ON THE NON-VANISHING OF THETA LIFTINGS OF TEMPERED REPRESENTATIONS OF $\mathrm{U}(p,q)$

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ABSTRACT. In this paper, we give an explicit determination of the non-vanishing of the theta liftings of tempered representations for unitary dual pairs $(\mathrm{U}(p,q),\mathrm{U}(r,s))$ for arbitrary non-negative integers p,q,r,s. For discrete series representations, in terms of Harish-Chandra parameters, we give a complete criterion when the theta lifts are nonzero. For tempered representations, we determine the non-vanishing in terms of the local Langlands correspondence assuming the local Gan–Gross–Prasad conjecture.

CONTENTS

1. INTRODUCTION

The theta lifting is an important tool in automorphic, real, and p-adic representation theories. In §1.1, we explain a global motivation of this paper. The local archimedean introduction is in §1.2. In §1.5, we recall the non-archimedean result in [AG]. Finally, in §1.6, we compare the proof of the archimedean result with the non-archimedean result. If a reader were not familiar with the automorphic or p-adic representation theory, he or she could see only §1.2–1.4.

1.1. **Global motivation.** In the theory of modular forms and automorphic representations, one of the most important problem is to construct (non-trivial) cusp forms or cuspidal representations.

Let k be a number field. We denote by $\mathbb{A}_k$ the adele ring of k. Let G and H be connected reductive groups over k. In this subsection, for an irreducible cuspidal automorphic representation π of $G(\mathbb{A}_k)$, we shall call Π a "lift" of π to $H(\mathbb{A}_k)$ if Π is an automorphic representation (or "packet") of $H(\mathbb{A}_k)$ such that an arithmetic property (e.g., an L-function) of Π is given explicitly by the one of π. There is a general problem.

Problem 1.1. *Let $\pi \mapsto \Pi$ be a "lift" from automorphic representations of $G(\mathbb{A}_k)$ to the ones of $H(\mathbb{A}_k)$.*

(1) *Determine when Π is nonzero.*
(2) *Determine when Π is irreducible and cuspidal.*
(3) *Determine the local components Π_v of Π for each place v of k.*

Now we consider this problem for global theta liftings, which contain many classical liftings, e.g., the Shimura correspondence and the Saito–Kurokawa lifting.

In this paper, we only consider theta liftings for unitary dual pairs. Let K/k be a quadratic extension of number fields and $\mathbb{A}_K$ be the adele ring of K. We denote by W_n an n-dimensional hermitian space over K. Also, we fix an anisotropic skew-hermitian space V_0 over K of dimension m_0, and denote by V_l the m-dimensional skew-hermitian space obtained from V_0 by addition of l hyperbolic planes, where $m = m_0 + 2l$.

Let $G = \mathrm{U}(W_n)$ and $H = \mathrm{U}(V_l)$ be the isometry groups of W_n and V_l, respectively. Then $\mathbb{W} = W_n \otimes_K V_l$ can be regarded as a $2mn$-dimensional symplectic space over k, and there exists a canonical map

$$\alpha_{V,W} \colon \mathrm{U}(W_n)(\mathbb{A}_k) \times \mathrm{U}(V_l)(\mathbb{A}_k) \to \mathrm{Sp}(\mathbb{W})(\mathbb{A}_k).$$

Let $\mathrm{Mp}(\mathbb{W})_{\mathbb{A}}$ be the metaplectic cover of $\mathrm{Sp}(\mathbb{W})(\mathbb{A}_k)$. Take a complete polarization $\mathbb{W} = \mathbb{X} + \mathbb{Y}$. Fixing a non-trivial additive character ψ of $\mathbb{A}_k/k$, we obtain the Weil representation ω_ψ of $\mathrm{Mp}(\mathbb{W})_{\mathbb{A}}$ associated to ψ which is realized on the space $\mathcal{S}(\mathbb{X}(\mathbb{A}_k))$ of Schwarts–Bruhat functions on $\mathbb{X}(\mathbb{A}_k)$. Let $\Theta \colon \mathcal{S}(\mathbb{X}(\mathbb{A}_k)) \to \mathbb{C}$ be the functional $\phi \mapsto \Theta(\phi) = \sum_{x \in \mathbb{X}(k)} \phi(x)$. There exists a unique canonical splitting $\mathrm{Sp}(\mathbb{W})(k) \to \mathrm{Mp}(\mathbb{W})_{\mathbb{A}}$ which satisfies that $\Theta(\omega_\psi(\gamma)\phi) = \Theta(\phi)$ for any $\gamma \in \mathrm{Sp}(\mathbb{W})(k)$ and $\phi \in \mathcal{S}(\mathbb{X}(\mathbb{A}_k))$.

Fix Hecke characters χ_W and χ_V of $\mathbb{A}_K^\times/K^\times$ such that $\chi_W|\mathbb{A}_k^\times = \eta_{K/k}^n$ and $\chi_V|\mathbb{A}_k^\times = \eta_{K/k}^m$, where $\eta_{K/k}$ is the quadratic Hecke character of $\mathbb{A}_k^\times/k^\times$ associated to K/k. Kudla gave an explicit splitting

$$\widetilde{\alpha}_{\chi_V,\chi_W} \colon \mathrm{U}(W_n)(\mathbb{A}_k) \times \mathrm{U}(V_l)(\mathbb{A}_k) \to \mathrm{Mp}(\mathbb{W})_{\mathbb{A}}.$$

Let $\omega_{\psi,V,W} = \omega_\psi \circ \widetilde{\alpha}_{\chi_V,\chi_W}$ be the pullback of the Weil representation of $\mathrm{Mp}(\mathbb{W})_{\mathbb{A}}$ to the product group $\mathrm{U}(W_n)(\mathbb{A}_k) \times \mathrm{U}(V_l)(\mathbb{A}_k)$. For each $\phi \in \mathcal{S}(\mathbb{X}(\mathbb{A}_k))$, we consider the theta function

$$\Theta(g, h; \phi) = \Theta(\omega_{\psi,V,W}(g,h)\phi)$$

for $g \in \mathrm{U}(W_n)(\mathbb{A}_k)$ and $h \in \mathrm{U}(V_l)(\mathbb{A}_k)$, which is an automorphic function on $\mathrm{U}(W_n)(\mathbb{A}_k) \times \mathrm{U}(V_l)(\mathbb{A}_k)$. If π is an irreducible cuspidal representation of $\mathrm{U}(W_n)(\mathbb{A}_k)$, the global theta lift $\theta_{\psi,V_l,W_n}(\pi)$ of π to $\mathrm{U}(V_l)(\mathbb{A}_k)$ is the automorphic representation of $\mathrm{U}(V_l)(\mathbb{A}_k)$ spanned by the functions

$$\theta(h; f, \phi) = \int_{\mathrm{U}(W_n)(k)\backslash \mathrm{U}(W_n)(\mathbb{A}_k)} \Theta(g, h; \phi)\overline{f(g)}dg,$$

where $f \in \pi$ and $\phi \in \mathcal{S}(\mathbb{X}(\mathbb{A}_k))$, and dg is the Tamagawa measure.

We may consider Problem 1.1 for $\theta_{\psi,V_l,W_n}(\pi)$. The cuspidality issue can be solved by the so-called tower property.

Theorem 1.2 (Tower property [R])**.** *When $\theta_{\psi,V_l,W_n}(\pi)$ is nonzero, $\theta_{\psi,V_{l+1},W_n}(\pi)$ is also nonzero. The theta lift $\theta_{\psi,V_l,W_n}(\pi)$ is cuspidal if and only if $\theta_{\psi,V_l,W_n}(\pi) \neq 0$ but $\theta_{\psi,V_j,W_n}(\pi) = 0$ for all $j < l$.*

The irreducibility and the local components are determined by Kudla–Rallis.

Theorem 1.3 ([KR, Corollary 7.1.3])**.** *Assume that $\theta_{\psi,V_l,W_n}(\pi)$ is nonzero and cuspidal. Then $\theta_{\psi,V_l,W_n}(\pi)$ is irreducible and $\theta_{\psi,V_l,W_n}(\pi) \cong \otimes_v \theta_{\psi,V_l,W_n}(\pi_v)$, where $\theta_{\psi,V_l,W_n}(\pi_v)$ is the "local theta lift" of the local component π_v of π.*

Therefore, the remaining issue is the non-vanishing problem. There is a local-global criterion for the non-vanishing of the global theta lifting $\theta_{\psi,V_l,W_n}(\pi)$ established by Yamana [Y] and Gan–Qiu–Takeda [GQT]. It is roughly stated as follows: When $\theta_{\psi,V_j,W_n}(\pi) = 0$ for any $j < l$ so that $\theta_{\psi,V_l,W_n}(\pi)$ is cuspidal (possibly zero),

- the global theta lifting $\theta_{\psi,V_l,W_n}(\pi)$ is nonzero

"if" and only if

- the local theta lifting $\theta_{\psi,V_l,W_n}(\pi_v)$ is nonzero for all places v of k;
- the "standard L-function $L(s,\pi)$" of π is non-vanishing or has a pole at a distinguished point s_0.

Since a local archimedean property is not proven, the "if" part is not completely established. For more precisions, see [GQT, Theorem 1.3] and a remark after this theorem.

In any case, Problem 1.1 for the global theta lifting is reduced to a local analogous problem. The purpose of this paper is to give a criterion for the non-vanishing of local theta liftings of tempered representations.

1.2. **Archimedean case.** The theory of local theta correspondence was initiated by Roger Howe. Since then, it has been a major theme in the representation theory.

In this paper, we only consider the theta liftings for unitary dual pairs $(\mathrm{U}(p,q), \mathrm{U}(r,s))$. More precisely, let $W_{p,q}$ and $V_{r,s}$ be a hermitian space and a skew hermitian space over $\mathbb{C}$ of signature (p,q) and (r,s) and of dimension $n = p+q$ and $m = r+s$, respectively. Then $\mathbb{W} = W_{p,q} \otimes_{\mathbb{C}} V_{r,s}$ is a symplectic space over

$\mathbb{R}$ of dimension $2mn$. The isometry group of $W_{p,q}$ (resp. $V_{r,s}$) is isomorphic to $\mathrm{U}(p,q)$ (resp. $\mathrm{U}(r,s)$). Fix characters χ_W and χ_V of $\mathbb{C}^\times$ such that $\chi_W|\mathbb{R}^\times = \mathrm{sgn}^n$ and $\chi_V|\mathbb{R}^\times = \mathrm{sgn}^m$, and a non-trivial additive character ψ of $\mathbb{R}$. Kudla [Ku2] gave an explicit splitting

$$\widetilde{\alpha}_{\chi_V,\chi_W} \colon \mathrm{U}(p,q) \times \mathrm{U}(r,s) \to \mathrm{Mp}(\mathbb{W})$$

of the natural map

$$\alpha_{V,W} \colon \mathrm{U}(p,q) \times \mathrm{U}(r,s) \to \mathrm{Sp}(\mathbb{W}),$$

where $\mathrm{Mp}(\mathbb{W})$ is the metaplectic cover of $\mathrm{Sp}(\mathbb{W})$. We assume that χ_W and χ_V depend only on $n \bmod 2$ and $m \bmod 2$, respectively. Let ω_ψ be the Weil representation of $\mathrm{Mp}(\mathbb{W})$ associated to ψ, which is a smooth genuine representation. Then we obtain the Weil representation $\omega_{\psi,V,W} = \omega_{p,q,r,s} = \omega_\psi \circ \widetilde{\alpha}_{\chi_V,\chi_W}$ of $\mathrm{U}(p,q) \times \mathrm{U}(r,s)$. For an irreducible (unitary) representation π of $\mathrm{U}(p,q)$, the maximal π-isotypic quotient of $\omega_{\psi,V,W}$ has the form

$$\pi \boxtimes \Theta_{r,s}(\pi)$$

for some representation $\Theta_{r,s}(\pi)$ of $\mathrm{U}(r,s)$ (known as the big theta lift of π). The most important result is the Howe duality correspondence stated as follows:

Theorem 1.4 (Howe duality correspondence [Ho, Theorem 2.1])**.** *If $\Theta_{r,s}(\pi)$ is nonzero, then it has a unique irreducible quotient $\theta_{r,s}(\pi)$.*

We shall interpret $\theta_{r,s}(\pi)$ to be zero if so is $\Theta_{r,s}(\pi)$. We call $\theta_{r,s}(\pi)$ the small theta lift (or simply, the local theta lift) of π to $\mathrm{U}(r,s)$. After the above theorem, it is natural to consider the following two basic problems:

Problem 1.5. (1) *Determine precisely when $\Theta_{r,s}(\pi)$ is nonzero.*
(2) *Determine $\theta_{r,s}(\pi)$ precisely if $\Theta_{r,s}(\pi)$ is nonzero.*

This problem is solved in the following cases:

- in the case where $q = 0$, i.e., $\mathrm{U}(p,q) = \mathrm{U}(p,0)$ is compact by Kashiwara–Vergne [KV] (see also [A, §6]);
- in the (almost) equal rank case by Paul [P1, P3];
- in special cases where π is discrete series by J.-S. Li [Li] and Paul [P2];
- in the case where π is one dimensional by Paul–Trapa [PT];

The theta correspondence for other reductive pairs was also considered by many persons. For example:

- He [He1] considered the non-vanishing problem for the $(\mathrm{O}(p,q), \mathrm{Sp}_{2n}(\mathbb{R}))$-case with $p+q \leq 2n+1$.
- Adams–Barbasch treated the complex case in [AB1] and the $(\mathrm{Mp}_{2n}(\mathbb{R}), \mathrm{O}_{2n+1})$-case in [AB2];
- Paul [P4] considered the (almost) equal rank case for $(\mathrm{Sp}_{2n}(\mathbb{R}), \mathrm{O}_{2m})$-case also;
- Li–Paul–Tan–Zhu [LPTZ] treated the $(\mathrm{Sp}(p,q), \mathrm{O}^*(2n))$-case;
- Mœglin studied the relation between the theta correspondence and A-packets in [Mœ2], and constructed special unipotent A-packets in [Mœ3].
- Zhu [Z] studied generalized Whittaker models of theta liftings, and obtained some non-vanishing results.

To formulate an answer to Problem 1.5, it is necessary to have some sort of classifications of irreducible representations of the groups $\mathrm{U}(p,q)$ and $\mathrm{U}(r,s)$. In this paper, we shall use the local Langlands correspondence (LLC) as a classification, and address Problem 1.5 (1) for tempered representations π.

The local Langlands correspondence for real reductive connected groups is well understood by the works of many mathematicians, including Adams, Arthur, Barbasch, Johnson, Langlands, Mœglin, Shelstad, and Vogan. For discrete series representations, it is essentially the same parametrization as using Harish-Chandra parameters (see Theorem 2.1 (4) below). In this introduction, we explain the main result (Theorem 4.2) only for discrete series representations in terms of their Harish-Chandra parameters.

Let π be a discrete series representation of $\mathrm{U}(p,q)$. Its Harish-Chandra parameter $\lambda = \mathrm{HC}(\pi)$ is of the form

$$\lambda = (\lambda_1, \ldots, \lambda_p; \lambda'_1, \ldots, \lambda'_q),$$

where $\lambda_i, \lambda'_j \in \mathbb{Z} + \frac{n-1}{2}$, $\lambda_1 > \cdots > \lambda_p$, $\lambda'_1 > \cdots > \lambda'_q$, and $\{\lambda_1, \dots, \lambda_p\} \cap \{\lambda'_1, \dots, \lambda'_q\} = \emptyset$. To state our main result, we define some terminologies.

Definition 1.6 (Definition 4.1). *Fix $\kappa \in \{1,2\}$ and suppose that χ_V is of the form*

$$\chi_V(ae^{\sqrt{-1}\theta}) = e^{\nu\sqrt{-1}\theta}$$

for $a > 0$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ with $\nu \in \mathbb{Z}$ such that $\nu \equiv n + \kappa \bmod 2$. Let π be a discrete series representation of $\mathrm{U}(p,q)$ with Harish-Chandra parameter $\lambda = (\lambda_1, \dots, \lambda_p; \lambda'_1, \dots, \lambda'_q)$.

(1) *Write*

$$\lambda - \left(\frac{\nu}{2}, \dots, \frac{\nu}{2}\right) = \left(a_1, \dots, a_x, \underbrace{\frac{k-1}{2}, \frac{k-3}{2}, \dots, \frac{-k+1}{2}}_{k}, b_1, \dots b_y; c_1, \dots, c_z, d_1, \dots, d_w\right)$$

or

$$\lambda - \left(\frac{\nu}{2}, \dots, \frac{\nu}{2}\right) = \left(a_1, \dots, a_x, b_1, \dots b_y; c_1, \dots, c_z, \underbrace{\frac{k-1}{2}, \frac{k-3}{2}, \dots, \frac{-k+1}{2}}_{k}, d_1, \dots, d_w\right)$$

with $a_x, -b_1, c_z, -d_1 \geq \frac{k+1}{2}$. We set k_λ to be the maximal choice of k unless $\lambda_i - \nu/2, \lambda'_j - \nu/2 \in \mathbb{Z}\setminus\{0\}$ for all i, j (i.e., $k = 0$ and $a_i, b_i, c_i, d_i \in \mathbb{Z} \setminus \{0\}$ for all i), in which case, we set $k_\lambda = -1$. Note that

$$2a_i \equiv 2b_i \equiv 2c_i \equiv 2d_i \equiv k_\lambda - 1 \bmod 2.$$

(2) *When k is chosen to be maximal in (1) (so that $k = 0$ if $k_\lambda \in \{-1, 0\}$), we put $r_\lambda = x + w$ and $s_\lambda = y + z$.*

(3) *Define $X_\lambda \subset \frac{1}{2}\mathbb{Z} \times \{\pm 1\}$ by*

$$X_\lambda = \left\{\left(\lambda_1 - \frac{\nu}{2}, +1\right), \dots, \left(\lambda_p - \frac{\nu}{2}, +1\right)\right\} \cup \left\{\left(\lambda'_1 - \frac{\nu}{2}, -1\right), \dots, \left(\lambda'_q - \frac{\nu}{2}, -1\right)\right\}.$$

(4) *We define a sequence $X_\lambda = X_\lambda^{(0)} \supset X_\lambda^{(1)} \supset \cdots \supset X_\lambda^{(n)} \supset \cdots$ as follows: Let $\{\beta_1, \dots, \beta_{u_j}\}$ be the image of $X_\lambda^{(j)}$ under the projection $\frac{1}{2}\mathbb{Z} \times \{\pm 1\} \to \frac{1}{2}\mathbb{Z}$ such that $\beta_1 > \cdots > \beta_{u_j}$. Set S to be the set of $i \in \{2, \dots, u_j\}$ such that one of the following holds:*
   - $\beta_{i-1} \in \{a_1, \dots, a_x\}$ *and* $\beta_i \in \{c_1, \dots, c_z\}$;
   - $\beta_{i-1} \in \{b_1, \dots, b_y\}$ *and* $\beta_i \in \{d_1, \dots, d_w\}$.

   Here, we assume that k is chosen to be maximal in (1) (so that $k = 0$ if $k_\lambda = \{-1, 0\}$). Then we define a subset $X_\lambda^{(j+1)}$ of $X_\lambda^{(j)}$ by

$$X_\lambda^{(j+1)} = X_\lambda^{(j)} \setminus \left(\bigcup_{i \in S} \{(\beta_{i-1}, +1), (\beta_i, -1)\}\right).$$

   Finally, we set $X_\lambda^{(\infty)} = X_\lambda^{(n)} = X_\lambda^{(n+1)}$.

(5) *For an integer T and $\epsilon \in \{\pm 1\}$, we define a set $\mathcal{C}_\lambda^\epsilon(T)$ by*

$$\mathcal{C}_\lambda^\epsilon(T) = \left\{(\alpha, \epsilon) \in X_\lambda^{(\infty)} \;\middle|\; 0 \leq \epsilon\alpha + \frac{k_\lambda - 1}{2} < T\right\}.$$

   In particular, if $T \leq 0$, then $\mathcal{C}_\lambda^\epsilon(T) = \emptyset$.

The main result for discrete series representations is stated as follows:

Theorem 1.7 (Theorem 4.2). *Let π be a discrete series representation of $\mathrm{U}(p,q)$ with Harish-Chandra parameter λ. Set $r = r_\lambda$ and $s = s_\lambda$.*

(1) *Suppose that $k_\lambda = -1$. Then for integers l and $t \geq 0$, the theta lift $\Theta_{r+2t+l+1, s+l}(\pi)$ is nonzero if and only if*

$$l \geq 0 \quad \text{and} \quad \#\mathcal{C}_\lambda^\epsilon(t + l) \leq l \quad \text{for each } \epsilon \in \{\pm 1\}.$$

(2) *Suppose that $k_\lambda \geq 0$. Then for integers l and $t \geq 1$, the theta lift $\Theta_{r+2t+l,s+l}(\pi)$ is nonzero if and only if*

$$l \geq k_\lambda \quad \text{and} \quad \#\mathcal{C}_\lambda^\epsilon(t+l) \leq l \quad \text{for each } \epsilon \in \{\pm 1\}.$$

Moreover, for an integer l, the theta lift $\Theta_{r+l,s+l}(\pi)$ is nonzero if and only if $l \geq 0$.

For the proof, we use the local Gan–Gross–Prasad conjecture (Conjecture 2.2 below). This conjecture gives a conjectural answer to restriction problems in terms of the local Langlands correspondence. For more precisions, see §2.3 below. To prove our main result for discrete series representations (Theorem 1.7), we use the local Gan–Gross–Prasad conjecture only for discrete series representations. This case has been established by He [He2] (in terms of Harish-Chandra parameters), so that the statement in Theorem 1.7 holds unconditionally. For tempered representations, only a weaker version of the local Gan–Gross–Prasad conjecture is proven by Beuzart-Plessis [BP].

When χ_V is of the form $\chi_V(ae^{\sqrt{-1}\theta}) = e^{\nu\sqrt{-1}\theta}$ for $a > 0$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ with $\nu \in \mathbb{Z}$ such that $\nu \equiv m \bmod 2$, it is known that $\Theta_{r,s}(\pi) \neq 0$ if and only if $\Theta_{s,r}(\pi^\vee \otimes \det^\nu) \neq 0$ (see Proposition 3.9 below). By this property together with our main result, we can determine the non-vanishing of all theta lifts of tempered representations (including the case where $t < 0$).

1.3. **Idea of the proof.** To study local theta correspondence, it is useful to consider "Howe's K-type of lowest degree". For the relation between this notion and Vogan's minimal K-type, see e.g., [P1, Lemma 1.4.5] and [Mœ1, §II]. In this paper, we use not these K-types (nor Blattner parameters) but L-parameters (Harish-Chandra parameters) and the local Gan–Gross–Prasad conjecture.

The proof of the main theorem consists of three steps: First, we show the sufficient condition for the non-vanishing of theta lifts when $t \geq 1$. This is proven by induction by using the local Gan–Gross–Prasad conjecture and seesaw identities (§5.2). The initial steps of the induction are the (almost) equal rank cases, which are established by Paul ([P1, P3]). Next, we show the necessary condition for the non-vanishing of theta lifts when $t \geq 1$. This is proven by using a non-vanishing result of theta lifts of one dimensional representations and seesaw identities (§5.3). Finally, by using the conservation relation (Theorem 3.7 below), we obtain the $t = 0$ case.

In this subsection, we explain the more detailed idea in the first two steps through a low rank case. For simplicity, we assume that χ_V is trivial, i.e., $\nu = 0$. Let us consider an irreducible discrete series representation π of $\mathrm{U}(1,1)$ with Harish-Chandra parameter λ satisfying

$$\lambda = (a; c)$$

with $a, c \in (1/2)\mathbb{Z} \setminus \mathbb{Z}$ and $a, c > 0$, and determine when the theta lift $\Theta_{3,1}(\pi)$ is nonzero. We note that

- $k_\lambda = 0$, $r_\lambda = s_\lambda = 1$;
- $X_\lambda = \{(a, +1), (c, -1)\}$;
- $X_\lambda^{(\infty)} = X_\lambda$ if $a < c$, and $X_\lambda^{(\infty)} = \emptyset$ if $a > c$;
- $\#\mathcal{C}_\lambda^\epsilon(T) = 1$ if $a < c$, $\epsilon = +1$ and $T > a - 1/2$, and $\#\mathcal{C}_\lambda^\epsilon(T) = 0$ otherwise.

In particular, $\#\mathcal{C}_\lambda^\epsilon(1) \leq 0$ for each $\epsilon \in \{\pm 1\}$ if and only if $a \neq 1/2$. Now we prove that $\Theta_{3,1}(\pi) \neq 0$ if and only if $a \neq 1/2$.

The first step is to show one direction $a \neq 1/2 \implies \Theta_{3,1}(\pi) \neq 0$ (see §5.2). To do this, we will look for an irreducible representation π' of $\mathrm{U}(2,1)$ such that

- $\mathrm{Hom}_{\mathrm{U}(1,1)}(\pi', \pi) \neq 0$; and
- $\Theta_{3,1}(\pi') \neq 0$.

If we were able to find such π', setting $\sigma' = \theta_{3,1}(\pi') \in \mathrm{Irr}(\mathrm{U}(3,1))$, we would have

$$0 \neq \mathrm{Hom}_{\mathrm{U}(1,1)}(\pi', \pi) \subset \mathrm{Hom}_{\mathrm{U}(1,1)}(\Theta_{2,1}(\sigma'), \pi) \cong \mathrm{Hom}_{\mathrm{U}(3,1)}(\Theta_{3,1}(\pi) \otimes \omega_{1,0,3,1}, \sigma')$$

by a seesaw identity (Proposition 3.11 (1)). In particular, we could obtain $\Theta_{3,1}(\pi) \neq 0$.

Following §5.1, choose integers $\alpha_1 > \alpha_2 > \alpha_3 = \alpha > \max\{a, c\} + 1$. Let π' be the irreducible discrete series representation of $\mathrm{U}(2,1)$ with Harish-Chandra parameter λ' satisfying

$$\lambda' = \begin{cases} (\alpha_1, \alpha_3; \alpha_2) & \text{if } a = c+1, \\ \left(\alpha, a - \dfrac{1}{2}; c + \dfrac{1}{2}\right) & \text{if } a \neq c+1. \end{cases}$$

Then using the local Gan–Gross–Prasad conjecture (proven in [He2]), we have $\mathrm{Hom}_{\mathrm{U}(1,1)}(\pi', \pi) \neq 0$. Moreover, since $a > 1/2$, by Paul's result [P3], the theta lift $\Theta_{3,1}(\pi')$ is nonzero. (In general, we will use the induction hypothesis here to obtain $\Theta_{r,s}(\pi') \neq 0$.) Therefore, we can deduce that $\Theta_{3,1}(\pi) \neq 0$.

The second step is to show the other direction $a = 1/2 \implies \Theta_{3,1}(\pi) = 0$ (see §5.3). The idea is the same as [P2, Theorem 3.14]. Consider the twist $\pi \otimes \det^{-1}_{\mathrm{U}(1,1)}$. This is a discrete series representation with Harish-Chandra parameter

$$(a-1; c-1) = \left(-\frac{1}{2}; c-1\right).$$

By Paul's result [P1], we have $\Theta_{0,2}(\pi \otimes \det^{-1}_{\mathrm{U}(1,1)}) \neq 0$ so that $\Theta_{2,0}(\pi^\vee \otimes \det_{\mathrm{U}(1,1)}) \neq 0$. Suppose for the sake of contradiction that $\Theta_{3,1}(\pi) \neq 0$. Then a seesaw identity (Proposition 3.11 (2)) implies that $\Theta_{5,1}(\det_{\mathrm{U}(1,1)}) \neq 0$. The theta liftings of one dimensional representations are well-understood (see Proposition 3.10), and it is known that $\Theta_{5,1}(\det_{\mathrm{U}(1,1)}) = 0$. This gives a contradiction so that $\Theta_{3,1}(\pi)$ must be zero.

1.4. **Examples.** We shall give some examples of the main theorem. First, we consider the case where $\mathrm{U}(p,q)$ is compact, i.e., $p = 0$ or $q = 0$.

Example 1.8 (Compact case)**.** *Suppose that $p = n$ and $q = 0$. Let π be an irreducible (continuous, discrete series) representation of $\mathrm{U}(n) = \mathrm{U}(n,0)$ with Harish-Chandra parameter λ satisfying*

$$\lambda - \left(\frac{\nu}{2}, \dots, \frac{\nu}{2}\right) = \left(a_1, \dots, a_x, \underbrace{\frac{k-1}{2}, \frac{k-3}{2}, \dots, \frac{-k+1}{2}}_{k}, b_1, \dots b_y\right),$$

where $k = \max\{0, k_\lambda\}$, $a_x, -b_1 \geq (k_\lambda + 1)/2$, $x + y + k = n$, and $\nu \in \mathbb{Z}$ such that $\chi_V(ae^{\sqrt{-1}\theta}) = e^{\nu\sqrt{-1}\theta}$ for $a > 0$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Note that $r_\lambda = x$ and $s_\lambda = y$. Then

$$X_\lambda = X_\lambda^{(\infty)} = \left\{(a_1, +1), \dots, (a_x, +1), \left(\frac{k-1}{2}, +1\right), \dots, \left(\frac{-k+1}{2}, +1\right), (b_1, +1), \dots (b_y, +1)\right\}.$$

Hence $\mathcal{C}^-_\lambda(T) = \emptyset$ for any T. Moreover, for $t \geq 1$ and $l \geq k$, we see that $\#\mathcal{C}^+_\lambda(t+l) \leq l$ if and only if $l - k \geq x$, or $l - k < x$ and

$$a_{x-l+k} \geq t + l - \frac{k_\lambda - 1}{2}.$$

Similarly, $\pi^\vee \otimes \det^\nu$ has the Harish-Chandra parameter $\lambda^\vee$ satisfying

$$\lambda^\vee - \left(\frac{\nu}{2}, \dots, \frac{\nu}{2}\right) = \left(-b_y, \dots, -b_1, \underbrace{\frac{k-1}{2}, \frac{k-3}{2}, \dots, \frac{-k+1}{2}}_{k}, -a_x, \dots - a_1\right).$$

Hence $\mathcal{C}^-_{\lambda^\vee}(T) = \emptyset$ for any T. Moreover, for $t \geq 1$ and $l \geq k$, we see that $\#\mathcal{C}^+_\lambda(t+l) \leq l$ if and only if $l - k \geq y$, or $l - k < y$ and

$$b_{1+l-k} \leq -\left(t + l - \frac{k_\lambda - 1}{2}\right).$$

Therefore, by Theorem 1.7, $\Theta_{r,s}(\pi)$ is nonzero if and only if one of the following holds:

- *$|(r-s) - (x-y)| \leq 1$, and $r \geq x$ and $s \geq y$;*
- *$(r-s) - (x-y) > 1$, and $s \geq n - x$, and if $n - x \leq s \leq n-1$, then*

$$a_{n-s} \geq \frac{m-n+1}{2};$$

- $(r-s)-(x-y)<-1$, *and* $r \geq n-y$, *and if* $n-y \leq r \leq n$, *then*

$$b_{1+y+r-n} \leq -\frac{m-n+1}{2}.$$

Here, we put $m = r+s$. *By using Blattner's formula, it is easy to check that this condition is compatible with results in* [KV] *and* [A, Proposition 6.6], *which are stated in terms of the highest weight.*

Second, we consider the case where π is holomorphic discrete series.

Example 1.9 (Holomorphic case)**.** *Suppose that* $p = q$ *so that* $n = 2p$. *Let* $k \geq n$ *be an even integer. Consider the holomorphic discrete series representation* π *of* $\mathrm{U}(p,p)$ *of weight* k. *Its Harish-Chandra parameter* λ *is given by*

$$\lambda = \left(\frac{k-1}{2}, \frac{k-3}{2}, \dots, \frac{k-n+1}{2}; \frac{-k+n-1}{2}, \dots, \frac{-k+3}{2}, \frac{-k+1}{2}\right).$$

For simplicity, we assume that χ_V *is trivial, i.e.,* $\nu = 0$. *Then we have* $k_\lambda = 0$, $r_\lambda = n$, $s_\lambda = 0$ *and*

$$X_\lambda = X_\lambda^{(\infty)} = \left\{\left(\frac{k-1}{2}, +1\right), \dots, \left(\frac{k-n+1}{2}, +1\right), \left(\frac{-k+n-1}{2}, -1\right), \dots, \left(\frac{-k+1}{2}, -1\right)\right\}.$$

Hence for integer T, *we have*

$$\#\mathcal{C}_\lambda^+(T) = \#\mathcal{C}_\lambda^-(T) = \begin{cases} 0 & \text{if } T \leq \dfrac{k-n}{2}, \\ T - \dfrac{k-n}{2} & \text{if } \dfrac{k-n}{2} \leq T \leq \dfrac{k}{2}, \\ p & \text{if } T \geq \dfrac{k}{2}. \end{cases}$$

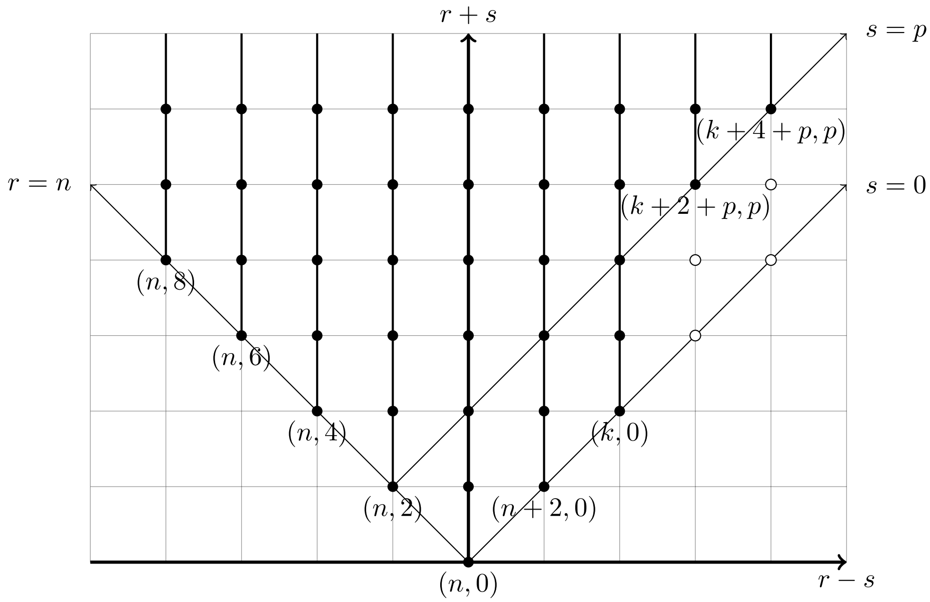
On the other hand, the Harish-Chandra parameter $\lambda^\vee$ *of* $\pi^\vee$ *is given by*

$$\lambda^\vee = \left(\frac{-k+n-1}{2}, \dots, \frac{-k+3}{2}, \frac{-k+1}{2}; \frac{k-1}{2}, \frac{k-3}{2}, \dots, \frac{k-n+1}{2}\right)$$

so that $\mathcal{C}_{\lambda^\vee}^\epsilon(T) = \emptyset$ *for any* $T \in \mathbb{Z}$ *and* $\epsilon \in \{\pm 1\}$. *Therefore, by Theorem 1.7,* $\Theta_{r,s}(\pi) \neq 0$ *with* $r+s \equiv 0 \bmod 2$ *if and only if one of the following holds:*

- $r-s \leq n$ *and* $r \geq n$;
- $n \leq r-s \leq k$ *and* $s \geq 0$;
- $r-s > k$ *and* $s \geq p$.

The following summarizes when $\Theta_{r,s}(\pi) \neq 0$ *with* $n = 2p = 4$ *and* $k = 8$. *Here, a black plot* (r,s) *means that* $\Theta_{r,s}(\pi) \neq 0$, *and a white plot* (r,s) *means that* $\Theta_{r,s}(\pi) = 0$.



Third, we consider the case where π is generic (large) discrete series.

Example 1.10 (Generic case)**.** *Suppose that $p=q+1$ so that $n=2p-1$. Consider a generic (large) discrete series representation π. Its Harish-Chandra parameter*

$$\lambda=(a_1,\dots,a_p;b_1,\dots,b_q)$$

satisfies that

$$a_1>b_1>a_2>b_2>\dots>a_q>b_q>a_p.$$

Hence

$$X_\lambda=\left\{\left(a_1-\frac{\nu}{2},+1\right),\left(b_1-\frac{\nu}{2},-1\right),\dots,\left(a_q-\frac{\nu}{2},+1\right),\left(b_q-\frac{\nu}{2},-1\right),\left(a_p-\frac{\nu}{2},+1\right)\right\}.$$

In particular, we have

$$k_\lambda=\begin{cases}1 & \text{if } \nu\in\{2a_1,\dots,2a_p\}\cup\{2b_1,\dots,2b_q\},\\ 0 & \text{if } \nu\not\equiv 2a_i\equiv 2b_j \bmod 2,\\ -1 & \text{otherwise,}\end{cases}$$

$$r_\lambda=\#\{i\in\{1,\dots,p\}\mid 2a_p>\nu\}+\#\{j\in\{1,\dots,q\}\mid 2b_j<\nu\},$$
$$s_\lambda=\#\{i\in\{1,\dots,p\}\mid 2a_p<\nu\}+\#\{j\in\{1,\dots,q\}\mid 2b_j>\nu\}.$$

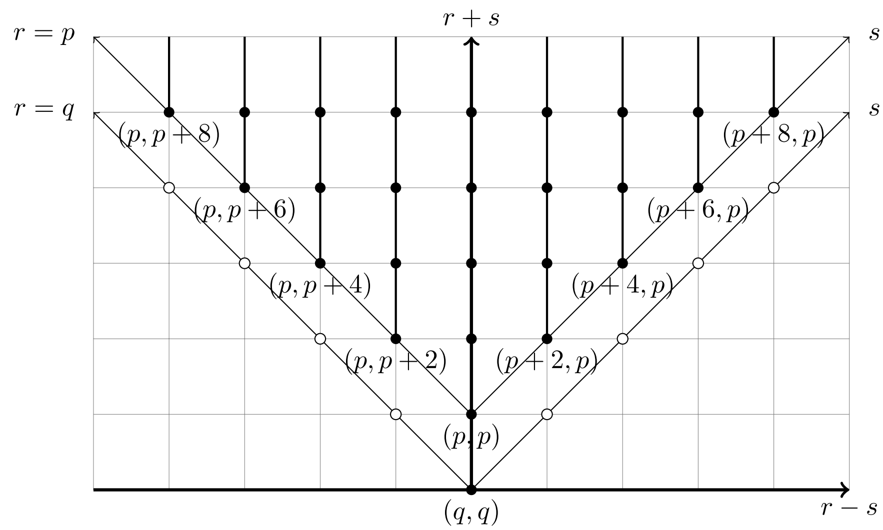
Now we further assume that $\nu=2a_{i_0}$ for some $1\le i_0\le p$. Then $k_\lambda=1$, $r_\lambda=s_\lambda=q$, and

$$X_\lambda^{(1)}=X_\lambda^{(\infty)}\subset\left\{(0,+1),\left(b_{i_0}-\frac{\nu}{2},-1\right),\left(a_p-\frac{\nu}{2},+1\right)\right\},$$

which is equal if $i_0<p$. Hence $\#\mathcal{C}_\lambda^\epsilon(T)\le 1$ for any $T\in\mathbb{Z}$ and $\epsilon\in\{\pm1\}$. Similarly, the Harish-Chandra parameter $\lambda^\vee$ of $\pi^\vee\otimes\det^\nu$ satisfies that

$$X_{\lambda^\vee}=\left\{\left(-a_p+\frac{\nu}{2},+1\right),\left(-b_q+\frac{\nu}{2},-1\right),\left(-a_q+\frac{\nu}{2},+1\right)\dots,\left(-b_1+\frac{\nu}{2},-1\right),\left(-a_1+\frac{\nu}{2},+1\right)\right\},$$
$$X_{\lambda^\vee}^{(\infty)}\subset\left\{(0,+1),\left(-b_{i_0-1}+\frac{\nu}{2},-1\right),\left(-a_1+\frac{\nu}{2},+1\right)\right\}.$$

The last inclusion is equal if $i_0>1$. Hence $\#\mathcal{C}_{\lambda^\vee}^\epsilon(T)\le 1$ for any $T\in\mathbb{Z}$ and $\epsilon\in\{\pm1\}$. Therefore, by Theorem 1.7, $\Theta_{r,s}(\pi)\neq 0$ with $r+s\equiv p+q+1 \bmod 2$ if and only if $(r,s)=(q,q)$ or $\min\{r-q,s-q\}\ge 1$. The following summarizes when $\Theta_{r,s}(\pi)\neq 0$. Here, a black plot (r,s) means that $\Theta_{r,s}(\pi)\neq 0$, and a white plot (r,s) means that $\Theta_{r,s}(\pi)=0$.



Finally, we give a more specific example.

Example 1.11. *Suppose that χ_V is trivial, i.e., $\nu = 0$. Let π be an irreducible discrete series representation of* U$(4,5)$ *with Harish-Chandra parameter*

$$\lambda = (6, 5, 4, -8; 3, 1, 0, -3, -7).$$

Then $k_\lambda = 1$, $r_\lambda = 5$, $s_\lambda = 3$, and

$$\begin{aligned} X_\lambda &= \{(6, +1), (5, +1), (4, +1), (3, -1), (1, -1), (0, -1), (-3, -1), (-7, -1), (-8, +1)\}, \\ X_\lambda^{(\infty)} &= \{(6, +1), \qquad\qquad\qquad\qquad\qquad\qquad (0, -1), (-3, -1), (-7, -1), (-8, +1)\}. \end{aligned}$$

Hence

$$\mathcal{C}_\lambda^+(T) = \begin{cases} \emptyset & \text{if } T \leq 6, \\ \{(6, +1)\} & \text{if } T > 6, \end{cases} \quad \mathcal{C}_\lambda^-(T) = \begin{cases} \{(0, -1)\} & \text{if } 0 < T \leq 3, \\ \{(0, -1), (-3, -1)\} & \text{if } 3 < T \leq 7, \\ \{(0, -1), (-3, -1), (-7, -1)\} & \text{if } T > 7. \end{cases}$$

Therefore, by Theorem 1.7, $\Theta_{5+2t+l,3+l}(\pi)$ is nonzero if and only if one of the following holds:

- *$t = 0$ and $l \geq 0$;*
- *$t \geq 1$, $l \geq 1$ and $t + 1 \leq 3$;*
- *$t \geq 1$, $l \geq 2$ and $t + 2 \leq 7$;*
- *$t \geq 1$ and $l \geq 3$.*

Similarly, the Harish-Chandra parameter of $\pi^\vee$ is given by

$$\lambda^\vee = (8, -4, -5, -6; 7, 3, 0, -1, -3).$$

We have $k_{\lambda^\vee} = 1$, $r_{\lambda^\vee} = 3$, $s_{\lambda^\vee} = 5$, and

$$\begin{aligned} X_{\lambda^\vee} &= \{(8, +1), (7, -1), (3, -1), (0, -1), (-1, -1), (-3, -1), (-4, +1), (-5, +1), (-6, +1)\}, \\ X_{\lambda^\vee}^{(\infty)} &= \{ \qquad\qquad\qquad (3, -1), (0, -1), (-1, -1), (-3, -1), (-4, +1), (-5, +1), (-6, +1)\}. \end{aligned}$$

Hence $\mathcal{C}_{\lambda^\vee}^+(T) = \emptyset$ for any T, and

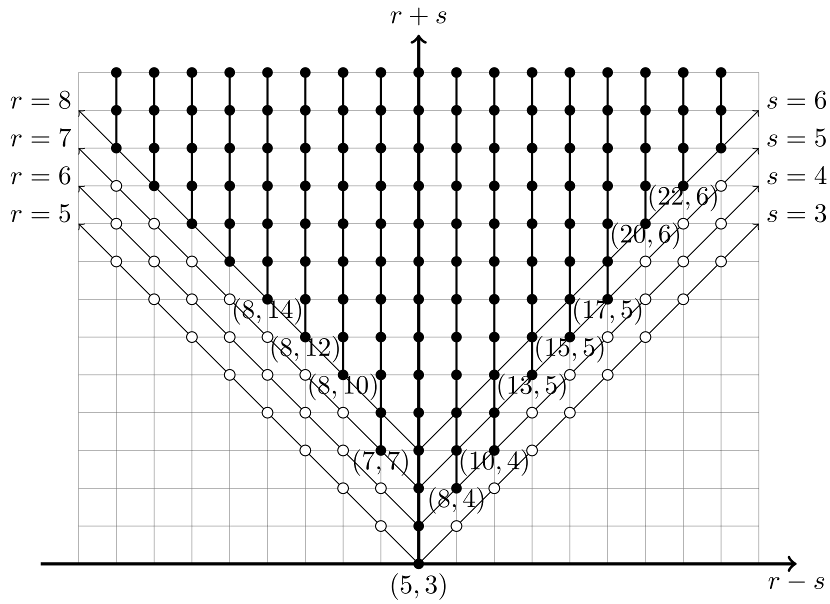
$$\mathcal{C}_{\lambda^\vee}^-(T) = \begin{cases} \{(0, -1)\} & \text{if } 0 < T \leq 1, \\ \{(0, -1), (-1, -1)\} & \text{if } 1 < T \leq 3, \\ \{(0, -1), (-1, -1), (-3, -1)\} & \text{if } T > 3. \end{cases}$$

Therefore, by Theorem 1.7, $\Theta_{5+l,3+2t+l}(\pi)$ is nonzero if and only if one of the following holds:

- *$t = 0$ and $l \geq 0$;*

- *$t \geq 0$, $l \geq 1$ and $t+1 \leq 1$;*
- *$t \geq 0$, $l \geq 2$ and $t+2 \leq 3$;*
- *$t \geq 0$ and $l \geq 3$.*

The following summarizes when $\Theta_{r,s}(\pi) \neq 0$. Here, a black plot (r,s) means that $\Theta_{r,s}(\pi) \neq 0$, and a white plot (r,s) means that $\Theta_{r,s}(\pi) = 0$.



1.5. **Non-archimedean case.** In this subsection, we recall a result in the non-archimedean case in [AG]. Let E/F be a quadratic extension of non-archimedean local fields of characteristic zero. We denote by W_n an n-dimensional hermitian space over E. Also, we fix an anisotropic skew-hermitian space V_0 over E of dimension m_0, and denote by V_l the m-dimensional skew-hermitian space obtained from V_0 by addition of l hyperbolic planes, where $m = m_0 + 2l$. As in §1.1, fixing a non-trivial additive character ψ of F and characters χ_W and χ_V of $E^\times$ such that $\chi_W|F^\times = \eta_{E/F}^n$ and $\chi_V|F^\times = \eta_{E/F}^m$, we obtain the Weil representation $\omega_{\psi,V,W}$ of $\mathrm{U}(W_n) \times \mathrm{U}(V_l)$. Here, $\eta_{E/F}$ is the quadratic character of $F^\times$ associated to E/F.

For an irreducible smooth representation π of $\mathrm{U}(W_n)$, the big theta lift $\Theta_{\psi,V_l,W_n}(\pi)$ is defined similarly as in §1.2. The Howe duality correspondence is proven by Waldspurger [W] when the residual characteristic of F is not two, and by Gan–Takeda [GT1, GT2] in general. Hence we can define the small theta lift (or simply, the local theta lift) $\theta_{\psi,V_l,W_n}(\pi)$ of π similarly as in §1.2, and we may consider Problem 1.5 in the non-archimedean case.

In [AG], we addressed both Problem 1.5 (1) and (2) in terms of the local Langlands correspondence established by Mok [Mo] and Kaletha–Mínguez–Shin–White [KMSW]. We recall only the result concerning Problem 1.5 (1).

The local Langlands correspondence classifies the irreducible tempered representations π of $\mathrm{U}(W_n)$ by their L-parameters $\lambda = (\phi, \eta)$, where

$$\phi\colon WD_E \to \mathrm{GL}_n(\mathbb{C})$$

is a conjugate self-dual representation of the Weil–Deligne group $WD_E = W_E \times \mathrm{SL}_2(\mathbb{C})$ of sign $(-1)^{n-1}$ with bounded image, and

$$\eta \in \mathrm{Irr}(A_\phi)$$

is an irreducible character of the component group A_ϕ associated to ϕ, which is an elementary two abelian group. More precisely, if we decompose

$$\phi = m_1\phi_1 \oplus \cdots \oplus m_u\phi_u \oplus (\phi' \oplus {}^c\phi'^\vee),$$

where

- ϕ_i is an irreducible conjugate self-dual representation of sign $(-1)^{n-1}$;
- $m_i > 0$ is the multiplicity of ϕ_i in ϕ;
- ϕ' is a sum of irreducible representations of WD_E which are not conjugate self-dual of sign $(-1)^{n-1}$;
- ${}^c\phi'^\vee$ is the conjugate dual of ϕ',

then
$$A_\phi = (\mathbb{Z}/2\mathbb{Z})e_{\phi_1} \oplus \cdots \oplus (\mathbb{Z}/2\mathbb{Z})e_{\phi_u}.$$
Namely, A_ϕ is a free $(\mathbb{Z}/2\mathbb{Z})$-module of rank u equipped with a canonical basis $\{e_{\phi_1}, \dots, e_{\phi_u}\}$ associated to $\{\phi_1, \dots, \phi_u\}$. We call the element $z_\phi = m_1 e_{\phi_1} + \cdots + m_u e_{\phi_u}$ in A_ϕ the central element of A_ϕ.

The tower property (Proposition 3.6 below) also holds in the non-archimedean case. It asserts that if $\Theta_{\psi,V_l,W_n}(\pi)$ is nonzero, then so is $\Theta_{\psi,V_{l+1},W_n}(\pi)$. Fix $\kappa \in \{1,2\}$ and $\delta \in E^\times$ such that $\mathrm{tr}_{E/F}(\delta) = 0$. There are exactly two anisotropic skew-hermitian spaces V_0^+ and V_0^- such that $\dim(V_0^\pm) \equiv n + \kappa \bmod 2$. For $\epsilon \in \{\pm 1\}$, when $V_0 = V_0^\epsilon$, we also write $V_l = V_l^\epsilon$. This is characterized by
$$\chi_V(\delta^{-m}(-1)^{(m-1)m/2}\det(V_l)) = \epsilon.$$
We call $\mathcal{V}_\epsilon = \{V_l^\epsilon \mid l \geq 0\}$ the ϵ-Witt tower, and
$$m_\epsilon(\pi) = \min\{\dim(V_l^\epsilon) \mid \Theta_{\psi,V_l^\epsilon,W_n}(\pi) \neq 0\}$$
the first occurrence index of π in the Witt tower $\mathcal{V}_\epsilon$.

The following theorem is one of the main results in [AG]. Here, we denote the unique k-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ by S_k.

Theorem 1.12 ([AG, Theorem 4.1]). *Fix $\kappa \in \{1,2\}$ and $\delta \in E^\times$ such that $\mathrm{tr}_{E/F}(\delta) = 0$. Let π be an irreducible tempered representation of $\mathrm{U}(W_n)$ with L-parameter $\lambda = (\phi, \eta)$.*

(1) *Consider the set $\mathcal{T}$ containing $\kappa - 2$ and all integers $k > 0$ with $k \equiv \kappa \bmod 2$ satisfying the following conditions:*
  (chain condition): *ϕ contains $\chi_V S_\kappa + \chi_V S_{\kappa+2} + \cdots + \chi_V S_k$;*
  (odd-ness condition): *the multiplicity of $\chi_V S_r$ in ϕ is odd for $r = \kappa, \kappa+2, \dots, k-2$;*
  (initial condition): *if $\kappa = 2$, then $\eta(e_{V,2}) = -1$;*
  (alternating condition): *$\eta(e_{V,r}) = -\eta(e_{V,r+2})$ for $r = \kappa, \kappa+2, \dots, k-2$.*
  Here, $e_{V,r}$ is the element in A_ϕ corresponding to $\chi_V S_r$. Let
$$k_\lambda = \max \mathcal{T}.$$
  Then
$$\min\{m_+(\pi), m_-(\pi)\} = n - k_\lambda \quad \text{and} \quad \max\{m_+(\pi), m_-(\pi)\} = n + 2 + k_\lambda.$$
(2) *If $k_\lambda = -1$, then $m_+(\pi) = m_-(\pi)$. Suppose that $k_\lambda \geq 0$. Then ϕ contains χ_V if $\kappa = 1$. Moreover, $\min\{m_+(\pi), m_-(\pi)\} = m_\alpha(\pi)$ if and only if*
$$\alpha = \begin{cases} \eta(z_\phi + e_{\chi_V}) & \text{if } \kappa = 1, \\ \eta(z_\phi) \cdot \varepsilon(\phi \otimes \chi_V^{-1}, \psi_2^E) & \text{if } \kappa = 2, \end{cases}$$
  where $\varepsilon(\phi \otimes \chi_V^{-1}, \psi_2^E) = \varepsilon(1/2, \phi \otimes \chi_V^{-1}, \psi_2^E)$ is the root number of $\phi \otimes \chi_V^{-1}$ with respect to the additive character ψ_2^E of E defined by $\psi_2^E(x) = \psi(\mathrm{tr}_{E/F}(\delta x))$.

It seems that this theorem closely resembles our main result (Theorem 4.2 below).

1.6. **Archimedean case v.s. non-archimedean case.** In this subsection, we explain some differences between the archimedean case and the non-archimedean case.

First of all, one of clear differences is the parametrization of Witt towers. Fixing $\kappa \in \{1,2\}$, in the non-archimedean case, there exist exactly two Witt towers
$$\mathcal{V}_\pm = \{V_l^\pm \mid l \geq 0,\ \dim(V_l^\pm) \equiv n + \kappa \bmod 2\},$$
whereas, in the archimedean case, for each integer d with $d \equiv \kappa \bmod 2$, there is a Witt tower
$$\mathcal{V}_d = \{V_{r,s} \mid r - s = d,\ r + s \equiv n + \kappa \bmod 2\}.$$

The first occurrence indices are defined similarly in §1.5 for each Witt towers. The conservation relation proven by Sun–Zhu [SZ2] asserts that $m_+(\pi) + m_-(\pi) = 2n + 2$ for any irreducible smooth representation π of $\mathrm{U}(W_n)$ in the non-archimedean case, whereas, it is more complicated in the archimedean case (see Theorem 3.7 below).

To study the theta correspondence, we often need to know a relation between theta lifts and induced representations. To show such a relation, in non-archimedean case, Kudla's filtration [Ku1] is useful. This is a finite explicit filtration of the Jacquet module of the Weil representation. Its archimedean analogue is the induction principle (see e.g., [P1, Theorem 4.5.5]). As a matter of fact, however, the induction principle is just an analogue of a corollary of Kudla's filtration, so that it is less useful than Kudla's filtration itself.

Both of the proofs of Theorem 1.12 and our main result (Theorem 4.2 below) use the local Gan–Gross–Prasad conjecture (for tempered representations). In the non-archimedean case, Beuzart-Plessis proved it completely, whereas, in the archimedean case, he proved only a weaker version of it ([BP]). For discrete series representations of $\mathrm{U}(p, q)$, He [He2] proved the local Gan–Gross–Prasad conjecture in terms of Harish-Chandra parameters.

Also, both of the proofs of two results (Theorem 1.12 and Theorem 4.2) use explicit descriptions of theta correspondence in the equal rank case and the almost equal rank case. In the archimedean case, they are Paul's results ([P1, P3]). In the non-archimedean case, they are two of Prasad's conjectures ([Pr]), both of which are proven by Gan–Ichino [GI].

1.7. **Organization of this paper.** This paper is organized as follows. In §2, we explain two parametrizations of irreducible representations of $\mathrm{U}(p, q)$, Harish-Chandra parameters and L-parameters, and we compare them. Also we state the local Gan–Gross–Prasad conjecture. In §3, we recall some basic results of theta correspondence, including the Howe duality correspondence, the induction principle, seesaw identities, and Paul's results ([P1, P3]). For the unitary dual pairs, we use Kudla's splitting, whereas, Paul uses double covers of unitary groups. We also compare them in §3. In §4, we state the main result (Theorem 4.2) and its corollary (Corollary 4.5). Finally, we prove the main result in §5. The relation between Harish-Chandra parameters and L-parameters (Theorem 2.1 (4)) might be well-known, but there seems to be no proper reference. For the convenience of the reader, we explain this relation in Appendix A.

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Notation. The symmetric group on n letters is denoted by S_n. For non-negative integers p and q, we set $n = p + q$ and define the unitary group $\mathrm{U}(p, q)$ of signature (p, q) by

$$\mathrm{U}(p,q) = \left\{ g \in \mathrm{GL}_n(\mathbb{C}) \;\middle|\; {}^t\overline{g} \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix} g = \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix} \right\}.$$

We also denote by $\mathrm{U}_n(\mathbb{R})$ a unitary group of size n, i.e., $\mathrm{U}_n(\mathbb{R})$ is a $\mathrm{U}(p, q)$ for some non-negative integers p, q such that $n = p + q$.

For a reductive Lie group G, we denote by $\mathrm{Irr}_{\mathrm{disc}}(G)$ (resp. $\mathrm{Irr}_{\mathrm{temp}}(G)$) the set of equivalence classes of discrete series representations (resp. tempered representations) of G.

Put $\mathbb{C}^1 = \{z \in \mathbb{C}^\times \mid z\overline{z} = 1\}$. For a symplectic space $\mathbb{W}$ over $\mathbb{R}$, we denote the $\mathbb{C}^1$-cover of $\mathrm{Sp}(\mathbb{W})$ by $\mathrm{Mp}(\mathbb{W})$, and the double cover of $\mathrm{Sp}(\mathbb{W})$ by $\widetilde{\mathrm{Sp}}(\mathbb{W})$, which is a closed subgroup of $\mathrm{Mp}(\mathbb{W})$. Namely,

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{C}^1 & \longrightarrow & \mathrm{Mp}(\mathbb{W}) & \longrightarrow & \mathrm{Sp}(\mathbb{W}) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \| & & \\
1 & \longrightarrow & \{\pm 1\} & \longrightarrow & \widetilde{\mathrm{Sp}}(\mathbb{W}) & \longrightarrow & \mathrm{Sp}(\mathbb{W}) & \longrightarrow & 1.
\end{array}$$

One should not confuse $\mathrm{Mp}(\mathbb{W})$ with $\widetilde{\mathrm{Sp}}(\mathbb{W})$. When we consider representations of covering groups, we always assume that they are unitary and genuine.

We denote representations of Lie groups by π and σ, and ones of covering groups by $\widetilde{\pi}$ and $\widetilde{\sigma}$. Also we denote the contragredient representations of π, σ, $\widetilde{\pi}$ and $\widetilde{\sigma}$ by $\pi^\vee$, $\sigma^\vee$, $\widetilde{\pi}^\vee$ and $\widetilde{\sigma}^\vee$, respectively. One should not confuse $\widetilde{\pi}$ with $\pi^\vee$.

2. Classifications of irreducible representations of U(p,q)

In this section, we give two parametrizations of irreducible representations of $\mathrm{U}(p,q)$, and we compare them. We also state the local Gan–Gross–Prasad conjecture in §2.3.

2.1. **Harish-Chandra parameters.** The Harish-Chandra parameters classify irreducible discrete series representations. Let $G = \mathrm{U}(p,q)$. We set $K \cong \mathrm{U}(p)\times\mathrm{U}(q)$ to be the maximal compact subgroup of G consisting of the usual block diagonal matrices, and T to be the maximal compact torus of G consisting of diagonal matrices. We denote the Lie algebras of G, K and T by $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{t}$, and its complexifications by $\mathfrak{g}_\mathbb{C}$, $\mathfrak{k}_\mathbb{C}$ and $\mathfrak{t}_\mathbb{C}$, respectively. The set Δ_c of compact roots of $\mathfrak{g}_\mathbb{C}$ with respect to $\mathfrak{t}_\mathbb{C}$ and the set Δ_n of non-compact roots are given by

$$\Delta_c = \{e_i - e_j \mid 1 \le i,j \le p\} \cup \{f_i - f_j \mid 1 \le i,j \le q\},$$
$$\Delta_n = \{\pm(e_i - f_j) \mid 1 \le i \le p,\ 1 \le j \le q\},$$

respectively. Here, $e_i, f_j \in \mathfrak{t}_\mathbb{C}^*$ are defined by

$$e_i\colon \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto t_i, \quad f_j\colon \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto t_{p+j}.$$

Note that e_i and f_j belong to $\sqrt{-1}\mathfrak{t}^*$, i.e., the images $e_i(\mathfrak{t})$ and $f_j(\mathfrak{t})$ are in $\sqrt{-1}\mathbb{R}$.

The Harish-Chandra parameter $\mathrm{HC}(\pi)$ of a discrete series representation π of $\mathrm{U}(p,q)$ is of the form

$$\mathrm{HC}(\pi) = (\lambda_1,\dots,\lambda_p;\lambda'_1,\dots,\lambda'_q) \in \sqrt{-1}\mathfrak{t}^*,$$

where

- $\lambda_i, \lambda'_j \in \mathbb{Z} + \frac{n-1}{2}$;
- $\lambda_i \neq \lambda'_j$ for $1 \le i \le p$ and $1 \le j \le q$;
- $\lambda_1 > \cdots > \lambda_p$ and $\lambda'_1 > \cdots > \lambda'_q$.

Here, using the basis $\{e_1,\dots,e_p,f_1,\dots,f_q\}$, we identify $\sqrt{-1}\mathfrak{t}^*$ with $\mathbb{R}^p\times\mathbb{R}^q$. Via this identification, we regard $\mathrm{HC}(\pi)$ as an element of $(\mathbb{Z}+\frac{n-1}{2})^p \times (\mathbb{Z}+\frac{n-1}{2})^q$. Hence we obtain an injection

$$\mathrm{HC}\colon \mathrm{Irr}_{\mathrm{disc}}(\mathrm{U}(p,q)) \hookrightarrow \left(\mathbb{Z}+\frac{n-1}{2}\right)^p \times \left(\mathbb{Z}+\frac{n-1}{2}\right)^q.$$

The infinitesimal character τ_λ of π is the W_G-orbit of $\lambda = \mathrm{HC}(\pi)$, where W_G is the Weyl group of G relative to T. As well as $\lambda = \mathrm{HC}(\pi)$ is regarded as an element of $(\mathbb{Z}+\frac{n-1}{2})^p \times (\mathbb{Z}+\frac{n-1}{2})^q$, we regard τ_λ as an element of $(\mathbb{Z}+\frac{n-1}{2})^n/S_n$. Note that given $\tau \in (\mathbb{Z}+\frac{n-1}{2})^n/S_n$, there are exactly $n!/(p!q!)$ discrete series representations of $\mathrm{U}(p,q)$ whose infinitesimal characters are equal to τ.

2.2. **L-parameters.** The local Langlands correspondence is a parametrization of irreducible tempered representations of $\mathrm{U}(p,q)$ in terms of L-parameters.

For $\alpha \in \frac{1}{2}\mathbb{Z}$, we define a unitary character $\chi_{2\alpha}$ of $\mathbb{C}^\times$ by

$$\chi_{2\alpha}(z) = \overline{z}^{-2\alpha}(z\overline{z})^\alpha = (z/\overline{z})^\alpha.$$

Note that $\chi_{2\alpha}(\overline{z}) = \chi_{2\alpha}(z)^{-1} = \chi_{-2\alpha}(z)$. When $a > 0$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, we have

$$\chi_{2\alpha}(ae^{\sqrt{-1}\theta}) = e^{2\alpha\sqrt{-1}\theta}.$$

Define $\Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R}))$ by

$$\Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R})) = \left\{ \chi_{2\alpha_1} \oplus \cdots \oplus \chi_{2\alpha_n} \;\middle|\; \alpha_i \in \frac{1}{2}\mathbb{Z},\ 2\alpha_i \equiv n-1 \bmod 2,\ \alpha_1 > \cdots > \alpha_n \right\}.$$

For $\phi = \chi_{2\alpha_1} \oplus \cdots \oplus \chi_{2\alpha_n} \in \Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R}))$, we define a component group A_ϕ of ϕ by

$$A_\phi = (\mathbb{Z}/2\mathbb{Z})e_{2\alpha_1} \oplus \cdots \oplus (\mathbb{Z}/2\mathbb{Z})e_{2\alpha_n}.$$

Namely, A_ϕ is a free $(\mathbb{Z}/2\mathbb{Z})$-module of rank n equipped with a canonical basis $\{e_{2\alpha_1}, \dots, e_{2\alpha_n}\}$ associated to $\{\chi_{2\alpha_1}, \dots, \chi_{2\alpha_n}\}$.

More generally, we define $\Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$ by the set of representations ϕ of $\mathbb{C}^\times$ of the form

$$\phi = (m_1\chi_{2\alpha_1} \oplus \cdots \oplus m_u\chi_{2\alpha_u}) \oplus (\xi_1 \oplus \cdots \oplus \xi_v) \oplus ({}^c\xi_1^{-1} \oplus \cdots \oplus {}^c\xi_v^{-1}),$$

where

- $\alpha_i \in \frac{1}{2}\mathbb{Z}$ satisfies $2\alpha_i \equiv n-1 \bmod 2$ and $\alpha_1 > \cdots > \alpha_u$;
- $m_i \geq 1$ is the multiplicity of $\chi_{2\alpha_i}$ in ϕ;
- $m_1 + \cdots + m_u + 2v = n$;
- ξ_i is a unitary character of $\mathbb{C}^\times$, which is not of the form $\chi_{2\alpha}$ with $2\alpha \equiv n-1 \bmod 2$;
- ${}^c\xi_i^{-1}$ is the unitary character of $\mathbb{C}^\times$ defined by ${}^c\xi_i^{-1}(z) = \xi_i(\overline{z}^{-1})$.

For such ϕ, we define a component group A_ϕ of ϕ by

$$A_\phi = (\mathbb{Z}/2\mathbb{Z})e_{2\alpha_1} \oplus \cdots \oplus (\mathbb{Z}/2\mathbb{Z})e_{2\alpha_u}.$$

We denote the Pontryagin dual of A_ϕ by $\widehat{A_\phi}$. For $\eta \in \widehat{A_\phi}$, define $-\eta \in \widehat{A_\phi}$ by $(-\eta)(e_{2\alpha_i}) = -\eta(e_{2\alpha_i})$ for $i = 1, \dots, u$.

We define an additive character $\psi^{\mathbb{C}}_{-2}$ of $\mathbb{C}$ by

$$\psi^{\mathbb{C}}_{-2}(z) = \exp(2\pi(\overline{z} - z))$$

for $z \in \mathbb{C}$. For a (continuous, completely reducible) representation ϕ of $\mathbb{C}^\times$, let $\varepsilon(s, \phi, \psi^{\mathbb{C}}_{-2})$ be the ε-factor of ϕ (see [T]). It satisfies that

- $\varepsilon(s, \phi_1 \oplus \phi_2, \psi^{\mathbb{C}}_{-2}) = \varepsilon(s, \phi_1, \psi^{\mathbb{C}}_{-2}) \cdot \varepsilon(s, \phi_2, \psi^{\mathbb{C}}_{-2})$;
- $\varepsilon(1/2, \xi \oplus {}^c\xi^{-1}, \psi^{\mathbb{C}}_{-2}) = 1$ for any character ξ of $\mathbb{C}^\times$;
- $\varepsilon(1/2, \chi_{2\alpha}, \psi^{\mathbb{C}}_{-2}) = 1$ for $\alpha \in \mathbb{Z}$;
- When $\alpha \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$,

$$\varepsilon\left(\frac{1}{2}, \chi_{2\alpha}, \psi^{\mathbb{C}}_{-2}\right) = \begin{cases} -1 & \text{if } \alpha > 0, \\ +1 & \text{if } \alpha < 0. \end{cases}$$

For the last equation, see e.g., [GGP2, Proposition 2.1]. We call the central value $\varepsilon(1/2, \phi, \psi^{\mathbb{C}}_{-2})$ the root number of ϕ with respect to $\psi^{\mathbb{C}}_{-2}$, and we denote it simply by $\varepsilon(\phi, \psi^{\mathbb{C}}_{-2})$.

The local Langlands correspondence is a parametrization of $\mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(p,q))$ as follows:

Theorem 2.1 ([L], [V3], [S1, S2, S3]). (1) *There is a canonical surjection*

$$\bigsqcup_{p+q=n} \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(p,q)) \to \Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R})).$$

For $\phi \in \Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$, we denote by Π_ϕ the inverse image of ϕ under this map, and call Π_ϕ the L-packet associated to ϕ.

(2) *There is a bijection*

$$J \colon \Pi_\phi \to \widehat{A_\phi}.$$

If $\pi \in \Pi_\phi$ corresponds to $\eta \in \widehat{A_\phi}$ via this bijection, we write $\pi = \pi(\phi, \eta)$ and call (ϕ, η) the L-parameter of π.

(3) *If $\phi = \chi_{2\alpha_1} \oplus \cdots \oplus \chi_{2\alpha_n} \in \Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R}))$, the L-packet Π_ϕ consists of discrete series representations of various $\mathrm{U}(p,q)$ whose infinitesimal characters are $(\alpha_1, \dots, \alpha_n) \in (\mathbb{Z} + \frac{n-1}{2})^n/S_n$.*

(4) *If* $\phi = \chi_{2\alpha_1} \oplus \cdots \oplus \chi_{2\alpha_n} \in \Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R}))$ *with* $\alpha_1 > \cdots > \alpha_n$*, the Harish-Chandra parameter*

$$\mathrm{HC}(\pi(\phi, \eta)) = (\lambda_1, \ldots, \lambda_p; \lambda'_1, \ldots, \lambda'_q)$$

of $\pi(\phi, \eta) \in \Pi_\phi$ *is given so that*

- $\{\lambda_1, \ldots, \lambda_p, \lambda'_1, \ldots, \lambda'_q\} = \{\alpha_1, \ldots, \alpha_n\}$*;*
- $\alpha_i \in \{\lambda_1, \ldots, \lambda_p\}$ *if and only if* $\eta(e_{2\alpha_i}) = (-1)^{i-1}$*.*

In particular, $\pi(\phi, \eta) \in \mathrm{Irr}_{\mathrm{disc}}(\mathrm{U}(p, q))$ *with*

$$p = \#\{i \mid \eta(e_{2\alpha_i}) = (-1)^{i-1}\}, \quad q = \#\{i \mid \eta(e_{2\alpha_i}) = (-1)^{i}\}.$$

(5) *If* $\phi = \xi \oplus \phi_0 \oplus {}^c\xi^{-1}$ *with a unitary character* ξ *of* $\mathbb{C}^\times$ *and an element* ϕ_0 *in* $\Phi_{\mathrm{temp}}(\mathrm{U}_{n-2}(\mathbb{R}))$*, for any* $\pi(\phi_0, \eta_0) \in \Pi_{\phi_0} \cap \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(p-1, q-1))$*, the induced representation* $\mathrm{Ind}_P^{\mathrm{U}(p,q)}(\xi \otimes \pi(\phi_0, \eta_0))$ *decomposes as follows:*

$$\mathrm{Ind}_P^{\mathrm{U}(p,q)}(\xi \otimes \pi(\phi_0, \eta_0)) = \bigoplus_{\substack{\eta \in \widehat{A_\phi}, \\ \eta | A_{\phi_0} = \eta_0}} \pi(\phi, \eta).$$

Here, P *is a parabolic subgroup of* $\mathrm{U}(p, q)$ *with Levi subgroup* $M_P = \mathbb{C}^\times \times \mathrm{U}(p-1, q-1)$*.*

(6) *The contragredient representation of* $\pi(\phi, \eta)$ *is given by* $\pi(\phi^\vee, \eta^\vee)$*, where* $\eta^\vee \colon A_{\phi^\vee} \to \{\pm 1\}$ *is defined by*

$$\eta^\vee(e_{-2\alpha_i}) = \begin{cases} \eta(e_{2\alpha_i}) & \text{if } n \text{ is odd}, \\ -\eta(e_{2\alpha_i}) & \text{if } n \text{ is even} \end{cases}$$

for any $e_{-2\alpha_i} \in A_{\phi^\vee}$*.*

(7) *If* $\pi = \pi(\phi, \eta) \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(p, q))$*, then* $\pi(\phi, -\eta) \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(q, p))$ *is the same representation as* π *via the canonical identification* $\mathrm{U}(p, q) = \mathrm{U}(q, p)$ *as subgroups of* $\mathrm{GL}_n(\mathbb{C})$*.*

(8) *If* $\phi \in \Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$*, then* $\phi\chi_2 = \phi \otimes \chi_2 \in \Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$ *and there is a canonical identification* $A_\phi = A_{\phi\chi_2}$*. If* $\pi = \pi(\phi, \eta)$*, the corresponding representation* $\pi(\phi\chi_2, \eta)$ *is the determinant twist* $\pi \otimes \det$*.*

In fact, the bijection $J \colon \Pi_\phi \to \widehat{A_\phi}$ in Theorem 2.1 (2) is characterized by endoscopic character identities as in [S3]. Also J depends on a choice of a pair of a quasi-split $\mathrm{U}(p, q)$ (i.e., $|p - q| \leq 1$) and a Whittaker datum of $\mathrm{U}(p, q)$. There are exactly two such pairs. Through this paper, we take a specific choice of a pair. (see Appendix A). Theorem 2.1 (4) highly depends on this choice, and there seems to be no proper reference. We will discuss this part in Appendix A below.

In Theorem 2.1 (4), we see that

$$(-1)^{i-1} = \varepsilon(\phi \otimes \chi_{-2\alpha_i} \otimes \chi_{-1}, \psi^{\mathbb{C}}_{-2}).$$

Hence the Harish-Chandra parameter of π and the unitary group $\mathrm{U}(p, q)$ which acts on π can be determined by the L-parameter $\lambda = (\phi, \eta)$ of π and certain root numbers.

2.3. **Local Gan–Gross–Prasad conjecture.** To prove the main result, we use the local Gan–Gross–Prasad conjecture [GGP1], which gives an answer to restriction problems.

Suppose that $(\mathrm{U}_n(\mathbb{R}), \mathrm{U}_{n+1}(\mathbb{R})) = (\mathrm{U}(p, q), \mathrm{U}(p+1, q))$ or $(\mathrm{U}(p, q), \mathrm{U}(p, q+1))$. Then there is a canonical injection $\mathrm{U}_n(\mathbb{R}) \hookrightarrow \mathrm{U}_{n+1}(\mathbb{R})$, so that we have a diagonal map

$$\Delta \colon \mathrm{U}_n(\mathbb{R}) \to \mathrm{U}_n(\mathbb{R}) \times \mathrm{U}_{n+1}(\mathbb{R}).$$

By a result of Sun–Zhu [SZ1], for $\pi_n \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$ and $\pi_{n+1} \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}_{n+1}(\mathbb{R}))$, we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\Delta \mathrm{U}_n(\mathbb{R})}(\pi_n \otimes \pi_{n+1}, \mathbb{C}) \leq 1.$$

We call a pair $(\mathrm{U}_n(\mathbb{R}), \mathrm{U}_{n+1}(\mathbb{R}))$ relevant if

$$(\mathrm{U}_n(\mathbb{R}), \mathrm{U}_{n+1}(\mathbb{R})) = \begin{cases} (\mathrm{U}(p, q), \mathrm{U}(p+1, q)) & \text{if } n \text{ is even}, \\ (\mathrm{U}(p, q), \mathrm{U}(p, q+1)) & \text{if } n \text{ is odd} \end{cases}$$

for some (p,q) such that $p+q=n$. Note that if $(\mathrm{U}(p,q),\mathrm{U}(p',q'))$ with $p'+q'=p+q+1$ and $0\leq p'-q, q'-q\leq 1$ is not relevant, then $(\mathrm{U}(q,p),\mathrm{U}(q',p'))$ is relevant. Gan, Gross and Prasad predicted when the Hom space

$$\mathrm{Hom}_{\Delta \mathrm{U}_n(\mathbb{R})}(\pi_n\otimes\pi_{n+1},\mathbb{C})$$

is nonzero for relevant pairs.

Conjecture 2.2 (local Gan–Gross–Prasad conjecture (GGP) [GGP1, Conjecture 17.3])**.** *Let $\phi_n\in\Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$ and $\phi_{n+1}\in\Phi_{\mathrm{temp}}(\mathrm{U}_{n+1}(\mathbb{R}))$ such that*

$$\phi_n=(m_1\chi_{2\alpha_1}\oplus\cdots\oplus m_u\chi_{2\alpha_u})\oplus(\xi_1\oplus\cdots\oplus\xi_v)\oplus({}^c\xi_1^{-1}\oplus\cdots\oplus{}^c\xi_v^{-1}),$$
$$\phi_{n+1}=(m'_1\chi_{2\beta_1}\oplus\cdots\oplus m'_{u'}\chi_{2\beta_{u'}})\oplus(\xi'_1\oplus\cdots\oplus\xi'_{v'})\oplus({}^c\xi_1'^{-1}\oplus\cdots\oplus{}^c\xi_{v'}'^{-1}),$$

where

- $\alpha_i,\beta_j\in\frac{1}{2}\mathbb{Z}$ *such that* $2\alpha_i\equiv n-1 \bmod 2$ *and* $2\beta_j\equiv n\bmod 2$*;*
- $m_i\geq 1$ *(resp.* $m'_j\geq 1$*) is the multiplicity of* $\chi_{2\alpha_i}$ *(resp.* $\chi_{2\beta_j}$*) in* ϕ_n *(resp.* ϕ_{n+1}*);*
- $m_1+\cdots+m_u+2v=n$ *and* $m'_1+\cdots+m'_{u'}+2v'=n+1$*;*
- ξ_i *(resp.* ξ'_j*) is a unitary character of* $\mathbb{C}^\times$*, which is not of the form* $\chi_{2\alpha}$ *with* $2\alpha\equiv n-1\bmod 2$ *(resp.* $\chi_{2\beta}$ *with* $2\beta\equiv n\bmod 2$*).*

Then there exists a unique pair of representations $(\pi_n,\pi_{n+1})\in\Pi_{\phi_n}\times\Pi_{\phi_{n+1}}$ *such that*

- (π_n,π_{n+1}) *is a pair of representations of a relevant pair* $(\mathrm{U}_n(\mathbb{R}),\mathrm{U}_{n+1}(\mathbb{R}))$*;*
- $\mathrm{Hom}_{\Delta\mathrm{U}_n(\mathbb{R})}(\pi_n\otimes\pi_{n+1},\mathbb{C})\neq 0$.

Moreover,

$$J(\pi_n)(e_{2\alpha_i})=\varepsilon(\chi_{2\alpha_i}\otimes\phi_{n+1},\psi^{\mathbb{C}}_{-2}),$$
$$J(\pi_{n+1})(e_{2\beta_j})=\varepsilon(\phi_n\otimes\chi_{2\beta_j},\psi^{\mathbb{C}}_{-2})$$

for $e_{2\alpha_i}\in A_{\phi_n}$ *and* $e_{2\beta_j}\in A_{\phi_{n+1}}$.

When $\phi_n\in\Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R}))$ and $\phi_{n+1}\in\Phi_{\mathrm{disc}}(\mathrm{U}_{n+1}(\mathbb{R}))$, Conjecture 2.2 is proven by He [He2]. In fact, He [He2, Theorem 1.1] only treat the discrete spectrum of $\pi_{n+1}|\mathrm{U}_n(\mathbb{R})$ for $\pi_{n+1}\in\mathrm{Irr}_{\mathrm{disc}}(\mathrm{U}_{n+1}(\mathbb{R}))$. Conjecture 2.2 for discrete series representations follows from this result together with the multiplicity one statement with respect to the Hom-space $\mathrm{Hom}_{\Delta\mathrm{U}_n(\mathbb{R})}(\pi_n\otimes\pi_{n+1},\mathbb{C})$ for general tempered L-packets shown by Beuzart-Plessis [BP].

We use this conjecture as the following form.

Proposition 2.3. *Assume the local Gan–Gross–Prasad conjecture (Conjecture 2.2). Let* $\phi\in\Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$ *and* $\phi'\in\Phi_{\mathrm{temp}}(\mathrm{U}_{n+1}(\mathbb{R}))$ *such that*

$$\phi=\chi_{2\alpha_1}\oplus\cdots\oplus\chi_{2\alpha_u}\oplus(\xi_1\oplus\cdots\oplus\xi_v)\oplus({}^c\xi_1^{-1}\oplus\cdots\oplus{}^c\xi_v^{-1}),$$
$$\phi'=\chi_{2\beta_1}\oplus\cdots\oplus\chi_{2\beta_{u'}}\oplus(\xi'_1\oplus\cdots\oplus\xi'_{v'})\oplus({}^c\xi_1'^{-1}\oplus\cdots\oplus{}^c\xi_{v'}'^{-1}),$$

where

- $\alpha_i,\beta_j\in\frac{1}{2}\mathbb{Z}$ *such that* $2\alpha_i\equiv n-1\bmod 2$ *and* $2\beta_j\equiv n\bmod 2$*;*
- ξ_i *(resp.* ξ'_j*) is a unitary character of* $\mathbb{C}^\times$ *(which can be of the form* $\chi_{2\alpha}$ *(resp.* $\chi_{2\beta}$*));*
- $u+2v=n$ *(resp.* $u'+2v'=n+1$*).*

Then for $(\pi,\pi')\in\Pi_\phi\times\Pi_{\phi'}$*, the following are equivalent:*

- $(\pi,\pi')\in\mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(p,q))\times\mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(p+1,q))$ *for some* (p,q) *and* $\mathrm{Hom}_{\mathrm{U}(p,q)}(\pi',\pi)\neq 0$*;*
- $J(\pi)\in\widehat{A_\phi}$ *and* $J(\pi')\in\widehat{A_{\phi'}}$ *are given by*

$$J(\pi)(e_{2\alpha})=(-1)^{\#\{j\in\{1,\dots,u'\}\ |\ \beta_j<\alpha\}+n},$$
$$J(\pi')(e_{2\beta})=(-1)^{\#\{i\in\{1,\dots,u\}\ |\ \alpha_i<\beta\}+n}$$

for any $\chi_{2\alpha}\subset\phi$ *so that* $e_{2\alpha}\in A_\phi$ *and any* $\chi_{2\beta}\subset\phi'$ *so that* $e_{2\beta}\in A_{\phi'}$.

Proof. Note that $\mathrm{Hom}_{\mathrm{U}(p,q)}(\pi',\pi) \neq 0$ if and only if $\mathrm{Hom}_{\Delta\mathrm{U}(p,q)}(\pi' \otimes \pi^\vee, \mathbb{C}) \neq 0$ by Riesz representation theorem. First, we assume that $n = p+q$ is even. Then $(\mathrm{U}(p,q), \mathrm{U}(p+1,q))$ is a relevant pair. By the local Gan–Gross–Prasad conjecture (Conjecture 2.2), we should see that $\mathrm{Hom}_{\Delta\mathrm{U}(p,q)}(\pi' \otimes \pi^\vee, \mathbb{C}) \neq 0$ if and only if

$$\begin{aligned} J(\pi^\vee)(e_{-2\alpha}) &= \varepsilon(\chi_{-2\alpha} \otimes \phi', \psi^{\mathbb{C}}_{-2}) = (-1)^{\#\{j \in \{1,\dots,u'\} \mid -\alpha+\beta_j > 0\}}, \\ J(\pi')(e_{2\beta}) &= \varepsilon(\phi^\vee \otimes \chi_{2\beta}, \psi^{\mathbb{C}}_{-2}) = (-1)^{\#\{i \in \{1,\dots,u\} \mid -\alpha_i+\beta > 0\}} \end{aligned}$$

for any $e_{-2\alpha} \in A_{\phi^\vee}$ and any $e_{2\beta} \in A_{\phi'}$. By Theorem 2.1 (6), we see that

$$J(\pi)(e_{2\alpha}) = -J(\pi^\vee)(e_{-2\alpha}) = -(-1)^{\#\{j \in \{1,\dots,u'\} \mid \beta_j > \alpha\}} = (-1)^{\#\{j \in \{1,\dots,u'\} \mid \beta_j < \alpha\}}$$

for any $e_{2\alpha} \in A_\phi$ since $u' \equiv n+1 \equiv 1 \bmod 2$. Hence we obtain the assertion in the case where $n = p+q$ is even.

Next, we assume that $n = p+q$ is odd. Then $(\mathrm{U}(q,p), \mathrm{U}(q,p+1))$ is a relevant pair. By Theorem 2.1 (7), we see that $\mathrm{Hom}_{\mathrm{U}(p,q)}(\pi',\pi) \neq 0$ if and only if $\mathrm{Hom}_{\mathrm{U}(q,p)}(\pi(\phi', -J(\pi')), \pi(\phi, -J(\pi))) \neq 0$. By a similar argument to the first case, this is equivalent to saying that

$$\begin{aligned} -J(\pi)(e_{2\alpha}) &= (-1)^{\#\{j \in \{1,\dots,u'\} \mid \beta_j < \alpha\}}, \\ -J(\pi')(e_{2\beta}) &= (-1)^{\#\{i \in \{1,\dots,u\} \mid \alpha_i < \beta\}} \end{aligned}$$

for any $e_{2\alpha} \in A_\phi$ and any $e_{2\beta} \in A_{\phi'}$. Hence we obtain the assertion in the case where $n = p+q$ is odd. This completes the proof. □

3. Theta liftings

In this subsection, we review the theory of theta liftings. First, we recall Kudla's splitting [Ku2] of a unitary dual pair (§3.1). Then we can consider theta lifts of irreducible unitary representations of unitary groups as in §1.2. On the other hand, in various results, including Paul's ones ([P1, P2, P3]), irreducible genuine representations of certain double covers of unitary groups are used for theta lifts. We compare Kudla's splitting with double covers of unitary groups in §3.2. In §3.3 and §3.4, we recall basic properties on theta liftings and Paul's results [P1, P3], respectively.

3.1. **Kudla's splitting.** Let $W = W_{p,q}$ (resp. $V = V_{r,s}$) be a (right) complex vector space of dimension $n = p+q$ (resp. $m = r+s$) equipped with a hermitian form $\langle \cdot, \cdot \rangle_W$ (resp. a skew-hermitian form $\langle \cdot, \cdot \rangle_V$) of signature (p,q) (resp. (r,s)). Namely,

- the pairings $\langle \cdot, \cdot \rangle_W$ and $\langle \cdot, \cdot \rangle_V$ satisfy
$$\begin{aligned} \langle w_1 a, w_2 b \rangle_W &= a\bar{b} \langle w_1, w_2 \rangle_W, \quad \langle w_2, w_1 \rangle_W = \overline{\langle w_1, w_2 \rangle_W}, \\ \langle v_1 a, v_2 b \rangle_V &= a\bar{b} \langle v_1, v_2 \rangle_V, \quad \langle v_2, v_1 \rangle_V = -\overline{\langle v_1, v_2 \rangle_V} \end{aligned}$$
for $a, b \in \mathbb{C}$, $w_1, w_2 \in W$ and $v_1, v_2 \in V$;
- there exist $e_1, \dots, e_n \in W$ and $e'_1, \dots, e'_m \in V$ such that
$$\langle e_i, e_j \rangle_W = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j \leq p, \\ -1 & \text{if } i = j > p, \end{cases} \quad \langle e'_i, e'_j \rangle_V = \begin{cases} 0 & \text{if } i \neq j, \\ \sqrt{-1} & \text{if } i = j \leq r, \\ -\sqrt{-1} & \text{if } i = j > r. \end{cases}$$

The isometry group $\mathrm{U}(W)$ of $\langle \cdot, \cdot \rangle_W$ (resp. $\mathrm{U}(V)$ of $\langle \cdot, \cdot \rangle_V$), which has a left action on W (resp. on V), is isomorphic to $\mathrm{U}(p,q)$ (resp. $\mathrm{U}(r,s)$). Let $\mathbb{W} = V \otimes_{\mathbb{C}} W$ be the symplectic space over $\mathbb{R}$ of dimension $2mn$ equipped with the symplectic form

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle = \mathrm{tr}_{\mathbb{C}/\mathbb{R}}(\langle v_1, v_2 \rangle_V \cdot \langle w_1, w_2 \rangle_W)$$

for $v_1, v_2 \in V$ and $w_1, w_2 \in W$. The symplectic group $\mathrm{Sp}(\mathbb{W})$ acts on $\mathbb{W}$ on the left.

We note that the convention in [Ku2] and [HKS] differs from ours. They use the following:

- W' is a left vector space and the hermitian form $\langle\cdot,\cdot\rangle_{W'}$ on W' satisfies
$$\langle aw'_1, bw'_2\rangle_{W'} = a\langle w'_1, w'_2\rangle_{W'}\overline{b}$$
for $a, b \in \mathbb{C}$ and $w'_1, w'_2 \in W'$;
- V' is a right vector space and the skew-hermitian form $\langle\cdot,\cdot\rangle_{V'}$ on V' satisfies
$$\langle v'_1 a, v'_2 b\rangle_{V'} = \overline{a}\langle v'_1, v'_2\rangle_{V'} b$$
for $a, b \in \mathbb{C}$ and $v'_1, v'_2 \in V'$;
- the symplectic form on $\mathbb{W}' = V' \otimes_{\mathbb{C}} W'$ is defined by
$$\langle v'_1 \otimes w'_1, v'_2 \otimes w'_2\rangle' = \frac{1}{2}\cdot \mathrm{tr}_{\mathbb{C}/\mathbb{R}}(\langle v'_1, v'_2\rangle_{V'} \cdot \overline{\langle w'_1, w'_2\rangle_{W'}})$$
for $v'_1, v'_2 \in V'$ and $w'_1, w'_2 \in W'$;
- $\mathrm{U}(W')$, $\mathrm{U}(V')$ and $\mathrm{Sp}(\mathbb{W}')$ act on W', V' and $\mathbb{W}'$ on the right, left and right, respectively.

To use results in [Ku2] and [HKS], we have to compare these conventions.

First, we compare $\mathrm{U}(W)$ with $\mathrm{U}(W')$. Assume that W' has a basis $\{e_1, \dots, e_n\}$ satisfying the same conditions as above, so that $W = W'$. However, since W is a right $\mathbb{C}$-vector space, whereas, W' is a left $\mathbb{C}$-vector space, we obtain expressions
$$W = \left\{ \begin{pmatrix} e_1 & \dots & e_n \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \middle| \, a_i \in \mathbb{C} \right\} \cong \mathbb{C}^n \quad \text{(column vectors)},$$
$$W' = \left\{ \begin{pmatrix} a'_1 & \dots & a'_n \end{pmatrix} \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} \middle| \, a'_i \in \mathbb{C} \right\} \cong \mathbb{C}^n \quad \text{(row vectors)}.$$
Via the $\mathbb{C}$-linear isomorphism
$$W \ni w = \begin{pmatrix} e_1 & \dots & e_n \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \mapsto w' = \begin{pmatrix} a_1 & \dots & a_n \end{pmatrix} \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} \in W',$$
we identify W with W'. Then we have $\langle w'_1, w'_2\rangle_{W'} = \langle w_1, w_2\rangle_W$ for any $w_1, w_2 \in W$. Also, via the identifications $W \cong \mathbb{C}^n$ (row vectors) and $W' \cong \mathbb{C}^n$ (column vectors), we have expressions
$$\mathrm{U}(W) = \left\{ g \in \mathrm{GL}_n(\mathbb{C}) \,\middle|\, {}^t g \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix} \overline{g} = \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix} \right\},$$
$$\mathrm{U}(W') = \left\{ g' \in \mathrm{GL}_n(\mathbb{C}) \,\middle|\, g' \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix} {}^t\overline{g'} = \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix} \right\}.$$
The above identification $W = W'$ implies the map
$$\mathrm{U}(W) \ni g \mapsto {}^t g \in \mathrm{U}(W').$$

Next, we compare $\mathrm{U}(V)$ with $\mathrm{U}(V')$. Assume that V' has a basis $\{e'_1, \dots, e'_m\}$ so that $V = V'$ as right $\mathbb{C}$-vector spaces. Set the skew-hermitian pairing $\langle\cdot,\cdot\rangle_{V'}$ by
$$\langle v'_1, v'_2\rangle_{V'} = 2\overline{\langle v_1, v_2\rangle_V}$$
for elements $v_1 = v'_1$ and $v_2 = v'_2$ in $V = V'$. Note that the signature of V' is (s, r). We have expressions
$$\mathrm{U}(V) = \left\{ h \in \mathrm{GL}_m(\mathbb{C}) \,\middle|\, {}^t h \begin{pmatrix} \sqrt{-1}\mathbf{1}_r & 0 \\ 0 & -\sqrt{-1}\mathbf{1}_s \end{pmatrix} \overline{h} = \begin{pmatrix} \sqrt{-1}\mathbf{1}_p & 0 \\ 0 & -\sqrt{-1}\mathbf{1}_q \end{pmatrix} \right\},$$
$$\mathrm{U}(V') = \left\{ h' \in \mathrm{GL}_m(\mathbb{C}) \,\middle|\, {}^t\overline{h'} \begin{pmatrix} -2\sqrt{-1}\mathbf{1}_r & 0 \\ 0 & 2\sqrt{-1}\mathbf{1}_s \end{pmatrix} h' = \begin{pmatrix} -2\sqrt{-1}\mathbf{1}_r & 0 \\ 0 & 2\sqrt{-1}\mathbf{1}_s \end{pmatrix} \right\}.$$
Hence $\mathrm{U}(V')$ coincides with $\mathrm{U}(V)$ as subgroups of $\mathrm{GL}_m(\mathbb{C})$.

Since

$$\begin{aligned}\langle v_1' \otimes w_1', v_2' \otimes w_2' \rangle' &= \frac{1}{2} \cdot \mathrm{tr}_{\mathbb{C}/\mathbb{R}}(\langle v_1', v_2' \rangle_{V'} \cdot \overline{\langle w_1', w_2' \rangle_{W'}}) \\ &= \mathrm{tr}_{\mathbb{C}/\mathbb{R}}(\overline{\langle v_1, v_2 \rangle_V \cdot \langle w_1, w_2 \rangle_W}) = \langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle\end{aligned}$$

for $v_1, v_2 \in V$ and $w_1, w_2 \in W$, we see that the two symplectic forms on $\mathbb{W} = \mathbb{W}'$ agree.

Note that both $\mathrm{U}(W)$ and $\mathrm{Sp}(\mathbb{W})$ act on W and $\mathbb{W}$ on the left, respectively, whereas, $\mathrm{U}(W')$ and $\mathrm{Sp}(\mathbb{W}')$ act on W' and $\mathbb{W}'$ on the right, respectively. Hence the canonical embedding $\alpha = \alpha_V \colon \mathrm{U}(W) \to \mathrm{Sp}(\mathbb{W})$ coincides with the counterpart $\alpha' = \alpha_{V'} \colon \mathrm{U}(W') \to \mathrm{Sp}(\mathbb{W}')$ via the above identifications. Therefore the results in [Ku2] can be transferred to our convention.

Fix a non-trivial additive character ψ of $\mathbb{R}$. Let $\mathrm{Mp}(\mathbb{W})$ be the $\mathbb{C}^1$-cover of $\mathrm{Sp}(\mathbb{W})$, i.e.,

$$1 \longrightarrow \mathbb{C}^1 \longrightarrow \mathrm{Mp}(\mathbb{W}) \longrightarrow \mathrm{Sp}(\mathbb{W}) \longrightarrow 1.$$

Choosing a character $\chi = \chi_V$ of $\mathbb{C}^\times$ such that $\chi|\mathbb{R}^\times = \mathrm{sgn}^m$ with $m = r+s = \dim(V)$, Kudla gave a splitting

$$\begin{tikzcd} & \mathrm{Mp}(\mathbb{W}) \arrow[d] \\ \mathrm{U}(W) \arrow[ur, dashed, "\widetilde{\alpha}_\chi"] \arrow[r, "\alpha"] & \mathrm{Sp}(\mathbb{W}). \end{tikzcd}$$

Lemma 3.1. *We identify* $\mathrm{Mp}(\mathbb{W}) = \mathrm{Sp}(\mathbb{W}) \times \mathbb{C}^1$ *as sets as in* [Ku2]*. If we write* $\widetilde{\alpha}_\chi(g) = (\alpha(g), \beta_\chi(g))$ *for* $g \in \mathrm{U}(W)$*, then* $\beta_\chi(g)$ *satisfies that*

$$\beta_\chi(g)^8 = \chi(\det(g))^4.$$

Proof. Let W_- denote the space W with the hermitian form $-\langle \cdot, \cdot \rangle_W$. Consider the space $W \oplus W_-$. Now we have a canonical embedding

$$\Delta \colon \mathrm{U}(W) \times \mathrm{U}(W_-) \to \mathrm{U}(W \oplus W_-).$$

Let $x(\Delta(g,1))$ be Rao's function (see [Ku2, §1]). Note that $x(\Delta(g,1))$ is an element in $\mathbb{C}^\times/\mathbb{R}_{>0}$ ([Ku2, Corollary 1.5]). By [Ku2, Lemma 3.5], it satisfies that

$$\det(g)^2 = \left(x(\Delta(g,1)) \cdot \overline{x(\Delta(g,1))}^{-1}\right)^2.$$

By [Ku2, Theorems 3.1, 3.3], we have

$$\beta_\chi(g) = \chi(x(\Delta(g,1)))\zeta$$

for some 8-th root of unity ζ. Since $\chi^2|\mathbb{R}^\times = \mathbf{1}$, we have

$$\beta_\chi(g)^8 = \chi(x(\Delta(g,1)))^8 = \chi\left(x(\Delta(g,1)) \cdot \overline{x(\Delta(g,1))}^{-1}\right)^4 = \chi(\det(g))^4.$$

This completes the proof. □

3.2. **Double cover of** $\mathrm{U}(p,q)$**.** Let W, V and $\mathbb{W}$ be as in §3.1. Then we have a canonical map $\alpha = \alpha_V \colon \mathrm{U}(W) \to \mathrm{Sp}(\mathbb{W})$. Let $\widetilde{\mathrm{Sp}}(\mathbb{W})$ be the double cover of $\mathrm{Sp}(\mathbb{W})$, which is a closed subgroup of $\mathrm{Mp}(\mathbb{W})$, i.e.,

$$1 \longrightarrow \{\pm 1\} \longrightarrow \widetilde{\mathrm{Sp}}(\mathbb{W}) \longrightarrow \mathrm{Sp}(\mathbb{W}) \longrightarrow 1.$$

Fix $\nu \in \mathbb{Z}$ such that $\nu \equiv m = r+s \bmod 2$ and define the $\det^{\nu/2}$-cover of $\mathrm{U}(W)$ by

$$\widetilde{\mathrm{U}}(W) = \{(g,z) \in \mathrm{U}(W) \times \mathbb{C}^1 \mid z^2 = \det(g)^\nu\}.$$

It has a genuine character

$$\det{}^{\nu/2} \colon \widetilde{\mathrm{U}}(W) \to \mathbb{C}^\times, \ (g,z) \mapsto z.$$

Hence the set of genuine tempered representations of $\widetilde{\mathrm{U}}(W)$ is given by

$$\mathrm{Irr}_{\mathrm{temp}}(\widetilde{\mathrm{U}}(W)) = \{\pi \otimes \det{}^{-\nu/2} \mid \pi \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(W))\}.$$

As in [P1, §1.2], we have a homomorphism

$$\widetilde{\alpha} = \widetilde{\alpha}_V \colon \widetilde{\mathrm{U}}(W) \to \widetilde{\mathrm{Sp}}(\mathbb{W})$$

such that $\widetilde{\alpha}(\widetilde{\mathrm{U}}(W))$ is the inverse image of $\alpha(\mathrm{U}(W))$, and the diagram

$$\begin{array}{ccc} \widetilde{\mathrm{U}}(W) & \xrightarrow{\widetilde{\alpha}} & \widetilde{\mathrm{Sp}}(\mathbb{W}) \\ \downarrow & & \downarrow \\ \mathrm{U}(W) & \xrightarrow{\alpha} & \mathrm{Sp}(\mathbb{W}) \end{array}$$

is commutative, where the left arrow is the first projection. We write the composition of $\widetilde{\alpha}\colon \widetilde{\mathrm{U}}(W) \rightarrow \widetilde{\mathrm{Sp}}(\mathbb{W})$ with the inclusion map $\widetilde{\mathrm{Sp}}(\mathbb{W}) \hookrightarrow \mathrm{Mp}(\mathbb{W}) = \mathrm{Sp}(\mathbb{W}) \times \mathbb{C}^1$ as

$$(g, z) \mapsto (\alpha(g), \beta(g, z)).$$

Note that $\beta(g, -z) = -\beta(g, z)$. We put $\mu_8 = \{\zeta \in \mathbb{C}^\times \mid \zeta^8 = 1\}$ to be the set of 8-th roots of unity in $\mathbb{C}^\times$.

Lemma 3.2. *For any $(g, z) \in \widetilde{\mathrm{U}}(W)$, the value $\beta(g, z)$ belongs to μ_8.*

Proof. We write

$$\mathrm{Mp}(\mathbb{W}) = \mathrm{Sp}(\mathbb{W}) \times \mathbb{C}^1, \quad \widetilde{\mathrm{Sp}}(\mathbb{W}) = \mathrm{Sp}(\mathbb{W}) \times \{\pm 1\}$$

as sets. The multiplication law of $\mathrm{Mp}(\mathbb{W})$ is given by $(h_1, z_1) \cdot (h_2, z_2) = (h_1h_2, z_1z_2c(h_1, h_2))$, where $c(h_1, h_2)$ is Rao's 2-cocycle (see [Ra] and [Ku2]). By [Ra, Theorem 4.1 (5)] (cf. [Ku2, Theorem]), $c(g_1, g_2)$ is a Weil index, which is an 8-th root of unity. We write the inclusion map $\widetilde{\mathrm{Sp}}(\mathbb{W}) \hookrightarrow \mathrm{Mp}(\mathbb{W})$ as

$$(h, \epsilon) \mapsto (h, \gamma(h, \epsilon)).$$

When $(h_1, \epsilon_1) \cdot (h_2, \epsilon_2) = (h_1h_2, \epsilon_3)$ in $\widetilde{\mathrm{Sp}}(\mathbb{W})$, we see that

$$\gamma(h_1h_2, \epsilon_3) = \gamma(h_1, \epsilon_1)\gamma(h_2, \epsilon_2)c(h_1, h_2).$$

Since $\gamma(h, -\epsilon) = -\gamma(h, \epsilon)$, we see that $(h, \epsilon) \mapsto \gamma(h, \epsilon)^8$ factors through a group homomorphism of $\mathrm{Sp}(\mathbb{W})$. Since $\mathrm{Sp}(\mathbb{W})$ is semisimple, there exists a dense subset X of $\widetilde{\mathrm{Sp}}(\mathbb{W})$ such that $\gamma(h, \epsilon) \in \mu_8$ for any $(h, \epsilon) \in X$. Since $\widetilde{\mathrm{Sp}}(\mathbb{W})$ is closed in $\mathrm{Mp}(\mathbb{W})$, the closure of X in $\mathrm{Mp}(\mathbb{W})$, which is contained in $\mathrm{Sp}(\mathbb{W}) \times \mu_8 \subset \mathrm{Mp}(\mathbb{W})$, coincides with $\widetilde{\mathrm{Sp}}(\mathbb{W})$. This means that $\gamma(h, \epsilon) \in \mu_8$ for any $(h, \epsilon) \in \widetilde{\mathrm{Sp}}(\mathbb{W})$. In particular, since the map $(g, z) \mapsto (\alpha(g), \beta(g, z))$ factors through $\widetilde{\mathrm{Sp}}(\mathbb{W}) \hookrightarrow \mathrm{Mp}(\mathbb{W})$, we conclude that $\beta(g, z)$ is an 8-th root of unity for any $(g, z) \in \widetilde{\mathrm{U}}(W)$. □

Now we compare $\beta_\chi(g)$ with $\beta(g, z)$.

Proposition 3.3. *Fix $\nu \in \mathbb{Z}$ such that $\nu \equiv m \bmod 2$. Let $\chi = \chi_\nu$ be the character of $\mathbb{C}^\times$ given by*

$$\chi(ae^{\sqrt{-1}\theta}) = e^{\nu\sqrt{-1}\theta}$$

for $a > 0$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Suppose that $\widetilde{\mathrm{U}}(W)$ is the $\det^{\nu/2}$-cover of $\mathrm{U}(W)$, i.e., $z^2 = \det(g)^\nu$ when $(g, z) \in \widetilde{\mathrm{U}}(W)$. Then

$$\beta_\chi(g) = \beta(g, z)z$$

for any $(g, z) \in \widetilde{\mathrm{U}}(W)$.

Proof. Since the maps $\mathrm{U}(W) \rightarrow \mathrm{Mp}(\mathbb{W}),\ g \mapsto (\alpha(g), \beta_\chi(g))$ and $\widetilde{\mathrm{U}}(W) \rightarrow \mathrm{Mp}(\mathbb{W}),\ (g, z) \mapsto (\alpha(g), \beta(g, z))$ are homomorphism, we see that

$$\beta_\chi(g_1g_2)\beta_\chi(g_1)^{-1}\beta_\chi(g_2)^{-1} = c(\alpha(g_1), \alpha(g_2)) = \beta(g_1g_2, z_1z_2)\beta(g_1, z_1)^{-1}\beta(g_2, z_2)^{-1}$$

for any $(g_1, z_1), (g_2, z_2) \in \widetilde{\mathrm{U}}(W)$. This implies that the map

$$\eta\colon \widetilde{\mathrm{U}}(W) \ni (g, z) \mapsto \frac{\beta_\chi(g)}{\beta(g, z)z} \in \mathbb{C}^1$$

factors through a group homomorphism η of $\mathrm{U}(W)$. Since

$$\left(\frac{\beta_\chi(g)}{\beta(g, z)z}\right)^8 = \frac{\chi(\det(g))^4}{\det(g)^{4\nu}} = 1$$

for any $(g,z) \in \widetilde{\mathrm{U}}(W)$, we may regard this map as a group homomorphism

$$\eta\colon \mathrm{U}(W) \to \mu_8,\ g \mapsto \frac{\beta_\chi(g)}{\beta(g,z)z}.$$

Since $\mathrm{U}(W)$ is a Lie group, we can find an open neighborhood X of 1 such that for any $g \in X$, there exists $h \in \mathrm{U}(W)$ such that $g = h^8$. This means that X is contained in the kernel of η, so that η is continuous. In conclusion, η is a (continuous) character of $\mathrm{U}(W)$ of finite order. Since $\mathrm{U}(W)$ is connected, it must be the trivial character. Hence $\beta_\chi(g) = \beta(g,z)z$. □

3.3. **Basic properties of theta liftings.** Through this paper, for each hermitian space $W = W_{p,q}$ and each skew-hermitian space $V = V_{r,s}$ as in §3.1, we fix characters $\chi_W = \chi_{W_{p,q}}$ and $\chi_V = \chi_{V_{r,s}}$ such that

- $\chi_W|\mathbb{R}^\times = \mathrm{sgn}^n$ and $\chi_V|\mathbb{R}^\times = \mathrm{sgn}^m$ with $n = p+q = \dim(W)$ and $m = r+s = \dim(V)$, respectively;
- χ_W and χ_V depend only on n mod 2 and m mod 2, respectively,

and a non-trivial additive character ψ of $\mathbb{R}$.

Let $W = W_{p,q}$ and $V = V_{r,s}$, and set $\mathbb{W} = V \otimes_{\mathbb{C}} W$ as in §3.1. Then the isometry groups $\mathrm{U}(W)$ and $\mathrm{U}(V)$ are isomorphic to $\mathrm{U}(p,q)$ and $\mathrm{U}(r,s)$, respectively. We have a canonical map

$$\alpha_V \times \alpha_W\colon \mathrm{U}(W) \times \mathrm{U}(V) \to \mathrm{Sp}(\mathbb{W}).$$

As in §3.1, we have a splitting

$$\widetilde{\alpha}_{\chi_V} \times \widetilde{\alpha}_{\chi_W}\colon \mathrm{U}(W) \times \mathrm{U}(V) \to \mathrm{Mp}(\mathbb{W})$$

of $\alpha_V \times \alpha_W$. On the other hand, as in §3.2, there are two-fold covers $\widetilde{\mathrm{U}}(W)$ and $\widetilde{\mathrm{U}}(V)$ of $\mathrm{U}(W)$ and $\mathrm{U}(V)$, respectively, and a lifting

$$\widetilde{\alpha}_V \times \widetilde{\alpha}_W\colon \widetilde{\mathrm{U}}(W) \times \widetilde{\mathrm{U}}(V) \to \widetilde{\mathrm{Sp}}(\mathbb{W})$$

of $\alpha_V \times \alpha_W$.

For $a \in \mathbb{R}^\times$, we define an additive character $a\psi$ of $\mathbb{R}$ by

$$(a\psi)(x) = \psi(ax)$$

for $x \in \mathbb{R}$. Let $\omega_{a\psi}$ be the Weil representation of $\mathrm{Mp}(\mathbb{W})$ associated to $a\psi$. It is a smooth representation satisfying that $\omega_{a\psi}(z) = z \cdot \mathrm{id}$ for $z \in \mathbb{C}^1 \subset \mathrm{Mp}(\mathbb{W})$. Moreover, $\omega_{a\psi} \cong \omega_{a'\psi}$ if and only if $aa' > 0$. Hence there are exactly two Weil representations of $\mathrm{Mp}(\mathbb{W})$. By the restriction, $\omega_{a\psi}$ is regarded as a representation of $\widetilde{\mathrm{Sp}}(\mathbb{W})$ also. We choose ψ such that ω_ψ is the Weil representation of $\widetilde{\mathrm{Sp}}(\mathbb{W})$ which Paul has used in [P1, P2, P3] (c.f., [P1, Lemma 1.4.5]). We consider two representations $\omega_\psi \circ \widetilde{\alpha}_{\chi_V}$ of $\mathrm{U}(W)$ and $\omega_\psi \circ \widetilde{\alpha}_V$ of $\widetilde{\mathrm{U}}(W)$. For $\pi \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(W))$ and $\widetilde{\pi} \in \mathrm{Irr}_{\mathrm{temp}}(\widetilde{\mathrm{U}}(W))$, the maximal π-isotypic quotient of $\omega_\psi \circ \widetilde{\alpha}_{\chi_V}$ and the maximal $\widetilde{\pi}$-isotypic quotient of $\omega_\psi \circ \widetilde{\alpha}_V$ are of the form

$$\pi \boxtimes \Theta_{r,s}(\pi) \quad \text{and} \quad \widetilde{\pi} \boxtimes \Theta_{r,s}(\widetilde{\pi}),$$

where $\Theta_{r,s}(\pi)$ and $\Theta_{r,s}(\widetilde{\pi})$ are (genuine or possibly zero) representations of $\mathrm{U}(V)$ and $\widetilde{\mathrm{U}}(V)$, respectively. We call $\Theta_{r,s}(\pi)$ (resp. $\Theta_{r,s}(\widetilde{\pi})$) the big theta lift of π (resp. $\widetilde{\pi}$).

Theorem 3.4 (Howe duality correspondence [Ho, Theorem 2.1])**.** *If $\Theta_{r,s}(\pi)$ (resp. $\Theta_{r,s}(\widetilde{\pi})$) is nonzero, then it has a unique irreducible quotient $\theta_{r,s}(\pi)$ (resp. $\theta_{r,s}(\widetilde{\pi})$).*

We interpret $\theta_{r,s}(\pi)$ (resp. $\theta_{r,s}(\widetilde{\pi})$) to be zero if so is $\Theta_{r,s}(\pi)$ (resp. $\Theta_{r,s}(\widetilde{\pi})$). We call $\theta_{r,s}(\pi)$ (resp. $\theta_{r,s}(\widetilde{\pi})$) the small theta lift of π (resp. $\widetilde{\pi}$). Similarly, for $\sigma \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(V))$ (resp. $\widetilde{\sigma} \in \mathrm{Irr}_{\mathrm{temp}}(\widetilde{\mathrm{U}}(V))$), we can define the big and small theta lifts $\Theta_{p,q}(\sigma)$ and $\theta_{p,q}(\sigma)$ (resp. $\Theta_{p,s}(\widetilde{\sigma})$ and $\theta_{p,q}(\widetilde{\sigma})$). Note that there is a Harish-Chandra module version of theta correspondence. These two versions agree by [BS].

In this paper, we determine explicitly when $\Theta_{r,s}(\pi)$ (resp. $\Theta_{r,s}(\widetilde{\pi})$) is nonzero for $\pi \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(p,q))$ (resp. $\widetilde{\pi} \in \mathrm{Irr}_{\mathrm{temp}}(\widetilde{\mathrm{U}}(p,q))$). A relation between the non-vanishing of $\Theta_{r,s}(\pi)$ and the one of $\Theta_{r,s}(\widetilde{\pi})$ is given as follows:

Proposition 3.5. *Suppose that* $\chi_V = \chi_\nu$ *with* $\nu \in \mathbb{Z}$ *such that* $\nu \equiv m \bmod 2$. *Then for any* $\pi \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(W))$, *we have*

$$\mathrm{Hom}_{\mathrm{U}(W)}(\omega_\psi \circ \widetilde{\alpha}_{\chi_V}, \pi) \neq 0 \iff \mathrm{Hom}_{\widetilde{\mathrm{U}}(W)}(\omega_\psi \circ \widetilde{\alpha}_V, \pi \otimes \det^{-\nu/2}) \neq 0.$$

In particular, $\Theta_{r,s}(\pi) \neq 0$ *if and only if* $\Theta_{r,s}(\pi \otimes \det^{-\nu/2}) \neq 0$.

Proof. For $(g, z) \in \widetilde{\mathrm{U}}(W)$, by Proposition 3.3, we have

$$\begin{aligned}(\omega_\psi \circ \widetilde{\alpha}_V) \otimes \det^{\nu/2}(g,z) &= \omega_\psi(\alpha_V(g), \beta(g,z))z \\ &= \omega_\psi(\alpha_V(g), \beta(g,z)z) \\ &= \omega_\psi(\alpha_V(g), \beta_{\chi_V}(g)) \\ &= \omega_\psi \circ \widetilde{\alpha}_{\chi_V}(g).\end{aligned}$$

Hence $(\omega_\psi \circ \widetilde{\alpha}_V) \otimes \det^{\nu/2} = \omega_\psi \circ \widetilde{\alpha}_{\chi_V}$ as representations of $\mathrm{U}(W)$, so that

$$\mathrm{Hom}_{\mathrm{U}(W)}((\omega_\psi \circ \widetilde{\alpha}) \otimes \det^{\nu/2}, \pi) \neq 0 \iff \mathrm{Hom}_{\mathrm{U}(W)}(\omega_\psi \circ \widetilde{\alpha}_{\chi_V}, \pi) \neq 0.$$

This completes the proof. □

If $\chi_V = \chi_\nu$ with $\nu \in \mathbb{Z}$ such that $\nu \equiv m \bmod 2$, the genuine character $\det^{\nu/2}$ of $\widetilde{\mathrm{U}}(W)$ is also denoted by χ_V. Hence χ_V^2 is the character $\det^\nu$ of $\mathrm{U}(W)$. By Proposition 3.5, $\Theta_{r,s}(\pi) \neq 0$ if and only if $\Theta_{r,s}(\pi \otimes \chi_V^{-1}) \neq 0$.

Now we recall basic properties on the theta correspondence. First, we state a proposition which is called the tower property or Kudla's persistence principle.

Proposition 3.6 (Tower property [Ku1])**.** *If* $\Theta_{r,s}(\pi)$ *is nonzero, then* $\Theta_{r+l,s+l}(\pi)$ *is also nonzero for any* $l \geq 0$.

Next, we state the conservation relation. Fix $\pi \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(p,q))$. For each integer d, we consider a set of theta lifts $\{\Theta_{r,s}(\pi) \mid r - s = d\}$. We call this set the d-th Witt tower of theta lifts of π. Also we call

$$m_d(\pi) = \min\{r + s \mid \Theta_{r,s}(\pi) \neq 0,\ r - s = d\}$$

the first occurrence index of the d-th Witt tower of theta lifts of π.

Theorem 3.7 (Conservation relation [SZ2])**.** *Fix* $\delta \in \{0, 1\}$ *and set*

$$m_\pm(\pi) = \min\{m_d(\pi) \mid d \equiv \delta \bmod 2,\ (-1)^{\frac{d-\delta}{2}} = \pm 1\}.$$

Then

$$m_+(\pi) + m_-(\pi) = 2n + 2$$

for any $\pi \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(p,q))$ *with* $n = p + q$.

The following is a consequence of the induction principle ([P1, Theorem 4.5.5]).

Proposition 3.8 (Induction principle)**.** *Let* $\pi_0 \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(p_0,q_0))$. *Suppose that* $\Theta_{r_0,s_0}(\pi_0)$ *is nonzero for some* (r_0, s_0). *Let* $\xi_1, \dots, \xi_v$ *be unitary characters of* $\mathbb{C}^\times$. *Put* $p = p_0 + v$, $q = q_0 + v$, $r = r_0 + v$ *and* $s = s_0 + v$. *Then there exists an irreducible subquotient* π *of the induced representation*

$$\mathrm{Ind}_P^{\mathrm{U}(p,q)}(\xi_1 \otimes \cdots \otimes \xi_v \otimes \pi_0)$$

such that $\Theta_{r,s}(\pi)$ *is nonzero, where* P *is a parabolic subgroup of* $\mathrm{U}(p,q)$ *with Levi subgroup of the form* $(\mathbb{C}^\times)^v \times \mathrm{U}(p_0, q_0)$.

For a relation between theta lifts and contragredient representations, the following is known.

Proposition 3.9. *Let* $\pi \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(p,q))$. *If* $\Theta_{r,s}(\pi) \neq 0$, *then* $\Theta_{s,r}(\pi^\vee \otimes \chi_V^2) \neq 0$.

Proof. By Proposition 3.5, $\Theta_{r,s}(\pi) \neq 0$ if and only if $\Theta_{r,s}(\pi \otimes \chi_V^{-1}) \neq 0$. Similarly, $\Theta_{s,r}(\pi^\vee \otimes \chi_V^2) \neq 0$ if and only if $\Theta_{s,r}(\pi^\vee \otimes \chi_V) \neq 0$. Hence the proposition follows from [P1, Proposition 2.1]. □

There is a non-vanishing result of theta lifts of one dimensional representations.

Proposition 3.10. *Fix a positive integer t and a half integer l such that $2l \equiv t \bmod 2$ and $-t/2 < l < t/2$. Let $\det^l$ be a genuine character of the $\det^l$-cover $\widetilde{\mathrm{U}}(p,q)$. If $\Theta_{r,s}(\det^l \otimes \chi_{V_{r,s}}) \neq 0$ and $|r-s| = t$, then $\min\{r,s\} \geq p+q$.*

Proof. Note that $\det^l \otimes \chi_{V_{r,s}}$ is a character of $\mathrm{U}(p,q)$. By Proposition 3.5, $\Theta_{r,s}(\det^l \otimes \chi_{V_{r,s}}) \neq 0$ if and only if $\Theta_{r,s}(\det^l) \neq 0$. Hence the proposition is a restatement of [P2, Lemma 3.1]. $\square$

We denote by $\omega_{p,q,r,s}$ the Weil representation of $\widetilde{\mathrm{U}}(p,q) \times \widetilde{\mathrm{U}}(r,s)$, i.e., $\omega_{p,q,r,s} = \omega_\psi \circ (\widetilde{\alpha}_V \times \widetilde{\alpha}_W)$. The following are called seesaw identities, which are key properties to prove the main result (Theorem 4.2 below).

Proposition 3.11 (Seesaw identity)**.** (1) *For $\widetilde{\pi} \in \mathrm{Irr}_{\mathrm{temp}}(\widetilde{\mathrm{U}}(p,q))$ and $\widetilde{\sigma} \in \mathrm{Irr}_{\mathrm{temp}}(\widetilde{\mathrm{U}}(r,s))$, we have*

$$\mathrm{Hom}_{\widetilde{\mathrm{U}}(p,q)}(\Theta_{p+p',q+q'}(\widetilde{\sigma}), \widetilde{\pi}) \cong \mathrm{Hom}_{\widetilde{\mathrm{U}}(r,s)}(\Theta_{r,s}(\widetilde{\pi}) \otimes \omega_{p',q',r,s}, \widetilde{\sigma}).$$

In particular, if there is $\pi' \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(p+p',q+q'))$ such that $\mathrm{Hom}_{\mathrm{U}(p,q)}(\pi',\pi) \neq 0$ and $\Theta_{r,s}(\pi') \neq 0$, then $\Theta_{r,s}(\pi) \neq 0$.

(2) *For $\pi \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(p,q))$, $\widetilde{\sigma}_1 \in \mathrm{Irr}_{\mathrm{temp}}(\widetilde{\mathrm{U}}(r_1,s_1))$ and $\widetilde{\sigma}_2 \in \mathrm{Irr}_{\mathrm{temp}}(\widetilde{\mathrm{U}}(s_2,r_2))$ with $r_1+s_1 \equiv r_2+s_2 \bmod 2$, we have*

$$\mathrm{Hom}_{\mathrm{U}(p,q)}(\Theta_{p,q}(\widetilde{\sigma}_1) \otimes \Theta_{p,q}(\widetilde{\sigma}_2), \pi) \cong \mathrm{Hom}_{\widetilde{\mathrm{U}}(r_1,s_1) \times \widetilde{\mathrm{U}}(s_2,r_2)}(\Theta_{r_1+s_2,s_1+r_2}(\widetilde{\pi}), \widetilde{\sigma}_1 \otimes \widetilde{\sigma}_2),$$

where $\widetilde{\pi}$ is the genuine representation of the trivial cover $\widetilde{\mathrm{U}}(p,q) = \mathrm{U}(p,q) \times \{\pm 1\}$ defined by $\widetilde{\pi}|\mathrm{U}(p,q) = \pi$. In particular, for a unitary character χ of $\mathrm{U}(p,q)$, if there is $\pi \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(p,q))$ such that $\Theta_{r_1,s_1}(\pi) \neq 0$ and $\Theta_{r_2,s_2}(\pi \otimes \chi^{-1}) \neq 0$, then $\Theta_{r_1+s_2,s_1+r_2}(\widetilde{\chi} \cdot \chi_{V_{0,0}}) \neq 0$.

Proof. The first assertions of (1) and (2) immediately follow from [P1, Lemma 2.8].

We show the last assertion of (1). Set $\widetilde{\pi} = \pi \otimes \chi_V^{-1}$ and $\widetilde{\pi}' = \pi' \otimes \chi_V^{-1}$. If $\mathrm{Hom}_{\mathrm{U}(p,q)}(\pi',\pi) \neq 0$ and $\Theta_{r,s}(\pi') \neq 0$ then $\mathrm{Hom}_{\widetilde{\mathrm{U}}(p,q)}(\widetilde{\pi}', \widetilde{\pi}) \neq 0$ and $\Theta_{r,s}(\widetilde{\pi}') \neq 0$. If we put $\widetilde{\sigma} = \theta_{r,s}(\widetilde{\pi}')$, then $\widetilde{\pi}'$ is a quotient of $\Theta_{p+p',q+q'}(\widetilde{\sigma})$. Hence we have

$$0 \neq \mathrm{Hom}_{\widetilde{\mathrm{U}}(p,q)}(\widetilde{\pi}', \widetilde{\pi}) \subset \mathrm{Hom}_{\widetilde{\mathrm{U}}(p,q)}(\Theta_{p+p',q+q'}(\widetilde{\sigma}), \widetilde{\pi}) \cong \mathrm{Hom}_{\widetilde{\mathrm{U}}(r,s)}(\Theta_{r,s}(\widetilde{\pi}) \otimes \omega_{p',q',r,s}, \widetilde{\sigma}).$$

In particular, we have $\Theta_{r,s}(\widetilde{\pi}) \neq 0$ so that $\Theta_{r,s}(\pi) \neq 0$.

We show the last assertion of (2). We denote $\chi_V = \chi_{V_{r_1,s_1}} = \chi_{V_{r_2,s_2}}$ and $\chi_{V_{0,0}} = \chi_{V_{r_1+s_2,s_1+r_2}}$. By Proposition 3.9, we have $\Theta_{s_2,r_2}(\pi^\vee \otimes \chi\chi_V^2) \neq 0$. Set $\widetilde{\pi}_1 = \pi \otimes \chi_V^{-1}$ and $\widetilde{\pi}_2 = \pi^\vee \otimes \chi\chi_V$. Then $\widetilde{\sigma}_1 = \theta_{r_1,s_1}(\widetilde{\pi}_1) \neq 0$, $\widetilde{\sigma}_2 = \theta_{s_2,r_2}(\widetilde{\pi}_2) \neq 0$, and there exist surjections

$$\Theta_{p,q}(\widetilde{\sigma}_1) \otimes \Theta_{p,q}(\widetilde{\sigma}_2) \twoheadrightarrow \widetilde{\pi}_1 \otimes \widetilde{\pi}_2 \twoheadrightarrow \chi$$

as representations of $\mathrm{U}(p,q)$. Since

$$0 \neq \mathrm{Hom}_{\mathrm{U}(p,q)}(\Theta_{p,q}(\widetilde{\sigma}_1) \otimes \Theta_{p,q}(\widetilde{\sigma}_2), \chi) \cong \mathrm{Hom}_{\widetilde{\mathrm{U}}(r_1,s_1) \times \widetilde{\mathrm{U}}(s_2,r_2)}(\Theta_{r_1+s_2,s_1+r_2}(\widetilde{\chi}), \widetilde{\sigma}_1 \otimes \widetilde{\sigma}_2),$$

we have $\Theta_{r_1+s_2,s_1+r_2}(\widetilde{\chi}) \neq 0$. Hence $\Theta_{r_1+s_2,s_1+r_2}(\widetilde{\chi} \cdot \chi_{V_{0,0}}) \neq 0$. $\square$

We write Proposition 3.11 (1) as

$$\begin{tikzcd}
\mathrm{U}(p+p',q+q') \arrow[d, dash] \arrow[dr, dash] & \mathrm{U}(r,s) \times \mathrm{U}(r,s) \arrow[d, dash] \arrow[dl, dash] \\
\mathrm{U}(p,q) \times \mathrm{U}(p',q') & \mathrm{U}(r,s)
\end{tikzcd}$$

and (2) as

$$\begin{tikzcd}
\mathrm{U}(p,q) \times \mathrm{U}(p,q) \arrow[d, dash] \arrow[dr, dash] & \mathrm{U}(r_1+s_2,s_1+r_2) \arrow[d, dash] \arrow[dl, dash] \\
\mathrm{U}(p,q) & \mathrm{U}(r_1,s_1) \times \mathrm{U}(s_2,r_2),
\end{tikzcd}$$

respectively. Note that in Proposition 3.11 (2), if $\chi = \det^a$ for some integer a, then $\widetilde{\chi}$ is the genuine character $\det^a$ of the $\det^a$-cover $\widetilde{\mathrm{U}}(p,q)$.

3.4. **Equal rank case and almost equal rank case.** To prove our main result, we use non-trivial results established by Paul in [P1] and [P3]. These are results on theta lifts in the equal rank case and the almost equal rank case. In these results, Paul considered the theta lifts from irreducible genuine representations of double covers of unitary groups. In this subsection, we recall Paul's results, and translate them into results of the theta lifts from irreducible representations of $\mathrm{U}(p,q)$.

Recall that

$$\widetilde{\mathrm{U}}(p,q) \cong \{(g,z) \in \mathrm{U}(p,q) \times \mathbb{C}^1 \mid z^2 = \det(g)^\nu\},$$

where $\nu \in \mathbb{Z}$ satisfies that $\nu \equiv m \bmod 2$ (so that $\widetilde{\mathrm{U}}(p,q)$ depends not only on (p,q) but also on $m \bmod 2$). It has a genuine character $\det^{\nu/2} \colon (g,z) \mapsto z$. Hence

$$\mathrm{Irr}_{\mathrm{disc}}(\widetilde{\mathrm{U}}(p,q)) = \{\pi \otimes \det^{-\nu/2} \mid \pi \in \mathrm{Irr}_{\mathrm{disc}}(\mathrm{U}(p,q))\}.$$

Since $\pi \in \mathrm{Irr}_{\mathrm{disc}}(\mathrm{U}(p,q))$ is characterized by its Harish-Chandra parameter $\mathrm{HC}(\pi)$, which is an element of $(\mathbb{Z} + \frac{n-1}{2})^p \times (\mathbb{Z} + \frac{n-1}{2})^q$ with $n = p+q$, the representation $\widetilde{\pi} = \pi \otimes \det^{-\nu/2}$ is characterized by its Harish-Chandra parameter

$$\mathrm{HC}(\widetilde{\pi}) = \mathrm{HC}(\pi) - \left(\frac{\nu}{2}, \dots, \frac{\nu}{2}; \frac{\nu}{2}, \dots, \frac{\nu}{2}\right) \in \left(\mathbb{Z} + \frac{n+m-1}{2}\right)^p \times \left(\mathbb{Z} + \frac{n+m-1}{2}\right)^q.$$

First, we recall the result in the equal rank case ([P1]).

Theorem 3.12 ([P1, Theorems 0.1, 6.1 (a)])**.** *Let $\widetilde{\mathrm{U}}(p,q)$ be the $\det^{(p+q)/2}$-cover of $\mathrm{U}(p,q)$, and $\widetilde{\pi}$ be an irreducible tempered genuine representation of $\widetilde{\mathrm{U}}(p,q)$. Then there exists a unique pair (r,s) such that $r+s = p+q$ and $\Theta_{r,s}(\widetilde{\pi}) \neq 0$. Moreover, if $\widetilde{\pi}$ is a direct summand of induced representation*

$$\mathrm{Ind}_{\widetilde{P}}^{\widetilde{\mathrm{U}}(p,q)}(\xi_1 \otimes \cdots \otimes \xi_v \otimes \widetilde{\pi}_0),$$

where

- *$\widetilde{P}$ is a parabolic subgroup of $\widetilde{\mathrm{U}}(p,q)$ with Levi subgroup of the form $(\mathbb{C}^\times)^v \times \widetilde{\mathrm{U}}(p_0,q_0)$ with $p = p_0 + v$ and $q = q_0 + v$;*
- *$\widetilde{\pi}_0$ is an irreducible genuine discrete series such that*

$$\mathrm{HC}(\widetilde{\pi}_0) = (a_1, \dots, a_x, b_1, \dots b_y; c_1, \dots, c_z, d_1, \dots, d_w) \in \left(\mathbb{Z} + \frac{1}{2}\right)^{p_0} \times \left(\mathbb{Z} + \frac{1}{2}\right)^{q_0}$$

  with $a_1 > \cdots > a_x > 0 > b_1 > \cdots > b_y$ and $c_1 > \cdots > c_z > 0 > d_1 > \cdots > d_w$;
- *$\xi_1, \dots, \xi_v$ are unitary characters of $\mathbb{C}^\times$,*

then $r = x + w + v$ and $s = y + z + v$.

We translate this theorem in terms of L-parameters. Fix $m \bmod 2$. Let $\lambda = (\phi, \eta)$ be a pair of $\phi \in \Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$ and $\eta \in \widehat{A_\phi}$. Write

$$\phi\chi_V^{-1} = \chi_{2\alpha_1} + \cdots + \chi_{2\alpha_u} + (\xi_1 + \cdots + \xi_v) + ({}^c\xi_1^{-1} + \cdots + {}^c\xi_v^{-1}),$$

where

- $\alpha_i \in \frac{1}{2}\mathbb{Z}$ such that $2\alpha_i \equiv n + m - 1 \bmod 2$ and $\alpha_1 > \cdots > \alpha_u$;
- ξ_i is a unitary character of $\mathbb{C}^\times$ (which can be of the form $\chi_{2\alpha}$);
- $u + 2v = n$.

Then

$$A_\phi \supset (\mathbb{Z}/2\mathbb{Z})e_{V,2\alpha_1} \oplus \cdots \oplus (\mathbb{Z}/2\mathbb{Z})e_{V,2\alpha_u},$$

where $\{e_{V,2\alpha_1}, \dots, e_{V,2\alpha_u}\}$ is the canonical basis associated to $\{\chi_V\chi_{2\alpha_1}, \dots, \chi_V\chi_{2\alpha_u}\}$.

The following theorem is a translation of Theorem 3.12.

Theorem 3.13. *Suppose that $m = n = p + q$. Let $\lambda = (\phi, \eta)$ be as above, and set $\pi = \pi(\phi, \eta)$. Then there exists a unique pair (r,s) such that $r + s = p + q$ and $\Theta_{r,s}(\pi) \neq 0$. Moreover (r,s) is given by*

$$\begin{cases} r = \#\{i \in \{1, \dots, u\} \mid (-1)^{i-1}\eta(e_{V,2\alpha_i})\alpha_i > 0\} + v, \\ s = \#\{i \in \{1, \dots, u\} \mid (-1)^{i-1}\eta(e_{V,2\alpha_i})\alpha_i < 0\} + v. \end{cases}$$

Similarly, we can translate the result in the almost equal rank case ([P3]).

Theorem 3.14. *Suppose that $m \equiv n+1 \bmod 2$. Let $\lambda = (\phi, \eta)$ be as above, and set $\pi = \pi(\phi, \eta)$.*

(1) *Suppose that ϕ does not contain χ_V. Then there exist exactly two pairs (r,s) such that $r+s = p+q+1$ and $\Theta_{r,s}(\pi) \neq 0$. Moreover these two pairs of (r,s) are given by*

$$\begin{cases} r = \#\{i \in \{1,\dots,u\} \mid (-1)^{i-1}\eta(e_{V,2\alpha_i})\alpha_i > 0\} + v + 1, \\ s = \#\{i \in \{1,\dots,u\} \mid (-1)^{i-1}\eta(e_{V,2\alpha_i})\alpha_i < 0\} + v \end{cases}$$

and

$$\begin{cases} r = \#\{i \in \{1,\dots,u\} \mid (-1)^{i-1}\eta(e_{V,2\alpha_i})\alpha_i > 0\} + v, \\ s = \#\{i \in \{1,\dots,u\} \mid (-1)^{i-1}\eta(e_{V,2\alpha_i})\alpha_i < 0\} + v + 1. \end{cases}$$

(2) *Suppose that ϕ contains χ_V with odd multiplicity. Then there exists a unique pair (r,s) such that $r+s = p+q-1$ and $\Theta_{r,s}(\pi) \neq 0$. Moreover (r,s) is given by*

$$\begin{cases} r = \#\{i \in \{1,\dots,u\} \mid (-1)^{i-1}\eta(e_{V,2\alpha_i})\alpha_i > 0\} + v, \\ s = \#\{i \in \{1,\dots,u\} \mid (-1)^{i-1}\eta(e_{V,2\alpha_i})\alpha_i < 0\} + v. \end{cases}$$

(3) *Suppose that ϕ contains χ_V with even multiplicity. Then there exists a unique pair (r,s) such that $r+s = p+q-1$ and $\Theta_{r,s}(\pi) \neq 0$. Moreover (r,s) is given by*

$$\begin{cases} r = \#\{i \in \{1,\dots,u\} \mid (-1)^{i-1}\eta(e_{V,2\alpha_i})\alpha_i > 0\} + v - 1, \\ s = \#\{i \in \{1,\dots,u\} \mid (-1)^{i-1}\eta(e_{V,2\alpha_i})\alpha_i < 0\} + v \end{cases}$$

or

$$\begin{cases} r = \#\{i \in \{1,\dots,u\} \mid (-1)^{i-1}\eta(e_{V,2\alpha_i})\alpha_i > 0\} + v, \\ s = \#\{i \in \{1,\dots,u\} \mid (-1)^{i-1}\eta(e_{V,2\alpha_i})\alpha_i < 0\} + v - 1. \end{cases}$$

In fact, Paul ([P3, Theorem 3.4]) has determined (r,s) in Theorem 3.14 (3) exactly in terms of a system of positive roots.

4. The definition and the main result

Through this and next sections, we fix $\kappa \in \{1,2\}$ and choose a character χ_V of $\mathbb{C}^\times$ such that $\chi_V|\mathbb{R}^\times = \mathrm{sgn}^{\kappa+n}$. We will only consider theta lifting $\Theta_{r,s}(\pi)$ of $\pi \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(p,q))$ with $p+q \equiv n \bmod 2$ and $r+s \equiv n+\kappa \bmod 2$. In this section, we state the main result and its corollary.

4.1. Definition.

Before stating the main result, we define some notations.

Definition 4.1. *Let $\lambda = (\phi, \eta)$ be a pair of $\phi \in \Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$ and $\eta \in \widehat{A_\phi}$.*

(1) *Consider the set $\mathcal{T}$ containing $\kappa - 2$ and all integers $k > 0$ with $k \equiv \kappa \bmod 2$ satisfying the following conditions:*
  - **(chain condition):** *ϕ contains $\chi_V\chi_{k-1} + \chi_V\chi_{k-3} + \cdots + \chi_V\chi_{-k+1}$;*
  - **(odd-ness condition):** *the multiplicity of $\chi_V\chi_{k+1-2i}$ in ϕ is odd for $i = 1,\dots,k$;*
  - **(alternating condition):** *$\eta(e_{V,k+1-2i}) = -\eta(e_{V,k-1-2i})$ for $i = 1,\dots,k-1$.*

  Here, $e_{V,2\alpha}$ is the element in A_ϕ corresponding to $\chi_V\chi_{2\alpha}$. Set

$$k_\lambda = \max \mathcal{T}.$$

(2) *Write*

$$\phi\chi_V^{-1} = \chi_{2\alpha_1} + \cdots + \chi_{2\alpha_u} + (\xi_1 + \cdots + \xi_v) + ({}^c\xi_1^{-1} + \cdots + {}^c\xi_v^{-1}),$$

  where
  - *$\alpha_i \in \frac{1}{2}\mathbb{Z}$ such that $2\alpha_i \equiv \kappa - 1 \bmod 2$ and $\alpha_1 > \cdots > \alpha_u$;*
  - *ξ_j is a unitary character of $\mathbb{C}^\times$ (which can be of the form $\chi_{2\alpha}$);*
  - *$u + 2v = n$.*

Then $A_\phi \supset (\mathbb{Z}/2\mathbb{Z})e_{V,2\alpha_1} + \cdots + (\mathbb{Z}/2\mathbb{Z})e_{V,2\alpha_u}$. *Define* (r_λ, s_λ) *by*

$$\begin{cases} r_\lambda = \#\{i \in \{1,\dots,u\} \mid |\alpha_i| \geq (k_\lambda+1)/2,\ (-1)^{i-1}\eta(e_{V,\alpha_i})\alpha_i > 0\} + v, \\ s_\lambda = \#\{i \in \{1,\dots,u\} \mid |\alpha_i| \geq (k_\lambda+1)/2,\ (-1)^{i-1}\eta(e_{V,\alpha_i})\alpha_i < 0\} + v. \end{cases}$$

(3) *Write*

$$\phi\chi_V^{-1} = m_1\chi_{2\alpha_1} + \cdots + m_u\chi_{2\alpha_u} + m'_1\chi_{2\alpha'_1} + \cdots + m'_{u'}\chi_{2\alpha'_{u'}} + (\xi_1 + \cdots + \xi_v) + ({}^c\xi_1^{-1} + \cdots + {}^c\xi_v^{-1}),$$

where

- $\alpha_i, \alpha'_{i'} \in \frac{1}{2}\mathbb{Z}$ *such that* $2\alpha_i \equiv 2\alpha'_{i'} \equiv \kappa - 1 \bmod 2$, $\alpha_1 > \cdots > \alpha_u$, $\alpha'_1 > \cdots > \alpha'_{u'}$ *and* $\{\alpha_1,\dots,\alpha_u\} \cap \{\alpha'_1,\dots,\alpha'_{u'}\} = \emptyset$;
- $m_i \geq 1$ *(resp.* $m'_{i'} \geq 1$*) is the multiplicity of* $\chi_{2\alpha_1}$ *(resp.* $\chi_{2\alpha'_{i'}}$*) in* $\phi\chi_V^{-1}$ *such that* m_i *is odd (resp.* $m'_{i'}$ *is even);*
- ξ_j *is a unitary character of* $\mathbb{C}^\times$ *which is not of the form* $\chi_{2\alpha}$ *with* $2\alpha \equiv \kappa - 1 \bmod 2$*;*
- $(m_1 + \cdots + m_u) + (m'_1 + \cdots + m'_{u'}) + 2v = n$.

Define a subset X_λ *of* $\frac{1}{2}\mathbb{Z} \times \{\pm1\}$ *by*

$$\begin{aligned} X_\lambda = &\{(\alpha_i, (-1)^{i-1}\eta(e_{V,2\alpha_i})) \mid i = 1,\dots,u\} \\ &\cup \{(\alpha'_{i'}, +1), (\alpha'_{i'}, -1) \mid i' = 1,\dots,u',\ \eta(e_{V,\alpha'_{i'}}) \neq (-1)^{\#\{i \in \{1,\dots,u\} \mid \alpha_i > \alpha'_{i'}\}}\}. \end{aligned}$$

(4) *We define a sequence* $X_\lambda = X_\lambda^{(0)} \supset X_\lambda^{(1)} \supset \cdots \supset X_\lambda^{(n)} \supset \cdots$ *as follows: Let* $\{\beta_1,\dots,\beta_{u_j}\}$ *be the image of* $X_\lambda^{(j)}$ *under the projection* $\frac{1}{2}\mathbb{Z} \times \{\pm1\} \to \frac{1}{2}\mathbb{Z}$ *such that* $\beta_1 > \cdots > \beta_{u_j}$. *Set* S_j *to be the set of* $i \in \{2,\dots,u_j\}$ *such that*

- $(\beta_{i-1}, +1), (\beta_i, -1) \in X_\lambda^{(j)}$*;*
- $\min\{|\beta_{i-1}|, |\beta_i|\} \geq (k_\lambda+1)/2$*;*
- $\beta_{i-1}\beta_i \geq 0$.

Then we define a subset $X_\lambda^{(j+1)}$ *of* $X_\lambda^{(j)}$ *by*

$$X_\lambda^{(j+1)} = X_\lambda^{(j)} \setminus \left(\bigcup_{i \in S_j} \{(\beta_{i-1}, +1), (\beta_i, -1)\} \right).$$

Finally, we set $X_\lambda^{(\infty)} = X_\lambda^{(n)} = X_\lambda^{(n+1)}$.

(5) *For an integer* T *and* $\epsilon \in \{\pm1\}$*, we define a set* $\mathcal{C}_\lambda^\epsilon(T)$ *by*

$$\mathcal{C}_\lambda^\epsilon(T) = \left\{ (\alpha, \epsilon) \in X_\lambda^{(\infty)} \;\middle|\; 0 \leq \epsilon\alpha + \frac{k_\lambda - 1}{2} < T \right\}.$$

In particular, if $T \leq 0$*, then* $\mathcal{C}_\lambda^\epsilon(T) = \emptyset$.

4.2. **Main result.** The main result is the following:

Theorem 4.2. *Assume the local Gan–Gross–Prasad conjecture (Conjecture 2.2). Let* $\lambda = (\phi, \eta)$ *be a pair of* $\phi \in \Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$ *and* $\eta \in \widehat{A_\phi}$. *Set* $\pi = \pi(\phi, \eta)$. *Let* $r = r_\lambda$, $s = s_\lambda$ *and* $\mathcal{C}_\lambda^\epsilon(T)$ *be as in Definition 4.1.*

(1) *Suppose that* $k_\lambda = -1$. *Then for integers* l *and* $t \geq 1$*, the theta lift* $\Theta_{r+2t+l+1,s+l}(\pi)$ *is nonzero if and only if*

$$l \geq 0 \quad \text{and} \quad \#\mathcal{C}_\lambda^\epsilon(t+l) \leq l \quad \text{for each } \epsilon \in \{\pm1\}.$$

Moreover, for an integer l*, the theta lift* $\Theta_{r+l+1,s+l}(\pi)$ *is nonzero if and only if*

$$\begin{cases} l \geq 0 & \text{if } \phi \text{ does not contain } \chi_V, \\ l \geq -1 & \text{if } \phi \text{ contains } \chi_V \text{ and both } (0,1) \text{ and } (0,-1) \text{ are not in } X_\lambda, \\ l \geq 1 & \text{if } \phi \text{ contains } \chi_V \text{ and both } (0,1) \text{ and } (0,-1) \text{ are in } X_\lambda. \end{cases}$$

(2) *Suppose that $k_\lambda \geq 0$. Then for integers l and $t \geq 1$, the theta lift $\Theta_{r+2t+l,s+l}(\pi)$ is nonzero if and only if*

$$l \geq k_\lambda \quad \text{and} \quad \#\mathcal{C}_\lambda^\epsilon(t+l) \leq l \quad \text{for each } \epsilon \in \{\pm 1\}.$$

Moreover, we consider the following three conditions:

(chain condition 2): *$\phi\chi_V^{-1}$ contains both χ_{k+1} and $\chi_{-(k+1)}$, so that*

$$\phi\chi_V^{-1} \supset \underbrace{\chi_{k+1} + \chi_{k-1} + \cdots + \chi_{-(k-1)} + \chi_{-(k+1)}}_{k+2};$$

(even-ness condition): *at least one of χ_{k+1} and $\chi_{-(k+1)}$ is contained in $\phi\chi_V^{-1}$ with even multiplicity;*

(alternating condition 2): *$\eta(e_{V,k+1-2i}) \neq \eta(e_{V,k-1-2i})$ for $i = 0, \ldots, k$.*

Then for an integer l, the theta lift $\Theta_{r+l,s+l}(\pi)$ is nonzero if and only if

$$\begin{cases} l \geq -1 & \text{if } \lambda = (\phi, \eta) \text{ satisfies the three conditions,} \\ l \geq 0 & \text{otherwise.} \end{cases}$$

Remark 4.3. (1) *When $\phi \in \Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R}))$, we need the local Gan–Gross–Prasad conjecture only for discrete series representations. Since He* [He2] *has established the conjecture in this case, the statements in Theorem 4.2 for discrete series representations hold unconditionally.*

(2) *If $\phi \in \Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R}))$, then π is a discrete series representation of some $\mathrm{U}(p,q)$. By Theorem 2.1 (4), we can translate Definition 4.1 and Theorem 4.2 in terms of Harish-Chandra parameters, and we obtain Definition 1.6 and Theorem 1.7.*

(3) *When $\phi \in \Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R}))$ and $t = 0, 1$, Theorem 4.2 is a translation of results of Paul (*[P2, Proposition 3.4, Theorem 3.14]*).*

(4) *When π is a representation of a compact unitary group and $l = 0$, Theorem 4.2 is compatible with results of* [KV] *and* [Li] *(see also* [A, Proposition 6.6]*).*

(5) *When $k_\lambda = -1$, by Definition 4.1, χ_V appears in ϕ with even multiplicity. Hence $(0,1)$ and $(0,-1)$ are both in X_λ or both not in X_λ. In the former case (resp. in the latter case), we write $(0, \pm 1) \in X_\lambda$ (resp. $(0, \pm 1) \notin X_\lambda$).*

By Proposition 3.9 together with the following lemma, we can obtain the first occurrence index of any Witt tower of theta lifts of any irreducible tempered representation π of $\mathrm{U}(p,q)$.

Lemma 4.4. *Let $\lambda = (\phi, \eta)$ as in Definition 4.1, and set $\lambda^\vee = (\phi^\vee \otimes \chi_V^2, \eta^\vee)$.*

(1) *We have $k_{\lambda^\vee} = k_\lambda$ and $(r_{\lambda^\vee}, s_{\lambda^\vee}) = (s_\lambda, r_\lambda)$.*

(2) *Suppose that $\chi_{2\alpha}$ is contained in $\phi\chi_V^{-1}$ with odd multiplicity. Then $(\alpha, \epsilon) \in X_\lambda$ if and only if $(-\alpha, \epsilon) \in X_{\lambda^\vee}$.*

(3) *Suppose that $\chi_{2\alpha}$ is contained in $\phi\chi_V^{-1}$ with even multiplicity (possibly zero). Then $(\alpha, \pm 1) \in X_\lambda$ if and only if $(-\alpha, \pm 1) \notin X_{\lambda^\vee}$.*

(4) *In general, if $(\alpha, \epsilon) \in X_\lambda$, then $(-\alpha, -\epsilon) \notin X_{\lambda^\vee}$.*

Proof. Write

$$\phi\chi_V^{-1} = \chi_{2\alpha_1} + \cdots + \chi_{2\alpha_u} + (\xi_1 + \cdots + \xi_v) + ({}^c\xi_1^{-1} + \cdots + {}^c\xi_v^{-1})$$

as in Definition 4.1 (2). Then

$$\phi^\vee\chi_V = \chi_{-2\alpha_u} + \cdots + \chi_{-2\alpha_1} + (\xi_1^{-1} + \cdots + \xi_v^{-1}) + ({}^c\xi_1 + \cdots + {}^c\xi_v).$$

Suppose that $\chi_{2\alpha}$ is contained in $\phi\chi_V^{-1}$ with odd multiplicity. This means that $\alpha = \alpha_i$ for some i. Then

$$\begin{aligned} (\alpha, \epsilon) \in X_\lambda &\iff (-1)^{i-1}\eta(e_{V,2\alpha}) = \epsilon, \\ &\iff (-1)^{u-i}\eta^\vee(e_{V,-2\alpha}) = \epsilon \iff (-\alpha, \epsilon) \in X_{\lambda^\vee}. \end{aligned}$$

Hence we have (2). This easily implies that $k_{\lambda^\vee} = k_\lambda$ and $(r_{\lambda^\vee}, s_{\lambda^\vee}) = (s_\lambda, r_\lambda)$. Hence we have (1).

Now suppose that $\chi_{2\alpha}$ is contained in $\phi\chi_V^{-1}$ with even multiplicity. Then

$$(\alpha, \pm 1) \in X_\lambda \iff \eta(e_{V,2\alpha}) = (-1)^{\#\{i \in \{1,\ldots,u\} \mid \alpha_i > \alpha\}+1},$$

$$\iff \eta^\vee(e_{V,-2\alpha}) = (-1)^{\#\{i\in\{1,\dots,u\} \mid -\alpha_i>-\alpha\}} \iff (-\alpha, \pm1) \notin X_{\lambda^\vee}.$$

Hence we have (3). The assertion (4) follows from (2) and (3). □

4.3. **A corollary.** As a consequence of Theorem 4.2, we obtain a new relation between theta lifts and induced representations.

Corollary 4.5. *Assume the local Gan–Gross–Prasad conjecture (Conjecture 2.2). Let* $\pi \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(p,q))$ *and* $\pi_0 \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(p-1,q-1))$*. Suppose that there exists a unitary character* ξ *such that*

$$\pi \subset \mathrm{Ind}_P^{\mathrm{U}(p,q)}(\xi \otimes \pi_0),$$

where P *is a parabolic subgroup of* $\mathrm{U}(p,q)$ *with Levi subgroup* $M_P = \mathbb{C}^\times \times \mathrm{U}(p-1,q-1)$*. For* (r,s)*, we have the following:*

1. *Suppose that* $\Theta_{r-1,s-1}(\pi_0) \neq 0$*. If* $r+s \leq p+q$*, then* $\Theta_{r,s}(\pi) \neq 0$*.*
2. *Suppose that* $\Theta_{r,s}(\pi) \neq 0$*. If* $r+s \geq p+q+1$*, then* $\Theta_{r-1,s-1}(\pi_0) \neq 0$*. In general,* $\Theta_{r,s}(\pi_0) \neq 0$*.*

Proof. Let $\lambda = (\phi,\eta)$ and $\lambda_0 = (\phi_0,\eta_0)$ be the L-parameters of π and π_0, respectively. By Theorem 2.1 (5), we have $\phi = \phi_0 + \xi + {}^c\xi^{-1}$ and $\eta|A_{\phi_0} = \eta_0$. In particular, we see that $k_\lambda = k_{\lambda_0}$, $(r_\lambda, s_\lambda) = (r_{\lambda_0}+1, s_{\lambda_0}+1)$ and $X_{\lambda_0} \subset X_\lambda$.

We show (1). By Theorem 4.2, if $\Theta_{r-1,s-1}(\pi_0) \neq 0$ and $r+s \leq p+q$, then $|(r-s)-(r_{\lambda_0}-s_{\lambda_0})| \leq 1$. Hence (1) follows from the last assertions of Theorem 4.2 (1) and (2).

We show (2). To prove the first part, it suffices to show that $\#\mathcal{C}^\epsilon_{\lambda_0}(T) \leq \#\mathcal{C}^\epsilon_\lambda(T)$ for any T and $\epsilon \in \{\pm1\}$. If $X_{\lambda_0} = X_\lambda$, it is clear that $\mathcal{C}^\epsilon_{\lambda_0}(T) = \mathcal{C}^\epsilon_\lambda(T)$ for any T and $\epsilon \in \{\pm1\}$. Hence we may assume that $\xi = \chi_V\chi_{2\alpha}$ and $(\alpha, \pm1) \in X_\lambda \setminus X_{\lambda_0}$. First, we assume that $\alpha \geq (k_\lambda+1)/2$. Then $\mathcal{C}^-_\lambda(T) = \mathcal{C}^-_{\lambda_0}(T)$ for any T. We note that

$$\#\{(\alpha_0,+1) \in X^{(\infty)}_{\lambda_0} \mid (\alpha_0,+1) \notin X^{(\infty)}_\lambda\} \leq 1.$$

If $X^{(\infty)}_{\lambda_0} \subset X^{(\infty)}_\lambda$, then $\mathcal{C}^+_{\lambda_0}(T) \subset \mathcal{C}^+_\lambda(T)$. Suppose that $(\alpha_0,+1) \in X^{(\infty)}_{\lambda_0}$ but $(\alpha_0,+1) \notin X^{(\infty)}_\lambda$. Then one of the following holds:

- $X^{(\infty)}_\lambda = (X^{(\infty)}_{\lambda_0} \setminus \{(\alpha_0,+1)\}) \cup \{(\alpha',+1)\}$ for some $(\alpha',+1) \notin X^{(\infty)}_{\lambda_0}$ with $\alpha' \geq (k_\lambda+1)/2$;
- $X^{(\infty)}_\lambda = X^{(\infty)}_{\lambda_0} \setminus \{(\alpha_0,+1),(\alpha',-1)\}$ for some $(\alpha',-1) \in X^{(\infty)}_{\lambda_0}$ with $\alpha' \geq (k_\lambda+1)/2$.

In both of two cases, we must have $\alpha' < \alpha_0$. However, in the second case, $X^{(\infty)}_{\lambda_0}$ contains both $(\alpha_0,+1)$ and $(\alpha',-1)$ with $\alpha_0 > \alpha' \geq (k_\lambda+1)/2$. This contradicts the definition of $X^{(\infty)}_{\lambda_0}$ (see Definition 4.1 (4)). Hence we must have $X^{(\infty)}_\lambda = (X^{(\infty)}_{\lambda_0} \setminus \{(\alpha_0,+1)\}) \cup \{(\alpha',+1)\}$ for some $(\alpha',+1) \notin X^{(\infty)}_{\lambda_0}$ with $\alpha_0 > \alpha' \geq (k_\lambda+1)/2$. Then we have

$$\#\mathcal{C}^+_\lambda(T) = \begin{cases} \#\mathcal{C}^+_{\lambda_0}(T)+1 & \text{if } \alpha' - \dfrac{k_\lambda-1}{2} < T \leq \alpha_0 - \dfrac{k_\lambda-1}{2}, \\ \#\mathcal{C}^+_{\lambda_0}(T) & \text{otherwise.} \end{cases}$$

Therefore in any case, we have

$$\#\mathcal{C}^+_{\lambda_0}(T) \leq \#\mathcal{C}^+_\lambda(T).$$

Similarly, if $\alpha \leq -(k_\lambda+1)/2$, then $\mathcal{C}^+_\lambda(T) = \mathcal{C}^+_{\lambda_0}(T)$ and $\#\mathcal{C}^-_{\lambda_0}(T) \leq \#\mathcal{C}^-_\lambda(T)$ for any T. Hence we have the first part of (2). The last part follows from the last assertions of Theorem 4.2 (1) and (2). □

Remark 4.6. *A non-archimedean analogue also holds (see Theorem 1.12). In the non-archimedean case, the part (2) is a corollary of Kudla's filtration* [Ku1]*, whereas, in the archimedean case, it would not follow from the induction principle (*[P1, Theorem 4.5.5]*).*

5. Proof of Theorem 4.2

In this section, we shall prove Theorem 4.2. In §5.2, we show the sufficient conditions of the non-vanishing of theta lifts in Theorem 4.2 (1), (2) when $t \geq 1$. The proof is an induction using a seesaw identity (Proposition 3.11 (1)). To use such a seesaw identity, for given $\pi \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(p,q))$, we have to find a "good" representation π' of $\mathrm{U}(p+1,q)$ such that $\mathrm{Hom}_{\mathrm{U}(p,q)}(\pi',\pi) \neq 0$. To do this, we use the Gan–Gross–Prasad conjecture (Conjecture 2.2) in §5.1. The necessary conditions of the non-vanishing of theta lifts in Theorem 4.2 (1), (2) when $t \geq 1$ are proven in §5.3. For the proof, we use a seesaw identity (Proposition 3.11 (2)) and Proposition 3.10. Finally, using the conservation relation (Theorem 3.7), we show Theorem 4.2 (1), (2) when $t = 0$ in §5.4.

5.1. **Finding a GGP pair.** A main tool in the proof of the main result is a seesaw identity (Proposition 3.11 (1)). To use it, in this subsection, for given $\pi \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(p,q))$, we find a "good" representation π' of $\mathrm{U}(p+1,q)$ such that $\mathrm{Hom}_{\mathrm{U}(p,q)}(\pi',\pi) \neq 0$.

For a pair $\lambda = (\phi,\eta)$ of $\phi \in \Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$ and $\eta \in \widehat{A_\phi}$, let k_λ, r_λ, s_λ and X_λ be as in Definition 4.1. Consider a representation ϕ_0'' of $\mathbb{C}^\times$ defined so that

$$\phi_0''\chi_V^{-1} = \bigoplus_{(\alpha,\epsilon)\in X_\lambda} \chi_{2\alpha-\epsilon}.$$

Note that $\dim(\phi_0'') \equiv n \bmod 2$. For each $\beta \in \frac{1}{2}\mathbb{Z}$ with $2\beta \equiv \kappa \bmod 2$, the multiplicity of $\chi_{2\beta}$ in $\phi_0''\chi_V^{-1}$ is at most 2. Moreover, it is equal to 2 if and only if both $(\beta+1/2,+1)$ and $(\beta-1/2,-1)$ are contained in X_λ. Define a representation ϕ_0' of $\mathbb{C}^\times$ so that

$$\phi_0'\chi_V^{-1} = \phi_0''\chi_V^{-1} - \bigoplus\{2\chi_{2\beta} \mid (\beta+1/2,+1),(\beta-1/2,-1) \in X_\lambda\}.$$

Then ϕ_0' is multiplicity-free and $\dim(\phi_0') \equiv n \bmod 2$. Put $v' = (n - \dim(\phi_0'))/2$.

In this subsection, we choose arbitrary half integers $\beta_0, \beta_1, \dots$ such that $2\beta_j \equiv \kappa \bmod 2$ and

$$\max\{\alpha_1+1, 0\} < \beta_0 < \beta_1 < \cdots.$$

Here, when $u = 0$ so that α_1 does not appear, we understand that $\max\{\alpha_1+1,0\} = 0$.

To use a seesaw identity (Proposition 3.11 (1)), we need the following lemmas.

Lemma 5.1. *Assume the local Gan–Gross–Prasad conjecture (Conjecture 2.2). Let $\lambda = (\phi,\eta)$ be a pair of $\phi \in \Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$ and $\eta \in \widehat{A_\phi}$. Suppose that $k_\lambda = -1$. Set*

$$\phi' = \phi_0' + \chi_V(\chi_{2\beta_0} + \chi_{2\beta_1} + \cdots + \chi_{2\beta_{2v'}}) \in \Phi_{\mathrm{disc}}(\mathrm{U}_{n+1}(\mathbb{R})).$$

We define $\eta' \in \widehat{A_{\phi'}}$ by setting $\eta'(e_{V,2\beta_j}) = 1$ for $j = 0,1,\dots,2v'$, and

$$\eta'(e_{V,2\beta}) = -\eta(e_{V,2\alpha})$$

when $\beta = \alpha - \epsilon/2$ with $(\alpha,\epsilon) \in X_\lambda$. Let $\lambda' = (\phi',\eta')$ and $\pi' = \pi(\phi',\eta')$. Then

- *$\pi' \in \mathrm{Irr}_{\mathrm{disc}}(\mathrm{U}(p+1,q))$ and $\mathrm{Hom}_{\mathrm{U}(p,q)}(\pi',\pi) \neq 0$;*
- *$k_{\lambda'} = 0$ and*
$$(r_{\lambda'}, s_{\lambda'}) = \begin{cases} (r_\lambda, s_\lambda+1) & \text{if } (0,\pm1) \in X_\lambda, \\ (r_\lambda+1, s_\lambda) & \text{otherwise}; \end{cases}$$
- *for $T \leq \beta_0 - 1/2$ and $\epsilon \in \{\pm1\}$, the map $(\beta,\epsilon) \mapsto (\beta+\epsilon/2,\epsilon)$ gives an injection*
$$\mathcal{C}^\epsilon_{\lambda'}(T) \hookrightarrow \mathcal{C}^\epsilon_\lambda(T);$$
- *for fixed $l \geq 0$ and $1 \leq T \leq \beta_0 - 1/2$, if $\#\mathcal{C}^\epsilon_\lambda(T) \leq l$ for each $\epsilon \in \{\pm1\}$, then $\#\mathcal{C}^\epsilon_{\lambda'}(T) \leq l$ for each $\epsilon \in \{\pm1\}$.*

Lemma 5.2. *Assume the local Gan–Gross–Prasad conjecture (Conjecture 2.2). Let $\lambda = (\phi,\eta)$ be a pair of $\phi \in \Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$ and $\eta \in \widehat{A_\phi}$. Suppose that $k_\lambda = 0$. Put*

$$\alpha_+ = \min\{\alpha \mid \alpha > 0 \text{ and } \exists\epsilon \in \{\pm1\} \text{ s.t. } (\alpha,\epsilon) \in X_\lambda\},$$
$$\alpha_- = \max\{\alpha \mid \alpha < 0 \text{ and } \exists\epsilon \in \{\pm1\} \text{ s.t. } (\alpha,\epsilon) \in X_\lambda\}.$$

We consider the following cases separately:

Case 1: *Both* $(1/2, +1)$ *and* $(-1/2, -1)$ *are not contained in* X_λ.
Case 2: *Both* $(1/2, +1)$ *and* $(-1/2, -1)$ *are contained in* X_λ.
Case 3: *There is* $\delta \in \{\pm 1\}$ *such that*

$$\alpha_\delta = \delta/2, \quad (\alpha_\delta, \delta) \in X_\lambda \quad \text{and} \quad \alpha_{-\delta} \neq -\delta/2, \quad (\alpha_{-\delta}, -\delta) \in X_\lambda.$$

Case 4: *There is* $\delta \in \{\pm 1\}$ *such that*

$$\alpha_\delta = \delta/2, \quad (\alpha_\delta, \delta) \in X_\lambda \quad \text{and} \quad \alpha_{-\delta} \neq -\delta/2, \quad (\alpha_{-\delta}, -\delta) \notin X_\lambda.$$

Case 5: *There is* $\delta \in \{\pm 1\}$ *such that*

$$\alpha_\delta = \delta/2, \quad (\alpha_\delta, \delta) \in X_\lambda \quad \text{and} \quad \alpha_{-\delta} = -\delta/2, \quad (\alpha_{-\delta}, -\delta) \notin X_\lambda.$$

The case 5 *cannot occur if* $\phi \in \Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R}))$. *We set* $\phi' \in \Phi_{\mathrm{disc}}(\mathrm{U}_{n+1}(\mathbb{R}))$ *so that*

$$\phi'\chi_V^{-1} = \begin{cases} \phi'_0\chi_V^{-1} + (\chi_{2\beta_0} + \chi_{2\beta_1} + \cdots + \chi_{2\beta_{2v'}}) & \text{if } \lambda \text{ is in case } 1,\ 2 \text{ or } 5, \\ \phi'_0\chi_V^{-1} - (\chi_{2\beta_+} + \chi_{2\beta_-}) + (\chi_{2\beta_0} + \chi_{2\beta_1} + \cdots + \chi_{2\beta_{2v'+2}}) & \text{if } \lambda \text{ is in case } 3, \\ \phi'_0\chi_V^{-1} - \mathbf{1} + \chi_{-2\delta} + (\chi_{2\beta_0} + \chi_{2\beta_1} + \cdots + \chi_{2\beta_{2v'}}) & \text{if } \lambda \text{ is in case } 4. \end{cases}$$

Here, in the case 3, *we put* $\beta_\pm = \alpha_\pm \mp 1/2$. *Also we define* $\eta' \in \widehat{A_{\phi'}}$ *by setting* $\eta'(e_{V,2\beta_j}) = 1$ *for* $j = 0, 1, \ldots,$ *and*

$$\eta'(e_{V,2\beta}) = -\eta(e_{V,2\alpha})$$

when $\beta = \alpha - \epsilon/2$ *with* $(\alpha, \epsilon) \in X_\lambda$. *In the case* 4, *we set* $\eta'(e_{V,-2\delta}) = -\eta(e_{V,2\alpha_\delta})$. *Let* $\lambda' = (\phi', \eta')$ *and* $\pi' = \pi(\phi', \eta')$. *Then*

- $\pi' \in \mathrm{Irr}_{\mathrm{disc}}(\mathrm{U}(p+1, q))$ *and* $\mathrm{Hom}_{\mathrm{U}(p,q)}(\pi', \pi) \neq 0$;
- $k_{\lambda'}$ *and* $(r_{\lambda'}, s_{\lambda'})$ *are given by*

$$k_{\lambda'} = \begin{cases} -1 & \text{if } \lambda \text{ is in case } 1,\ 2,\ 3 \text{ or } 4, \\ 1 & \text{if } \lambda \text{ is in case } 5, \end{cases}$$

$$(r_{\lambda'}, s_{\lambda'}) = \begin{cases} (r_\lambda + 1, s_\lambda) & \text{if } \lambda \text{ is in case } 1, \\ (r_\lambda, s_\lambda + 1) & \text{if } \lambda \text{ is in case } 2,\ 3 \text{ or } 4, \\ (r_\lambda, s_\lambda) & \text{if } \lambda \text{ is in case } 5; \end{cases}$$

- *for* $T \leq \beta_0$ *and* $\epsilon \in \{\pm 1\}$, *the map* $(\beta, \epsilon) \mapsto (\beta + \epsilon/2, \epsilon)$ *gives an injection*

$$\begin{cases} \mathcal{C}^\epsilon_{\lambda'}(T-1) \hookrightarrow \mathcal{C}^\epsilon_\lambda(T) & \text{if } \lambda \text{ is in case } 1,\ 2,\ 3 \text{ or } 4, \\ \mathcal{C}^\epsilon_{\lambda'}(T) \hookrightarrow \mathcal{C}^\epsilon_\lambda(T) & \text{if } \lambda \text{ is in case } 5; \end{cases}$$

- *for fixed* $l \geq 0$ *and* $1 \leq T \leq \beta_0$, *if* $\#\mathcal{C}^\epsilon_\lambda(T) \leq l$ *for each* $\epsilon \in \{\pm 1\}$, *then*

$$\begin{cases} \#\mathcal{C}^\epsilon_{\lambda'}(T-1) \leq l & \text{if } \lambda \text{ is in case } 1, \\ \#\mathcal{C}^\epsilon_{\lambda'}(T-1) \leq l - 1 & \text{if } \lambda \text{ is in case } 2,\ 3 \text{ or } 4, \\ \#\mathcal{C}^\epsilon_{\lambda'}(T) \leq l & \text{if } \lambda \text{ is in case } 5 \end{cases}$$

  for each $\epsilon \in \{\pm 1\}$.

Lemma 5.3. *Assume the local Gan–Gross–Prasad conjecture (Conjecture 2.2). Let* $\lambda = (\phi, \eta)$ *be a pair of* $\phi \in \Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$ *and* $\eta \in \widehat{A_\phi}$. *Suppose that* $k = k_\lambda > 0$. *Put*

$$\alpha_+ = \min\left\{\alpha \;\middle|\; \alpha > \frac{k-1}{2} \text{ and } \exists \epsilon \in \{\pm 1\} \text{ s.t. } (\alpha, \epsilon) \in X_\lambda \right\},$$

$$\alpha_- = \max\left\{\alpha \;\middle|\; \alpha < -\frac{k-1}{2} \text{ and } \exists \epsilon \in \{\pm 1\} \text{ s.t. } (\alpha, \epsilon) \in X_\lambda \right\}.$$

There is unique $\delta \in \{\pm 1\}$ *such that* $((k-1)/2, \delta), \ldots, ((-k+1)/2, \delta) \in X_\lambda$. *We consider the following cases separately.*

Case 1: $\alpha_\delta \neq (k+1)\delta/2$ *or* $(\alpha_\delta, \delta) \notin X_\lambda$.
Case 2: $\alpha_\delta = (k+1)\delta/2$, $(\alpha_\delta, \delta) \in X_\lambda$ *and* $\alpha_{-\delta} = -(k+1)\delta/2$, $(\alpha_{-\delta}, -\delta) \in X_\lambda$.
Case 3: $\alpha_\delta = (k+1)\delta/2$, $(\alpha_\delta, \delta) \in X_\lambda$ *and* $\alpha_{-\delta} \neq -(k+1)\delta/2$, $(\alpha_{-\delta}, -\delta) \in X_\lambda$.
Case 4: $\alpha_\delta = (k+1)\delta/2$, $(\alpha_\delta, \delta) \in X_\lambda$ *and* $\alpha_{-\delta} \neq -(k+1)\delta/2$, $(\alpha_{-\delta}, -\delta) \notin X_\lambda$.
Case 5: $\alpha_\delta = (k+1)\delta/2$, $(\alpha_\delta, \delta) \in X_\lambda$ *and* $\alpha_{-\delta} = -(k+1)\delta/2$, $(\alpha_{-\delta}, -\delta) \notin X_\lambda$.

The case 5 *cannot occur if* $\phi \in \Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R}))$. *We set* $\phi' \in \Phi_{\mathrm{disc}}(\mathrm{U}_{n+1}(\mathbb{R}))$ *so that*

$$\phi'\chi_V^{-1} = \begin{cases} \phi'_0\chi_V^{-1} + (\chi_{2\beta_0} + \chi_{2\beta_1} + \cdots + \chi_{2\beta_{2v'}}) & \text{if } \lambda \text{ is in case } 1, 2 \text{ or } 5, \\ \phi'_0\chi_V^{-1} - (\chi_{2\beta_+} + \chi_{2\beta_-}) + (\chi_{2\beta_0} + \chi_{2\beta_1} + \cdots + \chi_{2\beta_{2v'+2}}) & \text{if } \lambda \text{ is in case } 3, \\ \phi'_0\chi_V^{-1} - \chi_{-k\delta} + \chi_V\chi_{-(k+2)\delta} + (\chi_{2\beta_0} + \chi_{2\beta_1} + \cdots + \chi_{2\beta_{2v'}}) & \text{if } \lambda \text{ is in case } 4. \end{cases}$$

Here, in the case 3, *we put* $\beta_\delta = -k\delta/2$ *and* $\beta_{-\delta} = \alpha_{-\delta} + \delta/2$. *Also we define* $\eta' \in \widehat{A_{\phi'}}$ *by setting* $\eta'(e_{V,2\beta_j}) = 1$ *for* $j = 0, 1, \ldots,$ *and*

$$\eta'(e_{V,2\beta}) = -\eta(e_{V,2\alpha})$$

when $\beta = \alpha - \epsilon/2$ *with* $(\alpha, \epsilon) \in X_\lambda$. *In the case* 4, *we set* $\eta'(e_{V,-(k+2)\delta}) = -\eta(e_{V,-(k-1)\delta})$. *Let* $\lambda' = (\phi', \eta')$ *and* $\pi' = \pi(\phi', \eta')$. *Then*

- $\pi' \in \mathrm{Irr}_{\mathrm{disc}}(\mathrm{U}(p+1,q))$ *and* $\mathrm{Hom}_{\mathrm{U}(p,q)}(\pi', \pi) \neq 0$*;*
- $k_{\lambda'}$ *and* $(r_{\lambda'}, s_{\lambda'})$ *are given by*

$$k_{\lambda'} = \begin{cases} k-1 & \text{if } \lambda \text{ is in case } 1, 2, 3 \text{ or } 4, \\ k+1 & \text{if } \lambda \text{ is in case } 5, \end{cases}$$

$$(r_{\lambda'}, s_{\lambda'}) = \begin{cases} (r_\lambda + 1, s_\lambda + 1) & \text{if } \lambda \text{ is in case } 1, 2, 3 \text{ or } 4, \\ (r_\lambda, s_\lambda) & \text{if } \lambda \text{ is in case } 5; \end{cases}$$

- *for* $T \leq \beta_0 + k/2$ *and* $\epsilon \in \{\pm 1\}$, *the map* $(\beta, \epsilon) \mapsto (\beta + \epsilon/2, \epsilon)$ *gives an injection*

$$\begin{cases} \mathcal{C}^\epsilon_{\lambda'}(T-1) \hookrightarrow \mathcal{C}^\epsilon_\lambda(T) & \text{if } \lambda \text{ is in case } 1, 2, 3 \text{ or } 4, \\ \mathcal{C}^\epsilon_{\lambda'}(T) \hookrightarrow \mathcal{C}^\epsilon_\lambda(T) & \text{if } \lambda \text{ is in case } 5; \end{cases}$$

- *for fixed* $l \geq 0$ *and* $1 \leq T \leq \beta_0 + k/2$, *if* $\#\mathcal{C}^\epsilon_\lambda(T) \leq l$ *for each* $\epsilon \in \{\pm 1\}$, *then*

$$\begin{cases} \#\mathcal{C}^\epsilon_{\lambda'}(T-1) \leq l-1 & \text{if } \lambda \text{ is in case } 1, 2, 3 \text{ or } 4, \\ \#\mathcal{C}^\epsilon_{\lambda'}(T) \leq l & \text{if } \lambda \text{ is in case } 5 \end{cases}$$

  for each $\epsilon \in \{\pm 1\}$.

The proofs of Lemmas 5.1, 5.2 and 5.3 are straightforward and similar to each other. So we only prove Lemma 5.1.

Proof of Lemma 5.1. We check the conditions in Proposition 2.3. Write

$$\phi\chi_V^{-1} = \chi_{2\alpha_1} + \cdots + \chi_{2\alpha_u} + (\xi_1 + \cdots + \xi_v) + ({}^c\xi_1^{-1} + \cdots + {}^c\xi_v^{-1}),$$

where

- $\alpha_i \in \frac{1}{2}\mathbb{Z}$ such that $2\alpha_i \equiv \kappa - 1 \bmod 2$ and $\alpha_1 > \cdots > \alpha_u$;
- ξ_i is a unitary character of $\mathbb{C}^\times$ (which can be of the form $\chi_{2\alpha}$);
- $u + 2v = n$.

Fix α such that $\chi_V\chi_{2\alpha} \subset \phi$. Then

$$\#\{\beta \mid \chi_V\chi_{2\beta} \subset \phi',\ \beta < \alpha\} \equiv \begin{cases} \#\{i \in \{1, \ldots, u\} \mid \alpha_i < \alpha\} + 1 \mod 2 & \text{if } (\alpha, +1) \in X_\lambda, \\ \#\{i \in \{1, \ldots, u\} \mid \alpha_i < \alpha\} \mod 2 & \text{otherwise.} \end{cases}$$

This implies that

$$(-1)^{\#\{\beta \ \mid\ \chi_V\chi_{2\beta} \subset \phi',\ \beta < \alpha\}} = (-1)^n \eta(e_{V,2\alpha}).$$

Similarly, for each β such that $\beta \neq \beta_j$ and $\chi_V \chi_{2\beta} \subset \phi'$, we have

$$\begin{aligned}&\#\{i \in \{1,\dots,u\} \mid \alpha_i < \beta\}\\ &\equiv \begin{cases} \#\{i \in \{1,\dots,u\} \mid \alpha_i < \beta + 1/2\} + 1 \mod 2 & \text{if } (\beta,-1) \in X_{\lambda'}, (\beta - 1/2, +1) \notin X_\lambda \\ \#\{i \in \{1,\dots,u\} \mid \alpha_i < \beta - 1/2\} \mod 2 & \text{otherwise.}\end{cases}\end{aligned}$$

so that

$$(-1)^{\#\{i \in \{1,\dots,u\} \mid \alpha_i < \beta\}} = (-1)^n \eta'(e_{V,2\beta}).$$

This equation also holds when $\beta = \beta_j$ for $j = 0, 1, \dots, 2v'$. By the local Gan–Gross–Prasad conjecture (Conjecture 2.2 and Proposition 2.3), we conclude that $\pi' \in \mathrm{Irr}_{\mathrm{disc}}(\mathrm{U}(p+1,q))$ and $\mathrm{Hom}_{\mathrm{U}(p,q)}(\pi', \pi) \neq 0$.

By the construction, if $X_{\lambda'}$ contains $(1/2,-1)$ (resp. $(-1/2,+1)$), then X_λ must contain $(0,-1)$ (resp. $(0,+1)$). Since $k_\lambda = -1$, in this case, we must have $(0,\pm 1) \in X_\lambda$, so that $X_{\lambda'}$ cannot contain $(-1/2,-1)$ (resp. $(1/2,+1)$). Hence $k_{\lambda'} = 0$. Note that for $(\alpha,\epsilon) \in X_\lambda$ with $\alpha \neq 0$,

$$\epsilon\alpha > 0 \iff \epsilon(\alpha - \frac{\epsilon}{2}) > 0.$$

This implies that when $(0,\pm1) \notin X_\lambda$, we have $(r_{\lambda'}, s_{\lambda'}) = (r_\lambda + 1, s_\lambda)$. If $(\beta,\epsilon) = (\pm 1/2, \mp 1)$, then $\epsilon\beta < 0$. This implies that when $(0,\pm1) \in X_\lambda$, we have $(r_{\lambda'}, s_{\lambda'}) = (r_\lambda, s_\lambda + 1)$.

Finally, by definition, we have

$$X_{\lambda'} \subset \{(\beta_j, (-1)^j) \mid j = 0, \dots, 2v'\} \cup \{(\beta,\epsilon) \mid (\beta + \epsilon/2, \epsilon) \in X_\lambda\}.$$

Moreover, by construction of λ', we see that if $(\beta,\epsilon) \in X_{\lambda'}^{(\infty)}$ with $\beta \neq \beta_0$, then $(\alpha,\epsilon) \in X_\lambda^{(\infty)}$ with $\alpha = \beta + \epsilon/2$. In this case,

$$\begin{aligned} 0 \le \epsilon\beta - \frac{1}{2} < T &\iff 0 \le \epsilon\left(\alpha - \frac{\epsilon}{2}\right) - \frac{1}{2} < T \\ &\iff 0 \le \epsilon\alpha - 1 < T. \end{aligned}$$

Hence if $(\beta,\epsilon) \in \mathcal{C}^\epsilon_{\lambda'}(T)$ with $\beta \neq \beta_0$, then $(\alpha,\epsilon) \in \mathcal{C}^\epsilon_\lambda(T)$ with $\alpha = \beta + \epsilon/2$. When $T < \beta_0$, we see that $(\beta_0, +1)$ is not contained in $\mathcal{C}^\epsilon_{\lambda'}(T)$, so that we may consider the map

$$\mathcal{C}^\epsilon_{\lambda'}(T) \to \mathcal{C}^\epsilon_\lambda(T),\ (\beta,\epsilon) \mapsto \left(\beta + \frac{\epsilon}{2}, \epsilon\right).$$

This map is clearly injective. Hence we have $\#\mathcal{C}^\epsilon_{\lambda'}(T) \le \#\mathcal{C}^\epsilon_\lambda(T)$. This completes the proof of Lemma 5.1. □

5.2. **Non-vanishing.** In this subsection, we prove sufficient conditions of the non-vanishing of theta lifts in Theorem 4.2.

Let $\lambda = (\phi,\eta)$ be a pair of $\phi \in \Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$ and $\eta \in \widehat{A_\phi}$. Set $r = r_\lambda$, $s = s_\lambda$, and $\pi = \pi(\phi,\eta)$. For non-negative integers t and l, consider the following statements:

$(S)_{-1,t,l}$**:** Suppose that $k_\lambda = -1$. If $\#\mathcal{C}^\epsilon_\lambda(t+l) \le l$ for each $\epsilon \in \{\pm1\}$, then

$$\begin{cases} \Theta_{r+2t+l+1,s+l}(\pi) \neq 0 & \text{if } (0,\pm1) \notin X_\lambda, \\ \Theta_{r+2t+l,s+l+1}(\pi) \neq 0 & \text{if } (0,\pm1) \in X_\lambda. \end{cases}$$

$(S)_{k,t,l}$**:** Suppose that $k_\lambda = k \ge 0$ and $l \ge k$. If $\#\mathcal{C}^\epsilon_\lambda(t+l) \le l$ for each $\epsilon \in \{\pm1\}$, then $\Theta_{r+2t+l,s+l}(\pi) \neq 0$.

First, we consider these statements for the discrete series representations. We have implications.

Proposition 5.4. *Consider the statement $(S)_{k,t,l}$ only when $\phi \in \Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R}))$.*

1. *For $t \ge 0$ and $l \ge 0$, we have $(S)_{0,t,l} \Rightarrow (S)_{-1,t,l}$.*
2. *For $t \ge 1$ and $l \ge 0$, we have $(S)_{-1,t-1,l} + (S)_{-1,t,l-1} \Rightarrow (S)_{0,t,l}$.*
3. *For $t \ge 0$ and $l \ge k > 0$, we have $(S)_{k-1,t,l-1} \Rightarrow (S)_{k,t,l}$.*

Here, we interpret $(S)_{k,t,-1}$ to be empty.

Proof. Suppose that $\lambda = (\phi, \eta)$ is a pair of $\phi \in \Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R}))$ and $\eta \in \widehat{A_\phi}$. Let $\lambda' = (\phi', \eta')$ be as in Lemma 5.1, 5.2 or 5.3. Here, we take β_0 so that $\beta_0 + k_\lambda/2 \geq t + l$. Since $\phi \in \Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R}))$ and $\phi' \in \Phi_{\mathrm{disc}}(\mathrm{U}_{n+1}(\mathbb{R}))$, the local Gan–Gross–Prasad conjecture for ϕ and ϕ' has been established by He [He2]. So we have $\mathrm{Hom}_{\mathrm{U}(p,q)}(\pi', \pi) \neq 0$ unconditionally.

We show (1). Suppose that $k_\lambda = -1$. By Lemma 5.1, we have $k_{\lambda'} = 0$, $(r_{\lambda'}, s_{\lambda'}) = (r_\lambda + 1, s_\lambda)$, and if $\#\mathcal{C}_\lambda^\epsilon(t+l) \leq l$, then $\#\mathcal{C}_{\lambda'}^\epsilon(t+l) \leq l$. Hence we can apply $(S)_{0,t,l}$ to λ', and we obtain that

$$\Theta_{(r_\lambda+1)+2t+l, s_\lambda+l}(\pi') \neq 0.$$

Since $\mathrm{Hom}_{\mathrm{U}(p,q)}(\pi', \pi) \neq 0$, by the seesaw

$$\begin{tikzcd}[column sep=small]
\mathrm{U}(p+1,q) \arrow[d, dash] \arrow[drr, dash] & & \mathrm{U}(r_\lambda+1+2t+l, s_\lambda+l) \times \mathrm{U}(r_\lambda+1+2t+l, s_\lambda+l) \arrow[d, dash] \arrow[dll, dash] \\
\mathrm{U}(p,q) \times \mathrm{U}(1,0) & & \mathrm{U}(r_\lambda+1+2t+l, s_\lambda+l),
\end{tikzcd}$$

we conclude that $\Theta_{r_\lambda+1+2t+l, s_\lambda+l}(\pi) \neq 0$. Therefore, we have $(S)_{0,t,l} \Rightarrow (S)_{-1,t,l}$.

The proofs of (2) and (3) are similar. Note that the cases 5 in Lemmas 5.2 and 5.3 cannot occur since $\phi \in \Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R}))$. We omit the detail. □

Corollary 5.5. *The statement $(S)_{k,t,l}$ is true for $\lambda = (\phi, \eta)$ such that $\phi \in \Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R}))$.*

Proof. When $k > 0$, the statement $(S)_{k,t,l}$ is reduced to $(S)_{0,t,l-k}$ by Proposition 5.4 (3). We prove $(S)_{k,t,l}$ for $k \leq 0$ by induction on $t + l$. For $k = -1, 0$ and $T \geq 0$, we consider the following statement:

$(S')_{k,T}$**:** the statement $(S)_{k,t,l}$ is true for any $t, l \geq 0$ such that $t + l \leq T$.

Note that $(S)_{0,0,0}$ is Paul's result (Theorem 3.13). This implies $(S)_{-1,0,0}$ (Proposition 5.4 (1)). In particular, $(S')_{k,0}$ is true. Also the tower property (Proposition 3.6) implies $(S)_{k,0,l}$ for $k = -1, 0$ and $l \geq 0$. By Proposition 5.4 (1) and (2), we have

$$(S')_{-1,T-1} \Rightarrow (S')_{0,T} \Rightarrow (S')_{-1,T}$$

for any $T > 0$. Hence by induction, we obtain $(S')_{k,T}$ for $k = -1, 0$ and $T \geq 0$. □

Now we obtain the sufficient conditions of the non-vanishing of theta lifts.

Corollary 5.6. *Assume the local Gan–Gross–Prasad conjecture (Conjecture 2.2). Then the statement $(S)_{k,t,l}$ is true in general.*

Proof. The statement $(S)_{0,0,l}$ follows from Paul's result (Theorem 3.13) and the tower property (Proposition 3.6). By a similar argument to the proof of Proposition 5.4, the other assertions follow from Corollary 5.5 by using a seesaw identity (Proposition 3.11 (1)) and Lemmas 5.1, 5.2 and 5.3. We omit the detail. □

Let $\lambda = (\phi, \eta)$ be a pair of $\phi \in \Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$ and $\eta \in \widehat{A_\phi}$. Set $r = r_\lambda$, $s = s_\lambda$, and $\pi = \pi(\phi, \eta)$. Recall as in Theorem 3.14 (3) that when $k_\lambda = -1$ but ϕ contains χ_V (with even multiplicity), exactly one of $\Theta_{r-1,s}(\pi)$ and $\Theta_{r,s-1}(\pi)$ is nonzero. Now we can determine which is nonzero in terms of λ.

Corollary 5.7. *Assume the local Gan–Gross–Prasad conjecture (Conjecture 2.2). Let $\lambda = (\phi, \eta)$ be a pair of $\phi \in \Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$ and $\eta \in \widehat{A_\phi}$. Set $r = r_\lambda$, $s = s_\lambda$, and $\pi = \pi(\phi, \eta)$. Suppose that $k_\lambda = -1$ but ϕ contains χ_V (with even multiplicity). Then $\Theta_{r,s-1}(\pi) \neq 0$ if and only if $(0, \pm 1) \notin X_\lambda$.*

Proof. Note that $r + s = p + q$, and that $(0,1)$ and $(0,-1)$ are both in X_λ or both not in X_λ. By a result of Paul (Theorem 3.14 (3)) and the conservation relation (Theorem 3.7), we see that exactly one of $\Theta_{r+1,s}(\pi)$ and $\Theta_{r,s+1}(\pi)$ is nonzero. Hence $\Theta_{r,s-1}(\pi) \neq 0$ if and only if $\Theta_{r+1,s}(\pi) \neq 0$. By the statement $(S)_{-1,0,0}$, we have

$$\begin{cases} \Theta_{r+1,s}(\pi) \neq 0 & \text{if } (0, \pm 1) \notin X_\lambda, \\ \Theta_{r,s+1}(\pi) \neq 0 & \text{if } (0, \pm 1) \in X_\lambda. \end{cases}$$

This completes the proof. □

5.3. **Vanishing.** In this subsection, we prove necessary conditions of non-vanishing of theta lifts in Theorem 4.2. First, we prove the following:

Proposition 5.8. *Let $\lambda = (\phi, \eta)$ be a pair of $\phi \in \Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$ and $\eta \in \widehat{A_\phi}$. Set $\pi = \pi(\phi, \eta)$. Let $k = k_\lambda$, $r = r_\lambda$, $s = s_\lambda$, X_λ and $\mathcal{C}^\epsilon_\lambda(T)$ be as in Definition 4.1.*

(1) *Suppose that $k = -1$. For $t \geq 1$ and $l \geq 0$, if $\mathcal{C}^+_\lambda(t+l) > l$ or $\mathcal{C}^-_\lambda(t+l) > l$, then $\Theta_{r+2t+l+1,s+l}(\pi) = 0$.*

(2) *Suppose that $k \geq 0$. For $t \geq 1$ and $l \geq k$, if $\mathcal{C}^+_\lambda(t+l) > l$ or $\mathcal{C}^-_\lambda(t+l) > l$, then $\Theta_{r+2t+l,s+l}(\pi) = 0$.*

Proof. The proof is similar to that of [P2, Theorem 3.14]. Suppose for the sake of contradiction that for some $t \geq 1$ and $l \geq \max\{0, k\}$,

$$\begin{cases} \Theta_{r+2t+l,s+l}(\pi) \neq 0 & \text{if } k \geq 0, \\ \Theta_{r+1+2t+l,s+l}(\pi) \neq 0 & \text{if } k = -1, \end{cases}$$

but there exists $\epsilon \in \{\pm 1\}$ such that $\mathcal{C}^\epsilon_\lambda(t+l) > l$. Write

$$\mathcal{C}^\epsilon_\lambda(t+l) = \{(\alpha_1, \epsilon), (\alpha_2, \epsilon), \dots, (\alpha_{l+1}, \epsilon), \dots\}$$

with $\epsilon\alpha_1 < \dots < \epsilon\alpha_{l+1} < \dots < t + l - (k-1)/2$. We set

$$a = \begin{cases} \alpha_{l+1} + \epsilon/2 & \text{if } k \text{ is even}, \\ \alpha_{l+1} & \text{if } k \text{ is odd}. \end{cases}$$

Then a is an integer. By the definition of $\mathcal{C}^\epsilon_\lambda(t+l)$ and an easy calculation, we have

$$\begin{aligned} &\#\{(\alpha, \epsilon) \in X_\lambda \mid \epsilon\alpha > 0,\ \epsilon\alpha < \epsilon a\} - \#\{(\alpha, -\epsilon) \in X_\lambda \mid -\epsilon\alpha < 0,\ -\epsilon\alpha > -\epsilon a\} \\ &= \begin{cases} l + 1 - \dfrac{k}{2} & \text{if } k \text{ is even}, \\ l - \dfrac{k+1}{2} & \text{if } k \text{ is odd, } k > 0 \text{ and } (0, \epsilon) \in X_\lambda, \\ l - \dfrac{k-1}{2} & \text{if } k \text{ is odd, } k > 0 \text{ and } (0, -\epsilon) \in X_\lambda, \\ l & \text{if } k = -1 \text{ and } (0, \pm 1) \notin X_\lambda, \\ l + 1 & \text{if } k = -1 \text{ and } (0, \pm 1) \in X_\lambda. \end{cases} \end{aligned}$$

Also, since $l \geq \max\{0, k\}$, we have $\epsilon a > 0$. Hence

$$\{(\alpha, -\epsilon) \in X_\lambda \mid -\epsilon\alpha > 0,\ -\epsilon\alpha < -\epsilon a\} = \{(\alpha, \epsilon) \in X_\lambda \mid \epsilon\alpha < 0,\ \epsilon\alpha > \epsilon a\} = \emptyset.$$

We set

$$\delta = \begin{cases} 1 & \text{if } k \text{ is odd, } k > 0 \text{ and } (\alpha_{l+1}, -\epsilon) \in X_\lambda, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$\begin{aligned} x &= \#\{(\alpha, \epsilon') \in X_\lambda \mid \epsilon'\alpha > 0,\ \epsilon'\alpha > \epsilon' a\}, \\ y &= \#\{(\alpha, \epsilon') \in X_\lambda \mid \epsilon'\alpha > 0,\ \epsilon'\alpha < \epsilon' a\}, \\ z &= \#\{(\alpha, \epsilon') \in X_\lambda \mid \epsilon'\alpha < 0,\ \epsilon'\alpha > \epsilon' a\}, \\ w &= \#\{(\alpha, \epsilon') \in X_\lambda \mid \epsilon'\alpha < 0,\ \epsilon'\alpha < \epsilon' a\}, \end{aligned}$$

and $v = (n - \#X_\lambda)/2$. Then by Definition 4.1, we have

$$(x + y + v, z + w + v) = \begin{cases} \left(r + \dfrac{k}{2}, s + \dfrac{k}{2}\right) & \text{if } k \text{ is even}, \\ \left(r + \dfrac{k-1}{2} - 1, s + \dfrac{k-1}{2} - \delta\right) & \text{if } k \text{ is odd and } k > 0, \\ (r - 1, s - \delta) & \text{if } k = -1 \text{ and } (0, \pm 1) \notin X_\lambda, \\ (r - 2, s - 1 - \delta) & \text{if } k = -1 \text{ and } (0, \pm 1) \in X_\lambda. \end{cases}$$

On the other hand, by the above calculation, we have

$$y-z=\begin{cases} l+1-\dfrac{k}{2} & \text{if } k \text{ is even,}\\ l-\dfrac{k+1}{2} & \text{if } k \text{ is odd, } k>0 \text{ and } (0,\epsilon)\in X_\lambda,\\ l-\dfrac{k-1}{2} & \text{if } k \text{ is odd, } k>0 \text{ and } (0,-\epsilon)\in X_\lambda,\\ l & \text{if } k=-1 \text{ and } (0,\pm1)\notin X_\lambda,\\ l+1 & \text{if } k=-1 \text{ and } (0,\pm1)\in X_\lambda.\end{cases}$$

Now we consider $\lambda_a=(\phi\otimes\chi_{-2a},\eta)$ and $\pi(\phi\otimes\chi_{-2a},\eta)=\pi\otimes\det^{-a}$. Note that $k_{\lambda_a}\equiv k \bmod 2$. There exists a bijection

$$X_\lambda\to X_{\lambda_a},\ (\alpha,\epsilon')\mapsto(\alpha-a,\epsilon').$$

We set

$$r'=\#\{(\alpha',\epsilon')\in X_{\lambda_a}\mid \epsilon'\alpha'>0\}+v+\delta,$$
$$s'=\#\{(\alpha',\epsilon')\in X_{\lambda_a}\mid \epsilon'\alpha'<0\}+v+\delta.$$

Note that

$$r'+s'=\begin{cases} n & \text{if } k \text{ is even,}\\ n-1+\delta & \text{if } k \text{ is odd.}\end{cases}$$

By the above bijection, we have

$$\begin{aligned} r'&=\begin{cases} x+z+v & \text{if } k \text{ is even,}\\ x+z+v+\delta & \text{if } k>0,\ k \text{ is odd and } (0,\epsilon)\in X_\lambda,\\ x+z+v+\delta+1 & \text{if } k>0,\ k \text{ is odd and } (0,-\epsilon)\in X_\lambda,\\ x+z+v+\delta & \text{if } k=-1 \text{ and } (0,\pm1)\notin X_\lambda,\\ x+z+v+\delta+1 & \text{if } k=-1 \text{ and } (0,\pm1)\in X_\lambda\end{cases}\\ &=\begin{cases} r+k-l-1 & \text{if } k \text{ is even,}\\ r+k-l-1+\delta & \text{if } k>0 \text{ and } k \text{ is odd,}\\ r-l-1+\delta & \text{if } k=-1 \text{ and } (0,\pm1)\notin X_\lambda,\\ r-l-2+\delta & \text{if } k=-1 \text{ and } (0,\pm1)\in X_\lambda\end{cases}\end{aligned}$$

and

$$\begin{aligned} s'&=\begin{cases} y+w+v & \text{if } k \text{ is even,}\\ y+w+v+\delta+1 & \text{if } k>0,\ k \text{ is odd and } (0,\epsilon)\in X_\lambda,\\ y+w+v+\delta & \text{if } k>0,\ k \text{ is odd and } (0,-\epsilon)\in X_\lambda,\\ y+w+v+\delta & \text{if } k=-1 \text{ and } (0,\pm1)\notin X_\lambda,\\ y+w+v+\delta+1 & \text{if } k=-1 \text{ and } (0,\pm1)\in X_\lambda\end{cases}\\ &=\begin{cases} s+l+1 & \text{if } k \text{ is even,}\\ s+l & \text{if } k>0 \text{ and } k \text{ is odd,}\\ s+l & \text{if } k=-1 \text{ and } (0,\pm1)\notin X_\lambda,\\ s+l+1 & \text{if } k=-1 \text{ and } (0,\pm1)\in X_\lambda.\end{cases}\end{aligned}$$

By Theorems 3.13, 3.14 (1) and Corollary 5.7, we see that

$$\begin{cases} \Theta_{r'-1,s'}(\pi\otimes\det^{-a})\neq0 & \text{if } \delta=1,\\ \Theta_{r',s'}(\pi\otimes\det^{-a})\neq0 & \text{otherwise,}\end{cases}$$

i.e., $\Theta_{r'-\delta,s'}(\pi \otimes \det^{-a}) \neq 0$. Hence $\Theta_{s',r'-\delta}(\pi^\vee \otimes \det^a \otimes \chi_V^2) \neq 0$ by Proposition 3.9. By the seesaw (Proposition 3.11 (2))

$$\begin{tikzcd} \mathrm{U}(p,q)\times\mathrm{U}(p,q) \arrow[d, dash] \arrow[dr, dash] & \mathrm{U}((r+2t+l)+s',(s+l)+(r'-\delta)) \arrow[d, dash] \arrow[dl, dash] \\ \mathrm{U}(p,q) & \mathrm{U}(r+2t+l,s+l)\times\mathrm{U}(s',r'-\delta), \end{tikzcd}$$

we deduce that

$$\begin{cases} \Theta_{(r+2t+l)+s',(s+l)+(r'-\delta)}(\det^a \cdot \chi_{V_{0,0}}) \neq 0 & \text{if } k \geq 0, \\ \Theta_{(r+1+2t+l)+s',(s+l)+(r'-\delta)}(\det^a \cdot \chi_{V_{0,0}}) \neq 0 & \text{if } k = -1. \end{cases}$$

Here, $\chi_{V_{0,,0}} = \chi_{V_{(r+2t+l)+s',(s+l)+(r'-\delta)}}$ if $k \geq 0$, and $\chi_{V_{0,,0}} = \chi_{V_{(r+1+2t+l)+s',(s+l)+(r'-\delta)}}$ if $k=-1$.

On the other hand, since

$$r+s = \begin{cases} n-k & \text{if } k \geq 0, \\ n & \text{if } k=-1, \end{cases}$$

we have

$$(r+2t+l)+s' = \begin{cases} (n-1)+2t+2l+2-k & \text{if } k \text{ is even}, \\ (n-1)+2t+2l-(k-1) & \text{if } k>0 \text{ and } k \text{ is odd}, \\ (n-1)+2t+2l+1 & \text{if } k=-1 \text{ and } (0,\pm1) \notin X_\lambda, \\ (n-2)+2t+2l+3 & \text{if } k=-1 \text{ and } (0,\pm1) \in X_\lambda, \end{cases}$$

$$(s+l)+(r'-\delta) = \begin{cases} n-2 & \text{if } k=-1 \text{ and } (0,\pm1) \in X_\lambda, \\ n-1 & \text{otherwise}. \end{cases}$$

In particular, $\min\{(r+2t+l)+s',(s+l)+(r'-\delta)\} = (s+l)+(r'-\delta) < n$ and

$$((r+2t+l)+s') - ((s+l)+(r'-\delta)) = \begin{cases} 2t+2l+2-k & \text{if } k \text{ is even}, \\ 2t+2l-(k-1) & \text{if } k>0 \text{ and } k \text{ is odd}, \\ 2t+2l+1 & \text{if } k=-1 \text{ and } (0,\pm1) \notin X_\lambda, \\ 2t+2l+3 & \text{if } k=-1 \text{ and } (0,\pm1) \in X_\lambda. \end{cases}$$

Moreover, since $\epsilon\alpha_{l+1} + (k-1)/2 < t+l$ and $\epsilon a > 0$, we have

$$\begin{cases} 0 < \epsilon a < t+l+1-\dfrac{k}{2} & \text{if } k \text{ is even}, \\ 0 < \epsilon a < t+l-\dfrac{k-1}{2} & \text{if } k \text{ is odd}. \end{cases}$$

By Proposition 3.10, we must have

$$\begin{cases} \Theta_{(r+2t+l)+s',(s+l)+(r'-\delta)}(\det^a \cdot \chi_{V_{0,0}}) = 0 & \text{if } k \geq 0, \\ \Theta_{(r+1+2t+l)+s',(s+l)+(r'-\delta)}(\det^a \cdot \chi_{V_{0,0}}) = 0 & \text{if } k = -1. \end{cases}$$

We obtain a contradiction. $\square$

By a similar argument, we obtain the following.

Proposition 5.9. *Let $\lambda = (\phi,\eta)$ be a pair of $\phi \in \Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$ and $\eta \in \widehat{A_\phi}$. Set $\pi = \pi(\phi,\eta)$. Let $k = k_\lambda$, $r = r_\lambda$ and $s = s_\lambda$ be as in Definition 4.1.*

1. *Suppose that $k=-1$. If $\Theta_{r+2t+l+1,s+l}(\pi) \neq 0$ for some $t \geq 1$, then $l \geq 0$.*
2. *Suppose that $k \geq 0$. If $\Theta_{r+2t+l,s+l}(\pi) \neq 0$ for some $t \geq 1$, then $l \geq k$.*

Proof. We give an outline of the proof. Suppose that

$$\begin{cases} \Theta_{r+2t+l,s+l}(\pi) \neq 0 & \text{if } k \geq 0, \\ \Theta_{r+1+2t+l,s+l}(\pi) \neq 0 & \text{if } k=-1 \end{cases}$$

for some $t \geq 1$. Set

$$a = \begin{cases} \dfrac{k}{2}\epsilon & \text{if } k \text{ is even, } k > 0 \text{ and } \left(\dfrac{k-1}{2}, \epsilon\right) \in X_\lambda, \\ \dfrac{k-1}{2}\epsilon & \text{if } k \text{ is odd, } k > 0 \text{ and } \left(\dfrac{k-1}{2}, \epsilon\right) \in X_\lambda, \\ 0 & \text{if } k = -1 \text{ or } k = 0. \end{cases}$$

By Theorems 3.13, 3.14 (1) and Corollary 5.7, we have

$$\begin{cases} \Theta_{r,s+k}(\pi \otimes \det^{-a}) \neq 0 & \text{if } k \text{ is even,} \\ \Theta_{r,s+k-1}(\pi \otimes \det^{-a}) \neq 0 & \text{if } k \text{ is odd and } k > 0, \\ \Theta_{r,s-1}(\pi \otimes \det^{-a}) \neq 0 & \text{if } k = -1 \text{ and } (0,\pm 1) \notin X_\lambda \text{ but } \chi_V \subset \phi, \\ \Theta_{r,s+1}(\pi \otimes \det^{-a}) \neq 0 & \text{if } k = -1 \text{ and } (0,\pm 1) \in X_\lambda \text{ (so } \chi_V \subset \phi). \end{cases}$$

By a similar argument to the proof of Proposition 5.8, a seesaw identity (Proposition 3.11 (2)) and Proposition 3.10 imply that $l \geq \max\{0, k\}$. We omit the detail. □

By Propositions 5.8 and 5.9, we have the necessary conditions.

Corollary 5.10. *Let $\lambda = (\phi, \eta)$ be a pair of $\phi \in \Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$ and $\eta \in \widehat{A_\phi}$. Set $\pi = \pi(\phi, \eta)$. Let $k = k_\lambda$, $r = r_\lambda$, $s = s_\lambda$, X_λ and $\mathcal{C}^\epsilon_\lambda(T)$ be as in Definition 4.1. Let t be a positive integer.*

(1) *Suppose that $k = -1$. If $\Theta_{r+2t+l+1,s+l}(\pi) \neq 0$, then $l \geq 0$ and $\mathcal{C}^\epsilon_\lambda(t+l) \leq l$ for each $\epsilon \in \{\pm 1\}$.*
(2) *Suppose that $k \geq 0$. If $\Theta_{r+2t+l,s+l}(\pi) \neq 0$, then $l \geq k$ and $\mathcal{C}^\epsilon_\lambda(t+l) \leq l$ for each $\epsilon \in \{\pm 1\}$.*

5.4. **Going-down towers.** By Corollaries 5.6 and 5.10, we can determine the first occurrence indices $m_d(\pi)$ of the d-th Witt tower of theta lifts of $\pi = \pi(\phi, \eta)$ when $d - (r_\lambda - s_\lambda) > 1$ with $\lambda = (\phi, \eta)$. By Lemma 4.4, we can also determine $m_d(\pi)$ when $d - (r_\lambda - s_\lambda) < -1$. In particular, if $|d - (r_\lambda - s_\lambda)| > 1$, then

$$m_d(\pi) \geq n + 2.$$

In this case, we call the d-th Witt tower a going-up tower with respect to π. When $|d-(r_\lambda - s_\lambda)| \leq 1$, we call the d-th Witt tower a going-down tower with respect to π. By the conservation relation, we can determine the first occurrence indices of the going-down Witt towers.

Proposition 5.11. *Assume the local Gan–Gross–Prasad conjecture (Conjecture 2.2). Let $\lambda = (\phi, \eta)$ be a pair of $\phi \in \Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$ and $\eta \in \widehat{A_\phi}$. Set $\pi = \pi(\phi, \eta)$. Let $k = k_\lambda$, $r = r_\lambda$, $s = s_\lambda$ and X_λ be as in Definition 4.1.*

(1) *Suppose that $k = -1$. Then for an integer l, we have*

$$\Theta_{r+1+l,s+l}(\pi) \neq 0 \iff \begin{cases} l \geq 0 & \text{if } \phi \text{ does not contain } \chi_V, \\ l \geq -1 & \text{if } \phi \text{ contains } \chi_V \text{ but } (0,\pm 1) \notin X_\lambda, \\ l \geq 1 & \text{if } \phi \text{ contains } \chi_V \text{ and } (0,\pm 1) \in X_\lambda, \end{cases}$$

and

$$\Theta_{r+l,s+1+l}(\pi) \neq 0 \iff \begin{cases} l \geq 0 & \text{if } \phi \text{ does not contain } \chi_V, \\ l \geq 1 & \text{if } \phi \text{ contains } \chi_V \text{ but } (0,\pm 1) \notin X_\lambda, \\ l \geq -1 & \text{if } \phi \text{ contains } \chi_V \text{ and } (0,\pm 1) \in X_\lambda. \end{cases}$$

(2) *Suppose that $k \geq 0$. Consider the following three conditions on $\lambda = (\phi, \eta)$:*

(chain condition 2): *$\phi\chi_V^{-1}$ contains both χ_{k+1} and $\chi_{-(k+1)}$, so that*

$$\phi\chi_V^{-1} \supset \underbrace{\chi_{k+1} + \chi_{k-1} + \cdots + \chi_{-(k-1)} + \chi_{-(k+1)}}_{k+2};$$

(even-ness condition): *at least one of χ_{k+1} and $\chi_{-(k+1)}$ is contained in $\phi\chi_V^{-1}$ with even multiplicity;*

(alternating condition 2): *$\eta(e_{V,k+1-2i}) \neq \eta(e_{V,k-1-2i})$ for $i = 0, \ldots, k$.*

Then for an integer l, we have

$$\Theta_{r+l,s+l}(\pi) \neq 0 \iff \begin{cases} l \geq -1 & \text{if } \lambda \text{ satisfies these three conditions,} \\ l \geq 0 & \text{otherwise.} \end{cases}$$

When $\phi \in \Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R}))$, this proposition has been proven by Paul [P2, Proposition 3.4] in terms of Harish-Chandra parameters (not using the Gan–Gross–Prasad conjecture).

Proof of Proposition 5.11. We show (1). Suppose that $k = -1$. If ϕ does not contain χ_V, then by Theorem 3.14 (1), we have $\Theta_{r+1,s}(\pi) \neq 0$ and $\Theta_{r,s+1}(\pi) \neq 0$. By the tower property (Proposition 3.6) and the conservation relation (Theorem 3.7), we see that $\Theta_{r+1+l,s+l}(\pi) \neq 0 \iff l \geq 0$, and $\Theta_{r+l,s+1+l}(\pi) \neq 0 \iff l \geq 0$.

Now we assume that ϕ contains χ_V (with even multiplicity). Then by Corollary 5.7, we have

$$\begin{cases} \Theta_{r,s-1}(\pi) \neq 0 & \text{if } (0, \pm 1) \notin X_\lambda, \\ \Theta_{r-1,s}(\pi) \neq 0 & \text{if } (0, \pm 1) \in X_\lambda. \end{cases}$$

When $(0, \pm 1) \in X_\lambda$, by Corollary 5.6, $\Theta_{r+2,s+1}(\pi) \neq 0$ if $\mathcal{C}^\epsilon_\lambda(1) = \emptyset$ for each $\epsilon \in \{\pm 1\}$. By Definition 4.1 (5), $\mathcal{C}^\epsilon_\lambda(1) = \emptyset$ if and only if $(\epsilon, \epsilon) \notin X^{(\infty)}_\lambda$. However, since $(0, \pm 1) \in X_\lambda$, by Definition 4.1 (4), we see that $X^{(\infty)}_\lambda$ cannot contain (ϵ, ϵ) for each $\epsilon \in \{\pm 1\}$. Hence we deduce that $\Theta_{r+2,s+1}(\pi) \neq 0$. By the tower property (Proposition 3.6) and the conservation relation (Theorem 3.7), we see that $\Theta_{r+1+l,s+l}(\pi) \neq 0 \iff l \geq 1$, and $\Theta_{r+l,s+1+l}(\pi) \neq 0 \iff l \geq -1$.

Now suppose that ϕ contains χ_V (with even multiplicity) but $(0, \pm 1) \notin X_\lambda$. Set $\lambda^\vee = (\phi^\vee \otimes \chi_V^2, \eta^\vee)$. By Lemma 4.4, we have $(0, \pm 1) \in X_{\lambda^\vee}$ so that $\Theta_{s+2,r+1}(\pi^\vee \otimes \chi_V^2) \neq 0$ by the above case. By Proposition 3.9, we deduce that $\Theta_{r+1,s+2}(\pi) \neq 0$. By the tower property (Proposition 3.6) and the conservation relation (Theorem 3.7), we see that $\Theta_{r+1+l,s+l}(\pi) \neq 0 \iff l \geq -1$, and $\Theta_{r+l,s+1+l}(\pi) \neq 0 \iff l \geq 1$. This completes the proof of (1).

We show (2). Suppose that $k \geq 0$. By Corollaries 5.6, 5.10 and Proposition 3.9, we see that

$$m_d(\pi) \geq r + s + 2k + 2 = n + k + 2$$

for any integer d such that $d \neq r - s$. Hence $\min\{m_+(\pi), m_-(\pi)\} = m_{r-s}(\pi)$. Moreover, if $|d - (r - s)| > 2$, then $m_d(\pi) \geq n + k + 4$. We compute $m_{r-s+2}(\pi)$ and $m_{r-s-2}(\pi)$. By Corollaries 5.6 and 5.10, $\Theta_{r+2+k,s+k}(\pi) \neq 0$ if and only if $\#\mathcal{C}^\epsilon_\lambda(1 + k) \leq k$ for each $\epsilon \in \{\pm 1\}$.

First, we consider the case where $k = 0$. Then by Definition 4.1 (5), $\mathcal{C}^\epsilon_\lambda(1) = \emptyset$ if and only if $(\epsilon/2, \epsilon) \notin X^{(\infty)}_\lambda$. Also by Definition 4.1 (4), $(\epsilon/2, \epsilon) \in X^{(\infty)}_\lambda$ if and only if $(\epsilon/2, \epsilon) \in X_\lambda$. Hence

$$\Theta_{r+2,s}(\pi) \neq 0 \iff \left(\frac{\epsilon}{2}, \epsilon\right) \notin X_\lambda \text{ for each } \epsilon \in \{\pm 1\}.$$

Similarly, by using Proposition 3.9, we see that

$$\Theta_{r,s+2}(\pi) \neq 0 \iff \left(\frac{\epsilon}{2}, \epsilon\right) \notin X_{\lambda^\vee} \text{ for each } \epsilon \in \{\pm 1\}.$$

Now we assume that both $\Theta_{r+2,s}(\pi)$ and $\Theta_{r,s+2}(\pi)$ are zero. By Lemma 4.4 (4), this condition is equivalent to saying that $(1/2, +1) \in X_\lambda \cap X_{\lambda^\vee}$ or $(-1/2, -1) \in X_\lambda \cap X_{\lambda^\vee}$. We check (chain condition 2) and (even-ness condition). For $\epsilon \in \{\pm 1\}$, if $\chi_{-\epsilon}$ were not contained in $\phi\chi_V^{-1}$, then we must have $(\epsilon/2, \epsilon) \in X_\lambda$ and $(-\epsilon/2, -\epsilon) \in X_{\lambda^\vee}$. This contradicts Lemma 4.4 (4). Hence $\phi\chi_V^{-1}$ contains both χ_1 and χ_{-1}. If both χ_1 and χ_{-1} were contained in $\phi\chi_V^{-1}$ with odd multiplicities, then by Lemma 4.4 (2), there must be $\epsilon \in \{\pm 1\}$ such that $(\epsilon/2, \epsilon) \in X_\lambda \cap X_{\lambda^\vee}$. This implies that $(1/2, \epsilon), (-1/2, \epsilon) \in X_\lambda$, which contradicts that $k_\lambda = 0$ (see Definition 4.1 (1)).

We claim that under assuming (chain condition 2) and (even-ness condition), both $\Theta_{r+2,s}(\pi)$ and $\Theta_{r,s+2}(\pi)$ are zero if and only if λ satisfies (alternating condition 2), which is equivalent to saying that $\eta(e_{V,1}) \neq \eta(e_{V,-1})$. Replacing λ with $\lambda^\vee$ if necessary, we may assume that χ_1 appears in $\phi\chi_V^{-1}$ with even multiplicity. Write

$$\phi\chi_V^{-1} = \chi_{2\alpha_1} + \cdots + \chi_{2\alpha_u} + (\xi_1 + \cdots + \xi_v) + ({}^c\xi_1^{-1} + \cdots + {}^c\xi_v^{-1}),$$

where

- $\alpha_i \in \frac{1}{2}\mathbb{Z}$ such that $2\alpha_i \equiv \kappa - 1 \bmod 2$ and $\alpha_1 > \cdots > \alpha_u$;
- ξ_i is a unitary character of $\mathbb{C}^\times$ (which can be of the form $\chi_{2\alpha}$);
- $u + 2v = n$.

Then we see that

$$\begin{aligned} \left(\frac{1}{2}, +1\right) \in X_\lambda &\iff \eta(e_{V,1}) = (-1)^{\#\{i \in \{1,\dots,u\} \ \mid\ \alpha_i > 1/2\}+1}, \\ \left(-\frac{1}{2}, -1\right) \in X_{\lambda^\vee} &\iff \eta(e_{V,1}) = (-1)^{\#\{i \in \{1,\dots,u\} \ \mid\ \alpha_i > 1/2\}}, \\ \left(-\frac{1}{2}, -1\right) \in X_\lambda &\iff \eta(e_{V,-1}) = (-1)^{\#\{i \in \{1,\dots,u\} \ \mid\ \alpha_i > -1/2\}+1}, \\ \left(\frac{1}{2}, +1\right) \in X_{\lambda^\vee} &\iff \eta(e_{V,-1}) = (-1)^{\#\{i \in \{1,\dots,u\} \ \mid\ \alpha_i > -1/2\}}. \end{aligned}$$

In particular, both $\Theta_{r+2,s}(\pi)$ and $\Theta_{r,s+2}(\pi)$ are zero, i.e., $(1/2, +1) \in X_\lambda \cap X_{\lambda^\vee}$ or $(-1/2, -1) \in X_\lambda \cap X_{\lambda^\vee}$ if and only if

$$\begin{cases} \eta(e_{V,1}) = (-1)^{\#\{i \in \{1,\dots,u\} \ \mid\ \alpha_i > 1/2\}+1}, \\ \eta(e_{V,-1}) = (-1)^{\#\{i \in \{1,\dots,u\} \ \mid\ \alpha_i > -1/2\}}, \end{cases} \quad \text{or} \quad \begin{cases} \eta(e_{V,1}) = (-1)^{\#\{i \in \{1,\dots,u\} \ \mid\ \alpha_i > 1/2\}}, \\ \eta(e_{V,-1}) = (-1)^{\#\{i \in \{1,\dots,u\} \ \mid\ \alpha_i > -1/2\}+1}. \end{cases}$$

Since $1/2 \notin \{\alpha_1, \dots, \alpha_u\}$, we see that

$$\#\{i \in \{1,\dots,u\} \mid \alpha_i > 1/2\} = \#\{i \in \{1,\dots,u\} \mid \alpha_i > -1/2\}.$$

Hence under assuming (chain condition 2) and (even-ness condition), both $\Theta_{r+2,s}(\pi)$ and $\Theta_{r,s+2}(\pi)$ are zero if and only if $\eta(e_{V,1}) \neq \eta(e_{V,-1})$.

Similarly, when $k > 0$, we see that

$$\Theta_{r+2+k,s+k}(\pi) \neq 0 \iff \left(\frac{k+1}{2}\epsilon, \epsilon\right) \notin X_\lambda$$

and

$$\Theta_{r+k,s+2+k}(\pi) \neq 0 \iff \left(\frac{k+1}{2}\epsilon, \epsilon\right) \notin X_{\lambda^\vee},$$

where ϵ is the unique element in $\{\pm 1\}$ such that $((k-1)/2, \epsilon), \dots, (-(k-1)/2, \epsilon) \in X_\lambda$. Also, it is easy to see that if both $\Theta_{r+2+k,s+k}(\pi)$ and $\Theta_{r+k,s+2+k}(\pi)$ are zero, then λ satisfies (chain condition 2) and (even-ness condition). Furthermore, when χ_{k+1} appears in $\phi\chi_V^{-1}$ with even multiplicity, we see that

$$\begin{aligned} \left(\frac{k+1}{2}\epsilon, \epsilon\right) \in X_\lambda &\iff \eta(e_{V,(k+1)\epsilon}) = (-1)^{\#\{i \in \{1,\dots,u\} \ \mid\ \alpha_i > (k+1)\epsilon/2\}+1}, \\ \left(\frac{k+1}{2}\epsilon, \epsilon\right) \in X_{\lambda^\vee} &\iff \eta(e_{V,-(k+1)\epsilon}) = (-1)^{\#\{i \in \{1,\dots,u\} \ \mid\ \alpha_i > -(k+1)\epsilon/2\}}. \end{aligned}$$

Since $((k-1)/2, \epsilon), \dots, (-(k-1)/2, \epsilon) \in X_\lambda$, we have

$$\eta(e_{V,k+1-2j}) = \epsilon(-1)^{\#\{i \in \{1,\dots,u\} \ \mid\ \alpha_i > (k+1-2j)\epsilon/2\}}$$

for any $j = 1, \dots, k$. Since χ_{k+1-2j} appears in $\phi\chi_V^{-1}$ with odd multiplicity for any $j = 1, \dots, k$ (see (odd-ness condition) in Definition 4.1 (1)), we see that

$$\#\{i \in \{1,\dots,u\} \mid \alpha_i > (k+1-2j)\epsilon/2\} - \#\{i \in \{1,\dots,u\} \mid \alpha_i > (k-1-2j)\epsilon/2\} = -1$$

for any $j = 1, \dots, k-1$. Hence under assuming (chain condition 2) and (even-ness condition), both $\Theta_{r+2+k,s}(\pi)$ and $\Theta_{r,s+2+k}(\pi)$ are zero if and only if

$$\eta(e_{V,k+1}) \neq \eta(e_{V,k-1}) \neq \cdots \neq \eta(e_{V,-(k-1)}) \neq \eta(e_{V,-(k+1)}),$$

which is (alternating condition 2).

We have shown that for $k \geq 0$, both $\Theta_{r+2+k,s}(\pi)$ and $\Theta_{r,s+2+k}(\pi)$ are zero if and only if λ satisfies (chain condition 2), (even-ness condition) and (alternating condition 2). In this case, there exists $\epsilon \in \{\pm 1\}$

such that $\chi_{(k+1)\epsilon}$ is contained in $\phi\chi_V^{-1}$ with even multiplicity. Suppose that $((k+1)\epsilon/2, \pm 1) \in X_\lambda$ (so that $((k-1)/2, \epsilon), \dots, (-(k-1)/2, \epsilon) \in X_\lambda$ when $k > 0$). Then by Definition 4.1 (4) and (5), we see that $((k+3)\epsilon/2, \epsilon) \notin X_\lambda^{(\infty)}$, so that

$$\mathcal{C}_\lambda^\epsilon(k+2) = \left\{ \left(\frac{k+1}{2}\epsilon, \epsilon\right), \left(\frac{k-1}{2}\epsilon, \epsilon\right), \dots, \left(-\frac{k-1}{2}\epsilon, \epsilon\right) \right\}.$$

Hence $\#\mathcal{C}_\lambda^\epsilon(k+2) = k+1$. Moreover, since $((k+1)\epsilon/2, \epsilon) \in X_{\lambda^\vee}$ so that $(-(k+1)\epsilon/2, -\epsilon) \notin X_\lambda$ by Lemma 4.4 (4), we see that $\#\mathcal{C}_\lambda^{-\epsilon}(k+2) \leq k+1$. Hence by Corollary 5.6, we have $\Theta_{r+3+k,s+1+k}(\pi) \neq 0$. Similarly, if $((k+1)\epsilon/2, \pm 1) \notin X_\lambda$ (so that $((k-1)/2, -\epsilon), \dots, (-(k-1)/2, -\epsilon) \in X_\lambda$ when $k > 0$), then $(-(k+1)\epsilon/2, \pm 1) \in X_{\lambda^\vee}$, so that we have $\Theta_{r+1+k,s+3+k}(\pi) \neq 0$. In any case, we have

$$\min\{m_{r-s+2}(\pi), m_{r-s-2}(\pi)\} = r + s + 4 + 2k = n + 4 + k.$$

By the conservation relation (Theorem 3.7), we conclude that

$$m_{r-s}(\pi) = \begin{cases} n-2-k & \text{if } \lambda \text{ satisfies the three conditions,} \\ n-k & \text{otherwise.} \end{cases}$$

By the tower property (Proposition 3.6), we obtain (2). □

By Corollaries 5.6, 5.10 and Proposition, 5.11, we obtain Theorem 4.2.

Appendix A. Explicit local Langlands correspondence for discrete series representations

In this appendix, we review the local Langlands correspondence established by Langlands himself [L], Vogan [V3] and Shelstad [S1, S2, S3], and explain the relation between the Harish-Chandra parameters and L-parameters for discrete series representations of unitary groups.

A.1. **Weil groups and representations.** Recall that the Weil group $W_\mathbb{R}$ of $\mathbb{R}$ (resp. $W_\mathbb{C}$ of $\mathbb{C}$) is defined by

$$W_\mathbb{C} = \mathbb{C}^\times, \quad W_\mathbb{R} = \mathbb{C}^\times \cup \mathbb{C}^\times j$$

with

$$j^2 = -1 \in \mathbb{C}^\times, \quad jzj^{-1} = \overline{z}$$

for $z \in \mathbb{C}^\times \subset W_\mathbb{R}$. Then we have an exact sequence

$$1 \longrightarrow W_\mathbb{C} \longrightarrow W_\mathbb{R} \longrightarrow \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \longrightarrow 1,$$

where $W_\mathbb{R} \to \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is defined so that $j \mapsto$ (the complex conjugate) $\in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Also, the map

$$j \mapsto -1, \quad \mathbb{C}^\times \ni z \mapsto z\overline{z}$$

gives an isomorphism $W_\mathbb{R}^{\mathrm{ab}} \to \mathbb{R}^\times$.

For $F = \mathbb{R}$ or $F = \mathbb{C}$, a representation of W_F is a semisimple continuous homomorphism $\varphi\colon W_F \to \mathrm{GL}_n(\mathbb{C})$. Hence φ decomposes into a direct sum of irreducible representations.

For $2\alpha \in \mathbb{Z}$ (i.e., $\alpha \in \frac{1}{2}\mathbb{Z}$), we define a character $\chi_{2\alpha}$ of $W_\mathbb{C} = \mathbb{C}^\times$ by

$$\chi_{2\alpha}(z) = \overline{z}^{-2\alpha}(z\overline{z})^\alpha$$

for $z \in \mathbb{C}^\times$. A representation $\phi\colon W_\mathbb{C} \to \mathrm{GL}_n(\mathbb{C})$ is called conjugate self-dual of sign $b \in \{\pm 1\}$ if there exists a non-degenerate bilinear form $B\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ such that

$$\begin{cases} B(\phi(z)x, \phi(\overline{z})y) = B(x, y), \\ B(y, \phi(-1)x) = b \cdot B(x, y) \end{cases}$$

for $x, y \in \mathbb{C}^n$ and $z \in W_\mathbb{C} = \mathbb{C}^\times$. Such a representation ϕ is of the form

$$\phi = \chi_{2\alpha_1} + \dots + \chi_{2\alpha_u} + (\xi_1 + \dots + \xi_v) + ({}^c\xi_1^{-1} + \dots + {}^c\xi_v^{-1}),$$

where

- $\alpha_i \in \frac{1}{2}\mathbb{Z}$ such that $(-1)^{2\alpha_i} = b$;
- ξ_i is a character of $\mathbb{C}^\times$ (which can be of the form $\chi_{2\alpha}$);

- $u + 2v = n$.

For more precisions, see e.g., [GGP1, §3].

A.2. **L-groups and local Langlands correspondence for unitary groups.** Let $G = \mathrm{U}_n$ be a unitary group of size n, which is regarded as a connected reductive algebraic group over $\mathbb{R}$. Hence $G(\mathbb{R}) = \mathrm{U}(p,q)$ for some (p,q) such that $p+q=n$. Its dual group $\widehat{G}$ is isomorphic to $\mathrm{GL}_n(\mathbb{C})$. The Weil group $W_\mathbb{R} = \mathbb{C}^\times \cup \mathbb{C}^\times j$ acts on $\widehat{G} = \mathrm{GL}_n(\mathbb{C})$ as follows: $\mathbb{C}^\times$ acts trivially, and j acts by

$$j\colon \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C}),\ g \mapsto \begin{pmatrix} & & 1 \\ & \iddots & \\ (-1)^{n-1} & & \end{pmatrix} {}^t g^{-1} \begin{pmatrix} & & 1 \\ & \iddots & \\ (-1)^{n-1} & & \end{pmatrix}^{-1}.$$

The L-group of G is the semi-direct product ${}^LG = \widehat{G} \rtimes W_\mathbb{R} = \mathrm{GL}_n(\mathbb{C}) \rtimes W_\mathbb{R}$.

An admissible homomorphism of $G(\mathbb{R}) = \mathrm{U}_n(\mathbb{R})$ is a homomorphism $\varphi\colon W_\mathbb{R} \to {}^LG$ such that the composition

$$\mathrm{pr} \circ \varphi\colon W_\mathbb{R} \to \mathrm{GL}_n(\mathbb{C}) \rtimes W_\mathbb{R} \twoheadrightarrow W_\mathbb{R}$$

is identity and the restriction of φ to $W_\mathbb{C} = \mathbb{C}^\times$ is continuous and semisimple. Let $\Phi(\mathrm{U}_n(\mathbb{R}))$ be the set of $\widehat{G}$-conjugacy classes of admissible homomorphisms of $\mathrm{U}_n(\mathbb{R})$. For $\varphi \in \Phi(\mathrm{U}_n(\mathbb{R}))$, we define the component group A_φ of φ by

$$A_\varphi = \pi_0(\mathrm{Cent}(\mathrm{Im}(\varphi), \widehat{G})).$$

This is an elementary two abelian group. For $\varphi \in \Phi(\mathrm{U}_n(\mathbb{R}))$, the restriction of φ gives a conjugate self-dual representation $\phi = \varphi|\mathbb{C}^\times$ of $W_\mathbb{C}$ of dimension n and sign $(-1)^{n-1}$. Via the map $\varphi \mapsto \phi = \varphi|\mathbb{C}^\times$, we obtain an identification

$$\Phi(\mathrm{U}_n(\mathbb{R})) = \{\text{conjugate self-dual representations of } W_\mathbb{C} \text{ of dimension } n \text{ and sign } (-1)^{n-1}\}.$$

When $\phi = \varphi|\mathbb{C}^\times$, we also put $A_\phi = A_\varphi$.

We say that $\phi \in \Phi(\mathrm{U}_n(\mathbb{R}))$ is discrete (resp. tempered) if ϕ is of the form $\phi = \chi_{2\alpha_1} \oplus \cdots \oplus \chi_{2\alpha_n}$ with $2\alpha_i \equiv n-1 \bmod 2$ and $\alpha_1 > \cdots > \alpha_n$ (resp. ϕ is a direct sum of unitary characters of $\mathbb{C}^\times$). We denote the subset of $\Phi(\mathrm{U}_n(\mathbb{R}))$ consisting of discrete elements (resp. tempered elements) by $\Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R}))$ (resp. $\Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$).

When $\phi = \varphi|\mathbb{C}^\times = \chi_{2\alpha_1} \oplus \cdots \oplus \chi_{2\alpha_n} \in \Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R}))$, there exists a unique semisimple element $s_{2\alpha_i} \in \mathrm{Cent}(\mathrm{Im}(\varphi), \mathrm{GL}_n(\mathbb{C}))$ such that $W_\mathbb{C}$ acts on the (-1)-eigenspace of $s_{2\alpha_i}$ by $\chi_{2\alpha_i}$. Let $e_{2\alpha_i}$ be the image of $s_{2\alpha_i}$ in $A_\phi = \pi_0(\mathrm{Cent}(\mathrm{Im}(\varphi), \mathrm{GL}_n(\mathbb{C})))$. Then we have

$$A_\phi = (\mathbb{Z}/2\mathbb{Z})e_{2\alpha_1} \oplus \cdots \oplus (\mathbb{Z}/2\mathbb{Z})e_{2\alpha_n}.$$

For more precisions, see [GGP1, §4]. In particular, we have $|A_\phi| = 2^n$ for each $\phi \in \Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R}))$.

The local Langlands correspondence for unitary groups is as follows:

Theorem A.1. (1) *There exists a canonical surjection*

$$\bigsqcup_{p+q=n} \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(p,q)) \to \Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R})).$$

For $\phi \in \Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$, we denote by Π_ϕ the inverse image of ϕ under this map, and call Π_ϕ the L-packet associated to ϕ.

(2) $\#\Pi_\phi = \#A_\phi$;

(3) *$\pi \in \Pi_\phi$ is discrete series if and only if ϕ is discrete.*

(4) *The map $\pi \mapsto \phi$ is compatible with parabolic inductions (c.f., Theorem 2.1 (5)).*

(5) *If $\phi = \chi_{2\alpha_1} \oplus \cdots \oplus \chi_{2\alpha_n} \in \Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R}))$, then Π_ϕ is the set of all discrete series representations of various $\mathrm{U}(p,q)$ whose infinitesimal characters are equal to $(\alpha_1, \ldots, \alpha_n)$ via the Harish-Chandra map.*

Note that there exist exactly $(p+q)!/(p! \cdot q!)$ discrete series representations of $\mathrm{U}(p,q)$ with a given infinitesimal character. Theorem 2.1 (2) and (5) are compatible with the well-known equation

$$\sum_{p+q=n} \frac{(p+q)!}{p! \cdot q!} = 2^n.$$

A.3. **Whittaker data and generic representations.** The unitary group $\mathrm{U}(p,q)$ with $p+q=n$ is quasi-split if and only if $|p-q|\leq 1$. For such (p,q), a Whittaker datum of $\mathrm{U}(p,q)$ is the conjugacy class of pairs $\mathfrak{w}=(B,\mu)$, where $B=TU$ is an $\mathbb{R}$-rational Borel subgroup of $\mathrm{U}(p,q)$, and $\mu\colon U(\mathbb{R})\to\mathbb{C}^\times$ is a unitary generic character. Here, T is a maximal $\mathbb{R}$-torus of B and U is the unipotent radical of B, and $T(\mathbb{R})$ acts on $U(\mathbb{R})$ by conjugation. A unitary character μ of $U(\mathbb{R})$ is called generic if the stabilizer of μ in $T(\mathbb{R})$ is equal to the center $Z(\mathbb{R})$ of $\mathrm{U}(p,q)$. We say that $\pi\in\mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(p,q))$ is $\mathfrak{w}$-generic if

$$\mathrm{Hom}_{\mathrm{U}(p,q)}(\pi, C^\infty_{\mathrm{mg}}(U(\mathbb{R})\backslash\mathrm{U}(p,q),\mu))\neq 0,$$

where $C^\infty_{\mathrm{mg}}(U(\mathbb{R})\backslash\mathrm{U}(p,q),\mu)$ is the set of C^∞-functions $W\colon\mathrm{U}(p,q)\to\mathbb{C}$ of moderate growth satisfying that $W(ug)=\mu(u)W(g)$ for $u\in U(\mathbb{R})$ and $g\in\mathrm{U}(p,q)$, and $\mathrm{U}(p,q)$ acts on $C^\infty_{\mathrm{mg}}(U(\mathbb{R})\backslash\mathrm{U}(p,q),\mu)$ by the right translation.

For each $\phi\in\Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$, the L-packet Π_ϕ is parametrized by the Pontryagin dual $\widehat{A_\phi}$ of A_ϕ if a quasi-split form $\mathrm{U}_n(\mathbb{R})$ and its Whittaker datum are fixed.

Theorem A.2. *Fix (p,q) such that $p+q=n$ and $|p-q|\leq 1$, and a Whittaker datum $\mathfrak{w}$ of $\mathrm{U}(p,q)$. Then*

(1) *for $\phi\in\Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$, there exists a bijection*

$$J_{\mathfrak{w}}\colon\Pi_\phi\to\widehat{A_\phi}$$

which satisfies certain character identities;

(2) *for each $\phi\in\Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$, the L-packet Π_ϕ has a unique $\mathfrak{w}$-generic representation $\pi_{\mathfrak{w}}$;*

(3) *in particular, the bijection $J_{\mathfrak{w}}$ requires satisfying that $J_{\mathfrak{w}}(\pi_{\mathfrak{w}})$ is the trivial character of A_ϕ.*

In the next subsection, we will review the definition of $J_{\mathfrak{w}}$ when $\phi\in\Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R}))$, and give an explicit relation between $J_{\mathfrak{w}}(\pi)$ and the Harish-Chandra parameter of π for $\pi\in\Pi_\phi$. To give a such relation, we need to specify which representation is $\mathfrak{w}$-generic.

Fix (p,q) such that $p+q=n$ and $|p-q|\leq 1$. By [V1, §6, Theorem 6.2], for $\pi\in\mathrm{Irr}_{\mathrm{disc}}(\mathrm{U}(p,q))$ with Harish-Chandra parameter λ, the following are equivalent:

- π is large;
- π is $\mathfrak{w}$-generic for some Whittaker datum $\mathfrak{w}$ of $\mathrm{U}(p,q)$;
- all simple roots in Δ^+_λ are non-compact, i.e., do not belong to $\Delta(K_{p,q},T_{p,q})$.

Here,

- $K_{p,q}\cong\mathrm{U}(p)\times\mathrm{U}(q)$ is the usual maximal compact subgroup of $\mathrm{U}(p,q)$;
- $T_{p,q}$ is the usual maximal compact torus of $\mathrm{U}(p,q)$;
- $\Delta(\mathrm{U}(p,q),T_{p,q})$ (resp. $\Delta(K_{p,q},T_{p,q})$) is the set of roots of $T_{p,q}$ in $\mathrm{U}(p,q)$ (resp. in $K_{p,q}$);
- Δ^+_λ is the unique positive system of $\Delta(\mathrm{U}(p,q),T_{p,q})$ for which λ is dominant.

When n is odd, there exist exactly two quasi-split forms $\mathrm{U}((n+1)/2,(n-1)/2)$ and $\mathrm{U}((n-1)/2,(n+1)/2)$. For $\epsilon\in\{\pm1\}$, we put $(p_\epsilon,q_\epsilon)=((n+\epsilon)/2,(n-\epsilon)/2)$. Then there exists a unique Whittaker datum $\mathfrak{w}_\epsilon$ of $\mathrm{U}(p_\epsilon,q_\epsilon)$. For integers $\alpha_1>\cdots>\alpha_n$, it is easy to see that the Harish-Chandra parameter of the unique large discrete series representation $\pi_{\mathfrak{w}_\pm}$ of $\mathrm{U}(p_\pm,q_\pm)$ with infinitesimal character $(\alpha_1,\dots,\alpha_n)$ is given by

$$\begin{cases}\mathrm{HC}(\pi_{\mathfrak{w}_+})=(\alpha_1,\alpha_3,\dots,\alpha_n;\alpha_2,\alpha_4,\dots,\alpha_{n-1}),\\ \mathrm{HC}(\pi_{\mathfrak{w}_-})=(\alpha_2,\alpha_4,\dots,\alpha_{n-1};\alpha_1,\alpha_3,\dots,\alpha_n).\end{cases}$$

When $n=2m$ is even, $\mathrm{U}(m,m)$ is the unique quasi-split form of size n. It has exactly two Whittaker data $\mathfrak{w}_\pm$ constructed as follows. First, we set

$$G_m=\left\{g\in\mathrm{GL}_{2m}(\mathbb{C})\;\middle|\;{}^t\overline{g}\begin{pmatrix}\mathbf{1}_m&0\\0&-\mathbf{1}_m\end{pmatrix}g=\begin{pmatrix}\mathbf{1}_m&0\\0&-\mathbf{1}_m\end{pmatrix}\right\},$$

$$G'_m=\left\{g'\in\mathrm{GL}_{2m}(\mathbb{C})\;\middle|\;{}^t\overline{g'}\begin{pmatrix}&&1\\&\iddots&\\1&&\end{pmatrix}g'=\begin{pmatrix}&&1\\&\iddots&\\1&&\end{pmatrix}\right\}.$$

Note that G_m is the usual coordinate of $\mathrm{U}(m,m)$, and these two groups are isomorphic to each other. Put

$$T_m = \frac{1}{\sqrt{2}}\left(\begin{array}{ccc|ccc} 1 & & & & & -1 \\ & \ddots & & & \iddots & \\ & & 1 & -1 & & \\ \hline & & 1 & 1 & & \\ & \iddots & & & \ddots & \\ 1 & & & & & 1 \end{array}\right) \in \mathrm{GL}_{2m}(\mathbb{C}),$$

so that

$$T_m^{-1} = \frac{1}{\sqrt{2}}\left(\begin{array}{ccc|ccc} 1 & & & & & 1 \\ & \ddots & & & \iddots & \\ & & 1 & 1 & & \\ \hline & & -1 & 1 & & \\ & \iddots & & & \ddots & \\ -1 & & & & & 1 \end{array}\right) \in \mathrm{GL}_{2m}(\mathbb{C}).$$

Then

$${}^t\overline{T_m}\begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} T_m = \begin{pmatrix} \mathbf{1}_m & 0 \\ 0 & -\mathbf{1}_m \end{pmatrix},$$

so that the map

$$f_{T_m} : G_m \to G'_m,\ g \mapsto g' := T_m g T_m^{-1}$$

gives an isomorphism. Let $B' = T'U'$ be the Borel subgroup of G'_m consisting of upper triangular matrices, where T' is the maximal torus of G'_m consisting of diagonal matrices and U' is the unipotent radical of B'. Define a generic character $\mu_\pm$ of $U'(\mathbb{R})$ by

$$\mu_\pm(u) = \exp(\mp\pi\sqrt{-1}\mathrm{tr}_{\mathbb{C}/\mathbb{R}}(\sqrt{-1}(u_{1,2} + \cdots + u_{m,m+1}))).$$

We set the Whittaker datum $\mathfrak{w}_\pm$ of G_m to be the conjugacy class of

$$\mathfrak{w}_\pm = (f_{T_m}^{-1}(B'), \mu_\pm \circ f_{T_m}).$$

Fix half-integers $\alpha_1 > \cdots > \alpha_n$. Let π and π' be the discrete series representations with Harish-Chandra parameters λ and λ' given by

$$\begin{cases} \lambda = (\alpha_1, \alpha_3, \dots, \alpha_{n-1}; \alpha_2, \alpha_4, \dots, \alpha_n), \\ \lambda' = (\alpha_2, \alpha_4, \dots, \alpha_n; \alpha_1, \alpha_3, \dots, \alpha_{n-1}), \end{cases}$$

respectively. Then π and π' are the two large discrete series representations with infinitesimal character $(\alpha_1, \dots, \alpha_n)$. Hence there exists $\epsilon \in \{\pm 1\}$ such that π (resp. π') is $\mathfrak{w}_\epsilon$-generic (resp. $\mathfrak{w}_{-\epsilon}$-generic).

To give an explicit description of the local Langlands correspondence for $\mathrm{U}_n(\mathbb{R})$, we have to determine ϵ. The following proposition says that $\epsilon = +1$. It seems to be well-known (c.f., [M2]), but we give a proof for the convenience of the reader.

Proposition A.3. *Assume that $n = 2m$ is even. Fixing half-integers $\alpha_1 > \cdots > \alpha_n$, we let π (resp. π') be the large discrete series representation of $\mathrm{U}(m,m)$ whose Harish-Chandra parameter λ (resp. λ') is given as above. Then π is $\mathfrak{w}_+$-generic (resp. π' is $\mathfrak{w}_-$-generic).*

Proof. We prove the proposition by induction on m. First suppose that $m = 1$ so that $G_1 = \mathrm{U}(1,1)$. Note that

$$\mathfrak{g} = \mathrm{Lie}(G_1) = \left\{ \begin{pmatrix} a\sqrt{-1} & b + c\sqrt{-1} \\ b - c\sqrt{-1} & d\sqrt{-1} \end{pmatrix} \in \mathrm{M}_2(\mathbb{C}) \;\middle|\; a, b, c, d \in \mathbb{R} \right\}.$$

Set
$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{g}_{\mathbb{C}} = \mathrm{M}_2(\mathbb{C}).$$
Then (H, X_+, X_-) is an $\mathfrak{sl}_2$-triple, i.e.,
$$[X_+, X_-] = H, \quad [H, X_+] = 2X_+, \quad [H, X_-] = -2X_-.$$
Set
$$n(z) = f_{T_1}^{-1}\left(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1+z/2 & z/2 \\ -z/2 & 1-z/2 \end{pmatrix}, \quad m(x) = f_{T_1}^{-1}\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} x+x^{-1} & -x+x^{-1} \\ -x+x^{-1} & x+x^{-1} \end{pmatrix}$$
for $z \in \sqrt{-1}\mathbb{R}$ and $x \in \mathbb{R}_{>0}$. By the Iwasawa decomposition, any $g \in G$ has a unique decomposition
$$g = n(z)m(x)\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$$
for $t_1, t_2 \in \mathbb{C}^1$, $z \in \sqrt{-1}\mathbb{R}$ and $x \in \mathbb{R}_{>0}$. Define a C^∞-function W of moderate growth on G_1 by
$$W\left(n(z)m(x)\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}\right) = \exp(-\pi\sqrt{-1}\mathrm{tr}_{\mathbb{C}/\mathbb{R}}(\sqrt{-1}z)) \cdot x^{\alpha_1-\alpha_2+1}e^{-2\pi x^2} \cdot t_1^{\alpha_1+1/2}t_2^{\alpha_2-1/2}$$
for $t_1, t_2 \in \mathbb{C}^1$, $z \in \sqrt{-1}\mathbb{R}$ and $x \in \mathbb{R}_{>0}$. Then it is easy to see that
$$X_- \cdot W = 0,$$
so that W generates the discrete series representation of G_1 whose Harish-Chandra parameter is $\lambda = (\alpha_1; \alpha_2)$. We conclude that π is $\mathfrak{w}_+$-generic, so that π' is $\mathfrak{w}_-$-generic.

Next, we assume that $m > 1$. Let $P = f_{T_m}^{-1}(P') = MN$ be the parabolic subgroup of $G_m = \mathrm{U}(m,m)$ given by
$$P' = \left\{ \begin{pmatrix} a & * & * \\ 0 & g_0' & * \\ 0 & 0 & \overline{a}^{-1} \end{pmatrix} \,\middle|\, a \in \mathbb{C}^\times,\ g_0' \in G_{m-1}' \right\}.$$
Here, $M = f_{T_m}^{-1}(M')$ is the Levi subgroup of P such that M' consists of the block diagonal matrices of P'. Let π_0 be the discrete series representation of $\mathrm{U}(m-1,m-1)$ whose Harish-Chandra parameter λ_0 is given by
$$\lambda_0 = (\alpha_3, \alpha_5, \dots, \alpha_{n-1}; \alpha_4, \alpha_6, \dots, \alpha_n),$$
and χ be the character of $\mathbb{C}^\times$ defined by
$$\chi(ae^{\theta\sqrt{-1}}) = a^{\alpha_1-\alpha_2}e^{(\alpha_1+\alpha_2)\theta\sqrt{-1}}$$
for $a > 0$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Consider the normalized induced representation $I(\pi_0) = \mathrm{Ind}_P^G(\chi \boxtimes \pi_0)$. We denote the Harish-Chandra characters of π and $\mathrm{Ind}_P^G(\chi \boxtimes \pi_0)$ by
$$\Theta_\lambda \quad \text{and} \quad \Theta_{I(\pi_0)},$$
respectively. Now we use Schmid's character identity ([Sc1, (9.4) Theorem], [Sc2, Theorem (b)]). This focuses on representations of semisimple groups, but since $\mathrm{U}(m,m)$ is generated by its center and a semisimple group $\mathrm{SU}(m,m)$, we can apply Schmid's character identity to π. It asserts that
$$\Theta_\lambda + \Theta'_\lambda = \Theta_{I(\pi_0)},$$
where $\mathbb{Z}^{2m} \ni \mu \mapsto \Theta'_\mu$ is the coherent continuation of Harish-Chandra characters satisfying that:

- Θ'_μ is a virtual character corresponding to the infinitesimal character μ via the Harish-Chandra map;
- if $\mu = (\mu_1, \dots, \mu_m; \mu_{m+1}, \dots, \mu_{2m}) \in \mathbb{Z}^{2m}$ satisfies that
$$\mu_{m+1} > \mu_1 > \mu_2, \quad \text{and} \quad \mu_2 > \mu_{m+2} > \mu_3 > \mu_{m+3} > \cdots > \mu_m > \mu_{2m},$$
then Θ'_μ is the character of the discrete series representation whose Harish-Chandra parameter is μ.

By the theory of the wall-crossing of coherent families (see e.g., [V2, Corollary 7.3.9]), we see that Θ'_λ is the character of a representation of G. This implies that π is a subquotient of $I(\pi_0)$. By the additivity of Whittaker model (see e.g., [M1, §3]) and Hashizume's result ([Ha, Theorem 1], [M2, Theorem 3.4.1]), we see that if π is $\mathfrak{w}_\epsilon$-generic, then so is π_0. By the induction hypothesis, we must have $\epsilon = +1$, as desired. □

A.4. **Explicit description of discrete L-packets.** In this subsection, we recall the definition of $J_{\mathfrak{w}_+}$ in Theorem A.2 when $\phi \in \Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R}))$ (see e.g., [Ka, §5.6]), and explain Theorem 2.1 (4).

We regard $G = \mathrm{U}(p,q)$ as an algebraic group defined over $\mathbb{R}$. Namely, $G(\mathbb{C}) = \mathrm{GL}_n(\mathbb{C})$ with $n = p+q$, and the complex conjugate c acts on $G(\mathbb{C})$ by

$$c(g) = \begin{pmatrix} \mathbf{1}_p & \\ & -\mathbf{1}_q \end{pmatrix} {}^t\overline{g}^{-1} \begin{pmatrix} \mathbf{1}_p & \\ & -\mathbf{1}_q \end{pmatrix}^{-1}, \quad g \in G(\mathbb{C}),$$

where $g \mapsto \overline{g}$ is the usual action of the complex conjugate on $\mathrm{GL}_n(\mathbb{C})$. We set $G^* = \mathrm{U}(m,m)$ and $(p_0,q_0) = (m,m)$ if $p+q = 2m$, and $G^* = \mathrm{U}(m+1,m)$ and $(p_0,q_0) = (m+1,m)$ if $p+q = 2m+1$. We choose an isomorphism $\psi_{p,q} \colon G^*(\mathbb{C}) \to G(\mathbb{C})$ over $\mathbb{C}$ and a 1-cocycle $z_{p,q} \in Z^1(\mathbb{R}, G^*)$ by

$$\begin{cases} \psi_{p,q} = \mathrm{Int}\begin{pmatrix} \mathbf{1}_p & & \\ & \sqrt{-1}\mathbf{1}_{p_0-p} & \\ & & \mathbf{1}_{q_0} \end{pmatrix}, \quad z_{p,q}(c) = \begin{pmatrix} \mathbf{1}_p & & \\ & -\mathbf{1}_{p_0-p} & \\ & & \mathbf{1}_{q_0} \end{pmatrix} & \text{if } p \leq p_0, \\ \psi_{p,q} = \mathrm{Int}\begin{pmatrix} \mathbf{1}_{p_0} & & \\ & \sqrt{-1}\mathbf{1}_{q_0-q} & \\ & & \mathbf{1}_{q} \end{pmatrix}, \quad z_{p,q}(c) = \begin{pmatrix} \mathbf{1}_{p_0} & & \\ & -\mathbf{1}_{q_0-q} & \\ & & \mathbf{1}_{q} \end{pmatrix} & \text{if } q \leq q_0. \end{cases}$$

Here, we regard c as the non-trivial element in $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Then we have

$$\psi_{p,q}^{-1} \cdot c(\psi_{p,q}) = \mathrm{Int}(z_{p,q}(c)).$$

Hence $(G, \psi_{p,q}, z_{p,q})$ is a pure inner twist of G^*. In particular, G^* is a quasi-split pure inner form of G.

Let $\phi = \chi_{2\alpha_1} \oplus \cdots \oplus \chi_{2\alpha_n} \in \Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R}))$ with $n = p+q$. We can realize ϕ as

$$\phi(ae^{\sqrt{-1}\theta}) = \begin{pmatrix} e^{2\alpha_1\sqrt{-1}\theta} & & \\ & \ddots & \\ & & e^{2\alpha_n\sqrt{-1}\theta} \end{pmatrix}$$

for $a \in \mathbb{R}_{>0}$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Note that if $\phi = \varphi|W_\mathbb{C}$ for an admissible homomorphism $\varphi \colon W_\mathbb{R} \to {}^L\mathrm{U}_n$, then $\varphi(j)$ acts on $\phi(ae^{\sqrt{-1}\theta})$ by the inverse.

We denote the canonical right action of S_n on $(\mathbb{C}^1)^n$ by

$$(t_1, \dots, t_n)^\sigma = (t_{\sigma(1)}, \dots, t_{\sigma(n)})$$

for $\sigma \in S_n$ and $(t_1, \dots, t_n) \in (\mathbb{C}^1)^n$. For $\sigma \in S_n$, we define an embedding

$$\eta_{p,q}^\sigma \colon (\mathbb{C}^1)^n \hookrightarrow G = \mathrm{U}(p,q)$$

by

$$\eta_{p,q}^\sigma(t_1, \dots, t_n) = \begin{pmatrix} t_{\sigma(1)} & & & & & \\ & \ddots & & & & \\ & & t_{\sigma(p)} & & & \\ & & & t_{\sigma(p+1)} & & \\ & & & & \ddots & \\ & & & & & t_{\sigma(n)} \end{pmatrix} \in G = \mathrm{U}(p,q)$$

for $t_1, \dots, t_n \in \mathbb{C}^1$. We call $\eta_{p,q}^\sigma$ an admissible embedding of $(\mathbb{C}^1)^n$ (see [Ka, §5.6]). The image of $\eta_{p,q}^\sigma$ is independent of σ, and is denoted by $T_{p,q}$. Note that $\eta_{p,q}^\sigma$ and $\eta_{p,q}^{\sigma'}$ are $\mathrm{U}(p,q)$-conjugate if and only if

$$\sigma^{-1}\sigma' \in S_p \times S_q.$$

Hence when we consider the $\mathrm{U}(p,q)$-conjugacy class of $\eta^\sigma_{p,q}$, we may assume that

$$\sigma(1) < \cdots < \sigma(p), \quad \sigma(p+1) < \cdots < \sigma(n).$$

For such $\eta^\sigma_{p,q}$, we denote by $\pi^\sigma_{p,q}$ the irreducible discrete series representation of $\mathrm{U}(p,q)$ with Harish-Chandra parameter

$$\lambda^\sigma = (\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(p)}; \alpha_{\sigma(p+1)}, \dots, \alpha_{\sigma(n)}).$$

Let $\Theta_{\pi^\sigma_{p,q}}$ be the Harish-Chandra character of $\pi^\sigma_{p,q}$, which is a real analytic function on the regular set $\mathrm{U}(p,q)^{\mathrm{reg}}$ of $\mathrm{U}(p,q)$. We put $K_{p,q} = \mathrm{U}(p) \times \mathrm{U}(q)$ to be the usual maximal compact subgroup of $\mathrm{U}(p,q)$, which contains $T_{p,q}$ as a maximal torus. On $T_{p,q,\mathrm{reg}} \coloneqq \mathrm{U}(p,q)^{\mathrm{reg}} \cap T_{p,q}$, we have

$$\Theta_{\pi^\sigma_{p,q}}(t) = (-1)^{\frac{1}{2}\dim(\mathrm{U}(p,q)/K_{p,q})} \frac{\sum_{w \in W_{K_{p,q}}} \mathrm{sgn}(w) t^{w(\lambda^\sigma)}}{\prod_{\alpha \in \Delta^+_{\lambda^\sigma}} (t^{\alpha/2} - t^{-\alpha/2})},$$

where $\Delta^+_{\lambda^\sigma}$ is the unique positive system of $\Delta(\mathrm{U}(p,q), T_{p,q})$ such that $\langle \lambda^\sigma, \alpha^\vee \rangle > 0$ for any $\alpha \in \Delta^+_{\lambda^\sigma}$. Note that $\dim(\mathrm{U}(p,q)/K_{p,q}) = 2pq$ so that $(-1)^{\frac{1}{2}\dim(\mathrm{U}(p,q)/K_{p,q})} = (-1)^{pq}$. We put

$$\Pi^{\mathrm{U}(p,q)}_\phi = \{\pi^\sigma_{p,q} \mid \sigma \in S_n/(S_p \times S_q)\} \subset \mathrm{Irr}_{\mathrm{disc}}(\mathrm{U}(p,q)).$$

Then the L-packet Π_ϕ associated to ϕ is defined by

$$\Pi_\phi = \bigsqcup_{p+q=n} \Pi^{\mathrm{U}(p,q)}_\phi.$$

Recall that

$$A_\phi = (\mathbb{Z}/2\mathbb{Z})e_{2\alpha_1} \oplus \cdots \oplus (\mathbb{Z}/2\mathbb{Z})e_{2\alpha_n}.$$

Let $\pi^\sigma_{p,q} \in \Pi^{\mathrm{U}(p,q)}_\phi$ with $\sigma \in S_n$ satisfying $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(n)$. We define $g^\sigma_{p,q} \in G^*(\mathbb{C}) = \mathrm{GL}_n(\mathbb{C})$ as follows. There is an element $h^\sigma_{p,q} \in G_0(\mathbb{R})$ with $G_0 \coloneqq \mathrm{U}(n,0)$ such that $\mathrm{Int}(h^\sigma_{p,q})$ is equal to

$$\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto \begin{pmatrix} t_{\sigma_+^{-1}\sigma(1)} & & \\ & \ddots & \\ & & t_{\sigma_+^{-1}\sigma(n)} \end{pmatrix}$$

on $T_{n,0}$. Here, $\sigma_+ \in S_n$ is defined by

$$\sigma_+(i) = \begin{cases} 2i-1 & \text{if } i \leq p_0, \\ 2(i-p_0) & \text{if } i > p_0 \end{cases}$$

so that

$$(\alpha_{\sigma_+(1)}, \dots, \alpha_{\sigma_+(p_0)}; \alpha_{\sigma_+(p_0+1)}, \dots, \alpha_{\sigma_+(n)}) = (\alpha_1, \alpha_3, \dots; \alpha_2, \alpha_4, \dots).$$

We put

$$g^\sigma_{p,q} = \begin{pmatrix} \mathbf{1}_{p_0} & 0 \\ 0 & \sqrt{-1}\mathbf{1}_{q_0} \end{pmatrix}^{-1} h^\sigma_{p,q} \begin{pmatrix} \mathbf{1}_{p_0} & 0 \\ 0 & \sqrt{-1}\mathbf{1}_{q_0} \end{pmatrix} \in \mathrm{GL}_n(\mathbb{C}) = G^*(\mathbb{C}).$$

Then we have

$$\eta^\sigma_{p,q} = \psi_{p,q} \circ \mathrm{Int}(g^\sigma_{p,q}) \circ \eta^{\sigma_+}_{p_0,q_0},$$

that is, $\eta_{p,q}$ is the composition

$$(\mathbb{C}^1)^n \xrightarrow{\eta^{\sigma^+}_{p_0,q_0}} G^*(\mathbb{C}) \xrightarrow{\psi_{n,0}} G_0(\mathbb{C}) \xrightarrow{\mathrm{Int}(h^\sigma_{p,q})} G_0(\mathbb{C}) \xrightarrow{\psi_{n,0}^{-1}} G^*(\mathbb{C}) \xrightarrow{\psi_{p,q}} G(\mathbb{C}).$$

We set $\mathrm{inv}(\pi^{\sigma_+}_{p_0,q_0}, \pi^\sigma_{p,q}) \in H^1(\mathbb{R}, T_{p_0,q_0})$ to be the class of

$$c \mapsto (g^\sigma_{p,q})^{-1} \cdot z_{p,q}(c) \cdot c(g^\sigma_{p,q}),$$

where $c(g^\sigma_{p,q})$ is the action of c on $g^\sigma_{p,q}$ in $G^*(\mathbb{C})$ so that

$$(g^\sigma_{p,q})^{-1} = \begin{pmatrix} \mathbf{1}_{p_0} & \\ & -\sqrt{-1}\mathbf{1}_{q_0} \end{pmatrix} (h^\sigma_{p,q})^{-1} \begin{pmatrix} \mathbf{1}_{p_0} & \\ & \sqrt{-1}\mathbf{1}_{q_0} \end{pmatrix},$$

$$c(g^\sigma_{p,q}) = \begin{pmatrix} \mathbf{1}_{p_0} & \\ & \sqrt{-1}\mathbf{1}_{q_0} \end{pmatrix} h^\sigma_{p,q} \begin{pmatrix} \mathbf{1}_{p_0} & \\ & -\sqrt{-1}\mathbf{1}_{q_0} \end{pmatrix}.$$

We can compute $(g^\sigma_{p,q})^{-1} \cdot z_{p,q}(c) \cdot c(g^\sigma_{p,q})$ explicitly. First, we have

$$\begin{pmatrix} \mathbf{1}_{p_0} & 0 \\ 0 & \sqrt{-1}\mathbf{1}_{q_0} \end{pmatrix} z_{p,q}(c) \begin{pmatrix} \mathbf{1}_{p_0} & 0 \\ 0 & \sqrt{-1}\mathbf{1}_{q_0} \end{pmatrix} = \begin{pmatrix} \mathbf{1}_{p} & 0 \\ 0 & -\mathbf{1}_{q} \end{pmatrix}.$$

For $i = 1, \dots, n$, we define $\epsilon_i \in \{\pm 1\}$ by

$$\epsilon_i = \begin{cases} +1 & \text{if } i \leq p, \\ -1 & \text{if } i > p. \end{cases}$$

Then $(g^\sigma_{p,q})^{-1} \cdot z_{p,q}(c) \cdot c(g^\sigma_{p,q})$ is equal to

$$\begin{pmatrix} \epsilon_{\sigma^{-1}\sigma_+(1)} &&&&& \\ & \ddots &&&& \\ && \epsilon_{\sigma^{-1}\sigma_+(p_0)} &&& \\ &&& -\epsilon_{\sigma^{-1}\sigma_+(p_0+1)} && \\ &&&& \ddots & \\ &&&&& -\epsilon_{\sigma^{-1}\sigma_+(n)} \end{pmatrix}.$$

Note that $H^1(\mathbb{R}, \mathbb{C}^1) \cong \{\pm 1\}$. The map $\eta^{\sigma_+}_{p_0,q_0} \colon (\mathbb{C}^1)^n \to T_{p_0,q_0}$ gives an isomorphism

$$(\eta^{\sigma_+}_{p_0,q_0})^* \colon H^1(\mathbb{R}, (\mathbb{C}^1)^n) \to H^1(\mathbb{R}, T_{p_0,q_0}).$$

For each $e_{2\alpha_i} \in A_\phi$, we denote by $\langle e_{2\alpha_i}, \cdot \rangle$ the i-th projection

$$H^1(\mathbb{R}, (\mathbb{C}^1)^n) \cong H^1(\mathbb{R}, \mathbb{C}^1)^n \cong \{\pm 1\}^n \to \{\pm 1\}.$$

Then $J_{\mathfrak{w}_+}(\pi^\sigma_{p,q})(e_{2\alpha_i}) \in \{\pm 1\}$ is defined by

$$J_{\mathfrak{w}_+}(\pi^\sigma_{p,q})(e_{2\alpha_i}) = \langle e_{2\alpha_i}, (\eta^{\sigma_+}_{p_0,q_0})^{*-1}(\mathrm{inv}(\pi^{\sigma_+}_{p_0,q_0}, \pi^\sigma_{p,q})) \rangle.$$

Hence

$$J_{\mathfrak{w}_+}(\pi^\sigma_{p,q})(e_{2\alpha_{\sigma_+(i)}}) = \begin{cases} \epsilon_{\sigma^{-1}\sigma_+(i)} & \text{if } i \leq p_0, \\ -\epsilon_{\sigma^{-1}\sigma_+(i)} & \text{if } i > p_0. \end{cases}$$

If we put $j = \sigma_+(i)$, we see that $1 \leq i \leq p_0$ if and only if j is odd. Therefore, we conclude that

$$J_{\mathfrak{w}_+}(\pi^\sigma_{p,q})(e_{2\alpha_j}) = \begin{cases} (-1)^{j-1} & \text{if } \sigma^{-1}(j) \leq p, \\ (-1)^j & \text{if } \sigma^{-1}(j) > p. \end{cases}$$

The other bijection $J_{\mathfrak{w}_-} \colon \Pi_\phi \to \widehat{A_\phi}$ is defined by using $\sigma_- \in S_n$ such that

$$(\alpha_{\sigma_-(1)}, \dots, \alpha_{\sigma_-(p_0)}; \alpha_{\sigma_-(p_0+1)}, \dots, \alpha_{\sigma_-(n)}) = (\alpha_2, \alpha_4, \dots; \alpha_1, \alpha_3, \dots)$$

in place of σ_+. By a similar calculation, we have

$$J_{\mathfrak{w}_-}(\pi^\sigma_{p,q})(e_{2\alpha_j}) = \begin{cases} (-1)^j & \text{if } \sigma^{-1}(j) \leq p, \\ (-1)^{j-1} & \text{if } \sigma^{-1}(j) > p. \end{cases}$$

In particular, the isomorphism $J_{\mathfrak{w}_-} \circ (J_{\mathfrak{w}_+})^{-1} \colon \widehat{A_\phi} \to \widehat{A_\phi}$ is given by

$$J_{\mathfrak{w}_-} \circ (J_{\mathfrak{w}_+})^{-1} = \cdot \otimes \eta_{-1},$$

where the character $\eta_{-1} \colon A_\phi \to \{\pm 1\}$ is defined by

$$\eta_{-1}(e_{2\alpha_j}) = -1$$

for any $1 \leq j \leq n$.

Hence we have the following theorem.

Theorem A.4. *Let*
$$\phi = \chi_{2\alpha_1} \oplus \cdots \oplus \chi_{2\alpha_n} \in \Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R}))$$
with $2\alpha_i \in \mathbb{Z}$ *satisfying* $\alpha_1 > \cdots > \alpha_n$ *and* $2\alpha_i \equiv n-1 \bmod 2$. *Then the L-packet* Π_ϕ *consists of all irreducible discrete series representations of* $\mathrm{U}(p,q)$ *with* $p+q=n$ *whose infinitesimal characters are equal to* $(\alpha_1, \dots, \alpha_n)$. *Moreover, for each Whittaker datum* $\mathfrak{w}_\pm$ *of* $\mathrm{U}(p_0,q_0)$ *with* $|p_0-q_0| \leq 1$, *there is a bijection*
$$J_{\mathfrak{w}_\pm} \colon \Pi_\phi \to \widehat{A_\phi}$$
such that the Harish-Chandra parameter of $\pi^\pm(\phi,\eta) \coloneqq (J_{\mathfrak{w}_\pm})^{-1}(\eta)$ *is given by* $(\lambda_1, \dots, \lambda_p; \lambda'_1, \dots, \lambda'_q)$, *where*
$$\{\lambda_1, \dots, \lambda_p\} = \{\alpha_i \mid \eta(e_{2\alpha_i}) = \pm(-1)^{i-1}\},$$
$$\{\lambda'_1, \dots, \lambda'_q\} = \{\alpha_i \mid \eta(e_{2\alpha_i}) = \pm(-1)^{i}\}.$$
In particular, if we put $p = \#\{i \mid \eta(e_{2\alpha_i}) = \pm(-1)^{i-1}\}$ *and* $q = \#\{i \mid \eta(e_{2\alpha_i}) = \pm(-1)^{i}\}$, *then* $\pi^\pm(\phi,\eta)$ *is a representation of* $\mathrm{U}(p,q)$. *There is a unique* $\mathfrak{w}_\pm$*-generic representation in* Π_ϕ *which corresponds to the trivial character of* A_ϕ *via* $J_{\mathfrak{w}_\pm}$. *The bijections* $J_{\mathfrak{w}_+}$ *and* $J_{\mathfrak{w}_-}$ *are related by*
$$J_{\mathfrak{w}_-}(\pi) = J_{\mathfrak{w}_+}(\pi) \otimes \eta_{-1}$$
for any $\pi \in \Pi_\phi$, *where* $\eta_{-1} \in \widehat{A_\phi}$ *is defined by* $\eta_{-1}(e_{2\alpha_i}) = -1$ *for any* $e_{2\alpha_i} \in A_\phi$.

In this paper, we always use $J = J_{\mathfrak{w}_+}$. By this theorem, we obtain Theorem 2.1 (4).

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