the desired Cahn-Hoffman field for a short time. See Figure 2, 3 for evolutions of u and ζ depending on profiles of u_0 around a faceted region (x_i^0, x_{i+1}^0).

As in the proof of Theorem 1.2, we are able to construct an evolution of admissible functions u which is firm for a short time and solves (2). Since u is continuous up to $t = 0$, this u is a unique solution of (2). The proof is now complete. □

Remark 4. (i) Of course, the statement of Theorem 7.1 is still valid if there are several points satisfying the some properties as α^0. However, if one removes non-degeneracy assumptions like $\eta_{xx}(\alpha^0) \neq 0$ or $\eta_{xxx}(\alpha^0 - 0) \neq \eta_{xxx}(\alpha^0 + 0)$, it is not clear such a facet-splitting actually occurs. Another difficult situation is in the case $\eta_x(x) = 0$ at $x \in S(u_0)$.

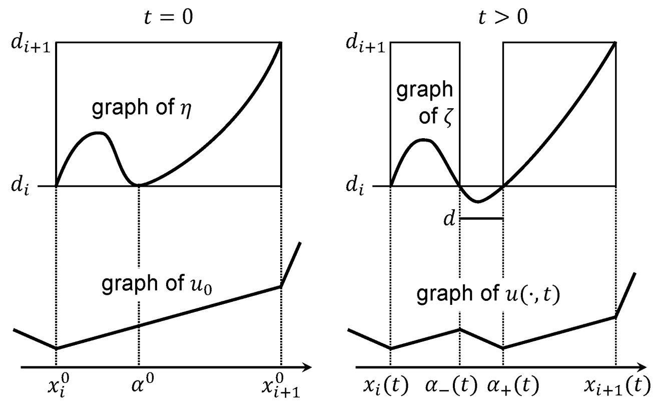

FIGURE 2. Evolution of profiles of u and ζ for a convex facet

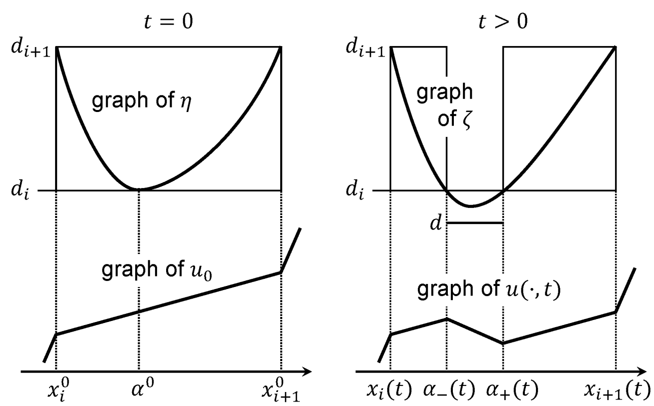

FIGURE 3. Evolution of profiles of u and ζ for an infletion facet

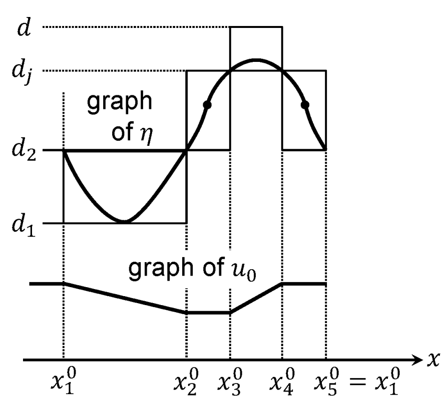

FIGURE 4. An example of global profile of u_0

(ii) It is an open problem whether or not a facet-splitting occurs during evolution when the initial data is firm. By Theorem 1.2, we know it stays firm at least for a short time. Numerical examples in [5] suggest a facet splitting may occur after some time even if it is initially firm.

(iii) It is not difficult to have an explicit example of initial profile of u_0 discussed in Theorem 7.1. An example including the profile given in Figure 3 is given in Figure 4. Let us explain Figure 4. We set $0 = (x_1^0 + x_2^0)/2$ and take $\eta(x) = |x|^2 x$ on (x_1^0, x_2^0) with $d_1 = 0$, $d_2 = (x_1^0)^3$. Since η in C^2 across nods, we see $\eta(x) = x^3 + \gamma(x - x_2^0)^3$ with some $\gamma \in \mathbf{R}$ on (x_2^0, x_3^0). Here η must be convex near $x = x_2^0$ and concave near $x = x_3^0$ so that $\gamma < 0$ and its inflection point $\hat{x} = \gamma x_2^0/(\gamma+1)$ must be $\hat{x} > x_2^0$, i.e., γ should satisfy $\gamma < -1$. Take such γ and take $x_3^0 > \hat{x}$ such that $\eta'(x_3^0) > 0$. We then set $d_3 = \eta(x_3^0 - 0)$ and extend η for $x > x_3^0$ as a quadratic function. Since η must be C^2 across $x = x_3^0$, this function is uniquely determined. We take $x_4^0 > x_3^0$ such that $\eta(x_4^0 - 0) = d_3$. We further take $x_5^0 > x_4^0$ so that $x_5^0 - \overline{x} = \overline{x} - x_2^0$ with $\overline{x} = (x_3^0 + x_4^0)/2$ and extend η to (x_4^0, x_5^0) so that η is even with respect to $x = \overline{x}$, i.e., $\eta(x) = \eta(2\overline{x} - x)$. Identifying $x_5^0 = x_1^0$, we obtain a periodic function u_0 having four faceted regions which is an explicit example of Theorem 7.1. Indeed, $\eta_{xxx}(+0) > \eta_{xxx}(-0)$ is fulfilled. We thus conclude that the facet-splitting in Theorem 7.1 actually occurs in some cases.

7.2. **An example of facet-merging.** There is a chance that some facet may disappear. This phenomena is well-known for the second-order problem; see e.g. [27]. We here give a simple example. We consider an admissible function u_0 such that $S(u_0) = \{0 = x_1 < x_2 < x_3 < \cdots < x_6 < x_7 = \omega\}$ with $x_7 = x_1$. We assume that u_0 is even with respect to x_4, i.e., $u_0(x - x_4) = u_0(-x + x_4)$ for $|x| < \omega/2$. Moreover, the slope $(u_{0x})_1 = (u_{0x})_3$ and $(u_{0x})_2 > 0$ and satisfies additional symmetry

$$u_0\left(x - \frac{x_2 + x_3}{2}\right) = -u_0\left(-x - \frac{x_2 + x_3}{2}\right) \quad \text{for} \quad |x| < \omega/4.$$

See Figure 5 for the graph of u_0. We assume that

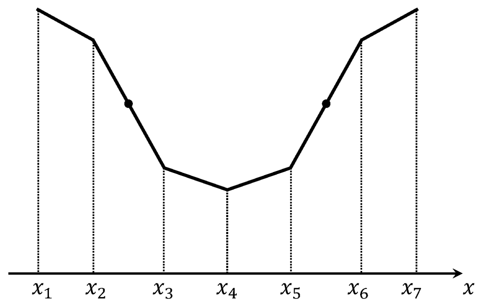

FIGURE 5. Graph of u_0

$$\partial W(p) = \begin{cases} [r,q], & p = p_2 := (u_{0x})_2 \\ r, & (u_{0x})_1 < p < (u_{0x})_2 \\ [0,r], & p = p_1 := (u_{0x})_1 \\ 0, & 0 \le p < (u_{0x})_1 \end{cases}$$

and $(\partial W)(p) = -(\partial W)(-p)$ for $p < 0$. Here $r, q \in \mathbf{R}$ is assumed to satisfy $0 < r < q$.

Theorem 7.2. *Let $u(x,t)$ be the solution of (2) with initial data u_0. Let $z_i(t)$ be i-th nod starting from x_i. Then, the faceted region $(z_2(t), z_3(t))$ and $(z_5(t), z_6(t))$ disappear in some time $t = t_*$ and $u(x,t_*)$ is also firm with two faceted regions. Moreover, $u(x,t) = u(x,t_*)$ for $t \ge t_*$. In other words, the motion stops for $t \ge t_*$.*

Remark 5. (i) Such u_0 is always firm. This symmetry assumption is preserved during the evolution, so u stays firm unless some faces disappear.
(ii) By symmetry, the velocity of u on (z_2, z_3) must be zero. This means that the (minimal) Cahn-Hoffman field ζ must be quadratic in (z_2, z_3).

Proof. By Theorem 1.2, there is $t_0 > 0$ such that $u(\cdot,t)$ stays firm for $t < t_0$. We observe that $L := (z_2 + z_3)/2$ is independent of t and $z_1(t) = 0$ for $0 < t < t_0$. We note that $4L = \omega$. Let ℓ denote the length of (z_1, z_2), i.e., $\ell = z_2 - z_1$. Let ζ be the minimal Cahn-Hoffman field of $u(\cdot,t)$. By symmetry,

$$\zeta_{xx}(z_1) = 0.$$

Since $u(\cdot,t)$ is firm,

$$b := \zeta_{xx}(z_2) = \zeta_{xx}(z_3) < 0.$$

Applying Proposition 5 (ii) with $i = 2$ so that $b_i = b$, $b_{i-1} = 0$, $\ell_{i-1} = \ell$, $\ell_i = 2(L-\ell)$, we obtain

$$\ell \cdot 0 + 2\left(\ell + 2(L-\ell)\right) b + 2(L-\ell)b = -6r/\ell$$

or

$$b = -\frac{6r}{2\ell(3L-2\ell)}.$$

We set $q = p_2 - p_1$ and observe by (8) that

$$\dot{z}_2(t) = \frac{V_2 - V_1}{p_1 - p_2} = \frac{V_1}{q}$$

since $V_2 = 0$ by Remark 5. By Proposition 5 (i), we have

$$V_1 = -\frac{b-0}{\ell}.$$

Since $\ell(t) = z_2(t)$, we now conclude that

$$\dot{\ell}(t) = \frac{V_1}{q} = \frac{1}{q\ell}\frac{3r}{(3L-2\ell)}.$$

Integrating this differential equation yields

$$k\left(\ell(t)\right) - k\left(\ell(0)\right) = \frac{3r}{q}t \quad \text{with} \quad k(\ell) = L\ell^3 - \frac{\ell^4}{2}.$$

Since $k'(\ell) > 0$ on $(0, 3L/2)$, k^{-1} exists so that

$$\ell(t) = k^{-1}\left(\frac{3r}{q}t + k\left(\ell(0)\right)\right).$$

If $\ell(0) < L$, then there is the maximal time $t_* > 0$ such that

$$\ell(t_*) = L, \quad \ell(t) < L \quad \text{for} \quad t \in (0, t_*).$$

It is given by $k(L) = \frac{3r}{q}t_* + k\left(\ell(0)\right)$. This implies that $z_2(t_*) = z_3(t_*)$, $z_5(t_*) = z_6(t_*)$. Then, there are only two faceted regions (z_1, z_4), (z_4, z_7) for $u(\cdot, t_*)$ with slope p_1 and $-p_1$, respectively. Then $u(\cdot, t_*)$ is still firm. The minimal Cahn-Hoffman field must be $\zeta \equiv 0$. Thus, $u(\cdot, t)$ stops for $t \geq t_*$. □

7.3. **Estimate of the speed.** In the second-order problem, the speed on i-th faceted region is determined only by a neighborhood of i-th facet. In our problem, it depends on whole profile. We derive an estimate of the speed of i-th facet by length of i-th facet.

Theorem 7.3. *Let v be an admissible function having at least two faceted region. Assume that v is firm. Let ζ be the minimal Cahn-Hoffman field. If $d_{i+1} = \zeta(x_{i+1})$ does not agree with $d_i = \zeta(x_i)$, then*

$$-3 \cdot 2^2 < \zeta_{xxx}(x)\ell_i^3/(d_{i+1} - d_i) < 3 \cdot 2^5 \quad \textit{for} \quad x \in (x_i, x_{i+1}).$$

If $d_i = d_{i+1}$ and the length of $\partial W(v_x(x))$ for $x \in (x, x_{i+1})$ equals $d > 0$, then

$$\left|\zeta_{xxx}(x)\ell_i^3/d\right| < 3^2\sqrt{3}/2 \quad \textit{for} \quad x \in (x_i, x_{i+1}).$$

These estimates are optimal. Here, $\ell_i = x_{i+1} - x_i$ and $i = 1, 2, \ldots, n$.

To prove Theorem 7.3, we recall several basic properties of a cubic polynominal.

Lemma 7.4. *Let $h_\sigma(x)$ be a cubic polynomial of the form*

$$h_\sigma(x) = \frac{x^3}{3} - \sigma^2 x, \ \sigma > 0,$$

which has a unique local maximum $2\sigma^3/3$ (resp. minimum $-2\sigma^3/3$) at $-\sigma$ $(+\sigma)$.

- (i) *$f_\sigma(s) := h_\sigma(s+1) - h_\sigma(s-1)$ is minimized at $s = 0$ for any $\sigma > 0$.*
- (ii) *Set $M_\sigma := \max_{|x|\le 1} h_\sigma(x)$. This value M_σ is minimized at $\sigma = 1/2$ under the constraint $M_\sigma = h_\sigma(1)$. The value $M_{1/2} = 1/12$.*
- (iii) *Assume that $-h_\sigma$ is monotone increasing in $[s-1, s+1]$ so that $s \le \sigma - 1$, $s \ge -\sigma + 1$ which implies $\sigma > 1$. Then,*

$$m_\sigma := \min_{-\sigma+1 \le s \le \sigma - 1} (-f_\sigma(s))$$

is attained at $s = \pm(\sigma - 1)$. The value $\min_{\sigma \ge 1} m_\sigma$ is attained only at $\sigma = 1$. Moreover, $\max_{|x|\le 1}(-h_1(x)) = m_1/2 = 2/3$.

Proof of Lemma 7.4. Since

$$f_\sigma(s) = 2s^2 + \frac{2}{3} - 2\sigma^2,$$

the first statement is clear. If $M_\sigma = h_\sigma(1)$ so that $\sigma \le 1$, then $M_\sigma \ge 2\sigma^3/3 = h_\sigma(-\sigma)$. Since $h_\sigma(-\sigma) = h_\sigma(2\sigma)$, this implies that $2\sigma \le 1$. Then $M_\sigma = h_\sigma(1) = \frac{1}{3} - \sigma^2$, which is minimized at $\sigma = 1/2$ for $\sigma \le 1/2$ with $M_{1/2} = 1/12$. This proves (ii).

It is clear that m_σ is attained at $s=\pm(\sigma-1)$. Then

$$m_\sigma/2 = (\sigma-1)^2 + \frac{1}{3} - \sigma^2 = -2\sigma + 4/3.$$

Thus, the minimum of $m_\sigma/2$ over $\sigma \geq 1$ is attained only at $\sigma=1$ and m_1 equals 2/3. □

Lemma 7.5. *Assume that $s_\pm$ is zero of $h_\sigma(s+1)-h_\sigma(s-1)$ so that $s_\pm = \pm(\sigma^2 - 1/3)^{1/2}$, where $\sigma^2 > 1/3$ is assumed. Assume that (s_+-1, s_++1) and (s_--1, s_-+1) has no intersection so that $s_+ \geq 1$ and $s_- \leq -1$. Then, the value*

$$A(\sigma) = \max_{x\in(s_--1,s_-+1)} \left(h_\sigma(x) - h_\sigma(s_-+1)\right)$$

is minimized only at $\sigma^2 = 4/3$, equivalently $s_\pm = \pm 1$. Its value $A(2/\sqrt{3}) = 2\sigma^3/3 = 16/(9\sqrt{3})$.

Proof of Lemma 7.5. By our assumption, $h_\sigma(x)$ takes its local maximum in $(s_- - 1, s_- + 1)$ so

$$A(\sigma) = 2\sigma^3/3 - h_\sigma(s_-+1).$$

Since $s_- = -(\sigma^2+1/3)^{1/2}$, we see

$$\begin{aligned} h_\sigma(s+1) &= (s+1)\left\{\frac{(s+1)^2}{3} - \left(s^2 + \frac{1}{3}\right)\right\} \\ &= \frac{2}{3}(1-s^2)s \quad \text{with} \quad s = s_-. \end{aligned}$$

Thus, we observe that

$$A\left(\sigma(s)\right) = \frac{2}{3}\left\{\left(s^2+\frac{1}{3}\right)^{3/2} + (s^2-1)s\right\} \quad \text{with} \quad s = s_-,\ \sigma(s) = \left(s^2+\frac{1}{3}\right)^{1/2}.$$

We differentiate

$$\begin{aligned} \frac{3}{2}\frac{d}{ds}A\left(\sigma(s)\right) &= 3s\left(s^2+\frac{1}{3}\right)^{1/2} + 3s^2 - 1 \\ &= 3s\left\{\left(s^2+\frac{1}{3}\right)^{1/2} + s\right\} - 1 < 0 \quad \text{for} \quad s < 0, \end{aligned}$$

since $(s^2 + 1/3) \geq -s$ for $s < 0$. Thus, $A(\sigma(s))$ is decreasing in s for $s < 0$. We now observe that $A(\sigma(s))$ for $s \leq -1$ is minimized at $s_- = -1$. If $s_- = 1$, then $\sigma^2 = \sigma(1)^2 = 4/3$ and $A(\sigma) = 2\sigma^3/3 = 16/9\sqrt{3}$. □

Proof of Lemma 7.3. To prove the first statement, we may assume that $d_{i+1} > d_i$. By scaling in x, we may assume that $\ell_i = 2$. By scaling in ζ, we may assume that $d_{i+1} - d_i = 2$. Assume that $\zeta_{xxx}(x) = a > 0$ in $x \in (x_i, x_{i+1})$. Then, ζ in (x_i, x_{i+1}) equals

$$\zeta(x) = \frac{a}{2} h_\sigma(x+s) + c \quad \text{with some} \quad s, c \in \mathbf{R} \quad \text{and} \quad \sigma > 0$$

such that $\zeta(x_i) = d_i$, $\zeta(x_{i+1}) = d_i + 2$. We may assume that $x_i = -1$ and $d_i = -1$ by translation. We shall maximize a under $\zeta(1) - \zeta(-1) = 2$. This means that we minimize $f_\sigma(s)$ for fixed σ. By Lemma 7.4 (i), s should be 0 and $c = 0$, so that the inflection point of ζ must be 0 to maximize a. Since $|\zeta| < 1$ on $(-1, 1)$, and $\zeta(1) = 1$, $a = 2/h_\sigma(1)$, we observe $M_\sigma = h_\sigma(1)$. By Lemma 7.4 (ii), $h_\sigma(1)$ is minimized at $\sigma = 1/2$. If $a < a_*$ with $a_* = 2/M_{1/2} = 24$, then $|\zeta| < 1$ on $(-1, 1)$. If $a = a_*$, ζ may take value 1 in $(0, 1)$. Thus, this upper bound is optimal. Since $\ell_i = 2$, $d_{i-1} - d_i = 2$ so that $a \cdot 2^3/2 < 24 \cdot 4 = 3 \cdot 2^5$. Thus, the upper bound has been proved.

To prove the lower bound, we next consider the case $a < 0$. We may assume $(x_i, x_{i+1}) = (-1, 1)$, $(d_i, d_{i-1}) = (-1, 1)$. Since $|\zeta| < 1$ on $(-1, 1)$, then $(-1, 1) \subset (s-1, s+1)$ so that $\sigma > 1$. Since $\zeta(\pm 1) = \pm 1$, a should be taken so that

$$\frac{a}{2} f_\sigma(s) = 2.$$

For fixed $a > 1$, we observe that

$$|a| \leq 4/m_\sigma$$

and by Lemma 7.4 (iii), m_σ is attained at $s = \pm(\sigma - 1)$. By Lemma 7.4 (iii), $\min_{\sigma \geq 1} m_\sigma$ is attained only at $\sigma = 1$, so this implies

$$|a| < \frac{4}{4/3} = 3.$$

In the case $\sigma = 1$, $\zeta_x(\pm 1) = 0$ which implies that the profile is not firm, so this inequality is optimal. We now conclude that $|a| 2^3/2 < 3 \cdot 4 = 12$.

It remains to prove the estimate for $d_i = d_{i+1}$. We may assume that $(x_i, x_{i+1}) = (-1, 1)$ and $\zeta > \zeta(\pm 1)$ on $(-1, 1)$ since the argument for $\zeta < \zeta(\pm 1)$ is symmetric.

The only restriction is that $\zeta \leq d + \zeta(\pm 1)$. We consider $a > 0$. In this case, the only restriction for a is

$$aA(\sigma) < d.$$

By Lemma 7.5, $A(\sigma)$ is maximized only at $\sigma^2 = 4/3$. Thus $a < d/A(2/\sqrt{3})$. We thus conclude $a < 9\sqrt{3}d/16$. Since $\ell_i^3 = 2^3$, this yields the desired estimate. The optimality of this bound also follows from above consideration. The case $a < 0$ can be handled in a similar way. □

8. **Appendix.** We give here a proof of lower semicontinuity of energy as well as a characterization of subdifferentials. These are well-known but we give their proofs for the reader's convenience and completeness.

We consider energy

$$E(u) = \begin{cases} \int_{\mathbf{T}} W(u_x)dx, & u \in W^{1,1}(\mathbf{T}) \cap H_{\mathrm{av}}^{-1}(\mathbf{T}), \\ \infty, & u \in H_{\mathrm{av}}^{-1}(\mathbf{T}) \setminus W^{1,1}(\mathbf{T}). \end{cases}$$

Proposition 7. *Assume that W is a real-valued convex function. Assume that W is coercive in the sense that*

$$\lim_{|p| \to \infty} W(p)/|p| = \infty.$$

Then the energy E is lower semicontinuous as a functional on H_{av}^{-1}.

Proof. The proof parallels that of L^2 case, which is found in [3]. We use a duality argument. Let W^* be a convex conjugate of W. Then we know that

$$W(p) = \sup_{q \in \mathbf{R}} \left\{ p \cdot q - W^*(q) \right\}.$$

Since $C^\infty(\mathbf{T})$ is dence in H_{av}^{-1}, we observe that

$$\int_{\mathbf{T}} W(u_x)dx = \sup \left\{ \int_{\mathbf{T}} \left(-u\varphi_x - W^*(\varphi) \right) dx, \ \varphi \in C^\infty(\mathbf{T}) \right\}.$$

Since $u_m \to u$ in H_{av}^{-1} implies $\int u_m \varphi_x \to \int u \varphi_x$ and since $\sup \underline{\lim} \leq \underline{\lim} \sup$, we conclude that

$$\int_{\mathbf{T}} W(u_x)dx \leq \underline{\lim}_{m \to \infty} \int_{\mathbf{T}} W(u_{mx})dx.$$

It remains to prove that $u \in W^{1,1}(\mathbf{T})$. For this purpose, it suffices to show that $|u_{mx}|$ is equi-absolutely continuous with respect to the Lebesgue measure provided that $E(u_m)$ is bounded, i.e., there is $M > 0$ such that

$$\int_{\mathbf{T}} W(u_{mx})dx \leq M.$$

We may assume W is positive by adding a constant. By coercivity, for any $\delta_1 > 0$ there is $k = k(\delta_1)$ such that

$$\delta_1 W(p) \geq |p| \quad \text{if} \quad |p| \geq k.$$

We shall estimate

$$\int_K |u_{mx}|dx = \left(\int_{K_k} + \int_{K_k^c} \right) |u_{mx}|dx \quad \text{with} \quad K_k = \left\{ x \in K \;\middle|\; |u_{mx}| \geq k \right\},$$

where the Lebesgue measure $|K|$ of K is dominated by $\delta > 0$. By coercivity,

$$\int_{K_k} |u_{mx}|dx \leq \delta_1 \int_{K_k} W(u_{mx})dx \leq \delta_1 M.$$

We thus observe that

$$\int_K |u_{mx}|dx \leq \delta_1 M + \delta k.$$

For a given $\varepsilon > 0$, we take $\delta_1 > 0$ small so that $\delta_1 M < \varepsilon/2$. We fix $k = k(\delta_1)$ and take $\delta > 0$ small so that $\delta k < \varepsilon/2$. Then our integral satisfies

$$\int_K |u_{mx}|dx < \varepsilon$$

provided that $|K| < \delta$. The choice of δ is independent of m. Thus we conclude that $|u_{mx}|$ is equi-absolutely continuous if $E(u_m)$ is bounded. The proof is now complete. □

We next prove a characterization of subdifferentials of E at u when W is piecewise linear and u is admissible. It is possible to extend such a result for more general function u by an approximation method; see e.g. [3] or very recent article [16]. However, we give here a simple direct and explicit proof.

We restrict E in $L^2(\mathbf{T})$ in the next proposition.

Proposition 8. *In addition to the assumption of Proposition 7, assume further that W is piecewise linear. Let u be an admissible piecewise linear function. Then the L^2-subdifferential of E at u is characterized as*

$$\partial_{L^2} E(u) = \Big\{ f = -\eta_x \in L^2(\mathbf{T}) \;\Big|\; \eta(x) \in \partial W\left(u_x(x)\right) \text{ for all } x \in \mathbf{T} \text{ at which } u \text{ is differentiable} \Big\}.$$

Proof. We set the right-hand side by G. It is easy to see that $G \subset \partial_{L^2} E(u)$ by a direct computation. What is non-trivial is the converse inclusion, i.e., $\partial_{L^2} E(u) \subset G$. For this purpose, we prepare an elementary lemma below whose proof is postponed at the end of this section.

Lemma 8.1. *Let σ be a real-valued continuous function in an interval $[a, b]$. Assume that*

$$\int_a^b |\varphi_x| dx \geq \int_a^b \sigma \varphi_x dx$$

for any $\varphi \in C_c^\infty(a, b)$. Then $\max \sigma - \min \sigma \leq 2$, where $\max$ and $\min$ is taken in $[a, b]$.

We use this lemma. By definition, $f \in \partial_{L^2} E(u)$ must satisfy

$$\int_{\mathbf{T}} \left(W(u_x + \varphi_x) - W(u_x)\right) dx \geq \int_{\mathbf{T}} f \varphi dx \tag{14}$$

for all $\varphi \in C^\infty(\mathbf{T})$. Let (a, b) be a facet of u so that the slope belongs to one of jumps of W'. We may assume that $W(u_x) = 0$ on (a, b) by adding a constant. Let $-\eta$ be a primitive of f. It is continuous in a neighborhood of $[a, b]$ unless $b = a + \omega$; in that case, u is a constant. We consider $\varphi \in C^\infty(\mathbf{T})$ supported in (a, b). Then the inequality (14) yields

$$\int_a^b (\alpha \varphi_{x+} - \beta \varphi_{x-}) \geq \int_a^b \eta \varphi_x dx$$

with some constants $\alpha > \beta$, where $p_+ = \max(p, 0)$, $p_- = \max(-p, 0)$. By adding a linear function, we may assume that $\alpha = -\beta$. Applying Lemma 8.1 for $\sigma = \eta$, we conclude that

$$\max_{(a,b)} \eta - \min_{(a,b)} \eta \leq \alpha - \beta.$$

Unless u is a constant function, continuity of σ near (a, b) and admissibility imply that $\eta(x) \in \partial W\left(u_x(x)\right)$ up to a constant function on $\mathbf{T}$. We now conclude that

$f = -\eta_x \in G$. If u is a constant, Lemma 8.1 implies that $\max_{\mathbf{T}} \eta - \min_{\mathbf{T}} \eta = \alpha - \beta$, so by adding a constant in $\mathbf{T}$, we conclude that $\eta \in \partial W(u_x(x))$. The proof is now complete. $\square$

Proof of Lemma 8.1. We may take φ as a Lipschitz function by a standard approximation. For given two points $q, r \in (a,b)$ with $q < r$, we consider

$$\varphi_m(x) = \min\{1, m(x-q)_+, m(r-x)_+\}$$

for sufficiently large m so that $\max \varphi_m = 1$. The first inequality in Lemma 8.1 implies that

$$2 \geq m \int_q^{q+1/m} \sigma dx - m \int_{r-1/m}^{r} \sigma dx.$$

Since σ is continuous, sending $m \to \infty$ implies

$$2 \geq \sigma(q) - \sigma(r) \quad \text{for any} \quad q, r \in (a,b)$$

with $q < r$. We may take $-\varphi_m$ to get $2 \geq \sigma(r) - \sigma(q)$. We thus conclude that $2 \geq \max \sigma - \min \sigma$. $\square$

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