ON AN ALGORITHM TO COMPUTE DERIVATIVES

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ABSTRACT. In this paper, we complete Jantzen's algorithm to compute the highest derivatives of irreducible representations of p-adic odd special orthogonal groups or symplectic groups. As an application, we give some examples of the Langlands data of the Aubert duals of irreducible representations, which are in the integral reducibility case.

CONTENTS

1. INTRODUCTION

In the theory of admissible representations of p-adic groups, *Jacquet modules* is one of most influential ingredients. Fix a p-adic field F. When G is a split odd special orthogonal group $\mathrm{SO}_{2n+1}(F)$, or a symplectic group $\mathrm{Sp}_{2n}(F)$, the author [2] gave a description for the semisimplifications of Jacquet modules of all irreducible tempered representations in terms of the local Langlands correspondence established by Arthur [1]. This result would relate the classical Mœglin–Tadić classification of discrete series representations with the local Langlands correspondence. However, even for tempered representations, to compute all Jacquet modules is a hard work. It should be difficult to extend the result in [1] to all irreducible representations.

In this paper, we treat the notion of *derivatives*, which are certain partial information of Jacquet modules of admissible representations. As before, let G be a split odd special orthogonal group $\mathrm{SO}_{2n+1}(F)$, or a symplectic group $\mathrm{Sp}_{2n}(F)$. Denote by $P_d = M_d N_d$ the standard parabolic subgroup of G with Levi $M_d = \mathrm{GL}_d(F) \times G_0$ for some classical group of the same type as G. Fix an irreducible unitary supercuspidal representation ρ of $\mathrm{GL}_d(F)$.

Definition 1.1. *Let π be a smooth admissible representation of G of finite length.*

(1) *If the semisimplification of the Jacquet module $\mathrm{Jac}_{P_d}(\pi)$ along P_d is of the form*

$$\mathrm{s.s.Jac}_{P_d}(\pi) = \bigoplus_i \tau_i \boxtimes \pi_i,$$

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*we define the **partial Jacquet module*** $\mathrm{Jac}_{\rho|\cdot|^x}(\pi)$ *with respect to* $\rho|\cdot|^x$ *for* $x \in \mathbb{R}$ *by*

$$\mathrm{Jac}_{\rho|\cdot|^x}(\pi) = \bigoplus_{\substack{i \\ \tau_i \cong \rho|\cdot|^x}} \pi_i.$$

(2) *For a positive integer* k*, the **k-th*** $\rho|\cdot|^x$***-derivative*** $D^{(k)}_{\rho|\cdot|^x}(\pi)$ *is defined by*

$$D^{(k)}_{\rho|\cdot|^x}(\pi) = \frac{1}{k!} \underbrace{\mathrm{Jac}_{\rho|\cdot|^x} \circ \cdots \circ \mathrm{Jac}_{\rho|\cdot|^x}}_{k}(\pi).$$

When $D^{(k)}_{\rho|\cdot|^x}(\pi) \neq 0$ *but* $D^{(k+1)}_{\rho|\cdot|^x}(\pi) = 0$*, we say that* $D^{(k)}_{\rho|\cdot|^x}(\pi)$ *is the **highest*** $\rho|\cdot|^x$***-derivative****.*

It is important that when $\rho|\cdot|^x$ is not self-dual, the highest derivative $D^{(k)}_{\rho|\cdot|^x}(\pi)$ of an irreducible representation π is also irreducible, and π is a unique irreducible subrepresentation of the parabolically induced representation $(\rho|\cdot|^x)^k \rtimes D^{(k)}_{\rho|\cdot|^x}(\pi)$ (see Proposition 2.6). In particular, π is recovered from its highest derivatives in this case. By these properties, the highest derivatives have many applications. For example:

- the proofs of the Howe duality conjecture by Mínguez [14] and Gan–Takeda [6];
- another proof of the classification the unitary dual of general linear groups by Lapid–Mínguez [12] using the analogous derivatives for these groups (see Definition 2.2);
- several results on the irreducibility of parabolically induced representations by Jantzen [9] and Lapid–Tadić [13].

Jantzen [7] and Mínguez [15] obtained a complete description of the highest derivatives of irreducible representations of general linear groups $\mathrm{GL}_n(F)$ independently. It gives an algorithm to compute the Zelevinsky involutions ([7, §3.3]). Similarly, if one were able to compute the highest derivatives of all irreducible representations, one might compute the Aubert dual of any irreducible representation (see Theorem 2.13 below). Jantzen [10] suggested an algorithm to compute the highest derivatives of irreducible representations. We will recall this algorithm in §2.5 below. According to this algorithm, the computation of the highest derivative of an arbitrary irreducible representation of a classical group is reduced to the ones of irreducible representations of the form $L((\rho|\cdot|^{-x})^a, \Delta_\rho[x-1,-x]^b; T)$ which is a unique irreducible (Langlands) subrepresentation of the standard module $(\rho|\cdot|^{-x})^a \times \Delta_\rho[x-1,-x]^b \rtimes T$, where ρ is an irreducible unitary supercuspidal representation of $\mathrm{GL}_d(F)$, x is a positive half-integer, $\Delta_\rho[x-1,-x]$ is a Steinberg representation of $\mathrm{GL}_{2dx}(F)$, and T is an irreducible tempered representation of a small classical group. For these notations, see §2 below.

For the problem to determine the highest derivative of $L((\rho|\cdot|^{-x})^a, \Delta_\rho[x-1,-x]^b; T)$, Jantzen gave an explicit formula for $x = 1/2$ ([10, Theorem 3.3]). Also, he suggested a strategy for $x > 1$ ([10, §3.4, Cases 2, 3]). This strategy is an induction on x, i.e., the computation for x is reduced to the one for $x-1$. Hence it would be possible to solve this problem when $x \in (1/2)\mathbb{Z} \setminus \mathbb{Z}$. Using this strategy, he computed some examples of the Aubert duals of certain irreducible representations in the half-integral reducibility case ([10, §4]). However, since [10, Theorem 3.3] is already complicated (there are 5 cases), it is very hard to proceed with this induction.

In this paper, we give an algorithm (Theorem 4.1) to compute the highest derivative of $L((\rho|\cdot|^{-x})^a, \Delta_\rho[x-1,-x]^b; T)$. One can also write down an explicit formula (Corollary 4.2). This corollary together with Corollary 4.4 completes Jantzen's algorithm. When $a = 0$, one might prove Corollary 4.2 by a similar argument to [10, Theorem 3.3], but we will give another argument. A new ingredient for the proof is two results of Xu on A-packets [22, 23]. The first is an estimation when derivatives of unitary representations of *Arthur type* are nonzero (Lemma 3.4). Namely, for a specific tuple (a, b, T), we will find a "good" A-parameter ψ such that $L((\rho|\cdot|^{-x})^a, \Delta_\rho[x-1,-x]^b; T)$ belongs to the A-packet Π_ψ (e.g., see Example 3.5 and Proposition 3.13). To compute the highest derivative, we will use Mœglin's construction of A-packets together with Xu's combinational result [23, Theorem 6.1].

In conclusion, we obtain an algorithm to compute the highest derivatives with respect to *non-self-dual* cuspidal representations $\rho|\cdot|^x$, and we can compute the Aubert duals of irreducible representations in almost all cases. However, the derivatives considered in this paper does not cover all cases because the derivatives with respect to *self-dual* ρ is mysterious and useless. See Remark 4.5 below. In the next paper [3], the author and Mínguez will give a new idea to overcame such a difficulty, which will lead a complete algorithm for the Aubert duality.

This paper is organized as follows. In §2, we review several results in representation theory for classical groups. In particular, we explain how the highest derivatives of an irreducible representation π determine its Langlands data almost completely (Theorem 2.13). Also we recall Jantzen's algorithm to compute the highest derivatives in §2.5. In §3, we review Arthur's theory including Xu's lemma (Lemma 3.4) and Mœglin's construction (§3.4). We also give a significant correction for an erratum to [2] for the highest derivatives of tempered representations (Proposition 3.6). In addition, some results on the irreducibility of parabolically induced representations are written in §3.3. In §4, we state the main results (Theorem 4.1 and Corollaries 4.2, 4.4). Also in §4.2, we give some examples of Aubert duals, which are in the integral case. Finally, we prove Theorem 4.1 in §5.

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Notation. Let F be a non-archimedean local field of characteristic zero. We denote by W_F the Weil group of F. The norm map $|\cdot|\colon W_F \to \mathbb{R}^\times$ is normalized so that $|\mathrm{Frob}| = q^{-1}$, where $\mathrm{Frob} \in W_F$ is a fixed (geometric) Frobenius element, and q is the cardinality of the residual field of F.

Each irreducible unitary supercuspidal representation ρ of $\mathrm{GL}_d(F)$ is identified with the irreducible bounded representation of W_F of dimension d via the local Langlands correspondence for $\mathrm{GL}_d(F)$. Through this paper, we fix such a ρ.

For a p-adic group G, we denote by $\mathrm{Rep}(G)$ (resp. $\mathrm{Irr}(G)$) the set of equivalence classes of smooth admissible (resp. irreducible) representations of G. For $\Pi \in \mathrm{Rep}(G)$, we write s.s.(Π) for the semisimplification of Π.

2. Representations of classical groups

In this section, we recall some results on parabolically induced representations and Jacquet modules.

2.1. **Representations of** $\mathrm{GL}_n(F)$**.** Let $P = MN$ be a standard parabolic subgroup of $\mathrm{GL}_n(F)$, i.e., P contains the Borel subgroup consisting of upper half triangular matrices. Then the Levi subgroup M is isomorphic to $\mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_r}(F)$ with $n_1 + \cdots + n_r = n$. For smooth representations $\tau_1, \dots, \tau_r$ of $\mathrm{GL}_{n_1}(F), \dots, \mathrm{GL}_{n_r}(F)$, respectively, we denote the normalized parabolically induced representation by

$$\tau_1 \times \cdots \times \tau_r = \mathrm{Ind}_P^{\mathrm{GL}_n(F)}(\tau_1 \boxtimes \cdots \boxtimes \tau_r).$$

When $\tau_1 = \cdots = \tau_r = \tau$, we write

$$\tau^r = \underbrace{\tau \times \cdots \times \tau}_{r}.$$

A **segment** is a symbol $[x, y]$, where $x, y \in \mathbb{R}$ with $x - y \in \mathbb{Z}$ and $x \geq y$. We identify $[x, y]$ with the set $\{x, x-1, \dots, y\}$ so that $\#[x, y] = x - y + 1$. Let ρ be an irreducible (unitary) supercuspidal representation of $\mathrm{GL}_d(F)$. Then the normalized parabolically induced representation

$$\rho|\cdot|^x \times \cdots \times \rho|\cdot|^y$$

of $\mathrm{GL}_{d(x-y+1)}(F)$ has a unique irreducible subrepresentation, which is denoted by

$$\Delta_\rho[x, y]$$

and is called a **Steinberg representation**. This is an essentially discrete series representation of $\mathrm{GL}_{d(x-y+1)}(F)$.

When $\tau_i = \Delta_{\rho_i}[x_i, y_i]$ with $x_1 + y_1 \leq \cdots \leq x_r + y_r$, the parabolically induced representation $\tau_1 \times \cdots \times \tau_r$ is called a **standard module**. The Langlands classification says that it has a unique irreducible subrepresentation, which is denoted by $L(\tau_1, \dots, \tau_r)$. This notation is used also for $\tau_1 \times \cdots \times \tau_r$ which is isomorphic to a standard module.

The definitions of Steinberg representations and standard modules might be unorthodox. However, these definitions seem to be better for the computation of Jacquet modules. For a partition $(n_1, \dots, n_r)$ of n, we denote by $\mathrm{Jac}_{(n_1,\dots,n_r)}$ the normalized Jacquet functor on $\mathrm{Rep}(\mathrm{GL}_n(F))$ with respect to the standard parabolic subgroup $P = MN$ with $M \cong \mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_r}(F)$. For a segment $[x, y]$, the Jacquet modules of $\Delta_\rho[x, y]$ and $L(\rho|\cdot|^y, \dots, \rho|\cdot|^x)$ are given by

$$\mathrm{Jac}_{(d,d(x-y))}(\Delta_\rho[x, y]) = \rho|\cdot|^x \boxtimes \Delta_\rho[x-1, y],$$
$$\mathrm{Jac}_{(d,d(x-y))}(L(\rho|\cdot|^y, \dots, \rho|\cdot|^x)) = \rho|\cdot|^y \boxtimes L(\rho|\cdot|^{y+1}, \dots, \rho|\cdot|^x),$$

respectively (see [24, Propositions 3.4, 9.5]). Here, we set $\Delta_\rho[x-1, x] = \mathbf{1}_{\mathrm{GL}_0(F)}$.

An irreducibility criterion for parabolically induced representations of Steinberg representations is given by Zelevinsky.

Theorem 2.1 (Zelevinsky [24, Theorem 9.7, Proposition 4.6])**.** *Let $[x, y]$ and $[x', y']$ be segments, and let ρ and ρ' be irreducible unitary supercuspidal representations of $\mathrm{GL}_d(F)$ and $\mathrm{GL}_{d'}(F)$, respectively. Then the parabolically induced representation*

$$\Delta_\rho[x, y] \times \Delta_{\rho'}[x', y']$$

is irreducible unless the following conditions hold:

- $\rho \cong \rho'$;
- $[x, y] \not\subset [x', y']$ *and* $[x', y'] \not\subset [x, y]$ *as sets;*
- $[x, y] \cup [x', y']$ *is also a segment.*

In this case, if $x + y < x' + y'$*, then there exists an exact sequence*

$$0 \longrightarrow L(\Delta_\rho[x, y], \Delta_\rho[x', y']) \longrightarrow \Delta_\rho[x, y] \times \Delta_\rho[x', y'] \longrightarrow \Delta_\rho[x', y] \times \Delta_\rho[x, y'] \longrightarrow 0.$$

Here, when $x = y' - 1$*, we omit* $\Delta_\rho[x, y']$.

Let $\mathcal{R}_n$ be the Grothendieck group of the category of smooth representations of $\mathrm{GL}_n(F)$ of finite length. By the semisimplification, we identify the objects in this category with elements in $\mathcal{R}_n$. Elements in $\mathrm{Irr}(\mathrm{GL}_n(F))$ form a $\mathbb{Z}$-basis of $\mathcal{R}_n$. Set $\mathcal{R} = \oplus_{n \geq 0} \mathcal{R}_n$. The parabolic induction functor gives a product

$$m \colon \mathcal{R} \otimes \mathcal{R} \to \mathcal{R}, \ \tau_1 \otimes \tau_2 \mapsto \mathrm{s.s.}(\tau_1 \times \tau_2).$$

This product makes $\mathcal{R}$ an associative commutative ring. On the other hand, the Jacquet functor gives a coproduct

$$m^* \colon \mathcal{R} \to \mathcal{R} \otimes \mathcal{R}$$

which is defined by the $\mathbb{Z}$-linear extension of

$$\mathrm{Irr}(\mathrm{GL}_n(F)) \ni \tau \mapsto \sum_{k=0}^{n} \mathrm{s.s.Jac}_{(k,n-k)}(\tau).$$

Then m and m^* make $\mathcal{R}$ a graded Hopf algebra, i.e., $m^* \colon \mathcal{R} \to \mathcal{R} \otimes \mathcal{R}$ is a ring homomorphism. An antipode is given by the signed Zelevinsky involution (see [24, 9.16, Proposition]).

Definition 2.2. *Let* π *be an irreducible representation of* $\mathrm{GL}_n(F)$.

(1) *Suppose that*

$$\mathrm{s.s.Jac}_{(d,n-d)}(\pi) = \bigoplus_{i \in I} \tau_i \boxtimes \pi_i, \quad \mathrm{s.s.Jac}_{(n-d,d)}(\pi) = \bigoplus_{j \in J} \pi'_j \boxtimes \tau'_j$$

with τ_i, τ'_j *and* π_i, π'_j *being irreducible representations of* $\mathrm{GL}_d(F)$ *and* $\mathrm{GL}_{n-d}(F)$*, respectively. Then for* $x \in \mathbb{R}$*, we define the* ***left*** $\rho|\cdot|^x$***-derivative*** $L_{\rho|\cdot|^x}(\pi)$ *and the* ***right*** $\rho|\cdot|^x$***-derivative*** $R_{\rho|\cdot|^x}(\pi)$

$$L_{\rho|\cdot|^x}(\pi) = \bigoplus_{\substack{i \in I \\ \tau_i \cong \rho|\cdot|^x}} \pi_i, \quad R_{\rho|\cdot|^x}(\pi) = \bigoplus_{\substack{j \in J \\ \tau'_j \cong \rho|\cdot|^x}} \pi'_j.$$

(2) *For a positive integer* k*, we define the* k***-th left and right*** $\rho|\cdot|^x$***-derivatives*** $L^{(k)}_{\rho|\cdot|^x}(\pi)$ *and* $R^{(k)}_{\rho|\cdot|^x}(\pi)$ *by*

$$L^{(k)}_{\rho|\cdot|^x}(\pi) = \frac{1}{k!} \underbrace{L_{\rho|\cdot|^x} \circ \cdots \circ L_{\rho|\cdot|^x}}_{k}(\pi), \quad R^{(k)}_{\rho|\cdot|^x}(\pi) = \frac{1}{k!} \underbrace{R_{\rho|\cdot|^x} \circ \cdots \circ R_{\rho|\cdot|^x}}_{k}(\pi).$$

We also set $L^{(0)}_{\rho|\cdot|^x}(\pi) = R^{(0)}_{\rho|\cdot|^x}(\pi) = \pi$.

(3) *When* $L^{(k)}_{\rho|\cdot|^x}(\pi) \neq 0$ *but* $L^{(k+1)}_{\rho|\cdot|^x}(\pi) = 0$*, we call* $L^{(k)}_{\rho|\cdot|^x}(\pi)$ *the* ***highest left derivative****. We also define the* ***highest right derivative*** *similarly.*

This notion is essentially due to Jantzen [7] and Mínguez [15]. One should not confuse our derivatives with Bernstein–Zelevinsky's ones, which are not used in this paper.

By [7, Lemma 2.1.2], for any irreducible representation π, its highest derivatives $L^{(k)}_{\rho|\cdot|^x}(\pi)$ and $R^{(k')}_{\rho|\cdot|^x}(\pi)$ are irreducible. The following result was obtained by Jantzen [7] and Mínguez [15, Théorème 7.5] independently. We adopt the statements of [7, Propositions 2.1.4, 2.4.3, Theorems 2.2.1, 2.4.5]. For another reformulation, see [12, Theorem 5.11].

Theorem 2.3 (Jantzen [7], Mínguez [15])**.** *Let* $\pi = L(\Delta_\rho[x_1, y_1], \dots, \Delta_\rho[x_r, y_r])$ *be an irreducible representation.*

(1) *We may assume that* $y_1 \leq \cdots \leq y_r$*, and that if* $y_j = y_{j+1}$*, then* $x_j \geq x_{j+1}$*. For* $1 \leq j \leq r$*, define* $n_x(j) = \#\{i \leq j \mid x_i = x\}$*, and set* $n_x(0) = 0$*. Then with* $k = \max_{j \geq 0}\{n_x(j) - n_{x-1}(j)\}$*, the left derivative* $L^{(k)}_{\rho|\cdot|^x}(\pi)$ *is highest. For* $1 \leq m \leq k$*, if we set* $j_m = \min\{j \mid n_x(j) - n_{x-1}(j) = m\}$*, then* $L^{(k)}_{\rho|\cdot|}(\pi)$ *is given from* $\pi = L(\Delta_\rho[x_1, y_1], \dots, \Delta_\rho[x_r, y_r])$ *by replacing* $x_j = x$ *with* $x - 1$ *for all* $j \in \{j_1, \dots, j_k\}$*.*

(2) *We may assume that* $x_1 \leq \cdots \leq x_r$*, and that if* $x_j = x_{j+1}$*, then* $y_j \geq y_{j+1}$*. For* $1 \leq j \leq r$*, define* $n'_y(j) = \#\{i \geq r - j + 1 \mid y_i = y\}$*, and set* $n'_y(0) = 0$*. Then with* $k = \max_{j \geq 0}\{n'_y(j) - n'_{y+1}(j)\}$*, the right derivative* $R^{(k)}_{\rho|\cdot|^y}(\pi)$ *is highest. For* $1 \leq m \leq k$*, if we set* $j_m = \min\{j \mid n'_y(j) - n'_{y+1}(j) = m\}$*, then* $R^{(k)}_{\rho|\cdot|^y}(\pi)$ *is given from* $\pi = L(\Delta_\rho[x_1, y_1], \dots, \Delta_\rho[x_r, y_r])$ *by replacing* $y_j = y$ *with* $y+1$ *for all* $j \in \{j_1, \dots, j_k\}$*.*

2.2. **Representations of** $\mathrm{SO}_{2n+1}(F)$ **and** $\mathrm{Sp}_{2n}(F)$**.** We set G_n to be split $\mathrm{SO}_{2n+1}(F)$ or $\mathrm{Sp}_{2n}(F)$, i.e., G_n is the group of F-points of the split algebraic group of type B_n or C_n. Fix a Borel subgroup of G_n, and let $P = MN$ be a standard parabolic subgroup of G_n. Then the Levi part M is of the form $\mathrm{GL}_{k_1}(F) \times \cdots \times \mathrm{GL}_{k_r}(F) \times G_{n_0}$ such that $k_1 + \cdots + k_r + n_0 = n$. For a smooth representation $\tau_1 \boxtimes \cdots \boxtimes \tau_r \boxtimes \pi_0$ of M, we denote the normalized parabolically induced representation by

$$\tau_1 \times \cdots \times \tau_r \rtimes \pi_0 = \mathrm{Ind}_P^{G_n}(\tau_1 \boxtimes \cdots \boxtimes \tau_r \boxtimes \pi_0).$$

The functor $\mathrm{Ind}_P^{G_n} \colon \mathrm{Rep}(M) \to \mathrm{Rep}(G_n)$ is exact.

On the other hand, for a smooth representation π of G_n, we denote the normalized Jacquet module with respect to P by

$$\mathrm{Jac}_P(\pi),$$

and its semisimplification by $\mathrm{s.s.Jac}_P(\pi)$. The functor $\mathrm{Jac}_P \colon \mathrm{Rep}(G_n) \to \mathrm{Rep}(M)$ is exact. The Frobenius reciprocity asserts that

$$\mathrm{Hom}_{G_n}(\pi, \mathrm{Ind}_P^{G_n}(\sigma)) \cong \mathrm{Hom}_M(\mathrm{Jac}_P(\pi), \sigma)$$

for $\pi \in \mathrm{Rep}(G_n)$ and $\sigma \in \mathrm{Rep}(M)$.

The maximal standard parabolic subgroup with Levi $\mathrm{GL}_k(F) \times G_{n-k}$ is denoted by $P_k = M_k N_k$ for $0 \leq k \leq n$. Let $\mathcal{R}(G_n)$ be the Grothendieck group of the category of smooth representations of G_n of finite length. Set $\mathcal{R}(G) = \oplus_{n \geq 0} \mathcal{R}(G_n)$. The parabolic induction defines a module structure

$$\rtimes \colon \mathcal{R} \otimes \mathcal{R}(G) \to \mathcal{R}(G), \ \tau \otimes \pi \mapsto \mathrm{s.s.}(\tau \rtimes \pi),$$

and the Jacquet functor defines a comodule structure

$$\mu^* \colon \mathcal{R}(G) \to \mathcal{R} \otimes \mathcal{R}(G)$$

by

$$\mathrm{Irr}(G_n) \ni \pi \mapsto \sum_{k=0}^{n} \mathrm{s.s.Jac}_{P_k}(\pi).$$

Tadić established a formula to compute μ^* for parabolically induced representations. The contragredient functor $\tau \mapsto \tau^\vee$ defines an automorphism $\vee \colon \mathcal{R} \to \mathcal{R}$ in a natural way. Let $s \colon \mathcal{R} \otimes \mathcal{R} \to \mathcal{R} \otimes \mathcal{R}$ be the homomorphism defined by $\sum_i \tau_i \otimes \tau_i' \mapsto \sum_i \tau_i' \otimes \tau_i$.

Theorem 2.4 (Tadić [20]). *Consider the composition*

$$M^* = (m \otimes \mathrm{id}) \circ (\vee \otimes m^*) \circ s \circ m^* \colon \mathcal{R} \to \mathcal{R} \otimes \mathcal{R}.$$

Then for the maximal parabolic subgroup $P_k = M_k N_k$ of G_n and for an admissible representation $\tau \boxtimes \pi$ of M_k, we have

$$\mu^*(\tau \rtimes \pi) = M^*(\tau) \rtimes \mu^*(\pi).$$

Here, the action $\rtimes$ of $\mathcal{R} \otimes \mathcal{R}$ on $\mathcal{R} \otimes \mathcal{R}(G)$ is defined by $(\tau_1 \otimes \tau_2) \rtimes (\tau \otimes \pi) = (\tau_1 \times \tau) \otimes (\tau_2 \rtimes \pi)$.

For a general reductive group G over F, irreducible smooth representations of $G(F)$ are classified by the **Langlands classification**. For a detail, see [11]. Here, we recall this for G_n.

The Langlands classification asserts that for any irreducible representation π of G_n, there exists an irreducible representation $\tau_1 \boxtimes \cdots \boxtimes \tau_r \boxtimes \sigma$ of a Levi subgroup $M = \mathrm{GL}_{k_1}(F) \times \cdots \times \mathrm{GL}_{k_r}(F) \times G_{n_0}$ of some standard parabolic subgroup P satisfying that

- $\tau_i = \Delta_{\rho_i}[x_i, y_i]$ for some irreducible unitary supercuspidal representation ρ_i of $\mathrm{GL}_{d_i}(F)$, and some segment $[x_i, y_i]$ with $x_i \geq y_i$;
- σ is an irreducible tempered representation of G_{n_0};
- $x_1 + y_1 \leq \cdots \leq x_r + y_r < 0$

such that π is a unique irreducible subrepresentation of the parabolically induced representation $\tau_1 \times \cdots \times \tau_r \rtimes \sigma$. In this case, we write

$$\pi = L(\tau_1, \ldots, \tau_r; \sigma),$$

and call it the **Langlands subrepresentation** of $\tau_1 \times \cdots \times \tau_r \rtimes \sigma$. Note that $L(\tau_1, \ldots, \tau_r; \sigma) \cong L(\tau_1', \ldots, \tau_{r'}'; \sigma')$ if and only if $\tau_1 \times \cdots \times \tau_r \rtimes \sigma \cong \tau_1' \times \cdots \times \tau_{r'}' \rtimes \sigma'$. We refer $(\tau_1, \ldots, \tau_r; \sigma)$ as the **Langlands data** of π.

2.3. **Derivatives.**

Definition 2.5. *Let π be a smooth representation of G_n.*

(1) *Consider* $\mathrm{s.s.Jac}_{P_d}(\pi)$ *(and a fixed irreducible supercuspidal unitary representation ρ of $\mathrm{GL}_d(F)$). If*

$$\mathrm{s.s.Jac}_{P_d}(\pi) = \bigoplus_{i \in I} \tau_i \boxtimes \pi_i$$

with τ_i *(resp.* π_i*) being an irreducible representation of* $\mathrm{GL}_d(F)$ *(resp.* G_{n-d}*), for* $x \in \mathbb{R}$*, we define a* ***partial Jacquet module*** $\mathrm{Jac}_{\rho|\cdot|^x}(\pi)$ *by*

$$\mathrm{Jac}_{\rho|\cdot|^x}(\pi) = \bigoplus_{\substack{i \in I \\ \tau_i \cong \rho|\cdot|^x}} \pi_i.$$

This is a representation of G_{n-d}*. For pairs* $(\rho_1, x_1), \dots, (\rho_t, x_t)$*, we also set* $\mathrm{Jac}_{\rho_1|\cdot|^{x_1}, \dots, \rho_t|\cdot|^{x_t}} = \mathrm{Jac}_{\rho_t|\cdot|^{x_t}} \circ \cdots \circ \mathrm{Jac}_{\rho_1|\cdot|^{x_1}}$*.*

(2) *For a non-negative integer* k*, the* ***k-th*** $\rho|\cdot|^x$***-derivative of*** π *is the k-th composition*

$$D^{(k)}_{\rho|\cdot|^x}(\pi) = \frac{1}{k!} \underbrace{\mathrm{Jac}_{\rho|\cdot|^x} \circ \cdots \circ \mathrm{Jac}_{\rho|\cdot|^x}}_{k}(\pi).$$

If $D^{(k)}_{\rho|\cdot|^x}(\pi) \neq 0$ *but* $D^{(k+1)}_{\rho|\cdot|^x}(\pi) = 0$*, we call* $D^{(k)}_{\rho|\cdot|^x}(\pi)$ ***the highest*** $\rho|\cdot|^x$***-derivative of*** π*.*

The derivative $D^{(k)}_{\rho|\cdot|^x}(\pi)$ is a representation of some group $G_{n'}$ of the same type as G_n. In fact, it is characterized so that

$$\mathrm{s.s.Jac}_{P_{dk}}(\pi) = (\rho|\cdot|^x)^k \otimes D^{(k)}_{\rho|\cdot|^x}(\pi) + \sum_{i \in I} \tau_i \otimes \pi_i$$

where τ_i is an irreducible representation of $\mathrm{GL}_{dk}(F)$ such that $\tau_i \not\cong (\rho|\cdot|^x)^k$. It follows from the transitivity of Jacquet modules and the fact that for any irreducible representation τ of $\mathrm{GL}_{dk}(F)$,

$$L^{(k)}_{\rho|\cdot|^x}(\tau) = \begin{cases} \mathbf{1}_{\mathrm{GL}_0(F)} & \text{if } \tau \cong (\rho|\cdot|^x)^k, \\ 0 & \text{otherwise.} \end{cases}$$

The following is essentially the same as [8, Lemma 3.1.3]. For the convenience of the readers, we give a proof.

Proposition 2.6. *Let* π *be an irreducible representation of* G_n*, and* $D^{(k)}_{\rho|\cdot|^x}(\pi)$ *be the highest derivative.*

(1) *There exists an irreducible representation* π' *of some group* $G_{n'}$ *of the same type as* G_n *such that* $D^{(k)}_{\rho|\cdot|^x}(\pi) = m \cdot \pi'$ *with a positive integer* m*. Moreover,* π *is an irreducible subrepresentation of the parabolically induced representation*

$$\underbrace{\rho|\cdot|^x \times \cdots \times \rho|\cdot|^x}_{k} \rtimes \pi'.$$

(2) *If* $\rho^\vee|\cdot|^{-x} \not\cong \rho|\cdot|^x$ *(in particular, if* $x \neq 0$*), then* $D^{(k)}_{\rho|\cdot|^x}(\pi)$ *is irreducible, i.e.,* $m = 1$*. Moreover, in this case,* π *is a unique irreducible subrepresentation of the above parabolically induced representation.*

Proof. If $D^{(k)}_{\rho|\cdot|^x}(\pi) \neq 0$, we can find an irreducible representation π' of some group $G_{n'}$ such that $\mathrm{Jac}_{P_{dk}}(\pi) \twoheadrightarrow (\rho|\cdot|^x)^k \otimes \pi'$, which is equivalent that

$$\pi \hookrightarrow \underbrace{\rho|\cdot|^x \times \cdots \times \rho|\cdot|^x}_{k} \rtimes \pi' = (\rho|\cdot|^x)^k \rtimes \pi'.$$

Note that $\mathrm{Jac}_{\rho|\cdot|^x}(\pi') = 0$ since $D^{(k)}_{\rho|\cdot|^x}(\pi)$ is the highest derivative. By Theorem 2.4, we see that

$$\mathrm{Jac}_{\rho|\cdot|^x}\left((\rho|\cdot|^x)^k \rtimes \pi'\right) = m_k \cdot (\rho|\cdot|^x)^{k-1} \rtimes \pi',$$

where

$$m_k = \begin{cases} k & \text{if } \rho^\vee|\cdot|^{-x} \not\cong \rho|\cdot|^x, \\ 2k & \text{if } \rho^\vee|\cdot|^{-x} \cong \rho|\cdot|^x. \end{cases}$$

This implies that

$$D^{(k)}_{\rho|\cdot|^x}\left((\rho|\cdot|^x)^k \rtimes \pi'\right) = c_k \cdot \pi'$$

with

$$c_k = \begin{cases} 1 & \text{if } \rho^\vee|\cdot|^{-x} \not\cong \rho|\cdot|^x, \\ 2^k & \text{if } \rho^\vee|\cdot|^{-x} \cong \rho|\cdot|^x. \end{cases}$$

Therefore $D^{(k)}_{\rho|\cdot|^x}(\pi) = m \cdot \pi'$ with a positive integer $m \leq c_k$. This shows (1) and the first assertion of (2). For the last assertion of (2), we note that if π_1 is an irreducible subrepresentation of $(\rho|\cdot|^x)^k \rtimes \pi'$, then $D^{(k)}_{\rho|\cdot|^x}(\pi_1) \neq 0$. However, when $\rho^\vee|\cdot|^{-x} \not\cong \rho|\cdot|^x$, we have

$$D^{(k)}_{\rho|\cdot|^x}\left((\rho|\cdot|^x)^k \rtimes \pi' - \pi\right) = 0.$$

This means that $(\rho|\cdot|^x)^k \rtimes \pi'$ contains π as a unique irreducible subrepresentation (with multiplicity one). □

Remark 2.7. *One might consider the same notions for* $\mathrm{SO}_{2n}(F)$ *or* $\mathrm{O}_{2n}(F)$. *However, the descriptions are more complicated in these cases since Tadić's formula needs a modification. See* [21, §5]. *We do not treat these cases in this paper.*

2.4. **ρ-data.** In this subsection, we introduce the ρ-data of irreducible representations. See §4.2 below for examples.

Definition 2.8. *Let* π *be an irreducible representation of* G_n. *For* $\epsilon \in \{\pm\}$, *the* ρ***-data*** *of* π *is of the form*

$$M^\epsilon_\rho(\pi) = [(x_1,k_1), \dots, (x_t,k_t); \pi_0],$$

where x_i *is a real number and* k_i *is a positive integer, defined inductively as follows.*

(1) *If* $\mathrm{Jac}_{\rho|\cdot|^x}(\pi) = 0$ *for any* $x \in \mathbb{R}$, *we set* $M^\epsilon_\rho(\pi) = [\pi]$ *(so that* $t = 0$ *and* $\pi_0 = \pi$*).*

(2) *If* $\mathrm{Jac}_{\rho|\cdot|^x}(\pi) \neq 0$ *for some* $x \in \mathbb{R}$, *we set*

$$x_1 = \begin{cases} \max\{x \in \mathbb{R} \mid \mathrm{Jac}_{\rho|\cdot|^x}(\pi) \neq 0\} & \text{if } \epsilon = +, \\ \min\{x \in \mathbb{R} \mid \mathrm{Jac}_{\rho|\cdot|^x}(\pi) \neq 0\} & \text{if } \epsilon = - \end{cases}$$

and $k_1 \geq 1$ *to be such that* $D^{(k_1)}_{\rho|\cdot|^{x_1}}(\pi)$ *is the highest* $\rho|\cdot|^{x_1}$*-derivative of* π. *By Proposition 2.6, we can write* $D^{(k_1)}_{\rho|\cdot|^{x_1}}(\pi) = m \cdot \pi'$ *for some irreducible representation* π'. *Then we define*

$$M^\epsilon_\rho(\pi) = \left[(x_1,k_1); M^\epsilon_\rho(\pi')\right].$$

Let π be an irreducible representation of G_n. Then one can define another irreducible representation $\hat{\pi}$, which is called the **Aubert dual** of π (see [4]). It is known by [4, Théorème 1.7] that

- $\hat{\hat{\pi}} = \pi$;
- if π is supercuspidal, then $\hat{\pi} = \pi$;
- $D^{(k)}_{\rho|\cdot|^x}(\hat{\pi})$ is the Aubert dual of $D^{(k)}_{\rho^\vee|\cdot|^{-x}}(\pi)$.

In particular, for $\epsilon \in \{\pm\}$, if

$$M^\epsilon_\rho(\pi) = [(x_1, k_1), \dots, (x_t, k_t); \pi_0],$$

then

$$M^{-\epsilon}_{\rho^\vee}(\hat{\pi}) = [(-x_1, k_1), \dots, (-x_t, k_t); \hat{\pi}_0].$$

We will use $M^-_\rho(\pi)$ mainly. By taking the Aubert dual, several properties of M^-_ρ can be translated into ones of $M^+_{\rho^\vee}$. The following is the most important property of M^-_ρ.

Theorem 2.9. *Let π be an irreducible representation of G_n. Then the ρ-data $M^-_\rho(\pi)$ can be rewritten as*

$$M^-_\rho(\pi) = \left[(x^{(1)}_1, k^{(1)}_1), \dots, (x^{(1)}_{t_1}, k^{(1)}_{t_1}), \dots, (x^{(r)}_1, k^{(r)}_1), \dots, (x^{(r)}_{t_r}, k^{(r)}_{t_r}); \pi_0\right]$$

such that

- $x^{(i)}_{j+1} = x^{(i)}_j - 1$ *and* $k^{(i)}_{j+1} \leq k^{(i)}_j$ *for* $1 \leq i \leq r$ *and* $1 \leq j \leq t_i - 1$;
- $x^{(1)}_1 < x^{(2)}_1 < \dots < x^{(r)}_1$.

Moreover, if we set $\tau^{(i)}_j = \Delta_\rho[x^{(i)}_1, x^{(i)}_j]$, then $\tau^{(i)}_j \times \tau^{(i)}_{j'} \cong \tau^{(i)}_{j'} \times \tau^{(i)}_j$ for any $1 \leq j, j' \leq t_r$, and π is an irreducible subrepresentation of

$$\left(\bigtimes_{j=1}^{t_1} \left(\tau^{(1)}_j\right)^{k^{(1)}_j - k^{(1)}_{j+1}}\right) \times \dots \times \left(\bigtimes_{j=1}^{t_r} \left(\tau^{(r)}_j\right)^{k^{(r)}_j - k^{(r)}_{j+1}}\right) \rtimes \pi_0,$$

where we set $k^{(i)}_{t_i+1} = 0$ for $1 \leq i \leq r$.

Proof. Write $M^-_\rho(\pi) = [(x_1, k_1), \dots, (x_t, k_t); \pi_0]$. Note that $x_{i+1} \neq x_i$ by definition. Suppose that $x_{i+1} < x_i$. Replacing π with the (unique) irreducible representation appearing in $D^{(k_{i-1})}_{\rho|\cdot|^{x_{i-1}}} \circ \dots \circ D^{(k_1)}_{\rho|\cdot|^{x_1}}(\pi)$, we may assume that $i = 1$. If we write $D^{(k_2)}_{\rho|\cdot|^{x_2}} \circ D^{(k_1)}_{\rho|\cdot|^{x_1}}(\pi) = m \cdot \pi'$, then

$$\pi \hookrightarrow \underbrace{\rho|\cdot|^{x_1} \times \dots \times \rho|\cdot|^{x_1}}_{k_1} \times \underbrace{\rho|\cdot|^{x_2} \times \dots \times \rho|\cdot|^{x_2}}_{k_2} \rtimes \pi'.$$

If $x_2 < x_1$ but $x_2 \neq x_1 - 1$, then $\rho|\cdot|^{x_1} \times \rho|\cdot|^{x_2} \cong \rho|\cdot|^{x_2} \times \rho|\cdot|^{x_1}$ so that $\mathrm{Jac}_{\rho|\cdot|^{x_2}}(\pi) \neq 0$. This contradicts the definition of x_1. Hence $x_2 = x_1 - 1$. In this case, by [24, §11.3], the above inclusion factors through

$$\pi \hookrightarrow L(\rho|\cdot|^{x_2}, \rho|\cdot|^{x_1})^{k'_0} \times \Delta_\rho[x_1, x_2]^{k_0} \times (\rho|\cdot|^{x_1})^{k'_1} \times (\rho|\cdot|^{x_2})^{k'_2} \rtimes \pi'$$

for some non-negative integers k_0, k'_0, k'_1, k'_2 with $\min\{k'_1, k'_2\} = 0$. Here, we note that $L(\rho|\cdot|^{x_2}, \rho|\cdot|^{x_1})$ commutes with all of $\rho|\cdot|^{x_1}$, $\rho|\cdot|^{x_2}$ and $\Delta_\rho[x_1, x_2]$. Since $\mathrm{Jac}_{\rho|\cdot|^{x_2}}(L(\rho|\cdot|^{x_2}, \rho|\cdot|^{x_1})) \neq 0$ but $\mathrm{Jac}_{\rho|\cdot|^{x_2}}(\pi) = 0$, we must have $k'_0 = 0$. Namely, we have

$$\pi \hookrightarrow (\rho|\cdot|^{x_1})^{k_1 - k_0} \times \Delta_\rho[x_1, x_2]^{k_0} \times (\rho|\cdot|^{x_2})^{k_2 - k_0} \rtimes \pi',$$

where $k_0 = \min\{k_1, k_2\}$. If $k_2 > k_1$ so that $k_0 = k_1$, by Theorem 2.1, we would have

$$\pi \hookrightarrow (\rho|\cdot|^{x_2})^{k_2 - k_1} \times \Delta_\rho[x_1, x_2]^{k_0} \rtimes \pi',$$

which implies that $\mathrm{Jac}_{\rho|\cdot|^{x_2}}(\pi) \neq 0$. This contradicts the definition of x_1. Hence we have $k_2 \leq k_1$.

We conclude that if $x_1 > \cdots > x_a$, then $x_{j+1} = x_j - 1$ and $k_{j+1} \leq k_j$ for $1 \leq j \leq a-1$. Moreover, in this case, π is a subrepresentation of

$$\Delta_\rho[x_1, x_1]^{k_1-k_2} \times \cdots \times \Delta_\rho[x_1, x_{a-1}]^{k_{a-1}-k_a} \times \Delta_\rho[x_1, x_a]^{k_a} \rtimes \pi'$$

for some irreducible representation π'. Now suppose that $x_a < x_{a+1}$.

- If $x_a < x_{a+1} < x_1$ and $x_{a+1} \neq x_b$ for any $1 \leq b < a$, by Theorem 2.1, we have $\mathrm{Jac}_{\rho|\cdot|^{x_{a+1}}}(\pi) \neq 0$. This contradicts the definition of x_1.
- If $x_{a+1} = x_b$ for some $1 \leq b < a$, by Theorem 2.1, we have

$$D^{(k_b+1)}_{\rho|\cdot|^{x_b}} \circ D^{(k_{b-1})}_{\rho|\cdot|^{x_{b-1}}} \circ \cdots \circ D^{(k_1)}_{\rho|\cdot|^{x_1}}(\pi) \neq 0.$$

This contradicts the definition of k_b.

Hence we must have $x_{a+1} > x_1$. Therefore $M^-_\rho(\pi)$ can be written as in the statement. The above argument also shows the other statements. □

To obtain some consequences, let us prepare a useful lemma.

Lemma 2.10. *Let π be an irreducible representation of G_n. Write $M^-_\rho(\pi) = [(x_1, k_1), \ldots, (x_t, k_t); \pi_0]$. Suppose that there exists an inclusion*

$$\pi \hookrightarrow \Delta_\rho[x, y] \rtimes \pi'$$

with some irreducible representation π' such that $x + y < 0$. Then there exists $i \leq t - (x - y)$ such that $x_i = x, x_{i+1} = x - 1, \ldots, x_{i+x-y} = y$. Moreover, if $i > 1$, then $x_{i-1} < x$.

Proof. Write $M^-_\rho(\pi') = \left[(x'_1, k'_1), \ldots, (x'_{t'}, k'_{t'}); \pi'_0\right]$. We prove the lemma by induction on t'. Note that $x_1 \leq x < -y$. If $t' = 0$ or $x'_1 \geq x$, then $i = 1$ and the assertion is trivial. Now we write $\pi \hookrightarrow (\rho|\cdot|^{x_1})^{k_1} \rtimes \pi_1$ and $\pi' \hookrightarrow (\rho|\cdot|^{x'_1})^{k'_1} \rtimes \pi'_1$ for some irreducible representations π_1 and π'_1. If $x'_1 < x$ and $x'_1 \neq y - 1$, by Theorem 2.1, we have

$$\pi \hookrightarrow (\rho|\cdot|^{x'_1})^{k'_1} \times \Delta_\rho[x, y] \rtimes \pi'_1$$

so that $x_1 = x'_1$, $k_1 = k'_1$ and $\pi_1 \hookrightarrow \Delta_\rho[x, y] \rtimes \pi'_1$. By applying the induction hypothesis, we obtain the assertion.

Finally, we assume that $x'_1 = y - 1$. Since we have an exact sequence

$$0 \longrightarrow \Delta_\rho[x, y-1] \longrightarrow \Delta_\rho[x, y] \times \rho|\cdot|^{y-1} \longrightarrow \rho|\cdot|^{y-1} \times \Delta_\rho[x, y],$$

by Theorem 2.1, we see that π can be embedded into

$$(\rho|\cdot|^{y-1})^{k'_1} \times \Delta_\rho[x, y] \rtimes \pi'_1, \quad \text{or}$$
$$(\rho|\cdot|^{y-1})^{k'_1-1} \times \Delta_\rho[x, y-1] \rtimes \pi'_1.$$

In the former case, we have $x_1 = y - 1$, $k_1 = k'_1$ and $\pi_1 \hookrightarrow \Delta_\rho[x, y] \rtimes \pi'_1$. In the latter case, we have one of the following:

- $(x_1, k_1) = (y - 1, k'_1 - 1)$ and $\pi_1 \hookrightarrow \Delta_\rho[x, y-1] \rtimes \pi'_1$;
- $k'_1 = 1$ and $\pi \hookrightarrow \Delta_\rho[x, y-1] \rtimes \pi'_1$.

In all cases, the induction hypothesis gives the assertion. □

Corollary 2.11. *Let π be an irreducible representation of G_n. Write*

$$M_\rho^-(\pi) = \left[(x_1^{(1)}, k_1^{(1)}), \dots, (x_{t_1}^{(1)}, k_{t_1}^{(1)}), \dots, (x_1^{(r)}, k_1^{(r)}), \dots, (x_{t_r}^{(r)}, k_{t_r}^{(r)}); \pi_0\right]$$

as in Theorem 2.9. Suppose that $(x, y) = (x_1^{(i)}, x_{t_i}^{(i)})$ satisfies that $x + y < 0$ and $x + y \leq x_1^{(j)} + x_{t_j}^{(j)}$ for any $j < i$. Then there exists an irreducible representation π' such that

$$\pi \hookrightarrow \Delta_\rho[x, y] \rtimes \pi'.$$

For any such π', the ρ-data $M_\rho^-(\pi')$ is obtained from $M_\rho^-(\pi)$ by replacing $k_1^{(i)}, \dots, k_{t_i}^{(i)}$ with $k_1^{(i)} - 1, \dots, k_{t_i}^{(i)} - 1$, respectively.

Proof. By the assumption, we notice that $x < -y$ and $x > x_1^{(j)} \geq x_{t_j}^{(j)} > y$ for any $j < i$. By Theorems 2.9 and 2.1, one can find an irreducible representation π' such that

$$\pi \hookrightarrow \Delta_\rho[x, y] \rtimes \pi'.$$

We compute the ρ-data $M_\rho^-(\pi') = [(x'_1, k'_1), \dots, (x'_t, k'_t); \pi'_0]$ by induction on $\sum_{j=1}^r t_j$ as in the proof of Lemma 2.10.

If $x'_1 \geq x$, then $x_1^{(1)} = x$ so that $i = 1$. In this case, the assertion is trivial. If $x'_1 < x$ and $x'_1 \neq y - 1$, then $i > 1$ and $(x'_1, k'_1) = (x_1^{(1)}, k_1^{(1)})$. Moreover, by Theorem 2.1, we have

$$D_{\rho|\cdot|^{x'_1}}^{(k'_1)}(\pi) \hookrightarrow \Delta_\rho[x, y] \rtimes D_{\rho|\cdot|^{x'_1}}^{(k'_1)}(\pi').$$

By the induction hypothesis, we obtain the assertion.

To complete the proof, it suffices to show that x'_1 never equals to $y - 1$. Suppose that $x'_1 = y - 1$. Then there exists an irreducible representation π'' such that $\pi' \hookrightarrow \rho|\cdot|^{y-1} \rtimes \pi''$ so that

$$\pi \hookrightarrow \Delta_\rho[x, y] \times \rho|\cdot|^{y-1} \rtimes \pi''.$$

Since we have an exact sequence

$$0 \longrightarrow \Delta_\rho[x, y-1] \longrightarrow \Delta_\rho[x, y] \times \rho|\cdot|^{y-1} \longrightarrow \rho|\cdot|^{y-1} \times \Delta_\rho[x, y],$$

we have $\mathrm{Jac}_{\rho|\cdot|^{y-1}}(\pi) \neq 0$ or $\pi \hookrightarrow \Delta_\rho[x, y-1] \rtimes \pi''$. Lemma 2.10 eliminates the latter case. In the former case, we must have $x_1^{(1)} \leq y - 1 < x$. This implies that $i > 1$ and

$$x_1^{(1)} + x_{t_1}^{(1)} \leq 2x_1^{(1)} \leq 2(y-1) < 2y \leq x + y,$$

which contradicts our assumption. This completes the proof. □

Also we can reformulate Casselman's tempered-ness criterion as follows.

Corollary 2.12. *Let π be an irreducible representation of G_n. Then π is tempered if and only if for any ρ, we can write*

$$M_\rho^-(\pi) = \left[(x_1^{(1)}, k_1^{(1)}), \dots, (x_{t_1}^{(1)}, k_{t_1}^{(1)}), \dots, (x_1^{(r)}, k_1^{(r)}), \dots, (x_{t_r}^{(r)}, k_{t_r}^{(r)}); \pi_0\right]$$

as in Theorem 2.9 such that $x_1^{(i)} + x_{t_i}^{(i)} \geq 0$ for any $1 \leq i \leq r$.

Proof. Suppose first that some ρ-data $M_\rho^-(\pi)$ of the above form has an index i such that $x_1^{(i)} + x_{t_i}^{(i)} < 0$. We take i so that $x_1^{(i)} + x_{t_i}^{(i)}$ achieves the minimum value. Then by Theorem 2.9, we find an irreducible representation π' such that

$$\pi \hookrightarrow \Delta_\rho[x_1^{(i)}, x_{t_i}^{(i)}] \rtimes \pi',$$

or equivalently,

$$\mathrm{Jac}_{P_{dt_i}}(\pi) \twoheadrightarrow \Delta_\rho[x_1^{(i)}, x_{t_i}^{(i)}] \otimes \pi'.$$

By the Casselman criterion, we see that π is not tempered.

Suppose conversely that π is not tempered. Then by the Casselman criterion, there exists an irreducible representation $\tau' \boxtimes \pi'$ of $\mathrm{GL}_k(F) \times G_{n-k}$ for some k such that $\pi \hookrightarrow \tau' \rtimes \pi'$, and such that the central character of τ' is of the form $\omega_u|\cdot|^s$ with ω_u being unitary and $s < 0$. If we write $\tau' = L(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_r}[x_r, y_r])$ where ρ_i is an irreducible unitary supercuspidal representation of $\mathrm{GL}_{d_i}(F)$ and $x_1+y_1 \leq \dots \leq x_r+y_r$, we have $s = \sum_i d_i(x_i+y_i)(x_i-y_i+1)/2$. Since $s < 0$, we must have $x_1 + y_1 < 0$. Hence we may replace τ' with $\Delta_{\rho_1}[x_1, y_1]$. In other words, there exist ρ, $[x, y]$ and π' such that $\pi \hookrightarrow \Delta_\rho[x, y] \rtimes \pi'$ with $x + y < 0$. By Lemma 2.10, we conclude that there exists i such that $x_1^{(i)} = x$ and $x_{t_i}^{(i)} \leq y$ so that $x_1^{(i)}+x_{t_i}^{(i)} \leq x+y < 0$. □

Now we compare the ρ-data with the Langlands data.

Theorem 2.13. *Let π be an irreducible representation of G_n. Write*

$$M_\rho^-(\pi) = \left[(x_1^{(1)}, k_1^{(1)}), \dots, (x_{t_1}^{(1)}, k_{t_1}^{(1)}), \dots, (x_1^{(r)}, k_1^{(r)}), \dots, (x_{t_r}^{(r)}, k_{t_r}^{(r)}); \pi_0\right]$$

and set $\tau_j^{(i)} = \Delta_\rho[x_1^{(i)}, x_j^{(i)}]$ as in Theorem 2.9. Suppose that $\pi = L(\tau_1, \dots, \tau_l; \sigma)$ with $\tau_i = \Delta_{\rho_i}[x_i, y_i]$. Then as multi-sets, $\{\tau_i \mid 1 \leq i \leq l,\ \rho_i \cong \rho\}$ is equal to the multi-set of consisting of $\tau_j^{(i)}$ with multiplicity $k_j^{(i)} - k_{j+1}^{(i)}$ for i, j such that $x_1^{(i)} + x_j^{(i)} < 0$. Moreover, $\{M_\rho^-(\sigma)\}_\rho$ can be computed from $\{M_\rho^-(\pi)\}_\rho$ by Corollary 2.11.

Proof. We prove the assertion by induction on n. If $x_1^{(i)}+x_j^{(i)} \geq 0$ for all i, j, then by Corollary 2.12, π is tempered so that $\sigma = \pi$. Otherwise, take the minimum i such that $x_1^{(i)} + x_{t_i}^{(i)} < 0$. Then by Corollary 2.11, we have

$$\pi \hookrightarrow \tau_{t_i}^{(i)} \rtimes \pi'$$

with an irreducible representation π' such that $M_\rho^-(\pi')$ is obtained from $M_\rho^-(\pi)$ by replacing $k_1^{(i)}, \dots, k_{t_i}^{(i)}$ with $k_1^{(i)} - 1, \dots, k_{t_i}^{(i)} - 1$, respectively. By the induction hypothesis, we can obtain the Langlands data of π' as in the assertion. If we write $\pi = L(\tau'_1, \dots, \tau'_{l'}; \sigma')$ with $\tau'_{i'} = \Delta_{\rho'_{i'}}[x'_{i'}, y'_{i'}]$, by the choice of i, we see that $x_1^{(i)} + x_j^{(i)} \leq x'_{i'} + y'_{i'}$ whenever $\rho'_{i'} \cong \rho$. Therefore, the Langlands data of π is obtained from that of π' by inserting $\tau_{t_i}^{(i)}$ in the relevant place. □

In fact, by Proposition 3.8 below, the tempered representation σ is determined almost completely by $\{M_\rho^-(\sigma)\}_\rho$.

Unfortunately, the map $\pi \mapsto M_\rho^\epsilon(\pi)$ is not injective. For example, when π_0 is supercuspidal and $\rho \rtimes \pi_0$ is reducible, this induced representation is semisimple of length two, i.e., $\rho \rtimes \pi_0 = \pi_1 \oplus \pi_2$ and $\pi_1 \not\cong \pi_2$. However $M_\rho^\epsilon(\pi_1) = M_\rho^\epsilon(\pi_2) = [(0, 1); \pi_0]$ for $\epsilon \in \{\pm\}$.

2.5. **Jantzen's algorithm.** Let $\pi = L(\tau_1, \dots, \tau_r; \sigma)$ be an irreducible representation of G_n. Suppose that ρ is self-dual and $x \in (1/2)\mathbb{Z}$. We recall Jantzen's algorithm ([10, §3.3]) to compute the highest derivative $D^{(k)}_{\rho|\cdot|^x}(\pi)$ with $x > 0$.

(1) Write $\tau_1 \times \dots \times \tau_r \cong \tau^{(1)}_1 \times \dots \times \tau^{(1)}_{r_1} \times \Delta_\rho[x-1, -x]^b$ with b maximal. (Hence, $\{\tau^{(1)}_1, \dots, \tau^{(1)}_{r_1}\} = \{\tau_1, \dots, \tau_r\} \setminus \{\Delta_\rho[x-1, -x]\}$.) Then

$$\pi \hookrightarrow L(\tau^{(1)}_1, \dots, \tau^{(1)}_{r_1}) \times \Delta_\rho[x-1, -x]^b \rtimes \sigma.$$

(2) Compute the right highest derivative $R^{(a)}_{\rho|\cdot|^{-x}}(L(\tau^{(1)}_1, \dots, \tau^{(1)}_{r_1})) = L(\tau^{(2)}_1, \dots, \tau^{(2)}_{r_2})$. Then Jantzen's Claim 1 says that

$$\pi \hookrightarrow L(\tau^{(2)}_1, \dots, \tau^{(2)}_{r_2}) \rtimes L((\rho|\cdot|^{-x})^a, \Delta_\rho[x-1, -x]^b, \sigma).$$

(3) Assume for a moment that we were able to compute the highest derivative $D^{(k_1)}_{\rho|\cdot|^x}(L((\rho|\cdot|^{-x})^a, \Delta_\rho[x-1, -x]^b, \sigma)) = \pi_1$. Then Jantzen's Claim 2 says that

$$\pi \hookrightarrow L(\tau^{(2)}_1, \dots, \tau^{(2)}_{r_2}, (\rho|\cdot|^x)^{k_1}) \rtimes \pi_1.$$

(4) Compute the left highest derivative $L^{(k)}_{\rho|\cdot|^x}(L(\tau^{(2)}_1, \dots, \tau^{(2)}_{r_2}, (\rho|\cdot|^x)^{k_1})) = L(\tau^{(3)}_1, \dots, \tau^{(3)}_{r_3})$. Then $D^{(k)}_{\rho|\cdot|^x}(\pi)$ is the highest derivative, and

$$D^{(k)}_{\rho|\cdot|^x}(\pi) \hookrightarrow L(\tau^{(3)}_1, \dots, \tau^{(3)}_{r_3}) \rtimes \pi_1.$$

(5) By Theorem 2.3, one can write $\tau^{(3)}_1 \times \dots \times \tau^{(3)}_{r_3} \cong \tau^{(4)}_1 \times \dots \times \tau^{(4)}_{r_4} \times (\rho|\cdot|^x)^{k_2}$ such that $\tau^{(4)}_i$ has a negative central exponent, and $k_2 \leq k_1$.

(6) There exists a unique irreducible representation π_2 of the form $L((\rho|\cdot|^{-x})^{a'}, \Delta_\rho[x-1, -x]^{b'}; \sigma')$ such that $D^{(k_2)}_{\rho|\cdot|^x}(\pi_2) = \pi_1$ is the highest derivative. Jantzen's Claim 3 says that

$$D^{(k)}_{\rho|\cdot|^x}(\pi) \hookrightarrow L(\tau^{(4)}_1, \dots, \tau^{(4)}_{r_4}) \rtimes \pi_2.$$

Assume for a moment that we could specify π_2.

(7) There exists a unique irreducible representation $L(\tau^{(5)}_1, \dots, \tau^{(5)}_{r_5})$ such that

$$R^{(a')}_{\rho|\cdot|^{-x}}(L(\tau^{(5)}_1, \dots, \tau^{(5)}_{r_5})) = L(\tau^{(4)}_1, \dots, \tau^{(4)}_{r_4})$$

is the highest right derivative. Moreover, $\tau^{(5)}_i$ has a negative central exponent. Then

$$D^{(k)}_{\rho|\cdot|^x}(\pi) = L(\tau^{(5)}_1, \dots, \tau^{(5)}_{r_5}, \Delta_\rho[x-1, -x]^{b'}; \sigma').$$

See §4.2 for examples. In conclusion, the computation of the highest derivative $D^{(k)}_{\rho|\cdot|^x}(\pi)$ is reduced to the one of $D^{(k)}_{\rho|\cdot|^x}(L((\rho|\cdot|^{-x})^a, \Delta_\rho[x-1, -x]^b, \sigma))$. Jantzen's strategy of this computation is an induction on x. Namely, this computation for the case $x \geq 3/2$ can be reduced to the case for $x-1$. Jantzen also gave an explicit formula in the case $x = 1/2$ ([10, Theorem 3.3]). As a consequence, he gave some examples to compute the Langlands data of the Aubert duals $\hat{\pi}$ of certain irreducible representations π in the half-integral reducibility case ([10, §4]).

In this paper, we will treat the general case. To do this, a key idea is to use certain **Arthur packets**.

3. Arthur packets

In his book [1], for each A-parameter ψ, Arthur defined a finite (multi-)set Π_ψ consisting of unitary representations of split $\mathrm{SO}_{2n+1}(F)$ or $\mathrm{Sp}_{2n}(F)$. In this section, we review his theory.

3.1. ***A*-parameters.** A homomorphism

$$\psi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$$

is called an **A-parameter for** $\mathrm{GL}_n(F)$ if

- $\psi(\mathrm{Frob}) \in \mathrm{GL}_n(\mathbb{C})$ is semisimple and all its eigenvalues have absolute value 1;
- $\psi|W_F$ is smooth, i.e., has an open kernel;
- $\psi|\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ is algebraic.

The local Langlands correspondence for $\mathrm{GL}_n(F)$ asserts that there is a canonical bijection between the set of irreducible unitary supercuspidal representations of $\mathrm{GL}_n(F)$ and the set of irreducible representations of W_F of bounded images. We identify these two sets, and use the symbol ρ for their elements.

Any irreducible representation of $W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ is of the form $\rho \boxtimes S_a \boxtimes S_b$, where S_a is the unique irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension a. We shortly write $\rho \boxtimes S_a = \rho \boxtimes S_a \boxtimes S_1$ and $\rho = \rho \boxtimes S_1 \boxtimes S_1$. For an A-parameter ψ, the multiplicity of $\rho \boxtimes S_a \boxtimes S_b$ in ψ is denoted by $m_\psi(\rho \boxtimes S_a \boxtimes S_b)$. When an A-parameter ψ for $\mathrm{GL}_n(F)$ is decomposed into a direct sum

$$\psi = \bigoplus_i \rho_i \boxtimes S_{a_i} \boxtimes S_{b_i},$$

we define an irreducible unitary representation τ_ψ of $\mathrm{GL}_n(F)$ by

$$\tau_\psi = \bigtimes_i L\left(\Delta_{\rho_i}[B_i, -A_i], \Delta_{\rho_i}[B_i + 1, -A_i + 1], \dots, \Delta_{\rho_i}[A_i, -B_i]\right),$$

where we set $A_i = (a_i + b_i)/2 - 1$ and $B_i = (a_i - b_i)/2$.

We say that an A-parameter $\psi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_k(\mathbb{C})$ is **symplectic** or **of symplectic type** (resp. **orthogonal** or **of orthogonal type**) if the image of ψ is in $\mathrm{Sp}_k(\mathbb{C})$ (so that k is even) (resp. in $\mathrm{O}_k(\mathbb{C})$). We call ψ an **A-parameter for** $\mathrm{SO}_{2n+1}(F)$ if it is an A-parameter for $\mathrm{GL}_{2n}(F)$ of symplectic type, i.e.,

$$\psi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_{2n}(\mathbb{C}).$$

Similarly, ψ is called an **A-parameter for** $\mathrm{Sp}_{2n}(F)$ if it is an A-parameter for $\mathrm{GL}_{2n+1}(F)$ of orthogonal type with the trivial determinant, i.e.,

$$\psi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{2n+1}(\mathbb{C}).$$

For $G_n = \mathrm{SO}_{2n+1}(F)$ (resp. $G_n = \mathrm{Sp}_{2n}(F)$), we let $\Psi(G_n)$ be the set of $\widehat{G_n}$-conjugacy classes of A-parameters for G_n, where $\widehat{G_n} = \mathrm{Sp}_{2n}(\mathbb{C})$ (resp. $\widehat{G_n} = \mathrm{SO}_{2n+1}(\mathbb{C})$). We say that

- $\psi \in \Psi(G_n)$ is **tempered** if the restriction of ψ to the second $\mathrm{SL}_2(\mathbb{C})$ is trivial;
- $\psi \in \Psi(G_n)$ is **of good parity** if ψ is a sum of irreducible self-dual representations of the same type as ψ;

We denote by $\Psi_{\mathrm{temp}}(G_n) = \Phi_{\mathrm{temp}}(G_n)$ (resp. $\Psi_{\mathrm{gp}}(G_n)$) the subset of $\Psi(G)$ consisting of tempered A-parameters (resp. A-parameters of good parity). Also, we put $\Phi_{\mathrm{gp}}(G_n) = \Phi_{\mathrm{temp}}(G_n) \cap \Psi_{\mathrm{gp}}(G_n)$. Set $\Psi_*(G) = \cup_{n \geq 0} \Psi_*(G_n)$ and $\Phi_*(G) = \cup_{n \geq 0} \Phi_*(G_n)$ for $* \in \{\emptyset, \mathrm{temp}, \mathrm{gp}\}$.

For $\psi \in \Psi(G_n)$, **the component group** is defined by $\mathcal{S}_\psi = \pi_0(\mathrm{Cent}_{\widehat{G_n}}(\mathrm{Im}(\psi))/Z(\widehat{G_n}))$. This is an elementary two abelian group. It can be described as follows. Let $\psi \in \Psi(G)$. For simplicity, we assume that ψ is of good parity. Hence we can decompose $\psi = \oplus_{i=1}^t \psi_i$, where ψ_i is an irreducible representation (which is self-dual of the same type as ψ). We define an **enhanced component group** $\mathcal{A}_\psi$ as

$$\mathcal{A}_\psi = \bigoplus_{i=1}^t (\mathbb{Z}/2\mathbb{Z})\alpha_{\psi_i}.$$

Namely, $\mathcal{A}_\psi$ is a free $\mathbb{Z}/2\mathbb{Z}$-module of rank t with a basis $\{\alpha_{\psi_i}\}$ associated with the irreducible components $\{\psi_i\}$. Then there exists a canonical surjection

$$\mathcal{A}_\psi \twoheadrightarrow \mathcal{S}_\psi$$

whose kernel is generated by the elements

- $z_\psi = \sum_{i=1}^t \alpha_{\psi_i}$; and
- $\alpha_{\psi_i} + \alpha_{\psi_{i'}}$ such that $\psi_i \cong \psi_{i'}$.

Let $\widehat{\mathcal{S}_\psi}$ and $\widehat{\mathcal{A}_\psi}$ be the Pontryagin duals of $\mathcal{S}_\psi$ and $\mathcal{A}_\psi$, respectively. Via the surjection $\mathcal{A}_\psi \twoheadrightarrow \mathcal{S}_\psi$, we may regard $\widehat{\mathcal{S}_\psi}$ as a subgroup of $\widehat{\mathcal{A}_\psi}$. For $\eta \in \widehat{\mathcal{A}_\psi}$, we write $\eta(\alpha_{\psi_i}) = \eta(\psi_i)$. By convention, we understand that $m_\psi(\rho \boxtimes S_0) = 1$ and $\eta(\rho \boxtimes S_0) = 1$.

Let $\psi, \psi' \in \Psi_{\mathrm{gp}}(G)$. When $\psi' \subset \psi$, we have a canonical inclusion $\mathcal{A}_{\psi'} \hookrightarrow \mathcal{A}_\psi$. When $\psi' = \psi - \psi_0 + \psi'_0$ with ψ_0, ψ'_0 being irreducible, we can define a map $\mathcal{A}_{\psi'} \hookrightarrow \mathcal{A}_\psi$ by sending $\alpha_{\psi'_0}$ to α_{ψ_0}. In these cases, for $\eta \in \widehat{\mathcal{A}_\psi}$, we denote its restriction by $\eta' \in \widehat{\mathcal{A}_{\psi'}}$. For example, in the second case, we set $\eta'(\psi'_0) = \eta(\psi_0)$.

Let $\mathrm{Irr}_{\mathrm{unit}}(G_n)$ (resp. $\mathrm{Irr}_{\mathrm{temp}}(G_n)$) be the set of equivalence classes of irreducible unitary (resp. tempered) representations of G_n. The local main theorem of Arthur's book is as follows.

Theorem 3.1 ([1, Theorem 2.2.1, Proposition 7.4.1])**.** *Let G_n be a split $\mathrm{SO}_{2n+1}(F)$ or $\mathrm{Sp}_{2n}(F)$.*

(1) *For each $\psi \in \Psi(G_n)$, there is a finite multi-set Π_ψ over $\mathrm{Irr}_{\mathrm{unit}}(G_n)$ with a map*

$$\Pi_\psi \to \widehat{\mathcal{S}_\psi},\ \pi \mapsto \langle \cdot, \pi \rangle_\psi$$

*satisfying certain (twisted and standard) endoscopic character identities. We call Π_ψ the A-**packet** for G_n associated with ψ.*

(2) *When $\psi = \phi \in \Phi_{\mathrm{temp}}(G_n)$, the A-packet Π_ϕ is in fact a subset of $\mathrm{Irr}_{\mathrm{temp}}(G_n)$. Moreover, the map $\Pi_\phi \ni \pi \mapsto \langle \cdot, \pi \rangle_\phi \in \widehat{\mathcal{S}_\phi}$ is bijective, $\Pi_\phi \cap \Pi_{\phi'} = \emptyset$ for $\phi \not\cong \phi'$, and*

$$\mathrm{Irr}_{\mathrm{temp}}(G_n) = \bigsqcup_{\phi \in \Phi_{\mathrm{temp}}(G_n)} \Pi_\phi.$$

When $\pi \in \Pi_\phi$ with $\eta = \langle \cdot, \pi \rangle_\phi \in \widehat{\mathcal{S}_\phi}$, we write $\pi = \pi(\phi, \eta)$.

(3) *If* $\psi = \oplus_i \rho_i \boxtimes S_{a_i} \boxtimes S_{b_i}$, *set*

$$\phi_{\psi,0} = \bigoplus_{\substack{i \\ b_i \equiv 1 \bmod 2}} \rho_i \boxtimes S_{a_i}.$$

Then for any $\sigma \in \Pi_{\phi_{\psi,0}}$, *the unique irreducible subrepresentation of*

$$\left(\bigtimes_{\substack{i \\ b_i \equiv 1 \bmod 2, b_i \neq 1}} L\left(\Delta_{\rho_i}[B_i, -A_i], \Delta_{\rho_i}[B_i+1, -A_i+1], \dots, \Delta_{\rho_i}[(a_i-3)/2, -(a_i-1)/2]\right) \right)$$
$$\times \left(\bigtimes_{\substack{i \\ b_i \equiv 0 \bmod 2}} L\left(\Delta_{\rho_i}[B_i, -A_i], \Delta_{\rho_i}[B_i+1, -A_i+1], \dots, \Delta_{\rho_i}[a_i/2-1, -a_i/2]\right) \right) \rtimes \sigma$$

belongs to Π_ψ, *where* $A_i = (a_i+b_i)/2 - 1$ *and* $B_i = (a_i - b_i)/2$. *Here, the products are taken so that the induced representation is isomorphic to a standard module.*

Remark 3.2. (1) *The map* $\Pi_\psi \ni \pi \mapsto \langle \cdot, \pi \rangle_\psi \in \widehat{\mathcal{S}_\psi}$ *is not canonical when* $G = \mathrm{Sp}_{2n}(F)$. *To specify this, we implicitly fix an* $F^{\times 2}$*-orbit of non-trivial additive characters of* F *through this paper.*

(2) *In general, the map* $\Pi_\psi \ni \pi \mapsto \langle \cdot, \pi \rangle_\psi \in \widehat{\mathcal{S}_\psi}$ *is neither injective nor surjective.*

(3) *In general,* Π_ψ *can intersect with* $\Pi_{\psi'}$ *even when* $\psi \not\cong \psi'$. *However,* [18, 4.2 Corollaire] *says that if* $\Pi_\psi \cap \Pi_{\psi'} \neq \emptyset$, *then* $\psi_d \cong \psi'_d$, *where* $\psi_d = \psi \circ \Delta$ *is the diagonal restriction of* ψ, *i.e.,* $\Delta\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ *is defined by* $\Delta(w, g) = (w, g, g)$.

The following is a deep result of Mœglin.

Theorem 3.3 (Mœglin [19])**.** *The A-packet* Π_ψ *is multiplicity-free, i.e., it is a subset of* $\mathrm{Irr}_{\mathrm{unit}}(G)$.

Xu proved the following key lemma, whose proof uses the theory of endoscopy.

Lemma 3.4 (Xu [22, Proposition 8.3 (ii)])**.** *Let* $\psi = \oplus_{i \in I} \rho_i \boxtimes S_{a_i} \boxtimes S_{b_i} \in \Psi_{\mathrm{gp}}(G)$. *For fixed* $x \in \mathbb{R}$, *if* $D^{(k)}_{\rho|\cdot|^x}(\pi) \neq 0$ *for some* $\pi \in \Pi_\psi$, *then*

$$k \leq \#\{i \in I \mid \rho_i \cong \rho,\ x = (a_i - b_i)/2\}.$$

The following is the first observation.

Example 3.5. *Let* $\phi \in \Phi_{\mathrm{gp}}(G)$ *and* $\eta \in \widehat{\mathcal{S}_\phi}$. *Fix* $x \in (1/2)\mathbb{Z}$ *with* $x > 0$. *Suppose that* $\rho \boxtimes S_{2x+1}$ *is self-dual of the same type as* ϕ.

(1) *Consider*

$$\pi = L(\Delta_\rho[x-1, -x]^b; \pi(\phi, \eta)).$$

By Theorem 3.1 (3), we have $\pi \in \Pi_\psi$ *with*

$$\psi = \phi + (\rho \boxtimes S_{2x} \boxtimes S_2)^{\oplus b}.$$

In particular, by Lemma 3.4, if $D^{(k)}_{\rho|\cdot|^x}(\pi) \neq 0$, *then* $k \leq m_\phi(\rho \boxtimes S_{2x+1})$.

(2) *Assume that* $x = 1$, *and consider*

$$\pi = L((\rho|\cdot|^{-1})^a, \Delta_\rho[0,-1]^b; \pi(\phi,\eta)).$$

By Theorem 3.1 (3), if $a \leq m_\phi(\rho)$, *then* $\pi \in \Pi_\psi$ *with*

$$\psi = \phi - \rho^{\oplus a} + (\rho \boxtimes S_1 \boxtimes S_3)^{\oplus a} + (\rho \boxtimes S_2 \boxtimes S_2)^{\oplus b}.$$

In particular, by Lemma 3.4, if $D^{(k)}_{\rho|\cdot|^1}(\pi) \neq 0$, *then* $k \leq m_\phi(\rho \boxtimes S_3)$.

3.2. **Highest derivatives of tempered representations.** In [8, 9, 10], Jantzen studied the derivatives of irreducible representations of G_n. To do this, he used the extended Mœglin–Tadić classification, which characterizes irreducible tempered representations by their cuspidal supports and the behavior of Jacquet modules. See [8]. Since the behavior of Jacquet modules of irreducible tempered representations is known by the previous paper [2], one can easily translate Jantzen's results in terms of the local Langlands correspondence (Theorem 3.1 (2)).

The highest derivatives of tempered representations are given as follows.

Proposition 3.6 ([9, Theorem 3.1])**.** *Let* $\phi \in \Phi_{\mathrm{gp}}(G)$ *and* $\eta \in \widehat{\mathcal{S}_\phi}$. *Fix a positive half-integer* $x \in (1/2)\mathbb{Z}$. *Suppose that* $\rho \boxtimes S_{2x+1}$ *is self-dual of the same type as* ϕ. *Denote* $m = m_\phi(\rho \boxtimes S_{2x+1}) \geq 0$ *by the multiplicity of* $\rho \boxtimes S_{2x+1}$ *in* ϕ.

(1) *When* $x > 0$, *we have*

$$D^{(m)}_{\rho|\cdot|^x}(\pi(\phi,\eta)) = \pi\left(\phi - (\rho \boxtimes S_{2x+1})^{\oplus m} + (\rho \boxtimes S_{2x-1})^{\oplus m}, \eta\right).$$

It is the highest derivative if it is nonzero. Moreover, $D^{(m)}_{\rho|\cdot|^x}(\pi(\phi,\eta)) = 0$ *if and only if*

- $m > 0$; *and*
- $\phi \supset \rho \boxtimes S_{2x-1}$ *and* $\eta(\rho \boxtimes S_{2x+1}) \neq \eta(\rho \boxtimes S_{2x-1})$,

where we understand that $\phi \supset \rho \boxtimes S_0$ *and* $\eta(\rho \boxtimes S_0) = 1$ *when* ρ *is self-dual of the opposite type to* ϕ.

(2) *Suppose that* $m > 0$, $x > 0$ *and* $D^{(m)}_{\rho|\cdot|^x}(\pi(\phi,\eta)) = 0$. *If* m *is odd, then*

$$D^{(m-1)}_{\rho|\cdot|^x}(\pi(\phi,\eta)) = \pi\left(\phi - (\rho \boxtimes S_{2x+1})^{\oplus m-1} + (\rho \boxtimes S_{2x-1})^{\oplus m-1}, \eta\right).$$

If m *is even,* $D^{(m-1)}_{\rho|\cdot|^x}(\pi(\phi,\eta))$ *is equal to*

$$L\left(\Delta_\rho[x-1,-x]; \pi\left(\phi - (\rho \boxtimes S_{2x+1})^{\oplus m} + (\rho \boxtimes S_{2x-1})^{\oplus m-2}, \eta\right)\right).$$

In particular, in both cases, $D^{(m-1)}_{\rho|\cdot|^x}(\pi(\phi,\eta))$ *is irreducible and the highest derivative. Moreover,* $D^{(m-1)}_{\rho|\cdot|^x}(\pi(\phi,\eta))$ *is tempered if and only if* m *is odd.*

(3) *When* $x = 0$, *set* $k = [m/2]$ *to be the largest integer which is not greater than* $m/2$. *Then*

$$D^{(k)}_{\rho}(\pi(\phi,\eta)) = c_k \cdot \pi\left(\phi - \rho^{\oplus 2k}, \eta\right)$$

with

$$c_k = \begin{cases} 2^{k-1} & \text{if } \rho \text{ is of the same type as } \phi \text{ and } m \text{ is even}, \\ 2^k & \text{otherwise}. \end{cases}$$

This is the highest derivative.

Remark 3.7. (1) *In [2], the highest derivatives of irreducible tempered representations was also considered, but Proposition 6.3 and Remark 6.4 in that paper are mistakes.*
(2) *One can also prove Proposition 3.6 by using results in [2].*
(3) *When ρ is not self-dual, $x \geq 0$ and $m = m_\phi(\rho \boxtimes S_{2x+1}) > 0$, we have $m_\phi(\rho^\vee \boxtimes S_{2x+1}) = m$ and*

$$\pi(\phi, \eta) = \Delta_\rho[x, -x]^m \rtimes \pi(\phi_0, \eta_0),$$

where $\phi_0 = \phi - (\rho \oplus \rho^\vee)^{\oplus m} \boxtimes S_{2x+1}$, and $\eta_0 = \eta | \mathcal{A}_{\phi_0}$. In this case, the highest derivative is

$$D^{(m)}_{\rho|\cdot|^x}(\pi(\phi, \eta)) = \Delta_\rho[x-1, x]^m \rtimes \pi(\phi_0, \eta_0).$$

Here, when $x = 0$, we omit $\Delta_\rho[x-1, x]$.

As a consequence, we have the following.

Proposition 3.8. *Let $\pi = \pi(\phi, \eta)$ be an irreducible tempered representation of G_n. Fix $x \in (1/2)\mathbb{Z}$ and assume that $\rho \boxtimes S_{2|x|+1}$ is self-dual of the same type as ϕ.*

(1) *Suppose that $M^-_\rho(\pi) = \left[(x_1, k_1), \dots, (x_t, k_t), (x, k); M^-_\rho(\pi_1)\right]$ where $x > 0$ and $\pi_1 = \pi(\phi_1, \eta_1)$ tempered (t can be zero). Then $x = 1/2$, or ϕ_1 contains $\rho \boxtimes S_{2x-1}$ with multiplicity greater than or equal to k. Moreover, if ϕ_1 contains $\rho \boxtimes S_{2x+1}$ and $\eta_1(\rho \boxtimes S_{2x+1}) \neq \eta_1(\rho \boxtimes S_{2x-1})$, then k is even. Set*

$$\phi_2 = \phi_1 - (\rho \boxtimes S_{2x-1})^{\oplus k} + (\rho \boxtimes S_{2x+1})^{\oplus k},$$

and define $\eta_2 \in \widehat{\mathcal{A}_{\phi_2}}$ so that $\eta_2(\rho' \boxtimes S_a) = \eta_1(\rho' \boxtimes S_a)$ for any $(\rho', a) \neq (\rho, 2x+1)$ and

$$\eta_2(\rho \boxtimes S_{2x+1}) = \begin{cases} \eta_1(\rho \boxtimes S_{2x+1}) & \text{if } \phi_1 \supset \rho \boxtimes S_{2x+1}, \\ \eta_1(\rho \boxtimes S_{2x-1}) & \text{otherwise.} \end{cases}$$

Here, we understand that $\phi_1 \supset \rho \boxtimes S_0$ and $\eta_1(\rho \boxtimes S_0) = 1$ when $x = 1/2$. Then

$$M^-_\rho(\pi) = \left[(x_1, k_1), \dots, (x_t, k_t); M^-_\rho(\pi(\phi_2, \eta_2))\right].$$

(2) *Suppose that $M^-_\rho(\pi) = \left[(x_1, k_1), \dots, (x_t, k_t), (x, k); M^-_\rho(\pi_1)\right]$ where $x = 0$ and $\pi_1 = \pi(\phi_1, \eta_1)$ tempered (t can be zero). Set*

$$\phi_2 = \phi_1 + \rho^{\oplus 2k}.$$

Then there exists $\eta_2 \in \widehat{\mathcal{A}_{\phi_2}}$ with $\eta_2 | \mathcal{A}_{\phi_1} = \eta_1$ such that

$$M^-_\rho(\pi) = \left[(x_1, k_1), \dots, (x_t, k_t); M^-_\rho(\pi(\phi_2, \eta_2))\right].$$

(3) *Suppose that $M^-_\rho(\pi) = \left[(x_1, k_1), \dots, (x_t, k_t), (-x, k); M^-_\rho(\pi_1)\right]$ where $x > 0$ and $\pi_1 = \pi(\phi_1, \eta_1)$ tempered. Then*
- *$k = 1$;*
- *there exists $1 \leq t' \leq t$ such that*

$$M^-_\rho(\pi) = \left[(x_1, k_1), \dots, (x_{t'}, k_{t'}), (x, k'), (x-1, 1), \dots, (-x, 1); M^-_\rho(\pi_1)\right]$$

  for some odd k';
- *ϕ_1 contains $\rho \boxtimes S_{2x-1}$ with multiplicity greater than or equal to k';*
- *ϕ_1 does not contain $\rho \boxtimes S_{2x+1}$.*

Set
$$\phi_2 = \phi_1 + (\rho \boxtimes S_{2x+1})^{\oplus k'+1} - (\rho \boxtimes S_{2x-1})^{\oplus k'-1},$$
and define $\eta_2 \in \widehat{\mathcal{A}_{\phi_2}}$ *so that* $\eta_2(\rho' \boxtimes S_a) = \eta_1(\rho' \boxtimes S_a)$ *for any* $(\rho', a) \neq (\rho, 2x+1)$ *and* $\eta_2(\rho \boxtimes S_{2x+1}) = -\eta_1(\rho \boxtimes S_{2x-1})$. *Then*
$$M_\rho^-(\pi) = \left[(x_1, k_1), \dots, (x_{t'}, k_{t'}); M_\rho^-(\pi(\phi_2, \eta_2))\right].$$

Proof. We show (1) and (2) so that $x \geq 0$. If an irreducible representation π_2 satisfies that $M_\rho^-(\pi) = \left[(x_1, k_1), \dots, (x_t, k_t); M_\rho^-(\pi_2)\right]$, then $M_\rho^-(\pi_2) = \left[(x, k); M_\rho^-(\pi_1)\right]$. By applying Corollary 2.12 to π_1 and π_2, we see that π_2 is also tempered. If $\pi_2 = \pi(\phi_2, \eta_2)$, the relation between (ϕ_1, η_1) and (ϕ_2, η_2) is given in Proposition 3.6. Hence we obtain (1) and (2).

We show (3) so that $M_\rho^-(\pi) = \left[(x_1, k_1), \dots, (x_t, k_t), (-x, k); M_\rho^-(\pi_1)\right]$ with $x > 0$. By applying Corollary 2.12 to π, we see that there exists $1 \leq t' \leq t$ such that $M_\rho^-(\pi) = \left[(x_1, k_1), \dots, (x_{t'}, k_{t'}), (x, k_1'), (x-1, k_2'), \dots, (-x, k_{2x+1}'); M_\rho^-(\pi_1)\right]$ with $k_1' \geq \dots \geq k_{2x+1}' = k$. Take an irreducible representation π_2 such that $M_\rho^-(\pi) = \left[(x_1, k_1), \dots, (x_{t'}, k_{t'}); M_\rho^-(\pi_2)\right]$. Then $M_\rho^-(\pi_2) = \left[(x, k_1'), (x-1, k_2'), \dots, (-x, k_{2x+1}'); M_\rho^-(\pi_1)\right]$. Since π_1 is tempered, by Corollary 2.12, we see that π_2 is also tempered. Write $\pi_2 = \pi(\phi_2, \eta_2)$. Then by Proposition 3.6, we have

- ϕ_2 contains $\rho \boxtimes S_{2x+1}$ with even multiplicity $2m > 0$;
- ϕ_2 contains $\rho \boxtimes S_{2x-1}$ and $\eta_2(\rho \boxtimes S_{2x+1}) \neq \eta_2(\rho \boxtimes S_{2x-1})$;
- $k_1' = 2m - 1$ and $k_2' = \dots = k_{2x+1}' = 1$;
- $\phi_1 = \phi_2 - (\rho \boxtimes S_{2x+1})^{\oplus 2m} + (\rho \boxtimes S_{2x-1})^{\oplus 2m-2}$ so that ϕ_1 contains $\rho \boxtimes S_{2x-1}$ with multiplicity greater than $2m - 2 = k_1' - 1$.

Hence we obtain (3). □

When $\rho \boxtimes S_{2|x|+1}$ is not self-dual of the same type as ϕ, a similar (and easier) statement holds. We leave the detail for readers. We note that:

- When $\rho \boxtimes S_{2|x|+1}$ is self-dual of the opposite type to ϕ, the case (3) cannot occur.
- When ρ is not self-dual, the case (1) cannot occur.

3.3. **Irreducibility of certain induced representations.** Using the highest derivatives, Jantzen proved some irreducibility of parabolically induced representations. For $\phi \in \Phi_{\mathrm{gp}}(G)$, we denote the multiplicity of $\rho \boxtimes S_a$ in ϕ by $m_\phi(\rho \boxtimes S_a)$. For consistency, we define $m_\phi(\rho \boxtimes S_0) = 1$ and $\eta(\rho \boxtimes S_0) = +1$ if ρ is self-dual of the opposite type to ϕ.

Theorem 3.9 ([9, Theorem 4.7])**.** *Let* $\phi \in \Phi_{\mathrm{gp}}(G)$ *and* $\eta \in \widehat{\mathcal{S}_\phi}$. *Fix* $a \in \mathbb{Z}$ *with* $a \geq 2$ *such that* $\rho \boxtimes S_a$ *is self-dual of the same type as* ϕ. *Consider* $\rho|\cdot|^{\frac{a-1}{2}} \rtimes \pi(\phi, \eta)$.

(1) *If* $m_\phi(\rho \boxtimes S_{a-2}) = 0$, *then* $\rho|\cdot|^{\frac{a-1}{2}} \rtimes \pi(\phi, \eta)$ *is irreducible.*

(2) *If* $m_\phi(\rho \boxtimes S_{a-2}) = 1$, *then*
  (a) $\rho|\cdot|^{\frac{a-1}{2}} \rtimes \pi(\phi, \eta)$ *is reducible if* $m_\phi(\rho \boxtimes S_a) = 0$ *or* $\eta(\rho \boxtimes S_a) = \eta(\rho \boxtimes S_{a-2})$;
  (b) $\rho|\cdot|^{\frac{a-1}{2}} \rtimes \pi(\phi, \eta)$ *is irreducible if* $m_\phi(\rho \boxtimes S_a) > 0$ *and* $\eta(\rho \boxtimes S_a) \neq \eta(\rho \boxtimes S_{a-2})$.

(3) *If* $m_\phi(\rho \boxtimes S_{a-2}) \geq 2$, *then* $\rho|\cdot|^{\frac{a-1}{2}} \rtimes \pi(\phi, \eta)$ *is reducible.*

We also need the irreducibility of other parabolically induced representations.

Proposition 3.10. *Let* $\phi \in \Phi_{\mathrm{gp}}(G)$ *and* $\eta \in \widehat{\mathcal{S}_\phi}$. *Fix* $x \in (1/2)\mathbb{Z}$ *with* $x \geq 1$ *such that* $\rho \boxtimes S_{2x+1}$ *is self-dual of the same type as* ϕ.

(1) *$\Delta_\rho[x-1,-x] \rtimes \pi(\phi,\eta)$ is irreducible if and only if ϕ contains both $\rho \boxtimes S_{2x+1}$ and $\rho \boxtimes S_{2x-1}$, and $\eta(\rho \boxtimes S_{2x+1}) \neq \eta(\rho \boxtimes S_{2x-1})$.*

(2) *If ϕ contains both $\rho \boxtimes S_{2x+1}$ and $\rho \boxtimes S_{2x-1}$, but if $\eta(\rho \boxtimes S_{2x+1}) = \eta(\rho \boxtimes S_{2x-1})$, then $\Delta_\rho[x-1,-x] \rtimes L(\Delta_\rho[x-1,-x];\pi(\phi,\eta))$ is irreducible.*

Proof. The only if part of (1) is proven in [2, Proposition 5.2]. The if part is similar to that proposition. Assume that ϕ contains both $\rho \boxtimes S_{2x+1}$ and $\rho \boxtimes S_{2x-1}$ and $\eta(\rho \boxtimes S_{2x+1}) \neq \eta(\rho \boxtimes S_{2x-1})$. Suppose that $\Delta_\rho[x-1,-x] \rtimes \pi(\phi,\eta)$ is reducible. Note that by Tadić's formula (Theorem 2.4) and by the Casselman criterion, this standard module has a unique irreducible non-tempered subquotient, which must be the Langlands subrepresentation. Take an irreducible quotient π' of $\Delta_\rho[x-1,-x] \rtimes \pi(\phi,\eta)$. Since the Langlands subrepresentation appears in the standard module as subquotients with multiplicity one, we see that π' is tempered. Moreover, $\pi' \in \Pi_{\phi'}$ with $\phi' = \phi + \rho \boxtimes (S_{2x+1} + S_{2x-1})$ by computing the Plancherel measure of π' (see [5, Lemma A.6]). We write $\pi' = \pi(\phi',\eta')$. When $D^{(k)}_{\rho|\cdot|^x}(\pi(\phi,\eta))$ is the highest derivative, by Proposition 3.6, we see that $D^{(k+1)}_{\rho|\cdot|^x}(\pi')$ is equal to the irreducible induction $\Delta_\rho[x-1,-(x-1)] \rtimes D^{(k)}_{\rho|\cdot|^x}(\pi(\phi,\eta))$. If $\eta'(\rho \boxtimes S_{2x+1}) = \eta'(\rho \boxtimes S_{2x-1})$, then $D^{(k+1)}_{\rho|\cdot|^x}(\pi')$ is tempered such that its L-parameter ϕ'' does not contain $\rho \boxtimes S_{2x+1}$, whereas, if $\Delta_\rho[x-1,-(x-1)] \rtimes D^{(k)}_{\rho|\cdot|^x}(\pi(\phi,\eta))$ is tempered, then its L-parameter must contain $\rho \boxtimes S_{2x+1}$. This is a contradiction, so that we have $\eta'(\rho \boxtimes S_{2x+1}) \neq \eta'(\rho \boxtimes S_{2x-1})$. In addition, we have equations of the highest derivatives

$$\begin{cases} D^{(l+2)}_{\rho|\cdot|^{x-1}} \circ D^{(k+1)}_{\rho|\cdot|^x}(\pi') = \Delta_\rho[x-2,-(x-2)] \rtimes D^{(l)}_{\rho|\cdot|^{x-1}} \circ D^{(k)}_{\rho|\cdot|^x}(\pi(\phi,\eta)) & \text{if } x \geq 2, \\ D^{(l+2)}_{\rho|\cdot|^{\frac{1}{2}}} \circ D^{(k+1)}_{\rho|\cdot|^{\frac{3}{2}}}(\pi') = D^{(l)}_{\rho|\cdot|^{\frac{1}{2}}} \circ D^{(k)}_{\rho|\cdot|^{\frac{3}{2}}}(\pi(\phi,\eta)) & \text{if } x = \dfrac{3}{2}, \\ D^{(l+1)}_{\rho|\cdot|^0} \circ D^{(k+1)}_{\rho|\cdot|^1}(\pi') = 2 \cdot D^{(l)}_{\rho|\cdot|^0} \circ D^{(k)}_{\rho|\cdot|^1}(\pi(\phi,\eta)) & \text{if } x = 1. \end{cases}$$

In each case, by Proposition 3.6, $\Delta_\rho[x-2,-x]$ appears in exactly one of the left or right hand side. This is a contradiction.

(2) is proven similarly to [10, Lemma 6.3]. We give a sketch of the proof.

- For $\phi' = \phi - \rho \boxtimes (S_{2x+1} + S_{2x-1})$, we have
$$\pi(\phi,\eta) \hookrightarrow \Delta_\rho[x,-(x-1)] \rtimes \pi(\phi',\eta').$$
- If we set $\pi = L(\Delta_\rho[x-1,-x];\pi(\phi,\eta))$, then $\pi \hookrightarrow \Delta_\rho[x-1,-x] \times \Delta_\rho[x,-(x-1)] \rtimes \pi(\phi',\eta')$. This implies that $\pi \hookrightarrow L(\Delta_\rho[x-1,-x],\Delta_\rho[x,-(x-1)]) \rtimes \pi(\phi',\eta')$ or $\pi \hookrightarrow \Delta_\rho[x,-x] \times \Delta_\rho[x-1,-(x-1)] \rtimes \pi(\phi',\eta')$. Since π is non-tempered, the former case must hold.
- Put $m = m_\phi(\rho \boxtimes S_{2x+1})$. Since $L_{\rho|\cdot|^x}(L(\Delta_\rho[x-1,-x],\Delta_\rho[x,-(x-1)])) = R_{\rho|\cdot|^{-x}}(L(\Delta_\rho[x-1,-x],\Delta_\rho[x,-(x-1)])) = 0$, and since $L(\Delta_\rho[x-1,-x],\Delta_\rho[x,-(x-1)])$ commutes with $\rho|\cdot|^x$, we see that $D^{(m-1)}_{\rho|\cdot|^x}(\pi)$ is the highest derivative, and
$$D^{(m-1)}_{\rho|\cdot|^x}(\pi) \hookrightarrow L(\Delta_\rho[x-1,-x],\Delta_\rho[x,-(x-1)]) \rtimes D^{(m-1)}_{\rho|\cdot|^x}(\pi(\phi',\eta')).$$
- Set $\lambda = L(\Delta_\rho[x-1,-x]^2;\pi(\phi,\eta))$. Since up to semisimplification
$$\Delta_\rho[x-1,-x]^2 \rtimes \pi(\phi,\eta) = \Delta_\rho[x-1,-x] \times \Delta_\rho[x,-(x-1)] \rtimes \pi(\phi,\eta)$$

$$=L(\Delta_\rho[x-1,-x],\Delta_\rho[x,-(x-1)])\rtimes\pi(\phi,\eta)$$
$$\oplus\,\Delta_\rho[x,-x]\times\Delta_\rho[x-1,-(x-1)]\rtimes\pi(\phi,\eta),$$

we see that λ is a subrepresentation of $L(\Delta_\rho[x-1,-x],\Delta_\rho[x,-(x-1)])\rtimes\pi(\phi,\eta)$. In particular,

$$D^{(m)}_{\rho|\cdot|^x}(\lambda)\hookrightarrow L(\Delta_\rho[x-1,-x],\Delta_\rho[x,-(x-1)])\rtimes D^{(m)}_{\rho|\cdot|^x}(\pi(\phi,\eta))$$

is the highest derivative.

- Since $\lambda\hookrightarrow\Delta_\rho[x-1,-x]\rtimes\pi$, we must have
$$D^{(m)}_{\rho|\cdot|^x}(\lambda)\subset\Delta_\rho[x-1,-(x-1)]\rtimes D^{(m-1)}_{\rho|\cdot|^x}(\pi).$$
In particular, since $\lambda\hookrightarrow(\rho|\cdot|^x)^m\times\Delta_\rho[x-1,-(x-1)]\rtimes D^{(m-1)}_{\rho|\cdot|^x}(\pi)$, we have $\lambda\hookrightarrow\Delta_\rho[x,-(x-1)]\times(\rho|\cdot|^x)^{m-1}\rtimes D^{(m-1)}_{\rho|\cdot|^x}(\pi)$ or $\lambda\hookrightarrow L(\Delta_\rho[x-1,-(x-1)],\rho|\cdot|^x)\times(\rho|\cdot|^x)^{m-1}\rtimes D^{(m-1)}_{\rho|\cdot|^x}(\pi)$. Since $x\geq1$ and $D^{(m)}_{\rho|\cdot|^x}(\lambda)\neq0$, the former case must hold. Moreover, we see that
$$\lambda\hookrightarrow\Delta_\rho[x,-(x-1)]\rtimes\pi.$$
- Since $\Delta_\rho[x,-(x-1)]\times\pi(\phi,\eta)\twoheadrightarrow\pi$, we see that λ is the unique irreducible (Langlands) quotient of $\Delta_\rho[x,-(x-1)]\rtimes\pi$. Since this quotient appear in subquotients with multiplicity one, $\Delta_\rho[x,-(x-1)]\rtimes\pi$ must be irreducible.

This completes the proof. □

3.4. **Mœglin's construction.** To obtain Theorem 3.3, Mœglin constructed A-packets Π_ψ concretely. In this subsection, we review her construction in a special case. For more precision, see [16, 17, 19] and [22].

Let $\psi\in\Psi_{\mathrm{gp}}(G_n)$. We assume that ψ is of the form

$$\psi=\phi_0+\left(\bigoplus_{i=1}^r\rho\boxtimes S_{a_i}\boxtimes S_{b_i}\right)$$

such that

(a) $\phi_0\in\Phi_{\mathrm{gp}}(G)$ such that $\phi_0\not\supset\rho\boxtimes S_d$ for any $d\geq1$;
(b) $a_i\geq b_i$ for any $1\leq i\leq r$;
(c) if $a_i-b_i>a_j-b_j$ and $a_i+b_i>a_j+b_j$, then $i>j$.

Note that the last condition may not determine an order on $\{(a_1,b_1),\dots,(a_r,b_r)\}$ uniquely. Once we fix such an order, we write $\rho\boxtimes S_{a_i}\boxtimes S_{b_i}<_\psi\rho\boxtimes S_{a_j}\boxtimes S_{b_j}$ if $i<j$.

Fix $\eta_0\in\widehat{\mathcal{A}_{\phi_0}}$ such that if $\phi_0=\oplus_i\phi_i$ is a decomposition into irreducible representations, then $\eta_0(\phi_i)=\eta_0(\phi_j)$ whenever $\phi_i\cong\phi_j$. Let $\underline{l}=(l_1,\dots,l_r)\in\mathbb{Z}^r$ and $\underline{\eta}=(\eta_1,\dots,\eta_r)\in\{\pm1\}^r$ such that $0\leq l_i\leq b_i/2$ and

$$\eta_0(z_{\phi_0})\prod_{i=1}^r(-1)^{[b_i/2]+l_i}\eta_i^{b_i}=1.$$

For these data, Mœglin constructed a representation $\pi_{<_\psi}(\psi,\underline{l},\underline{\eta},\eta_0)$ of G_n. If the order $<_\psi$ is fixed, we may write $\pi(\psi,\underline{l},\underline{\eta},\eta_0)=\pi_{<_\psi}(\psi,\underline{l},\underline{\eta},\eta_0)$.

Theorem 3.11 (Mœglin)**.** *Notation is as above.*

(1) *The representation* $\pi(\psi, \underline{l}, \underline{\eta}, \eta_0)$ *is irreducible or zero. Moreover,*
$$\Pi_\psi = \{\pi(\psi, \underline{l}, \underline{\eta}, \eta_0) \mid \underline{l}, \underline{\eta}, \eta_0 \text{ as above}\} \setminus \{0\}.$$
(2) *If* $\pi(\psi, \underline{l}, \underline{\eta}, \eta_0) \cong \pi(\psi, \underline{l'}, \underline{\eta'}, \eta'_0) \neq 0$, *then* $\underline{l} = \underline{l'}$, $\eta_0 = \eta'_0$, *and* $\eta_i = \eta'_i$ *unless* $l_i = b_i/2$.
(3) *Assume further that* $a_i - b_i \geq a_{i-1} + b_{i-1}$ *for any* $i > 1$. *Then* $\pi(\psi, \underline{l}, \underline{\eta}, \eta_0)$ *is nonzero and is a unique irreducible subrepresentation of*
$$\bigtimes_{i=1}^{r} L\left(\Delta_\rho[B_i, -A_i], \Delta_\rho[B_i+1, -A_i+1], \dots, \Delta_\rho[B_i+l_i-1, -A_i+l_i-1]\right) \rtimes \pi(\phi, \eta),$$
where $A_i = (a_i + b_i)/2 - 1$, $B_i = (a_i - b_i)/2$, *and*
$$\phi = \phi_0 + \left(\bigoplus_{i=1}^{r} \rho \boxtimes (S_{2B_i+2l_i+1} + \dots + S_{2A_i-2l_i+1})\right),$$
and $\eta \in \widehat{\mathcal{S}_\phi} \subset \widehat{\mathcal{A}_\phi}$ *is given by* $\eta | \mathcal{A}_{\phi_0} = \eta_0$ *and* $\eta(\rho \boxtimes S_{2B_i+2l_i+2c-1}) = (-1)^{c-1}\eta_i$ *for* $1 \leq c \leq b_i - 2l_i$.
(4) *Suppose that*
$$\psi' = \phi_0 + \left(\bigoplus_{i=1}^{r} \rho \boxtimes S_{a'_i} \boxtimes S_{b_i}\right) \in \Psi_{\mathrm{gp}}(G)$$
also satisfies the above assumptions (a)–(c). Assume further that $a'_i \geq a_i$ *for any* $i \geq 1$ *and that* $a'_i - b_i \geq a'_{i-1} + b_{i-1}$ *for any* $i > 1$. *Then*
$$\pi(\psi, \underline{l}, \underline{\eta}, \eta_0) = \mathrm{Jac}_{\rho|\cdot|^{\frac{a'_r-1}{2}}, \dots, \rho|\cdot|^{\frac{a_r+1}{2}}} \circ \dots \circ \mathrm{Jac}_{\rho|\cdot|^{\frac{a'_1-1}{2}}, \dots, \rho|\cdot|^{\frac{a_1+1}{2}}} (\pi(\psi', \underline{l}, \underline{\eta}, \eta_0)).$$

For the proof, see [22, §8].

Example 3.12. *Let us consider the situation of Example 3.5 with* $x \geq 1$. *Set* $\psi = \phi + (\rho \boxtimes S_{2x} \boxtimes S_2)^b$. *Write*
$$\psi = \phi_0 + \left(\bigoplus_{i=1}^{r} \rho \boxtimes S_{a_i} \boxtimes S_{b_i}\right)$$
as above such that $a_1 \leq \dots \leq a_r$ *and* $b_i \in \{1, 2\}$. *Define*

- $\underline{l} \in \mathbb{Z}^r$ *so that* $l_i = 1$ *if* $b_i = 2$, *and* $l_i = 0$ *if* $b_i = 1$;
- $\underline{\eta} \in \{\pm 1\}^r$ *so that* $\eta_i = \eta(\rho \boxtimes S_{a_i})$ *if* $b_i = 1$;
- $\eta_0 \in \widehat{\mathcal{A}_{\phi_0}}$ *by* $\eta_0 = \eta | \mathcal{A}_{\phi_0}$.

Then it is easy to see that $\pi(\psi, \underline{l}, \underline{\eta}, \eta_0) = L(\Delta_\rho[x-1, -x]^b; \pi(\phi, \eta))$.

Similarly, we have the following proposition, which is a key observation.

Proposition 3.13. *Let* $\phi \in \Phi_{\mathrm{gp}}(G)$ *and* $\eta \in \widehat{\mathcal{S}_\phi}$. *Fix* $x \in (1/2)\mathbb{Z}$ *with* $x \geq 1$. *Suppose that* $\rho \boxtimes S_{2x+1}$ *is self-dual of the same type as* ϕ. *Assume that* $m_\phi(\rho \boxtimes S_{2x+1}) \neq 0$, $m_\phi(\rho \boxtimes S_{2x-1}) \neq 0$ *and*
$$\eta(\rho \boxtimes S_{2x+1})\eta(\rho \boxtimes S_{2x-1}) = (-1)^{b+1}.$$
Then $L(\Delta_\rho[x-1, -x]^b; \pi(\phi, \eta)) \in \Pi_\psi$ *with*
$$\psi = \phi - \rho \boxtimes (S_{2x+1} + S_{2x-1}) + (\rho \boxtimes S_{2x} \boxtimes S_2)^{\oplus b+1}.$$

More precisely, when we write

$$\psi = \phi_0 + \left(\bigoplus_{i=1}^{r} \rho \boxtimes S_{a_i} \boxtimes S_{b_i} \right)$$

as above such that $a_1 \leq \cdots \leq a_r$ *and* $b_i \in \{1, 2\}$*, define*

- $\underline{l} \in \mathbb{Z}^r$ *so that* $l_i = 0$ *for any* $1 \leq i \leq r$*;*
- $\underline{\eta} \in \{\pm 1\}^r$ *so that*

$$\eta_i = \begin{cases} \eta(\rho \boxtimes S_{a_i}) & \text{if } b_i = 1, \\ (-1)^{\#\{j<i \mid b_j = 2\}} \eta(\rho \boxtimes S_{2x-1}) & \text{if } b_i = 2; \end{cases}$$

- $\eta_0 \in \widehat{\mathcal{A}_{\phi_0}}$ *by* $\eta_0 = \eta | \mathcal{A}_{\phi_0}$.

Then $\pi(\psi, \underline{l}, \underline{\eta}, \eta_0) = L(\Delta_\rho[x-1, -x]^b; \pi(\phi, \eta))$*. In particular, if* $D^{(k)}_{\rho|\cdot|^x}(L(\Delta_\rho[x-1, -x]^b; \pi(\phi, \eta))) \neq 0$*, then* $k \leq m_\phi(\rho \boxtimes S_{2x+1}) - 1$.

Proof. When $b = 0$ or $b = 1$, it follows from Theorem 3.11 together with Proposition 3.6. Suppose that $b \geq 2$. Take

$$\psi' = \psi - \rho \boxtimes S_{2x} \boxtimes S_2 + \rho \boxtimes S_{2x+2} \boxtimes S_2 - \left(\bigoplus_{\substack{1 \leq i \leq r \\ a_i \geq 2x+1}} \rho \boxtimes S_{a_i} \right) + \left(\bigoplus_{\substack{1 \leq i \leq r \\ a_i \geq 2x+1}} \rho \boxtimes S_{a'_i} \right)$$

such that $a'_r > a'_{r-1} > \cdots$ are sufficiently large so that $m_{\psi'}(\rho \boxtimes S_{2x+1}) = m_{\psi'}(\rho \boxtimes S_{2x+3}) = 0$. By Theorem 3.11 (4), we have $\pi(\psi, \underline{l}, \underline{\eta}, \eta_0) = J_2 \circ J_1(\pi(\psi', \underline{l}, \underline{\eta}, \eta_0))$ with

$$J_1 = \mathrm{Jac}_{\rho|\cdot|^x, \rho|\cdot|^{x+1}},$$

$$J_2 = \left(\mathrm{Jac}_{\rho|\cdot|^{\frac{a'_r - 1}{2}}, \dots, \rho|\cdot|^{\frac{a_r+1}{2}}} \right) \circ \left(\mathrm{Jac}_{\rho|\cdot|^{\frac{a'_{r-1}-1}{2}}, \dots, \rho|\cdot|^{\frac{a_{r-1}+1}{2}}} \right) \circ \cdots.$$

Since $l_i = 0$ for any i, we may apply the induction hypothesis to $\pi(\psi', \underline{l}, \underline{\eta}, \eta_0)$. Hence $\pi(\psi', \underline{l}, \underline{\eta}, \eta_0) = L(\Delta_\rho[x-1, -x]^{b-1}; \pi(\phi', \eta'))$, where

$$\begin{aligned} \phi' &= \psi' - (\rho \boxtimes S_{2x} \boxtimes S_2)^{\oplus b} - \rho \boxtimes S_{2x+2} \boxtimes S_2 + \rho \boxtimes (S_{2x-1} + S_{2x+1}^{\oplus 2} + S_{2x+3}) \\ &= \phi + \rho \boxtimes (S_{2x+1} + S_{2x+3}) - \left(\bigoplus_{\substack{1 \leq i \leq r \\ a_i \geq 2x+1}} \rho \boxtimes S_{a_i} \right) + \left(\bigoplus_{\substack{1 \leq i \leq r \\ a_i \geq 2x+1}} \rho \boxtimes S_{a'_i} \right). \end{aligned}$$

Note that $m_{\phi'}(\rho \boxtimes S_{2x+1}) = 2$, $m_{\phi'}(\rho \boxtimes S_{2x+3}) = 1$ and $\eta'(\rho \boxtimes S_{2x-1})\eta'(\rho \boxtimes S_{2x+1}) = (-1)^b$.

Take $\kappa \in \{0, 1\}$ such that $b \equiv \kappa \bmod 2$. By Proposition 3.10, $\Delta_\rho[x-1, -x] \rtimes L(\Delta_\rho[x-1, -x]^{1-\kappa}; \pi(\phi', \eta'))$ is irreducible. Note that any irreducible subquotient of $\Delta_\rho[x-1, -x]^{2c}$ is of the form

$$L(\Delta_\rho[x-1, -x]; \Delta_\rho[x, -(x-1)])^\alpha \times (\Delta_\rho[x, -x] \times \Delta_\rho[x-1, -(x-1)])^\beta$$

with $\alpha + \beta = c$. Considering the Langlands data of $\pi(\psi', \underline{l}, \underline{\eta}, \eta_0)$, we see that it is a subrepresentation of

$$L(\Delta_\rho[x-1, -x]; \Delta_\rho[x, -(x-1)])^{\frac{b-2+\kappa}{2}} \rtimes L(\Delta_\rho[x-1, -x]^{1-\kappa}; \pi(\phi', \eta')).$$

Note that this induced representation is irreducible. Indeed, it is unitary so that it is semisimple, but since it is subrepresentation of a standard module, it has a unique irreducible subrepresentation. Hence $\pi(\psi, \underline{l}, \underline{\eta}, \eta_0)$ is equal to

$$L(\Delta_\rho[x-1,-x];\Delta_\rho[x,-(x-1)])^{\frac{b-2+\kappa}{2}} \rtimes J_2 \circ J_1(L(\Delta_\rho[x,-(x-1)]^{1-\kappa};\pi(\phi',\eta'))).$$

Since $\mathrm{Jac}_{\rho|\cdot|^{x+1}}(\pi(\phi',\eta')) = 0$, we have

$$J_2 \circ J_1(L(\Delta_\rho[x,-(x-1)]^{1-\kappa};\pi(\phi',\eta'))) = L(\Delta_\rho[x,-(x-1)]^{1-\kappa}; J_2 \circ J_1(\pi(\phi',\eta'))).$$

By [2, Theorem 4.2], we have

$$J_2 \circ J_1(\pi(\phi',\eta')) = L(\Delta_\rho[x,-(x-1)];\pi(\phi,\eta)).$$

Therefore,

$$\begin{aligned}\pi(\psi, \underline{l}, \underline{\eta}, \eta_0) &= L(\Delta_\rho[x-1,-x];\Delta_\rho[x,-(x-1)])^{\frac{b-2+\kappa}{2}} \rtimes L(\Delta_\rho[x,-(x-1)]^{2-\kappa};\pi(\phi,\eta)) \\ &\cong L(\Delta_\rho[x,-(x-1)]^{b};\pi(\phi,\eta)),\end{aligned}$$

as desired. □

When $x = 1$, we have the following more general version.

Proposition 3.14. *Let $\phi \in \Phi_{\mathrm{gp}}(G)$ and $\eta \in \widehat{\mathcal{S}_\phi}$. Suppose that ρ is self-dual of the same type as ϕ. Assume that $m_\phi(\rho) \neq 0$, $m_\phi(\rho \boxtimes S_3) \neq 0$ and*

$$\eta(\rho)\eta(\rho \boxtimes S_3) = (-1)^{b+1}.$$

If $a \leq m_\phi(\rho) - 1$, then $L((\rho|\cdot|^{-1})^a, \Delta_\rho[0,-1]^b; \pi(\phi,\eta)) \in \Pi_\psi$ with

$$\begin{aligned}\psi = \phi - \rho^{\oplus a} &+ (\rho \boxtimes S_1 \boxtimes S_3)^{\oplus a} \\ &- (\rho + \rho \boxtimes S_3) + (\rho \boxtimes S_2 \boxtimes S_2)^{\oplus b+1}.\end{aligned}$$

In particular, if $D^{(k)}_{\rho|\cdot|^1}(\pi) \neq 0$, then $k \leq m_\phi(\rho \boxtimes S_3) - 1$.

The proof is the same as the one of Proposition 3.13, but it requires Mœglin's construction more generally since ψ contains $\rho \boxtimes S_1 \boxtimes S_3$. We omit the detail.

4. Derivatives of certain representations

Let $\phi \in \Phi_{\mathrm{gp}}(G)$ and $\eta \in \widehat{\mathcal{S}_\phi}$. Fix $x \in (1/2)\mathbb{Z}$ with $x > 0$ such that $\rho \boxtimes S_{2x+1}$ is self-dual of the same type as ϕ. Let $\pi = L((\rho|\cdot|^{-x})^a, \Delta_\rho[x-1,-x]^b; \pi(\phi,\eta))$. By Jantzen's algorithm recalled in §2.5, the computation of the highest derivatives of arbitrary irreducible representations is reduced to the one of $D^{(k)}_{\rho|\cdot|^x}(\pi)$. In this section, we give an algorithm to compute this for $x \geq 1$.

4.1. Statements.

The following is an algorithm.

Theorem 4.1. *Suppose that $x \geq 1$.*

(1) *If $m_\phi(\rho \boxtimes S_{2x+1}) \neq 0$, $m_\phi(\rho \boxtimes S_{2x-1}) \neq 0$ and $\eta(\rho \boxtimes S_{2x+1})\eta(\rho \boxtimes S_{2x-1}) = (-1)^{b+1}$, set*

$$\psi = \phi - \rho \boxtimes (S_{2x+1} + S_{2x-1}) + (\rho \boxtimes S_{2x} \boxtimes S_2)^{\oplus b+1}.$$

Otherwise, set $\psi = \phi + (\rho \boxtimes S_{2x} \boxtimes S_2)^{\oplus b}$. Then $L(\Delta_\rho[x-1,-x]^b; \pi(\phi,\eta)) \in \Pi_\psi$.

(2) *Put* $m = m_\psi(\rho \boxtimes S_{2x+1})$, $m' = m_\psi(\rho \boxtimes S_{2x-1})$, *and*
$$l = \min\{a - m', 0\}.$$
Then $D^{(l+m)}_{\rho|\cdot|^x}(L((\rho|\cdot|^{-x})^a, \Delta_\rho[x-1,-x]^b; \pi(\phi,\eta)))$ *is the highest derivative. Moreover, it is a unique irreducible subrepresentation of*
$$(\rho|\cdot|^{-x})^{a-l} \rtimes D^{(m)}_{\rho|\cdot|^x}(L(\Delta_\rho[x-1,-x]^b; \pi(\phi,\eta))).$$

(3) *If we set* $\psi' = \psi - (\rho \boxtimes S_{2x+1})^{\oplus m} + (\rho \boxtimes S_{2x-1})^{\oplus m}$, *then* $D^{(m)}_{\rho|\cdot|^x}(L(\Delta_\rho[x-1,-x]^b; \pi(\phi,\eta))) \in \Pi_{\psi'}$. *In particular, if we write*
$$\psi = \phi_0 + \left(\bigoplus_{i=1}^r \rho \boxtimes S_{a_i} \boxtimes S_{b_i}\right), \quad \psi' = \phi_0 + \left(\bigoplus_{i=1}^r \rho \boxtimes S_{a'_i} \boxtimes S_{b'_i}\right)$$
such that $a_1 \leq \cdots \leq a_r$, $a'_1 \leq \cdots \leq a'_r$, *and such that* $\phi_0 \not\supset \rho \boxtimes S_d$ *for any* $d > 0$, *then*
$$\pi(\psi, \underline{l}, \underline{\eta}, \eta_0) = L(\Delta_\rho[x-1,-x]^b; \pi(\phi,\eta)),$$
$$\pi(\psi', \underline{l'}, \underline{\eta'}, \eta'_0) = D^{(m)}_{\rho|\cdot|^x}(L(\Delta_\rho[x-1,-x]^b; \pi(\phi,\eta)))$$
for some data. These are related as follows:
- $\eta_0 = \eta'_0 = \eta|\mathcal{A}_{\phi_0}$;
- *if* $b'_i = 1$, *then* $l'_i = 0$;
- *if* $b'_i = b_j = 2$, *then* $l'_i = l_j$ *if and only if* m *is even;*
- *if* $b'_i = 1$ *and* $a'_i \neq 2x-1$, *then* $\eta'_i = \eta(\rho \boxtimes S_{a_i})$;
- *if* $b'_i = 1$ *and* $a'_i = 2x-1$, *or if* $b'_i = 2$ *and* $l'_i = 0$, *then*
$$\eta'_i = \eta(\rho \boxtimes S_{2x-1}) \quad \text{or} \quad \eta'_i = (-1)^{m_\psi(\rho \boxtimes S_{2x} \boxtimes S_2)} \eta(\rho \boxtimes S_{2x+1}).$$
These values are equal to each other if $\phi \supset \rho \boxtimes (S_{2x-1} + S_{2x+1})$.

(4) *We can compute the Langlands data of* $\pi(\psi', \underline{l'}, \underline{\eta'}, \eta'_0)$ *by using Example 3.12 and Proposition 3.13.*

We can also write down this theorem more explicitly. Let $\kappa \in \{0,1\}$ such that $\kappa \equiv b \bmod 2$. Set $m_d = m_\phi(\rho \boxtimes S_d)$ for $d \geq 1$. For $d \geq 3$, define
$$\delta_d = \begin{cases} 1 & \text{if } m_d m_{d-2} \neq 0 \text{ and } \eta(\rho \boxtimes S_d) \neq \eta(\rho \boxtimes S_{d-2}), \\ 0 & \text{otherwise.} \end{cases}$$
Put
$$\phi' = \phi - (\rho \boxtimes S_{2x+1})^{\oplus m_{2x+1}} + (\rho \boxtimes S_{2x-1})^{\oplus m_{2x+1}}.$$
Let $\eta' \in \widehat{\mathcal{A}_{\phi'}}$ be the pullback of η via the canonical map $\mathcal{A}_{\phi'} \hookrightarrow \mathcal{A}_\phi \twoheadrightarrow \mathcal{S}_\phi$. The following is a reformulation of Theorem 4.1, which follows from Example 3.12 and Proposition 3.13 immediately.

Corollary 4.2. *The notation is as above. Let* $\pi = L((\rho|\cdot|^{-x})^a, \Delta_\rho[x-1,-x]^b; \pi(\phi,\eta))$. *Then* $D^{(k)}_{\rho|\cdot|^x}(\pi)$ *is the highest derivative with*
$$k = \begin{cases} \max\{a - m_{2x-1} + m_{2x+1}, m_{2x+1} - 1\} & \text{if } m_{2x-1}m_{2x+1} \neq 0, \kappa \neq \delta_{2x+1}, \\ \max\{a - m_{2x-1} + m_{2x+1}, m_{2x+1}\} & \text{otherwise.} \end{cases}$$
Moreover:

(1) *Suppose that* $m_{2x+1} = 0$. *Set* $l = \max\{a - m_{2x-1}, 0\}$. *Then*

$$D^{(l)}_{\rho|\cdot|^x}(\pi) = L((\rho|\cdot|^{-x})^{a-l}, \Delta_\rho[x-1,-x]^b; \pi(\phi,\eta)).$$

(2) *Suppose that* $m_{2x-1}m_{2x+1} \neq 0$. *Set*

$$l = \begin{cases} \max\{a - m_{2x-1}, 0\} & \text{if } \kappa = \delta_{2x+1}, \\ \max\{a - m_{2x-1} + 1, 0\} & \text{if } \kappa \neq \delta_{2x+1}. \end{cases}$$

(a) *If* $\kappa \neq \delta_{2x+1}$, *and if* m_{2x+1} *is odd, then*

$$D^{(l+m_{2x+1}-1)}_{\rho|\cdot|^x}(\pi) = L\left((\rho|\cdot|^{-x})^{a-l}, \Delta_\rho[x-1,-x]^b; \pi(\phi' + \rho \boxtimes (S_{2x+1} - S_{2x-1}), \eta')\right).$$

(b) *If* $\kappa \neq \delta_{2x+1}$, *and if* m_{2x+1} *is even, then*

$$D^{(l+m_{2x+1}-1)}_{\rho|\cdot|^x}(\pi) = L\left((\rho|\cdot|^{-x})^{a-l}, \Delta_\rho[x-1,-x]^{b+1}; \pi(\phi' - (\rho \boxtimes S_{2x-1})^{\oplus 2}, \eta')\right).$$

(c) *If* $\kappa = \delta_{2x+1}$, *and if* m_{2x+1} *is odd and* $b > 0$, *then*

$$D^{(l+m_{2x+1})}_{\rho|\cdot|^x}(\pi) = L\left((\rho|\cdot|^{-x})^{a-l}, \Delta_\rho[x-1,-x]^{b-1}; \pi(\phi' + \rho \boxtimes (S_{2x+1} + S_{2x-1}), \eta')\right).$$

(d) *If* $\kappa = \delta_{2x+1}$, *and if* m_{2x+1} *is even or* $b = 0$, *then*

$$D^{(l+m_{2x+1})}_{\rho|\cdot|^x}(\pi) = L\left((\rho|\cdot|^{-x})^{a-l}, \Delta_\rho[x-1,-x]^b; \pi(\phi', \eta')\right).$$

(3) *Suppose that* $m_{2x-1} = 0$.

(a) *If* m_{2x+1} *is even or* $b = 0$, *then*

$$D^{(a+m_{2x+1})}_{\rho|\cdot|^x}(\pi) = L(\Delta_\rho[x-1,-x]^b; \pi(\phi', \eta'_b)),$$

where $\eta'_b(\rho \boxtimes S_{2x-1}) = (-1)^b \eta(\rho \boxtimes S_{2x+1})$.

(b) *If* m_{2x+1} *is odd and* $b > 0$, *then*

$$D^{(a+m_{2x+1})}_{\rho|\cdot|^x}(\pi) = L(\Delta_\rho[x-1,-x]^{b-1}; \pi(\phi' + \rho \boxtimes (S_{2x-1} + S_{2x+1}), \eta'_b)),$$

where $\eta'_b(\rho \boxtimes S_{2x-1}) = (-1)^b \eta(\rho \boxtimes S_{2x+1})$ *and* $\eta'_b(\rho \boxtimes S_{2x+1}) = \eta(\rho \boxtimes S_{2x+1})$.

Remark 4.3. *When* $x = 1/2$, *we formally understand that* $m_0 = 1$, $\eta(\rho \boxtimes S_0) = +1$ *and* $a = 0$. *Then Corollary 4.2 still holds for* $x = 1/2$ *(*[10, Theorem 3.3]*). Note that when* $x = 1/2$, *the case (3) does not appear.*

Recall that for $k' \geq 0$, the parabolically induced representation $(\rho|\cdot|^x)^{k'} \rtimes D^{(k)}_{\rho|\cdot|^x}(\pi)$ has a unique irreducible subrepresentation π'. When $k' = 0$ (resp. $k' = k$), we have $\pi' = D^{(k)}_{\rho|\cdot|^x}(\pi)$ (resp. $\pi' = \pi$). When $0 < k' < k$, the following formula follows from Corollary 4.2 immediately.

Corollary 4.4. *Let* $\pi = L((\rho|\cdot|^{-x})^a, \Delta_\rho[x-1,-x]^b; \pi(\phi,\eta))$ *be as in Corollary 4.2, and* $D^{(k)}_{\rho|\cdot|^x}(\pi)$ *be its highest derivative. Set*

$$a_0 = \begin{cases} \min\{a, m_{2x-1}\} & \text{if } \kappa = \delta_{2x+1}, \\ \min\{a, m_{2x-1} - 1\} & \text{if } \kappa \neq \delta_{2x+1}. \end{cases}$$

Suppose that $0 < k' < k$. *Then the unique irreducible subrepresentation* π' *of* $(\rho|\cdot|^x)^{k'} \rtimes D^{(k)}_{\rho|\cdot|^x}(\pi)$ *is of the form* $L((\rho|\cdot|^{-x})^{a_{k'}}, \Delta_\rho[x-1,-x]^{b_{k'}}; \pi(\phi_{k'}, \eta_{k'}))$, *where* $(a_{k'}, b_{k'}, \phi_{k'}, \eta_{k'})$ *is given as follows.*

(1) *Suppose that* $m_{2x+1} = 0$. *Then* $k = a - m_{2x-1} > 0$ *and*

$$(a_{k'}, b_{k'}, \phi_{k'}, \eta_{k'}) = (k' + m_{2x-1}, b, \phi, \eta).$$

(2) *Suppose that* $m_{2x-1}m_{2x+1} \neq 0$.

(a) *If* $\kappa \neq \delta_{2x+1}$, *then* $(a_{k'}, b_{k'}, \phi_{k'}, \eta_{k'})$ *is equal to*

$$\begin{cases} (a_0, b, \phi' + (\rho \boxtimes S_{2x+1})^{\oplus k'+1} - (\rho \boxtimes S_{2x-1})^{\oplus k'+1}, \eta') & \text{if } k' < m_{2x+1} - 1, k' \not\equiv m_{2x+1} \bmod 2, \\ (a_0, b+1, \phi' + (\rho \boxtimes S_{2x+1})^{\oplus k'} - (\rho \boxtimes S_{2x-1})^{\oplus k'+2}, \eta') & \text{if } k' < m_{2x+1} - 1, k' \equiv m_{2x+1} \bmod 2, \\ (k' - m_{2x+1} + m_{2x-1}, b, \phi, \eta) & \text{if } k' \geq m_{2x+1} - 1. \end{cases}$$

(b) *If* $\kappa = \delta_{2x+1}$ *and* $b > 0$, *then* $(a_{k'}, b_{k'}, \phi_{k'}, \eta_{k'})$ *is equal to*

$$\begin{cases} (a_0, b, \phi' + (\rho \boxtimes S_{2x+1})^{\oplus k'} - (\rho \boxtimes S_{2x-1})^{\oplus k'}, \eta') & \text{if } k' < m_{2x+1}, k' \equiv m_{2x+1} \bmod 2, \\ (a_0, b-1, \phi' + (\rho \boxtimes S_{2x+1})^{\oplus k'+1} - (\rho \boxtimes S_{2x-1})^{\oplus k'-1}, \eta') & \text{if } k' < m_{2x+1}, k' \not\equiv m_{2x+1} \bmod 2, \\ (k' - m_{2x+1} + m_{2x-1}, b, \phi, \eta) & \text{if } k' \geq m_{2x+1}. \end{cases}$$

Similarly, if $\kappa = \delta_{2x+1}$ *and* $b = 0$, *then* $(a_{k'}, b_{k'}, \phi_{k'}, \eta_{k'})$ *is equal to*

$$\begin{cases} (a_0, 0, \phi' + (\rho \boxtimes S_{2x+1})^{\oplus k'} - (\rho \boxtimes S_{2x-1})^{\oplus k'}, \eta') & \text{if } k' < m_{2x+1}, \\ (k' - m_{2x+1} + m_{2x-1}, 0, \phi, \eta) & \text{if } k' \geq m_{2x+1}. \end{cases}$$

(3) *Suppose that* $m_{2x-1} = 0$. *If* $b > 0$, *then* $(a_{k'}, b_{k'}, \phi_{k'}, \eta_{k'})$ *is equal to*

$$\begin{cases} (0, b, \phi' + (\rho \boxtimes S_{2x+1})^{\oplus k'} - (\rho \boxtimes S_{2x-1})^{\oplus k'}, \eta'_b) & \text{if } k' < m_{2x+1}, k' \equiv m_{2x+1} \bmod 2, \\ (0, b-1, \phi' + (\rho \boxtimes S_{2x+1})^{\oplus k'+1} - (\rho \boxtimes S_{2x-1})^{\oplus k'-1}, \eta'_b) & \text{if } k' < m_{2x+1}, k' \not\equiv m_{2x+1} \bmod 2, \\ (k' - m_{2x+1}, b, \phi, \eta) & \text{if } k' \geq m_{2x+1}. \end{cases}$$

Similarly, if $b = 0$, *then* $(a_{k'}, b_{k'}, \phi_{k'}, \eta_{k'})$ *is equal to*

$$\begin{cases} (0, 0, \phi' + (\rho \boxtimes S_{2x+1})^{\oplus k'} - (\rho \boxtimes S_{2x-1})^{\oplus k'}, \eta'_b) & \text{if } k' < m_{2x+1}, \\ (k' - m_{2x+1}, 0, \phi, \eta) & \text{if } k' \geq m_{2x+1}. \end{cases}$$

4.2. **Examples of the Aubert duals.** Let π be an irreducible representation of G_n. By Theorem 2.13 and Proposition 3.8, if one could compute $M^+_{\rho^\vee}(\pi)$ for all ρ, one would obtain the Langlands data of the Aubert dual $\hat{\pi}$ of π almost explicitly. In this subsection, we give some specific examples of explicit calculations of $M^+_{\rho^\vee}(\pi)$ and $\hat{\pi}$ for some irreducible representations π of $G_n = \mathrm{Sp}_{2n}(F)$. When $\phi = S_{d_1} \oplus \cdots \oplus S_{d_t} \in \Phi(G)$ and $\eta_i = \eta(S_{d_i})$, where $d_1, \ldots, d_t$ are all odd, we write

$$\pi(\phi, \eta) = \pi(d_1^{\eta_1}, \ldots, d_t^{\eta_t}).$$

We also write $\mathbf{1} = \mathbf{1}_{\mathrm{GL}_1(F)}$, and $\Delta[x,y] = \Delta_{\mathbf{1}}[x,y]$.

Let us consider $\pi = L(\Delta[0,-2], \Delta[0,-1]; \pi(3^-, 3^-, 5^+)) \in \mathrm{Irr}(\mathrm{Sp}_{20}(F))$. Note that $\mathrm{Jac}_{|\cdot|^x}(\pi) \neq 0 \implies x \in \{0, 1, 2\}$. We compute the highest derivatives by applying Jantzen's algorithm recalled in §2.5. For $x = 2$, we have:

(1) $\pi \hookrightarrow L(\Delta[0,-2], \Delta[0,-1]) \rtimes \pi(3^-, 3^-, 5^+)$;

(2) $\pi \hookrightarrow \Delta[0,-1]^2 \rtimes L(|\cdot|^{-2}; \pi(3^-,3^-,5^+))$;
(3) by Corollary 4.2, we have $\mathrm{Jac}_{|\cdot|^2}(L(|\cdot|^{-2}; \pi(3^-,3^-,5^+))) = 0$;
(4) we conclude that $\mathrm{Jac}_{|\cdot|^2}(\pi) = 0$.

For $x = 1$, we have:

(1)–(2) $\pi \hookrightarrow \Delta[0,-2] \rtimes L(\Delta[0,-1]; \pi(3^-,3^-,5^+))$;
(3) by Corollary 4.2 (3)-(a), we have $D^{(2)}_{|\cdot|^1}(L(\Delta[0,-1]; \pi(3^-,3^-,5^+))) = L(\Delta[0,-1]; \pi(1^+,1^+,5^+))$ so that

$$\pi \hookrightarrow L(\Delta[0,-2], (|\cdot|^1)^2) \rtimes L(\Delta[0,-1]; \pi(1^+,1^+,5^+));$$

(4)–(5) $D_{|\cdot|^1}(\pi) \hookrightarrow L(\Delta[0,-2], |\cdot|^1) \rtimes L(\Delta[0,-1]; \pi(1^+,1^+,5^+))$;
(6)–(7) by Corollary 4.4 (3), we find that $D^{(1)}_{|\cdot|^1}(\pi(1^+,1^+,3^-,3^-,5^+)) = L(\Delta[0,-1]; \pi(1^+,1^+,5^+))$ so that

$$D_{|\cdot|^1}(\pi) \hookrightarrow \Delta[0,-2] \rtimes \pi(1^+,1^+,3^-,3^-,5^+).$$

Hence

$$M^+_{\mathbf{1}}(\pi) = \left[(1,1); M^+_{\mathbf{1}}\left(L(\Delta[0,-2]; \pi(1^+,1^+,3^-,3^-,5^+))\right)\right].$$

Set $\pi_1 = L(\Delta[0,-2]; \pi(1^+,1^+,3^-,3^-,5^+))$. Note that $\mathrm{Jac}_{|\cdot|^x}(\pi_1) \neq 0 \implies x \in \{0,2\}$. By a similar computation as above, we have $\mathrm{Jac}_{|\cdot|^2}(\pi_1) = 0$. Since

$$\begin{aligned}\pi_1 &\hookrightarrow \Delta[0,-2] \rtimes \pi(1^+,1^+,3^-,3^-,5^+)\\ &\hookrightarrow \mathbf{1}^2 \times \Delta[-1,-2] \rtimes \pi(3^-,3^-,5^+),\end{aligned}$$

we have

$$M^+_{\mathbf{1}}(\pi_1) = \left[(0,2); M^+_{\mathbf{1}}\left(L(\Delta[-1,-2]; \pi(3^-,3^-,5^+))\right)\right].$$

Set $\pi_2 = L(\Delta[-1,-2]; \pi(3^-,3^-,5^+))$. Note that $\mathrm{Jac}_{|\cdot|^x}(\pi_2) \neq 0 \implies x \in \{-1,1,2\}$. By a similar computation as above, we have $\mathrm{Jac}_{|\cdot|^2}(\pi_2) = 0$. Since

$$\begin{aligned}\pi_2 &\hookrightarrow \Delta[-1,-2] \rtimes \pi(3^-,3^-,5^+)\\ &\hookrightarrow \Delta[-1,2] \times (|\cdot|^1)^2 \rtimes \pi(1^-,1^-,5^+)\\ &\hookrightarrow (|\cdot|^1)^2 \times \Delta[-1,2] \rtimes \pi(1^-,1^-,5^+),\end{aligned}$$

we have

$$M^+_{\mathbf{1}}(\pi_2) = \left[(1,2); M^+_{\mathbf{1}}\left(L(\Delta[-1,-2]; \pi(1^-,1^-,5^+))\right)\right].$$

Set $\pi_3 = L(\Delta[-1,-2]; \pi(1^-,1^-,5^+))$. Note that $\mathrm{Jac}_{|\cdot|^x}(\pi_3) \neq 0 \implies x \in \{-1,0,2\}$. By applying Jantzen's algorithm for $x = 2$, we have:

(1) $\pi_3 \hookrightarrow \Delta[-1,-2] \rtimes \pi(1^-,1^-,5^+)$;
(2) $\pi_3 \hookrightarrow |\cdot|^{-1} \rtimes L(|\cdot|^{-2}; \pi(1^-,1^-,5^+))$;
(3) by Corollary 4.2 (3)-(a), we have $D^{(2)}_{|\cdot|^2}(L(|\cdot|^{-2}; \pi(1^-,1^-,5^+))) = \pi(1^-,1^-,3^+)$ so that

$$\pi_3 \hookrightarrow L(|\cdot|^{-1}, (|\cdot|^2)^2) \rtimes \pi(1^-,1^-,3^+);$$

(4)–(7) $D^{(2)}_{|\cdot|^2}(\pi_3) \hookrightarrow |\cdot|^{-1} \rtimes \pi(1^-,1^-,3^+)$.

Hence

$$M_{\mathbf{1}}^{+}(\pi_3) = \left[(2,2); M_{\mathbf{1}}^{+}\left(L(|\cdot|^{-1};\pi(1^-,1^-,3^+))\right)\right].$$

Set $\pi_4 = L(|\cdot|^{-1};\pi(1^-,1^-,3^+))$. Note that $\mathrm{Jac}_{|\cdot|^x}(\pi_4) \neq 0 \implies x \in \{-1,0,1\}$. Since we have now

$$M_{\mathbf{1}}^{+}(\pi) = [(1,1),(0,2),(1,2),(2,2); M_{\mathbf{1}}^{+}(\pi_4)],$$

by applying Theorem 2.9 to π, we see that $\mathrm{Jac}_{|\cdot|^1}(\pi_4) = \mathrm{Jac}_{|\cdot|^0}(\pi_4) = 0$. Therefore,

$$M_{\mathbf{1}}^{+}(\pi_4) = \left[(-1,1); M_{\mathbf{1}}^{+}\left(\pi(1^-,1^-,3^+)\right)\right].$$

Finally, it is easy to see from Proposition 3.6 that $M_{\mathbf{1}}^{+}(\pi(1^-,1^-,3^+)) = [(0,1),(1,1);\pi(1^+)]$. Hence we conclude that

$$M_{\mathbf{1}}^{+}(\pi) = [(1,1),(0,2),(1,2),(2,2),(-1,1),(0,1),(1,1);\pi(1^+)].$$

Therefore,

$$M_{\mathbf{1}}^{-}(\hat{\pi}) = [(-1,1),(0,2),(-1,2),(-2,2),(1,1),(0,1),(-1,1);\pi(1^+)].$$

By Theorem 2.13 and Proposition 3.8, we have

$$\hat{\pi} \hookrightarrow |\cdot|^{-1} \times \Delta[0,-2]^2 \rtimes \pi(1^+,3^-,3^-).$$

Remark 4.5. *This method does not always determine $\hat{\pi}$. For example, let us consider $\pi_\epsilon = L(\Delta[0,-2];\pi(1^\epsilon,1^\epsilon,3^+)) \in \mathrm{Irr}(\mathrm{Sp}_{10}(F))$ for a sign $\epsilon \in \{\pm\}$. Then for any $\epsilon \in \{\pm\}$, we have*

$$M_{\mathbf{1}}^{+}(\pi_\epsilon) = \left[(0,2),(1,1),(2,1),(-1,1);\pi(1^+)\right].$$

Using Theorem 2.13 and Proposition 3.8, this implies that $\hat{\pi}_\epsilon = \pi_{\hat{\epsilon}}$ for some $\hat{\epsilon} \in \{\pm\}$. However, the correspondence $\epsilon \mapsto \hat{\epsilon}$ is not determined in this stage. In the next paper with Mínguez [3], we will give an another idea. As a conclusion, one can find that $\hat{\epsilon} = \epsilon$, i.e., π_ϵ is fixed by the Aubert involution.

5. Proof of Theorem 4.1

In this section, we prove Theorem 4.1.

5.1. The case $a = 0$.

First, we consider the case $a = 0$.

Proof of Theorem 4.1 when $a = 0$. Let $\pi = L(\Delta_\rho[x-1,-x]^b;\pi(\phi,\eta))$. The assertions (1) and (4) follow from Example 3.12 and Proposition 3.13. For (2) and (3), we note that $D^{(k)}_{\rho|\cdot|^x}(\pi) \neq 0 \implies k \leq m$ by Lemma 3.4.

To determine $D^{(m)}_{\rho|\cdot|^x}(\pi)$, we will use Mœglin's construction. Write

$$\begin{aligned}\psi = \phi_0 &+ \rho \boxtimes (S_{a_{-t'}} + \cdots + S_{a_{-1}}) \\ &+ (\rho \boxtimes S_{2x-1})^{\oplus m'} + (\rho \boxtimes S_{2x} \boxtimes S_2)^{\oplus b'} + (\rho \boxtimes S_{2x+1})^{\oplus m} \\ &+ \rho \boxtimes (S_{a_1} + \cdots + S_{a_t}),\end{aligned}$$

where $a_{-t'} \leq \cdots \leq a_{-1} < 2x-1$ and $2x+1 < a_1 \leq \cdots \leq a_t$, and $\phi_0 \not\supset \rho \boxtimes S_d$ for any $d > 0$. Take an A-parameter of the form

$$\begin{aligned}\psi_> = \phi_0 &+ \rho \boxtimes (S_{a_{-t'}} + \cdots + S_{a_{-1}}) \\ &+ (\rho \boxtimes S_{2x-1})^{\oplus m'} + (\rho \boxtimes S_{2x} \boxtimes S_2)^{\oplus b'} + \rho \boxtimes (S_{2y_1+1} + \cdots + S_{2y_m+1}) \\ &+ \rho \boxtimes (S_{a'_1} + \cdots + S_{a'_t}),\end{aligned}$$

where $2y_i+1 \equiv a'_i \equiv 2x+1 \mod 2$ such that $2x+1 < 2y_1+1 < \cdots < 2y_m+1 < a'_1 < \cdots < a'_t$. When $\pi = \pi(\psi, \underline{l}, \underline{\eta}, \eta_0)$, we set $\pi_> = \pi(\psi_>, \underline{l}, \underline{\eta}, \eta_0)$. Then Mœglin's construction says that $\pi = J_2 \circ J_1(\pi_>)$, where

$$J_1 = \mathrm{Jac}_{\rho|\cdot|^{y_m},\dots,\rho|\cdot|^{x+1}} \circ \cdots \circ \mathrm{Jac}_{\rho|\cdot|^{y_1},\dots,\rho|\cdot|^{x+1}},$$
$$J_2 = \mathrm{Jac}_{\rho|\cdot|^{\frac{a'_t-1}{2}},\dots,\rho|\cdot|^{\frac{a_t+1}{2}}} \circ \cdots \circ \mathrm{Jac}_{\rho|\cdot|^{\frac{a'_1-1}{2}},\dots,\rho|\cdot|^{\frac{a_1+1}{2}}}.$$

Set $\pi' = J_2 \circ J'_1(\pi_>)$ with

$$J'_1 = \mathrm{Jac}_{\rho|\cdot|^{y_m},\dots,\rho|\cdot|^{x}} \circ \cdots \circ \mathrm{Jac}_{\rho|\cdot|^{y_1},\dots,\rho|\cdot|^{x}}.$$

Then $\pi' = \pi_{<_\psi}(\psi', \underline{l}, \underline{\eta}, \eta_0)$, where

$$\begin{aligned}\psi' = \phi_0 &+ \rho \boxtimes (S_{d_{-t'}} + \cdots + S_{d_{-1}}) \\ &+ (\rho \boxtimes S_{2x-1})^{\oplus m'} + (\rho \boxtimes S_{2x} \boxtimes S_2)^{\oplus b'} + (\rho \boxtimes S_{2x-1})^{\oplus m} \\ &+ \rho \boxtimes (S_{d_1} + \cdots + S_{d_t}).\end{aligned}$$

However, the order $<_\psi$ for $(\rho \boxtimes S_{2x-1})^{m'+m}$ and $(\rho \boxtimes S_{2x} \boxtimes S_2)^{b'}$ is

$$\underbrace{\rho \boxtimes S_{2x-1} <_\psi \cdots <_\psi \rho \boxtimes S_{2x-1}}_{m'} <_\psi \rho \boxtimes S_{2x} \boxtimes S_2 <_\psi \underbrace{\rho \boxtimes S_{2x-1} <_\psi \cdots <_\psi \rho \boxtimes S_{2x-1}}_{m}.$$

To change the order so that $\rho \boxtimes S_{2x-1} <_{\psi'} \rho \boxtimes S_{2x} \boxtimes S_2$ for all $\rho \boxtimes S_{2x-1}$ and $\rho \boxtimes S_{2x} \boxtimes S_2$, we use a result of Xu [23, Theorem 6.1]. By this theorem, we see that $\pi' = \pi(\psi', \underline{l'}, \underline{\eta'}, \eta_0)$, where $\underline{l'}$ and $\underline{\eta'}$ are given in Theorem 4.1 (3). In particular, $\pi' \neq 0$ by Example 3.12 and Proposition 3.13.

Now, by [21, Corollary 5.4, Lemma 5.7] and Lemma 3.4, one can write

$$\pi_> \hookrightarrow \bigtimes_{i=1}^{m} \Delta_\rho[y_i, x] \rtimes J'_1(\pi_>).$$

Hence $J_1(\pi_>) \hookrightarrow (\rho|\cdot|^x)^m \rtimes J'_1(\pi_>)$ so that we conclude that

$$\pi \hookrightarrow (\rho|\cdot|^x)^m \rtimes \pi'.$$

Since $D^{(m)}_{\rho|\cdot|^x}(\pi)$ is irreducible or zero by Proposition 2.6, we see that $D^{(m)}_{\rho|\cdot|^x}(\pi) = \pi'$. □

5.2. **The case $x > 1$.** In this subsection, we will prove the following proposition.

Proposition 5.1. *Assume that $x > 1$. Consider $\pi = L((\rho|\cdot|^{-x})^a, \Delta_\rho[x-1,-x]^b; \pi(\phi,\eta))$ and $\pi_0 = L(\Delta_\rho[x-1,-x]^b; \pi(\phi,\eta))$. Set*

$$l = \begin{cases} \max\{a - m_{2x-1} + 1, 0\} & \text{if } m_{2x-1}m_{2x+1} \neq 0, \delta_{2x+1} \neq \kappa, \\ \max\{a - m_{2x-1}, 0\} & \text{otherwise.} \end{cases}$$

If $D^{(k_0)}_{\rho|\cdot|^x}(\pi_0)$ and $D^{(k)}_{\rho|\cdot|^x}(\pi)$ are the highest derivatives, then $k - k_0 = l$.

Assume this proposition for a moment. If $l = 0$ so that $k = k_0$, since $x \geq 1$, we have

$$\begin{aligned}\pi &\hookrightarrow (\rho|\cdot|^{-x})^a \rtimes \pi_0 \\ &\hookrightarrow (\rho|\cdot|^x)^{k_0} \times (\rho|\cdot|^{-x})^a \rtimes D^{(k_0)}_{\rho|\cdot|^x}(\pi_0).\end{aligned}$$

Hence we have a non-zero map

$$D^{(k)}_{\rho|\cdot|^x}(\pi) \to (\rho|\cdot|^{-x})^a \rtimes D^{(k_0)}_{\rho|\cdot|^x}(\pi_0).$$

Since $D^{(k)}_{\rho|\cdot|^x}(\pi)$ is irreducible, this map must be an injection. If $l > 0$, then

$$\pi \hookrightarrow (\rho|\cdot|^{-x})^l \rtimes L((\rho|\cdot|^{-x})^{a-l}, \Delta_\rho[x-1,-x]^b; \pi(\phi,\eta)).$$

Since $D^{(k_0)}_{\rho|\cdot|^x}(L((\rho|\cdot|^{-x})^{a-l}, \Delta_\rho[x-1,-x]^b; \pi(\phi,\eta)))$ is the highest derivative, we must have

$$\begin{aligned} D^{(k)}_{\rho|\cdot|^x}(\pi) &= D^{(k_0)}_{\rho|\cdot|^x}(L((\rho|\cdot|^{-x})^{a-l}, \Delta_\rho[x-1,-x]^b; \pi(\phi,\eta))) \\ &\hookrightarrow (\rho|\cdot|^{-x})^{a-l} \rtimes D^{(k_0)}_{\rho|\cdot|^x}(\pi_0). \end{aligned}$$

Therefore, Proposition 5.1 implies Theorem 4.1 (for $x > 1$).

Now we prove Proposition 5.1. The proof uses Jantzen's strategy.

Proof of Proposition 5.1. We note that $x-1>0$. As explained in [10, §3.4], if $\pi_1 = D^{(\alpha)}_{\rho|\cdot|^{x-1}}(\pi)$, $\pi_2 = D^{(\beta)}_{\rho|\cdot|^x}(\pi_1)$ and $\pi_3 = D^{(\gamma)}_{\rho|\cdot|^{x-1}}(\pi_2)$ are the highest derivatives, then $\mathrm{Jac}_{\rho|\cdot|^x}(\pi_3) = 0$ and

$$\pi \hookrightarrow (\rho|\cdot|^x)^{\max\{\beta-\alpha-\gamma,0\}} \times (\rho|\cdot|^{x-1})^{\max\{\alpha-\beta+\gamma,0\}} \times L(\rho|\cdot|^{x-1}, \rho|\cdot|^x)^{\min\{\alpha,\beta-\gamma\}} \times \Delta_\rho[x,x-1]^\gamma \rtimes \pi_3.$$

Hence we see that $D^{(k)}_{\rho|\cdot|^x}(\pi)$ is the highest derivative with

$$k = \max\{\beta-\alpha-\gamma,0\} + \gamma.$$

We compute α,β,γ by a case-by-case consideration using Jantzen's algorithm (§2.5). We have to consider the following cases separately:

(1) $m_{2x-1} = 0$;
(2) $\delta_{2x+1} = 0$, $\delta_{2x-1} = 0$ and $m_{2x-1} > 0$;
(3) $\delta_{2x+1} = 1$ and $\delta_{2x-1} = 0$;
(4) $\delta_{2x+1} = 0$, $\delta_{2x-1} = 1$ and $m_{2x-1} \equiv 1 \bmod 2$;
(5) $\delta_{2x+1} = 1$, $m_{2x+1} \equiv 1 \bmod 2$, $\delta_{2x-1} = 1$ and $m_{2x-1} \equiv 1 \bmod 2$;
(6) $\delta_{2x+1} = 1$, $m_{2x+1} \equiv 0 \bmod 2$, $\delta_{2x-1} = 1$ and $m_{2x-1} \equiv 1 \bmod 2$;
(7) $\delta_{2x+1} = 0$, $\delta_{2x-1} = 1$ and $m_{2x-1} \equiv 0 \bmod 2$;
(8) $\delta_{2x+1} = 1$, $\delta_{2x-1} = 1$ and $m_{2x-1} \equiv 0 \bmod 2$.

For example, suppose that $\delta_{2x+1} = 1$, $\delta_{2x-1} = 1$ and $m_{2x-1} \equiv 0 \bmod 2$. Then with $\phi_1 = \phi - (\rho \boxtimes S_{2x-1})^{\oplus m_{2x-1}} + (\rho \boxtimes S_{2x-3})^{\oplus m_{2x-1}-2}$, we see that

$$\pi_1 = D^{(b+m_{2x-1}-1)}_{\rho|\cdot|^{x-1}}(\pi) = L\left((\rho|\cdot|^{-x})^a, \Delta_\rho[x-2,-x]^b, \Delta_\rho[x-2,-(x-1)]; \pi(\phi_1,\eta_1)\right)$$

is the highest derivative, so that $\alpha = b + m_{2x-1} - 1$. Note that

$$\pi_1 \hookrightarrow L(\rho|\cdot|^{-x}, \Delta_\rho[x-2,-(x-1)]) \times \Delta_\rho[x-2,-(x-1)]^b \times (\rho|\cdot|^{-x})^{a+b-1} \rtimes \pi(\phi_1,\eta_1)$$

if $a > 0$. By [10, Proposition 3.4 (1)], with $\phi_2 = \phi_1 - (\rho \boxtimes S_{2x+1})^{\oplus m_{2x+1}} + (\rho \boxtimes S_{2x-1})^{\oplus m_{2x+1}}$, we see that

$$\pi_2 = D^{(a+b-1+m_{2x+1})}_{\rho|\cdot|^x}(\pi_1) = L(\rho|\cdot|^{-x}, \Delta_\rho[x-2,-(x-1)]^{b+1}; \pi(\phi_2,\eta_2))$$

is the highest derivative, so that $\beta = a + b - 1 + m_{2x+1}$. By Corollary 4.2 for $a = 0$, we have

$$\gamma = k_0 = \begin{cases} m_{2x+1} & \text{if } b \equiv 1 \bmod 2, \\ m_{2x+1} - 1 & \text{if } b \equiv 0 \bmod 2, \end{cases}$$

Therefore, $k - k_0 = \max\{a - m_{2x+1} - \kappa + 1, 0\}$, where $\kappa \in \{0, 1\}$ such that $\kappa \equiv b \bmod 2$.

In every case, a similar argument implies that $k = k_0 + l$ with l being in the assertion. □

5.3. **The case $x = 1$.** In this subsection, we prove:

Proposition 5.2. *Proposition 5.1 holds even when $x = 1$.*

As explained after Proposition 5.1, this proposition also implies Theorem 4.1 for $x = 1$. Note that Jantzen's strategy to determine k from α, β, γ cannot be applied to the case $x = 1$. Instead of this, we use the following proposition.

Proposition 5.3. *Let $\phi \in \Phi_{\mathrm{gp}}(G)$ and $\eta \in \widehat{\mathcal{S}_\phi}$. Suppose that ρ is self-dual of the same type as ϕ. Set*

$$\delta_3 = \begin{cases} 1 & \text{if } \phi \supset \rho \boxtimes (S_1 + S_3) \text{ and } \eta(\rho \boxtimes S_1) \neq \eta(\rho \boxtimes S_3), \\ 0 & \text{otherwise.} \end{cases}$$

Consider $\pi = L((\rho|\cdot|^{-1})^a; \pi(\phi, \eta))$. Then $\rho|\cdot|^{-1} \rtimes \pi$ is irreducible if and only if $a \geq m_\phi(\rho) - \delta_3$.

Proof. Write $m_1 = m_\phi(\rho)$ and $m_3 = m_\phi(\rho \boxtimes S_3)$. By Theorem 3.9, we may assume that $a > 0$. When $a \leq m_1$, by Example 3.5, we have $\pi \in \Pi_{\psi_0}$ with $\psi_0 = \phi - \rho^{\oplus a} + (\rho \boxtimes S_1 \boxtimes S_3)^{\oplus a}$. Also, if $\delta_3 = 1$ and $a \leq m_1 - 1$, by Proposition 3.14, we have $\pi \in \Pi_{\psi_1}$ with

$$\begin{aligned} \psi_1 = \phi - \rho^{\oplus a} &+ (\rho \boxtimes S_1 \boxtimes S_3)^{\oplus a} \\ &- (\rho + \rho \boxtimes S_3) + \rho \boxtimes S_2 \boxtimes S_2. \end{aligned}$$

In particular, when $a \leq m_1 - \delta_3$, if $D^{(k)}_{\rho|\cdot|^1}(\pi) \neq 0$, then $k \leq m_3 - \delta_3$ by Lemma 3.4. Since

$$\begin{aligned} \pi &\hookrightarrow (\rho|\cdot|^{-1})^a \rtimes \pi(\phi, \eta) \\ &\hookrightarrow (\rho|\cdot|^1)^{m_3 - \delta_3} \times (\rho|\cdot|^{-1})^a \rtimes D^{(m_3-\delta_3)}_{\rho|\cdot|^1}(\pi(\phi, \eta)), \end{aligned}$$

we have $D^{(m_3-\delta_3)}_{\rho|\cdot|^1}(\pi) \neq 0$. Since it is irreducible, we have

$$D^{(m_3-\delta_3)}_{\rho|\cdot|^1}(\pi) \hookrightarrow (\rho|\cdot|^{-1})^a \rtimes D^{(m_3-\delta_3)}_{\rho|\cdot|^1}(\pi(\phi, \eta)).$$

This implies the reducibility of $\rho|\cdot|^{-1} \rtimes \pi$ when $a < m_1 - \delta_3$.

Suppose that $a = m_1 - \delta_3 > 0$. We prove the irreducibility of $\rho|\cdot|^{-1} \rtimes \pi$ by induction on a. Suppose that $\rho|\cdot|^{-1} \rtimes \pi$ is reducible and choose an irreducible quotient σ. Note that $D^{(a)}_{\rho|\cdot|^{-1}}(\sigma) \neq 0$. By [9, Proposition 4.1], $\sigma = L((\rho|\cdot|^{-1})^a; \pi(\phi_1, \eta_1))$ or $\sigma = L((\rho|\cdot|^{-1})^a, \Delta_\rho[0, -1]; \pi(\phi_2, \eta_2))$ with

$$\phi_1 = \phi - \rho + \rho \boxtimes S_3, \quad \phi_2 = \phi - \rho^{\oplus 2}.$$

We consider the former case $\sigma = L((\rho|\cdot|^{-1})^a; \pi(\phi_1, \eta_1))$ with $\phi_1 = \phi - \rho + \rho \boxtimes S_3$. Note that $m_{\phi_1}(\rho) = m_1 - 1 \geq \delta_3$. By the induction hypothesis, $\sigma = \rho|\cdot|^{-1} \rtimes L((\rho|\cdot|^{-1})^{a-1}; \pi(\phi_1, \eta_1))$ is an irreducible induction. In particular, $D^{(m_3+2-\delta_3)}_{\rho|\cdot|^1}(\sigma) \neq 0$. Hence $D^{(m_3+2-\delta_3)}_{\rho|\cdot|^1}(\rho|\cdot|^{-1} \rtimes \pi) \neq 0$, which implies that $D^{(m_3+1-\delta_3)}_{\rho|\cdot|^1}(\pi) \neq 0$. This contradicts Lemma 3.4. Now, we consider the latter case $\sigma = L((\rho|\cdot|^{-1})^a, \Delta_\rho[0,-1]; \pi(\phi_2, \eta_2))$ with $\phi_2 = \phi - \rho^{\oplus 2}$ so that $m_1 \geq 2$. Note that $\mathrm{Jac}_\rho(\sigma)$ is nonzero and irreducible (up to multiplicity) since $D_\rho^{(2)}(\pi) = 0$ by Lemma 3.4. If σ' is the unique irreducible component of $\mathrm{Jac}_\rho(\sigma)$, then

$$\sigma' = L((\rho|\cdot|^{-1})^{a+1}; \pi(\phi_2, \eta_2)).$$

Recall that $a+1 = m_1 + 1 - \delta_3$. By the induction hypothesis,

$$\sigma' = \begin{cases} (\rho|\cdot|^{-1})^3 \rtimes L((\rho|\cdot|^{-1})^{m_1-2-\delta_3}; \pi(\phi_2, \eta_2)) & \text{if } m_1 > 2, \\ (\rho|\cdot|^{-1})^{3-\delta_3} \rtimes L((\rho|\cdot|^{-1})^{m_1-2}; \pi(\phi_2, \eta_2)) & \text{if } m_1 = 2 \end{cases}$$

is an irreducible induction. Hence we have $D_{\rho|\cdot|^1}^{(m_3+3-\delta_3)}(\sigma') \neq 0$. Therefore we have

$$\begin{aligned} &D_{\rho|\cdot|^1}^{(m_3+3-\delta_3)}(\mathrm{Jac}_\rho(\rho|\cdot|^{-1} \rtimes \pi)) \neq 0 \\ &\implies D_{\rho|\cdot|^1}^{(m_3+2-\delta_3)}(\rho|\cdot|^{-1} \rtimes \pi) \neq 0 \\ &\implies D_{\rho|\cdot|^1}^{(m_3+1-\delta_3)}(\pi) \neq 0, \end{aligned}$$

which contradicts Lemma 3.4.

The case where $a > m_1 - \delta_3$ follows from the case $a = m_1 - \delta_3$. This completes the proof. □

Now we prove Proposition 5.2

Proof of Proposition 5.2. Recall that $\pi = L((\rho|\cdot|^{-1})^a, \Delta_\rho[0,-1]^b; \pi(\phi,\eta))$ and $\pi_0 = L(\Delta_\rho[0,-1]^b; \pi(\phi,\eta))$, and that $D_{\rho|\cdot|^1}^{(k)}(\pi)$ and $D_{\rho|\cdot|^1}^{(k_0)}(\pi_0)$ are the highest derivatives. Set

$$l = \begin{cases} \max\{a - m_1 + 1, 0\} & \text{if } m_1 m_3 \neq 0, \delta_3 \neq \kappa, \\ \max\{a - m_1, 0\} & \text{otherwise.} \end{cases}$$

We will prove that $k - k_0 = l$.

Note that

$$\pi \hookrightarrow (\rho|\cdot|^{-1})^l \rtimes L((\rho|\cdot|^{-1})^{a-l}, \Delta_\rho[0,-1]^b; \pi(\phi,\eta)).$$

By applying Example 3.5 and Proposition 3.14 to $L((\rho|\cdot|^{-1})^{a-l}, \Delta_\rho[0,-1]^b; \pi(\phi,\eta))$, we see that $D_{\rho|\cdot|^1}^{(k_0)}(L((\rho|\cdot|^{-1})^{a-l}, \Delta_\rho[0,-1]^b; \pi(\phi,\eta)))$ is the highest derivative. Hence we have $k \leq k_0 + l$. Also, we see that if $l = 0$, then $k = k_0$.

Assume that $l > 0$. Since π is a unique irreducible subrepresentation of

$$(\rho|\cdot|^{-1})^a \times \Delta_\rho[0,-1]^b \rtimes \pi(\phi,\eta) \cong \Delta_\rho[0,-1]^b \times (\rho|\cdot|^{-1})^a \rtimes \pi(\phi,\eta),$$

by Proposition 5.3, we see that

$$\begin{aligned} \pi &\hookrightarrow \Delta_\rho[0,-1]^b \times (\rho|\cdot|^{-1})^{a-m_1+\delta_3} \rtimes L((\rho|\cdot|^{-1})^{m_1-\delta_3}; \pi(\phi,\eta)) \\ &\hookrightarrow \rho^b \times (\rho|\cdot|^{-1})^{b+a-m_1+\delta_3} \rtimes L((\rho|\cdot|^{-1})^{m_1-\delta_3}; \pi(\phi,\eta)) \\ &\cong \rho^b \times (\rho|\cdot|^{1})^{b+a-m_1+\delta_3} \rtimes L((\rho|\cdot|^{-1})^{m_1-\delta_3}; \pi(\phi,\eta)) \\ &\hookrightarrow \rho^b \times (\rho|\cdot|^{1})^{b+a-m_1+m_3} \rtimes D_{\rho|\cdot|^1}^{(m_3-\delta_3)}(L((\rho|\cdot|^{-1})^{m_1-\delta_3}; \pi(\phi,\eta))). \end{aligned}$$

This implies that $k \geq a - m_1 + m_3$. Since $k_0 + l = a - m_1 + m_3$, we conclude that $k = k_0 + l$. □

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