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Doctoral Dissertation

A Proof-Theoretic Study of Term-Sequence-Dyadic Deontic Logic and Common Sense Modal Predicate Logic

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Dedicated to my family

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Takahiro Sawasaki

Contents

Chapter 1
Introduction

1.1 Background and Motivation

This thesis has two aims. The first aim is to provide, *term-sequence-modal logic*, an expansion of a modal predicate logic called *term-modal logic* and investigate them from a proof-theoretic viewpoint. As an application of the expansions, we will propose a deontic logic which we call *term-sequence-dyadic deontic logic*. The second aim is to study, *common sense modal predicate logic*, another modal predicate logic with a novel varying domain semantics, from a proof-theoretic viewpoint.

Modal logic stemming from Aristotle's works enables us to express logical properties of a variety of modalities like necessity and possibility. The first modern proof theory for modal logic was provided in 1959 by Lewis and Langford [51], whereas a modern semantics for modal logic was established in 1963 by Kripke [47]. One can consult Goldblatt [28] for historical developments of modal logic. The semantics called *Kripke semantics*, *possible world semantics*, or *relational semantics*, defines the satisfaction relations of propositions involving modality in the following way. Let R be a binary relation on worlds which is often called *accessibility relation*. Intuitively, wRv represents that v is accessible from w. Let us also read $\Box\varphi$ and $\Diamond\varphi$ as "it is necessary that φ" and "it is possible that φ", respectively. Then the satisfaction relations of a necessary proposition $\Box\varphi$ and a possible proposition $\Diamond\varphi$ are given as follows.

$\Box\varphi$ is true at a world w	iff	for *all* worlds v, if wRv then φ is true at v
$\Diamond\varphi$ is true at a world w	iff	for *some* worlds v, wRv and φ is true at v

Other familiar modalities are formalized in a similar vein. For example, when K_a is a modal operator to express knowledge of an agent a, a proposition "a knows that φ" is formalized as $K_a\varphi$. In this case, we often call a binary relation R_a relative to a *indistinguishability relation* and let wR_av represent that v is indistinguishable from w to a (see e.g. [80]). When $\Box$ is a modal operator to express provability in some fixed system, a proposition "φ is provable in the system" is formalized as $\Box\varphi$ (e.g. [98]). How a modality should behave is determined by adding modal axioms. Whatever modality is under consideration, axiom (K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ is usually adopted together with an inference rule called necessitation rule that allows to infer $\Box\varphi$ from φ. In addition to these axiom and rule, axioms such as (D) $\Box\varphi \to \Diamond\varphi$, (T) $\Box\varphi \to \varphi$, (4) $\Box\varphi \to \Box\Box\varphi$ and (B) $\varphi \to \Box\Diamond\varphi$ are often added according to the intended behavior of the modality. For contemporary developments of modal logic, see Blackburn et al. [6].

As for modal predicate logic, there are many options regarding how we should combine first order logic with modality. Modal predicate logic has often been presented as either assuming constant domains or increasing domains. The first assumption is that whatever exists in a world exists in every world and the second assumption is that whatever exists in a world exists in every world accessible from the world. However, these assumptions are less acceptable from a philosophical viewpoint. It is because, for example, the earth might not have existed. An exception is a sound and strongly complete modal predicate logic with respect to a varying domain semantics introduced e.g. in Fitting and Mendelsohn [16, pp. 132–136]. Let us call it the Kripke-style logic since it derives from Kripke [48]. In the semantics, what exists in each world may be thoroughly different, so neither of constant domains nor increasing domains are required. Accordingly, the Kripke-style logic fits with our intuition in this regard. However, it is built on a restricted first order logic in such a way that neither free variables nor constant symbols occur in formulas. For example, consider two formulas $\forall y(\forall x Px \to Py)$ and $\forall x Px \to Py$ provable in first order logic. The former is provable in the Kripke-style logic as well, but the latter is not provable in it. Therefore, the Kripke-style logic is not available unless we will give up fully retaining first order logic. For comprehensive discussions on the topic, see, e.g., Hughes and Cresswell [37, pp. 274–311], Fitting and Mendelsohn [16, pp. 81–185] and Garson [22].

Term-modal logic (TML) developed by Thalmann [92] and Fitting et al. [17] is a modal predicate logic with an increasing domain semantics, which is one of the modal predicate logics that have been rapidly developed in recent years (e.g. [43, 70, 103, 53, 18]). Besides formulas in ordinary modal predicate logic, TML can form a formula with a modal operator indexed by a term in which variables can be quantified. For example, where K is a modal operator indexed by a term to express knowledge of an agent, $K_t Pt$ and $\forall x K_x Px$ are well-formed formulas in TML that can be read as "t knows that t is P" and "everyone knows that they are P", respectively.

There are still two aspects of TML which are not studied enough.

First, TML can be generalized so that modal operators can be indexed by a finite sequence of terms instead of a single term. This generalization which we call *term-sequence-modal logic* (TSML) can have, for example, not only a modal operator $[t]$ indexed by a single term t, but also a modal operator $[t, s, u]$ indexed by a finite sequence (t, s, u) of terms t, s, u. TSML is mathematically quite natural, just as predicate symbols in first order logic are allowed to take (a finite sequence of) terms more than one. It is also a useful generalization to express a modality relative to multiple agents like an obligation of someone towards someone. For example, where $[t, s]$ is a modal operator indexed by a sequence (t, s) of terms t, s to express obligation of an agent t towards s, $\mathbf{O}_{ts}Pt$ and $\forall x\mathbf{O}_{xs}Px$ are well-formed formulas in TSML that can be read as "t has an obligation towards s to see to it that t is P" and "everyone has an obligation towards s to see to it that they are P", respectively.

Second, it is worthwhile to study well-behaved cut-free sequent calculi for TSMLs subsuming TMLs. Roughly speaking, sequent calculus is a proof system which operates on syntactic objects called *sequents* and which consists of a few axioms and many rules. In contrast, a proof system called Hilbert system consists of many axioms and a few rules. One of the advantages of sequent calculus over Hilbert system is that one can relatively easily prove the unprovability of a formula (strictly speaking, sequent) by *finite* means. The proof also requires only purely syntactic manipulations, so does not involve any semantics given to proof systems. As reviewed in Section 1.1.1, TML has been mainly studied from a semantic viewpoint, and there are not so many sequent calculi for TML. Few exceptions are [92, 17, 70], but their sequent calculi are not given in the form of ordinary sequent calculus (Section 1.1.1). Also, as reviewed in Section 1.1.2, there is a philosophical thesis in deontic logic that norms are neither true nor false. Taking into account such a thesis, whatever deontic logic is considered, it is desirable to be able to prove consistency of the logic without ascribing truth values to formulas in the logic. This desideratum is satisfied by a cut-free sequent calculus for the logic. Therefore, it is worthwhile to study well-behaved cut-free sequent calculi for TSMLs.

Accordingly, the first question we are to ask is as follows.

Question 1. How can we provide well-behaved cut-free sequent calculi for TSMLs subsuming TMLs?

To answer the question, we will propose two-sided and non-labelled cut-free sequent calculi for TSMLs (Chapter 3).

As for an application of TSML, deontic logic is of interest. As explained in Section 1.1.2, the standard deontic logic has a difficulty making normative conflicts impossible. We can use TSML to develop a deontic logic such that it overcomes the difficulty to some extent in a way compatible with the thesis that norms are neither true nor false. Thus our second question is as follows.

Question 2. How can we develop a deontic logic that accommodates normative conflicts in a way compatible with the thesis that norms are neither true nor false?

We try to answer this by developing what we call term-sequence-dyadic deontic logic (Chapter 4).

On the other hand, common sense modal predicate logic (CMPC)[1], developed by Seligman [87, 89, 88], is an **S5**-like modal predicate logic which is sound and strongly complete with respect to a varying domain semantics with no accessibility relation. The logic is obtained from (non-restricted) first order logic by adding the ordinary necessitation rule and modal axioms (T), (4) and (B), as well as a modal axiom ($\mathtt{K}_{\mathtt{inv}}$) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ where all free variables in φ are also free in ψ. As for the semantics given to it, van Benthem [95, 94, 93] first put forth the same semantics except that an accessibility relation is given as an arbitrary binary relation on worlds. Then Seligman independently provided the above semantics. Remarkably, their varying domain semantics is significantly different from the one introduced in Fitting and Mendelsohn [16].

CMPC is not studied enough in many regards. In what follows, for ease of reference, let **cK** be the logic obtained from first order logic by adding only the ordinary necessitation rule and ($\mathtt{K}_{\mathtt{inv}}$). Furthermore, we shall broaden the meaning of CMPC so as to cover any expansion of **cK**.

One of the directions worth studying is to develop sound and strongly complete CMPCs other than Seligman's CMPC. Thus our third question is as follows.

Question 3. Are there any sound and strongly complete CMPCs other than the original one?

To this question, we will provide sound and strongly complete expansions of **cK** by modal axioms like (T) and (B) (Section 5.1 in Chapter 5).

Another direction to be pursued, similarly as for TSML, is to study well-behaved cut-free sequent calculi for CMPCs. Since there are still no sequent calculi for CMPCs, it is worthwhile as such. Therefore, the last question we will ask is as follows.

Question 4. How can we provide well-behaved cut-free sequent calculi for CMPCs?

As an answer to this, we will build two-sided and non-labelled cut-free sequent calculi for expansions of **cK** by modal axioms like (T) and (4) (Section 5.2 in Chapter 5).

In what follows, we shall explain each question and our proposals in detail via reviews of characteristics and recent developments of the relating logics.

1.1.1 Term-Modal Logic

Characteristics and Developments

TML was first presented in Thalmann [92] and Fitting et al. [17]. In their papers, term-modal expansions of modal predicate logic **K**, **KD**, **KT**, **K4**, **KD4** and **S4** without Barcan

[1] The acronym CMPC comes from the fact that common sense modal predicate *logic* is sometimes called common sense modal predicate *calculus* by J. Seligman.

formula are considered. The given Kripke semantics is an increasing domain semantics, and the given proof systems are one-sided sequent calculi and tableau systems.

The very idea of TML is already found in von Wright's *An Essay in Modal Logic* [99]. In [99, p. 35] he says as follows.

> [w]e could develop an alternative system in which the epistemic modalities are treated as "relative" to persons. [...] Introducing quantifiers we should get a combined system dealing with expressions like "known to somebody", "unknown to everybody", *etc.*

Similar comments are also found in Hintikka's *Knowledge and Belief* [36]. For a sentence that a knows that P, he says "a is a name of a person or [...] a definite description referring to a human being" [36, p. 3] and considers substitution of such names by equality axioms [36, ch. 6]. For other similar works prior to TML, see Thalmann [92, pp. 7–9] and Liberman et al. [53, pp. 22–3].

We can find further developments of TML in the literature. The first direction of developments involves domains of models. The original semantics of TML given in [92, 17] is an increasing domain semantics requiring that whatever exists in a world exists in every world accessible from the world, but this requirement is less acceptable from philosophical view points. Orlandelli and Corsi [70] provide a varying domain semantics of TML making this requirement optional to develop labelled sequent calculi of a variety of TMLs. In addition to the problem of increasing domain semantics, unless all elements of domains are regarded as agents, the original semantics suffers from the inability to give a decent meaning to an epistemic operator K_a when a denotes a non-agent. Rendsvig [81] takes a solution that such a modality is alway interpreted as a global modality. Achen [1] and Liberman et al. [53] more straightforwardly provide two-sorted versions of TML such that a in K_a designates only an agent.

The second direction is to add dynamics into TML. Kooi [43] develops dynamic TML based on **S5** to introduce a term-modal version of update models. The language is that of first-order dynamic logic with wildcard assignments instead of quantifiers, where the set of atomic programs is the set of first-order terms. Wang and Seligman [103] investigate a fragment of Kooi's dynamic TML by considering a quantifier-free (and wildcard assignment-free) term-modal language with only the basic assignment modalities. This fragment can still make the distinction between *de dicto* and *de re*, similarly as Kooi's dynamic TML. Liberman et al. [53] present an epistemic planning framework via dynamic epistemic TML, which can be based on normal modal logics including **S5**. Liberman and Rendsvig [54] suggest how it may be used for logical studies of epistemic social network dynamics.

The third direction is to find (un)decidable fragments from a variety of TMLs. Because a number of studies in this direction have been done in [43, 86, 103, 70, 75, 73, 71, 72, 74, 55] etc., we will review just a few studies. See Liberman et al. [53, p. 24] for a more detailed survey. Kooi [43] invokes a result of Kripke [46] to point out that even the monadic fragment of his dynamic TML, i.e., a fragment of dynamic TML with unary predicate symbols as predicate symbols and without any function symbols, is undecidable. Orlandelli and Corsi [70] consider two decidable fragments of TML. The first decidable fragment has 0-ary predicate symbols (i.e., propositional letters) and quantified term-modal formulas $\exists x[x]\varphi$ and $\forall x\langle x\rangle\varphi$, where $[x]$ and $\langle x\rangle$ are modal operators $\Box$ and $\Diamond$ indexed by x, respectively. The second decidable fragment has 0-ary predicate symbols, unary predicate symbols, term-modal formulas $[x]\varphi$, and quantified term-modal formulas $\forall y(Py \to [y]\varphi)$ and $\exists y(Py \land [y]\varphi)$, where φ does not contain any unary predicate symbols and the set of variables is finite. Padmanabha and Ramanujam [72] show that even the TML having only propositional letters as atoms still remains undecidable. As a decidable fragment, Padmanabha and Ramanujam [72] identify the *monodic* fragment of the TML having only propositional letters as atoms, where the monodic fragment is a fragment such that only a variable x can occur in the scope of a term-modality $[x]$. Furthermore, Padmanabha and Ramanujam [74] identify the two variable fragment of TML. For the other results on decidability of various fragments of TML, see Padmanabha [71, p. 106].

Finally, there is the fourth direction of developments close to our direction in this thesis, i.e., a direction to index modal operators by terms more than one. Naumov and Tao [63] present a modal propositional logic with modal operators indexed by a finite set of terms. In their syntax, for example, $\forall x[\{x, y\}]\varphi$ is a well-formed formula. The logic presented is a sound and complete logic for reasoning about distributed knowledge with quantifiers over agents. As an application to deontic logic, Frijters and Van De Putte [20] develop non-normal term-modal deontic logic with modal operators indexed by a finite sequence of terms. In their syntax, $\forall x[x, y]\varphi$ is a well-formed formula where "x, y" denotes a sequence. In addition to some characteristics of non-normality, they also investigate interactions between sequences of terms by additional axioms such as $\forall x\forall y([x, y]\varphi \to [x]\varphi)$ and $\forall x\forall y([x]\varphi \to \langle y\rangle\varphi)$. Frijters [18] further develops a variety of term-modal deontic logics with modal operators indexed by a finite sequence of terms, such as a term-modal dyadic deontic logic and a term-modal deontic logic capturing relations of rights and duties. Frijters and De Coninck [19] use the framework of [18] to argue that an extension of Frijters' term-modal deontic logic by adaptive logic can accommodate normative conflicts.

Not so much proof-theoretic studies on TMLs have been done, however. In particular, no one has presented ordinary sequent calculi for TMLs, i.e., *two-sided* and *non-labelled* sequent calculi for TMLs. Roughly speaking, sequent calculus is a calculus of inferences which operates on syntactic objects called *sequents* consisting of (labelled) formulas. A sequent is two-sided if it is divided into the left side and the right side of a central symbol $\Rightarrow$, and it is one-sided if it is not divided at all. An example of two-sided sequent is $\{ P, Q \} \Rightarrow \{ P \}$ and an example of one-sided sequent is $\{ P, Q, \neg P \}$, where curly braces $\{, \}$ are often omitted. Each sequent is intuitively read as "P follows from P, Q" and "It is impossible to have all of $\{ P, Q, \neg P \}$," respectively. A sequent calculus is two-sided if each sequent is two-sided, and it is one-sided if each sequent is one-sided. In addition to the distinction between one-sided and two-sided sequent calculi, there is also a distinction between labelled and non-labelled sequent calculi. The basic idea of a labelled sequent calculus is to use *labelled formulas* to capture the notion of "truth at worlds" in a syntactic way. For example, a labelled formula $w : \varphi$ represents "a formula φ is true at a world w." Here labelled formulas are expressions of the form $w : \varphi$ in the object language, where w is a label and φ is an ordinary formula. A sequent calculus (for modal predicate logic) is labelled if each sequent consists of labelled formulas $w : \varphi$, *relational atoms* wRv and *domain atoms* $x \in D_w$. Just like labelled formulas, relational atoms wRv and domain atoms $x \in D_w$ are also expressions in the object language, and represent "v is accessible from w" and "x is in the domain of w," respectively. A sequent calculus is non-labelled if each sequent consists of ordinary formulas.

The sequent calculi presented in Thalmann [92] and Fitting et al. [17] are non-labelled but *one-sided* calculi. To use their one-sided sequent calculi, we need to transform a given formula into its negation normal form, i.e., a formula equivalent to the given formula in which the negation operator $\neg$ can be prefixed only to atomic formulas. For example, suppose we want to check that a formula $\exists x Px$ syntactically follows from a formula $\forall x Px$ in a one-sided sequent calculi given by them. To do this, we need to consider the negation $\neg\exists x Px$ of $\exists x Px$, then translating it into the negation normal form $\forall x \neg Px$, finally checking that a one-sided sequent $\{ \exists x Px, \forall x \neg Px \}$ is derivable in the calculus. This translation procedure is not difficult, but does not capture how $\exists x Px$ syntactically follows from $\forall x Px$. Two-sided sequent calculi can capture such a process by a two-sided sequent $\forall x Px \Rightarrow \exists x Px$.

The sequent calculi that Orlandelli and Corsi [70] provided are two-sided but *labelled* calculi. Consequently, their sequent calculi are a kind of syntactic representations of Kripke semantics. Such sequent calculi admit the cut elimination theorems and have a number of applications, but are not suitable for an application to deontic logic. It is because, when providing a sequent calculus for deontic logic, we do not want to involve whatever is reminiscent of Kripke semantics due to the thesis that norms are neither true nor false. Non-labelled sequent calculi do not at least intimate semantic notions that are specific to Kripke semantics.

Therefore, it is still worthwhile to develop ordinary sequent calculi for TSMLs subsuming TMLs.

Our Proposal: Term-Sequence-Modal Logic

Recall that **Question 1** asked how we can provide well-behaved cut-free sequent calculi for TSMLs subsuming TMLs. To answer this, we will first develop TSMLs subsuming TMLs (Section 3.1) and then present cut-free ordinary sequent calculi for TSMLs, i.e., two-sided and non-labelled cut-free sequent calculi for TSMLs (Section 3.2). Since TSMLs subsume TMLs, our sequent calculi for TSMLs are also those for TMLs. We will also develop TSMLs with equality for our theoretical interest (Section 3.3).

Our TSMLs have at least three novelties. First, our TSMLs can have various modal axioms as additional axioms. For example, term-sequence-modal expansions of, not only modal predicate logics **K**, **KD**, **KT**, **K4**, **KD4** and **S4** without Barcan formula, but also **KB**, **K5** and **S5** with Barcan formula, are presented in this thesis. As far as we know, except for a term-sequence-modal expansion of **KD** with Barcan formula which is given by Frijters [18], no one has developed our logics. Second, our sequent calculi are ordinary sequent calculi. Thus ours are distinct from both of Thalmann [92] and Fitting et al. [17]'s sequent calculi and Orlandelli and Corsi [70]'s sequent. Third, our TSMLs with equality provide us TMLs with equality which were not studied in both of Thalmann [92] and Fitting et al. [17].

1.1.2 Deontic Logic

Characteristics and Developments

Deontic logic is a logic that enables us to express logical properties of normative notions such as obligation and permission. It has currently been often studied as a variation of modal logic. The idea of formalizing such properties is said to at least date back to the fourteen century [42], but contemporary studies of deontic logic have begun from Mally [58] and von Wright [100]. For the details of their deontic logics, see Supplements A and B of [60].

The most familiar contemporary deontic logic is the normal modal propositional logic **KD** introduced e.g. in [35], which is often called the *standard deontic logic* **SDL**. It is obtained from the minimal normal modal propositional logic **K** by adding a modal axiom (D) $\mathbf{O}\varphi \to \mathbf{P}\varphi$, where $\mathbf{O}$ is a modal operator and $\mathbf{P}$ is the dual of $\mathbf{O}$, i.e., $\neg\mathbf{O}\neg$. We have many options as to which readings of $\mathbf{O}\varphi$ and $\mathbf{P}\varphi$ to choose. Here we read them as "it ought to be the case that φ" and "it may be the case that φ," respectively. Semantically, when Kripke semantics is under consideration, **SDL** is characterized as follows. A tuple (W, R) of a set W of worlds and a binary relation R on W is called *frame*. Also a frame is said to be serial if it is the case that for all worlds w there exists some world v such that wRv. Based on these terminologies, **SDL** is characterized as the logic of the class of all the serial frames. In this semantics, the satisfaction relations of an obligation $\mathbf{O}\varphi$ and a permission $\mathbf{P}\varphi$ are defined in the same way as $\Box\varphi$ and $\Diamond\varphi$. That is to say:

$$
\begin{array}{rll}
\mathbf{O}\varphi \text{ is true at a world } w & \text{iff} & \text{for all worlds } v, \text{ if } wRv \text{ then } \varphi \text{ is true at } v \\
\mathbf{P}\varphi \text{ is true at a world } w & \text{iff} & \text{for some worlds } v,\ wRv \text{ and } \varphi \text{ is true at } v
\end{array}
$$

where wRv is often read in such a way as "v is acceptable from the standpoint of w" or "v is an ideal world related to w" as pointed out in [60].

Contrary to its name, **SDL** has been said to fail in many respects to capture our intuitions involving obligation and permission. One of the well-known characteristics is that **SDL** makes normative conflicts impossible. Roughly speaking, a normative conflict is a situation in which it ought to be the case that φ and it ought to be the case that $\neg\varphi$. A natural formalization of such a situation in **SDL** would be $\mathbf{O}\varphi \wedge \mathbf{O}\neg\varphi$, which implies a contradictory formula $\bot$ by (D) $\mathbf{O}\varphi \rightarrow \mathbf{P}\varphi$. This means that normative conflicts never happen. As far as moral obligations are involved, one might accept this consequence and take a philosophical position in which normative conflicts do not exist in the first place, as I. Kant apparently did so in *Die Metaphisyk der Sitten* [39, pp. 378–379]:

> A *conflict of duties* (*collisio officiourum s. obligationum*) would be a relation between them in which one would cancel the other (wholly or in part). – But since duty and obligation are concepts that express the objective practical *necessity* of certain actions and two rules opposed to each other cannot be necessary at the same time, if it is a duty to act in accordance with one rule, to act in accordance with the opposite rule is not a duty but even contrary to duty; so a *collision of duties* and obligations is inconceivable (*obligationes non colliduntur*).

Against this position, we will take a philosophical position in which normative conflicts can exist even when moral obligations are involved. Although we cannot attempt to demonstrate the existence of normative conflicts in the thesis, we will stick to the position admitting the existence of normative conflicts. Thus we need to modify **SDL**.

The literature on conflict-tolerant deontic logics is vast. We just review some of them following Goble [26]. The first possible approach is to relativize obligations to agents or the like. In this approach, for example, a normative conflict can be accommodated by a formalization such as $\mathbf{O}_a\varphi \wedge \mathbf{O}_b\neg\varphi$, where $\mathbf{O}_a$ and $\mathbf{O}_b$ stand for distinct obligations given to agents a and b, respectively. The very idea of making obligations relative to something is often found in the literature (e.g. [8, 11, 21, 96, 106, 44, 45, 5, 91, 25, 20]), but only a small number of authors seem to take the first approach. As far as we know, obvious examples are Kooi [44, 45], Glavanicova [25] and Yamada [107]. The second approach is to introduce A. Ross' distinction between all-things-considered and *prima facie* obligations. In a traditional version of this approach, the all-things-considered obligation is defined as the *prima facie* obligation "outweighing all competing *prima facie* obligations," and the former cannot allow for conflicts but the latter can. A more formal illustration is found in Goble [26, pp. 256–296]. Finally, the third approach is to reject some of what Goble calls the core principles of deontic logic:

(C) $(\mathbf{O}\varphi \wedge \mathbf{O}\psi) \to \mathbf{O}(\varphi \wedge \psi)$
(D) $\mathbf{O}\varphi \to \mathbf{P}\varphi$
(P) From $\neg\varphi$, we may infer $\neg\mathbf{O}\varphi$
(RM) From $\varphi \to \psi$, we may infer $\mathbf{O}\varphi \to \mathbf{O}\psi$

Examples are Lemmon [50], van Fraassen [97] and Marcus [59] etc. Unfortunately, there does not seem to be a commonly accepted solution in the literature. As found in Section 4.3.1, our solution to the problem of normative conflict is categorized into the first approach.

On the other hand, there is a more philosophical criticism against deontic logic including **SDL**. As we saw above, **SDL** uses Kripke semantics to typically ascribe *truth* or *falsity* to obligation and permission. Some contemporary philosophers have argued against such ascriptions in favor of a thesis that norms such as obligation and permission cannot be essentially true nor false. For example, there is a metaethical position called non-cognitivism that stems from emotivism of a logical positivist A. J. Ayer. According to non-cognitivism, what is expressed by a moral judgment or statement of a person is just a non-cognitive state distinct from a belief, and such a non-cognitive state cannot be true nor false. Thus, if they consider moral judgments or statements to be a kind of norms, they cannot admit the use of Kripke semantics for deontic logic. Furthermore, one can find the thesis that norms are neither true nor false even in the literature on deontic logic. A relatively straightforward example can be found in von Wright [101, p. 104].

> That prescriptions lack truth-value we can, I think, safely accept. Or would anyone who wish to maintain that the *permission*, given by the words 'You may park your car in front of my house', or the *command* formulated 'Open the door', or the *prohibition* 'No through traffic', are true or false?

Since he uses the word "prescription" to mean norms as commands, permissions and prohibitions [101, p. 15], we can conclude that at least on the surface he is in favor of the thesis on norms. For other straightforward examples found in the literature on deontic logic, see e.g. Makinson [57, pp. 29–30] and Hansen et al. [32, p. 3]. For attempts to develop deontic logics in accordance with the thesis that norms neither true nor false, see e.g. [2, 57, 76, 108, 29]. A comprehensive discussion on this topic is found in Hansen [31].

Taking into account the thesis that norms are neither true nor false, it is preferable to present deontic logic in a way compatible with the thesis. However, it is not so clear what deontic logic is compatible with the thesis. In what follows, let us call formulas like $\mathbf{O}\varphi$ *deontic formulas*.

On the one hand, one might think that even **SDL** *is* compatible in a sense with the thesis by interpreting deontic formulas as formulations of *normative proposition*[2].

[2] I owe this point to T. Yamada's comment in my defense.

Normative propositions are propositions that there are norms like commands, permissions, or prohibitions ([101, p. viii]). This means that, unlike norms, normative propositions are capable of being true or false. Thus, if we interpret deontic formulas as formulations of *normative propositions*, then even deontic logics given to which semantics need to ascribe truth values to such formulas *are* compatible with the thesis that *norms* are neither true nor false. Hansson [33] and Opałek and Woleński [69] etc are clear examples. A difficult example to categorize is von Wright [101]. In [101], von Wright intentionally leaves an ambiguity on whether we should regard deontic formulas as formulations of norms or those of normative propositions [101, p. 132]. For this reason, he ascribes truth values to deontic formulas when those are seen as formulations of normative proposition [101, pp. 166-7].

On the other hand, there are also deontic logicians who interpret deontic formulas as formulations of *norms*. And if we interpret deontic formulas as formulations of norms, then any deontic logics given to which semantics need to ascribe truth values to such formulas would *not* be compatible with the thesis that norms are neither true nor false. Alchourrón and Bulygin [2, 3] and Alchourrón and Martino [4] are clear examples. Amongst them, for example, Alchourrón and Bulygin [3] takes what they call the hyletic conception of norm, i.e., a conception that norms are "abstract entities analogous to propositions, though they, unlike propositions, lack truth values" [3, p. 274]. Based on this conception, they propose a formal logic of norm [3, pp. 285–287]. Another possible example is G. Kalinowski [38]. First, he notes in a footnote of [38, p. 143] that normative propositions in his terminology are *not* "statements about norms." This does not necessarily imply that he identifies norms with normative propositions in his paper, but if so then the following claim found in the same page would be an evidence that he claims that norms *are* true or false.

> In conclusion, the norm-propositions, otherwise called normative propositions, do designate and therefore fall within the categories of the true and the false. [...] The norm "Baudoin and Fabiola owe fidelity to each other" is true in the strong sense just as much as the norm "Penelope and Ulysses owe each other fidelity is true" in the weak sense.

Other possible examples are von Wright [102], Makinson [57] and Parent and van der Torre [76] etc. We also note that there are also objections against the interpretation of formulas as formulations of normative propositions [102, 105].

In this thesis, we take the interpretation of deontic formulas as formulations of normative propositions. However, we also want to leave the possibility of the interpretation of deontic formulas as formulations of norms since not a few deontic logicians are motivated to take this interpretation.

Now the problem is this. When we want to present a deontic logic compatible with the thesis on norms under the interpretation of deontic formulas as formulations of norms, what conditions must the presented deontic logic satisfy without ascribing truth values to deontic formulas of the logic? One reasonable condition is that the presented deontic logic must be consistent, since the consistency is often considered as an indispensable property for logic. Thus, proving the consistency of the presented deontic logic without the truth-ascription would be a first step towards the development of deontic logics compatible with the thesis that norms are neither true nor false.

Our Proposal: Term-Sequence Dyadic Deontic Logic

Recall first that **Question 2** asked how we can develop a deontic logic that accommodates normative conflicts in a way compatible with the thesis that norms are neither true nor false. Towards the development of such a deontic logic, we will develop a first order dyadic deontic logic which we call *term-sequence-dyadic deontic logic* (TDDL) (Section 4.1). Our TDDL is a combination of TSML and the minimal normal conditional propositional logic **CK** introduced in Chellas [10, p. 269]. This logic has, for example, $\mathbf{O}_{ts}\varphi$ and $\mathbf{O}_{ts}(\varphi|\psi)$ as well-formed formulas, and their intended readings are "agent t has an obligation towards agent s to see to it that φ" and "agent t has an obligation towards agent s to see to it that φ given that ψ," respectively. We will prove the soundness and strong completeness of the Hilbert system for TDDL (Section 4.1.2), and then present an cut-free ordinary sequent calculus for TDDL (Section 4.2).

We first note that TDDL can accommodate normative conflicts of the following two kinds (Section 4.3.1). Normative conflicts of the first kind are situations in which incompatible obligations are directed towards different agents. One example is a situation in which Adam has obligations towards Barbara and Charles to be with her and him, respectively, but cannot be with both. Deontic logics with no indices, for example **SDL** or the conditional propositional logic **CK** seen as a deontic logic, are difficult to accommodate such a situation. Our TDDL is easy to accommodate such a situation with a formula $\mathbf{O}_{ab}Wab \wedge \mathbf{O}_{ac}\neg Wab$, where Wab and Wac represent "Adam is with Barbara" and "Adam is with Charles," respectively. Normative conflicts of the second kind are situations in which incompatible obligations are directed towards the same agent under different conditions. One example is a situation in which Adam has an obligation towards Barbara to be with her given that she is old and in which he has another obligation towards her not to be with her given that COVID-19 is still spreading. Deontic logics with no conditionals, for example **SDL** or multi-agent propositional deontic logics found e.g. in [107], are not so straightforward to accommodate such a situation. Our TDDL can accommodate such a situation with a formula $\mathbf{O}_{ab}(Wab|Ob) \wedge \mathbf{O}_{ab}(\neg Wab|C)$, where Wab, Ob and C represent "Adam is with Barbara," "Barbara is old" and "COVID-19 is spreading," respectively. Moreover, as we will see in Section 4.3.1, TDDL has an advantage over, say, multi-agent conditional propositional deontic logics in that TDDL can use *quantifiers* to formalize even situations implying what we call derived normative conflicts.

We also need to note that it is still an open question whether or not TDDL is compatible with the thesis that norms are neither true nor false. It is because it is not so clear what deontic logic is compatible with the thesis. Still, we can prove the consistency of TDDL without any semantics via the cut-free sequent calculus for TDDL given in Section 4.2. This implies that we can prove the consistency of TDDL without the truth-ascription to (deontic) formulas of TDDL[3]. In fact, Alchourron and Martino [4] take a similar (but probably stronger) position. In the paper they avoid giving a semantics to their deontic logic, instead take the following position [4, p. 48].

> [The alternative they take] accepts that norms, even though they are not true or false, have a logic and explains this by assuming the notion of consequence as primitive rather than by starting off from the concepts of truth and falsity. The meaning of both deontic and other logical connectives is expressed by the use made of these connectives in a deductive context.

Thus, we can say that TDDL is a first step towards the development of deontic logics compatible with the thesis on norms. In addition to the consistency without the truth-ascription, we can prove even the fact that TDDL can accommodate normative conflicts of the first and second kinds without any formal semantics (thus without the truth-ascription) (Sectoin 4.3.2).

Our TDDL should be compared with Frijters' term-modal dyadic deontic logic **NCL** given in [18, pp. 130–133]. Let $[\mathbf{U}]$ be the universal operator, i.e., a modal operator such that $[\mathbf{U}]\varphi$ is true at a world w iff φ is true at v for all world v (that may not be accessible from w) [30]. His logic **NCL** is a combination of TSML and **CK** with $[\mathbf{U}]$ such that each modal operator is indexed by a formula and a pair of agents. Thus, except that $[\mathbf{U}]$ is supplemented to it, it is quite similar to TDDL. He also develops a term-modal dyadic deontic logic **CCL** weaker than **NCL**, which is based on the minimal classical conditional logic **CE** introduced in Chellas [10, pp. 269–70]. However, our TDDL still has a novel formal aspect, i.e., a cut-free sequent calculus.

1.1.3 Common Sense Modal Predicate Logic

Characteristics and Developments

As we have explained at the beginning, modal predicate logics have often been presented as either assuming constant domains or increasing domains. The first assumption is that whatever exists in a world exists in every world and the second assumption is that whatever exists in a world exists in every world accessible from the world. The Kripke-style logic is sound and strongly complete with respect to a varying domain

[3]Giving a semantics to a logic and ascribing truth values in the semantics to formulas of the logic are probably different matters. However, the latter seems to imply the former so its contraposition also seems to hold.

semantics without such assumptions, but built on a restricted first order logic. For example, $\forall y(\forall x Px \to Py)$ is provable in it, but $\forall x Px \to Py$ is not.

The original CMPC is an **S5**-like modal predicate logic which is sound and strongly complete with respect to another varying domain semantics with no accessibility relation. As we commented above, it is obtained from (non-restricted) first order logic by adding the ordinary necessitation rule and (T), (4), (B) as well as ($\mathtt{K_{inv}}$) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$, where all free variables in φ are also free in ψ. Accordingly, for example, $\forall y(\forall x Px \to Py)$ and $\forall x Px \to Py$ are both provable in CMPCs including the original, but $\Box(Px \to \exists x Px) \to (\Box Px \to \Box\exists x Px)$ is not since the free variable x in Px is not free in $\exists x Px$. On the one hand, the main characteristics of the varying domain semantics given to CMPCs lies in the satisfaction relations of $\Box\varphi$ and $\Diamond\varphi$. Consider as φ a formula Px, and let $\overline{x}$ be the value of a variable x and D_v the domain of a world v. Roughly speaking, the satisfaction relations of $\Box Px$ and $\Diamond Px$ are given as follows.[4]

$\Box Px$ is true at a world w	iff	for all worlds v such that $\overline{x}$ *is in* D_v,
		if wRv then Px is true at v
$\Diamond Px$ is true at a world w	iff	for some worlds v such that $\overline{x}$ *is in* D_v,
		wRv and Px is true at v

The clause "$\overline{x}$ is in D_v" intuitively means that things which we are mentioning *do* exist in the world v. Notably, this varying domain semantics does not validate (K).

The probably first development of the semantics given to CMPCs is van Benthem [95, 94, 93]. He put forth the same semantics as the one given to the original except that an accessibility relation is given as an arbitrary binary relation on worlds. His works are done not for philosophical reasons but for a mathematical reason of seeing "what proposed axioms mean in terms of correspondence" [95, p. 121]. For example, he identifies in it that $\exists x\Box Px \to \Box\exists x Px$ corresponds to the class of all the increasing domain frames. After his works, Seligman [87, 89, 88] independently developed the original CMPC and the semantics with respect to which it is sound and strongly complete. Contrary to van Benthem's works, Seligman's works are done for a philosophical reason of taking the existential quantifier $\exists$ "to mean just 'exists' while denying the Constant Domain thesis" [87, p. 8]. It should also be pointed out that A. Hazen was aware in [34] that D. Lewis' counterpart-theoretic semantics does not validate (K) for much the same reason as above. Interestingly, unlike Seligman, Hazen considered the invalidity of (K) to be a "serious failing" in Lewis' semantics [34, p. 326].

What should be noted here is that neither van Benthem nor Seligman consider CMPCs other than the original, much less their sequent calculi. Thus it is a theoretically interesting task to develop sound and strongly complete CMPCs other than the original

[4] The satisfaction relations with no accessibility relation can be rewritten as the satisfaction relations with the total relation R on the set W of worlds, i.e. $R = W \times W$.

as well as their sequent calculi. It is also worthwhile in terms of applications, because it enables us to develop, say, a deontic logic based on the logic **cK**, i.e., the logic obtained from first order logic by adding only the ordinary necessitation rule and ($\mathtt{K_{inv}}$).

Our Proposal: Weak Common Sense Modal Predicate Logics

Recall first that **Question 3** asked whether or not there are any sound and strongly complete CMPCs other than the original one. To this question, we will provide sound and strongly complete Hilbert systems for weaker CMPCs than Seligman's CMPC with respect to the provability, i.e., expansions of **cK** by ($\mathtt{T}$), ($\mathtt{B}$), ($\mathtt{B}^-$) and ($\mathtt{D}_{\leqslant n}$) (Section 5.1), where

($\mathtt{B}^-$)	$\varphi \to \Box\Diamond\varphi$	where φ has a free variable;
($\mathtt{D}_{\leqslant n}$)	$\Box\varphi \to \Diamond\varphi$	where the number of free variables in φ is at most n.

In the same place, we also prove that modal axioms ($\mathtt{K}$), ($\mathtt{4}$), ($\mathtt{4}^-$), ($\mathtt{5}$) and ($\mathtt{5}^-$) are *not* canonical in the sense defined there, where

($\mathtt{4}^-$)	$\Box\varphi \to \Box\Box\varphi$	where φ has a free variable;
($\mathtt{5}^-$)	$\Diamond\varphi \to \Box\Diamond\varphi$	where φ has a free variable.

Recall then that **Question 4** asked how we can provide well-behaved cut-free sequent calculi for CMPCs. As an answer to this, we will build two-sided and non-labelled cut-free sequent calculi for expansions of **cK** by ($\mathtt{T}$), ($\mathtt{4}$) and ($\mathtt{D}_{\leqslant n}$) (Section 5.2).

Our studies on CMPCs have clear novelties, since no works addressing **Question 3** and **Question 4** have been done.

1.2 Outline

This thesis proceeds as follows.

In Chapter 2, we will see the technical details of two logics, (multi-)modal propositional logic and modal predicate logic. Section 2.1 describes modal propositional logic to introduce necessary notions and notations throughout the thesis. Based on this description, Section 2.2 illustrates ordinary modal predicate logic of increasing domain semantics. The illustration makes easy comparisons of it to term-sequence-modal logic in Chapter 3 and common sense modal predicate logic in Chapter 5.

In Chapter 3, we will build TSML subsuming TML provided by Thalmann [92] and Fitting et al. [17]. Section 3.1 develops sound and strongly complete Hilbert systems of TSMLs with respect to the semantics given there. Section 3.2 presents cut-free ordinary sequent calculi for some of the given logics and also proves the Craig interpolation theorem. Section 3.3 further considers TSMLs with equality to show soundness and strong completeness of Hilbert systems for them.

In Chapter 4, we will see how we can apply TSML in order to develop a deontic logic that accommodates normative conflicts in a way compatible with the thesis that norms are neither true nor false. Towards the development of such a deontic logic, we will first present TDDL in Section 4.1. Our TDDL is a combination of TSML and the minimal normal conditional logic **CK** introduced in Chellas [10, p. 269]. We will further provide a cut-free ordinary sequent calculus for TDDL in Section 4.2. In Section 4.3, we will use these results to argue that TDDL can accommodate two kinds of normative conflicts. We then claim that TDDL is consistent without the truth-ascription to (deontic) formulas of TDDL, as well as that TDDL can accommodate normative conflicts of the above kinds without the truth-ascription.

In Chapter 5, we will examine another modal predicate logic, CMPC. The main proof-theoretic difference of CMPC and TSML/TDDL is that CMPC has axiom ($\mathtt{K_{inv}}$) instead of axiom ($\mathtt{K}$). Section 5.1 begins with the examination of a family of CMPCs other than Seligman's CMPC. Then, Section 5.2 examines proof-theoretic properties of these logics by presenting cut-free ordinary sequent calculi.

Throughout this thesis, we usually follow the notations in Table 1.1.

variable	$x, y, z, \ldots$
term	$t, s, u, \ldots$
sequence of terms	$\vec{t}, \vec{s}$
predicate symbol	$P, Q, \ldots$
formula	$\varphi, \psi, \gamma, \delta$
set of formulas	Γ, Δ, Ξ, Π
sequent	$\Gamma \Rightarrow \Delta$
substitution of a term s for a variable x in a term t	$t(s/x)$
substitution of a term s for a variable x in a formula φ	$\varphi(s/x)$
world	$w, v, \ldots$
object in domains	$d, e, \ldots$
frame	$\mathfrak{F}, \mathfrak{G}$
model	$\mathfrak{M}$

Table 1.1: Notations used in the thesis

Chapter 2
Preliminaries

In this chapter, we will see the technical details of two logics, (multi-)modal propositional logic and modal predicate logic. Section 2.1 describes modal propositional logic to introduce necessary notions and notations throughout the thesis. Based on this description, Section 2.2 illustrates ordinary modal predicate logic of increasing domain semantics. The illustration makes easy comparisons of it to term-sequence-modal logic in Chapter 3 and common sense modal predicate logic in Chapter 5.

2.1 Modal Propositional Logic

Modal propositional logic is an expansion of propositional logic with modal operators which are intended to express modalities such as necessity, obligation, belief, knowledge, provability and so on. This section in particular introduces normal modal propositional logic and describes basic well-known results on it[1]. The descriptions of soundness and strong completeness of Hilbert systems (2.1.2, 2.2.2) mainly rely on Hughes and Cresswell [37] and Blackburn et al. [6]. The descriptions of sequent calculi (2.1.3, 2.2.3) mainly rely on Ono [67].

2.1.1 Syntax and Kripke Semantics

Let us first provide a syntax of normal modal propositional logic.

Definition 1. The *language* L_{ML} of *modal propositional logic* (*ML*) consists of a countably infinite set $\mathsf{Prop} = \{\, P, Q, \cdots \}$ of *propositional letters*, the set $\{\, \bot, \rightarrow \}$ of *logical symbols*, and a countable set Mod of *unary modal operators* $\Box$. A *formula* φ *in* L_{ML} is

[1]For normality of modal logic, see Blackburn [6, pp. 33–37].

recursively defined in Backus-Naur form, as below[2].

$$\varphi ::= P \mid \bot \mid (\varphi \to \varphi) \mid \Box\varphi$$

where P is a propositional letter from Prop and $\Box$ is a modal operator from Mod.

As usual, $\bot$ and $\varphi \to \psi$ are intended to represent a contradiction and an implication in the metalanguage, and parentheses are omitted whenever no confusion arises with respect to the reading of a formula. Furthermore, $\neg\varphi$ (*negation*), $\varphi \wedge \psi$ (*conjunction*), $\varphi \vee \psi$ (*disjunction*), $\varphi \leftrightarrow \psi$ (*logical equivalence*), $\top$ (*truth*) and $\Diamond\varphi$ (the dual of $\Box$) are defined as abbreviations as follows, where $A := B$ means A is defined as B.

$$\begin{array}{lcllcl}
\neg\varphi & := & \varphi \to \bot & \varphi \leftrightarrow \psi & := & (\varphi \to \psi) \wedge (\psi \to \varphi) \\
\varphi \wedge \psi & := & \neg(\varphi \to \neg\psi) & \top & := & \bot \to \bot \\
\varphi \vee \psi & := & \neg\varphi \to \psi & \Diamond\varphi & := & \neg\Box\neg\varphi
\end{array}$$

Given a finite set $\Gamma = \{ \gamma_1, \ldots, \gamma_n \}$ of formulas, the conjunction and disjunction of all the formulas in Γ are denoted by $\bigwedge \Gamma$ or $\bigwedge_{i \leqslant n} \gamma_i$ and $\bigvee \Gamma$ or $\bigvee_{i \leqslant n} \gamma_i$, respectively, where $\bigwedge \emptyset := \top$ and $\bigvee \emptyset := \bot$. As noted in Chapter 1, formulas $\Box\varphi$ and $\Diamond\varphi$ are, for example, read as “it is necessary that φ” and “it is possible that φ,” respectively.

Definition 2. The *length* $l(\varphi)$ *of a formula* φ is inductively defined as follows.

1. $l(P) = l(\bot) = 0$.
2. $l(\varphi \to \psi) = l(\varphi) + l(\psi) + 1$.
3. $l(\Box\varphi) = l(\varphi) + 1$.

In addition to the syntax above, we need a formal semantics to obtain the precise definitions of semantic notions. For this purpose, we introduce a *Kripke semantics* for $\mathsf{L_{ML}}$ introduced e.g. in [37, 6].

Definition 3. A *frame* (*for* $\mathsf{L_{ML}}$) is a tuple $\mathfrak{F} = (W, (R_\Box)_{\Box \in \mathsf{Mod}})$, where W is a non-empty set of elements called *worlds* and $R_\Box$ is a binary relation on W, i.e., $R_\Box \subseteq W \times W$. Given a frame $\mathfrak{F}$, a *model* (*for* $\mathsf{L_{ML}}$) is a tuple $\mathfrak{M} = (\mathfrak{F}, I)$, where I is a function called *interpretation* that maps each propositional letter P and each world w to a *truth value* $I(P, w) \in \{ \mathtt{True}, \mathtt{False} \}$.

Definition 4. Let $\mathfrak{M} = (W, (R_\Box)_{\Box \in \mathsf{Mod}}, I)$ be a model, w a world, and φ a formula. The *satisfaction relation* $\mathfrak{M}, w \models \varphi$ *between* $\mathfrak{M}, w$ *and* φ is defined as follows.

[2] Note that the Greek letter φ of this definition stands for the *syntactic category* of formula. Thus this definition says that a propositional letter P is a formula; a logical symbol $\bot$ is a formula; $\varphi \to \varphi$ is a formula whenever any two formulas are connected by the logical symbol $\to$; $\Box\varphi$ is a formula whenever any formula is prefixed by a modal operator $\Box$. Since φ stands for the syntactic category of formula, for example $P \to Q$ is also a formula.

$$\begin{array}{lll}
\mathfrak{M}, w \models P & \text{iff} & I(P, w) = \mathtt{True} \\
\mathfrak{M}, w \not\models \bot & & \\
\mathfrak{M}, w \models \varphi \to \psi & \text{iff} & \mathfrak{M}, w \models \varphi \text{ implies } \mathfrak{M}, w \models \psi \\
\mathfrak{M}, w \models \Box\varphi & \text{iff} & \text{for all worlds } v \in W,\ wR_\Box v \text{ implies } \mathfrak{M}, v \models \varphi
\end{array}$$

For a set Γ of formulas, $\mathfrak{M}, w \models \Gamma$ means that $\mathfrak{M}, w \models \psi$ for all formulas $\psi \in \Gamma$.

Definition 5. Let φ be a formula and Γ a set of formulas.

- φ is *valid in a frame* $\mathfrak{F}$ (*for* L_{ML}), denoted by $\mathfrak{F} \models \varphi$, if $(\mathfrak{F}, I), w \models \varphi$ for all interpretations I and worlds w.
- Γ is *valid in a frame* $\mathfrak{F}$ (*for* L_{ML}), denoted by $\mathfrak{F} \models \Gamma$, if $\mathfrak{F} \models \gamma$ for all formulas $\gamma \in \Gamma$.
- φ is *valid in a class* $\mathbb{F}$ *of frames* (*for* L_{ML}), denoted by $\mathbb{F} \models \varphi$, if $\mathfrak{F} \models \varphi$ for all frames $\mathfrak{F} \in \mathbb{F}$.
- φ is a *consequence from* Γ *in a class* $\mathbb{F}$ *of frames* (*for* L_{ML}) if $(\mathfrak{F}, I), w \models \Gamma$ implies $(\mathfrak{F}, I), w \models \varphi$ for all frames $\mathfrak{F} \in \mathbb{F}$, interpretations I and worlds w.

It is well known in the literature that there are a kind of correspondences between formulas and frames. Here is an example. Let us consider a formula $\Box P \to P$. As we see below, it “corresponds” to the class of all the reflexive frames in a sense that the following equivalence

$$\Box P \to P \text{ is valid in } \mathfrak{F} \quad \text{iff} \quad \mathfrak{F} \in \{\, (W, (R_\Box)_{\Box\in\mathsf{Mod}}) \mid wR_\Box w \text{ for all worlds } w \in W \,\}$$

holds for all frames $\mathfrak{F}$. Generally speaking, the notion of frame correspondence in modal propositional logic is defined as follows [6, p. 125].

Definition 6. A set Γ of formulas in L_{ML} *corresponds* to a class $\mathbb{F}$ of frames if the equivalence

$$\mathfrak{F} \models \Gamma \quad \text{iff} \quad \mathfrak{F} \in \mathbb{F}$$

holds for all frames $\mathfrak{F}$. If $\Gamma = \{\, \varphi \,\}$, we just say that φ corresponds to $\mathbb{F}$.

The following frame correspondences in modal propositional logic are well known in the literature.

Definition 7. Let $\mathfrak{F} = (W, (R_\Box)_{\Box\in\mathsf{Mod}})$ be a frame and $\Box \in \mathsf{Mod}$.

1. $\mathfrak{F}$ is $\Box$-*reflexive* if for all $w \in W$, $wR_\Box w$.
2. $\mathfrak{F}$ is $\Box$-*serial* if for all $w \in W$, there exists some $v \in W$ such that $wR_\Box v$.
3. $\mathfrak{F}$ is $\Box$-*symmetric* if for all $w, v \in W$, if $wR_\Box v$ then $vR_\Box w$.

4. $\mathfrak{F}$ is $\Box$-*transitive* if for all $w, v, u \in W$, if $wR_\Box v$ and $vR_\Box u$ then $wR_\Box u$.
5. $\mathfrak{F}$ is $\Box$-*euclidean* if for all $w, v, u \in W$, if $wR_\Box v$ and $wR_\Box u$ then $vR_\Box u$.

Proposition 8. (Frame correspondence) Let P be a propositional letter and $\Box \in \mathsf{Mod}$.

1. $\mathrm{T}_\Box := \Box P \to P$ corresponds to the class of all the $\Box$-reflexive frames.
2. $\mathrm{D}_\Box := \Box P \to \Diamond P$ corresponds to the class of all the $\Box$-serial frames.
3. $\mathrm{B}_\Box := P \to \Box\Diamond P$ corresponds to the class of all the $\Box$-symmetric frames.
4. $4_\Box := \Box P \to \Box\Box P$ corresponds to the class of all the $\Box$-transitive frames.
5. $5_\Box := \Diamond P \to \Box\Diamond P$ corresponds to the class of all the $\Box$-euclidean frames.

Every formula in Proposition 8 is also known as a *Sahlqvist formula*. This means, Proposition 8 follows from a more general theorem called *Sahlqvist Theorem*. See [6, ch. 3] for more details.

2.1.2 Hilbert System $\mathsf{H}(\mathbf{K}\Sigma)$

A Hilbert system $\mathsf{H}(\mathbf{K})$ for ML consists of axioms and inference rules in Table 2.1. Note that axioms and inference rules of $\mathsf{H}(\mathbf{K})$ are presented as *schemas*. It means that, for example,

- $\Box(P \to Q) \to (\Box P \to \Box Q)$;
- $\Box(\neg P \to \neg Q) \to (\Box\neg P \to \Box\neg Q)$;
- $\Box(\Box P \to \Box Q) \to (\Box\Box P \to \Box\Box Q)$

are all axioms in $\mathsf{H}(\mathbf{K})$, because they are instances of $(\mathtt{K}_\Box)$.

(Taut1)	$\varphi \to (\psi \to \varphi)$	
(Taut2)	$(\varphi \to (\psi \to \gamma)) \to ((\varphi \to \psi) \to (\varphi \to \gamma))$	
(Taut3)	$(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$	
$(\mathtt{K}_\Box)$	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	$(\Box \in \mathsf{Mod})$
(MP)	From $\varphi \to \psi$ and φ, we may infer ψ	
$(\mathtt{Nec}_\Box)$	From φ, we may infer $\Box\varphi$	$(\Box \in \mathsf{Mod})$

Table 2.1: Hilbert system $\mathsf{H}(\mathbf{K})$

Expansions of $\mathsf{H}(\mathbf{K})$ are obtained as follows. Put

$$\mathsf{Axiom}_{\mathsf{ML}} := \{\, \mathrm{T}_\Box, \mathrm{D}_\Box, \mathrm{B}_\Box, 4_\Box, 5_\Box \mid \Box \in \mathsf{Mod} \,\}.$$

Formulas	Schemas
$\mathrm{T}_{\Box} = \Box P \to P$	$(\mathrm{T}_{\Box}) \coloneqq \Box\varphi \to \varphi$
$\mathrm{D}_{\Box} = \Box P \to \Diamond P$	$(\mathrm{D}_{\Box}) \coloneqq \Box\varphi \to \Diamond\varphi$
$\mathrm{B}_{\Box} = P \to \Box\Diamond P$	$(\mathrm{B}_{\Box}) \coloneqq \varphi \to \Box\Diamond\varphi$
$4_{\Box} = \Box P \to \Box\Box P$	$(4_{\Box}) \coloneqq \Box\varphi \to \Box\Box\varphi$
$5_{\Box} = \Diamond P \to \Box\Diamond P$	$(5_{\Box}) \coloneqq \Diamond\varphi \to \Box\Diamond\varphi$

Table 2.2: The schemas corresponding to formulas of Σ

For a set $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{ML}}$, we mean by $\mathsf{Inst}(\Sigma)$ the set of all instances of the schema corresponding to a formula of Σ which is listed in Table 2.2.

Definition 9. Given a set Φ of formulas, the Hilbert system $\mathsf{H}(\mathbf{K}\oplus\Phi)$ is the system obtained from $\mathsf{H}(\mathbf{K})$ by adding all formulas of Φ as axioms. Given $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{ML}}$, $\mathsf{H}(\mathbf{K}\Sigma)$ is the system $\mathsf{H}(\mathbf{K} \oplus \mathsf{Inst}(\Sigma))$. We sometimes write $\mathsf{H}(\mathbf{K}\{\mathrm{T}_{\Box}, 4_{\Box}, \mathrm{B}_{\Box}\})$ as $\mathsf{H}(\mathbf{S5}_{\Box})$.

Remark 10. Contrary to axioms and inference rules of $\mathsf{H}(\mathbf{K})$, additional axioms from Σ are *formulas* and *not schemas*. This treatment of additional axioms makes it easy to talk about canonicity of a formula in common sense modal predicate logic introduced in Chapter 5.

Definition 11. Let $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{ML}}$. A *proof* in $\mathsf{H}(\mathbf{K}\Sigma)$ is a finite sequence of formulas consisting of an instance of an axiom of $\mathsf{H}(\mathbf{K})$ or the result of an application of an inference rule of $\mathsf{H}(\mathbf{K})$ to preceding formulas. A formula φ is *provable in* $\mathsf{H}(\mathbf{K}\Sigma)$, denoted by $\vdash_{\mathsf{H}(\mathbf{K}\Sigma)} \varphi$, if there exists a proof in $\mathsf{H}(\mathbf{K}\Sigma)$ whose last formula is φ. Given a set Γ of formulas, φ is *provable from* Γ *in* $\mathsf{H}(\mathbf{K}\Sigma)$, denoted by $\Gamma \vdash_{\mathsf{H}(\mathbf{K}\Sigma)} \varphi$, if there exists some finite subset Δ of Γ such that $\bigwedge \Delta \to \varphi$ is provable in $\mathsf{H}(\mathbf{K}\Sigma)$.

The soundness theorem of $\mathsf{H}(\mathbf{K}\Sigma)$ is by induction on the length of a proof.

Theorem 12. (Soundness of $\mathsf{H}(\mathbf{K}\Sigma)$) Let $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{ML}}$ and $\mathbb{F}_{\Sigma}$ be the class of all the frames to which Σ corresponds. For all formulas φ, if φ is provable in $\mathsf{H}(\mathbf{K}\Sigma)$, then φ is valid in $\mathbb{F}_{\Sigma}$.

On the other hand, the completeness theorem of $\mathsf{H}(\mathbf{K}\Sigma)$ needs more tricks to prove. It is the converse of the soundness theorem, i.e., if φ is valid in $\mathbb{F}_{\Sigma}$, then φ is provable in $\mathsf{H}(\mathbf{K}\Sigma)$. In what follows, we will present a stronger form of the completeness theorem called the *strong* completeness theorem, i.e., if φ is a consequence from Γ in $\mathbb{F}_{\Sigma}$, then φ is provable from Γ in $\mathsf{H}(\mathbf{K}\Sigma)$. We prove this via construction of the *canonical model*.

The following propositions and lemmas are found in [37, ch. 6] and [6, ch. 4].

Definition 13. Let $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{ML}}$. Given a set Γ of formulas, Γ is $\mathsf{H}(\mathbf{K}\Sigma)$*-inconsistent* if $\Gamma \vdash_{\mathsf{H}(\mathbf{K}\Sigma)} \bot$; Γ is $\mathsf{H}(\mathbf{K}\Sigma)$*-consistent* if Γ is not $\mathsf{H}(\mathbf{K}\Sigma)$-inconsistent; Γ is *maximal* if $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$ for all formulas φ; Γ is a *maximal* $\mathsf{H}(\mathbf{K}\Sigma)$*-consistent set* ($\mathsf{H}(\mathbf{K}\Sigma)$*-MCS*) if Γ is $\mathsf{H}(\mathbf{K}\Sigma)$-consistent and maximal.

Proposition 14. Let $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{ML}}$, Γ a $\mathsf{H}(\mathbf{K}\Sigma)$-MCS and φ, ψ formulas.

1. $\Gamma \vdash_{\mathsf{H}(\mathbf{K}\Sigma)} \varphi \quad$ iff $\quad \varphi \in \Gamma$.
2. If $\varphi \in \Gamma$ and $\vdash_{\mathsf{H}(\mathbf{K}\Sigma)} \varphi \to \psi$, then $\psi \in \Gamma$.
3. $\bot \notin \Gamma$.
4. $\varphi \to \psi \in \Gamma \quad$ iff $\quad \varphi \notin \Gamma$ or $\psi \in \Gamma$.

In what follows, we abbreviate $\mathsf{H}(\mathbf{K}\Sigma)$ as Λ for some fixed $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{ML}}$.

Lemma 15. (Lindenbaum Lemma) Let Γ be a Λ-consistent set of formulas. There exists a Λ-MCS Γ^+ such that $\Gamma \subseteq \Gamma^+$.

Definition 16. The *canonical* Λ*-frame* is a tuple $\mathfrak{F}^\Lambda = (W^\Lambda, (R^\Lambda_\Box)_{\Box \in \mathsf{Mod}})$, where

- $W^\Lambda := \{ \Gamma \mid \Gamma \text{ is a } \Lambda\text{-MCS} \}$;
- $\Gamma R^\Lambda_\Box \Delta \quad$ iff $\quad \Box\varphi \in \Gamma$ implies $\varphi \in \Delta$ for all formulas φ.

The *canonical* Λ*-model* is a tuple $\mathfrak{M}^\Lambda = (\mathfrak{F}^\Lambda, I^\Lambda)$, where

- $\mathfrak{F}^\Lambda$ is the canonical Λ-frame;
- $I^\Lambda(P, \Gamma) = \mathtt{True} \quad$ iff $\quad P \in \Gamma$.

Lemma 17. (Existence Lemma) If $\neg\Box\varphi \in \Gamma \in W^\Lambda$, there exists some $\Delta \in W^\Lambda$ such that $\neg\varphi \in \Delta$ and $\Gamma R^\Lambda_\Box \Delta$.

Lemma 18. (Truth Lemma) Let $\mathfrak{M}^\Lambda$ be the canonical Λ-model. For all Λ-MCSs $\Gamma \in W^\Lambda$ and formulas φ,

$$\mathfrak{M}^\Lambda, \Gamma \models \varphi \quad \text{iff} \quad \varphi \in \Gamma$$

Proposition 19. Let $\mathfrak{F}^\Lambda$ be the canonical Λ-frame and $\Box \in \mathsf{Mod}$.

1. If $\mathrm{T}_\Box \in \Sigma$ then $\mathfrak{F}^\Lambda$ is $\Box$-reflexive.
2. If $\mathrm{D}_\Box \in \Sigma$ then $\mathfrak{F}^\Lambda$ is $\Box$-serial.
3. If $\mathrm{B}_\Box \in \Sigma$ then $\mathfrak{F}^\Lambda$ is $\Box$-symmetric.
4. If $4_\Box \in \Sigma$ then $\mathfrak{F}^\Lambda$ is $\Box$-transitive.
5. If $5_\Box \in \Sigma$ then $\mathfrak{F}^\Lambda$ is $\Box$-euclidean.

Theorem 20. (Strong completeness of $\mathsf{H}(\mathbf{K}\Sigma)$) Let $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{ML}}$ and $\mathbb{F}_{\Sigma}$ be the class of all the frames to which Σ corresponds. For all formulas φ and sets Γ of formulas, if φ is a consequence from Γ in $\mathbb{F}_{\Sigma}$, then φ is provable from Γ in $\mathsf{H}(\mathbf{K}\Sigma)$.

Proof. By contraposition. Let $\Lambda = \mathsf{H}(\mathbf{K}\Sigma)$ for short. Suppose φ is not provable from Γ in Λ. We show φ is not a consequence from Γ in $\mathbb{F}_{\Sigma}$, i.e., there exist a frame $\mathfrak{F} \in \mathbb{F}_{\Sigma}$, an interpretation I and a world w such that $(\mathfrak{F}, I), w \models \Gamma$ and $(\mathfrak{F}, I), w \not\models \varphi$. Note first that $\Gamma \cup \{ \neg\varphi \}$ is Λ-consistent. By Lindenbaum Lemma (Lemma 15), we obtain a Λ-MCS Γ^{+} such that $\Gamma \cup \{ \neg\varphi \} \subseteq \Gamma^{+}$. It then follows from Truth Lemma (Lemma 18) that $\mathfrak{M}^{\Lambda}, \Gamma^{+} \models \Gamma$ and $\mathfrak{M}^{\Lambda}, \Gamma^{+} \not\models \varphi$, where $\mathfrak{M}^{\Lambda} = (\mathfrak{F}^{\Lambda}, I^{\Lambda})$ is the canonical Λ-model. We need to check that $\mathfrak{F}^{\Lambda} \in \mathbb{F}_{\Sigma}$, which is established by Proposition 19. ■

2.1.3 Sequent Calculus $\mathsf{G}(\mathbf{K}\Sigma)$

Before introducing sequent calculus for modal propositional logic, we illustrate a notion of sequent following Ono [67, p. 7]. Given finite *multisets* Γ, Δ of formulas, an expression $\Gamma \Rightarrow \Delta$ we call a *sequent*[3]. Intuitively, it means that some formulas in Δ follow from all formulas in Γ. When $\Gamma = \emptyset$, the sequent $\Rightarrow \Delta$ means that some formulas in Δ follow without any assumptions. When $\Delta = \emptyset$, the sequent $\Gamma \Rightarrow$ means that a contradiction follows from all of the formulas in Γ. Roughly speaking, sequent calculus is a calculus of inferences which operates on sequents as basic components.

The first development of sequent calculi of classical and intuitionistic first order logics is by Gentzen [23, 24]. As surveyed in Wansing [104], a sequent calculus for normal modal (classical) propositional logic **K** is provided e.g. in Leivant [49], Sambin and Valentini [82], and Mints [61]. In addition Sambin and Valentini [82] consider a sequent calculus for **K4**. Sequent calculi for **KD** and **KD4** are studied by Goble [27], and sequent calculi for **KT**, **T4** and **S5** are given by Ohnishi and Matsumoto [64].

A sequent calculus for propositional logic can be obtained from a sequent calculus for first order logic by removing logical rules involving quantifiers. As such a sequent calculus, we introduce $\mathsf{G}(\mathbf{PC})$ consisting of *initial sequents* $(id), (\bot)$, *structural rules* $(\Rightarrow w), (w \Rightarrow), (\Rightarrow c), (c \Rightarrow), (Cut)$ and *logical rules* $(\Rightarrow \to), (\to \Rightarrow)$ in Table 2.3, where they are presented as schemas. For simplicity, let $\mathsf{Mod} = \{ \Box \}$. A sequent calculus $\mathsf{G}(\mathbf{K})$ and its expansions are obtained from $\mathsf{G}(\mathbf{PC})$ as follows. Put

$$\mathsf{Axiom}^{-}_{\mathsf{ML}} := \{ \mathrm{D}_{\Box}, \mathrm{T}_{\Box}, 4_{\Box} \}.$$

Definition 21. Given $\Sigma \subseteq \mathsf{Axiom}^{-}_{\mathsf{ML}}$, the sequent calculus $\mathsf{G}(\mathbf{K}\Sigma)$ is the calculus obtained from $\mathsf{G}(\mathbf{K})$ by adding all logical rules for Σ in Table 2.3, where we define logical rules for $\{ \mathrm{D}_{\Box}, \mathrm{T}_{\Box} \}$ and $\{ \mathrm{D}_{\Box}, \mathrm{T}_{\Box}, 4_{\Box} \}$ by those for $\{ \mathrm{T}_{\Box} \}$ and $\{ \mathrm{T}_{\Box}, 4_{\Box} \}$, respectively.

[3] A multiset is a set which allows for multiple instances for each of its elements.

Irrespective of Σ, the sequent calculus $\mathrm{G}(\mathbf{S5})$ is the sequent calculus obtained from $\mathrm{G}(\mathbf{K})$ by adding logical rules $(\Box\mathrm{S5})$ and $(\Box\mathrm{T})$ for $\{ \mathrm{T}_\Box, \mathrm{B}_\Box, 4_\Box \}$ in Table 2.3.

Definition 22. Let $\Sigma \subseteq \mathsf{Axiom}^{-}_{\mathsf{ML}}$. A *derivation in* $\mathrm{G}(\mathbf{K}\Sigma)$ is a finite tree generated from initial sequents by applying structural or logical rules of $\mathrm{G}(\mathbf{K}\Sigma)$. A sequent $\Gamma \Rightarrow \Delta$ is *derivable in* $\mathrm{G}(\mathbf{K}\Sigma)$, denoted by $\vdash_{\mathrm{G}(\mathbf{K}\Sigma)} \Gamma \Rightarrow \Delta$, if there exists a derivation in $\mathrm{G}(\mathbf{K}\Sigma)$ whose root is $\Gamma \Rightarrow \Delta$. The same definition is also used for $\mathrm{G}(\mathbf{S5})$.

We first note that $\mathsf{H}(\mathbf{K}\Sigma)$ and $\mathrm{G}(\mathbf{K}\Sigma)$ are *equipollent*, i.e., a formula φ is provable in $\mathsf{H}(\mathbf{K}\Sigma)$ iff a sequent $\Rightarrow \varphi$ is derivable in $\mathrm{G}(\mathbf{K}\Sigma)$. In other words, they are the same with respect to provability. The same thing also holds for $\mathsf{H}(\mathbf{S5})$ and $\mathrm{G}(\mathbf{S5})$. The proofs of them are done as in [40].

Proposition 23. (Equipollence of $\mathsf{H}(\mathbf{K}\Sigma)$ and $\mathrm{G}(\mathbf{K}\Sigma)$) Let $\Sigma \subseteq \mathsf{Axiom}^{-}_{\mathsf{ML}}$. A formula φ is provable in $\mathsf{H}(\mathbf{K}\Sigma)$ iff a sequent $\Rightarrow \varphi$ is derivable in $\mathrm{G}(\mathbf{K}\Sigma)$.

Proposition 24. (Equipollence of $\mathsf{H}(\mathbf{S5})$ and $\mathrm{G}(\mathbf{S5})$) A formula φ is provable in $\mathsf{H}(\mathbf{S5})$ iff a sequent $\Rightarrow \varphi$ is derivable in $\mathrm{G}(\mathbf{S5})$.

We then introduce the cut elimination theorem. The cut elimination theorem of a sequent calculus says that *any derivable sequent in the sequent calculus is derivable in it without any application of* (Cut). In other words, it says that (Cut) is dispensable in the sequent calculus. The cut elimination of a sequent calculus is of importance since it implies that the sequent calculus has *subformula property*, i.e., every formula occurring in a derivation of a sequent is a subformula of a formula in the sequent ([67, p. 32]). For example, consider the following derivation $\mathfrak{D}$ of $\Box P, \Box(P \to Q) \Rightarrow \Box Q$ in $\mathrm{G}(\mathbf{K})$:

$$\dfrac{\dfrac{\dfrac{}{P \Rightarrow P}\,(id) \qquad \dfrac{}{Q \Rightarrow Q}\,(id)}{P, P \to Q \Rightarrow Q}\,(\to\Rightarrow)}{\Box P, \Box(P \to Q) \Rightarrow \Box Q}\,(\Box\mathrm{K})$$

One can observe that every formula occurring in $\mathfrak{D}$ of $\Box P, \Box(P \to Q) \Rightarrow \Box Q$ is a subformula of a formula in the sequent.

Not all sequent calculi enjoy the cut elimination theorem. A famous example is the sequent calculus $\mathrm{G}(\mathbf{S5})$ for modal propositional logic $\mathbf{S5}$ which is first provided in Ohnishi and Matsumoto [64, 65]. Consider a sequent $P \Rightarrow \Box\neg\Box\neg P$ derivable in $\mathrm{G}(\mathbf{S5})$ as below.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Box\neg P \Rightarrow \Box\neg P}\,(id)}{\Box\neg P \Rightarrow \Box\neg P, \bot}\,(\Rightarrow w)}{\Rightarrow \neg\Box\neg P, \Box\neg P}\,(\Rightarrow\to)}{\Rightarrow \Box\neg\Box\neg P, \Box\neg P}\,(\Rightarrow\Box_0) \qquad \dfrac{\dfrac{\dfrac{}{P \Rightarrow P}\,(id) \quad \dfrac{}{\bot \Rightarrow}\,(\bot)}{\neg P, P \Rightarrow}\,(\Rightarrow\to)}{\Box\neg P, P \Rightarrow}\,(\Box_0 \Rightarrow)}{P \Rightarrow \Box\neg\Box\neg P}\,(Cut)$$

Sequent Calculus G(**PC**)

$$\frac{}{\varphi \Rightarrow \varphi}\ (id) \qquad \frac{}{\bot \Rightarrow}\ (\bot)$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}\ (\Rightarrow w) \qquad \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (w \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}\ (\Rightarrow c) \qquad \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (c \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Xi \Rightarrow \Sigma}{\Gamma, \Xi \Rightarrow \Delta, \Sigma}\ (Cut)$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}\ (\Rightarrow\to) \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Xi \Rightarrow \Sigma}{\varphi \to \psi, \Gamma, \Xi \Rightarrow \Delta, \Sigma}\ (\to\Rightarrow)$$

Σ	Logical Rules for Σ
$\emptyset$	$\frac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}\ (\Box\mathrm{K})$
$\{\,\mathrm{D}_\Box\,\}$	$\frac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}\ (\Box\mathrm{K}) \qquad \frac{\Gamma \Rightarrow}{\Box\Gamma \Rightarrow}\ (\Box\mathrm{D})$
$\{\,\mathrm{T}_\Box\,\}$	$\frac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}\ (\Box\mathrm{K}) \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Box\varphi, \Gamma \Rightarrow \Delta}\ (\Box\mathrm{T})$
$\{\,4_\Box\,\}$	$\frac{\Gamma, \Box\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}\ (\Box 4)$
$\{\,\mathrm{D}_\Box, 4_\Box\,\}$	$\frac{\Gamma, \Box\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}\ (\Box 4) \qquad \frac{\Gamma, \Box\Gamma \Rightarrow}{\Box\Gamma \Rightarrow}\ (\Box\mathrm{D})$
$\{\,\mathrm{T}_\Box, 4_\Box\,\}$	$\frac{\Box\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}\ (\Box\mathrm{S4}) \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Box\varphi, \Gamma \Rightarrow \Delta}\ (\Box\mathrm{T})$

where $\Box\Gamma := \{\,\Box\varphi \mid \varphi \in \Gamma\,\}$.

	Logical Rules for $\{\,\mathrm{T}_\Box, \mathrm{B}_\Box, 4_\Box\,\}$
$\{\,\mathrm{T}_\Box, \mathrm{B}_\Box, 4_\Box\,\}$	$\frac{\Box\Gamma \Rightarrow \Box\Delta, \varphi}{\Box\Gamma \Rightarrow \Box\Delta, \Box\varphi}\ (\Box\mathrm{S5}) \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Box\varphi, \Gamma \Rightarrow \Delta}\ (\Box\mathrm{T})$

where $\Box\Gamma := \{\,\Box\varphi \mid \varphi \in \Gamma\,\}$.

Table 2.3: Sequent Calculus G(**K**Σ)

To admit cut elimination, there must be a derivation of $P \Rightarrow \Box\neg\Box\neg P$ in $G(\mathbf{S5})$ with no applications of (Cut). However, as pointed out in [64, p. 124], any derivation of $P \Rightarrow \Box\neg\Box\neg P$ needs an application of (Cut). Thus, $P \Rightarrow \Box\neg\Box\neg P$ is a counterexample against the cut elimination of $G(\mathbf{S5})$.

Except for $G(\mathbf{S5})$, all sequent calculi that we presented above admit the cut elimination theorem [67, p. 50]. Recall $\mathsf{Axiom}^{-}_{\mathsf{ML}} = \{ D_{\Box}, T_{\Box}, 4_{\Box} \}$. Let $G^{-}(\mathbf{K}\Sigma)$ be the sequent calculus obtained from $G(\mathbf{K}\Sigma)$ by removing (Cut).

Theorem 25. (Cut elimination) Let $\Sigma \subseteq \mathsf{Axiom}^{-}_{\mathsf{ML}}$. If a sequent $\Gamma \Rightarrow \Delta$ is derivable in $G(\mathbf{K}\Sigma)$, then $\Gamma \Rightarrow \Delta$ is also derivable in $G^{-}(\mathbf{K}\Sigma)$.

The cut elimination theorem of a sequent calculus brings us a lot of valuable results. One of such propositions is that the sequent calculus is consistent, i.e., a sequent $\Rightarrow \bot$ is not derivable in the sequent calculus. It is valuable particularly when subformula property is obtained in a purely proof-theoretic way, because then we obtain the consistency of the relevant sequent calculus without invoking any semantic notions.

Corollary 26. Let $\Sigma \subseteq \mathsf{Axiom}^{-}_{\mathsf{ML}}$. A sequent $\Rightarrow \bot$ is not derivable in $G(\mathbf{K}\Sigma)$.

Another well-known application of the cut elimination theorem is the Craig interpolation theorem [67, p. 41–45]. It is proved via the method introduced by Maehara [56] together with the cut elimination theorem [67, p. 50]. Below, $\mathsf{Prop}(\Gamma)$ means the set of all propositional letters in a set Γ of formulas.

Definition 27. A *partition* of a sequent $\Gamma \Rightarrow \Delta$ is a pair $((\Gamma_1, \Delta_1), (\Gamma_2, \Delta_2))$ of pairs of finite multisets of formulas such that $\Gamma = \Gamma_1, \Gamma_2$ and $\Delta = \Delta_1, \Delta_2$. A partition $((\Gamma_1, \Delta_1), (\Gamma_2, \Delta_2))$ is denoted by the notation $(\Gamma_1 : \Delta_1), (\Gamma_2 : \Delta_2)$.

Lemma 28. Let $\Sigma \subseteq \mathsf{Axiom}^{-}_{\mathsf{ML}}$ and $\Gamma \Rightarrow \Delta$ a sequent derivable in $G(\mathbf{K}\Sigma)$. If $(\Gamma_1 : \Delta_1)$, $(\Gamma_2 : \Delta_2)$ is a partition of $\Gamma \Rightarrow \Delta$, there is an *interpolant* of it, i.e., a formula φ such that

- $\vdash_{G(\mathbf{K}\Sigma)} \Gamma_1 \Rightarrow \Delta_1, \varphi$ and $\vdash_{G(\mathbf{K}\Sigma)} \varphi, \Gamma_2 \Rightarrow \Delta_2$;
- $\mathsf{Prop}(\varphi) \subseteq \mathsf{Prop}(\Gamma_1, \Delta_1) \cap \mathsf{Prop}(\Gamma_2, \Delta_2)$;

Theorem 29. (Craig interpolation theorem) Let $\Sigma \subseteq \mathsf{Axiom}^{-}_{\mathsf{ML}}$. If $\Rightarrow \varphi \to \psi$ is derivable in $G(\mathbf{K}\Sigma)$, then there exists a formula γ such that

- $\vdash_{G(\mathbf{K}\Sigma)} \varphi \to \gamma$ and $\vdash_{G(\mathbf{K}\Sigma)} \Rightarrow \gamma \to \psi$;
- $\mathsf{Prop}(\gamma) \subseteq \mathsf{Prop}(\varphi) \cap \mathsf{Prop}(\psi)$.

Remark 30. As we saw above, the sequent calculus $G(\mathbf{S5})$ does not enjoy the cut elimination theorem. However, Takano [90] proved in a proof-theoretic way that $G(\mathbf{S5})$ has the subformula property and thus is consistent. In addition, it is proved in Ono [66, pp. 245–6] that $G(\mathbf{S5})$ enjoys the Craig interpolation theorem.

2.2 Modal Predicate Logic

Modal predicate logic is an expansion of first order logic with modal operators. This section first provides a syntax of modal predicate logic and then introduces an increasing domain (Kripke) semantics. After that, Hilbert systems for modal predicate logics are presented which are sound and strongly complete with respect to the given semantics. As a frame in the given semantics, a (locally) constant domain frame is also defined. The following descriptions mainly rely on Hughes and Cresswell [37], Fitting and Mendelsohn [16] and Garson [22].

2.2.1 Syntax and Kripke Semantics

We cannot enter here into a philosophical discussion about how we should interpret constant and function symbols in modal predicate logic. To avoid such a discussion and make simple a syntax for modal predicate logic, we confine ourselves to modal predicate logic with one and the same modal operator $\Box$ and without any constant and function symbols. We here read $\Box$ as representing necessity. Thus, a syntax of modal predicate logic is specified as follows.

Definition 31. The *language* $\mathsf{L}_{\mathsf{QML}}$ *of modal predicate logic* (*QML*) consists of a countably infinite set $\mathsf{Var} = \{ x, y, \dots \}$ of *variables*, the union $\mathsf{Pred} = \bigcup_{n \in \mathbb{N}} \mathsf{Pred}_n$ of countably infinite sets $\mathsf{Pred}_n = \{ P, Q, \dots \}$ of *predicate symbols with arity* n, the set $\{ \bot, \to, \forall \}$ of logical symbols, and the set $\mathsf{Mod} = \{ \Box \}$ of the unary modal operator $\Box$. A *formula* φ *in* $\mathsf{L}_{\mathsf{QML}}$ is recursively defined as follows.

$$\varphi ::= Px_1 \dots x_n \mid \bot \mid (\varphi \to \varphi) \mid \forall x \varphi \mid \Box \varphi,$$

where P is a predicate symbol with arity n and $x_1, \dots, x_n, x$ are variables. Instead of $\mathsf{L}_{\mathsf{QML}}$, we often write L when it is clear from the context.

As usual, we call $\forall$ the *universal quantifier* and define the *existential quantifier* $\exists$ by $\exists x := \neg \forall x \neg$. Note that Pred includes the countably infinite set Pred_0 of predicate symbols with arity 0. Note also that predicate symbols with arity 0 are formulas of the form P with no variables by the definition of formulas. Since we can identify such predicate symbols with propositional letters, $\mathsf{L}_{\mathsf{QML}}$ subsumes L_{ML} whose Mod is $\{ \Box \}$.

Definition 32. The *length* $l(\varphi)$ *of a formula* φ is inductively defined as follows.

1. $l(Px_1 \dots x_n) = l(\bot) = 0$.
2. $l(\varphi \to \psi) = l(\varphi) + l(\psi) + 1$.
3. $l(\forall x \varphi) = l(\Box \varphi) = l(\varphi) + 1$.

Definition 33. Let φ be a formula. The set $\mathsf{FV}(\varphi)$ of *free variables in* φ is recursively defined as follows.

1. $\mathsf{FV}(Px_1 \dots x_n) = \{ x_1, \dots, x_n \}$.
2. $\mathsf{FV}(\bot) = \emptyset$.
3. $\mathsf{FV}(\varphi \to \psi) = \mathsf{FV}(\varphi) \cup \mathsf{FV}(\psi)$.
4. $\mathsf{FV}(\forall x\varphi) = \mathsf{FV}(\varphi) \backslash \{ x \}$.
5. $\mathsf{FV}(\Box\varphi) = \mathsf{FV}(\varphi)$.

The set $\mathsf{FV}(\Gamma)$ of *free variables in a set* Γ *of formulas* is defined as $\{ x \mid x \in \mathsf{FV}(\varphi)$ for some $\varphi \in \Gamma \}$. The notations $\mathsf{FV}(\Gamma, \varphi)$ and $\mathsf{FV}(\Gamma, \Delta)$ mean $\mathsf{FV}(\Gamma \cup \{ \varphi \})$ and $\mathsf{FV}(\Gamma \cup \Delta)$, respectively. We also say that a variable is *fresh* in variables or formulas if it does not syntactically occur in them.

Definition 34. Let x, y, z be variables and φ a formula. The *substitution* $z(y/x)$ *of* y *for* x *in* z is defined by

$$z(y/x) = \begin{cases} y & \text{if} \quad z = x; \\ z & \text{if} \quad z \neq x. \end{cases}$$

The *substitution* $\varphi(y/x)$ *of* y *for* x *in* φ is recursively defined as follows.

1. $(Px_1 \dots x_n)(y/x) = Px_1(y/x) \dots x_n(y/x)$.
2. $\bot(y/x) = \bot$.
3. $(\varphi \to \psi)(y/x) = \varphi(y/x) \to \psi(y/x)$.
4. $(\forall z\varphi)(y/x) =$

$$\begin{cases} \forall z\varphi & \text{if} \quad z = x; \\ \forall z(\varphi(y/x)) & \text{if} \quad z \neq x \text{ and } z \neq y; \\ \forall u(\varphi(u/z)(y/x)) & \text{if} \quad z \neq x \text{ and } z = y, \text{ where } u \text{ is fresh in } \forall z\varphi, y. \end{cases}$$

5. $(\Box\varphi)(y/x) = \Box(\varphi(y/x))$.

Compared with Kripke semantics for L_{ML}, we have many options as regards which Kripke semantics to choose for $\mathsf{L}_{\mathsf{QML}}$. Roughly speaking, there are varying domain semantics, increasing domain semantics, and constant domain semantics as options [22]. For comparisons with the semantics defined in Chapters 3, 5, we introduce an increasing domain semantics. What is introduced here is a slight adaptation of the semantics found in Hughes and Cresswell [37, p. 275].

Definition 35. A *frame* (*for* $\mathsf{L}_{\mathsf{QML}}$) is a tuple $\mathfrak{F} = (W, R, (D_w)_{w \in W})$, where W is a non-empty set of worlds; R is a binary relation on W; D_w is a non-empty set called the *domain of a world* w and each element of D_w is called *object*; $\mathfrak{F}$ has *increasing domains* (*in QML*):

$$\text{for all } w, v \in W, \text{ if } wRv \text{ then } D_w \subseteq D_v.$$

Given a frame $\mathfrak{F}$, a *model* (*for* $\mathsf{L}_{\mathsf{QML}}$) is a tuple $\mathfrak{M} = (\mathfrak{F}, I)$, where I is an interpretation that maps each predicate symbol P with arity n and world w to a subset $I(P, w)$ of D_w^n. We define D as $\bigcup_{w \in W} D_w$. We sometimes write `True` and `False` instead of $D_w^0 = \{ \emptyset \}$ and $\emptyset$, respectively.

It should be noted that an interpretation I maps each predicate symbol P with arity 0 and each world w to a subset $I(P, w)$ of D_w^0. Together with the notations `True` and `False`, it follows that $I(P, w) =$ `True` or $I(P, w) =$ `False`.

Definition 36. A function from Var to D is said to be an *assignment*. The assignment $\alpha(x|d)$ stands for the same assignment as α except for assigning d to x. The notation $\alpha(x_1, \ldots, x_n)$ denotes the sequence $(\alpha(x_1), \ldots, \alpha(x_n))$ of $\alpha(x_i)$s. A formula φ is called an *α_w-formula* if $\alpha(x) \in D_w$ for all variables $x \in \mathsf{FV}(\varphi)$.

Definition 37. Let $\mathfrak{M} = (W, R, (D_w)_{w \in W}, I)$ be a model, w a world, α an assignment, and φ a formula such that φ is an α_w-formula. The *satisfaction relation* $\mathfrak{M}, w, \alpha \models \varphi$ *between* $\mathfrak{M}, w, \alpha$ *and* φ is defined as follows.

$$\begin{array}{lll}
\mathfrak{M}, w, \alpha \models Px_1 \ldots x_n & \text{iff} & \alpha(x_1, \ldots, x_n) \in I(P, w) \\
\mathfrak{M}, w, \alpha \not\models \bot & & \\
\mathfrak{M}, w, \alpha \models \varphi \to \psi & \text{iff} & \mathfrak{M}, w, \alpha \models \varphi \text{ implies } \mathfrak{M}, w, \alpha \models \psi \\
\mathfrak{M}, w, \alpha \models \forall x \varphi & \text{iff} & \text{for all objects } d \in D_w,\ \ \mathfrak{M}, w, \alpha(x|d) \models \varphi \\
\mathfrak{M}, w, \alpha \models \Box \varphi & \text{iff} & \text{for all worlds } v \in W,\ \ wRv \text{ implies } \mathfrak{M}, v, \alpha \models \varphi
\end{array}$$

For a set Γ of formulas, $\mathfrak{M}, w, \alpha \models \Gamma$ means that $\mathfrak{M}, w, \alpha \models \psi$ for all formulas $\psi \in \Gamma$.

Definition 38. Let φ be a formula and Γ a set of formulas.

- φ is *valid in a frame* $\mathfrak{F}$ (*for* $\mathsf{L}_{\mathsf{QML}}$), denoted by $\mathfrak{F} \models \varphi$, if $(\mathfrak{F}, I), w, \alpha \models \varphi$ for all interpretations I, worlds w and assignments α such that φ is an α_w-formula.
- Γ is *valid in a frame* $\mathfrak{F}$ (*for* $\mathsf{L}_{\mathsf{QML}}$), denoted by $\mathfrak{F} \models \Gamma$, if $\mathfrak{F} \models \gamma$ for all formulas $\gamma \in \Gamma$.
- φ is *valid in a class* $\mathbb{F}$ *of frames* (*for* $\mathsf{L}_{\mathsf{QML}}$), denoted by $\mathbb{F} \models \varphi$, if $\mathfrak{F} \models \varphi$ for all frames $\mathfrak{F} \in \mathbb{F}$.
- φ is a *consequence from* Γ *in a class* $\mathbb{F}$ *of frames* (*for* $\mathsf{L}_{\mathsf{QML}}$) if $(\mathfrak{F}, I), w, \alpha \models \Gamma$ implies $(\mathfrak{F}, I), w, \alpha \models \varphi$ for all frames $\mathfrak{F} \in \mathbb{F}$, interpretations I, worlds w, assignments α such that ψ is an α_w-formula for all formulas ψ in $\Gamma \cup \{ \varphi \}$.

Similarly to Definition 6, we define the notion of frame correspondence in modal predicate logic, as below.

Definition 39. A set Γ of formulas in $\mathsf{L}_{\mathsf{QML}}$ *corresponds* to a class $\mathbb{F}$ of frames for $\mathsf{L}_{\mathsf{QML}}$ if the equivalence

$$\mathfrak{F} \models \Gamma \quad \text{iff} \quad \mathfrak{F} \in \mathbb{F}$$

holds for all frames $\mathfrak{F}$ for $\mathsf{L}_{\mathsf{QML}}$. If $\Gamma = \{ \varphi \}$, we just say that φ corresponds to $\mathbb{F}$.

The frame properties introduced in Definition 7 are, *mutatis mutandis*, also defined in frames for QML. For example, a frame $\mathfrak{F} = (W, R, (D_w)_{w \in W})$ is *reflexive* if for all $w \in W$, wRw. In addition to the aforementioned frame properties, let us say

$\mathfrak{F}$ has *decreasing domains* if for all $w, v \in W$, if wRv then $D_w \supseteq D_v$.

Proposition 40 which states frame correspondence results in QML seems to be folklore in the literature. The last formula is often referred to as (an instance of) *Barcan formula.* Recall $\mathsf{Mod} = \{ \Box \}$ in $\mathsf{L}_{\mathsf{QML}}$.

Proposition 40. (Frame correspondence) Let P, Q be predicate symbols with arities 0 and 1, respectively.

1. $\mathrm{T} := \Box P \to P$ corresponds to the class of all the reflexive frames.
2. $\mathrm{D} := \Box P \to \Diamond P$ corresponds to the class of all the serial frames.
3. $\mathrm{B} := P \to \Box\Diamond P$ corresponds to the class of all the symmetric frames.
4. $4 := \Box P \to \Box\Box P$ corresponds to the class of all the transitive frames.
5. $5 := \Diamond P \to \Box\Diamond P$ corresponds to the class of all the euclidean frames.
6. $\mathrm{BF} := \forall x \Box Qx \to \Box \forall x Qx$ corresponds to the class of all the decreasing domain frames.

Remark 41. Given a frame $\mathfrak{F} = (W, R, (D_w)_{w \in W})$, we say

$\mathfrak{F}$ has *constant domains* if for all $w, v \in W$, $D_w = D_v$;
$\mathfrak{F}$ has *locally constant domains* if for all $w, v \in W$, if wRv then $D_w = D_v$.

Recall also that every frame has increasing domains by the definition of frames. This means that any symmetric frame has decreasing domains and that any decreasing domain frame has locally constant domains. In short, any symmetric frame has locally constant domains. However, even if a frame is symmetric it does not necessarily have constant domains, because if wRv fails then it can be the case that $D_w \neq D_v$.

(Taut1)	$\varphi \to (\psi \to \varphi)$	
(Taut2)	$(\varphi \to (\psi \to \gamma)) \to ((\varphi \to \psi) \to (\varphi \to \gamma))$	
(Taut3)	$(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$	
(U)	$\forall x\varphi \to \varphi(y/x)$	
(K)	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	
(MP)	From $\varphi \to \psi$ and φ, we may infer ψ	
(Gen)	From $\varphi \to \psi(y/x)$, we may infer $\varphi \to \forall x\psi$	if $y \notin \mathsf{FV}(\varphi, \forall x\psi)$
(Nec)	From φ, we may infer $\Box\varphi$	

Table 2.4: Hilbert system $\mathsf{H}(\mathbf{QK})$

2.2.2 Hilbert System $\mathsf{H}(\mathbf{QK}\Sigma)$

A Hilbert system $\mathsf{H}(\mathbf{QK})$ for QML consists of axioms and inference rules in Table 2.4, where all axioms and inference rules are presented as schemas.

Expansions of $\mathsf{H}(\mathbf{QK})$ for QML are obtained similarly to $\mathsf{H}(\mathbf{K}\Sigma)$. Put

$$\mathsf{Axiom}_{\mathsf{QML}} := \{\mathrm{T}, \mathrm{D}, \mathrm{B}, 4, 5, \mathrm{BF}\}.$$

For a set $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{QML}}$, we mean by $\mathsf{Inst}(\Sigma)$ the set of all instances of the schema corresponding to a formula of Σ which is listed in Table 2.5.

Formulas	Schemas
$\mathrm{T} = \Box P \to P$	$(\mathrm{T}) := \Box\varphi \to \varphi$
$\mathrm{D} = \Box P \to \Diamond P$	$(\mathrm{D}) := \Box\varphi \to \Diamond\varphi$
$\mathrm{B} = P \to \Box\Diamond P$	$(\mathrm{B}) := \varphi \to \Box\Diamond\varphi$
$4 = \Box P \to \Box\Box P$	$(4) := \Box\varphi \to \Box\Box\varphi$
$5 = \Diamond P \to \Box\Diamond P$	$(5) := \Diamond\varphi \to \Box\Diamond\varphi$
$\mathrm{BF} = \forall x\Box Qx \to \Box\forall xQx$	$(\mathrm{BF}) := \forall x\Box\varphi \to \Box\forall x\varphi$

Table 2.5: The schemas corresponding to formulas of Σ

Definition 42. Given a set Φ of formulas, the Hilbert system $\mathsf{H}(\mathbf{QK}\oplus\Phi)$ is the system obtained from $\mathsf{H}(\mathbf{QK})$ by adding all formulas of Φ as axioms. Given $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{QML}}$, $\mathsf{H}(\mathbf{QK}\Sigma)$ denotes the system $\mathsf{H}(\mathbf{QK} \oplus \mathsf{Inst}(\Sigma))$. By $\mathsf{H}(\mathbf{QKX}\Sigma)$ we mean the system $\mathsf{H}(\mathbf{QK}\{\mathrm{X}\} \cup \Sigma)$. We sometimes write $\mathsf{H}(\mathbf{QK}\{\mathrm{T}, 4, \mathrm{B}\})$ as $\mathsf{H}(\mathbf{QS5})$.

For example, $\mathsf{H}(\mathbf{QK}\{\mathrm{T}\})$ denotes $\mathsf{H}(\mathbf{QK}\oplus\mathsf{Inst}(\{\mathrm{T}\}))$. On the other hand, $\mathsf{H}(\mathbf{QKBF}\{\mathrm{T}\})$ means $\mathsf{H}(\mathbf{QK}\{\mathrm{BF}, \mathrm{T}\})$, which denotes $\mathsf{H}(\mathbf{QK} \oplus \mathsf{Inst}(\{\mathrm{BF}, \mathrm{T}\}))$.

The notion of proof in $\mathsf{H}(\mathbf{QK\Sigma})$ is defined as in $\mathsf{H}(\mathbf{K\Sigma})$. As well known in the literature, it is proved in Prior [78, p. 146] that if $\mathrm{B} \in \Sigma$ then all formulas of $\mathsf{Inst}(\mathrm{BF})$ are provable in $\mathsf{H}(\mathbf{QK\Sigma})$.

Proposition 43. Let $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{QML}}$ such that $\mathrm{B} \in \Sigma$. A formula $\forall x \Box \varphi \to \Box \forall x \varphi$ is provable in $\mathsf{H}(\mathbf{QK\Sigma})$.

The soundness theorem of $\mathsf{H}(\mathbf{QK\Sigma})$ is proved by induction on the length of a proof.

Theorem 44. (Soundness of $\mathsf{H}(\mathbf{QK\Sigma})$) Let $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{QML}}$ and $\mathbb{F}_\Sigma$ be the class of all the frames to which Σ corresponds. For all formulas φ, if φ is provable in $\mathsf{H}(\mathbf{QK\Sigma})$, then φ is valid in $\mathbb{F}_\Sigma$.

As for the strong completeness theorem of $\mathsf{H}(\mathbf{QK\Sigma})$, we will follow strategies introduced in Hughes and Cresswell [37, chs. 14–5]. Their strategies provide us the strong completeness theorems of two classes of the Hilbert systems given here. They are divided in terms of the existence of BF. The first class is without BF, i.e., when Σ is any subset of $\mathsf{Axiom}_{\mathsf{QML}} \backslash \{ \mathrm{BF}, 5 \}$. For example, $\mathsf{H}(\mathbf{QK}\{ \mathrm{T}, 4 \})$ and $\mathsf{H}(\mathbf{QK}\{ \mathrm{T}, 4, \mathrm{B} \})$ belong to the first class. The second class is with BF, i.e., when Σ is any subset of $\mathsf{Axiom}_{\mathsf{QML}}$ with $\mathrm{BF} \in \Sigma$. For example, $\mathsf{H}(\mathbf{QK}\{ 5, \mathrm{BF} \})$ and $\mathsf{H}(\mathbf{QK}\{ \mathrm{T}, 4, \mathrm{B}, \mathrm{BF} \})$ belong to the second class. Notice that, for *any* subset Σ of $\mathsf{Axiom}_{\mathsf{QML}}$, when $\mathrm{B} \in \Sigma$, $\mathsf{H}(\mathbf{QK\Sigma})$ is equipollent to $\mathsf{H}(\mathbf{QKBF\Sigma})$ by Proposition 43. This means that, say, strong completeness of $\mathsf{H}(\mathbf{QK}\{ \mathrm{T}, 4, \mathrm{B} \})$ of the first class is obtained by strong completeness of $\mathsf{H}(\mathbf{QK}\{ \mathrm{T}, 4, \mathrm{B}, \mathrm{BF} \})$ of the second class. In this thesis, we do *not* prove strong completeness of the class of all Hilbert systems $\mathsf{H}(\mathbf{QK5\Sigma})$ such that Σ is a subset of $\mathsf{Axiom}_{\mathsf{QML}} \backslash \{ \mathrm{B}, \mathrm{BF} \}$.

Let us first introduce new languages obtained from $\mathsf{L}_{\mathsf{QML}}$ and sets of variables.

Definition 45. We define Var^+ as $\mathsf{Var} \cup \mathsf{Var}'$, where Var' is a new countably infinite set of variables disjoint from Var. Given $V \subseteq \mathsf{Var}^+$, the language $\mathsf{L}_{\mathsf{QML}}(V)$ denotes the language obtained from $\mathsf{L}_{\mathsf{QML}}$ by replacing Var with V. Given a set Γ of formulas, $\mathsf{L}_{\mathsf{QML}}(\Gamma)$ denotes the language $\mathsf{L}_{\mathsf{QML}}(\mathsf{FV}(\Gamma))$. Given $V, V' \subseteq \mathsf{Var}^+$, by a notation $V \sqsubset V'$ we mean that $V \subseteq V'$ and $V' \backslash V$ is countably infinite.

Definition 46. Let $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{QML}}$. Given a set Γ of formulas, Γ is $\mathsf{H}(\mathbf{QK\Sigma})$*-inconsistent* if $\Gamma \vdash_{\mathsf{H}(\mathbf{QK\Sigma})} \bot$; Γ is $\mathsf{H}(\mathbf{QK\Sigma})$*-consistent* if Γ is not $\mathsf{H}(\mathbf{QK\Sigma})$-inconsistent; Γ is *maximal* if $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$ for all formulas φ in $\mathsf{L}(\Gamma)$; Γ is a *maximal* $\mathsf{H}(\mathbf{QK\Sigma})$*-consistent set* ($\mathsf{H}(\mathbf{QK\Sigma})$*-MCS*) if Γ is $\mathsf{H}(\mathbf{QK\Sigma})$-consistent and maximal; Γ is *witnessed* if, for all formulas of the form $\forall x \varphi$ in $\mathsf{L}(\Gamma)$, there exists some $y \in \mathsf{FV}(\Gamma)$ such that $\varphi(y/x) \to \forall x \varphi \in \Gamma$.

Proposition 47. Let $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{QML}}$, Γ a $\mathsf{H}(\mathbf{QK\Sigma})$-MCS in $\mathsf{L}(\mathsf{Var}^+)$ and φ, ψ formulas in $\mathsf{L}(\Gamma)$.

1. $\Gamma \vdash_{\mathsf{H}(\mathbf{QK}\Sigma)} \varphi$ iff $\varphi \in \Gamma$.
2. If $\varphi \in \Gamma$ and $\vdash_{\mathsf{H}(\mathbf{QK}\Sigma)} \varphi \to \psi$, then $\psi \in \Gamma$.
3. $\bot \notin \Gamma$.
4. $\varphi \to \psi \in \Gamma$ iff $\varphi \notin \Gamma$ or $\psi \in \Gamma$.

Strong Completeness of $\mathsf{H}(\mathbf{QK}\Sigma)$

Recall $\mathsf{Axiom}_{\mathsf{QML}} = \{\, \mathrm{T}, \mathrm{D}, \mathrm{B}, 4, 5, \mathrm{BF} \,\}$. Below, we abbreviate $\mathsf{H}(\mathbf{QK}\Sigma)$ as Λ for some fixed set $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{QML}} \backslash \{\, \mathrm{B}, 5, \mathrm{BF} \,\}$. The proofs of the following propositions, lemmas and theorem are found in [37, ch. 15].

Lemma 48. (Lindenbaum Lemma) Let Γ be a Λ-consistent set in $\mathsf{L}(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma) \sqsubset \mathsf{Var}^+$. There exists a witnessed Λ-MCS Γ^+ in $\mathsf{L}(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma^+) \sqsubset \mathsf{Var}^+$ and $\Gamma \subseteq \Gamma^+$.

Definition 49. The *canonical Λ-frame* is a tuple $\mathfrak{F}^\Lambda = (W^\Lambda, R^\Lambda, (D^\Lambda_\Gamma)_{\Gamma \in W^\Lambda})$, where

- $W^\Lambda := \{\, \Gamma \mid \Gamma$ is a witnessed Λ-MCS in $\mathsf{L}(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma) \sqsubset \mathsf{Var}^+ \,\}$;
- $\Gamma R^\Lambda \Delta$ iff $\Box\varphi \in \Gamma$ implies $\varphi \in \Delta$ for all formulas φ;
- $D^\Lambda_\Gamma := \mathsf{FV}(\Gamma)$.

The *canonical Λ-model* is a tuple $\mathfrak{M}^\Lambda = (\mathfrak{F}^\Lambda, I^\Lambda)$, where

- $\mathfrak{F}^\Lambda$ is the canonical Λ-frame;
- $(x_1, \ldots, x_n) \in I^\Lambda(P, \Gamma)$ iff $Px_1 \ldots x_n \in \Gamma$.

The *canonical assignment* is the assignment $\iota : \mathsf{Var}^+ \to D^\Lambda$ defined by $\iota(x) = x$.

Proposition 50. The canonical Λ-frame has increasing domains, i.e., for all $\Gamma, \Delta \in W^\Lambda$, if $\Gamma R^\Lambda \Delta$ then $D^\Lambda_\Gamma \subseteq D^\Lambda_\Delta$. Therefore, the canonical Λ-model *is* a model.

Lemma 51. (Existence Lemma) If $\neg\Box\varphi \in \Gamma \in W^\Lambda$, there exists some $\Delta \in W^\Lambda$ such that $\neg\varphi \in \Delta$ and $\Gamma R^\Lambda \Delta$.

Lemma 52. (Truth Lemma) Let $\mathfrak{M}^\Lambda$ be the canonical Λ-model and ι the canonical assignment. For all Λ-MCS $\Gamma \in W^\Lambda$ and formulas φ in $\mathsf{L}(\Gamma)$,

$$\mathfrak{M}^\Lambda, \Gamma, \iota \models \varphi \quad \text{iff} \quad \varphi \in \Gamma$$

Proposition 53. Let $\mathfrak{F}^\Lambda$ be the canonical Λ-frame.

1. If $\mathrm{T} \in \Sigma$ then $\mathfrak{F}^\Lambda$ is reflexive.

2. If $\mathrm{D} \in \Sigma$ then $\mathfrak{F}^\Lambda$ is serial.
3. If $4 \in \Sigma$ then $\mathfrak{F}^\Lambda$ is transitive.

Theorem 54. (Strong completeness of $\mathsf{H}(\mathbf{QK\Sigma})$) Let $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{QML}} \backslash \{ \mathrm{B}, 5, \mathrm{BF} \}$ and $\mathbb{F}_\Sigma$ be the class of all the frames to which Σ corresponds. For all formulas φ and sets Γ of formulas, if φ is a consequence from Γ in $\mathbb{F}_\Sigma$, then φ is provable from Γ in $\mathsf{H}(\mathbf{QK\Sigma})$.

Proof. By contraposition. Let $\Lambda = \mathsf{H}(\mathbf{QK\Sigma})$ for short. Suppose φ is not provable from Γ in Λ. We show φ is not a consequence from Γ in $\mathbb{F}_\Sigma$, i.e., there exist a frame $\mathfrak{F} \in \mathbb{F}_\Sigma$, an interpretation I, a world w and an assignment α such that ψ is an α_w-formula for all formulas ψ in $\Gamma \cup \{ \varphi \}$, and $(\mathfrak{F}, I), w, \alpha \models \Gamma$ but $(\mathfrak{F}, I), w, \alpha \not\models \varphi$. Note first that $\Gamma \cup \{ \neg\varphi \}$ is Λ-consistent in L. We claim $\Gamma \cup \{ \neg\varphi \}$ is Λ-consistent in $\mathsf{L}(\mathsf{Var}^+)$. By Lindenbaum Lemma (Lemma 48), we obtain a witnessed Λ-MCS Γ^+ in $\mathsf{L}(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma^+) \sqsubset \mathsf{Var}^+$ and $\Gamma \cup \{ \neg\varphi \} \subseteq \Gamma^+$. It then follows from Truth Lemma (Lemma 52) that

$$\mathfrak{M}^\Lambda, \Gamma^+, \iota \models \Gamma \quad \text{and} \quad \mathfrak{M}^\Lambda, \Gamma^+, \iota \not\models \varphi,$$

where $\mathfrak{M}^\Lambda = (\mathfrak{F}^\Lambda, I^\Lambda)$ is the canonical Λ-model and ι is the canonical assignment. We must further show that $\mathfrak{F}^\Lambda \in \mathbb{F}_\Sigma$, which is established by Proposition 53. Hence, in $\mathsf{L}(\mathsf{Var}^+)$, φ is not a consequence from Γ in $\mathbb{F}_\Sigma$. By restricting $\mathsf{L}(\mathsf{Var}^+)$ to L, we conclude in L that φ is not a consequence from Γ in $\mathbb{F}_\Sigma$. ∎

Strong Completeness of $\mathsf{H}(\mathbf{QKBF\Sigma})$

Recall $\mathsf{Axiom}_{\mathsf{QML}} = \{ \mathrm{T}, \mathrm{D}, \mathrm{B}, 4, 5, \mathrm{BF} \}$. Below, we abbreviate $\mathsf{H}(\mathbf{QKBF\Sigma})$ as Λ for some fixed set $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{QML}}$. The proofs of the following propositions, lemmas and theorem are found in [37, ch. 14].

Lemma 55. (Lindenbaum Lemma) Let Γ be a Λ-consistent set in $\mathsf{L}(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma) \sqsubset \mathsf{Var}^+$. There exists a witnessed Λ-MCS Γ^+ in $\mathsf{L}(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma^+) = \mathsf{Var}^+$ and $\Gamma \subseteq \Gamma^+$.

Definition 56. The *canonical* Λ*-frame* is a tuple $\mathfrak{F}^\Lambda = (W^\Lambda, R^\Lambda, (D^\Lambda_\Gamma)_{\Gamma \in W^\Lambda})$, where

- $W^\Lambda := \{ \Gamma \mid \Gamma \text{ is a witnessed } \Lambda\text{-MCS in } \mathsf{L}(\mathsf{Var}^+) \text{ such that } \mathsf{L}(\Gamma) = \mathsf{L}(\mathsf{Var}^+) \}$;
- $\Gamma R^\Lambda \Delta \quad \text{iff} \quad \Box\varphi \in \Gamma \text{ implies } \varphi \in \Delta \quad \text{for all formulas } \varphi$;
- $D^\Lambda_\Gamma := \mathsf{Var}^+$.

The *canonical* Λ*-model* is a tuple $\mathfrak{M}^\Lambda = (\mathfrak{F}^\Lambda, I^\Lambda)$, where

- $\mathfrak{F}^\Lambda$ is the canonical Λ-frame;

- $(x_1, \ldots, x_n) \in I^\Lambda(P, \Gamma)$ iff $Px_1 \ldots x_n \in \Gamma$.

The *canonical assignment* is the assignment $\iota : \mathsf{Var}^+ \to D^\Lambda$ defined by $\iota(x) = x$.

Proposition 57. The canonical Λ-frame has increasing domains, i.e., for all $\Gamma, \Delta \in W^\Lambda$, if $\Gamma R^\Lambda \Delta$ then $D^\Lambda_\Gamma \subseteq D^\Lambda_\Delta$. Therefore, the canonical Λ-model *is* a model.

Lemma 58. (Existence Lemma) If $\neg\Box\varphi \in \Gamma \in W^\Lambda$, there exists some $\Delta \in W^\Lambda$ such that $\neg\varphi \in \Delta$ and $\Gamma R^\Lambda \Delta$.

Lemma 59. (Truth Lemma) Let $\mathfrak{M}^\Lambda$ be the canonical Λ-model and ι the canonical assignment. For all Λ-MCS $\Gamma \in W^\Lambda$ and formulas φ in $\mathsf{L}(\mathsf{Var}^+)$,

$$\mathfrak{M}^\Lambda, \Gamma, \iota \models \varphi \quad \text{iff} \quad \varphi \in \Gamma$$

Proposition 60. Let $\mathfrak{F}^\Lambda$ be the canonical Λ-frame.

1. $\mathfrak{F}^\Lambda$ has constant domains.
2. If $\mathrm{T} \in \Sigma$ then $\mathfrak{F}^\Lambda$ is reflexive.
3. If $\mathrm{D} \in \Sigma$ then $\mathfrak{F}^\Lambda$ is serial.
4. If $\mathrm{B} \in \Sigma$ then $\mathfrak{F}^\Lambda$ is symmetric.
5. If $4 \in \Sigma$ then $\mathfrak{F}^\Lambda$ is transitive.
6. If $5 \in \Sigma$ then $\mathfrak{F}^\Lambda$ is euclidean.

Definition 61. By $\mathbb{CD}$ and $\mathbb{LCD}$, we mean the class of all the constant domain frames and the class of all the locally constant domain frames, respectively.

Theorem 62. (Strong completeness of $\mathsf{H}(\mathbf{QKBF}\Sigma)$) Let $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{QML}}$ and $\mathbb{F}_\Sigma$ be the class of all the frames to which Σ corresponds. For all formulas φ and sets Γ of formulas, if φ is a consequence from Γ in $\mathbb{F}_\Sigma \cap \mathbb{LCD}$, then φ is provable from Γ in $\mathsf{H}(\mathbf{QKBF}\Sigma)$.

Proof. By contraposition. Let $\Lambda = \mathsf{H}(\mathbf{QKBF}\Sigma)$ for short. Suppose φ is not provable from Γ in Λ. Then $\Gamma \cup \{\neg\varphi\}$ is Λ-consistent in L. We claim $\Gamma \cup \{\neg\varphi\}$ is Λ-consistent in $\mathsf{L}(\mathsf{Var}^+)$. Similarly as in the proof of Theorem 54, we obtain by Lindenbaum Lemma (Lemma 55) a witnessed Λ-MCS Γ^+ in $\mathsf{L}(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma^+) = \mathsf{Var}^+$ and $\Gamma \cup \{\neg\varphi\} \subseteq \Gamma^+$. It follows from Truth Lemma (Lemma 59) that

$$\mathfrak{M}^\Lambda, \Gamma^+, \iota \models \Gamma \quad \text{and} \quad \mathfrak{M}^\Lambda, \Gamma^+, \iota \not\models \varphi,$$

where $\mathfrak{M}^\Lambda = (\mathfrak{F}^\Lambda, I^\Lambda)$ is the canonical Λ-model and ι is the canonical assignment. We also need to establish that $\mathfrak{F}^\Lambda \in \mathbb{F}_\Sigma \cap \mathbb{LCD}$ holds. This is established since $\mathfrak{F}^\Lambda \in$

$\mathbb{F}_\Sigma \cap \mathbb{CD}$ by Proposition 60 and $\mathbb{F}_\Sigma \cap \mathbb{CD} \subseteq \mathbb{F}_\Sigma \cap \mathbb{LCD}$. Therefore, in $\mathsf{L}(\mathsf{Var}^+)$, φ is not a consequence from Γ in $\mathbb{F}_\Sigma \cap \mathbb{LCD}$. By restricting $\mathsf{L}(\mathsf{Var}^+)$ to L, we conclude in L that φ is not a consequence from Γ in $\mathbb{F}_\Sigma \cap \mathbb{LCD}$. ∎

We will make a final remark in this chapter. Taking into account the fact that every frame has increasing domains, it holds that BF corresponds to $\mathbb{LCD}$. Contrary to this, there is no formula which corresponds to $\mathbb{CD}$. This follows from the following stronger proposition.

Proposition 63. There exists no set of formulas which corresponds to $\mathbb{CD}$.

Proof. Suppose not. There exists a set Γ of formulas such that $\mathfrak{F} \models \Gamma$ iff $\mathfrak{F} \in \mathbb{CD}$ for all frames. Then $\mathbb{CD} \models \Gamma$. With the help of strong completeness of $\mathsf{H}(\mathbf{QKBF})$ obtained from completeness of $\mathsf{H}(\mathbf{QKBF})$ given in [37, ch. 14], we can deduce that φ is provable in $\mathsf{H}(\mathbf{QKBF})$ for all $\varphi \in \Gamma$. Take a frame $\mathfrak{G}$ which has locally constant domains but does not have constant domains. By soundness (Theorem 44), we have $\mathfrak{G} \models \Gamma$. Since Γ corresponds to $\mathbb{CD}$, this implies $\mathfrak{G} \in \mathbb{CD}$. However, $\mathfrak{G}$ does not have constant domains so a contradiction. ∎

2.2.3 Sequent Calculus $\mathsf{G}(\mathbf{QK}\Sigma)$

We can present a sequent calculus for QML in a similar way as for ML.

As such a sequent calculus for first order logic, we introduce $\mathsf{G}(\mathbf{FOL})$ consisting of initial sequents (id), $(\bot)$, structural rules $(\Rightarrow w)$, $(w \Rightarrow)$, $(\Rightarrow c)$, $(c \Rightarrow)$, (Cut) and logical rules $(\Rightarrow \to)$, $(\to \Rightarrow)$, $(\Rightarrow \forall)$, $(\forall \Rightarrow)$ in Table 2.6, where they are presented as schemas. A sequent calculus $\mathsf{G}(\mathbf{QK})$ and its expansions are obtained from $\mathsf{G}(\mathbf{FOL})$ as follows. Put

$$\mathsf{Axiom}^-_{\mathsf{QML}} := \{\, \mathrm{D}, \mathrm{T}, 4 \,\}.$$

Definition 64. Given $\Sigma \subseteq \mathsf{Axiom}^-_{\mathsf{ML}}$, the sequent calculus $\mathsf{G}(\mathbf{QK}\Sigma)$ is the calculus obtained from $\mathsf{G}(\mathbf{QK})$ by adding all logical rules for Σ in Table 2.6, where we define logical rules for $\{\, \mathrm{D}, \mathrm{T} \,\}$ and $\{\, \mathrm{D}, \mathrm{T}, 4 \,\}$ by those for $\{\, \mathrm{T} \,\}$ and $\{\, \mathrm{T}, 4 \,\}$, respectively. Irrespective of Σ, the sequent calculus $\mathsf{G}(\mathbf{QS5})$ is the calculus obtained from $\mathsf{G}(\mathbf{QK})$ by adding logical rules $(\Box \mathrm{S5})$ and $(\Box \mathrm{T})$ for $\{\, \mathrm{T}, \mathrm{B}, 4 \,\}$ in Table 2.6.

The notion of derivation in $\mathsf{G}(\mathbf{QK}\Sigma)$ and $\mathsf{G}(\mathbf{QS5})$ is defined as in $\mathsf{G}(\mathbf{K}\Sigma)$.

Equipollence is obtained in a similar vein as in $\mathsf{G}(\mathbf{K}\Sigma)$ and $\mathsf{G}(\mathbf{S5})$ [40].

Proposition 65. (Equipollence of $\mathsf{H}(\mathbf{QK}\Sigma)$ and $\mathsf{G}(\mathbf{QK}\Sigma)$) Let $\Sigma \subseteq \mathsf{Axiom}^-_{\mathsf{QML}}$. A formula φ is provable in $\mathsf{H}(\mathbf{QK}\Sigma)$ iff a sequent $\Rightarrow \varphi$ is derivable in $\mathsf{G}(\mathbf{QK}\Sigma)$.

Sequent Calculus G(**FOL**)

$$\frac{}{\varphi \Rightarrow \varphi}\ (id) \qquad \frac{}{\bot \Rightarrow}\ (\bot)$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}\ (\Rightarrow w) \qquad \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (w \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}\ (\Rightarrow c) \qquad \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (c \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Xi \Rightarrow \Sigma}{\Gamma, \Xi \Rightarrow \Delta, \Sigma}\ (Cut)$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}\ (\Rightarrow \to) \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Xi \Rightarrow \Sigma}{\varphi \to \psi, \Gamma, \Xi \Rightarrow \Delta, \Sigma}\ (\to \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi(y/x)}{\Gamma \Rightarrow \Delta, \forall x \varphi}\ (\Rightarrow \forall)^{\dagger} \qquad \frac{\varphi(t/x), \Gamma \Rightarrow \Delta}{\forall x \varphi, \Gamma \Rightarrow \Delta}\ (\forall \Rightarrow)$$

where †: y is not a free variable in $\Gamma, \Delta, \forall x \varphi$.

Σ	Logical Rules for Σ	
$\emptyset$	$\dfrac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}\ (\Box\mathrm{K})$	
$\{\mathrm{D}\}$	$\dfrac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}\ (\Box\mathrm{K})$	$\dfrac{\Gamma \Rightarrow}{\Box\Gamma \Rightarrow}\ (\Box\mathrm{D})$
$\{\mathrm{T}\}$	$\dfrac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}\ (\Box\mathrm{K})$	$\dfrac{\varphi, \Gamma \Rightarrow \Delta}{\Box\varphi, \Gamma \Rightarrow \Delta}\ (\Box\mathrm{T})$
$\{4\}$	$\dfrac{\Gamma, \Box\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}\ (\Box 4_{\mathrm{inv}})$	
$\{\mathrm{D}, 4\}$	$\dfrac{\Gamma, \Box\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}\ (\Box 4_{\mathrm{inv}})$	$\dfrac{\Gamma, \Box\Gamma \Rightarrow}{\Box\Gamma \Rightarrow}\ (\Box\mathrm{D})$
$\{\mathrm{T}, 4\}$	$\dfrac{\Box\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}\ (\Box\mathrm{S4})$	$\dfrac{\varphi, \Gamma \Rightarrow \Delta}{\Box\varphi, \Gamma \Rightarrow \Delta}\ (\Box\mathrm{T})$

where $\Box\Gamma := \{\Box\varphi \mid \varphi \in \Gamma\}$.

	Logical Rules for $\{\mathrm{T}, \mathrm{B}, 4\}$	
$\{\mathrm{T}, \mathrm{B}, 4\}$	$\dfrac{\Box\Gamma \Rightarrow \Box\Delta, \varphi}{\Box\Gamma \Rightarrow \Box\Delta, \Box\varphi}\ (\Box\mathrm{S5})$	$\dfrac{\varphi, \Gamma \Rightarrow \Delta}{\Box\varphi, \Gamma \Rightarrow \Delta}\ (\Box\mathrm{T})$

where $\Box\Gamma := \{\Box\varphi \mid \varphi \in \Gamma\}$.

Table 2.6: Sequent Calculus G(**QK**Σ)

Proposition 66. (Equipollence of $\mathsf{H}(\mathbf{QS5})$ and $\mathrm{G}(\mathbf{QS5})$) A formula φ is provable in $\mathsf{H}(\mathbf{QS5})$ iff a sequent $\Rightarrow \varphi$ is derivable in $\mathrm{G}(\mathbf{QS5})$.

The cut elimination theorem, the consistency and the Craig interpolation theorem of $\mathrm{G}(\mathbf{QK}\Sigma)$ are also obtained in a similar vein as in $\mathrm{G}(\mathbf{K}\Sigma)$ [67]. In what follows, we do not take $\mathrm{G}(\mathbf{QS5})$ into account. Let $\mathrm{G}^{-}(\mathbf{QK}\Sigma)$ be the sequent calculus obtained from $\mathrm{G}(\mathbf{QK}\Sigma)$ by removing (Cut). Also, let $\mathrm{Pred}(\Gamma)$ be the set of all predicate symbols in a set Γ of formulas.

Theorem 67. (Cut elimination) Let $\Sigma \subseteq \mathsf{Axiom}^{-}_{\mathsf{QML}}$. If a sequent $\Gamma \Rightarrow \Delta$ is derivable in $\mathrm{G}(\mathbf{QK}\Sigma)$, then $\Gamma \Rightarrow \Delta$ is also derivable in $\mathrm{G}^{-}(\mathbf{QK}\Sigma)$.

Corollary 68. Let $\Sigma \subseteq \mathsf{Axiom}^{-}_{\mathsf{QML}}$. A sequent $\Rightarrow \bot$ is not derivable in $\mathrm{G}(\mathbf{QK}\Sigma)$.

Theorem 69. (Craig interpolation theorem) Let $\Sigma \subseteq \mathsf{Axiom}^{-}_{\mathsf{QML}}$. If $\Rightarrow \varphi \to \psi$ is derivable in $\mathrm{G}(\mathbf{QK}\Sigma)$, then there exists a formula γ such that

- $\vdash_{\mathrm{G}(\mathbf{QK}\Sigma)} \varphi \to \gamma$ and $\vdash_{\mathrm{G}(\mathbf{QK}\Sigma)} \Rightarrow \gamma \to \psi$;
- $\mathsf{Pred}(\gamma) \subseteq \mathsf{Pred}(\varphi) \cap \mathsf{Pred}(\psi)$;
- $\mathsf{FV}(\gamma) \subseteq \mathsf{FV}(\varphi) \cap \mathsf{FV}(\psi)$.

Remark 70. In Remark 30, we remarked that $\mathrm{G}(\mathbf{S5})$ enjoys the Craig interpolation theorem. Contrary to this fact, it is proved in Fine [15] that $\mathrm{G}(\mathbf{QS5})$ *does not* enjoy the Craig interpolation theorem.

Chapter 3
Term-Sequence-Modal Logic

In this chapter, we build term-sequence-modal logic subsuming the original term-modal logic provided by Thalmann [92] and Fitting et al. [17]. Term-sequence-modal logic can have, for example, not only a modal operator $[t]$ indexed by a single term t, but also a modal operator $[t, s, u]$ indexed by a sequence (t, s, u) of terms t, s, u. By combining such modal operators with quantifiers, this logic can form a formula like $\forall x \forall y \forall z [x, y, z] Pxy$. Section 3.1 develops sound and strongly complete Hilbert systems of term-sequence-modal logics with respect to given semantics. Section 3.2 presents cut-free ordinary sequent calculi, i.e., non-labelled and two-sided sequent calculi in which (Cut) is dispensable, for some of the given logics. In the same place, it is also proved that these sequent calculi admit the Craig interpolation theorems. Section 3.3 further considers term-sequence-modal logics with equality to show soundness and strong completeness of Hilbert systems for them.

This chapter is based on Sawasaki et al. [85] "Term-Sequence-Modal Logic".

3.1 From Term to Term-Sequence

Term-sequence-modal logic has advantages over the original term-modal logic. First, term-sequence-modal logic is mathematically quite natural, just as predicate symbols in first order logic are allowed to take (a finite sequence of) terms more than one. It is also a useful generalization to express a modality relative to multiple agents like an obligation of someone towards someone.

We first describe a syntax and a Kripke semantics for term-sequence-modal logic (3.1.1). In the description, we examine some frame correspondences in this logic. We then put forth Hilbert systems for term-sequence-modal logics, proving soundness and strong completeness of some class of them (3.1.2).

3.1.1 Syntax and Kripke Semantics

Definition 71. The *language* $\mathsf{L}_{\mathsf{TSML}}$ of *term-sequence-modal logic* (TSML) consists of a countably infinite set $\mathsf{Var} = \{ x, y, \dots \}$ of variables, a countable set $\mathsf{Con} = \{ c, d, \dots \}$ of *constant symbols*, a countable set $\mathsf{Func} = \{ f, g, \dots \}$ of *function symbols* each of which has a fixed finite arity more than zero, the union $\mathsf{Pred} = \bigcup_{n \in \mathbb{N}} \mathsf{Pred}_n$ of countably infinite sets $\mathsf{Pred}_n = \{ P, Q, \dots \}$ of predicate symbols with arity n, the set $\{ \bot, \to, \forall \}$ of logical constants, and the set $\mathsf{Mod} = \{ [\cdot] \}$ of the binary modal operator $[\cdot]$. A *term* t *in* $\mathsf{L}_{\mathsf{TSML}}$ is recursively defined by

$$t ::= x \mid c \mid f(t_1, \dots, t_n)$$

where x is a variable, c is a constant symbol and f is a function symbol with arity n. A *formula* φ *in* $\mathsf{L}_{\mathsf{TSML}}$ is recursively defined by

$$\varphi ::= Pt_1 \dots t_n \mid \bot \mid (\varphi \to \varphi) \mid \forall x \varphi \mid [t_1, \dots, t_n]\varphi$$

where P is a predicate symbol with arity n, x is a variable, and $t_1, \dots, t_n$ are terms. A *sublanguage* L_n of $\mathsf{L}_{\mathsf{TSML}}$ is the language $\mathsf{L}_{\mathsf{TSML}}$ whose formulas are defined such that any term-sequence in the modal operator $[\cdot]$ is of length n. Instead of $\mathsf{L}_{\mathsf{TSML}}$, we often write L when it is clear from the context. We also write $\mathsf{L}_{\mathsf{TSML}}(V)$ to denote $\mathsf{L}_{\mathsf{TSML}}$ in which the set of variables is V.

Formulas $\langle t_1, \dots, t_n \rangle \varphi$, $\Box\varphi$ and $\Diamond\varphi$ are defined by $\neg[t_1, \dots, t_n]\neg\varphi$, $[\varepsilon]\varphi$ and $\langle\varepsilon\rangle\varphi$, respectively, where ε is an empty sequence. It should be noted that the sublanguages L_0 and L_1 can be seen as the language $\mathsf{L}_{\mathsf{QML}}$ of QML in Section 2.2 and a language of the original term-modal logic in Thalmann [92] and Fitting et al. [17], respectively. The other Boolean connectives and the existential quantifier are defined as in Section 2.2.

Definition 72. The set $\mathsf{V}(t)$ of *variables in a term* t is recursively defined as follows.

1. $\mathsf{V}(x) = \{ x \}$.
2. $\mathsf{V}(c) = \emptyset$.
3. $\mathsf{V}(f(t_1, \dots, t_n)) = \mathsf{V}(t_1) \cup \dots \cup \mathsf{V}(t_n)$.

The notation $\mathsf{V}(t_1, \dots, t_n)$ means $\mathsf{V}(t_1) \cup \dots \cup \mathsf{V}(t_n)$.

Let φ be a formula, Γ a set of formulas, t, s terms, x a variable. The length $l(\varphi)$ of φ is defined as in $\mathsf{L}_{\mathsf{QML}}$ except that

$$l([t_1, \dots, t_n]\varphi) = l(\varphi) + 1.$$

The sets $\mathsf{FV}(\varphi)$ and $\mathsf{FV}(\Gamma)$ of free variables in φ and Γ are defined as in $\mathsf{L}_{\mathsf{QML}}$ except that

$$\mathsf{FV}([t_1, \dots, t_n]\varphi) = \mathsf{V}(t_1, \dots, t_n) \cup \mathsf{FV}(\varphi).$$

The substitutions $t(s/x)$ and $\varphi(s/x)$ of s for x in t and φ are defined as in $\mathsf{L}_{\mathsf{QML}}$ except that

$$([t_1, \ldots, t_n]\varphi)(s/x) = [t_1(s/x), \ldots, t_n(s/x)]\varphi(s/x).$$

We often write $\vec{t}$ instead of $(t_1, \ldots, t_n)$ for short.

The Kripke semantics for $\mathsf{L}_{\mathsf{TSML}}$ is almost the same as the increasing domain semantics for $\mathsf{L}_{\mathsf{QML}}$ given in Section 2.2. For simplicity, we interpret both of constant and function symbols rigidly. In other words, their interpretations are invariant across the worlds. The key modification is to relativize binary relations on W to finite sequences of objects.

Definition 73. A *frame* (*for* $\mathsf{L}_{\mathsf{TSML}}$) is a tuple $\mathfrak{F} = (W, R, (D_w)_{w\in W})$, where W is a non-empty set of worlds; D_w is a non-empty domain of a world w[1]; R is a function that maps each $\vec{d} \in \bigcup_{w\in W}(D_w^{<\omega})$ to a binary relation $R_{\vec{d}}$ on W, where $\vec{d}$ is a finite sequence of objects and $D_w^{<\omega}$ is the set of all finite sequences of D_w; $\mathfrak{F}$ has *increasing domains* (*in TSML*):

$$\text{for all } w, v \in W \text{ and } \vec{d} \in D_w^{<\omega}, \text{ if } wR_{\vec{d}}v \text{ then } D_w \subseteq D_v.^2$$

Let $\mathfrak{F}$ be a frame and define D as $\bigcup_{w\in W} D_w$. A *model* (*for* $\mathsf{L}_{\mathsf{TSML}}$) is a tuple $\mathfrak{M} = (\mathfrak{F}, I)$, where I is an interpretation that maps each predicate symbol P with arity n and world w to a subset $I(P, w)$ of D_w^n; each constant symbol to an object $I(c) \in \bigcap_{w\in W} D_w$; each function symbol with arity n to an n-place function $I(f) : D^n \to D$ such that $I(f)(d_1, \ldots, d_n) \in D_w$ for all $w \in W$ and $(d_1, \ldots, d_n) \in D_w^n$. We sometimes write `True` and `False` instead of D_w^0 and $\emptyset$, respectively.

The notion of assignment in $\mathsf{L}_{\mathsf{QML}}$ is carried over with one modification. That is, we extend the domain of an assignment α to the set of terms by letting $\alpha(c) := I(c)$ and $\alpha(f(t_1, \ldots, t_n)) := I(f)(\alpha(t_1), \ldots, \alpha(t_n))$. Given an assignment α and a sequence $\vec{t} = (t_1, \ldots, t_n)$ of terms, we write $\alpha(\vec{t})$ to mean $\alpha(t_1, \ldots, t_n)$, i.e., $(\alpha(t_1), \ldots, \alpha(t_n))$.

Definition 74. Let $\mathfrak{M} = (W, R, (D_w)_{w\in W}, I)$ be a model, w a world, α an assignment, and φ a formula such that φ is an α_w-formula. The *satisfaction relation* $\mathfrak{M}, w, \alpha \models \varphi$ *between* $\mathfrak{M}, w, \alpha$ *and* φ is defined as follows.

$$\begin{array}{lll}
\mathfrak{M}, w, \alpha \models Pt_1 \ldots t_n & \text{iff} & \alpha(t_1, \ldots, t_n) \in I(P, w) \\
\mathfrak{M}, w, \alpha \not\models \bot & & \\
\mathfrak{M}, w, \alpha \models \varphi \to \psi & \text{iff} & \mathfrak{M}, w, \alpha \models \varphi \text{ implies } \mathfrak{M}, w, \alpha \models \psi \\
\mathfrak{M}, w, \alpha \models \forall x\varphi & \text{iff} & \text{for all objects } d \in D_w,\ \ \mathfrak{M}, w, \alpha(x|d) \models \varphi \\
\mathfrak{M}, w, \alpha \models [\vec{t}]\varphi & \text{iff} & \text{for all worlds } v \in W,\ \ wR_{\alpha(\vec{t})}v \text{ implies } \mathfrak{M}, v, \alpha \models \varphi
\end{array}$$

[1]When Con $\neq \emptyset$, it is also required that $\bigcap_{w\in W} D_w \neq \emptyset$.

[2]This corresponds to the weakened monotonicity condition mentioned in the footnote 1 of Thalmann [92, p. 19] and Fitting et al. [17, p. 139].

For a set Γ of formulas, $\mathfrak{M}, w, \alpha \models \Gamma$ means that $\mathfrak{M}, w, \alpha \models \psi$ for all formulas $\psi \in \Gamma$.

Remark 75. Consider the modal operators $\Box = [\varepsilon]$ and $[t]$. Then the satisfaction relations of $\Box\varphi$ and $[t]\varphi$ are

$$\begin{array}{lll} \mathfrak{M}, w, \alpha \models \Box\varphi & \text{iff} & \text{for all worlds } v \in W, \ \ wR_{\varepsilon}v \text{ implies } \mathfrak{M}, v, \alpha \models \varphi; \\ \mathfrak{M}, w, \alpha \models [t]\varphi & \text{iff} & \text{for all worlds } v \in W, \ \ wR_{\alpha(t)}v \text{ implies } \mathfrak{M}, v, \alpha \models \varphi \end{array}$$

respectively, which are essentially the same as the satisfaction relations of $\Box\varphi$ in $\mathsf{L}_{\mathsf{QML}}$ and $[t]\varphi$ in a language of TML.

Definition 76. Let φ be a formula and Γ a set of formulas.

- φ is *valid in a frame* $\mathfrak{F}$ (*for* $\mathsf{L}_{\mathsf{TSML}}$), denoted by $\mathfrak{F} \models \varphi$, if $(\mathfrak{F}, I), w, \alpha \models \varphi$ for all interpretations I, worlds w and assignments α such that φ is an α_w-formula.
- Γ is *valid in a frame* $\mathfrak{F}$ (*for* $\mathsf{L}_{\mathsf{TSML}}$), denoted by $\mathfrak{F} \models \Gamma$, if $\mathfrak{F} \models \gamma$ for all formulas $\gamma \in \Gamma$.
- φ is *valid in a class* $\mathbb{F}$ *of frames* (*for* $\mathsf{L}_{\mathsf{TSML}}$), denoted by $\mathbb{F} \models \varphi$, if $\mathfrak{F} \models \varphi$ for all frames $\mathfrak{F} \in \mathbb{F}$.
- φ is a *consequence from* Γ *in a class* $\mathbb{F}$ *of frames* (*for* $\mathsf{L}_{\mathsf{TSML}}$) if $(\mathfrak{F}, I), w, \alpha \models \Gamma$ implies $(\mathfrak{F}, I), w, \alpha \models \varphi$ for all frames $\mathfrak{F} \in \mathbb{F}$, interpretations I, worlds w, assignments α such that ψ is an α_w-formula for all formulas ψ in $\Gamma \cup \{\varphi\}$.

As before, the notion of frame correspondence in term-sequence-modal logic is defined as follows.

Definition 77. A set Γ of formulas in $\mathsf{L}_{\mathsf{TSML}}$ *corresponds* to a class $\mathbb{F}$ of frames if the equivalence

$$\mathfrak{F} \models \Gamma \quad \text{iff} \quad \mathfrak{F} \in \mathbb{F}$$

holds for all frames $\mathfrak{F}$. If $\Gamma = \{\varphi\}$, we just say that φ corresponds to $\mathbb{F}$.

Definition 78 and Proposition 79 are analogues of Definition 7 and Proposition 40, respectively.

Definition 78. Let $\mathfrak{F} = (W, R, (D_w)_{w\in W})$ be a frame.

1. $\mathfrak{F}$ is *n-reflexive* if for all $w \in W$ and $\vec{d} \in D_w^n$, $wR_{\vec{d}}w$.
2. $\mathfrak{F}$ is *n-serial* if for all $w \in W$ and $\vec{d} \in D_w^n$, there exists some $v \in W$ such that $wR_{\vec{d}}v$.
3. $\mathfrak{F}$ is *n-symmetric* if for all $w, v \in W$ and $\vec{d} \in D_w^n$, if $wR_{\vec{d}}v$ then $vR_{\vec{d}}w$.
4. $\mathfrak{F}$ is *n-transitive* if for all $w, v, u \in W$ and $\vec{d} \in D_w^n$, if $wR_{\vec{d}}v$ and $vR_{\vec{d}}u$ then $wR_{\vec{d}}u$.

5. $\mathfrak{F}$ is *n-euclidean* if for all $w, v, u \in W$ and $\vec{d} \in D_w^n$, if $wR_{\vec{d}}v$ and $wR_{\vec{d}}u$ then $vR_{\vec{d}}u$.
6. $\mathfrak{F}$ has *n-decreasing domains* if for all $w, v \in W$ and $\vec{d} \in D_w^n$, if $wR_{\vec{d}}v$ then $D_w \supseteq D_v$

Proposition 79. (Frame correspondence) Let $x_1, \ldots, x_n, y$ be pairwise distinct variables, $\vec{x} = (x_1, \ldots, x_n)$, and P, Q predicate symbols with arities 0 and 1, respectively.

1. $\mathrm{T}_n := [\vec{x}]P \to P$ corresponds to the class of all the n-reflexive frames.
2. $\mathrm{D}_n := [\vec{x}]P \to \langle\vec{x}\rangle P$ corresponds to the class of all the n-serial frames.
3. $\mathrm{B}_n := P \to [\vec{x}]\langle\vec{x}\rangle P$ corresponds to the class of all the n-symmetric frames.
4. $4_n := [\vec{x}]P \to [\vec{x}][\vec{x}]P$ corresponds to the class of all the n-transitive frames.
5. $5_n := \langle\vec{x}\rangle P \to [\vec{x}]\langle\vec{x}\rangle P$ corresponds to the class of all the n-euclidean frames.
6. $\mathrm{BF}_n := \forall y[\vec{x}]Qy \to [\vec{x}]\forall yQy$ corresponds to the class of all the n-decreasing domain frames, where y is distinct from each x_i of $\vec{x}$.

Proof.

1. For one direction, suppose $\mathfrak{F}$ is n-reflexive, i.e., for all $w \in W$ and $\vec{d} \in D_w^n$, $wR_{\vec{d}}w$ holds. We show $\mathfrak{F} \models [\vec{x}]P \to P$. Take any interpretation I, world w, assignment α such that $[\vec{x}]P \to P$ is an α_w-formula. Suppose $(\mathfrak{F}, I), w, \alpha \models [\vec{x}]P$. Since $\alpha(x_i) \in D_w$ and $\mathfrak{F}$ is n-reflexive, we have $wR_{\alpha(\vec{x})}w$. Hence $(\mathfrak{F}, I), w, \alpha \models P$, as required.

   For the other direction, suppose $\mathfrak{F} \models [\vec{x}]P \to P$. Take any $w \in W$, $\vec{d} \in D_w^n$. We show $wR_{\vec{d}}w$. Let I and α be an interpretation and an assignment such that $I(P, w) = \mathtt{False}$; $I(P, w') = \mathtt{True}$ for all $w' \neq w$; $\alpha(x_i) = d_i$. Then $(\mathfrak{F}, I), w, \alpha \not\models P$. Together with $\mathfrak{F} \models [\vec{x}]P \to P$, we have $(\mathfrak{F}, I), w, \alpha \not\models [\vec{x}]P$. It follows from $\alpha(\vec{x}) = \vec{d}$ that there exists some world v such that $wR_{\vec{d}}v$ and $(\mathfrak{F}, I), v, \alpha \not\models P$. Such a v must be w since $(\mathfrak{F}, I), w', \alpha \models P$ for all $w' \neq w$. Hence $wR_{\vec{d}}w$.

2. For one direction, suppose $\mathfrak{F}$ is n-serial, i.e., for all $w \in W$ and $\vec{d} \in D_w^n$, there exists some $v \in W$ such that $wR_{\vec{d}}v$. We show $\mathfrak{F} \models [\vec{x}]P \to \langle\vec{x}\rangle P$. Take any interpretation I, world w, assignment α such that $[\vec{x}]P \to \langle\vec{x}\rangle P$ is an α_w-formula. Suppose $(\mathfrak{F}, I), w, \alpha \models [\vec{x}]P$. Our goal is $(\mathfrak{F}, I), w, \alpha \models \langle\vec{x}\rangle P$. Since $\alpha(x_i) \in D_w$ and $\mathfrak{F}$ is n-serial, we have some world v such that $wR_{\alpha(\vec{x})}v$. Together with $(\mathfrak{F}, I), w, \alpha \models [\vec{x}]P$, we also have $(\mathfrak{F}, I), v, \alpha \models P$. Hence $(\mathfrak{F}, I), w, \alpha \models \langle\vec{x}\rangle P$.

   For the other direction, suppose $\mathfrak{F} \models [\vec{x}]P \to \langle\vec{x}\rangle P$. Take any $w \in W$, $\vec{d} \in D_w^n$. We show there exists some $v \in W$ such that $wR_{\vec{d}}v$. Let I and α be an interpretation and an assignment such that $I(P, v) = \texttt{True}$ for all $v \in W$; $\alpha(x_i) = d_i$. Then $(\mathfrak{F}, I), w, \alpha \models [\vec{x}]P$. Together with $\mathfrak{F} \models [\vec{x}]P \to \langle\vec{x}\rangle P$, we have $(\mathfrak{F}, I), w, \alpha \models \langle\vec{x}\rangle P$. It follows from $\alpha(\vec{x}) = \vec{d}$ that there exists some $v \in W$ such that $wR_{\vec{d}}v$.

3. For one direction, suppose $\mathfrak{F}$ is n-symmetric, i.e., for all $w, v \in W$ and $\vec{d} \in D_w^n$, if $wR_{\vec{d}}v$ then $vR_{\vec{d}}w$. We show $\mathfrak{F} \models P \to [\vec{x}]\langle\vec{x}\rangle P$. Take any interpretation I, world w, assignment α such that $P \to [\vec{x}]\langle\vec{x}\rangle P$ is an α_w-formula. Suppose $(\mathfrak{F}, I), w, \alpha \models P$ and take any world v such that $wR_{\alpha(\vec{x})}v$. Our goal is to show$(\mathfrak{F}, I), v, \alpha \models \langle\vec{x}\rangle P$. Since $\alpha(x_i) \in D_w$ and $\mathfrak{F}$ is n-symmetric, we have $vR_{\alpha(\vec{x})}w$. Together with $(\mathfrak{F}, I), w, \alpha \models P$, we obtain $(\mathfrak{F}, I), v, \alpha \models \langle\vec{x}\rangle P$.

   For the other direction, suppose $\mathfrak{F} \models P \to [\vec{x}]\langle\vec{x}\rangle P$. Take any $w, v \in W$, $\vec{d} \in D_w^n$ such that $wR_{\vec{d}}v$. We show $vR_{\vec{d}}w$. Let I and α be an interpretation and an assignment such that $I(P, w) = \texttt{True}$; $I(P, w') = \texttt{False}$ for all $w' \neq w$; $\alpha(x_i) = d_i$. Then $(\mathfrak{F}, I), w, \alpha \models P$. Together with $\mathfrak{F} \models P \to [\vec{x}]\langle\vec{x}\rangle P$, we have $(\mathfrak{F}, I), w, \alpha \models [\vec{x}]\langle\vec{x}\rangle P$. It follows from $wR_{\vec{d}}v$ and $\vec{d} = \alpha(\vec{x})$ that $(\mathfrak{F}, I), v, \alpha \models \langle\vec{x}\rangle P$, which implies there exists some $u \in W$ such that $vR_{\vec{d}}u$ and $(\mathfrak{F}, I), u, \alpha \models P$. Such an u must be w since $(\mathfrak{F}, I), w', \alpha \not\models P$ for all $w' \neq w$. Hence $vR_{\vec{d}}w$.

4. For one direction, suppose $\mathfrak{F}$ is n-transitive, i.e., for all $w, v, u \in W$ and $\vec{d} \in D_w^n$, if $wR_{\vec{d}}v$ and $vR_{\vec{d}}u$ then $wR_{\vec{d}}u$. We show $\mathfrak{F} \models [\vec{x}]P \to [\vec{x}][\vec{x}]P$. Take any interpretation I, world w, assignment α such that $[\vec{x}]P \to [\vec{x}][\vec{x}]P$ is an α_w-formula. Suppose $(\mathfrak{F}, I), w, \alpha \models [\vec{x}]P$. Take any worlds v, u such that $wR_{\alpha(\vec{x})}v$ and $vR_{\alpha(\vec{x})}u$. Our goal is to show$(\mathfrak{F}, I), u, \alpha \models P$. Since $\alpha(x_i) \in D_w$ and $\mathfrak{F}$ is n-transitive, we have $wR_{\alpha(\vec{x})}u$. Together with $(\mathfrak{F}, I), w, \alpha \models [\vec{x}]P$, we obtain $(\mathfrak{F}, I), u, \alpha \models P$.

   For the other direction, suppose $\mathfrak{F} \models [\vec{x}]P \to [\vec{x}][\vec{x}]P$. Take any $w, v, u \in W$ and $\vec{d} \in D_w^n$ such that $wR_{\vec{d}}v$ and $vR_{\vec{d}}u$. We show $wR_{\vec{d}}u$. Let I and α be an interpretation and an assignment such that $I(P, u) = \texttt{False}$; $I(P, u') = \texttt{True}$ for all $u' \neq u$; $\alpha(x_i) = d_i$. Then $(\mathfrak{F}, I), w, \alpha \not\models [\vec{x}][\vec{x}]P$ as $\alpha(\vec{x}) = \vec{d}$. Together with $\mathfrak{F} \models [\vec{x}]P \to [\vec{x}][\vec{x}]P$, we have $(F, I), w, \alpha \not\models [\vec{x}]P$. It follows from $\alpha(\vec{x}) = \vec{d}$ that there exists some $s \in W$ such that $wR_{\vec{d}}s$ and $(\mathfrak{F}, I), s, \alpha \not\models P$. Such an s must be u since $(\mathfrak{F}, I), u', \alpha \models P$ for all $u' \neq u$. Hence $wR_{\vec{d}}u$.

5. For one direction, suppose $\mathfrak{F}$ is n-euclidean, i.e., for all $w, v, u \in W$ and $\vec{d} \in D_w^n$, if $wR_{\vec{d}}v$ and $wR_{\vec{d}}u$ then $vR_{\vec{d}}u$. We show $\mathfrak{F} \models \langle\vec{x}\rangle P \to [\vec{x}]\langle\vec{x}\rangle P$. Take any interpretation I, world w, assignment α such that $\langle\vec{x}\rangle P \to [\vec{x}]\langle\vec{x}\rangle P$ is an α_w-formula. Suppose $(\mathfrak{F}, I), w, \alpha \models \langle\vec{x}\rangle P$ and take any world v such that $wR_{\alpha(\vec{x})}v$. Our goal is to show$(\mathfrak{F}, I), v, \alpha \models \langle\vec{x}\rangle P$. By $(\mathfrak{F}, I), w, \alpha \models \langle\vec{x}\rangle P$, we have some world u such that $wR_{\alpha(\vec{x})}u$ and $(\mathfrak{F}, I), u, \alpha \models P$. Thus, as $\alpha(x_i) \in D_w$, $wR_{\alpha(\vec{x})}v$ and $\mathfrak{F}$ is n-euclidean, we have $vR_{\alpha(\vec{x})}u$. Hence $(\mathfrak{F}, I), v, \alpha \models \langle\vec{x}\rangle P$.

For the other direction, suppose $\mathfrak{F} \models \langle\vec{x}\rangle P \rightarrow [\vec{x}]\langle\vec{x}\rangle P$. Take any $w, v, u \in W$ and $\vec{d} \in D^n_w$ such that $wR_{\vec{d}}v$ and $wR_{\vec{d}}u$. We show $vR_{\vec{d}}u$. Let I and α be an interpretation and an assignment such that $I(P, u) = \texttt{True}$; $I(P, u') = \texttt{False}$ for all $u' \neq u$; $\alpha(x_i) = d_i$. Then $(\mathfrak{F}, I), w, \alpha \models \langle\vec{x}\rangle P$ as $\alpha(\vec{x}) = \vec{d}$. Together with $\mathfrak{F} \models \langle\vec{x}\rangle P \rightarrow [\vec{x}]\langle\vec{x}\rangle P$, we have $(\mathfrak{F}, I), w, \alpha \models [\vec{x}]\langle\vec{x}\rangle P$. It follows from $wR_{\vec{d}}v$ and $\alpha(\vec{x}) = \vec{d}$ that $(\mathfrak{F}, I), v, \alpha \models \langle\vec{x}\rangle P$, which implies there exists some $s \in W$ such that $vR_{\vec{d}}s$ and $(\mathfrak{F}, I), s, \alpha \models P$. Such an s must be u since $(\mathfrak{F}, I), u', \alpha \not\models P$ for all $u' \neq u$. Hence $vR_{\vec{d}}u$.

6. For one direction, suppose $\mathfrak{F}$ has n-decreasing domains, i.e., for all $w, v \in W$ and $\vec{d} \in D^n_w$, if $wR_{\vec{d}}v$ then $D_w \supseteq D_v$. We show $\mathfrak{F} \models \forall y[\vec{x}]Qy \rightarrow [\vec{x}]\forall yQy$, where y is distinct from each x_i of $\vec{x}$. Take any interpretation I, world w, assignment α such that $\forall y[\vec{x}]Qy \rightarrow [\vec{x}]\forall yQy$ is an α_w-formula. Suppose $(\mathfrak{F}, I), w, \alpha \models \forall y[\vec{x}]Qy$. Take any world v such that $wR_{\alpha(\vec{x})}v$ and any object $d \in D_v$. Our goal is to show$(\mathfrak{F}, I), v, \alpha(y|d) \models Qy$. Since $\alpha(\vec{x}) \in D^n_w$ and $\mathfrak{F}$ is an n-decreasing domain frame, by $wR_{\alpha(\vec{x})}v$, we have $D_w \supseteq D_v$. Then $d \in D_w$. Together with $(\mathfrak{F}, I), w, \alpha \models \forall y[\vec{x}]Qy$, we obtain $(\mathfrak{F}, I), w, \alpha(y|d) \models [\vec{x}]Qy$. Also, $\alpha(y|d)(\vec{x}) = \alpha(\vec{x})$ since y is distinct from x_i ($1 \leqslant i \leqslant n$), from which $wR_{\alpha(y|d)(\vec{x})}v$ follows. Hence $(\mathfrak{F}, I), v, \alpha(y|d) \models Qy$.

   For the other direction, suppose $\mathfrak{F} \models \forall y[\vec{x}]Qy \rightarrow [\vec{x}]\forall yQy$, where y is distinct from each x_i of $\vec{x}$. Take any $w, v \in W$ and $\vec{d} \in D^n_w$ such that $wR_{\vec{d}}v$. To show $D_w \supseteq D_v$, take any $d \in D_v$ and show $d \in D_w$. Let I and α be an interpretation and an assignment such that $I(Q, u) = D_u \backslash \{ d \}$ for all $u \in W$; $\alpha(x_i) = d_i$. We claim $(\mathfrak{F}, I), w, \alpha \not\models [\vec{x}]\forall yQy$ since $wR_{\alpha(\vec{x})}v$ and $(\mathfrak{F}, I), v, \alpha(y|d) \not\models Qy$. We also have $(\mathfrak{F}, I), w, \alpha \models \forall y[\vec{x}]Qy \rightarrow [\vec{x}]\forall yQy$ by our initial supposition. Hence $(\mathfrak{F}, I), w, \alpha \not\models \forall y[\vec{x}]Qy$ follows from the claim above. This implies there exist some $e \in D_w$ and $u \in W$ such that $wR_{\alpha(\vec{x})}u$ and $(\mathfrak{F}, I), u, \alpha(y|e) \not\models Qy$. Such an object e is also in D_u since $\mathfrak{F}$ has increasing domains. Therefore, $d = e \in D_w$ follows from $(\mathfrak{F}, I), u, \alpha(y|e) \not\models Qy$.

■

Remark 80. Since n may be zero, Proposition 79 can be seen as a generalization of Proposition 40. For example, $\mathrm{D}_0 = \Box P \supset \Diamond P$ corresponds to the class of all the 0-serial frames for $\mathsf{L}_{\mathsf{TSML}}$, which can be seen as the class of all the serial frames for $\mathsf{L}_{\mathsf{QML}}$. Moreover, Proposition 79 provides frame correspondence results of the original term-modal logic. For example, Proposition 79 tells us that $\{ \mathrm{T}_1, 4_1, \mathrm{B}_1 \}$ corresponds to the class of all the 1-reflexive, 1-transitive and 1-symmetric frames for $\mathsf{L}_{\mathsf{TSML}}$, which can be seen as the class of all the reflexive, transitive and symmetric frames for a language of the original term-modal logic of Thalmann [92] and Fitting et al. [17].

3.1.2 Hilbert System $\mathsf{H}(\mathbf{tK}\Sigma)$

A Hilbert system $\mathsf{H}(\mathbf{tK})$ consists of axioms and inference rules in Table 3.1.

(Taut1)	$\varphi \to (\psi \to \varphi)$	
(Taut2)	$(\varphi \to (\psi \to \gamma)) \to ((\varphi \to \psi) \to (\varphi \to \gamma))$	
(Taut3)	$(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$	
(U)	$\forall x\varphi \to \varphi(y/x)$	
(K)	$[\vec{t}](\varphi \to \psi) \to ([\vec{t}]\varphi \to [\vec{t}]\psi)$	
(MP)	From $\varphi \to \psi$ and φ, we may infer ψ	
(Gen)	From $\varphi \to \psi(y/x)$, we may infer $\varphi \to \forall x\psi$	if $y \notin \mathsf{FV}(\varphi, \forall x\psi)$
(Nec)	From φ, we may infer $[\vec{t}]\varphi$	

Table 3.1: Hilbert system $\mathsf{H}(\mathbf{tK})$

Expansions of $\mathsf{H}(\mathbf{tK})$ are obtained as follows. Put

$$\mathsf{Axiom}_{\mathsf{TSML}} := \{\, \mathrm{T}_n, \mathrm{D}_n, \mathrm{B}_n, 4_n, 5_n, \mathrm{BF}_n \mid n \in \mathbb{N} \,\}.$$

For a set $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{QML}}$, we mean by $\mathsf{Inst}(\Sigma)$ the set of all instances of the schema corresponding to a formula of Σ which is listed in Table 3.2.

Formulas	Schemas
$\mathrm{T}_n = [\vec{x}]P \to P$	$(\mathrm{T}_n) := [\vec{t}]\varphi \to \varphi$
$\mathrm{D}_n = [\vec{x}]P \to \langle\vec{x}\rangle P$	$(\mathrm{D}_n) := [\vec{t}]\varphi \to \langle\vec{t}\rangle\varphi$
$\mathrm{B}_n = P \to [\vec{x}]\langle\vec{x}\rangle P$	$(\mathrm{B}_n) := \varphi \to [\vec{t}]\langle\vec{t}\rangle\varphi$
$4_n = [\vec{x}]P \to [\vec{x}][\vec{x}]P$	$(4_n) := [\vec{t}]\varphi \to [\vec{t}][\vec{t}]\varphi$
$5_n = \langle\vec{x}\rangle P \to [\vec{x}]\langle\vec{x}\rangle P$	$(5_n) := \langle\vec{t}\rangle\varphi \to [\vec{t}]\langle\vec{t}\rangle\varphi$
$\mathrm{BF}_n = \forall y[\vec{x}]Qy \to [\vec{x}]\forall yQy$	$(\mathrm{BF}_n)^{\dagger} := \forall x[\vec{t}]\varphi \to [\vec{t}]\forall x\varphi$
where the length of $\vec{t}$ is n and †: $x \notin \mathsf{V}(t_1, \ldots, t_n)$.	

Table 3.2: The schemas corresponding to formulas of Σ

Definition 81. Given a set Φ of formulas, the Hilbert system $\mathsf{H}(\mathbf{tK} \oplus \Phi)$ is the system obtained from $\mathsf{H}(\mathbf{tK})$ by adding all formulas of Φ as axioms. Given $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{TSML}}$, $\mathsf{H}(\mathbf{tK}\Sigma)$ denotes $\mathsf{H}(\mathbf{tK} \oplus \mathsf{Inst}(\Sigma))$. Let also $\Sigma_n = \{\, \mathrm{X}_n \mid \mathrm{X}_n \in \Sigma \,\}$. The Hilbert system $\mathsf{H}(\mathbf{tK}\Sigma_n \restriction \mathsf{L}_n)$ is the restriction of $\mathsf{H}(\mathbf{tK}\Sigma_n)$ to the sublanguage L_n of L. By $\mathsf{H}(\mathbf{tKX}_n\Sigma)$ we mean the systems $\mathsf{H}(\mathbf{tK}\{\, \mathrm{X}_n \,\} \cup \Sigma)$. We sometimes write $\mathsf{H}(\mathbf{tK}\{\, \mathrm{T}_n, 4_n, \mathrm{B}_n \,\} \restriction \mathsf{L}_n)$ as $\mathsf{H}(\mathbf{tS5}_n \restriction \mathsf{L}_n)$.

The notion of proof in $\mathsf{H}(\mathbf{tK}\Sigma)$ is defined as in $\mathsf{H}(\mathbf{K}\Sigma)$.

Example 82. A subset Σ of Axiom may contain formulas indexed by different numbers. For example, let $\Sigma = \{ \mathrm{D}_0, \mathrm{T}_1 \}$ and read $\Box$ and $[t]$ as deontic and epistemic modalities, respectively. The resulting system $\mathsf{H}(\mathbf{tKD_0T_1})$ has all instances of $\mathsf{Inst}(\{ \mathrm{D}_0, \mathrm{T}_1 \})$ as axioms. In this system, say, a formula $[c]\Box Pc \to \Diamond Pc$ ("if c knows that it ought to be the case that Pc, then it may be the case that Pc") is provable, but not so in $\mathsf{H}(\mathbf{tKD_0})$ or $\mathsf{H}(\mathbf{tKT_1})$. Furthermore, we can obtain Hilbert systems of modal predicate logic and term-modal logic from term-sequence-modal logic by restricting the language $\mathsf{L}_{\mathsf{TSML}}$. For example, $\mathsf{H}(\mathbf{tKD_0} \upharpoonright \mathsf{L}_0)$ is the system $\mathsf{H}(\mathbf{QKD})$ for QML introduced in Chapter 2, and $\mathsf{H}(\mathbf{tK}\{ \mathrm{T}_1, 4_1, \mathrm{B}_1 \} \upharpoonright \mathsf{L}_1)$ is a Hilbert system of a term-modal expansion of **S5**. The term-modal expansion of **S5** is found e.g. in [43, 70, 53], but was not considered in Thalmann [92] and Fitting et al. [17].

For convenience, let us list some formulas provable in $\mathsf{H}(\mathbf{tK}\Sigma)$. Proposition 84 tells us that a Barcan-like formula is provable in $\mathsf{H}(\mathbf{tK}\Sigma)$ with some (B_n). As B. Kooi pointed out in [43], the condition $x \notin \mathsf{V}(t_1, \ldots, t_n)$ is crucial in Proposition 84. Note that Proposition 43 follows from Proposition 84 under the stipulation of $\Box = [\varepsilon]$.

Proposition 83. Let $\Sigma \subseteq$ Axiom.

1. $\vdash_{\mathsf{H}(\mathbf{tK}\Sigma)} [\vec{t}](\varphi_1 \wedge \cdots \wedge \varphi_m) \leftrightarrow [\vec{t}]\varphi_1 \wedge \cdots \wedge [\vec{t}]\varphi_m$.
2. $\vdash_{\mathsf{H}(\mathbf{tK}\Sigma)} \langle\vec{t}\rangle(\varphi_1 \vee \cdots \vee \varphi_m) \leftrightarrow \langle\vec{t}\rangle\varphi_1 \vee \cdots \vee \langle\vec{t}\rangle\varphi_m$.
3. $\vdash_{\mathsf{H}(\mathbf{tK}\Sigma)} \langle\vec{t}\rangle\bot \leftrightarrow \bot$.
4. $\vdash_{\mathsf{H}(\mathbf{tK}\Sigma)} \langle\vec{t}\rangle[\vec{t}]\varphi \to \varphi$ where $\vec{t}$ has the length n and $\mathrm{B}_n \in \Sigma$.
5. $\vdash_{\mathsf{H}(\mathbf{tK}\Sigma)} \langle\vec{t}\rangle\varphi \to \psi$ iff $\vdash_{\mathsf{H}(\mathbf{tK}\Sigma)} \varphi \to [\vec{t}]\psi$ where $\vec{t}$ has the length n and $\mathrm{B}_n \in \Sigma$.
6. $\vdash_{\mathsf{H}(\mathbf{tK}\Sigma)} \langle\vec{t}\rangle[\vec{t}]\varphi \to [\vec{t}]\varphi$ where $\vec{t}$ has the length n and $5_n \in \Sigma$.
7. $\vdash_{\mathsf{H}(\mathbf{tK}\Sigma)} \langle\vec{t}\rangle[\vec{t}]\varphi \to [\vec{t}]\varphi$ where $\vec{t}$ has the length n and $4_n, \mathrm{B}_n \in \Sigma$.

Proof. We check only item 5 since the other items are straightforward to establish. Note that $\vec{t}$ has the length n and $\mathrm{B}_n \in \Sigma$. The left-to-right direction is shown as follows.

1.	$\vdash \langle\vec{t}\rangle\varphi \to \psi$	Assumption
2.	$\vdash [\vec{t}]\langle\vec{t}\rangle\varphi \to [\vec{t}]\psi$	1, (`Nec`), (`K`)
3.	$\vdash \varphi \to [\vec{t}]\langle\vec{t}\rangle\varphi$	(B_n)
4.	$\vdash \varphi \to [\vec{t}]\psi$	2, 3, PC

The right-to-left direction is shown as follows.

1.	$\vdash \varphi \to [\vec{t}]\psi$	Assumption

2. $\vdash \neg[\vec{t}]\psi \to \neg\varphi$ 1, PC
3. $\vdash [\vec{t}]\neg[\vec{t}]\psi \to [\vec{t}]\neg\varphi$ (`Nec`), (`K`)
4. $\vdash \langle\vec{t}\rangle\varphi \to \langle\vec{t}\rangle[\vec{t}]\psi$ 3, PC
5. $\vdash \langle\vec{t}\rangle[\vec{t}]\psi \to \psi$ item 4 of Proposition 83
6. $\vdash \langle\vec{t}\rangle\varphi \to \psi$ 4, 5, PC

■

Proposition 84. Let $\vec{t}$ be a term-sequence of the length n and $\Sigma \subseteq$ Axiom such that $\mathrm{B}_n \in \Sigma$. A formula $\forall x[\vec{t}]\varphi \to [\vec{t}]\forall x\varphi$ is provable in $\mathsf{H}(\mathbf{tK}\Sigma)$ if $x \notin \mathsf{V}(t_1, \dots, t_n)$.

Proof.

1. $\vdash_{\mathsf{H}(\mathbf{tK}\Sigma)} \forall x[\vec{t}]\varphi \to [\vec{t}]\varphi$ FOL
2. $\vdash_{\mathsf{H}(\mathbf{tK}\Sigma)} \langle\vec{t}\rangle\forall x[\vec{t}]\varphi \to \varphi$ 1, item 5 of Proposition 83
3. $\vdash_{\mathsf{H}(\mathbf{tK}\Sigma)} \langle\vec{t}\rangle\forall x[\vec{t}]\varphi \to \forall x\varphi$ 2, $x \notin \mathsf{FV}(t_1, \dots, t_n)$, (`Gen`)
4. $\vdash_{\mathsf{H}(\mathbf{tK}\Sigma)} [\vec{t}]\langle\vec{t}\rangle\forall x[\vec{t}]\varphi \to [\vec{t}]\forall x\varphi$ 3, (`Nec`), (`K`)
5. $\vdash_{\mathsf{H}(\mathbf{tK}\Sigma)} \forall x[\vec{t}]\varphi \to [\vec{t}]\langle\vec{t}\rangle\forall x[\vec{t}]\varphi$ ($\mathtt{B}_n$)
6. $\vdash_{\mathsf{H}(\mathbf{tK}\Sigma)} \forall x[\vec{t}]\varphi \to [\vec{t}]\forall x\varphi$ 4, 5, PC

■

Theorem 85. (Soundness of $\mathsf{H}(\mathbf{tK}\Sigma)$) Let $\Sigma \subseteq$ Axiom and $\mathbb{F}_\Sigma$ be the class of all the frames to which Σ corresponds. For all formulas φ, if φ is provable in $\mathsf{H}(\mathbf{tK}\Sigma)$, then φ is valid in $\mathbb{F}_\Sigma$.

Proof. By induction on the length of a proof of φ. One can easily establish that each axiom of $\mathsf{H}(\mathbf{tK})$ is valid in $\mathbb{F}_\Sigma$ and each inference rule of $\mathsf{H}(\mathbf{tK})$ preserves the validity in $\mathbb{F}_\Sigma$. The validity of additional axioms from Σ are shown similarly as in Proposition 79. ■

As before, we will prove strong completeness of Hilbert systems for TSML via construction of the canonical model for $\mathsf{L}_{\mathsf{TSML}}$ defined below. The canonical models for term-modal logic and its analogues are introduced e.g. in [1, 103, 85, 20, 53]. Amongst them, only Wang and Seligman [103] and Sawasaki et al. [85] construct the canonical models having increasing domains.[3] The whole proof strategy is the same as in QML.

[3]Wang and Seligman [103] consider a constant domain semantics, but for the lack of Barcan-like formulas, construct the canonical pseudo model having increasing domains via the techniques inspired by Corsi [12].

The strong completeness theorems are, roughly speaking, given for a class without Barcan formula in the full language $\mathsf{L}_{\mathsf{TSML}}$ and a class with Barcan formula in a sublanguage L_n. More precisely, the first class is the class of all Hilbert systems $\mathsf{H}(\mathbf{tK}\Sigma)$ such that $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{TSML}} \setminus \{\mathrm{B}_n, 5_n, \mathrm{BF}_n \mid n \in \mathbb{N}\}$. For example, $\mathsf{H}(\mathbf{tK}\{\mathrm{T}_1, 4_1\})$ and $\mathsf{H}(\mathbf{tK}\{\mathrm{T}_1, 4_2\})$ belong to the first class. The second class is the class of all Hilbert systems $\mathsf{H}(\mathbf{tKBF}_n\Sigma_n \upharpoonright \mathsf{L}_n)$ such that $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{TSML}}$, where $\Sigma_n := \{\mathrm{X}_n \mid \mathrm{X}_n \in \Sigma\}$. For example, $\mathsf{H}(\mathbf{tKBF}_1\{5_1\} \upharpoonright \mathsf{L}_1)$ and $\mathsf{H}(\mathbf{tKBF}_1\{\mathrm{T}_1, 4_1, \mathrm{B}_1\} \upharpoonright \mathsf{L}_1)$ belong to the second class. Similarly as in QML, when $\mathrm{B}_n \in \Sigma$, strong completeness of $\mathsf{H}(\mathbf{tK}\Sigma_n \upharpoonright \mathsf{L}_n)$ of the first class follows from strong completeness of $\mathsf{H}(\mathbf{tKBF}_n\Sigma_n \upharpoonright \mathsf{L}_n)$ of the second class in terms of Proposition 84. For example, strong completeness of $\mathsf{H}(\mathbf{tK}\{\mathrm{T}_1, 4_1, \mathrm{B}_1\} \upharpoonright \mathsf{L}_1)$ follows from strong completeness of $\mathsf{H}(\mathbf{tKBF}_1\{\mathrm{T}_1, 4_1, \mathrm{B}_1\} \upharpoonright \mathsf{L}_1)$. On the other hand, for example, strong completeness of $\mathsf{H}(\mathbf{tKB}_n)$, $\mathsf{H}(\mathbf{tKBF}_n)$ and $\mathsf{H}(\mathbf{tK5}_n \upharpoonright \mathsf{L}_n)$ are not proved here.

Recall that we write $\mathsf{L}_{\mathsf{TSML}}(V)$ to denote $\mathsf{L}_{\mathsf{TSML}}$ in which the set of variables is V.

Definition 86. We define Var^+ as $\mathsf{Var} \cup \mathsf{Var}'$, where Var' is a *fresh* countably infinite set of variables disjoint from Var. Given $V \subseteq \mathsf{Var}^+$, the set $\mathsf{Term}(V)$ refers to the set of all terms in $\mathsf{L}_{\mathsf{TSML}}(V)$. Given a set Γ of formulas, $\mathsf{L}_{\mathsf{TSML}}(\Gamma)$ and $\mathsf{Term}(\Gamma)$ denote $\mathsf{L}_{\mathsf{TSML}}(\mathsf{FV}(\Gamma))$ and $\mathsf{Term}(\mathsf{FV}(\Gamma))$, respectively. Given $V, V' \subseteq \mathsf{Var}^+$, by a notation $V \sqsubset V'$ we mean that $V \subseteq V'$ and $V' \backslash V$ is countably infinite. The same definition and notations are also used for sublanguages of $\mathsf{L}_{\mathsf{TSML}}$.

Definition 87. Let $\Sigma \subseteq \mathsf{Axiom}$. Given a set Γ of formulas, Γ is $\mathsf{H}(\mathbf{tK}\Sigma)$-*inconsistent* if $\Gamma \vdash_{\mathsf{H}(\mathbf{tK}\Sigma)} \bot$; Γ is $\mathsf{H}(\mathbf{tK}\Sigma)$-*consistent* if Γ is not $\mathsf{H}(\mathbf{tK}\Sigma)$-inconsistent; Γ is *maximal* if $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$ for all formulas φ in $\mathsf{L}(\Gamma)$; Γ is a *maximal* $\mathsf{H}(\mathbf{tK}\Sigma)$-*consistent set* ($\mathsf{H}(\mathbf{tK}\Sigma)$-*MCS*) if Γ is $\mathsf{H}(\mathbf{tK}\Sigma)$-consistent and maximal; Γ is *witnessed* if, for all formulas of the form $\forall x\varphi$ in $\mathsf{L}(\Gamma)$, there exists some $t \in \mathsf{Term}(\Gamma)$ such that $\varphi(t/x) \to \forall x\varphi \in \Gamma$.

Proposition 88. Let $\Sigma \subseteq \mathsf{Axiom}$, Γ a $\mathsf{H}(\mathbf{tK}\Sigma)$-MCS in $\mathsf{L}(\mathsf{Var}^+)$, φ, ψ formulas in $\mathsf{L}(\Gamma)$.

1. $\Gamma \vdash_{\mathsf{H}(\mathbf{tK}\Sigma)} \varphi$ iff $\varphi \in \Gamma$.
2. If $\varphi \in \Gamma$ and $\vdash_{\mathsf{H}(\mathbf{tK}\Sigma)} \varphi \to \psi$, then $\psi \in \Gamma$.
3. $\bot \notin \Gamma$.
4. $\varphi \to \psi \in \Gamma$ iff $\varphi \notin \Gamma$ or $\psi \in \Gamma$.

Strong Completeness of $\mathsf{H}(\mathbf{tK}\Sigma)$

Recall $\mathsf{Axiom}_{\mathsf{TSML}} = \{\mathrm{T}_n, \mathrm{D}_n, \mathrm{B}_n, 4_n, 5_n, \mathrm{BF}_n \mid n \in \mathbb{N}\}$. Until Theorem 95, we abbreviate $\mathsf{H}(\mathbf{tK}\Sigma)$ as Λ for some fixed subset $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{TSML}} \setminus \{\mathrm{B}_n, 5_n, \mathrm{BF}_n \mid n \in \mathbb{N}\}$.

Lemma 89. (Lindenbaum Lemma) Let Γ be a Λ-consistent set in $\mathsf{L}(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma) \sqsubset \mathsf{Var}^+$. There exists a witnessed Λ-MCS Γ^+ in $\mathsf{L}(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma^+) \sqsubset \mathsf{Var}^+$ and $\Gamma \subseteq \Gamma^+$.

Proof. The proof proceeds similarly to the standard proof of Lindenbaum Lemma for QML (Lemma 48). Take some set Var' of variables in $\mathsf{L}(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma) \sqsubset \mathsf{Var}' \sqsubset \mathsf{Var}^+$. Let g be an enumeration of Var' and h an enumeration of all formulas in $\mathsf{L}(\mathsf{Var}')$. We define a sequence $(\Gamma_m)_{m\in\mathbb{N}}$ of Λ-consistent sets Γ_m as follows.

- $\Gamma_0 := \Gamma$;
- $\Gamma_{m+1} :=$

$$\begin{cases} \Gamma_m \cup \{\, \varphi \,\} & \text{if } \Gamma_m \cup \{\, \varphi \,\} \nvdash \bot; \\ \Gamma_m \cup \{\, \neg\varphi \,\} & \text{if } \Gamma_m \cup \{\, \varphi \,\} \vdash \bot \text{ and } \varphi \text{ is not of the form } \forall x\psi \\ \Gamma_m \cup \{\, \neg\forall x\psi, \neg\psi(y/x) \,\} & \text{if } \Gamma_m \cup \{\, \varphi \,\} \vdash \bot \text{ and } \varphi \text{ is of the form } \forall x\psi; \end{cases}$$

where φ is the m-th formula in the enumeration h and y is the first variable in the enumeration g which is fresh in $\Gamma_m, \neg\forall x\psi$. It is not difficult to check that each Γ_m is Λ-consistent. Put Γ^+ as $\bigcup_{m\in\mathbb{N}} \Gamma_m$. From the construction of Γ^+, we find that Γ^+ is a witnessed Λ-MCS in $\mathsf{L}(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma^+) \sqsubset \mathsf{Var}^+$ and $\Gamma \subseteq \Gamma^+$. ∎

Definition 90. The *canonical* Λ*-frame* is a tuple $\mathfrak{F}^\Lambda = (W^\Lambda, R^\Lambda, (D^\Lambda_\Gamma)_{\Gamma\in W^\Lambda})$, where

- $W^\Lambda := \{\, \Gamma \mid \Gamma \text{ is a witnessed } \Lambda\text{-MCS in } \mathsf{L}(\mathsf{Var}^+) \text{ such that } \mathsf{FV}(\Gamma) \sqsubset \mathsf{Var}^+ \,\}$;
- $D^\Lambda_\Gamma := \mathsf{Term}(\Gamma)$;
- $\Gamma R^\Lambda_{\vec{t}} \Delta$ iff $[\vec{t}\,]\varphi \in \Gamma$ implies $\varphi \in \Delta$ for all formulas φ in $\mathsf{L}_n(\mathsf{Var}^+)$.

The *canonical* Λ*-model* is a tuple $\mathfrak{M}^\Lambda = (\mathfrak{F}^\Lambda, I^\Lambda)$, where

- $\mathfrak{F}^\Lambda$ is the canonical Λ-frame;
- $(t_1, \ldots, t_n) \in I^\Lambda(P, \Gamma)$ iff $Pt_1 \ldots t_n \in \Gamma$;
- $I^\Lambda(c) := c$;
- $I^\Lambda(f)(t_1, \ldots, t_n) := f(t_1, \ldots, t_n)$.

The *canonical assignment* is the assignment $\iota : \mathsf{Var}^+ \to D^\Lambda$ defined by $\iota(x) := x$.

Proposition 91. The canonical Λ-model *is* a model.

Proof. Let $\mathfrak{F}^\Lambda$ be the canonical Λ-frame of the canonical Λ-model. We confirm only that $\mathfrak{F}^\Lambda$ has increasing domains, i.e., for all $\Gamma, \Delta \in W^\Lambda$ and $\vec{t} \in (D^\Lambda_\Gamma)^{<\omega}$, if $\Gamma R^\Lambda_{\vec{t}} \Delta$ then $D^\Lambda_\Gamma \subseteq D^\Lambda_\Delta$. Suppose $\Gamma R^\Lambda_{\vec{t}} \Delta$ and $s \in D^\Lambda_\Gamma$. Then, as $s \in D^\Lambda_\Gamma$ and $t_i \in D^\Lambda_\Gamma$ for each t_i of $\vec{t}$, it holds that $[\vec{t}\,](Ps \to Ps) \in \Gamma$ by item 1 of Proposition 88. Thus it follows from $\Gamma R^\Lambda_{\vec{t}} \Delta$ that $Ps \to Ps \in \Delta$, which implies $s \in D^\Lambda_\Delta$. ∎

Lemma 92. (Existence Lemma) If $\neg[\vec{t}\,]\varphi \in \Gamma \in W^\Lambda$, there exists some $\Delta \in W^\Lambda$ such that $\neg\varphi \in \Delta$ and $\Gamma R^\Lambda_{\vec{t}} \Delta$.

Proof. The proof is analogous to the standard proof of Existence Lemma for QML (Lemma 51). Suppose $\neg[\vec{t}\,]\varphi \in \Gamma \in W^\Lambda$. To establish $\Delta_0 := \{\neg\varphi\} \cup \{\gamma \mid [\vec{t}\,]\gamma \in \Gamma\}$ is Λ-consistent, suppose not. Then (1) $\vdash \bigwedge_{i \leqslant m} \gamma_i \to \varphi$ for some $\gamma_1, \ldots, \gamma_m$ such that (2) $[\vec{t}\,]\gamma_i \in \Gamma$. We obtain $\Gamma \vdash \bot$ as follows.

1. $\vdash \bigwedge_{i \leqslant m} \gamma_i \to \varphi$ — (1)
2. $\vdash [\vec{t}\,] \bigwedge_{i \leqslant m} \gamma_i \to [\vec{t}\,]\varphi$ — 1, (Nec), (K),
3. $\vdash (\bigwedge_{i \leqslant m} [\vec{t}\,]\gamma_i) \to [\vec{t}\,]\varphi$ — 2, item 1 of Proposition 83
4. $\Gamma \vdash [\vec{t}\,]\varphi$ — 3, (2), PC
5. $\Gamma \vdash \neg[\vec{t}\,]\varphi$ — $\neg[\vec{t}\,]\varphi \in \Gamma$
6. $\Gamma \vdash \bot$ — 4, 5, PC

However, Γ should be Λ-consistent, so a contradiction. Thus Δ_0 is Λ-consistent.

By Lindenbaum Lemma (Lemma 89), we obtain some $\Delta \in W^\Lambda$ such that $\Delta_0 \subseteq \Delta$. Since $\Delta_0 = \{\neg\varphi\} \cup \{\gamma \mid [\vec{t}\,]\gamma \in \Gamma\}$, it holds that $\neg\varphi \in \Delta$ and $\Gamma R^\Lambda_{\vec{t}} \Delta$. ∎

Lemma 93. (Truth Lemma) Let $\mathfrak{M}^\Lambda$ be the canonical Λ-model and ι the canonical assignment. For all Λ-MCSs $\Gamma \in W^\Lambda$ and formulas φ in $\mathsf{L}(\Gamma)$,

$$\mathfrak{M}^\Lambda, \Gamma, \iota \models \varphi \quad \text{iff} \quad \varphi \in \Gamma$$

Proof. By induction on the length of φ. We check only cases in which φ is $Pt_1 \ldots t_n$, in which φ is $\forall x\psi$, and in which φ is $[\vec{t}\,]\psi$.

Case in which φ is $Pt_1 \ldots t_n$. By the satisfaction relation of $Pt_1 \ldots t_n$ and the definitions of ι and I^Λ, our goal immediately follows from the equivalence $\mathfrak{M}^\Lambda, \Gamma, \iota \models Pt_1 \ldots t_n$ iff $\iota(t_1, \ldots, t_n) \in I^\Lambda(P, \Gamma)$ iff $(t_1, \ldots, t_n) \in I^\Lambda(P, \Gamma)$ iff $Pt_1 \ldots t_n \in \Gamma$.

Case in which φ is $\forall x\psi$. For the right-to-left direction, suppose $\forall x\psi \in \Gamma$ and take any $t \in D_\Gamma$. We show $\mathfrak{M}, \Gamma, \iota(x|t) \models \psi$. It follows from our supposition that $\psi(t/x) \in \Gamma$, so by inductive hypothesis, we have $\mathfrak{M}, \Gamma, \iota \models \psi(t/x)$. From this we claim $\mathfrak{M}, \Gamma, \iota(x|\iota(t)) \models \psi$. Since $\iota(t) = t$, we obtain $\mathfrak{M}, \Gamma, \iota(x|t) \models \psi$. For the left-to-right direction, we prove the contraposition. Suppose $\forall x\psi \notin \Gamma$. We show $\mathfrak{M}, \Gamma, \iota \not\models \forall x\psi$. Since Γ is witnessed, it follows that there is some term $t \in \mathsf{Term}(\Gamma)$ such that $\varphi(t/x) \notin \Gamma$. Then by inductive hypothesis, we have $\mathfrak{M}, \Gamma, \iota \not\models \varphi(t/x)$. From this and $\iota(t) = t$ we claim $\mathfrak{M}, \Gamma, \iota(x|t) \not\models \varphi$. Since $t \in \mathsf{Term}(\Gamma) = D_\Gamma$, our claim implies $\mathfrak{M}, \Gamma, \iota \not\models \forall x\psi$.

Case in which φ is $[\vec{t}]\psi$. Note that $\iota(\vec{t}) = \vec{t}$ holds from the definition of ι. The right-to-left direction is as follows. Suppose $[\vec{t}]\psi \in \Gamma$. Take any $\Delta \in W^\Lambda$ such that $\Gamma R^\Lambda_{\vec{t}}\Delta$. It holds that $\psi \in \Delta$, so by inductive hypothesis $\mathfrak{M}^\Lambda, \Delta, \iota \models \psi$, as required. For the left-to-right direction, we prove the contraposition. Suppose $[\vec{t}]\psi \notin \Gamma$. We show $\mathfrak{M}^\Lambda, \Gamma, \iota \not\models [\vec{t}]\psi$. Since $[\vec{t}]\psi$ is a formula in $\mathsf{L}(\Gamma)$, it holds that $\neg[\vec{t}]\psi \in \Gamma$. By Existence Lemma (Lemma 92), we obtain some $\Delta \in W^\Lambda$ such that $\psi \notin \Delta$ and $\Gamma R^\Lambda_{\vec{t}}\Delta$. Then $\mathfrak{M}^\Lambda, \Delta, \iota \not\models \psi$ holds by inductive hypothesis. Therefore $\mathfrak{M}^\Lambda, \Gamma, \iota \not\models [\vec{t}]\psi$.

■

Proposition 94. Let $\mathfrak{F}^\Lambda$ be the canonical Λ-frame, where $\Lambda = \mathsf{H}(\mathbf{tK}\Sigma)$.

1. If $\mathrm{T}_n \in \Sigma$ then $\mathfrak{F}^\Lambda$ is n-reflexive.
2. If $\mathrm{D}_n \in \Sigma$ then $\mathfrak{F}^\Lambda$ is n-serial.
3. If $4_n \in \Sigma$ then $\mathfrak{F}^\Lambda$ is n-transitive.

Proof. Note that each $\vec{t}$ has the length n in the following proofs.

1. Suppose $\mathrm{T}_n \in \Sigma$. We show $\mathfrak{F}^\Lambda$ is n-reflexive, i.e., for all $\Gamma \in W^\Lambda$ and $\vec{t} \in (D^\Lambda_\Gamma)^n$, $\Gamma R^\Lambda_{\vec{t}}\Gamma$. Take any $\Gamma \in W^\Lambda$ and $\vec{t} \in (D^\Lambda_\Gamma)^n$. Suppose $[\vec{t}]\varphi \in \Gamma$. It then follows from $(\mathtt{T}_n)$ that $\varphi \in \Gamma$.
2. Suppose $\mathrm{D}_n \in \Sigma$. We show $\mathfrak{F}^\Lambda$ is n-serial, i.e., for all $\Gamma \in W^\Lambda$ and $\vec{t} \in (D^\Lambda_\Gamma)^n$, there exists some $\Delta \in W^\Lambda$ such that $\Gamma R^\Lambda_{\vec{t}}\Delta$. Take any $\Gamma \in W^\Lambda$ and $\vec{t} \in (D^\Lambda_\Gamma)^n$. We claim that $\neg[\vec{t}]\bot \in \Gamma$ since $[\vec{t}]\bot \to \bot$ is provable in $\Lambda = \mathsf{H}(\mathbf{tK}\Sigma)$ by $(\mathtt{D}_n)$ and item 3 of Proposition 83 and thus $[\vec{t}]\bot \to \bot \in \Gamma$ holds. By Existence Lemma (Lemma 92), there exists some $\Delta \in W^\Lambda$ such that $\Gamma R^\Lambda_{\vec{t}}\Delta$.
3. Suppose $4_n \in \Sigma$. We show $\mathfrak{F}^\Lambda$ is n-transitive, i.e., for all $\Gamma, \Delta, \Xi \in W^\Lambda$ and $\vec{t} \in (D^\Lambda_\Gamma)^n$, if $\Gamma R^\Lambda_{\vec{t}}\Delta$ and $\Delta R^\Lambda_{\vec{t}}\Xi$ then $\Gamma R^\Lambda_{\vec{t}}\Xi$. Take any $\Gamma, \Delta, \Xi \in W^\Lambda$ and $\vec{t} \in (D^\Lambda_\Gamma)^n$. Suppose $\Gamma R^\Lambda_{\vec{t}}\Delta$, $\Delta R^\Lambda_{\vec{t}}\Xi$ and assume $[\vec{t}]\varphi \in \Gamma$. It then follows from (4_n) that $[\vec{t}][\vec{t}]\varphi \in \Gamma$. Together with $\Gamma R^\Lambda_{\vec{t}}\Delta$ and $\Delta R^\Lambda_{\vec{t}}\Xi$, this implies $\varphi \in \Xi$.

■

Theorem 95. (Strong completeness of $\mathsf{H}(\mathbf{tK}\Sigma)$) Let $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{TSML}} \setminus \{\mathrm{B}_n, 5_n, \mathrm{BF}_n \mid n \in \mathbb{N}\}$ and $\mathbb{F}_\Sigma$ be the class of all the frames to which Σ corresponds. For all formulas φ and sets Γ of formulas, if φ is a consequence from Γ in $\mathbb{F}_\Sigma$, then φ is provable from Γ in $\mathsf{H}(\mathbf{tK}\Sigma)$.

Proof. The proof proceeds similarly to the proof of Theorem 54. Let $\Lambda = \mathsf{H}(\mathbf{tK}\Sigma)$ for short. Suppose φ is not provable from Γ in Λ. We show φ is not a consequence from Γ in $\mathbb{F}_\Sigma$. Note first that $\Gamma \cup \{\neg\varphi\}$ is Λ-consistent in L. We claim $\Gamma \cup \{\neg\varphi\}$ is Λ-consistent in $\mathsf{L}(\mathsf{Var}^+)$. By Lindenbaum Lemma (Lemma 89), we obtain a witnessed Λ-MCS Γ^+ in $\mathsf{L}(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma^+) \sqsubset \mathsf{Var}^+$ and $\Gamma \cup \{\neg\varphi\} \subseteq \Gamma^+$. It then follows from Truth Lemma (Lemma 93) that

$$\mathfrak{M}^\Lambda, \Gamma^+, \iota \models \Gamma \quad \text{and} \quad \mathfrak{M}^\Lambda, \Gamma^+, \iota \not\models \varphi,$$

where $\mathfrak{M}^\Lambda = (\mathfrak{F}^\Lambda, I^\Lambda)$ is the canonical Λ-model and ι is the canonical assignment. We must further show that $\mathfrak{F}^\Lambda \in \mathbb{F}_\Sigma$, which is established by Proposition 94. Hence, in $\mathsf{L}(\mathsf{Var}^+)$, φ is not a consequence from Γ in $\mathbb{F}_\Sigma$. By restricting $\mathsf{L}(\mathsf{Var}^+)$ to L, we conclude in L that φ is not a consequence from Γ in $\mathbb{F}_\Sigma$. ■

Strong Completeness of $\mathsf{H}(\mathbf{tKBF}_n\Sigma_n{\restriction}\mathsf{L}_n)$

Recall $\mathsf{Axiom}_{\mathsf{TSML}} = \{\mathrm{T}_n, \mathrm{D}_n, \mathrm{B}_n, 4_n, 5_n, \mathrm{BF}_n \mid n \in \mathbb{N}\}$ and $\Sigma_n = \{\mathrm{X}_n \mid \mathrm{X}_n \in \Sigma\}$. Until Theorem 104, we abbreviate $\mathsf{H}(\mathbf{tKBF}_n\Sigma_n \restriction \mathsf{L}_n)$ as Λ for some fixed subset $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{TSML}}$. In addition, we always mean by $\vec{t}$ a term-sequence of the length n.

Lemmas 96, 100 and Proposition 98 are established as before. Thus, we will describe only proofs of Lemma 99 and Proposition 102.

Lemma 96. (Lindenbaum Lemma) Let Γ be a Λ-consistent set in $\mathsf{L}_n(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma) \sqsubset \mathsf{Var}^+$.

1. There exists a Λ-MCS Γ^+ in $\mathsf{L}_n(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma^+) = \mathsf{FV}(\Gamma)$ and $\Gamma \subseteq \Gamma^+$.
2. There exists a witnessed Λ-MCS Γ^+ in $\mathsf{L}_n(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma^+) = \mathsf{Var}^+$ and $\Gamma \subseteq \Gamma^+$.

Proof. Item 1 is proved as follows. Let g be an enumeration of all formulas in $\mathsf{L}(\Gamma)$. We define a sequence $(\Gamma_m)_{m\in\mathbb{N}}$ of Λ-consistent sets Γ_m as follows.

- $\Gamma_0 := \Gamma$;

- $\Gamma_{m+1} :=$

$$\begin{cases} \Gamma_m \cup \{\varphi\} & \text{if } \Gamma_m \cup \{\varphi\} \nvdash \bot; \\ \Gamma_m \cup \{\neg\varphi\} & \text{if } \Gamma_m \cup \{\varphi\} \vdash \bot \end{cases}$$

where φ is the m-th formula in the enumeration g. It is not difficult to check that each Γ_m is Λ-consistent. Put Γ^+ as $\bigcup_{m \in \mathbb{N}} \Gamma_m$. From the construction of Γ^+, we find that Γ^+ is a Λ-MCS in $\mathsf{L}(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma^+) = \mathsf{FV}(\Gamma)$ and $\Gamma \subseteq \Gamma^+$.

Item 2 is proved as follows. Enumerate Var^+ and all formulas in $\mathsf{L}(\mathsf{Var}^+)$, and construct a sequence $(\Gamma_m)_{m \in \mathbb{N}}$ of Λ-consistent sets Γ_m similarly as in the proof of Lindenbaum Lemma for TSML without (BF_n)s (Lemma 89). Put Γ^+ as $\bigcup_{m \in \mathbb{N}} \Gamma_m$. We first show $\mathsf{Var}^+ \subseteq \mathsf{FV}(\Gamma^+)$. Suppose $z \in \mathsf{FV}(\Gamma^+)$. Consider a formula $Pz \to Pz$ with some index m in the enumeration on formulas. Then $Pz \to Pz \in \Gamma_{m+1}$, which implies $z \in \mathsf{FV}(\Gamma^+)$. Moreover, it is evident that $\mathsf{FV}(\Gamma^+) \subseteq \mathsf{Var}^+$. Thus we have $\mathsf{FV}(\Gamma^+) = \mathsf{Var}^+$. From the construction of Γ^+, we find that Γ^+ is a witnessed Λ-MCS in $\mathsf{L}(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma^+) = \mathsf{Var}^+$ and $\Gamma \subseteq \Gamma^+$. ■

Definition 97. The *canonical Λ-frame* is a tuple $\mathfrak{F}^\Lambda = (W^\Lambda, R^\Lambda, (D^\Lambda_\Gamma)_{\Gamma \in W^\Lambda})$, where

- $W^\Lambda := \{\, \Gamma \mid \Gamma \text{ is a witnessed } \Lambda\text{-MCS in } \mathsf{L}_n(\mathsf{Var}^+) \text{ such that } \mathsf{FV}(\Gamma) = \mathsf{Var}^+ \,\}$;
- $D^\Lambda_\Gamma := \mathsf{Term}(\mathsf{Var}^+)$;
- $\Gamma R^\Lambda_{\vec{t}} \Delta$ iff $[\vec{t}\,]\varphi \in \Gamma$ implies $\varphi \in \Delta$ for all formulas φ in $\mathsf{L}(\mathsf{Var}^+)$.

The *canonical Λ-model* is a tuple $\mathfrak{M}^\Lambda = (\mathfrak{F}^\Lambda, I^\Lambda)$, where

- $\mathfrak{F}^\Lambda$ is the canonical Λ-frame;
- $(t_1, \ldots, t_n) \in I^\Lambda(P, \Gamma)$ iff $Pt_1 \ldots t_n \in \Gamma$;
- $I^\Lambda(c) := c$;
- $I^\Lambda(f)(t_1, \ldots, t_n) := f(t_1, \ldots, t_n)$.

The *canonical assignment* is the assignment $\iota : \mathsf{Var}^+ \to D^\Lambda$ defined by $\iota(x) := x$.

Proposition 98. The canonical Λ-model *is* a model.

Lemma 99. (Existence Lemma) If $\neg[\vec{t}\,]\varphi \in \Gamma \in W^\Lambda$, there exists some $\Delta \in W^\Lambda$ such that $\neg\varphi \in \Delta$ and $\Gamma R^\Lambda_{\vec{t}} \Delta$.

Proof. The proof proceeds analogously to the standard proof of Existence Lemma for QML (Lemma 58). Suppose $\neg[\vec{t}\,]\varphi \in \Gamma \in W^\Lambda$. Let g be an enumeration of $\mathsf{Term}(\mathsf{Var}^+)$ and h an enumeration of all formulas of the form $\forall x \psi$ in $\mathsf{L}_n(\mathsf{Var}^+)$. We define a sequence $(\Delta_m)_{m \in \mathbb{N}}$ of Λ-consistent sets Δ_m by induction on m, as below.

For the basis, put

$$\Delta_0 := \{\neg\varphi\} \cup \left\{\gamma \mid [\vec{t}]\gamma \in \Gamma\right\}.$$

Similarly as in the proof of Existence Lemma for TSML without ($\mathtt{BF}_n$)s (Lemma 92), we can claim that Δ_0 is Λ-consistent.

For the induction step, assume that we have already defined a Λ-consistent set Δ_m. In order to define a Λ-consistent set Δ_{m+1}, let $\forall x\psi$ be the m-th formula in the enumeration h. We first establish that there is a term $t \in \mathsf{Term}(\mathsf{Var}^+)$ such that $\Delta_m \cup \{\psi(t/x) \to \forall x\psi\} \nvdash \bot$.

Suppose for contradiction that, for all $t \in \mathsf{Term}(\mathsf{Var}^+)$, $\Delta_m \cup \{\psi(t/x) \to \forall x\psi\} \vdash \bot$. Note that $\Delta_m^- := \Delta_m \setminus \left\{\gamma \mid [\vec{t}]\gamma \in \Gamma\right\}$ is finite. Then it follows that

(1) for all $t \in \mathsf{Term}(\mathsf{Var}^+)$, there are some formulas $\gamma_1, \ldots, \gamma_k \in \left\{\gamma \mid [\vec{t}]\gamma \in \Gamma\right\}$ such that $\vdash \bigwedge_{i \leqslant k} \gamma_i \to (\bigwedge \Delta_m^- \to \neg(\psi(t/x) \to \forall x\psi))$,

from which it follows by ($\mathtt{Nec}$) and ($\mathtt{K}$) that

(2) for all $t \in \mathsf{Term}(\mathsf{Var}^+)$, there are some formulas $\gamma_1, \ldots, \gamma_k \in \left\{\gamma \mid [\vec{t}]\gamma \in \Gamma\right\}$ such that $\vdash [\vec{t}] \bigwedge_{i \leqslant k} \gamma_i \to [\vec{t}](\bigwedge \Delta_m^- \to \neg(\psi(t/x) \to \forall x\psi))$.

By $[\vec{t}]\gamma_i \in \Gamma$ and item 1 of Proposition 83, we have $\Gamma \vdash [\vec{t}] \bigwedge_{i \leqslant k} \gamma_i$. Thus (2) implies

(3) for all $t \in \mathsf{Term}(\mathsf{Var}^+)$, $\Gamma \vdash [\vec{t}](\bigwedge \Delta_m^- \to \neg(\psi(t/x) \to \forall x\psi))$.

Let z be a variable fresh in $\vec{t}, \Delta_m^-, \forall x\psi$. Let also

$$\delta := \bigwedge \Delta_m^- \to \neg(\psi(z/x) \to \forall x\psi).$$

Since $\forall z[\vec{t}]\delta$ is a formula in $\mathsf{L}_n(\mathsf{Var}^+) = \mathsf{L}_n(\Gamma)$ and Γ is witnessed, there exists some $t \in \mathsf{Term}(\mathsf{Var}^+)$ such that

(4) $([\vec{t}]\delta)(t/z) \to \forall z[\vec{t}]\delta \in \Gamma$.

Fix such a term t. Since

$$([\vec{t}]\delta)(t/z) = [\vec{t}](\bigwedge \Delta_m^- \to \neg(\psi(t/x) \to \forall x\psi)),$$

we obtain $\Gamma \vdash \forall z[\vec{t}]\delta$ by (3) and (4). Recall that $\Lambda = \mathsf{H}(\mathbf{tKBF}_n\Sigma_n \upharpoonright \mathsf{L}_n)$ is now under consideration. Since the length of $\vec{t}$ is n and z is fresh in $\vec{t}$, we also have $\vdash \forall z[\vec{t}]\delta \to [\vec{t}]\forall z\delta$ by ($\mathtt{BF}_n$). Thus, it holds that $\Gamma \vdash [\vec{t}]\forall z\delta$, i.e.,

(5) $\Gamma \vdash [\vec{t}]\forall z(\bigwedge \Delta_m^- \to \neg(\psi(z/x) \to \forall x\psi))$

We can now deduce $\Gamma \vdash [\vec{t}]\neg \bigwedge \Delta_m^-$ as follows.

1. $\vdash \forall z(\bigwedge \Delta_m^- \to \neg(\psi(z/x) \to \forall x\psi)) \to (\bigwedge \Delta_m^- \to \forall z\neg(\psi(z/x) \to \forall x\psi))$

FOL, z: fresh in Δ_m^-

2. $\vdash \neg\forall z\neg(\psi(z/x) \to \forall x\psi)$ FOL, z: fresh in $\forall x\psi$
3. $\vdash \forall z(\bigwedge \Delta_m^- \to \neg(\psi(z/x) \to \forall x\psi)) \to \neg \bigwedge \Delta_m^-$ 1, 2, PC
4. $\vdash [\vec{t}\,]\forall z(\bigwedge \Delta_m^- \to \neg(\psi(z/x) \to \forall x\psi)) \to [\vec{t}\,]\neg \bigwedge \Delta_m^-$ 3, (Nec), (K)
5. $\Gamma \vdash [\vec{t}\,]\neg \bigwedge \Delta_m^-$ 4, (5), PC

Then $[\vec{t}\,]\neg \bigwedge \Delta_m^- \in \Gamma$. This implies $\neg \bigwedge \Delta_m^- \in \{\gamma \mid [\vec{t}\,]\gamma \in \Gamma\} \subseteq \Delta_m$ so $\Delta_m \vdash \neg \bigwedge \Delta_m^-$. However, as $\Delta_m^- \subseteq \Delta_m$, we also have $\Delta_m \vdash \bigwedge \Delta_m^-$. Thus $\Delta_m \vdash \bot$, which contradicts our inductive hypothesis that $\Delta_m \nvdash \bot$. Therefore, there is a term $t \in \mathsf{Term}(\mathsf{Var}^+)$ such that $\Delta_m \cup \{\psi(t/x) \to \forall x\psi\}$ is Λ-consistent.

Now put

$$\Delta_{m+1} := \Delta_m \cup \{\psi(t/x) \to \forall x\psi\},$$

where $\forall x\psi$ is the m-th formula in the enumeration g and t is the first term in the enumeration h such that $\Delta_m \cup \{\psi(t/x) \to \forall x\psi\}$ is Λ-consistent. Since Δ_{m+1} is obviously Δ-consistent, we have finished defining a sequence $(\Delta_m)_{m\in\mathbb{N}}$ of Λ-consistent sets Δ_m.

We now use $(\Delta_m)_{m\in\mathbb{N}}$ to obtain some $\Delta \in W^\Lambda$ such that $\neg\varphi \in \Delta$ and $\Gamma R^\Lambda_{\vec{t}}\Delta$, as below. We first show $\mathsf{Var}^+ \subseteq \mathsf{FV}(\bigcup_{m\in\mathbb{N}} \Delta_m)$ as follows. Suppose $z \in \mathsf{Var}^+$. Take a variable x distinct from z and consider a formula $\forall xPz$ with some index m in the enumeration g. Then $Pz \to \forall xPz \in \Delta_{m+1}$, which implies $z \in \mathsf{FV}(\bigcup_{m\in\mathbb{N}} \Delta_m)$. Moreover, it is evident that $\mathsf{FV}(\bigcup_{m\in\mathbb{N}} \Delta_m) \subseteq \mathsf{Var}^+$. Thus we have $\mathsf{FV}(\bigcup_{m\in\mathbb{N}} \Delta_m) = \mathsf{Var}^+$. We also claim that $\bigcup_{m\in\mathbb{N}} \Delta_m$ is a witnessed Λ-consistent set. By item 1 of Lindenbaum Lemma (Lemma 96), we have a Λ-MCS Δ in $\mathsf{L}_n(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Delta) = \mathsf{FV}(\bigcup_{m\in\mathbb{N}} \Delta_m) = \mathsf{Var}^+$ and $\bigcup_{m\in\mathbb{N}} \Delta_m \subseteq \Delta$. This Δ is also witnessed from the construction of $\bigcup_{m\in\mathbb{N}} \Delta_m$. Therefore, since

$$\{\neg\varphi\} \cup \{\gamma \mid [\vec{t}\,]\gamma \in \Gamma\} = \Delta_0 \subseteq \textstyle\bigcup_{m\in\mathbb{N}} \Delta_m \subseteq \Delta,$$

this Δ satisfies that $\Delta \in W^\Lambda$, $\neg\varphi \in \Delta$ and $\Gamma R^\Lambda_{\vec{t}}\Delta$. ∎

Lemma 100. (Truth Lemma) Let $\mathfrak{M}^\Lambda$ be the canonical Λ-model and ι the canonical assignment. For all Λ-MCSs $\Gamma \in W^\Lambda$ and formulas φ in $\mathsf{L}_n(\mathsf{Var}^+)$,

$$\mathfrak{M}^\Lambda, \Gamma, \iota \models \varphi \quad \text{iff} \quad \varphi \in \Gamma$$

Proof. By induction on the length of φ. Since the proof is done similarly to the proof of Truth Lemma for TSML without $(\mathtt{BF}_n)$s (Lemma 93), we check only a case in which φ is $[\vec{t}\,]\psi$.

Note that $\iota(\vec{t}\,) = \vec{t}$ holds from the definition of ι. The right-to-left direction is as follows. Suppose $[\vec{t}\,]\psi \in \Gamma$. Take any $\Delta \in W^\Lambda$ such that $\Gamma R^\Lambda_{\vec{t}}\Delta$. It holds that $\psi \in \Delta$, so by inductive hypothesis $\mathfrak{M}^\Lambda, \Delta, \iota \models \psi$, as required. For the left-to-right direction, we prove the contraposition. Suppose $[\vec{t}]\psi \notin \Gamma$. We show $\mathfrak{M}^\Lambda, \Gamma, \iota \not\models [\vec{t}]\psi$. Since $[\vec{t}]\psi$ is a formula in $\mathsf{L}(\Gamma)$, it holds that $\neg[\vec{t}]\psi \in \Gamma$. By Existence Lemma (Lemma 99), we obtain some $\Delta \in W^\Lambda$ such that $\psi \notin \Delta$ and $\Gamma R^\Lambda_{\vec{t}} \Delta$. Then $\mathfrak{M}^\Lambda, \Delta, \iota \not\models \psi$ holds by inductive hypothesis. Therefore, $\mathfrak{M}^\Lambda, \Gamma, \iota \not\models [\vec{t}]\psi$. ∎

Definition 101. Let $\mathfrak{F} = (W, R, (D_w)_{w \in W})$ be a frame.

1. $\mathfrak{F}$ has *constant domains* if for all $w, v \in W$, $D_w = D_v$.
2. $\mathfrak{F}$ has *locally constant domains* (*in TSML*) if for all $w, v \in W$ and all $\vec{d} \in D_w^{<\omega}$, if $wR_{\vec{d}}v$ then $D_w \subseteq D_v$.

Proposition 102. Let $\mathfrak{F}^\Lambda$ be the canonical Λ-frame, where $\Lambda = \mathsf{H}(\mathbf{tKBF}_n\Sigma_n{\upharpoonright}\mathsf{L}_n)$.

1. $\mathfrak{F}^\Lambda$ has constant domains.
2. If $\mathrm{T}_n \in \Sigma$ then $\mathfrak{F}^\Lambda$ is n-reflexive.
3. If $\mathrm{D}_n \in \Sigma$ then $\mathfrak{F}^\Lambda$ is n-serial.
4. If $\mathrm{B}_n \in \Sigma$ then $\mathfrak{F}^\Lambda$ is n-symmetric.
5. If $4_n \in \Sigma$ then $\mathfrak{F}^\Lambda$ is n-transitive.
6. If $5_n \in \Sigma$ then $\mathfrak{F}^\Lambda$ is n-euclidean.

Proof. Item 1 is obvious and items 2, 3, 5 are shown as in the proof of Proposition 94. We show the other items. Note that each $\vec{t}$ has the length n in the following proofs.

4. Suppose $\mathrm{B}_n \in \Sigma$. We show $\mathfrak{F}^\Lambda$ is n-symmetric, i.e., for all $\Gamma, \Delta \in W^\Lambda$ and $\vec{t} \in (D^\Lambda_\Gamma)^n$, if $\Gamma R^\Lambda_{\vec{t}}\Delta$ then $\Delta R^\Lambda_{\vec{t}}\Gamma$. Take any $\Gamma, \Delta \in W^\Lambda$ and $\vec{t} \in (D^\Lambda_\Gamma)^n$. Suppose $\Gamma R^\Lambda_{\vec{t}}\Delta$ and $[\vec{t}]\varphi \in \Delta$. We claim $\langle\vec{t}\rangle[\vec{t}]\varphi \in \Gamma$. By item 4 of Proposition 83, we obtain $\varphi \in \Gamma$.
6. Suppose $5_n \in \Sigma$. We show $\mathfrak{F}^\Lambda$ is n-euclidean, i.e., for all $\Gamma, \Delta, \Xi \in W^\Lambda$ and $\vec{t} \in (D^\Lambda_\Gamma)^n$, if $\Gamma R^\Lambda_{\vec{t}}\Delta$ and $\Gamma R^\Lambda_{\vec{t}}\Xi$ then $\Delta R^\Lambda_{\vec{t}}\Xi$. Take any $\Gamma, \Delta, \Xi \in W^\Lambda$ and $\vec{t} \in (D^\Lambda_\Gamma)^n$. Suppose $\Gamma R^\Lambda_{\vec{t}}\Delta$ and $\Gamma R^\Lambda_{\vec{t}}\Xi$, and assume $[\vec{t}]\varphi \in \Delta$. We claim $\langle\vec{t}\rangle[\vec{t}]\varphi \in \Gamma$. By item 6 of Proposition 83, we have $[\vec{t}]\varphi \in \Gamma$. Thus, we obtain $\varphi \in \Xi$ by $\Gamma R^\Lambda_{\vec{t}}\Xi$.

∎

Definition 103. By $\mathbb{CD}$ and $\mathbb{LCD}$, we mean the class of all the constant domain frames and the class of all the locally constant domain frames, respectively.

Theorem 104. (Strong completeness of $\mathsf{H}(\mathbf{tKBF}_n\Sigma_n{\restriction}\mathsf{L}_n)$) Let $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{TSML}}$ and $\mathbb{F}_{\Sigma_n}$ be the class of all the frames to which Σ_n corresponds. For all formulas φ and sets Γ of formulas in L_n, if φ is a consequence from Γ in $\mathbb{F}_{\Sigma_n} \cap \mathbb{LCD}$, then φ is provable from Γ in $\mathsf{H}(\mathbf{tKBF}_n\Sigma_n{\restriction}\mathsf{L}_n)$.

Proof. The proof proceeds similarly to the proof of Theorem 62. Let $\Lambda = \mathsf{H}(\mathbf{QKBF}\Sigma)$ for short. Suppose φ is not provable from Γ in Λ. We show φ is not a consequence from Γ in $\mathbb{F}_{\Sigma_n} \cap \mathbb{LCD}$. Note first that $\Gamma \cup \{\neg\varphi\}$ is Λ-consistent in L. We claim $\Gamma \cup \{\neg\varphi\}$ is Λ-consistent in $\mathsf{L}(\mathsf{Var}^+)$. Similarly as in the proof of Theorem 62, we obtain by item 2 of Lindenbaum Lemma (Lemma 96) a witnessed Λ-MCS Γ^+ in $\mathsf{L}(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma^+) = \mathsf{Var}^+$ and $\Gamma \cup \{\neg\varphi\} \subseteq \Gamma^+$. It follows from Truth Lemma (Lemma 100) that

$$\mathfrak{M}^\Lambda, \Gamma^+, \iota \models \Gamma \quad \text{and} \quad \mathfrak{M}^\Lambda, \Gamma^+, \iota \not\models \varphi,$$

where $\mathfrak{M}^\Lambda = (\mathfrak{F}^\Lambda, I^\Lambda)$ is the canonical Λ-model and ι is the canonical assignment. We also need to establish that $\mathfrak{F}^\Lambda \in \mathbb{F}_{\Sigma_n} \cap \mathbb{LCD}$ holds. This is established since $\mathfrak{F}^\Lambda \in \mathbb{F}_{\Sigma_n} \cap \mathbb{CD}$ by Proposition 102 and $\mathbb{F}_{\Sigma_n} \cap \mathbb{CD} \subseteq \mathbb{F}_{\Sigma_n} \cap \mathbb{LCD}$. Therefore, in $\mathsf{L}(\mathsf{Var}^+)$, φ is not a consequence from Γ in $\mathbb{F}_{\Sigma_n} \cap \mathbb{LCD}$. By restricting $\mathsf{L}(\mathsf{Var}^+)$ to L, we conclude in L that φ is not a consequence from Γ in $\mathbb{F}_{\Sigma_n} \cap \mathbb{LCD}$. ■

From the theorem above, we can immediately obtain the strong completeness theorem of a term-modal version $\mathsf{H}(\mathbf{tS5}_1{\restriction}\mathsf{L}_1)$ of $\mathsf{H}(\mathbf{QS5})$.

Corollary 105. (Strong completeness of $\mathsf{H}(\mathbf{tS5}_1{\restriction}\mathsf{L}_1)$) Let $\mathbb{F}$ be the class of all frames to which $\{\mathrm{T}_1, 4_1, \mathrm{B}_1\}$ corresponds. For all formulas φ and sets Γ of formulas in L_1, if φ is a consequence from Γ in $\mathbb{F} \cap \mathbb{LCD}$, then φ is provable from Γ in $\mathsf{H}(\mathbf{tS5}_1{\restriction}\mathsf{L}_1)$.

3.2 Proof-Theoretic Analysis

In this section we first present sequent calculi equipollent to some of Hilbert systems for TSML (3.2.1). We then prove that almost all sequent calculi presented here admit the cut elimination theorems and the Craig interpolation theorems (3.2.2).

3.2.1 Sequent Calculus $\mathsf{G}(\mathbf{tK}\Sigma)$

As for term-modal logic, Thalmann [92] and Fitting et al. [17] develop one-sided sequent calculi for the original term-modal logics, and then Orlandelli and Corsi [70] provide labelled sequent calculi for more than the originals. Contrary to them, the sequent calculi that we present for TSML are one-sided and non-labelled sequent calculi.

A sequent calculus $\mathsf{G}(\mathbf{tK})$ and its expansions are obtained from $\mathsf{G}(\mathbf{QK})$ as follows. Put

$$\mathsf{Axiom}^{-}_{\mathsf{TSML}} := \{\, \mathrm{D}_n, \mathrm{T}_n, 4_n \mid n \in \mathbb{N} \,\}.$$

For a set $\Sigma \subseteq \mathsf{Axiom}^{-}_{\mathsf{TSML}}$, we also mean by Σ_n the set $\{\, \mathrm{X}_n \mid \mathrm{X}_n \in \Sigma \,\}$.

Definition 106. Given $\Sigma \subseteq \mathsf{Axiom}^{-}_{\mathsf{TSML}}$, the sequent calculus $\mathsf{G}(\mathbf{tK\Sigma})$ is the calculus obtained from $\mathsf{G}(\mathbf{QK})$ by adding all logical rules for Σ_n in Table 3.3 for each $n \in \mathbb{N}$, where we define logical rules for $\{\, \mathrm{D}_n, \mathrm{T}_n \,\}$ and $\{\, \mathrm{D}_n, \mathrm{T}_n, 4_n \,\}$ by those for $\{\, \mathrm{T}_n \,\}$ and $\{\, \mathrm{T}_n, 4_n \,\}$, respectively. The sequent calculus $\mathsf{G}(\mathbf{tK\Sigma}_n \upharpoonright \mathsf{L}_n)$ means the restriction of $\mathsf{G}(\mathbf{tK\Sigma}_n)$ to the sublanguage L_n. Irrespective of Σ, the sequent calculus $\mathsf{G}(\mathbf{tS5}_n \upharpoonright \mathsf{L}_n)$ is the restriction of the sequent calculus $\mathsf{G}(\mathbf{tKT}_n\mathbf{4}_n\mathbf{B}_n)$ to the sublanguage L_n, where $\mathsf{G}(\mathbf{tKT}_n\mathbf{4}_n\mathbf{B}_n)$ is the sequent calculus obtained from $\mathsf{G}(\mathbf{QK})$ by adding logical rules $(\Box\mathrm{S5}_n)$ and $(\Box\mathrm{T}_n)$ for $\{\, \mathrm{T}_n, \mathrm{B}_n, 4_n \,\}$ in Table 3.3.

The notion of derivation in $\mathsf{G}(\mathbf{tK\Sigma})$ and $\mathsf{G}(\mathbf{tS5}_n \upharpoonright \mathsf{L}_n)$ is defined as in $\mathsf{G}(\mathbf{K\Sigma})$.

Example 107. Similarly to $\mathsf{H}(\mathbf{tK\Sigma})$, $\mathsf{G}(\mathbf{tK\Sigma})$ allows Σ to contain formulas indexed by different numbers. As an example, consider the case of $\Sigma = \{\, \mathrm{D}_0, \mathrm{T}_1 \,\}$ which was considered in Example 82. The resulting calculus $\mathsf{G}(\mathbf{tKD}_0\mathbf{T}_1)$ is the sequent calculus G with logical rules $(\Box\mathrm{D}_0)$, $(\Box\mathrm{T}_1)$ and $(\Box\mathrm{K}_n)$ $(n \in \mathbb{N})$. In this calculus, a sequent $\Rightarrow [c]\Box Pc \to \Diamond Pc$ is derivable, corresponding to the fact that a formula $[c]\Box Pc \to \Diamond Pc$ is provable in $\mathsf{H}(\mathbf{tKD}_0\mathbf{T}_1)$. Restrictions of language $\mathsf{L}_{\mathsf{TSML}}$ to sublanguages also supply sequent calculi for modal predicate logic and term-modal logic. Consider the sublanguages L_0 and L_1 of $\mathsf{L}_{\mathsf{TSML}}$. Then $\mathsf{G}(\mathbf{tKD}_0 \upharpoonright \mathsf{L}_0)$ is a sequent calculus equipollent to $\mathsf{H}(\mathbf{QKD})$, and $\mathsf{G}(\mathbf{tS5}_n \upharpoonright \mathsf{L}_n)$ is a sequent calculus for a term-modal expansion of S5. Besides $\mathsf{G}(\mathbf{tS5}_n \upharpoonright \mathsf{L}_n)$, Orlandelli and Corsi [70] present a labelled sequent calculus for the expansion.

In the remaining of this subsection, we establish the two equipollences of Hilbert systems and sequent calculi for TSML, i.e., the equipollence of $\mathsf{H}(\mathbf{tK\Sigma})$ and $\mathsf{G}(\mathbf{tK\Sigma})$ for any $\Sigma \subseteq \mathsf{Axiom}^{-}_{\mathsf{TSML}}$, and the equipollence of $\mathsf{H}(\mathbf{tS5}_n \upharpoonright \mathsf{L}_n)$ and $\mathsf{G}(\mathbf{tS5}_n \upharpoonright \mathsf{L}_n)$. These results are established similarly as in [40].

Proposition 108. Let $\Sigma \subseteq \mathsf{Axiom}^{-}_{\mathsf{TSML}}$.

1. $\Rightarrow [\vec{t}](\varphi \to \psi) \to ([\vec{t}]\varphi \to [\vec{t}]\psi)$ is derivable in $\mathsf{G}(\mathbf{tK\Sigma})$.
2. $\Rightarrow [\vec{t}](\varphi \to \psi) \to ([\vec{t}]\varphi \to [\vec{t}]\psi)$ is derivable in $\mathsf{G}(\mathbf{tS5}_n \upharpoonright \mathsf{L}_n)$ if $\vec{t}$ has the length n.

Proof.

1. The proof is done depending on what Σ_n is like.

   Case in which either $\Sigma_n = \emptyset$, $\Sigma_n = \{\, \mathrm{D}_n \,\}$ or $\Sigma_n = \{\, \mathrm{T}_n \,\}$.

Sequent Calculus $G(\mathbf{QK})$

$$\frac{}{\varphi \Rightarrow \varphi}\ (id) \qquad \frac{}{\bot \Rightarrow}\ (\bot)$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}\ (\Rightarrow w) \qquad \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (w \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}\ (\Rightarrow c) \qquad \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (c \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Xi \Rightarrow \Sigma}{\Gamma, \Xi \Rightarrow \Delta, \Sigma}\ (Cut)$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}\ (\Rightarrow \to) \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Xi \Rightarrow \Sigma}{\varphi \to \psi, \Gamma, \Xi \Rightarrow \Delta, \Sigma}\ (\to \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi(y/x)}{\Gamma \Rightarrow \Delta, \forall x \varphi}\ (\Rightarrow \forall)^{\dagger} \qquad \frac{\varphi(t/x), \Gamma \Rightarrow \Delta}{\forall x \varphi, \Gamma \Rightarrow \Delta}\ (\forall \Rightarrow)$$

where †: y is not a free variable in $\Gamma, \Delta, \forall x \varphi$.

Σ_n	Logical Rules for Σ_n	
$\emptyset$	$\dfrac{\Gamma \Rightarrow \varphi}{[\vec{t}]\Gamma \Rightarrow [\vec{t}]\varphi}\ (\Box K_n)$	
$\{ D_n \}$	$\dfrac{\Gamma \Rightarrow \varphi}{[\vec{t}]\Gamma \Rightarrow [\vec{t}]\varphi}\ (\Box K_n)$	$\dfrac{\Gamma \Rightarrow}{[\vec{t}]\Gamma \Rightarrow}\ (\Box D_n)$
$\{ T_n \}$	$\dfrac{\Gamma \Rightarrow \varphi}{[\vec{t}]\Gamma \Rightarrow [\vec{t}]\varphi}\ (\Box K_n)$	$\dfrac{\varphi, \Gamma \Rightarrow \Delta}{[\vec{t}]\varphi, \Gamma \Rightarrow \Delta}\ (\Box T_n)$
$\{ 4_n \}$	$\dfrac{\Gamma, [\vec{t}]\Gamma \Rightarrow \varphi}{[\vec{t}]\Gamma \Rightarrow [\vec{t}]\varphi}\ (\Box 4_n)$	
$\{ D_n, 4_n \}$	$\dfrac{\Gamma, [\vec{t}]\Gamma \Rightarrow \varphi}{[\vec{t}]\Gamma \Rightarrow [\vec{t}]\varphi}\ (\Box 4_n)$	$\dfrac{\Gamma, [\vec{t}]\Gamma \Rightarrow}{[\vec{t}]\Gamma \Rightarrow}\ (\Box D4_n)$
$\{ T_n, 4_n \}$	$\dfrac{[\vec{t}]\Gamma \Rightarrow \varphi}{[\vec{t}]\Gamma \Rightarrow [\vec{t}]\varphi}\ (\Box S4_n)$	$\dfrac{\varphi, \Gamma \Rightarrow \Delta}{[\vec{t}]\varphi, \Gamma \Rightarrow \Delta}\ (\Box T_n)$

where $[\vec{t}]\Gamma := \{ [\vec{t}]\varphi \mid \varphi \in \Gamma \}$ and the length of $\vec{t}$ is n.

	Logical Rules for $\{ T_n, B_n, 4_n \}$	
$\{ T_n, B_n, 4_n \}$	$\dfrac{[\vec{t}]\Gamma \Rightarrow [\vec{t}]\Delta, \varphi}{[\vec{t}]\Gamma \Rightarrow [\vec{t}]\Delta, [\vec{t}]\varphi}\ (\Box S5_n)$	$\dfrac{\varphi, \Gamma \Rightarrow \Delta}{[\vec{t}]\varphi, \Gamma \Rightarrow \Delta}\ (\Box T_n)$

where $[\vec{t}]\Gamma := \{ [\vec{t}]\varphi \mid \varphi \in \Gamma \}$ and the length of $\vec{t}$ is n.

Table 3.3: Sequent Calculus $G(\mathbf{tK\Sigma})$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(id) \qquad \dfrac{}{\psi \Rightarrow \psi}\,(id)}{\varphi \to \psi, \varphi \Rightarrow \psi}\,(\to\Rightarrow)}{[\overrightarrow{t}](\varphi \to \psi), [\overrightarrow{t}]\varphi \Rightarrow [\overrightarrow{t}]\psi}\,(\Box \mathrm{K}_n)}{\Rightarrow [\overrightarrow{t}](\varphi \to \psi) \to ([\overrightarrow{t}]\varphi \to [\overrightarrow{t}]\psi)}\,(\Rightarrow\to)$$

Case in which either $\Sigma_n = \{ 4_n \}$ or $\Sigma_n = \{ \mathrm{D}_n, 4_n \}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(id) \qquad \dfrac{}{\psi \Rightarrow \psi}\,(id)}{\varphi \to \psi, \varphi \Rightarrow \psi}\,(\to\Rightarrow)}{\varphi \to \psi, \varphi, [\overrightarrow{t}](\varphi \to \psi), [\overrightarrow{t}]\varphi \Rightarrow \psi}\,(w\Rightarrow)}{[\overrightarrow{t}](\varphi \to \psi), [\overrightarrow{t}]\varphi \Rightarrow [\overrightarrow{t}]\psi}\,(\Box 4_n)}{\Rightarrow [\overrightarrow{t}](\varphi \to \psi) \to ([\overrightarrow{t}]\varphi \to [\overrightarrow{t}]\psi)}\,(\Rightarrow\to)$$

Case in which $\Sigma_n = \{ \mathrm{T}_n, 4_n \}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(id) \qquad \dfrac{}{\psi \Rightarrow \psi}\,(id)}{\varphi \to \psi, \varphi \Rightarrow \psi}\,(\to\Rightarrow)}{[\overrightarrow{t}](\varphi \to \psi), [\overrightarrow{t}]\varphi \Rightarrow \psi}\,(\Box \mathrm{T}_n)}{[\overrightarrow{t}](\varphi \to \psi), [\overrightarrow{t}]\varphi \Rightarrow [\overrightarrow{t}]\psi}\,(\Box \mathrm{S}4_n)}{\Rightarrow [\overrightarrow{t}](\varphi \to \psi) \to ([\overrightarrow{t}]\varphi \to [\overrightarrow{t}]\psi)}\,(\Rightarrow\to)$$

2. The proof is done along the same lines as the case in which $\Sigma_n = \{ \mathrm{T}_n, 4_n \}$.

■

Proposition 109. (Equipollence of $\mathsf{H}(\mathbf{tK}\Sigma)$ and $\mathsf{G}(\mathbf{tK}\Sigma)$) Let $\Sigma \subseteq \mathsf{Axiom}^{-}_{\mathsf{TSML}}$. A formula φ is provable in $\mathsf{H}(\mathbf{tK}\Sigma)$ iff a sequent $\Rightarrow \varphi$ is derivable in $\mathsf{G}(\mathbf{tK}\Sigma)$.

Proof. The left-to-right direction is by induction on the length of a proof of φ. We skip cases involving first order logic and a case using ($\mathtt{Nec}$). A case in which a proved formula is an instance of ($\mathtt{K}_{\mathtt{inv}}$) is shown by item 1 of Proposition 108. The remaining cases are cases in which a proved formula belongs to $\mathsf{Inst}(\Sigma)$. Thus, it suffices to show cases in which it belongs to $\mathsf{Inst}(\Sigma_n)$ for each $n \in \mathbb{N}$.

Case in which $\Sigma_n = \{ \mathrm{D}_n \}$. The proved formula belongs to $\mathsf{Inst}(\mathrm{D}_n)$ so has the form $[\overrightarrow{t}]\varphi \to \langle \overrightarrow{t} \rangle\varphi$. A derivation of it in $\mathsf{G}(\mathbf{tK}\Sigma)$ is as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(id) \qquad \dfrac{}{\bot \Rightarrow}\,(\bot)}{\varphi, \neg\varphi \Rightarrow}\,(\to\Rightarrow)}{[\overrightarrow{t}]\varphi, [\overrightarrow{t}]\neg\varphi \Rightarrow}\,(\Box \mathrm{D}_n)}{[\overrightarrow{t}]\varphi, [\overrightarrow{t}]\neg\varphi \Rightarrow \bot}\,(\Rightarrow w)}{\Rightarrow [\overrightarrow{t}]\varphi \to \langle \overrightarrow{t} \rangle\varphi}\,(\Rightarrow\to)$$

Case in which $\Sigma_n = \{ \mathrm{T}_n \}$. The proved formula belongs to $\mathsf{Inst}(\mathrm{T}_n)$ so has the form $[\vec{t}\,]\varphi \to \varphi$. A derivation of it in $\mathrm{G}(\mathbf{tK}\Sigma)$ is as follows.

$$\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(id)}{[\vec{t}\,]\varphi \Rightarrow \varphi}\,(\Box \mathrm{T}_n)}{\Rightarrow [\vec{t}\,]\varphi \to \varphi}\,(\Rightarrow \to)$$

Case in which $\Sigma_n = \{ 4_n \}$. The proved formula belongs to $\mathsf{Inst}(4_n)$ so has the form $[\vec{t}\,]\varphi \to [\vec{t}\,][\vec{t}\,]\varphi$. A derivation of it in $\mathrm{G}(\mathbf{tK}\Sigma)$ is as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{[\vec{t}\,]\varphi \Rightarrow [\vec{t}\,]\varphi}\,(id)}{\varphi, [\vec{t}\,]\varphi \Rightarrow [\vec{t}\,]\varphi}\,(w \Rightarrow)}{[\vec{t}\,]\varphi \Rightarrow [\vec{t}\,][\vec{t}\,]\varphi}\,(\Box 4_n)}{\Rightarrow [\vec{t}\,]\varphi \to [\vec{t}\,][\vec{t}\,]\varphi}\,(\Rightarrow \to)$$

Case in which $\Sigma_n = \{ \mathrm{D}_n, 4_n \}$. The proved formula belongs to $\mathsf{Inst}(\mathrm{D}_n, 4_n)$.

Case in which the proved formula belongs to $\mathsf{Inst}(\mathrm{D}_n)$. The proved formula has the form $[\vec{t}\,]\varphi \to \langle \vec{t}\,\rangle\varphi$. A derivation of it in $\mathrm{G}(\mathbf{tK}\Sigma)$ is as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(id) \quad \dfrac{}{\bot \Rightarrow}\,(\bot)}{\varphi, \neg\varphi \Rightarrow}\,(\to \Rightarrow)}{\varphi, \neg\varphi, [\vec{t}\,]\varphi, [\vec{t}\,]\neg\varphi \Rightarrow}\,(w \Rightarrow)}{[\vec{t}\,]\varphi, [\vec{t}\,]\neg\varphi \Rightarrow}\,(\Box \mathrm{D}4_n)}{[\vec{t}\,]\varphi, [\vec{t}\,]\neg\varphi \Rightarrow \bot}\,(\Rightarrow w)}{\Rightarrow [\vec{t}\,]\varphi \to \langle \vec{t}\,\rangle\varphi}\,(\Rightarrow \to)$$

Case in which the proved formula belongs to $\mathsf{Inst}(4_n)$. Same as the case in which $\Sigma_n = \{ 4_n \}$.

Case in which $\Sigma_n = \{ \mathrm{T}_n, 4_n \}$. The proved formula belongs to $\mathsf{Inst}(\mathrm{T}_n, 4_n)$.

Case in which the proved formula belongs to $\mathsf{Inst}(\mathrm{T}_n)$. Same as the case in which $\Sigma_n = \{ \mathrm{T}_n \}$.

Case in which the proved formula belongs to $\mathsf{Inst}(4_n)$. The proved formula has the form $[\vec{t}\,]\varphi \to [\vec{t}\,][\vec{t}\,]\varphi$. A derivation of it in $\mathrm{G}(\mathbf{tK}\Sigma)$ is as follows.

$$\dfrac{\dfrac{\dfrac{}{[\vec{t}\,]\varphi \Rightarrow [\vec{t}\,]\varphi}\,(id)}{[\vec{t}\,]\varphi \Rightarrow [\vec{t}\,][\vec{t}\,]\varphi}\,(\Box \mathrm{S}4_n)}{\Rightarrow [\vec{t}\,]\varphi \to [\vec{t}\,][\vec{t}\,]\varphi}\,(\Rightarrow \to)$$

Therefore, the proof of the left-to-right direction was done.

The right-to-left direction is obtained from the following two claims:

- $\vdash_{\mathsf{G}(\mathbf{tK\Sigma})} \Gamma \Rightarrow \Delta$ implies $\vdash_{\mathsf{H}(\mathbf{tK\Sigma})} \bigwedge \Gamma \rightarrow \bigvee \Delta$;
- $\vdash_{\mathsf{H}(\mathbf{tK\Sigma})} (\bigwedge \emptyset \rightarrow \bigvee \{ \varphi \}) \rightarrow \varphi$.

The latter is easy to establish, so we show the former by induction on the height of a derivation of $\Gamma \Rightarrow \Delta$ in $\mathsf{G}(\mathbf{tK\Sigma})$. We skip cases in which the last rule is a rule from G, since they are straightforward. The remaining cases are cases in which it is a logical rule from Σ. Thus, it suffices to show cases in which it is a logical rule from Σ_n for each $n \in \mathbb{N}$.

Case in which $\Sigma_n = \emptyset$. The last applied rule is $(\Box\mathrm{K}_n)$ so the derivation is of the form

$$\dfrac{\begin{matrix}\vdots \\ \Gamma \Rightarrow \varphi\end{matrix}}{[\vec{t}\,]\Gamma \Rightarrow [\vec{t}\,]\varphi}\ (\Box\mathrm{K}_n)$$

where $\vec{t}$ has the length n. We can obtain $\vdash \bigwedge [\vec{t}\,]\Gamma \rightarrow [\vec{t}\,]\varphi$ as follows.

1.	$\vdash \bigwedge \Gamma \rightarrow \varphi$	Inductive hypothesis
2.	$\vdash [\vec{t}\,] \bigwedge \Gamma \rightarrow [\vec{t}\,]\varphi$	1, $(\mathtt{Nec})$, $(\mathtt{K})$
3.	$\vdash \bigwedge [\vec{t}\,]\Gamma \rightarrow [\vec{t}\,]\varphi$	2, Item 1 of Proposition 83

Case in which $\Sigma_n = \{ \mathrm{D}_n \}$. The last applied rule is $(\Box\mathrm{K}_n)$ or $(\Box\mathrm{D}_n)$.

Case of $(\Box\mathrm{K}_n)$. Same as the case in which $\Sigma_n = \emptyset$.

Case of $(\Box\mathrm{D}_n)$. The derivation is of the form

$$\dfrac{\begin{matrix}\vdots \\ \Gamma \Rightarrow\end{matrix}}{[\vec{t}\,]\Gamma \Rightarrow}\ (\Box\mathrm{D}_n)$$

where $\vec{t}$ has the length n. We can obtain $\vdash \bigwedge [\vec{t}\,]\Gamma \rightarrow \bot$ as follows.

1.	$\vdash \bigwedge \Gamma \rightarrow \bot$	Inductive hypothesis
2.	$\vdash [\vec{t}\,] \bigwedge \Gamma \rightarrow [\vec{t}\,]\bot$	1, $(\mathtt{Nec})$, $(\mathtt{K})$
3.	$\vdash [\vec{t}\,]\bot \rightarrow \langle \vec{t}\,\rangle\bot$	$(\mathtt{D}_n)$
4.	$\vdash [\vec{t}\,] \bigwedge \Gamma \rightarrow \bot$	2, 3, Item 3 of Proposition 83
5.	$\vdash \bigwedge [\vec{t}\,]\Gamma \rightarrow \bot$	4, Item 1 of Proposition 83

Case in which $\Sigma_n = \{ \mathrm{T}_n \}$. The last applied rule is ($\Box\mathrm{K}_n$) or ($\Box\mathrm{T}_n$).

Case of ($\Box\mathrm{K}_n$). Same as the case in which $\Sigma_n = \emptyset$.

Case of ($\Box\mathrm{T}_n$). The derivation is of the form

$$\dfrac{\begin{array}{c}\vdots\\ \varphi, \Gamma \Rightarrow \Delta\end{array}}{[\vec{t}\,]\varphi, \Gamma \Rightarrow \Delta}\ (\Box\mathrm{T}_n)$$

where $\vec{t}$ has the length n. We can obtain $\vdash ([\vec{t}\,]\varphi \wedge \bigwedge \Gamma) \to \bigvee \Delta$ as follows.

1.	$\vdash \varphi \wedge \bigwedge \Gamma \to \bigvee \Delta$	Inductive hypothesis
2.	$\vdash [\vec{t}\,]\varphi \to \varphi$	($\mathtt{T}_n$)
3.	$\vdash ([\vec{t}\,]\varphi \wedge \bigwedge \Gamma) \to \bigvee \Delta$	1, 2, PC

Case in which $\Sigma_n = \{ 4_n \}$. The last applied rule is ($\Box 4_n$) so the derivation is of the form

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma, [\vec{t}\,]\Gamma \Rightarrow \varphi\end{array}}{[\vec{t}\,]\Gamma \Rightarrow [\vec{t}\,]\varphi}\ (\Box 4_n)$$

where $\vec{t}$ has the length n. We can obtain $\vdash \bigwedge[\vec{t}\,]\Gamma \to [\vec{t}\,]\varphi$ as follows.

1.	$\vdash \bigwedge \Gamma \wedge \bigwedge[\vec{t}\,]\Gamma \to \varphi$	Inductive hypothesis
2.	$\vdash [\vec{t}\,](\bigwedge \Gamma \wedge \bigwedge[\vec{t}\,]\Gamma) \to [\vec{t}\,]\varphi$	1, ($\mathtt{Nec}$), ($\mathtt{K}$)
3.	$\vdash (\bigwedge[\vec{t}\,]\Gamma \wedge \bigwedge[\vec{t}\,][\vec{t}\,]\Gamma) \to [\vec{t}\,]\varphi$	2, Item 1 of Proposition 83
4.	$\vdash [\vec{t}\,] \bigwedge \Gamma \to [\vec{t}\,][\vec{t}\,] \bigwedge \Gamma$	(4_n)
5.	$\vdash \bigwedge[\vec{t}\,]\Gamma \to \bigwedge[\vec{t}\,][\vec{t}\,]\Gamma$	4, Item 1 of Proposition 83
6.	$\vdash \bigwedge[\vec{t}\,]\Gamma \to [\vec{t}\,]\varphi$	3, 5, PC

Case in which $\Sigma_n = \{ \mathrm{D}_n, 4_n \}$. The last applied rule is ($\Box 4_n$) or ($\Box\mathrm{D}4_n$).

Case of ($\Box 4_n$). Same as the case in which $\Sigma_n = \{ 4_n \}$.

Case of ($\Box\mathrm{D}4_n$). The derivation is of the form

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma, [\vec{t}\,]\Gamma \Rightarrow\end{array}}{[\vec{t}\,]\Gamma \Rightarrow}\ (\Box\mathrm{D}4_n)$$

where $\vec{t}$ has the length n. We can obtain $\vdash \bigwedge[\vec{t}]\Gamma \to \bot$ as follows.

1. $\vdash \bigwedge\Gamma \wedge \bigwedge[\vec{t}]\Gamma \to \bot$	Inductive hypothesis
2. $\vdash [\vec{t}](\bigwedge\Gamma \wedge \bigwedge[\vec{t}]\Gamma) \to [\vec{t}]\bot$	1, $(\mathtt{Nec})$, $(\mathtt{K})$
3. $\vdash (\bigwedge[\vec{t}]\Gamma \wedge \bigwedge[\vec{t}][\vec{t}]\Gamma) \to [\vec{t}]\bot$	2, Item 1 of Proposition 83
4. $\vdash [\vec{t}]\bot \to \langle\vec{t}\rangle\bot$	$(\mathtt{D}_n)$
5. $\vdash (\bigwedge[\vec{t}]\Gamma \wedge \bigwedge[\vec{t}][\vec{t}]\Gamma) \to \bot$	3, 4, Item 3 of Proposition 83
6. $\vdash [\vec{t}]\bigwedge\Gamma \to [\vec{t}][\vec{t}]\bigwedge\Gamma$	(4_n)
7. $\vdash \bigwedge[\vec{t}]\Gamma \to \bigwedge[\vec{t}][\vec{t}]\Gamma$	6, Item 1 of Proposition 83
8. $\vdash \bigwedge[\vec{t}]\Gamma \to \bot$	5, 7, PC

Case in which $\Sigma_n = \{\mathrm{T}_n, 4_n\}$. The last applied rule is $(\Box\mathrm{S}4_n)$ or $(\Box\mathrm{T}_n)$.

Case of $(\Box\mathrm{S}4_n)$. The derivation is of the form

$$\cfrac{\begin{array}{c}\vdots\\ [\vec{t}]\Gamma \Rightarrow \varphi\end{array}}{[\vec{t}]\Gamma \Rightarrow [\vec{t}]\varphi}\ (\Box\mathrm{S}4_n)$$

where $\vec{t}$ has the length n. We can obtain $\vdash \bigwedge[\vec{t}]\Gamma \to [\vec{t}]\varphi$ as follows.

1. $\vdash \bigwedge[\vec{t}]\Gamma \to \varphi$	Inductive hypothesis
2. $\vdash [\vec{t}]\bigwedge[\vec{t}]\Gamma \to [\vec{t}]\varphi$	1, $(\mathtt{Nec})$, $(\mathtt{K})$
3. $\vdash \bigwedge[\vec{t}][\vec{t}]\Gamma \to [\vec{t}]\varphi$	2, Item 1 of Proposition 83
4. $\vdash [\vec{t}]\bigwedge\Gamma \to [\vec{t}][\vec{t}]\bigwedge\Gamma$	(4_n)
5. $\vdash \bigwedge[\vec{t}]\Gamma \to \bigwedge[\vec{t}][\vec{t}]\Gamma$	4, Item 1 of Proposition 83
6. $\vdash \bigwedge[\vec{t}]\Gamma \to [\vec{t}]\varphi$	3, 5, PC

Case of $(\Box\mathrm{T}_n)$. Same as the case in which $\Sigma_n = \{\mathrm{T}_n\}$.

Thus the proof of the right-to-left direction is also done. ∎

Proposition 110. (Equipollence of $\mathsf{H}(\mathbf{tS5}_n{\restriction}\mathsf{L}_n)$ and $\mathsf{G}(\mathbf{tS5}_n{\restriction}\mathsf{L}_n)$) A formula φ is provable in $\mathsf{H}(\mathbf{tS5}_n{\restriction}\mathsf{L}_n)$ iff a sequent $\Rightarrow \varphi$ is derivable in $\mathsf{G}(\mathbf{tS5}_n{\restriction}\mathsf{L}_n)$.

Proof. The proof is done similarly to the proof of Proposition 109. The left-to-right direction is by induction on the length of a proof of φ. We skip cases involving first order logic and a case using $(\mathtt{Nec})$. A case in which a proved formula is an instance of $(\mathtt{K})$ is shown by item 2 of Proposition 108. Thus it suffices to check only cases in which a proved formula belongs to $\mathsf{Inst}(\{\mathrm{T}_n, 4_n, \mathrm{B}_n\})$.

Case in which the proved formula belongs to $\mathsf{Inst}(\mathrm{T}_n)$**.** The proved formula has the form $[\vec{t}\,]\varphi \to \varphi$. A derivation of it in $\mathrm{G}(\mathbf{tS5}_n{\restriction}\mathsf{L}_n)$ is as follows.

$$\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(id)}{[\vec{t}\,]\varphi \Rightarrow \varphi}\,(\Box \mathrm{T}_n)}{\Rightarrow [\vec{t}\,]\varphi \to \varphi}\,(\Rightarrow\to)$$

Case in which the proved formula belongs to $\mathsf{Inst}(4_n)$**.** The proved formula has the form $[\vec{t}\,]\varphi \to [\vec{t}\,][\vec{t}\,]\varphi$. A derivation of it in $\mathrm{G}(\mathbf{tS5}_n{\restriction}\mathsf{L}_n)$ is as follows.

$$\dfrac{\dfrac{\dfrac{}{[\vec{t}\,]\varphi \Rightarrow [\vec{t}\,]\varphi}\,(id)}{[\vec{t}\,]\varphi \Rightarrow [\vec{t}\,][\vec{t}\,]\varphi}\,(\Box \mathrm{S5}_n)}{\Rightarrow [\vec{t}\,]\varphi \to [\vec{t}\,][\vec{t}\,]\varphi}\,(\Rightarrow\to)$$

Case in which the proved formula belongs to $\mathsf{Inst}(\mathrm{B}_n)$**.** The proved formula has the form $\varphi \to [\vec{t}\,]\langle\vec{t}\,\rangle\varphi$. A derivation of it in $\mathrm{G}(\mathbf{tS5}_n{\restriction}\mathsf{L}_n)$ is as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[\vec{t}\,]\neg\varphi \Rightarrow [\vec{t}\,]\neg\varphi}\,(id)}{[\vec{t}\,]\neg\varphi \Rightarrow [\vec{t}\,]\neg\varphi, \bot}\,(\Rightarrow w)}{\Rightarrow \langle\vec{t}\,\rangle\varphi, [\vec{t}\,]\neg\varphi}\,(\Rightarrow\to)}{\Rightarrow [\vec{t}\,]\langle\vec{t}\,\rangle\varphi, [\vec{t}\,]\neg\varphi}\,(\Box \mathrm{S5}_n) \quad \dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(id) \quad \dfrac{}{\bot \Rightarrow}\,(\bot)}{\neg\varphi, \varphi \Rightarrow}\,(\Rightarrow\to)}{[\vec{t}\,]\neg\varphi, \varphi \Rightarrow}\,(\Box \mathrm{T}_n)}{\varphi \Rightarrow [\vec{t}\,]\langle\vec{t}\,\rangle\varphi}\,(Cut)}{\Rightarrow \varphi \to [\vec{t}\,]\langle\vec{t}\,\rangle\varphi}\,(\Rightarrow\to)$$

Therefore, the proof of the left-to-right direction was done.

As before, the right-to-left direction is obtained from the claims that $\vdash_{\mathrm{G}(\mathbf{tS5}_n\restriction\mathsf{L}_n)} \Gamma \Rightarrow \Delta$ implies $\vdash_{\mathrm{G}(\mathbf{tS5}_n\restriction\mathsf{L}_n)} \bigwedge\Gamma \to \bigvee\Delta$ and that $\vdash_{\mathrm{H}(\mathbf{tS5}_n\restriction\mathsf{L}_n)} (\bigwedge\emptyset \to \bigvee\{\varphi\}) \to \varphi$. The latter is straightforward. The former is by induction on the height of a derivation of $\Gamma \Rightarrow \Delta$. We skip cases in which the last applied rule is a rule from G, and show a case in which it is $(\Box\mathrm{S5}_n)$ or $(\Box\mathrm{T}_n)$.

Case of $(\Box\mathrm{S5}_n)$**.** The derivation is of the form

$$\dfrac{\begin{array}{c}\vdots\\ [\vec{t}\,]\Gamma \Rightarrow [\vec{t}\,]\Delta, \varphi\end{array}}{[\vec{t}\,]\Gamma \Rightarrow [\vec{t}\,]\Delta, [\vec{t}\,]\varphi}\,(\Box \mathrm{S5}_n)$$

where $\vec{t}$ has the length n. We can obtain $\vdash \bigwedge[\vec{t}\,]\Gamma \to (\bigvee[\vec{t}\,]\Delta \vee [\vec{t}\,]\varphi)$ as follows.

1. $\vdash \bigwedge[\vec{t}\,]\Gamma \to (\bigvee[\vec{t}\,]\Delta \vee \varphi)$ Inductive hypothesis

2. $\vdash [\vec{t}\,] \bigwedge [\vec{t}\,]\Gamma \to (\langle \vec{t}\,\rangle \bigvee [\vec{t}\,]\Delta \lor [\vec{t}\,]\varphi)$ 1, (Nec), (K), PC
3. $\vdash \bigwedge [\vec{t}\,][\vec{t}\,]\Gamma \to (\bigvee \langle \vec{t}\,\rangle [\vec{t}\,]\Delta \lor [\vec{t}\,]\varphi)$ 2, Items 1, 2 of Proposition 83
4. $\vdash [\vec{t}\,] \bigwedge \Gamma \to [\vec{t}\,][\vec{t}\,] \bigwedge \Gamma$ (4_n)
5. $\vdash \bigwedge [\vec{t}\,]\Gamma \to \bigwedge [\vec{t}\,][\vec{t}\,]\Gamma$ 4, Item 1 of Proposition 83
6. $\vdash \bigwedge [\vec{t}\,]\Gamma \to (\bigvee \langle \vec{t}\,\rangle [\vec{t}\,]\Delta \lor [\vec{t}\,]\varphi)$ 3, 5, PC
7. $\vdash \langle \vec{t}\,\rangle [\vec{t}\,]\delta \to [\vec{t}\,]\delta$ $\delta \in \Delta$, Item 7 of Proposition 83
8. $\vdash \bigvee \langle \vec{t}\,\rangle [\vec{t}\,]\Delta \to \bigvee [\vec{t}\,]\Delta$ 7, PC
9. $\vdash \bigwedge [\vec{t}\,]\Gamma \to (\bigvee [\vec{t}\,]\Delta \lor [\vec{t}\,]\varphi)$ 6, 8, PC

Case of $(\Box \mathrm{T}_n)$**.** Same as the case of $\Sigma_n = \{ \mathrm{T}_n \}$ in the proof of Proposition 109.

Thus the right-to-left direction was also done. ■

3.2.2 Cut Elimination

Let Σ be some fixed subset of $\mathsf{Axiom}^-_{\mathsf{TSML}} = \{ \mathrm{D}_n, \mathrm{T}_n, 4_n \mid n \in \mathbb{N} \}$ throughout this subsection. In this subsection, we prove the cut elimination theorem of $\mathrm{G}(\mathbf{tK}\Sigma)$. Note that we do not take $\mathrm{G}(\mathbf{tS5}_n \restriction \mathsf{L}_n)$ into account here. The proof of the cut elimination theorem is done by using the extended rule (Cut^*) of (Cut) introduced in [68, 41, 67]:

$$\frac{\Gamma \Rightarrow \Delta, \varphi^l \qquad \varphi^m, \Xi \Rightarrow \Pi}{\Gamma, \Xi \Rightarrow \Delta, \Pi}\ (Cut^*)$$

where l, m can be zero and φ is called the *cut-formula*. As we see in the proof of the cut elimination theorem, (Cut^*) plays a similar role as the rule (Mix) in Gentzen [23]. In what follows, we assume that free variables and bound variables in derivations are thoroughly separated.

Definition 111. Let r be a logical rule of $\mathrm{G}(\mathbf{tK}\Sigma)$. Then *principal formulas in r* are defined as follows.

- if $r = \begin{cases} \dfrac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}\ (\Rightarrow \to) \text{ or} \\ \dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Xi \Rightarrow \Pi}{\varphi \to \psi, \Gamma, \Xi \Rightarrow \Delta, \Pi}\ (\to \Rightarrow), \end{cases}$ then $\varphi \to \psi$ is principal in r.

- if $r = \begin{cases} \dfrac{\Gamma \Rightarrow \Delta, \varphi(y/x)}{\Gamma \Rightarrow \Delta, \forall x \varphi}\ (\Rightarrow \forall) \text{ or} \\ \dfrac{\varphi(t/x), \Gamma \Rightarrow \Delta}{\forall x \varphi, \Gamma \Rightarrow \Delta}\ (\forall \Rightarrow), \end{cases}$ then $\forall x \varphi$ is principal in r.

- if $r = \dfrac{\Gamma \Rightarrow \varphi}{[\vec{t}\,]\Gamma \Rightarrow [\vec{t}\,]\varphi}$ ($\Box\mathrm{K}_n$), then $[\vec{t}\,]\Gamma$ and $[\vec{t}\,]\varphi$ are principal in r.

- if $r = \begin{cases} \dfrac{\Gamma \Rightarrow}{[\vec{t}\,]\Gamma \Rightarrow} \ (\Box\mathrm{D}_n) \text{ or} \\ \dfrac{\Gamma, [\vec{t}\,]\Gamma \Rightarrow}{[\vec{t}\,]\Gamma \Rightarrow} \ (\Box\mathrm{D4}_n), \end{cases}$ then $[\vec{t}\,]\Gamma$ are principal in r.

- if $r = \dfrac{\varphi, \Gamma \Rightarrow \Delta}{[\vec{t}\,]\varphi, \Gamma \Rightarrow \Delta}$ ($\Box\mathrm{T}_n$), then $[\vec{t}\,]\varphi$ is principal in r.

- if $r = \begin{cases} \dfrac{[\vec{t}\,]\Gamma \Rightarrow \varphi}{[\vec{t}\,]\Gamma \Rightarrow [\vec{t}\,]\varphi} \ (\Box\mathrm{S4}_n) \text{ or} \\ \dfrac{\Gamma, [\vec{t}\,]\Gamma \Rightarrow \varphi}{[\vec{t}\,]\Gamma \Rightarrow [\vec{t}\,]\varphi} \ (\Box 4_n), \end{cases}$ then $[\vec{t}\,]\Gamma$ and $[\vec{t}\,]\varphi$ are principal in r.

Definition 112. The sequent calculus $\mathrm{G}^-(\mathbf{tK\Sigma})$ is the calculus obtained from $\mathrm{G}(\mathbf{tK\Sigma})$ by removing (Cut) of $\mathrm{G}(\mathbf{tK\Sigma})$. The sequent calculus $\mathrm{G}^*(\mathbf{tK\Sigma})$ is the calculus obtained from $\mathrm{G}(\mathbf{tK\Sigma})$ by replacing (Cut) of $\mathrm{G}(\mathbf{tK\Sigma})$ with (Cut^*).

Definition 113. A derivation $\mathfrak{D}$ in $\mathrm{G}^*(\mathbf{tK\Sigma})$ is of the (Cut^*)*-bottom form* if the last applied rule in $\mathfrak{D}$ is (Cut^*) and there are no other applications of (Cut^*) in $\mathfrak{D}$, depicted as follows:

$$\dfrac{\begin{matrix}\mathfrak{D}_1 \\ \Gamma \Rightarrow \Delta, \varphi^l\end{matrix} \qquad \begin{matrix}\mathfrak{D}_2 \\ \varphi^m, \Xi \Rightarrow \Pi\end{matrix}}{\Gamma, \Xi \Rightarrow \Delta, \Pi}\ (Cut^*)$$

where $\mathfrak{D}_1$, $\mathfrak{D}_2$ are derivations of $\Gamma \Rightarrow \Delta, \varphi^l$ and $\varphi^m, \Xi \Rightarrow \Pi$, respectively, with no applications of (Cut^*).

Definition 114. Let σ be an application of (Cut^*) in a derivation $\mathfrak{D}$ in $\mathrm{G}^*(\mathbf{tK\Sigma})$. Let also $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be the sub-derivations of $\mathfrak{D}$ whose roots are the left and right upper sequents of σ, respectively. We define the *grade* $g(\sigma)$ and the *weight* $w(\sigma)$ of σ by

$$g(\sigma) = l(\varphi) \quad \text{where } \varphi \text{ is the cut-formula of } \sigma \text{ and } l(\varphi) \text{ is the length of } \varphi;$$
$$w(\sigma) = |\mathfrak{D}_1| + |\mathfrak{D}_2| \quad \text{where } |\mathfrak{D}'| \text{ denotes the number of sequents in } \mathfrak{D}'.$$

In addition, we define a lexicographical order $<$ on $\mathbb{N} \times \mathbb{N}$ by

$$(i, j) < (i', j') \quad \text{iff} \quad i < i', \text{ or } i = i' \text{ and } j < j'.$$

Theorem 115. (Cut elimination) If a sequent $\Gamma \Rightarrow \Delta$ is derivable in $\mathrm{G}(\mathbf{tK\Sigma})$, then $\Gamma \Rightarrow \Delta$ is also derivable in $\mathrm{G}^-(\mathbf{tK\Sigma})$.

Proof. Since (Cut) is an instance of (Cut^*), it suffices to show $\vdash_{\mathrm{G}^*(\mathbf{tK\Sigma})} \Gamma \Rightarrow \Delta$ implies $\vdash_{\mathrm{G}^-(\mathbf{tK\Sigma})} \Gamma \Rightarrow \Delta$. Furthermore, it is obtained immediately from the following claim.

> If there is a derivation $\mathfrak{D}$ of the (Cut^*)-bottom form of a sequent $\Gamma \Rightarrow \Delta$ in $\mathrm{G}^*(\mathbf{tK\Sigma})$, there is also a derivation of $\Gamma \Rightarrow \Delta$ in $\mathrm{G}^-(\mathbf{tK\Sigma})$.

We show this claim by double induction on a pair $(g(\sigma), w(\sigma))$ of the grade $g(\sigma)$ and the weight $w(\sigma)$ of the only application σ of (Cut^*) in a derivation $\mathfrak{D}$ of the (Cut^*)-bottom form. Assume we are given a derivation $\mathfrak{D}$ of the (Cut^*)-bottom form of a sequent $\Gamma \Rightarrow \Delta$ in $\mathrm{G}^*(\mathbf{tK\Sigma})$. Then $\mathfrak{D}$ is of the form

$$\dfrac{\dfrac{\mathfrak{D}_1}{\Gamma \Rightarrow \Delta, \varphi^l}\ \rho(L) \qquad \dfrac{\mathfrak{D}_2}{\varphi^m, \Xi \Rightarrow \Pi}\ \rho(R)}{\Gamma, \Xi \Rightarrow \Delta, \Pi}\ (Cut^*)$$

where $\rho(L)$ and $\rho(R)$ are the last applied rule to $\mathfrak{D}_1$ and $\mathfrak{D}_2$, respectively. We may also assume that both of the numbers l, m are more than zero, because if not we can obtain a derivation of $\Gamma \Rightarrow \Delta$ in $\mathrm{G}^-(\mathbf{tK\Sigma})$ by applying $(\Rightarrow w)$ and $(w \Rightarrow)$ repeatedly to $\mathfrak{D}_1$ or $\mathfrak{D}_2$.

Regardless of our choice of Σ, we divide our proof into the following four cases, though our arguments for the third and fourth cases depend on our choice of Σ.

- $\rho(L)$ or $\rho(R)$ is an initial sequent.
- $\rho(L)$ or $\rho(R)$ is an application of a structural rule.
- $\rho(L)$ or $\rho(R)$ is an application of a logical rule but the cut-formula is not principal in the logical rule, respectively.
- Both of $\rho(L)$ and $\rho(R)$ are applications of logical rules and the cut-formula is principal in both of the logical rules.

We skip the first case since it is easy to show. For the second case, one should pay attention to subcases in which $\rho(L) = (\Rightarrow c)$ and $\rho(R) = (c \Rightarrow)$, since they are places where (Cut^*) plays a similar role as (Mix). We confirm only the subcase in which $\rho(L) = (\Rightarrow c)$. For proofs of the first case and the other subcases of the second case, see Ono [67, p. 28].

Case in which $\rho(L) = (\Rightarrow c)$. There are two cases depending on whether or not $(\Rightarrow c)$ is applied to the cut-formula. The case in which $(\Rightarrow c)$ is applied to the cut-formula is a place where (Cut^*) plays a role.

Case in which $(\Rightarrow c)$ is not applied to the cut-formula. Then $\mathfrak{D} =$

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \Gamma \Rightarrow \Delta, \psi, \psi, \varphi^l\end{matrix}}{\Gamma \Rightarrow \Delta, \psi, \varphi^l}(\Rightarrow c) \qquad \begin{matrix}\mathfrak{D}_2\\ \varphi^m, \Xi \Rightarrow \Pi\end{matrix}}{\Gamma, \Xi \Rightarrow \Delta, \Pi, \psi}(Cut^*)$$

which can be transformed into a derivation $\mathfrak{D}' =$

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \Gamma \Rightarrow \Delta, \psi, \psi, \varphi^l\end{matrix} \qquad \begin{matrix}\mathfrak{D}_2\\ \varphi^m, \Xi \Rightarrow \Pi\end{matrix}}{\Gamma, \Xi \Rightarrow \Delta, \Pi, \psi, \psi}(Cut^*)}{\Gamma, \Xi \Rightarrow \Delta, \Pi, \psi}(\Rightarrow c)$$

The application σ_1 of (Cut^*) in $\mathfrak{D}'$ is eliminable by inductive hypothesis, because $g(\sigma_1) = g(\sigma)$ and $w(\sigma_1) < w(\sigma)$ hold.

Case in which $(\Rightarrow c)$ **is applied to the cut-formula.** Then $\mathfrak{D} =$

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \Gamma \Rightarrow \Delta, \varphi, \varphi^l\end{matrix}}{\Gamma \Rightarrow \Delta, \varphi^l}(\Rightarrow c) \qquad \begin{matrix}\mathfrak{D}_2\\ \varphi^m, \Xi \Rightarrow \Pi\end{matrix}}{\Gamma, \Xi \Rightarrow \Delta, \Pi}(Cut^*)$$

which can be transformed into a derivation $\mathfrak{D}' =$

$$\dfrac{\begin{matrix}\mathfrak{D}_1\\ \Gamma \Rightarrow \Delta, \varphi, \varphi^l\end{matrix} \qquad \begin{matrix}\mathfrak{D}_2\\ \varphi^m, \Xi \Rightarrow \Pi\end{matrix}}{\Gamma, \Xi \Rightarrow \Delta, \Pi}(Cut^*)$$

The application σ_1 of (Cut^*) in $\mathfrak{D}'$ is eliminable by inductive hypothesis, because $g(\sigma_1) = g(\sigma)$ and $w(\sigma_1) < w(\sigma)$ hold.

We have done the first and second cases so far. For the third case, we divide two subcases depending on whether $\mathrm{T}_n \in \Sigma$ for some $n \in \mathbb{N}$ or not.

Consider first a subcase in which $\mathrm{T}_n \notin \Sigma$ for all $n \in \mathbb{N}$. Recall the definition of principality and our assumption that both of the number l, m of the cut-formula are more than zero. These tell us that it cannot be the case that $\rho(L)$ or $\rho(R)$ is a logical rule from Σ. Thus, in this subcase, it suffices to check when $\rho(L)$ or $\rho(R)$ is a logical rule from $\mathrm{G}(\mathbf{QK})$. This is shown by a usual argument found e.g. in [67, p. 29]. Therefore, this subcase is done.

Consider then the other subcase in which $\mathrm{T}_n \in \Sigma$ for some $n \in \mathbb{N}$. In this subcase, in addition to the cases in which $\rho(L)$ or $\rho(R)$ is a logical rule from $\mathrm{G}(\mathbf{QK})$, we must check cases in which $\rho(L)$ or $\rho(R)$ is $(\Box\mathrm{T}_n)$. Since the former cases are proved by the argument found in [67, p. 29], we check the cases in which $\rho(L)$ or $\rho(R)$ is $(\Box\mathrm{T}_n)$.

Case in which $\rho(L) = (\Box \mathrm{T}_n)$. Then $\mathfrak{D}$ =

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_1\\ \psi, \Gamma \Rightarrow \Delta, \varphi^l\end{array}}{[\vec{t}]\psi, \Gamma \Rightarrow \Delta, \varphi^l}\,(\Box \mathrm{T}_n) \qquad \begin{array}{c}\mathfrak{D}_2\\ \varphi^m, \Xi \Rightarrow \Pi\end{array}}{[\vec{t}]\psi, \Gamma, \Xi \Rightarrow \Delta, \Pi}\,(Cut^*)$$

which can be transformed into a derivation $\mathfrak{D}'$ =

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_1\\ \psi, \Gamma \Rightarrow \Delta, \varphi^l\end{array} \qquad \begin{array}{c}\mathfrak{D}_2\\ \varphi^m, \Xi \Rightarrow \Pi\end{array}}{\psi, \Gamma, \Xi \Rightarrow \Delta, \Pi}\,(Cut^*)}{[\vec{t}]\psi, \Gamma, \Xi \Rightarrow \Delta, \Pi}\,(\Box \mathrm{T}_n)$$

The application σ_1 of (Cut^*) in $\mathfrak{D}'$ is eliminable by inductive hypothesis, because $g(\sigma_1) = g(\sigma)$ and $w(\sigma_1) < w(\sigma)$ hold. Thus, there is a derivation of $[\vec{t}]\psi, \Gamma, \Xi \Rightarrow \Delta, \Pi$ in $\mathrm{G}^-(\mathbf{tK\Sigma})$.

Case in which $\rho(R) = (\Box \mathrm{T}_n)$. If the cut-formula is principal in $(\Box \mathrm{T}_n)$, as the third case is now under consideration, it has to be the case that $\rho(L)$ is an application of a logical rule but the cut-formula is not principal in the logical rule. When the logical rule comes from $\mathrm{G}(\mathbf{QK})$, the current case is established by the argument in [67, p. 29]. When the logical rule comes from Σ, the current case must be the case of $\rho(L) = (\Box \mathrm{T}_n)$, which we just saw above. Therefore, we may assume the cut-formula is not principal in $(\Box \mathrm{T}_n)$. Then, $\mathfrak{D}$ =

$$\dfrac{\begin{array}{c}\mathfrak{D}_1\\ \Gamma \Rightarrow \Delta, \varphi^l\end{array} \qquad \dfrac{\begin{array}{c}\mathfrak{D}_2\\ \varphi^m, \psi, \Xi \Rightarrow \Pi\end{array}}{\varphi^m, [\vec{t}]\psi, \Xi \Rightarrow \Pi}\,(\Box \mathrm{T}_n)}{[\vec{t}]\psi, \Gamma, \Xi \Rightarrow \Delta, \Pi}\,(Cut^*)$$

which can be transformed into a derivation $\mathfrak{D}'$ =

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_1\\ \Gamma \Rightarrow \Delta, \varphi^l\end{array} \qquad \begin{array}{c}\mathfrak{D}_2\\ \varphi^m, \psi, \Xi \Rightarrow \Pi\end{array}}{\psi, \Gamma, \Xi \Rightarrow \Delta, \Pi}\,(Cut^*)}{[\vec{t}]\psi, \Gamma, \Xi \Rightarrow \Delta, \Pi}\,(\Box \mathrm{T}_n)$$

The application σ_1 of (Cut^*) in $\mathfrak{D}'$ is eliminable by inductive hypothesis, because $g(\sigma_1) = g(\sigma)$ and $w(\sigma_1) < w(\sigma)$ hold. Thus, there is a derivation of $[\vec{t}]\psi, \Gamma, \Xi \Rightarrow \Delta, \Pi$ in $\mathrm{G}^-(\mathbf{tK\Sigma})$.

Now that we have done the first, second and third cases, the remaining case is the fourth case. Recall that the fourth case is the case in which both of $\rho(L)$ and $\rho(R)$ are applications of logical rules and the cut-formula is principal in both of the logical rules. One can establish cases in which $\rho(L)$ and $\rho(R)$ are logical rules from G(**QK**) by a usual argument found e.g. in [67, pp. 29–30]. We show only cases in which $\rho(L)$ and $\rho(R)$ are logical rules from Σ. Note that the current cases arise only when the the two same term-sequences $\vec{t}$ index the modal operator $[\vec{t}]$ of the cut formulas of the form $[\vec{t}]\varphi$. Thus it suffices to consider when $\rho(L)$ and $\rho(R)$ are logical rules from $\Sigma_n = \{ \mathrm{X}_n \mid \mathrm{X}_n \in \Sigma \}$.

Case in which $\rho(L) = \rho(R) = (\Box\mathrm{K}_n)$. Then $\mathfrak{D}$ =

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \Gamma \Rightarrow \varphi\end{matrix}}{[\vec{t}]\Gamma \Rightarrow [\vec{t}]\varphi}\,(\Box\mathrm{K}_n) \qquad \dfrac{\begin{matrix}\mathfrak{D}_2\\ \varphi^m, \Xi \Rightarrow \psi\end{matrix}}{([\vec{t}]\varphi)^m, [\vec{t}]\Xi \Rightarrow [\vec{t}]\psi}\,(\Box\mathrm{K}_n)}{[\vec{t}]\Gamma, [\vec{t}]\Xi \Rightarrow [\vec{t}]\psi}\,(Cut^*)$$

which can be transformed into a derivation $\mathfrak{D}'$ =

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \Gamma \Rightarrow \varphi\end{matrix} \qquad \begin{matrix}\mathfrak{D}_2\\ \varphi^m, \Xi \Rightarrow \psi\end{matrix}}{\Gamma, \Xi \Rightarrow \psi}\,(Cut^*)}{[\vec{t}]\Gamma, [\vec{t}]\Xi \Rightarrow [\vec{t}]\psi}\,(\Box\mathrm{K}_n)$$

The application σ_1 of (Cut^*) in $\mathfrak{D}'$ is eliminable by inductive hypothesis, because $g(\sigma_1) < g(\sigma)$ holds by $l(\varphi) < l([\vec{t}]\varphi)$. Thus, there is a derivation of $[\vec{t}]\Gamma, [\vec{t}]\Xi \Rightarrow [\vec{t}]\psi$ in $\mathrm{G}^-(\mathbf{tK}\Sigma)$.

Case in which $\rho(L) = (\Box\mathrm{K}_n)$ and $\rho(R) = (\Box\mathrm{D}_n)$. Then $\mathfrak{D}$ =

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \Gamma \Rightarrow \varphi\end{matrix}}{[\vec{t}]\Gamma \Rightarrow [\vec{t}]\varphi}\,(\Box\mathrm{K}_n) \qquad \dfrac{\begin{matrix}\mathfrak{D}_2\\ \varphi^m, \Xi \Rightarrow\end{matrix}}{([\vec{t}]\varphi)^m, [\vec{t}]\Xi \Rightarrow}\,(\Box\mathrm{D}_n)}{[\vec{t}]\Gamma, [\vec{t}]\Xi \Rightarrow}\,(Cut^*)$$

which can be transformed into a derivation $\mathfrak{D}'$ =

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \Gamma \Rightarrow \varphi\end{matrix} \qquad \begin{matrix}\mathfrak{D}_2\\ \varphi^m, \Xi \Rightarrow\end{matrix}}{\Gamma, \Xi \Rightarrow}\,(Cut^*)}{[\vec{t}]\Gamma, [\vec{t}]\Xi \Rightarrow}\,(\Box\mathrm{D}_n)$$

The application σ_1 of (Cut^*) in $\mathfrak{D}'$ is eliminable by inductive hypothesis, because $g(\sigma_1) < g(\sigma)$ holds by $l(\varphi) < l([\vec{t}\,]\varphi)$. Thus, there is a derivation of $[\vec{t}\,]\Gamma, [\vec{t}\,]\Xi \Rightarrow$ in $\mathrm{G}^-(\mathbf{tK\Sigma})$.

Case in which $\rho(L) = (\Box\mathrm{K}_n)$ and $\rho(R) = (\Box\mathrm{T}_n)$. Then $\mathfrak{D} =$

$$
\begin{prooftree}
\AxiomC{$\mathfrak{D}_1$}
\noLine
\UnaryInfC{$\Gamma \Rightarrow \varphi$}
\RightLabel{$(\Box\mathrm{K}_n)$}
\UnaryInfC{$[\vec{t}\,]\Gamma \Rightarrow [\vec{t}\,]\varphi$}
\AxiomC{$\mathfrak{D}_2$}
\noLine
\UnaryInfC{$\varphi, ([\vec{t}\,]\varphi)^{m-1}, \Xi \Rightarrow \Delta$}
\RightLabel{$(\Box\mathrm{T}_n)$}
\UnaryInfC{$[\vec{t}\,]\varphi, ([\vec{t}\,]\varphi)^{m-1}, \Xi \Rightarrow \Delta$}
\RightLabel{(Cut^*)}
\BinaryInfC{$[\vec{t}\,]\Gamma, [\vec{t}\,]\Xi \Rightarrow \Delta$}
\end{prooftree}
$$

which can be transformed into a derivation $\mathfrak{D}' =$

$$
\begin{prooftree}
\AxiomC{$\mathfrak{D}_1$}
\noLine
\UnaryInfC{$\Gamma \Rightarrow \varphi$}
\AxiomC{$\mathfrak{D}_1$}
\noLine
\UnaryInfC{$\Gamma \Rightarrow \varphi$}
\RightLabel{$(\Box\mathrm{K}_n)$}
\UnaryInfC{$[\vec{t}\,]\Gamma \Rightarrow [\vec{t}\,]\varphi$}
\AxiomC{$\mathfrak{D}_2$}
\noLine
\UnaryInfC{$\varphi, ([\vec{t}\,]\varphi)^{m-1}, \Xi \Rightarrow \Delta$}
\RightLabel{(Cut^*)}
\BinaryInfC{$\varphi, [\vec{t}\,]\Gamma, \Xi \Rightarrow \Delta$}
\RightLabel{(Cut^*)}
\BinaryInfC{$\Gamma, [\vec{t}\,]\Gamma, \Xi \Rightarrow \Delta$}
\doubleLine
\RightLabel{$(\Box\mathrm{T}_n)$}
\UnaryInfC{$[\vec{t}\,]\Gamma, [\vec{t}\,]\Gamma, [\vec{t}\,]\Xi \Rightarrow \Delta$}
\doubleLine
\RightLabel{$(c \Rightarrow)$}
\UnaryInfC{$[\vec{t}\,]\Gamma, [\vec{t}\,]\Xi \Rightarrow \Delta$}
\end{prooftree}
$$

The upper application σ_1 of (Cut^*) in $\mathfrak{D}'$ is eliminable by inductive hypothesis, because $g(\sigma_1) = g(\sigma)$ and $w(\sigma_1) < w(\sigma)$ hold. The lower application σ_2 of (Cut^*) in $\mathfrak{D}'$ is also eliminable by inductive hypothesis, because $g(\sigma_1) < g(\sigma)$ holds by $l(\varphi) < l([\vec{t}\,]\varphi)$. Thus, there is a derivation of $[\vec{t}\,]\Gamma, [\vec{t}\,]\Xi \Rightarrow \Delta$ in $\mathrm{G}^-(\mathbf{tK\Sigma})$.

Case in which $\rho(L) = \rho(R) = (\Box 4_n)$. Then $\mathfrak{D} =$

$$
\begin{prooftree}
\AxiomC{$\mathfrak{D}_1$}
\noLine
\UnaryInfC{$\Gamma, [\vec{t}\,]\Gamma \Rightarrow \varphi$}
\RightLabel{$(\Box 4_n)$}
\UnaryInfC{$[\vec{t}\,]\Gamma \Rightarrow [\vec{t}\,]\varphi$}
\AxiomC{$\mathfrak{D}_2$}
\noLine
\UnaryInfC{$\varphi^m, \Xi, ([\vec{t}\,]\varphi)^m, [\vec{t}\,]\Xi \Rightarrow \psi$}
\RightLabel{$(\Box 4_n)$}
\UnaryInfC{$([\vec{t}\,]\varphi)^m, [\vec{t}\,]\Xi \Rightarrow [\vec{t}\,]\psi$}
\RightLabel{(Cut^*),}
\BinaryInfC{$[\vec{t}\,]\Gamma, [\vec{t}\,]\Xi \Rightarrow [\vec{t}\,]\psi$}
\end{prooftree}
$$

which can be transformed into a derivation $\mathfrak{D}' =$

$$
\begin{prooftree}
\AxiomC{$\mathfrak{D}_1$}
\noLine
\UnaryInfC{$\Gamma, [\vec{t}\,]\Gamma \Rightarrow \varphi$}
\AxiomC{$\mathfrak{D}_1$}
\noLine
\UnaryInfC{$\Gamma, [\vec{t}\,]\Gamma \Rightarrow \varphi$}
\RightLabel{$(\Box 4_n)$}
\UnaryInfC{$[\vec{t}\,]\Gamma \Rightarrow [\vec{t}\,]\varphi$}
\AxiomC{$\mathfrak{D}_2$}
\noLine
\UnaryInfC{$\varphi^m, \Xi, ([\vec{t}\,]\varphi)^m, [\vec{t}\,]\Xi \Rightarrow \psi$}
\RightLabel{(Cut^*)}
\BinaryInfC{$[\vec{t}\,]\Gamma, \varphi^m, \Xi, [\vec{t}\,]\Xi \Rightarrow \psi$}
\RightLabel{(Cut^*)}
\BinaryInfC{$\Gamma, [\vec{t}\,]\Gamma, [\vec{t}\,]\Gamma, \Xi, [\vec{t}\,]\Xi \Rightarrow \psi$}
\doubleLine
\RightLabel{$(c \Rightarrow)$}
\UnaryInfC{$\Gamma, \Xi, [\vec{t}\,]\Gamma, [\vec{t}\,]\Xi \Rightarrow \psi$}
\RightLabel{$(\Box 4_n)$}
\UnaryInfC{$[\vec{t}\,]\Gamma, [\vec{t}\,]\Xi \Rightarrow [\vec{t}\,]\psi$}
\end{prooftree}
$$

The upper application σ_1 of (Cut^*) in $\mathfrak{D}'$ is eliminable by inductive hypothesis, because $g(\sigma_1) = g(\sigma)$ and $w(\sigma_1) < w(\sigma)$ hold. The lower application σ_2 of (Cut^*) in $\mathfrak{D}'$ is also eliminable by inductive hypothesis, because $g(\sigma_1) < g(\sigma)$ holds by $l(\varphi) < l([\vec{t}]\varphi)$. Thus, there is a derivation of $[\vec{t}]\Gamma, [\vec{t}]\Xi \Rightarrow [\vec{t}]\psi$ in $\mathrm{G}^-(\mathbf{tK}\Sigma)$.

Case in which $\rho(L) = (\Box 4_n)$ **and** $\rho(R) = (\Box \mathrm{D}4_n)$**.** Then $\mathfrak{D}$ =

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_1\\ \Gamma, [\vec{t}]\Gamma \Rightarrow \varphi\end{array}}{[\vec{t}]\Gamma \Rightarrow [\vec{t}]\varphi}\,(\Box 4_n) \quad \dfrac{\begin{array}{c}\mathfrak{D}_2\\ \varphi^m, \Xi, ([\vec{t}]\varphi)^m, [\vec{t}]\Xi \Rightarrow\end{array}}{([\vec{t}]\varphi)^m, [\vec{t}]\Xi \Rightarrow}\,(\Box \mathrm{D}4_n)}{[\vec{t}]\Gamma, [\vec{t}]\Xi \Rightarrow}\,(Cut^*)$$

which can be transformed into a derivation $\mathfrak{D}'$ =

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_1\\ \Gamma, [\vec{t}]\Gamma \Rightarrow \varphi\end{array} \quad \dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_1\\ \Gamma, [\vec{t}]\Gamma \Rightarrow \varphi\end{array}}{[\vec{t}]\Gamma \Rightarrow [\vec{t}]\varphi}\,(\Box 4_n) \quad \begin{array}{c}\mathfrak{D}_2\\ \varphi^m, \Xi, ([\vec{t}]\varphi)^m, [\vec{t}]\Xi \Rightarrow\end{array}}{[\vec{t}]\Gamma, \varphi^m, \Xi, [\vec{t}]\Xi \Rightarrow}\,(Cut^*)}{\Gamma, [\vec{t}]\Gamma, [\vec{t}]\Gamma, \Xi, [\vec{t}]\Xi \Rightarrow}\,(Cut^*)}{\Gamma, \Xi, [\vec{t}]\Gamma, [\vec{t}]\Xi \Rightarrow}\,(c \Rightarrow)}{[\vec{t}]\Gamma, [\vec{t}]\Xi \Rightarrow}\,(\Box 4_n)$$

The upper application σ_1 of (Cut^*) in $\mathfrak{D}'$ is eliminable by inductive hypothesis, because $g(\sigma_1) = g(\sigma)$ and $w(\sigma_1) < w(\sigma)$ hold. The lower application σ_2 of (Cut^*) in $\mathfrak{D}'$ is also eliminable by inductive hypothesis, because $g(\sigma_1) < g(\sigma)$ holds by $l(\varphi) < l([\vec{t}]\varphi)$. Thus, there is a derivation of $[\vec{t}]\Gamma, [\vec{t}]\Xi \Rightarrow$ in $\mathrm{G}^-(\mathbf{tK}\Sigma)$.

Case in which $\rho(L) = \rho(R) = (\Box \mathrm{S}4_n)$**.** Then $\mathfrak{D}$ =

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_1\\ [\vec{t}]\Gamma \Rightarrow \varphi\end{array}}{[\vec{t}]\Gamma \Rightarrow [\vec{t}]\varphi}\,(\Box \mathrm{S}4_n) \quad \dfrac{\begin{array}{c}\mathfrak{D}_2\\ ([\vec{t}]\varphi)^m, [\vec{t}]\Xi \Rightarrow \psi\end{array}}{([\vec{t}]\varphi)^m, [\vec{t}]\Xi \Rightarrow [\vec{t}]\psi}\,(\Box \mathrm{S}4_n)}{[\vec{t}]\Gamma, [\vec{t}]\Xi \Rightarrow [\vec{t}]\psi}\,(Cut^*)$$

which can be transformed into a derivation $\mathfrak{D}'$ =

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_1\\ [\vec{t}]\Gamma \Rightarrow \varphi\end{array}}{[\vec{t}]\Gamma \Rightarrow [\vec{t}]\varphi}\,(\Box \mathrm{S}4_n) \quad \begin{array}{c}\mathfrak{D}_2\\ ([\vec{t}]\varphi)^m, [\vec{t}]\Xi \Rightarrow \psi\end{array}}{[\vec{t}]\Gamma, [\vec{t}]\Xi \Rightarrow \psi}\,(Cut^*)}{[\vec{t}]\Gamma, [\vec{t}]\Xi \Rightarrow [\vec{t}]\psi}\,(\Box \mathrm{S}4_n)$$

The application σ_1 of (Cut^*) in $\mathfrak{D}'$ is eliminable by inductive hypothesis, because $g(\sigma_1) = g(\sigma)$ and $w(\sigma_1) < w(\sigma)$ hold. Thus, there is a derivation of $[\vec{t}]\Gamma, [\vec{t}]\Xi \Rightarrow [\vec{t}]\psi$ in $\mathrm{G}^-(\mathbf{tK\Sigma})$.

Case in which $\rho(L) = (\Box\mathrm{S4}_n)$ and $\rho(R) = (\Box\mathrm{T}_n)$. Then $\mathfrak{D}$ =

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ [\vec{t}]\Gamma \Rightarrow \varphi\end{matrix}}{[\vec{t}]\Gamma \Rightarrow [\vec{t}]\varphi}\,(\Box\mathrm{S4}_n) \quad \dfrac{\begin{matrix}\mathfrak{D}_2\\ \varphi, ([\vec{t}]\varphi)^{m-1}, \Xi \Rightarrow \Delta\end{matrix}}{[\vec{t}]\varphi, ([\vec{t}]\varphi)^{m-1}, \Xi \Rightarrow \Delta}\,(\Box\mathrm{T}_n)}{[\vec{t}]\Gamma, \Xi \Rightarrow \Delta}\,(Cut^*)$$

which can be transformed into a derivation $\mathfrak{D}'$ =

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ [\vec{t}]\Gamma \Rightarrow \varphi\end{matrix} \quad \dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ [\vec{t}]\Gamma \Rightarrow \varphi\end{matrix}}{[\vec{t}]\Gamma \Rightarrow [\vec{t}]\varphi}\,(\Box\mathrm{S4}_n) \quad \begin{matrix}\mathfrak{D}_2\\ \varphi, ([\vec{t}]\varphi)^{m-1}, \Xi \Rightarrow \Delta\end{matrix}}{\varphi, [\vec{t}]\Gamma, \Xi \Rightarrow \Delta}\,(Cut^*)}{[\vec{t}]\Gamma, [\vec{t}]\Gamma, \Xi \Rightarrow \Delta}\,(Cut^*)}{[\vec{t}]\Gamma, \Xi \Rightarrow \Delta}\,(c \Rightarrow)$$

The upper application σ_1 of (Cut^*) in $\mathfrak{D}'$ is eliminable by inductive hypothesis, because $g(\sigma_1) = g(\sigma)$ and $w(\sigma_1) < w(\sigma)$ hold. The lower application σ_2 of (Cut^*) in $\mathfrak{D}'$ is also eliminable by inductive hypothesis, because $g(\sigma_1) < g(\sigma)$ holds by $l(\varphi) < l([\vec{t}]\varphi)$. Thus, there is a derivation of $[\vec{t}]\Gamma, \Xi \Rightarrow \Delta$ in $\mathrm{G}^-(\mathbf{tK\Sigma})$.

The above argument finished showing all the four cases. ■

As well known in the literature, the cut elimination theorem for a sequent calculus supplies us a *purely proof-theoretic* proof of the consistency of the calculus.

Corollary 116. A sequent $\Rightarrow \bot$ is not derivable in $\mathrm{G}(\mathbf{tK\Sigma})$.

Proof. Suppose for contradiction that $\vdash_{\mathrm{G}(\mathbf{tK\Sigma})} \Rightarrow \bot$. Then it follows from $\vdash_{\mathrm{G}(\mathbf{tK\Sigma})} \bot \Rightarrow$ and (Cut) that $\vdash_{\mathrm{G}(\mathbf{tK\Sigma})} \Rightarrow$. Thus the cut elimination theorem (Theorem 115) tells us that $\vdash_{\mathrm{G}^-(\mathbf{tK\Sigma})} \Rightarrow$, which cannot be the case from all the rules of $\mathrm{G}^-(\mathbf{tK\Sigma})$. ■

As another application of the cut elimination theorem, we can show that $\mathrm{G}(\mathbf{tK\Sigma})$ enjoys the Craig interpolation theorem by Maehara method [56]. Lemma 118 and Theorem 119 might be generalized so that $\mathsf{Func}(\varphi) \subseteq \mathsf{Func}(\Gamma_1, \Delta_1) \cap \mathsf{Func}(\Gamma_2, \Delta_2)$ and $\mathsf{Func}(\chi) \subseteq \mathsf{Func}(\varphi) \cap \mathsf{Func}(\psi)$, respectively, where $\mathsf{Func}(\Gamma)$ means the set of all function symbols in a set Γ of formulas. However, such a generalization is not done in this thesis. For such a possible generalization, see [62]. In what follows, $\mathsf{Pred}(\Gamma)$ and $\mathsf{Con}(\Gamma)$ mean the sets of all predicate symbols and all constant symbols in a set Γ of formulas, respectively.

Definition 117. A *partition* of a sequent $\Gamma \Rightarrow \Delta$ is a pair $((\Gamma_1, \Delta_1), (\Gamma_2, \Delta_2))$ of pairs of finite multisets of formulas such that $\Gamma = \Gamma_1, \Gamma_2$ and $\Delta = \Delta_1, \Delta_2$. A partition $((\Gamma_1, \Delta_1), (\Gamma_2, \Delta_2))$ is denoted by the notation $(\Gamma_1 : \Delta_1), (\Gamma_2 : \Delta_2)$.

Lemma 118. Let $\Gamma \Rightarrow \Delta$ be a sequent derivable in $\mathrm{G}(\mathbf{tK\Sigma})$. If $(\Gamma_1 : \Delta_1), (\Gamma_2 : \Delta_2)$ is a partition of $\Gamma \Rightarrow \Delta$, there is an *interpolant* of it, i.e., a formula φ such that

- $\vdash_{\mathrm{G}(\mathbf{tK\Sigma})} \Gamma_1 \Rightarrow \Delta_1, \varphi$ and $\vdash_{\mathrm{G}(\mathbf{tK\Sigma})} \varphi, \Gamma_2 \Rightarrow \Delta_2$;
- $\mathsf{Pred}(\varphi) \subseteq \mathsf{Pred}(\Gamma_1, \Delta_1) \cap \mathsf{Pred}(\Gamma_2, \Delta_2)$;
- $\mathsf{FV}(\varphi) \subseteq \mathsf{FV}(\Gamma_1, \Delta_1) \cap \mathsf{FV}(\Gamma_2, \Delta_2)$;
- $\mathsf{Con}(\varphi) \subseteq \mathsf{Con}(\Gamma_1, \Delta_1) \cap \mathsf{Con}(\Gamma_2, \Delta_2)$.

Proof. By the cut elimination theorem (Theorem 115), we have a derivation of $\Gamma \Rightarrow \Delta$ in $\mathrm{G}^-(\mathbf{tK\Sigma})$. Our proof is done by induction on the height of the derivation of $\Gamma \Rightarrow \Delta$ in $\mathrm{G}^-(\mathbf{tK\Sigma})$. In what follows, we exclude partitions of the form $(\emptyset : \emptyset), (\Gamma : \Delta)$ or $(\Gamma : \Delta), (\emptyset : \emptyset)$, since $\top$ and $\bot$ are easily seen to be interpolants of the former and the latter, respectively. For first order cases, one can consult [67, pp. 41–44]. We show only cases in which the last applied rule is a logical rule from Σ.

Case in which the last applied rule is $(\Box \mathrm{K}_n)$. Then the derivation ends with

$$\dfrac{\begin{matrix}\vdots \\ \Gamma_1, \Gamma_2 \Rightarrow \varphi\end{matrix}}{[\vec{t}\,]\Gamma_1, [\vec{t}\,]\Gamma_2 \Rightarrow [\vec{t}\,]\varphi}\ (\Box \mathrm{K}_n)$$

There are two cases depending on which side of a partition of the last sequent contains $[\vec{t}\,]\varphi$.

Case of $([\vec{t}\,]\Gamma_1 : [\vec{t}\,]\varphi), ([\vec{t}\,]\Gamma_2 : \emptyset)$. By inductive hypothesis, we have an interpolant γ of a partition $(\Gamma_1 : \varphi), (\Gamma_2 : \emptyset)$ of $\Gamma_1, \Gamma_2 \Rightarrow \varphi$. Then

- $\vdash [\vec{t}\,]\Gamma_1 \Rightarrow [\vec{t}\,]\varphi, \langle \vec{t}\, \rangle\gamma$ and $\vdash \langle \vec{t}\, \rangle\gamma, [\vec{t}\,]\Gamma_2 \Rightarrow$

are established by the following derivations, respectively:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma_1 \Rightarrow \varphi, \gamma \qquad \dfrac{}{\bot \Rightarrow}\,(\bot)}{\neg\gamma, \Gamma_1 \Rightarrow \varphi}\,(\to\Rightarrow)}{[\vec{t}\,]\neg\gamma, [\vec{t}\,]\Gamma_1 \Rightarrow [\vec{t}\,]\varphi}\,(\Box \mathrm{K}_n)}{[\vec{t}\,]\neg\gamma, [\vec{t}\,]\Gamma_1 \Rightarrow [\vec{t}\,]\varphi, \bot}\,(\Rightarrow w)}{[\vec{t}\,]\Gamma_1 \Rightarrow [\vec{t}\,]\varphi, \langle \vec{t}\, \rangle\gamma}\,(\Rightarrow\to)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\gamma, \Gamma_2 \Rightarrow}{\gamma, \Gamma_2 \Rightarrow \bot}\ (\Rightarrow w)}{\Gamma_2 \Rightarrow \neg\gamma}\ (\Rightarrow\to)}{[\vec{t}]\Gamma_2 \Rightarrow [\vec{t}]\neg\gamma}\ (\Box\mathrm{K}_n) \quad \dfrac{}{\bot \Rightarrow}\ (\bot)}{\langle\vec{t}\rangle\gamma, [\vec{t}]\Gamma_2 \Rightarrow}\ (\to\Rightarrow)$$

where $\Gamma_1 \Rightarrow \varphi, \gamma$ and $\gamma, \Gamma_2 \Rightarrow$ are derivable since γ is an interpolant of $(\Gamma_1 : \varphi)$, $(\Gamma_2 : \emptyset)$. Note also $[\vec{t}]\Gamma_2 \neq \emptyset$, otherwise the current partition becomes one of the partitions we excluded at the beginning of the proof. Thus we can claim

- $\mathsf{Pred}(\langle\vec{t}\rangle\gamma) \subseteq \mathsf{Pred}([\vec{t}]\Gamma_1, [\vec{t}]\varphi) \cap \mathsf{Pred}([\vec{t}]\Gamma_2)$
- $\mathsf{FV}(\langle\vec{t}\rangle\gamma) \subseteq \mathsf{FV}([\vec{t}]\Gamma_1, [\vec{t}]\varphi) \cap \mathsf{FV}([\vec{t}]\Gamma_2)$
- $\mathsf{Con}(\langle\vec{t}\rangle\gamma) \subseteq \mathsf{Con}([\vec{t}]\Gamma_1, [\vec{t}]\varphi) \cap \mathsf{Con}([\vec{t}]\Gamma_2)$

Therefore $\langle\vec{t}\rangle\gamma$ is an interpolant of $([\vec{t}]\Gamma_1 : [\vec{t}]\varphi), ([\vec{t}]\Gamma_2 : \emptyset)$.

Case of $([\vec{t}]\Gamma_1 : \emptyset), ([\vec{t}]\Gamma_2 : [\vec{t}]\varphi)$**.** By inductive hypothesis, we have an interpolant γ of a partition $(\Gamma_1 : \emptyset), (\Gamma_2 : \varphi)$ of $\Gamma_1, \Gamma_2 \Rightarrow \varphi$. Then

- $\vdash [\vec{t}]\Gamma_1 \Rightarrow [\vec{t}]\gamma$ and $\vdash [\vec{t}]\gamma, [\vec{t}]\Gamma_2 \Rightarrow [\vec{t}]\varphi$

are established by the following derivations, respectively:

$$\dfrac{\Gamma_1 \Rightarrow \gamma}{[\vec{t}]\Gamma_1 \Rightarrow [\vec{t}]\gamma}\ (\Box\mathrm{K}_n)$$

$$\dfrac{\gamma, \Gamma_2 \Rightarrow \varphi}{[\vec{t}]\gamma, [\vec{t}]\Gamma_2 \Rightarrow [\vec{t}]\varphi}\ (\Box\mathrm{K}_n)$$

where $\Gamma_1 \Rightarrow \gamma$ and $\gamma, \Gamma_2 \Rightarrow \varphi$ are derivable since γ is an interpolant of $(\Gamma_1 : \emptyset)$, $(\Gamma_2 : \varphi)$. Note also $[\vec{t}]\Gamma_1 \neq \emptyset$, otherwise the current partition becomes one of the partitions we excluded at the beginning of the proof. Thus we can claim

- $\mathsf{Pred}([\vec{t}]\gamma) \subseteq \mathsf{Pred}([\vec{t}]\Gamma_1) \cap \mathsf{Pred}([\vec{t}]\Gamma_2, [\vec{t}]\varphi)$
- $\mathsf{FV}([\vec{t}]\gamma) \subseteq \mathsf{FV}([\vec{t}]\Gamma_1) \cap \mathsf{FV}([\vec{t}]\Gamma_2, [\vec{t}]\varphi)$
- $\mathsf{Con}([\vec{t}]\gamma) \subseteq \mathsf{Con}([\vec{t}]\Gamma_1) \cap \mathsf{Con}([\vec{t}]\Gamma_2, [\vec{t}]\varphi)$

Therefore $[\vec{t}]\varphi$ is an interpolant of $([\vec{t}]\Gamma_1 : \emptyset), ([\vec{t}]\Gamma_2 : [\vec{t}]\varphi)$.

Case in which the last applied rule is $(\Box\mathrm{D}_n)$**.** Then the derivation ends with

$$\dfrac{\begin{array}{c}\vdots \\ \Gamma_1, \Gamma_2 \Rightarrow\end{array}}{[\vec{t}]\Gamma_1, [\vec{t}]\Gamma_2 \Rightarrow}\ (\Box\mathrm{D}_n)$$

A partition of the last sequent should be $([\vec{t}]\Gamma_1 : \emptyset), ([\vec{t}]\Gamma_2 : \emptyset)$, so we need to find an interpolant of it. By inductive hypothesis, we have an interpolant γ of a partition $(\Gamma_1 : \emptyset), (\Gamma_2 : \emptyset)$ of $\Gamma_1, \Gamma_2 \Rightarrow$. Then

- $\vdash [\vec{t}]\Gamma_1 \Rightarrow [\vec{t}]\gamma$ and $\vdash [\vec{t}]\gamma, [\vec{t}]\Gamma_2 \Rightarrow$

are established by the following derivations, respectively:

$$\dfrac{\Gamma_1 \Rightarrow \gamma}{[\vec{t}]\Gamma_1 \Rightarrow [\vec{t}]\gamma}\ (\Box\mathrm{K}_n)$$

$$\dfrac{\gamma, \Gamma_2 \Rightarrow}{[\vec{t}]\gamma, [\vec{t}]\Gamma_2 \Rightarrow}\ (\Box\mathrm{D}_n)$$

where $\Gamma_1 \Rightarrow \gamma$ and $\gamma, \Gamma_2 \Rightarrow$ are derivable since γ is an interpolant of $(\Gamma_1 : \emptyset)$, $(\Gamma_2 : \emptyset)$. Note also $[\vec{t}]\Gamma_1 \neq \emptyset$ and $[\vec{t}]\Gamma_2 \neq \emptyset$, otherwise the current partition becomes one of the partitions we excluded at the beginning of the proof. Thus we can claim

- $\mathsf{Pred}([\vec{t}]\gamma) \subseteq \mathsf{Pred}([\vec{t}]\Gamma_1) \cap \mathsf{Pred}([\vec{t}]\Gamma_2)$
- $\mathsf{FV}([\vec{t}]\gamma) \subseteq \mathsf{FV}([\vec{t}]\Gamma_1) \cap \mathsf{FV}([\vec{t}]\Gamma_2)$
- $\mathsf{Con}([\vec{t}]\gamma) \subseteq \mathsf{Con}([\vec{t}]\Gamma_1) \cap \mathsf{Con}([\vec{t}]\Gamma_2)$

Therefore $[\vec{t}]\gamma$ is an interpolant of $([\vec{t}]\Gamma_1 : \emptyset), ([\vec{t}]\Gamma_2 : \emptyset)$.

Case in which the last applied rule is $(\Box\mathrm{T}_n)$. Then the derivation ends with

$$\dfrac{\begin{array}{c}\vdots\\ \varphi, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2\end{array}}{[\vec{t}]\varphi, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (\Box\mathrm{T}_n)$$

There are two cases depending on which side of a partition of the last sequent contains $[\vec{t}]\varphi$.

Case of $([\vec{t}]\varphi, \Gamma_1 : \Delta_1), (\Gamma_2 : \Delta_2)$. By inductive hypothesis, we have an interpolant γ of a partition $(\varphi, \Gamma_1 : \Delta_1), (\Gamma_2 : \Delta_2)$ of $\varphi, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$. Then, with the use of $(\Box\mathrm{T}_n)$,

- $\vdash [\vec{t}]\varphi, \Gamma_1 \Rightarrow \Delta_1, \gamma$ and $\vdash \gamma, \Gamma_2 \Rightarrow \Delta_2$

are immediately obtained. We can also claim

- $\mathsf{Pred}(\gamma) \subseteq \mathsf{Pred}([\vec{t}]\varphi, \Gamma_1, \Delta_1) \cap \mathsf{Pred}(\Gamma_2, \Delta_2)$

- $\mathsf{FV}(\gamma) \subseteq \mathsf{FV}([\vec{t}]\varphi, \Gamma_1, \Delta_1) \cap \mathsf{FV}(\Gamma_2, \Delta_2)$
- $\mathsf{Con}(\gamma) \subseteq \mathsf{Con}([\vec{t}]\varphi, \Gamma_1, \Delta_1) \cap \mathsf{Con}(\Gamma_2, \Delta_2)$

Therefore γ is an interpolant of $([\vec{t}]\varphi, \Gamma_1 : \Delta_1), (\Gamma_2 : \Delta_2)$.

Case of $(\Gamma_1 : \Delta_1), ([\vec{t}]\varphi, \Gamma_2 : \Delta_2)$. By inductive hypothesis, we have an interpolant γ of a partition $(\Gamma_1 : \Delta_1), (\varphi, \Gamma_2 : \Delta_2)$ of $\varphi, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$. Then, with the use of $(\Box\mathrm{T}_n)$,

- $\vdash \Gamma_1 \Rightarrow \Delta_1, \gamma$ and $\vdash \gamma, [\vec{t}]\varphi, \Gamma_2 \Rightarrow \Delta_2$

are immediately obtained. We can also claim

- $\mathsf{Pred}(\gamma) \subseteq \mathsf{Pred}(\Gamma_1, \Delta_1) \cap \mathsf{Pred}([\vec{t}]\varphi, \Gamma_2, \Delta_2)$
- $\mathsf{FV}(\gamma) \subseteq \mathsf{FV}(\Gamma_1, \Delta_1) \cap \mathsf{FV}([\vec{t}]\varphi, \Gamma_2, \Delta_2)$
- $\mathsf{Con}(\gamma) \subseteq \mathsf{Con}(\Gamma_1, \Delta_1) \cap \mathsf{Con}([\vec{t}]\varphi, \Gamma_2, \Delta_2)$

Therefore γ is an interpolant of $(\Gamma_1 : \Delta_1), ([\vec{t}]\varphi, \Gamma_2 : \Delta_2)$.

Case in which the last applied rule is $(\Box 4_n)$**.** Then the derivation ends with

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma_1, \Gamma_2, [\vec{t}]\Gamma_1, [\vec{t}]\Gamma_2 \Rightarrow \varphi\end{array}}{[\vec{t}]\Gamma_1, [\vec{t}]\Gamma_2 \Rightarrow [\vec{t}]\varphi}\ (\Box 4_n)$$

There are two cases depending on which side of a partition of the last sequent contains $[\vec{t}]\varphi$.

Case of $([\vec{t}]\Gamma_1 : \emptyset), ([\vec{t}]\Gamma_2 : [\vec{t}]\varphi)$**.** By inductive hypothesis, we have an interpolant γ of a partition $(\Gamma_1, [\vec{t}]\Gamma_1 : \emptyset), (\Gamma_2, [\vec{t}]\Gamma_2 : \varphi)$ of $\Gamma_1, \Gamma_2, [\vec{t}]\Gamma_1, [\vec{t}]\Gamma_2 \Rightarrow \varphi$. Then

- $\vdash [\vec{t}]\Gamma_1 \Rightarrow [\vec{t}]\gamma$ and $\vdash [\vec{t}]\gamma, [\vec{t}]\Gamma_2 \Rightarrow [\vec{t}]\varphi$

are established by the following derivations, respectively:

$$\dfrac{\Gamma_1, [\vec{t}]\Gamma_1 \Rightarrow \gamma}{[\vec{t}]\Gamma_1 \Rightarrow [\vec{t}]\gamma}\ (\Box 4_n)$$

$$\dfrac{\dfrac{\gamma, \Gamma_2, [\vec{t}]\Gamma_2 \Rightarrow \varphi}{\gamma, \Gamma_2, [\vec{t}]\gamma, [\vec{t}]\Gamma_2 \Rightarrow \varphi}\ (w \Rightarrow)}{[\vec{t}]\gamma, [\vec{t}]\Gamma_2 \Rightarrow [\vec{t}]\varphi}\ (\Box 4_n)$$

where $\Gamma_1, [\vec{t}]\Gamma_1 \Rightarrow \gamma$ and $\gamma, \Gamma_2, [\vec{t}]\Gamma_2 \Rightarrow \varphi$ are derivable since γ is an interpolant of $(\Gamma_1, [\vec{t}]\Gamma_1 : \emptyset), (\Gamma_2, [\vec{t}]\Gamma_2 : \varphi)$. Note also $[\vec{t}]\Gamma_1 \neq \emptyset$, otherwise the current partition becomes one of the partitions we excluded at the beginning of the proof. Thus we can claim

- $\mathsf{Pred}([\vec{t}]\gamma) \subseteq \mathsf{Pred}([\vec{t}]\Gamma_1) \cap \mathsf{Pred}([\vec{t}]\Gamma_2, [\vec{t}]\varphi)$
- $\mathsf{FV}([\vec{t}]\gamma) \subseteq \mathsf{FV}([\vec{t}]\Gamma_1) \cap \mathsf{FV}([\vec{t}]\Gamma_2, [\vec{t}]\varphi)$
- $\mathsf{Con}([\vec{t}]\gamma) \subseteq \mathsf{Con}([\vec{t}]\Gamma_1) \cap \mathsf{Con}([\vec{t}]\Gamma_2, [\vec{t}]\varphi)$

Therefore γ is an interpolant of $([\vec{t}]\Gamma_1 : \emptyset), ([\vec{t}]\Gamma_2 : [\vec{t}]\varphi)$.

Case of $([\vec{t}]\Gamma_1 : [\vec{t}]\varphi), ([\vec{t}]\Gamma_2 : \emptyset)$**.** By inductive hypothesis, we have an interpolant γ of a partition $(\Gamma_1, [\vec{t}]\Gamma_1 : \varphi), (\Gamma_2, [\vec{t}]\Gamma_2 : \emptyset)$ of $\Gamma_1, \Gamma_2, [\vec{t}]\Gamma_1, [\vec{t}]\Gamma_2 \Rightarrow \varphi$. Then

- $\vdash [\vec{t}]\Gamma_1 \Rightarrow [\vec{t}]\varphi, \langle\vec{t}\rangle\gamma$ and $\vdash \langle\vec{t}\rangle\gamma, [\vec{t}]\Gamma_2 \Rightarrow$

are established by the following derivations, respectively:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma_1, [\vec{t}]\Gamma_1 \Rightarrow \varphi, \gamma \qquad \dfrac{}{\bot \Rightarrow}(\bot)}{\neg\gamma, \Gamma_1, [\vec{t}]\Gamma_1 \Rightarrow \varphi}(\to\Rightarrow)}{\neg\gamma, \Gamma_1, [\vec{t}]\neg\gamma, [\vec{t}]\Gamma_1 \Rightarrow \varphi}(w\Rightarrow)}{[\vec{t}]\neg\gamma, [\vec{t}]\Gamma_1 \Rightarrow [\vec{t}]\varphi}(\Box 4_n)}{[\vec{t}]\neg\gamma, [\vec{t}]\Gamma_1 \Rightarrow [\vec{t}]\varphi, \bot}(\Rightarrow w)}{[\vec{t}]\Gamma_1 \Rightarrow [\vec{t}]\varphi, \langle\vec{t}\rangle\gamma}(\Rightarrow\to)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\gamma, \Gamma_2, [\vec{t}]\Gamma_2 \Rightarrow}{\gamma, \Gamma_2, [\vec{t}]\Gamma_2 \Rightarrow \bot}(\Rightarrow w)}{\Gamma_2, [\vec{t}]\Gamma_2 \Rightarrow \neg\gamma}(\Rightarrow\to)}{[\vec{t}]\Gamma_2 \Rightarrow [\vec{t}]\neg\gamma}(\Box 4_n) \qquad \dfrac{}{\bot \Rightarrow}(\bot)}{\langle\vec{t}\rangle\gamma, [\vec{t}]\Gamma_2 \Rightarrow}(\to\Rightarrow)$$

where $\Gamma_1, [\vec{t}]\Gamma_1 \Rightarrow \varphi, \gamma$ and $\gamma, \Gamma_2, [\vec{t}]\Gamma_2 \Rightarrow$ are derivable since γ is an interpolant of $(\Gamma_1, [\vec{t}]\Gamma_1 : \varphi), (\Gamma_2, [\vec{t}]\Gamma_2 : \emptyset)$. Note also $[\vec{t}]\Gamma_2 \neq \emptyset$, otherwise the current partition becomes one of the partitions we excluded at the beginning of the proof. Thus we can claim

- $\mathsf{Pred}(\langle\vec{t}\rangle\gamma) \subseteq \mathsf{Pred}([\vec{t}]\Gamma_1, [\vec{t}]\varphi) \cap \mathsf{Pred}([\vec{t}]\Gamma_2)$
- $\mathsf{FV}(\langle\vec{t}\rangle\gamma) \subseteq \mathsf{FV}([\vec{t}]\Gamma_1, [\vec{t}]\varphi) \cap \mathsf{FV}([\vec{t}]\Gamma_2)$
- $\mathsf{Con}(\langle\vec{t}\rangle\gamma) \subseteq \mathsf{Con}([\vec{t}]\Gamma_1, [\vec{t}]\varphi) \cap \mathsf{Con}([\vec{t}]\Gamma_2)$

Therefore $\langle\vec{t}\rangle\gamma$ is an interpolant of $([\vec{t}]\Gamma_1 : [\vec{t}]\varphi), ([\vec{t}]\Gamma_2 : \emptyset)$.

Case in which the last applied rule is ($\Box$D4_n)**.** Then the derivation ends with

$$\dfrac{\vdots \atop \Gamma_1, \Gamma_2, [\vec{t}]\Gamma_1, [\vec{t}]\Gamma_2 \Rightarrow}{[\vec{t}]\Gamma_1, [\vec{t}]\Gamma_2 \Rightarrow}\ (\Box\mathrm{D}4_n)$$

A partition of the last sequent should be $([\vec{t}]\Gamma_1 : \emptyset)$, $([\vec{t}]\Gamma_2 : \emptyset)$, so we need to find an interpolant of it. By inductive hypothesis, we have an interpolant γ of a partition $(\Gamma_1, [\vec{t}]\Gamma_1 : \emptyset)$, $(\Gamma_2, [\vec{t}]\Gamma_2 : \emptyset)$ of $\Gamma_1, \Gamma_2, [\vec{t}]\Gamma_1, [\vec{t}]\Gamma_2 \Rightarrow$. Then

- $\vdash [\vec{t}]\Gamma_1 \Rightarrow [\vec{t}]\gamma$ and $\vdash [\vec{t}]\gamma, [\vec{t}]\Gamma_2 \Rightarrow$

are established by the following derivations, respectively:

$$\dfrac{\Gamma_1, [\vec{t}]\Gamma_1 \Rightarrow \gamma}{[\vec{t}]\Gamma_1 \Rightarrow [\vec{t}]\gamma}\ (\Box 4_n)$$

$$\dfrac{\dfrac{\gamma, \Gamma_2, [\vec{t}]\Gamma_2 \Rightarrow}{\gamma, \Gamma_2, [\vec{t}]\gamma, [\vec{t}]\Gamma_2 \Rightarrow}\ (w \Rightarrow)}{[\vec{t}]\gamma, [\vec{t}]\Gamma_2 \Rightarrow}\ (\Box\mathrm{D}_n)$$

where $\Gamma_1, [\vec{t}]\Gamma_1 \Rightarrow \gamma$ and $\gamma, \Gamma_2, [\vec{t}]\Gamma_2 \Rightarrow$ are derivable since γ is an interpolant of $(\Gamma_1, [\vec{t}]\Gamma_1 : \emptyset)$, $(\Gamma_2, [\vec{t}]\Gamma_2 : \emptyset)$. Note also $\Gamma_1, [\vec{t}]\Gamma_1 \neq \emptyset$ and $\Gamma_2, [\vec{t}]\Gamma_2 \neq \emptyset$, otherwise the current partition becomes one of the partitions we excluded at the beginning of the proof. Thus we can claim

- $\mathsf{Pred}([\vec{t}]\gamma) \subseteq \mathsf{Pred}([\vec{t}]\Gamma_1) \cap \mathsf{Pred}([\vec{t}]\Gamma_2)$
- $\mathsf{FV}([\vec{t}]\gamma) \subseteq \mathsf{FV}([\vec{t}]\Gamma_1) \cap \mathsf{FV}([\vec{t}]\Gamma_2)$
- $\mathsf{Con}([\vec{t}]\gamma) \subseteq \mathsf{Con}([\vec{t}]\Gamma_1) \cap \mathsf{Con}([\vec{t}]\Gamma_2)$

Therefore $[\vec{t}]\gamma$ is an interpolant of $([\vec{t}]\Gamma_1 : \emptyset)$, $([\vec{t}]\Gamma_2 : \emptyset)$.

Case in which the last applied rule is ($\Box$S4_n)**.** Then the derivation ends with

$$\dfrac{\vdots \atop [\vec{t}]\Gamma_1, [\vec{t}]\Gamma_2 \Rightarrow \varphi}{[\vec{t}]\Gamma_1, [\vec{t}]\Gamma_2 \Rightarrow [\vec{t}]\varphi}\ (\Box\mathrm{S}4_n)$$

There are two cases depending on which side of a partition of the last sequent contains $[\vec{t}]\varphi$.

Case of $([\vec{t}]\Gamma_1 : [\vec{t}]\varphi), ([\vec{t}]\Gamma_2 : \emptyset)$**.** By inductive hypothesis, we have an interpolant γ of a partition $([\vec{t}]\Gamma_1 : \varphi), ([\vec{t}]\Gamma_2 : \emptyset)$ of $[\vec{t}]\Gamma_1, [\vec{t}]\Gamma_2 \Rightarrow \varphi$. Then

- $\vdash [\vec{t}]\Gamma_1 \Rightarrow [\vec{t}]\varphi, \langle \vec{t} \rangle\gamma$ and $\vdash \langle \vec{t} \rangle\gamma, [\vec{t}]\Gamma_2 \Rightarrow$

are established by the following derivations, respectively:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\vec{t}]\Gamma_1 \Rightarrow \varphi, \gamma \qquad \dfrac{}{\bot \Rightarrow}\,(\bot)}{\neg\gamma, [\vec{t}]\Gamma_1 \Rightarrow \varphi}\,(\to\Rightarrow)}{[\vec{t}]\neg\gamma, [\vec{t}]\Gamma_1 \Rightarrow \varphi}\,(\Box \mathrm{T}_n)}{[\vec{t}]\neg\gamma, [\vec{t}]\Gamma_1 \Rightarrow [\vec{t}]\varphi}\,(\Box \mathrm{S4}_n)}{[\vec{t}]\neg\gamma, [\vec{t}]\Gamma_1 \Rightarrow [\vec{t}]\varphi, \bot}\,(\Rightarrow w)}{[\vec{t}]\Gamma_1 \Rightarrow [\vec{t}]\varphi, \langle \vec{t} \rangle\gamma}\,(\Rightarrow\to)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\gamma, [\vec{t}]\Gamma_2 \Rightarrow}{\gamma, [\vec{t}]\Gamma_2 \Rightarrow \bot}\,(\Rightarrow w)}{[\vec{t}]\Gamma_2 \Rightarrow \neg\gamma}\,(\Rightarrow\to)}{[\vec{t}]\Gamma_2 \Rightarrow [\vec{t}]\neg\gamma}\,(\Box \mathrm{S4}_n) \qquad \dfrac{}{\bot \Rightarrow}\,(\bot)}{\langle \vec{t} \rangle\gamma, [\vec{t}]\Gamma_2 \Rightarrow}\,(\to\Rightarrow)$$

where $[\vec{t}]\Gamma_1 \Rightarrow \varphi, \gamma$ and $\gamma, [\vec{t}]\Gamma_2 \Rightarrow$ are derivable since γ is an interpolant of $([\vec{t}]\Gamma_1 : \varphi), ([\vec{t}]\Gamma_2 : \emptyset)$. Note also $[\vec{t}]\Gamma_2 \neq \emptyset$, otherwise the current partition becomes one of the partitions we excluded at the beginning of the proof. Thus we can claim

- $\mathsf{Pred}(\langle \vec{t} \rangle\gamma) \subseteq \mathsf{Pred}([\vec{t}]\Gamma_1, [\vec{t}]\varphi) \cap \mathsf{Pred}([\vec{t}]\Gamma_2)$
- $\mathsf{FV}(\langle \vec{t} \rangle\gamma) \subseteq \mathsf{FV}([\vec{t}]\Gamma_1, [\vec{t}]\varphi) \cap \mathsf{FV}([\vec{t}]\Gamma_2)$
- $\mathsf{Con}(\langle \vec{t} \rangle\gamma) \subseteq \mathsf{Con}([\vec{t}]\Gamma_1, [\vec{t}]\varphi) \cap \mathsf{Con}([\vec{t}]\Gamma_2)$

Therefore $\langle \vec{t} \rangle$ is an interpolant of $([\vec{t}]\Gamma_1 : [\vec{t}]\varphi), ([\vec{t}]\Gamma_2 : \emptyset)$.

Case of $([\vec{t}]\Gamma_1 : \emptyset), ([\vec{t}]\Gamma_2 : [\vec{t}]\varphi)$**.** By inductive hypothesis, we have an interpolant γ of a partition $([\vec{t}]\Gamma_1 : \emptyset), ([\vec{t}]\Gamma_2 : \varphi)$ of $[\vec{t}]\Gamma_1, [\vec{t}]\Gamma_2 \Rightarrow \varphi$. Then

- $\vdash [\vec{t}]\Gamma_1 \Rightarrow [\vec{t}]\gamma$ and $\vdash [\vec{t}]\gamma, [\vec{t}]\Gamma_2 \Rightarrow [\vec{t}]\varphi$

are established by the following derivations, respectively:

$$\dfrac{[\vec{t}]\Gamma_1 \Rightarrow \gamma}{[\vec{t}]\Gamma_1 \Rightarrow [\vec{t}]\gamma}\,(\Box \mathrm{S4}_n)$$

$$\dfrac{\gamma, [\vec{t}]\Gamma_2 \Rightarrow [\vec{t}]\varphi}{[\vec{t}]\gamma, [\vec{t}]\Gamma_2 \Rightarrow [\vec{t}]\varphi}\,(\Box \mathrm{T}_n)$$

where $[\vec{t}]\Gamma_1 \Rightarrow \gamma$ and $\gamma, [\vec{t}]\Gamma_2 \Rightarrow [\vec{t}]\varphi$ are derivable since γ is an interpolant of $([\vec{t}]\Gamma_1 : \emptyset), ([\vec{t}]\Gamma_2 : \varphi)$. Note also $[\vec{t}]\Gamma_1 \neq \emptyset$, otherwise the current partition becomes one of the partitions we excluded at the beginning of the proof. Thus we can claim

- $\mathsf{Pred}([\vec{t}]\gamma) \subseteq \mathsf{Pred}([\vec{t}]\Gamma_1) \cap \mathsf{Pred}([\vec{t}]\Gamma_2, [\vec{t}]\varphi)$
- $\mathsf{FV}([\vec{t}]\gamma) \subseteq \mathsf{FV}([\vec{t}]\Gamma_1) \cap \mathsf{FV}([\vec{t}]\Gamma_2, [\vec{t}]\varphi)$
- $\mathsf{Con}([\vec{t}]\gamma) \subseteq \mathsf{Con}([\vec{t}]\Gamma_1) \cap \mathsf{Con}([\vec{t}]\Gamma_2, [\vec{t}]\varphi)$

Therefore $[\vec{t}]\gamma$ is an interpolant of $([\vec{t}]\Gamma_1 : \emptyset), ([\vec{t}]\Gamma_2 : [\vec{t}]\varphi)$.

■

Theorem 119. (Craig interpolation theorem) If $\Rightarrow \varphi \rightarrow \psi$ is derivable in $\mathrm{G}(\mathbf{tK}\Sigma)$), then there exists a formula γ such that

- $\vdash_{\mathrm{G}(\mathbf{tK}\Sigma)} \varphi \rightarrow \gamma$ and $\vdash_{\mathrm{G}(\mathbf{tK}\Sigma)} \Rightarrow \gamma \rightarrow \psi$;
- $\mathsf{Pred}(\gamma) \subseteq \mathsf{Pred}(\varphi) \cap \mathsf{Pred}(\psi)$;
- $\mathsf{FV}(\gamma) \subseteq \mathsf{FV}(\varphi) \cap \mathsf{FV}(\psi)$;
- $\mathsf{Con}(\gamma) \subseteq \mathsf{Con}(\varphi) \cap \mathsf{Con}(\psi)$.

Proof. By the following derivation:

$$\dfrac{\Rightarrow \varphi \rightarrow \psi \qquad \dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(id) \qquad \dfrac{}{\psi \Rightarrow \psi}\,(id)}{\varphi \rightarrow \psi, \varphi \Rightarrow \psi}\,(\rightarrow\Rightarrow)}{\varphi \Rightarrow \psi}\,(Cut)$$

the sequent $\varphi \Rightarrow \psi$ is derivable in $\mathrm{G}(\mathbf{tK}\Sigma)$, as well as in $\mathrm{G}^-(\mathbf{tK}\Sigma)$ by the cut elimination theorem (Theorem 115). It follows from Lemma 118 that there exists an interpolant γ of a partition $(\varphi : \emptyset)$, $(\emptyset : \psi)$ of $\varphi \Rightarrow \psi$. Then, $\vdash_{\mathrm{G}(\mathbf{tK}\Sigma)} \varphi \rightarrow \gamma$ and $\vdash_{\mathrm{G}(\mathbf{tK}\Sigma)} \Rightarrow \gamma \rightarrow \psi$, $\mathsf{Pred}(\gamma) \subseteq \mathsf{Pred}(\varphi) \cap \mathsf{Pred}(\psi)$, $\mathsf{FV}(\gamma) \subseteq \mathsf{FV}(\varphi) \cap \mathsf{FV}(\psi)$, and $\mathsf{Con}(\gamma) \subseteq \mathsf{Con}(\varphi) \cap \mathsf{Con}(\psi)$. ■

3.3 Adding Equality

In this section, we further develop term-sequence-modal logic with equality. Our proof strategy uses the techniques introduced in Corsi [12].

The language $\mathsf{L}^{=}_{\mathsf{TSML}}$ of *term-sequence-modal logic with equality* (*TSML*$^=$) is the language obtained from $\mathsf{L}_{\mathsf{TSML}}$ by adding $=$ as a logical symbol and letting $\mathsf{Func} = \emptyset$. A term and a formula in $\mathsf{L}^{=}_{\mathsf{TSML}}$ is defined as in $\mathsf{L}_{\mathsf{TSML}}$, where $t = s$ is a formula for all terms t, s. The other syntactic notions in TSML are straightforwardly carried over, but $\mathsf{FV}(t = s)$ is defined as $\mathsf{V}(t, s)$. The Kripke semantics for $\mathsf{L}^{=}_{\mathsf{TSML}}$ is obtained from the semantics for $\mathsf{L}_{\mathsf{TSML}}$ by giving the satisfaction relation of $t = s$ as follows.

$$\mathfrak{M}, w, \alpha \models t = s \quad \text{iff} \quad \alpha(t) = \alpha(s)$$

where $t = s$ is an α_w-formula. Recall $\mathsf{Axiom}_{\mathsf{TSML}} = \{ \mathrm{T}_n, \mathrm{D}_n, \mathrm{B}_n, 4_n, 5_n, \mathrm{BF}_n \mid n \in \mathbb{N} \}$. The Hilbert system $\mathsf{H}(\mathbf{tK\Sigma}^{=})$ for TSML$^{=}$ is the system obtained from $\mathsf{H}(\mathbf{tK\Sigma})$ by adding all axioms in Table 3.4 as schemas. We sometimes use a notation $\mathsf{H}(\mathbf{tS5}^{=}_n \restriction \mathsf{L}^{=}_n)$ in a similar way to $\mathsf{H}(\mathbf{tS5}_n \restriction \mathsf{L}_n)$.

(EQ1)	$t = t$
(EQ2)	$t = s \to (\varphi(t/z) \to \varphi(s/z))$
(EQ3)	$t \neq s \to [\vec{u}]t \neq s$

Table 3.4: Axioms on Equality

Proposition 120. Let $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{TSML}}$. A formula $t = s \to [\vec{u}]t = s$ is provable in $\mathsf{H}(\mathbf{tK\Sigma}^{=})$.

Proof. Note that, when z is fresh in t, u, $([\vec{u}]t = t) = ([\vec{u}]t = z)(t/z)$ and $([\vec{u}]t = s) = ([\vec{u}]t = z)(s/z))$.

1. $\vdash t = s \to (([\vec{u}]t = z)(t/z) \to ([\vec{u}]t = z)(s/z))$ — (EQ2), z: fresh in t, u
2. $\vdash ([\vec{u}]t = z)(t/z) \to (t = s \to [\vec{u}](t = z)(s/z))$ — 1, PC
3. $\vdash t = t$ — (EQ1)
4. $\vdash [\vec{u}]t = t$ — 3, (Nec)
5. $\vdash t = s \to [\vec{u}]t = s$ — 2, 4, (MP)

■

Theorem 121. (Soundness of $\mathsf{H}(\mathbf{tK\Sigma}^{=})$) Let $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{TSML}}$ and $\mathbb{F}_\Sigma$ be the class of all the frames for TSML$^{=}$ to which Σ corresponds. For all formulas φ in $\mathsf{L}^{=}_{\mathsf{TSML}}$, if φ is provable in $\mathsf{H}(\mathbf{tK\Sigma}^{=})$, then φ is valid in $\mathbb{F}_\Sigma$.

Recall that we have proved, in Section 3.1, the strong completeness of Hilbert systems $\mathsf{H}(\mathbf{tK\Sigma})$ such that $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{TSML}} \setminus \{ \mathrm{B}_n, 5_n, \mathrm{BF}_n \mid n \in \mathbb{N} \}$ and Hilbert systems $\mathsf{H}(\mathbf{tKBF}_n\Sigma_n \restriction \mathsf{L}_n)$ such that $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{TSML}}$. In what follows, we will prove strong completeness of $\mathsf{H}(\mathbf{tK}\Sigma^{=})$ and $\mathsf{H}(\mathbf{tKBF}_n\Sigma^{=}_n \restriction \mathsf{L}^{=}{}_n)$. For this purpose, we define notions such as variable-extended language and witnessing in TSML$^{=}$ in a similar vein as in TSML. It should be noted that strong completeness of $\mathsf{H}(\mathbf{tK}\Sigma^{=})$ are harder to prove than those of $\mathsf{H}(\mathbf{tKBF}_n\Sigma^{=}_n \restriction \mathsf{L}^{=}{}_n)$. It is because the canonical models for the former do not have (locally) constant domains in general, nevertheless the satisfaction relation of $t = s$ requires that $t = s \in \Gamma$ iff $t = s \in \Delta$ for all $\mathsf{H}(\mathbf{tK}\Sigma^{=})$-MCSs Γ, Δ, where the values of t, s are in the domain of a "staring world" which can access Γ and Δ in finite steps. To overcome this difficulty, we will use the techniques introduced in Corsi [12]. Let us define $|t|_\Gamma$ as $\{ s \in \mathsf{Term}(\mathsf{Var}^+) \mid t = s \in \Gamma \}$ for each $\mathsf{H}(\mathbf{tK}\Sigma^{=})$-MCS Γ. Then we can express the key property in Corsi-style canonical model construction as follows.

$$\Gamma R \Delta \quad \text{implies} \quad |u|_\Gamma = |u|_\Delta \quad \text{for all } u \in \mathsf{Term}(\Gamma)$$

As seen below, this property correctly gives the satisfaction relation of $t = s$ in the canonical models.

Strong Completeness of $\mathsf{H}(\mathbf{tK}\Sigma^{=})$

Recall $\mathsf{Axiom}_{\mathsf{TSML}} = \{ \mathrm{T}_n, \mathrm{D}_n, \mathrm{B}_n, 4_n, 5_n, \mathrm{BF}_n \mid n \in \mathbb{N} \}$. Until Theorem 130, we abbreviate $\mathsf{H}(\mathbf{tK}\Sigma^{=})$ as Λ for some fixed subset $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{TSML}} \setminus \{ \mathrm{B}_n, 5_n, \mathrm{BF}_n \mid n \in \mathbb{N} \}$.

Lemma 122. (Lindenbaum Lemma) Let Γ be a Λ-consistent set in $\mathsf{L}^{=}(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma) \sqsubset \mathsf{Var}^+$.

1. There exists a Λ-MCS Γ^+ in $\mathsf{L}^{=}(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma^+) = \mathsf{FV}(\Gamma)$ and $\Gamma \subseteq \Gamma^+$.
2. There exists a witnessed Λ-MCS Γ^+ in $\mathsf{L}^{=}(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma^+) \sqsubset \mathsf{Var}^+$ and $\Gamma \subseteq \Gamma^+$.

Proof. Item 1 is proved similarly as in the proof of item 1 of Lemma 96. Item 2 is proved similarly as in the proof of Lemma 89. ■

Definition 123. Consider the tuple $(W, R, (D_\Gamma)_{\Gamma \in W})$ where

- $W := \{ \Gamma \mid \Gamma \text{ is a witnessed } \Lambda\text{-MCS in } \mathsf{L}^{=}(\mathsf{Var}^+) \text{ such that } \mathsf{FV}(\Gamma) \sqsubset \mathsf{Var}^+ \}$;
- $D_\Gamma := \{ |t|_\Gamma \mid t \in \mathsf{Term}(\Gamma) \}$;
- $\Gamma R_{\overrightarrow{|t|_\Gamma}} \Delta \quad \text{iff} \quad \begin{cases} [\vec{t}\,]\varphi \in \Gamma \text{ implies } \varphi \in \Delta & \text{for all formulas } \varphi \text{ in } \mathsf{L}^{=}(\mathsf{Var}^+) \text{ and} \\ |u|_\Gamma = |u|_\Delta & \text{for all terms } u \in \mathsf{Term}(\Gamma). \end{cases}$

Let $\Gamma_0 \in W$ and $R^{(*)}$ be the union $\bigcup_{m \in \mathbb{N}} R^{(m)}$ of binary relations $R^{(m)}$ on W defined by

- $\Gamma R^{(0)} \Delta \quad \text{iff} \quad \Gamma = \Delta$;

- $\Gamma R^{(1)}\Delta$ iff $\Gamma R_{\overrightarrow{|t|_{\Gamma_0}}}\Delta$ for some $\overrightarrow{|t|_{\Gamma_0}} \in D^{<\omega}_{\Gamma_0}$;
- $\Gamma R^{(m+1)}\Delta$ iff $\Gamma R^{(m)}\Xi$ and $\Xi R_{\overrightarrow{|t|_\Xi}}\Delta$ for some $\Xi \in W$ and some $\overrightarrow{|t|_\Xi} \in D^{<\omega}_{\Xi}$

where $D^{<\omega}_\Gamma$ is the set of all finite sequences of D_Γ. The *canonical* Λ*-frame* (*over* Γ_0) is the tuple $\mathfrak{F}^\Lambda = (W^\Lambda, R^\Lambda, (D^\Lambda_\Gamma)_{\Gamma\in W^\Lambda})$, where

- $W^\Lambda := \{\Gamma \in W \mid \Gamma_0 R^{(*)}\Gamma\}$;
- $D^\Lambda_\Gamma := D_\Gamma$;
- $R^\Lambda_{\overrightarrow{|t|_\Gamma}} := R_{\overrightarrow{|t|_\Gamma}} \cap (W^\Lambda \times W^\Lambda)$.

The *canonical* Λ*-model* (*over* Γ_0) is the tuple $\mathfrak{M}^\Lambda = (\mathfrak{F}^\Lambda, I^\Lambda)$, where

- $\mathfrak{F}^\Lambda$ is the canonical Λ-frame;
- $(|t_1|_\Gamma, \dots, |t_n|_\Gamma) \in I^\Lambda(P, \Gamma)$ iff $Pt_1 \dots t_n \in \Gamma$;
- $I^\Lambda(c) := |c|_{\Gamma_0}$.

The *canonical assignment* (*over* Γ_0) is the assignment $\iota : \mathsf{FV}(\Gamma_0) \to D^\Lambda$ defined by $\iota(x) := |x|_{\Gamma_0}$.

Proposition 124. In Definition 123, R and $I^\Lambda(P,\Gamma)$ are well defined.

Proof. We first show $I^\Lambda(P,\Gamma)$ is well-defined, i.e., for all terms $t_1, \dots, t_n, s_1, \dots, s_n$, if $|t_i|_\Gamma = |s_i|_\Gamma$ holds and $Pt_1 \dots t_n \in \Gamma$, then $Ps_1 \dots s_n \in \Gamma$. Suppose $|t_i|_\Gamma = |s_i|_\Gamma$ and $Pt_1 \dots t_n \in \Gamma$. Then $t_i \in \mathsf{Term}(\Gamma)$, so $t_i \in \{u \in \mathsf{Term}(\mathsf{Var}^+) \mid t = u \in \Gamma\} = |t_i|_\Gamma$. By $|t_i|_\Gamma = |s_i|_\Gamma$ we have $t = s \in \Gamma$. Since $Pt_1 \dots t_n \in \Gamma$, by ($\mathsf{EQ2}$) we obtain $Ps_1 \dots s_n \in \Gamma$.

We then show R is well defined, i.e., for all finite sequences $\vec{t}, \vec{s}$ of terms, if $|t_i|_\Gamma = |s_i|_\Gamma$ holds and for all formulas φ in $\mathsf{L}(\mathsf{Var}^+)$ $[\vec{t}]\varphi \in \Gamma$ implies $\varphi \in \Delta$, then for all formulas φ in $\mathsf{L}(\mathsf{Var}^+)$ $[\vec{s}]\varphi \in \Gamma$ implies $\varphi \in \Delta$. Suppose that $|t_i|_\Gamma = |s_i|_\Gamma$ holds and that for all formulas φ in $\mathsf{L}(\mathsf{Var}^+)$ $[\vec{t}]\varphi \in \Gamma$ implies $\varphi \in \Delta$. Take any formula $\varphi \in \mathsf{L}(\mathsf{Var}^+)$ such that $[\vec{s}]\varphi \in \Gamma$. Since $s_i = t_i \in \Gamma$ by our supposition, it follows from $[\vec{s}]\varphi \in \Gamma$ and ($\mathsf{EQ2}$) that $[\vec{t}]\varphi \in \Gamma$. Therefore, $\varphi \in \Delta$ holds by our supposition. ■

Below, let Γ_0 be some fixed witnessed Λ-MCS in $\mathsf{L}^=(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma_0) \sqsubset \mathsf{Var}^+$.

Proposition 125. Let $\mathfrak{M}^\Lambda$ be the canonical Λ-model over Γ_0. For all terms $t \in \mathsf{Term}(\Gamma_0)$ and worlds $\Gamma \in W^\Lambda$, it holds that $|t|_{\Gamma_0} = |t|_\Gamma$.

Proof. Take any term $t \in \mathsf{Term}(\Gamma_0)$ and any world $\Gamma \in W^\Lambda$. Since $\Gamma_0 R^{(*)}\Gamma$,

$$\Gamma_0 R_{\overrightarrow{|s_0|_{\Gamma_0}}}\Gamma_1 \cdots \Gamma_m R_{\overrightarrow{|s_m|_{\Gamma_m}}}\Gamma_{m+1} = \Gamma \quad \text{for some } m, \Gamma_{i+1}, \overrightarrow{|s_i|_{\Gamma_i}} \in (D^\Lambda_{\Gamma_i})^{<\omega}\ (0 \leqslant i \leqslant m).$$

Fix such m and such Γ_{i+1}, $\overrightarrow{|s_i|_{\Gamma_i}}$ for each i. Since $t \in \mathsf{Term}(\Gamma_0)$ and $|u|_{\Gamma_i} = |u|_{\Gamma_{i+1}}$ for all terms $u \in \mathsf{Term}(\Gamma_i)$,

$$|t|_{\Gamma_0} = |t|_{\Gamma_1} = \cdots = |t|_{\Gamma_{n-1}} = |t|_{\Gamma_n} = |t|_{\Gamma}.$$

■

Proposition 126. The canonical Λ-model over Γ_0 *is* a model.

Proof. Recall that the canonical Λ-model is the tuple $(W^\Lambda, R^\Lambda, (D^\Lambda_\Gamma)_{\Gamma \in W^\Lambda}, I^\Lambda)$. We confirm that $I^\Lambda(c) \in \bigcap D^\Lambda_\Gamma$ and that the canonical Λ-frame has increasing domains, i.e., for all $\Gamma, \Delta \in W^\Lambda$ and $\overrightarrow{|t|_\Gamma} \in (D^\Lambda_\Gamma)^{<\omega}$, if $\Gamma R^\Lambda_{\overrightarrow{|t|_\Gamma}} \Delta$ then $D^\Lambda_\Gamma \subseteq D^\Lambda_\Delta$. The former follows from Proposition 125. For the latter, take any $\Gamma, \Delta \in W^\Lambda$ and $\overrightarrow{|t|_\Gamma} \in (D^\Lambda_\Gamma)^{<\omega}$ such that $\Gamma R^\Lambda_{\overrightarrow{|t|_\Gamma}} \Delta$. Assume $|s|_\Gamma \in D^\Lambda_\Gamma$. Our goal is to show $|s|_\Gamma \in D^\Lambda_\Delta$. We now have that $s \in \mathsf{Term}(\Gamma)$ and that $t_i \in \mathsf{Term}(\Gamma)$ for each t_i of $\vec{t}$, so it holds that $[\vec{t}\,](Ps \to Ps) \in \Gamma$. Thus $s \in \mathsf{Term}(\Delta)$ follows from $\Gamma R^\Lambda_{\overrightarrow{|t|_\Gamma}} \Delta$, which implies $|s|_\Delta \in D^\Lambda_\Delta$. Also, $|s|_\Gamma = |s|_\Delta$ follows from $\Gamma R^\Lambda_{\overrightarrow{|t|_\Gamma}} \Delta$ and $s \in \mathsf{Term}(\Gamma)$. Therefore, $|s|_\Gamma = |s|_\Delta \in D^\Lambda_\Delta$. ■

Lemma 127. (Existence Lemma) If $\neg[\vec{t}\,]\varphi \in \Gamma \in W^\Lambda$, there exists some $\Delta \in W^\Lambda$ such that $\neg\varphi \in \Delta$ and $\Gamma R^\Lambda_{|\vec{t}|_\Gamma} \Delta$.

Proof. Suppose $\neg[\vec{t}\,]\varphi \in \Gamma \in W^\Lambda$. As in the proof of Existence Lemma for TSML without (BF_n)s (Lemma 92), we claim $\Delta_0 := \{\neg\varphi\} \cup \left\{\psi \mid [\vec{t}\,]\psi \in \Gamma\right\} \nvdash \bot$. In what follows, we use Δ_0 to construct a Λ-MCS Δ in $\mathsf{L}^=(\mathsf{Var}^+)$ such that (a) Δ is witnessed; (b) $\mathsf{FV}(\Delta) \sqsubset \mathsf{Var}^+$; (c) $\Gamma_0 R^{(*)} \Delta$; (d) $\neg\varphi \in \Delta$; (e) $\Gamma R_{\overrightarrow{|t|_\Gamma}} \Delta$.

Take some set Var' of variables in $\mathsf{L}^=(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma) \sqsubset \mathsf{Var}' \sqsubset \mathsf{Var}^+$. Let h be an enumeration of Var' and g an enumeration of all formulas of the form $\forall x\psi$ in $\mathsf{L}^=(\mathsf{Var}')$. We define a sequence $(\Delta_m)_{m \in \mathbb{N}}$ of Λ-consistent sets Δ_m starting from Δ_0 such that $\mathsf{FV}(\Delta_m) \sqsubset \mathsf{Var}'$, as below. Assume we have defined Λ-consistent sets Δ_m. Let $\forall x\psi$ be the first formula in the enumeration g such that $\mathsf{FV}(\forall x\psi) \subseteq \mathsf{FV}(\Delta_m)$ and $\forall x\psi$ does not occur in $\Delta_m \backslash \Delta_0$. Let also y be the first variable in the enumeration h which does not occur in $\forall x\psi, \Delta_m$. Define Δ_{m+1} by

- if $\Delta_m \cup \{\psi(t/x) \to \forall x\psi\} \nvdash \bot$ for some term $t \in \mathsf{Term}(\Delta_m)$,

$$\Delta_{m+1} := \Delta_m \cup \{\psi(t/x) \to \forall x\psi\};$$

- if $\Delta_m \cup \{\psi(t/x) \to \forall x\psi\} \vdash \bot$ for all terms $t \in \mathsf{Term}(\Delta_m)$,

$$\Delta_{m+1} := \Delta_m \cup \{\psi(y/x) \to \forall x\psi\} \cup \{t \neq y \mid t \in \mathsf{Term}(\Delta_m)\}.$$

We establish $\Delta_{m+1} \nvdash \bot$. If $\Delta_m \cup \{\, \psi(t/x) \to \forall x\psi \,\} \nvdash \bot$ for some term $t \in \mathsf{Term}(\Delta_m)$, then $\Delta_{n+1} \nvdash \bot$ by definition. Thus we may assume $\Delta_m \cup \{\, \psi(t/x) \to \forall x\psi \,\} \vdash \bot$ for all terms $t \in \mathsf{Term}(\Delta_m)$. Our goal is to show

$$\Delta_{m+1} = \Delta_m \cup \{\, \psi(y/x) \to \forall x\psi \,\} \cup \{\, t \neq y \mid t \in \mathsf{Term}(\Delta_m) \,\} \nvdash \bot.$$

Suppose for contradiction that $\Delta_{m+1} \vdash \bot$. Together with our assumption, we have

(1) $\Delta_m \vdash (\psi(y/x) \to \forall x\psi) \to (t_1 = y \vee \cdots \vee t_k = y)$

(2) $\Delta_m \vdash \psi(t_j/x)$

(3) $\Delta_m \vdash \neg\forall x\psi$

where $1 \leqslant j \leqslant k$ and $t_1, \ldots, t_k \in \mathsf{Term}(\Delta_m)$. We now obtain $\Delta_m \vdash \bot$ as follows.

1. $\Delta_m \vdash (\psi(y/x) \to \forall x\psi) \to \bigvee_{j\leqslant k} t_j = y$ (1)
2. $\Delta_m \vdash (\psi(y/x) \to \forall x\psi) \to \bigvee_{j\leqslant k}(t_j = y \wedge \psi(t_j/x))$ 1, (2), PC
3. $\vdash (t_j = y \wedge \psi(t_j/x)) \to \psi(y/x)$ FOL
4. $\Delta_m \vdash (\psi(y/x) \to \forall x\psi) \to \psi(y/x)$ 2, 3, PC
5. $\Delta_m \vdash \psi(y/x)$ 4, PC
6. $\Delta_m \vdash \forall x\psi$ 5, (`Gen`), y: fresh in $\forall x\psi, \Delta_m$
7. $\Delta_m \vdash \neg\forall x\psi$ (3)
8. $\Delta_m \vdash \bot$ 6, 7, PC

However, we also have $\Delta_m \nvdash \bot$ by inductive hypothesis, so a contradiction.

We claim $\bigcup_{m\in\mathbb{N}} \Delta_m$ is a Λ-consistent set and $\mathsf{FV}(\bigcup_{m\in\mathbb{N}} \Delta_m) \subseteq \mathsf{Var}'$. By item 1 of Lindenbaum Lemma for TSML (Lemma 122), we have a Λ-MCS Δ in $\mathsf{L}^{=}(\mathsf{Var}^{+})$ such that $\mathsf{FV}(\Delta) = \mathsf{FV}(\bigcup_{m\in\mathbb{N}} \Delta_m)$ and $\bigcup_{m\in\mathbb{N}} \Delta_m \subseteq \Delta$. We need to show that Δ satisfies the aforementioned conditions, i.e., (a) Δ is witnessed; (b) $\mathsf{FV}(\Delta) \sqsubset \mathsf{Var}^{+}$; (c) $\Gamma_0 R^{(*)} \Delta$; (d) $\neg\varphi \in \Delta$; (e) $\Gamma R_{\overrightarrow{|t|_\Gamma}} \Delta$. It is not difficult to check (b) and (d). Furthermore, (c) follows from (e) and $\Gamma_0 R^{(*)} \Gamma$. Thus we show the other conditions (a) and (e).

For (a), take any formula of the form $\forall x\psi$ in $\mathsf{L}^{=}(\Delta)$. Since $\mathsf{FV}(\Delta) = \mathsf{FV}(\bigcup_{m\in\mathbb{N}} \Delta_m) = \bigcup_{m\in\mathbb{N}} \mathsf{FV}(\Delta_m) \subseteq \mathsf{Var}'$, $\forall x\psi$ is a formula in the enumeration g and also $\mathsf{FV}(\forall x\psi) \subseteq \mathsf{FV}(\Delta_m)$ holds for some m. We may assume that $\forall x\psi$ does not occur in $\Delta_m \backslash \Delta_0$, otherwise there is some term $t \in \mathsf{Term}(\Delta_m) \subseteq \mathsf{Term}(\Delta)$ such that $\psi(t/x) \to \forall x\psi \in \Delta_m \subseteq \Delta$. Therefore, for some $l \geqslant m$, either $\Delta_{l+1} = \Delta_l \cup \{\, \psi(t/x) \to \forall x\psi \,\}$ for some $t \in \mathsf{Term}(\Delta_l)$, or $\Delta_{l+1} = \Delta_l \cup \{\, \psi(y/x) \to \forall x\psi \,\} \cup \{\, t \neq y \mid t \in \mathsf{Term}(\Delta_l) \,\}$ where y is the first variable in the enumeration h which does not occur in $\Delta_l, \forall x\psi$. In either case, as $\Delta_{l+1} \subseteq \Delta$, there exists some $t \in \mathsf{Term}(\Delta)$ such that $\psi(t/x) \to \forall x\psi \in \Delta$.

For (e), we have to establish (e_1) $[\vec{t}\,]\varphi \in \Gamma$ implies $\varphi \in \Delta$ for all formulas φ and (e_2) $|u|_\Gamma = |u|_\Delta$ for all terms $u \in \mathsf{Term}(\Gamma)$. The former follows from the construction of Δ. The latter is established as follows. Take any term $u \in \mathsf{Term}(\Gamma)$. It suffices to show the following: for any term $s \in \mathsf{Term}(\mathsf{Var}^+)$, $u = s \in \Gamma$ iff $u = s \in \Delta$. Note that $t_i \in \mathsf{Term}(\Gamma)$ holds for each t_i of $\vec{t}$ since we have $\neg[\vec{t}]\varphi \in \Gamma$ as an assumption of this lemma.

For the left-to-right direction, take any term s such that $u = s \in \Gamma$. Since $[\vec{t}]u = s$ is a formula in $\mathsf{L}^=(\Gamma)$, we have $[\vec{t}]u = s \in \Gamma$ by Proposition 120. Therefore, $u = s \in \Delta$ holds by (e_1).

For the right-to-left direction, take any term s. We show the contraposition, i.e., if $u = s \notin \Gamma$ then $u = s \notin \Delta$. Suppose $u = s \notin \Gamma$. When $s \in \mathsf{Term}(\Gamma)$, $[\vec{t}]u = s$ is a formula in $\mathsf{L}^=(\Gamma)$. Since $u \neq s \in \Gamma$, we have $[\vec{t}]u \neq s \in \Gamma$ by ($\mathtt{EQ3}$). By (e_1), this implies $u \neq s \in \Delta$ so $u = s \notin \Delta$, as required. Consider when $s \notin \mathsf{Term}(\Gamma)$. We may suppose $s \in \mathsf{Term}(\Delta)$ since otherwise obviously $u = s \notin \Delta$. Then, as any constant symbol is in $\mathsf{Term}(\Gamma)$, s must be a variable in $\mathsf{FV}(\Delta) = \bigcup_{m \in \mathbb{N}} \mathsf{FV}(\Delta_m)$. Also, $s \notin \mathsf{FV}(\Gamma)$ holds by $s \notin \mathsf{Term}(\Gamma)$. These imply, for some $m \in \mathbb{N}$ and some formula $\forall x\psi$, $\Delta_{m+1} = \Delta_m \cup \{\, \psi(s/x) \to \forall x\psi \,\} \cup \{\, t \neq s \mid t \in \mathsf{Term}(\Delta_m) \,\}$. Note that $u \in \mathsf{Term}(\Delta_m)$ holds by (e_1) since $[\vec{t}](Pu \to Pu) \in \Gamma$. Thus, $u \neq s \in \Delta_{m+1} \subseteq \Delta$ so $u = s \notin \Delta$, as required. ■

Lemma 128. (Truth Lemma) Let $\mathfrak{M}^\Lambda$ be the canonical Λ-model and ι the canonical assignment over Γ_0. For all Λ-MCSs $\Gamma \in W^\Lambda$ and formulas φ in $\mathsf{L}^=(\Gamma_0)$,

$$\mathfrak{M}^\Lambda, \Gamma, \iota \models \varphi \quad \text{iff} \quad \varphi \in \Gamma$$

Proof. By induction on the length of φ. The proof proceeds similarly to the proof of Truth Lemma for TSML without $(\mathtt{BF}_n)$s (Lemma 93). We show only a case in which φ is $t = s$. Note that $t, s \in \mathsf{Term}(\Gamma_0)$ and thus $t, s \in \mathsf{Term}(\Gamma)$.

$$\begin{aligned}
\mathfrak{M}^\Lambda, \Gamma, \iota \models t = s \quad & \text{iff} \quad \iota(t) = \iota(s) & \\
& \text{iff} \quad |t|_{\Gamma_0} = |s|_{\Gamma_0} & \text{the definition of } \iota \\
& \text{iff} \quad |t|_\Gamma = |s|_\Gamma & \text{Proposition 125} \\
& \text{iff} \quad t = s \in \Gamma &
\end{aligned}$$

■

Proposition 129. Let $\mathfrak{F}^\Lambda$ be the canonical Λ-frame, where $\Lambda = \mathsf{H}(\mathbf{tK}\Sigma^=)$.

1. If $\mathrm{T}_n \in \Sigma$ then $\mathfrak{F}^\Lambda$ is n-reflexive.
2. If $\mathrm{D}_n \in \Sigma$ then $\mathfrak{F}^\Lambda$ is n-serial.
3. If $4_n \in \Sigma$ then $\mathfrak{F}^\Lambda$ is n-transitive.

Proof. The whole proof proceeds similarly to the proof of Proposition 94. Note that each $\vec{t}$ has the length n in the following proofs.

1. Suppose $\mathrm{T}_n \in \Sigma$. We show $\mathfrak{F}^\Lambda$ is n-reflexive, i.e., for all $\Gamma \in W^\Lambda$ and $\overrightarrow{|t|_\Gamma} \in (D^\Lambda_\Gamma)^n$, $\Gamma R^\Lambda_{\overrightarrow{|t|_\Gamma}} \Gamma$. Take any $\Gamma \in W^\Lambda$ and $\overrightarrow{|t|_\Gamma} \in (D^\Lambda_\Gamma)^n$. Our goal is to show $\Gamma R^\Lambda_{\overrightarrow{|t|_\Gamma}} \Gamma$, i.e., (a) $[\vec{t}]\varphi \in \Gamma$ implies $\varphi \in \Gamma$ for all formulas φ in $\mathsf{L}^=(\mathsf{Var}^+)$; (b) $|u|_\Gamma = |u|_\Gamma$ for all terms $u \in \mathsf{Term}(\Gamma)$; (c) $\Gamma \in W^\Lambda$. Since (b) and (c) are obvious, we show (a). Suppose $[\vec{t}]\varphi \in \Gamma$. It then follows from $(\mathtt{T}_n)$ that $\varphi \in \Gamma$.
2. Suppose $\mathrm{D}_n \in \Sigma$. We show $\mathfrak{F}^\Lambda$ is n-serial, i.e., for all $\Gamma \in W^\Lambda$ and $\overrightarrow{|t|_\Gamma} \in (D^\Lambda_\Gamma)^n$, there exists some $\Delta \in W^\Lambda$ such that $\Gamma R^\Lambda_{\overrightarrow{|t|_\Gamma}} \Delta$. Take any $\Gamma \in W^\Lambda$ and $\overrightarrow{|t|_\Gamma} \in (D^\Lambda_\Gamma)^n$. We claim that $\neg[\vec{t}]\bot \in \Gamma$ since $[\vec{t}]\bot \to \bot$ is provable in $\Lambda = \mathsf{H}(\mathbf{tK}\Sigma^=)$ by $(\mathtt{D}_n)$ and item 3 of Proposition 83 and thus $[\vec{t}]\bot \to \bot \in \Gamma$ holds. By Existence Lemma (Lemma 127), there exists some $\Delta \in W^\Lambda$ such that $\Gamma R^\Lambda_{|\vec{t}|_\Gamma} \Delta$.
3. Suppose $4_n \in \Sigma$. We show $\mathfrak{F}^\Lambda$ is n-transitive, i.e., for all $\Gamma, \Delta, \Xi \in W^\Lambda$ and $\overrightarrow{|t|_\Gamma} \in (D^\Lambda_\Gamma)^n$, if $\Gamma R^\Lambda_{\overrightarrow{|t|_\Gamma}} \Delta$ and $\Delta R^\Lambda_{\overrightarrow{|t|_\Delta}} \Xi$ then $\Gamma R^\Lambda_{\overrightarrow{|t|_\Gamma}} \Xi$. Take any $\Gamma, \Delta, \Xi \in W^\Lambda$ and $\overrightarrow{|t|_\Gamma} \in (D^\Lambda_\Gamma)^n$. Suppose $\Gamma R^\Lambda_{\overrightarrow{|t|_\Gamma}} \Delta$ and $\Delta R^\Lambda_{\overrightarrow{|t|_\Delta}} \Xi$. Our goal is to show $\Gamma R^\Lambda_{\overrightarrow{|t|_\Gamma}} \Xi$, i.e., (a) $[\vec{t}]\varphi \in \Gamma$ implies $\varphi \in \Xi$ for all formulas φ in $\mathsf{L}^=(\mathsf{Var}^+)$; (b) $|u|_\Gamma = |u|_\Xi$ for all terms $u \in \mathsf{Term}(\Gamma)$; (c) $\Gamma, \Xi \in W^\Lambda$. Since (c) is obvious, we show (a) and (b). For (a), suppose $[\vec{t}]\varphi \in \Gamma$. It then follows from (4_n) that $[\vec{t}][\vec{t}]\varphi \in \Gamma$. Together with $\Gamma R^\Lambda_{\vec{t}} \Delta$ and $\Delta R^\Lambda_{\vec{t}} \Xi$, this implies $\varphi \in \Xi$. For (b), take any term $u \in \mathsf{Term}(\Gamma)$. Since the canonical Λ-frame has increasing domains, $u \in \mathsf{Term}(\Delta)$ holds by $\Gamma R^\Lambda_{\overrightarrow{|t|_\Gamma}} \Delta$ and $|u|_\Gamma \in D^\Lambda_\Gamma$. Therefore, by $\Gamma R^\Lambda_{\overrightarrow{|t|_\Gamma}} \Delta$ and $\Delta R^\Lambda_{\overrightarrow{|t|_\Delta}} \Xi$, $|u|_\Gamma = |u|_\Delta = |u|_\Xi$.

■

Theorem 130. (Strong completeness of $\mathsf{H}(\mathbf{tK}\Sigma^=)$) Let $\Sigma \subsetneq \mathsf{Axiom}_{\mathsf{TSML}} \setminus \{ \mathrm{B}_n, 5_n, \mathrm{BF}_n \mid n \in \mathbb{N} \}$ and $\mathbb{F}_\Sigma$ be the class of all the frames for TSML$^=$ to which Σ corresponds. For all formulas φ and sets Γ of formulas in $\mathsf{L}^=_{\mathsf{TSML}}$, if φ is a consequence from Γ in $\mathbb{F}_\Sigma$, then φ is provable from Γ in $\mathsf{H}(\mathbf{tK}\Sigma^=)$.

Proof. Let $\Lambda = \mathsf{H}(\mathbf{tK}\Sigma)$ for short. Suppose φ is not provable from Γ in Λ. We show φ is not a consequence from Γ in $\mathbb{F}_\Sigma$. Note first that $\Gamma \cup \{ \neg\varphi \}$ is Λ-consistent in L. We claim $\Gamma \cup \{ \neg\varphi \}$ is Λ-consistent in $\mathsf{L}(\mathsf{Var}^+)$. By item 2 of Lindenbaum Lemma (Lemma 122), we obtain a witnessed Λ-MCS Γ^+ in $\mathsf{L}(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma^+) \sqsubset \mathsf{Var}^+$ and $\Gamma \cup \{ \neg\varphi \} \subseteq \Gamma^+$. It then follows from Truth Lemma (Lemma 128) that

$$\mathfrak{M}^\Lambda, \Gamma^+, \iota \models \Gamma \quad \text{and} \quad \mathfrak{M}^\Lambda, \Gamma^+, \iota \not\models \varphi,$$

where $\mathfrak{M}^\Lambda = (\mathfrak{F}^\Lambda, I^\Lambda)$ is the canonical Λ-model and ι is the canonical assignment. We must further show that $\mathfrak{F}^\Lambda \in \mathbb{F}_\Sigma$, which is established by Proposition 129. Hence, in $\mathsf{L}(\mathsf{Var}^+)$, φ is not a consequence from Γ in $\mathbb{F}_\Sigma$. By restricting $\mathsf{L}(\mathsf{Var}^+)$ to L, we conclude in L that φ is not a consequence from Γ in $\mathbb{F}_\Sigma$. ∎

Strong Completeness of $\mathsf{H}(\mathbf{tKBF}_n\Sigma_n^= \restriction \mathsf{L}_n^=)$

Recall $\mathsf{Axiom}_{\mathsf{TSML}} = \{ \mathrm{T}_n, \mathrm{D}_n, \mathrm{B}_n, 4_n, 5_n, \mathrm{BF}_n \mid n \in \mathbb{N} \}$ and $\Sigma_n = \{ \mathrm{X}_n \mid \mathrm{X}_n \in \Sigma \}$. Until Theorem 139, we abbreviate $\mathsf{H}(\mathbf{tKBF}_n\Sigma_n^= \restriction \mathsf{L}_n^=)$ as Λ for some fixed subset $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{TSML}}$. In addition, we always mean by $\overrightarrow{t}$ a term-sequence of the length n.

Lemma 131. (Lindenbaum Lemma) Let Γ be a Λ-consistent set in $\mathsf{L}^=(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma) \sqsubset \mathsf{Var}^+$.

1. There exists a Λ-MCS Γ^+ in $\mathsf{L}^=(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma^+) = \mathsf{FV}(\Gamma)$ and $\Gamma \subseteq \Gamma^+$.
2. There exists a witnessed Λ-MCS Γ^+ in $\mathsf{L}^=(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma^+) = \mathsf{Var}^+$ and $\Gamma \subseteq \Gamma^+$.

Proof. Item 1 is proved similarly as in the proof of item 1 of Lemma 96. Item 2 is proved similarly as in the proof of item 2 of Lemma 96. ∎

Definition 132. Consider the tuple $(W, R, (D_\Gamma)_{\Gamma \in W})$ where

- $W := \{ \Gamma \mid \Gamma \text{ is a witnessed } \Lambda\text{-MCS in } \mathsf{L}_n^=(\mathsf{Var}^+) \text{ such that } \mathsf{FV}(\Gamma) = \mathsf{Var}^+ \}$;
- $D_\Gamma := \{ |t|_\Gamma \mid t \in \mathsf{Term}(\mathsf{Var}^+) \}$;
- $\Gamma R_{\overrightarrow{|t|_\Gamma}} \Delta$ iff $[\overrightarrow{t}]\varphi \in \Gamma$ implies $\varphi \in \Delta$ for all formulas φ in $\mathsf{L}_n^=(\mathsf{Var}^+)$.

Let $\Gamma_0 \in W$ and $R^{(*)}$ be the union $\bigcup_{m \in \mathbb{N}} R^{(m)}$ of binary relations $R^{(m)}$ on W defined by

- $\Gamma R^{(0)} \Delta$ iff $\Gamma = \Delta$;
- $\Gamma R^{(1)} \Delta$ iff $\Gamma R_{\overrightarrow{|t|_{\Gamma_0}}} \Delta$ for some $\overrightarrow{|t|_{\Gamma_0}} \in D_{\Gamma_0}^{<\omega}$;
- $\Gamma R^{(m+1)} \Delta$ iff $\Gamma R^{(m)} \Xi$ and $\Xi R_{\overrightarrow{|t|_\Xi}} \Delta$ for some $\Xi \in W$ and some $\overrightarrow{|t|_\Xi} \in D_\Xi^{<\omega}$

where $D_\Gamma^{<\omega}$ is the set of all finite sequences of D_Γ. The *canonical Λ-frame* (*over* Γ_0) is the tuple $\mathfrak{F}^\Lambda = (W^\Lambda, R^\Lambda, (D_\Gamma^\Lambda)_{\Gamma \in W^\Lambda})$, where

- $W^\Lambda := \{ \Gamma \in W \mid \Gamma_0 R^{(*)} \Gamma \}$;
- $D_\Gamma^\Lambda := D_\Gamma$;

- $R^{\Lambda}_{\overrightarrow{|t|_\Gamma}} := R_{\overrightarrow{|t|_\Gamma}} \cap (W^{\Lambda} \times W^{\Lambda})$.

The *canonical* Λ-*model* (*over* Γ_0) is the tuple $\mathfrak{M}^{\Lambda} = (\mathfrak{F}^{\Lambda}, I^{\Lambda})$, where

- $\mathfrak{F}^{\Lambda}$ is the canonical Λ-frame;
- $(|t_1|_\Gamma, \ldots, |t_n|_\Gamma) \in I^{\Lambda}(P, \Gamma) \quad$ iff $\quad Pt_1 \ldots t_n \in \Gamma$;
- $I^{\Lambda}(c) := |c|_{\Gamma_0}$.

The *canonical assignment* (*over* Γ_0) is the assignment $\iota : \mathsf{Var}^+ \to D^{\Lambda}$ defined by $\iota(x) := |x|_{\Gamma_0}$.

Proposition 133. In Definition 123, R and $I^{\Lambda}(P, \Gamma)$ are well defined.

Proof. Similarly done to the proof of Proposition 124. ∎

Below, let Γ_0 be some fixed witnessed Λ-MCS in $\mathsf{L}_n^{=}(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma_0) = \mathsf{Var}^+$).

Proposition 134. Let $\mathfrak{M}^{\Lambda}$ be the canonical Λ-model over Γ_0. For all terms $t \in \mathsf{Term}(\mathsf{Var}^+)$ and worlds $\Gamma \in W^{\Lambda}$, it holds that $|t|_{\Gamma_0} = |t|_\Gamma$.

Proof. Take any term $t \in \mathsf{Term}(\mathsf{Var}^+)$ and any world $\Gamma \in W^{\Lambda}$. Since $\Gamma_0 R^{(*)} \Gamma$, we have

$$(\dagger) \quad \Gamma_0 R_{\overrightarrow{|s_0|_{\Gamma_0}}} \Gamma_1 \cdots \Gamma_m R_{\overrightarrow{|s_m|_{\Gamma_m}}} \Gamma_{m+1} = \Gamma \quad \text{for some } m, \Gamma_{i+1}, \overrightarrow{|s_i|_{\Gamma_i}} \in D_{\Gamma_i}\ (0 \leqslant i \leqslant m).$$

Fix such m and such $\Gamma_{i+1}, \overrightarrow{|s_i|_{\Gamma_i}}$ for each i. We show that, for any term $s \in \mathsf{Term}(\mathsf{Var}^+)$, $t = s \in \Gamma_0$ iff $t = s \in \Gamma$. For the left-to-right direction, suppose $t = s \in \Gamma_0$. By Proposition 120, we have $[\vec{s_0}] \cdots [\vec{s_m}] t = s \in \Gamma$. Together with ($\dagger$), it follows that $t = s \in \Gamma$. For the right-to-left direction, we show the contraposition. Suppose $t = s \notin \Gamma_0$. Then $t \neq s \in \Gamma_0$. By ($\mathtt{EQ3}$), we have $[\vec{s_0}] \cdots [\vec{s_m}] t \neq s \in \Gamma$. Together with ($\dagger$), it follows that $t \neq s \in \Gamma$, which implies $t = s \notin \Gamma$. ∎

Proposition 135. The canonical Λ-model over Γ_0 *is* a model.

Lemma 136. (Existence Lemma) If $\neg[\vec{t}]\varphi \in \Gamma \in W^{\Lambda}$, there exists some $\Delta \in W^{\Lambda}$ such that $\neg\varphi \in \Delta$ and $\Gamma R^{\Lambda}_{|\vec{t}|_\Gamma} \Delta$.

Proof. The proof is similarly done to the proof of Existence Lemma for TSML with ($\mathtt{BF}_n$) (Lemma 99) with the help of item 1 of Lemma 131. ∎

Lemma 137. (Truth Lemma) Let $\mathfrak{M}^\Lambda$ be the canonical Λ-model and ι the canonical assignment over Γ_0. For all Λ-MCS $\Gamma \in W^\Lambda$ and formulas φ in $\mathsf{L}^=_n(\mathsf{Var}^+)$,

$$\mathfrak{M}^\Lambda, \Gamma, \iota \models \varphi \quad \text{iff} \quad \varphi \in \Gamma$$

Proof. The proof is similarly done to the proof of Truth Lemma for TSML with ($\mathtt{BF}_n$) (Lemma 100) with the help of Proposition 134. ■

Proposition 138. Let $\mathfrak{F}^\Lambda$ be the canonical Λ-frame, where $\Lambda = \mathsf{H}(\mathbf{tKBF}_n\Sigma^=_n \upharpoonright \mathsf{L}^=_n)$.

1. $\mathfrak{F}^\Lambda$ has constant domains.
2. If $\mathrm{T}_n \in \Sigma$ then $\mathfrak{F}^\Lambda$ is n-reflexive.
3. If $\mathrm{D}_n \in \Sigma$ then $\mathfrak{F}^\Lambda$ is n-serial.
4. If $\mathrm{B}_n \in \Sigma$ then $\mathfrak{F}^\Lambda$ is n-symmetric.
5. If $4_n \in \Sigma$ then $\mathfrak{F}^\Lambda$ is n-transitive.
6. If $5_n \in \Sigma$ then $\mathfrak{F}^\Lambda$ is n-euclidean.

Proof. Item 1 is obvious and items 2, 3, 5 are shown as in the proof of Proposition 129. We show the other items. Note that each $\vec{t}$ has the length n in the following proofs.

4. Suppose $\mathrm{B}_n \in \Sigma$. We show $\mathfrak{F}^\Lambda$ is n-symmetric, i.e., for all $\Gamma, \Delta \in W^\Lambda$ and $\overrightarrow{|t|_\Gamma} \in (D^\Lambda_\Gamma)^n$, if $\Gamma R^\Lambda_{\overrightarrow{|t|_\Gamma}} \Delta$ then $\Delta R^\Lambda_{\overrightarrow{|t|_\Delta}} \Gamma$. Take any $\Gamma, \Delta \in W^\Lambda$ and $\overrightarrow{|t|_\Gamma} \in (D^\Lambda_\Gamma)^n$. Suppose $\Gamma R^\Lambda_{\overrightarrow{|t|_\Gamma}} \Delta$. Our goal is to show $\Delta R^\Lambda_{\overrightarrow{|t|_\Delta}} \Gamma$, i.e., (a) $[\vec{t}]\varphi \in \Delta$ implies $\varphi \in \Gamma$ for all formulas φ in $\mathsf{L}^={}_n(\mathsf{Var}^+)$ and (b) $\Delta, \Gamma \in W^\Lambda$. Since (b) is obvious, we show (a). Suppose $[\vec{t}]\varphi \in \Delta$. We claim $\langle \vec{t} \rangle [\vec{t}]\varphi \in \Gamma$. By item 4 of Proposition 83, we obtain $\varphi \in \Gamma$.
6. Suppose $5_n \in \Sigma$. We show $\mathfrak{F}^\Lambda$ is n-euclidean, i.e., for all $\Gamma, \Delta, \Xi \in W^\Lambda$ and $\overrightarrow{|t|_\Gamma} \in (D^\Lambda_\Gamma)^n$, if $\Gamma R^\Lambda_{\overrightarrow{|t|_\Gamma}} \Delta$ and $\Gamma R^\Lambda_{\overrightarrow{|t|_\Gamma}} \Xi$ then $\Delta R^\Lambda_{\overrightarrow{|t|_\Delta}} \Xi$. Take any $\Gamma, \Delta, \Xi \in W^\Lambda$ and $\overrightarrow{|t|_\Gamma} \in (D^\Lambda_\Gamma)^n$. Suppose $\Gamma R^\Lambda_{\overrightarrow{|t|_\Gamma}} \Delta$ and $\Gamma R^\Lambda_{\overrightarrow{|t|_\Gamma}} \Xi$. Our goal is to show $\Delta R^\Lambda_{\overrightarrow{|t|_\Delta}} \Xi$, i.e., (a) $[\vec{t}]\varphi \in \Delta$ implies $\varphi \in \Xi$ for all formulas φ in $\mathsf{L}^={}_n(\mathsf{Var}^+)$ and (b) $\Delta, \Xi \in W^\Lambda$. Since (b) is obvious, we show (a). Suppose $[\vec{t}]\varphi \in \Delta$. We claim $\langle \vec{t} \rangle [\vec{t}]\varphi \in \Gamma$. By item 6 of Proposition 83, we have $[\vec{t}]\varphi \in \Gamma$. Thus, we obtain $\varphi \in \Xi$ by $\Gamma R^\Lambda_{\overrightarrow{|t|_\Gamma}} \Xi$.

■

Recall that $\mathbb{CD}$ and $\mathbb{LCD}$ mean the class of all the constant domain frames and the class of all the locally constant domain frames, respectively.

Theorem 139. (Strong completeness of $\mathsf{H}(\mathbf{tKBF}_n\Sigma_n^{=}\restriction \mathsf{L}_n^{=})$) Let $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{TSML}}$ and $\mathbb{F}_{\Sigma_n}$ be the class of all the frames to which Σ_n corresponds. For all formulas φ and sets Γ of formulas in $\mathsf{L}_n^{=}$, if φ is a consequence from Γ in $\mathbb{F}_{\Sigma_n} \cap \mathbb{LCD}$, then φ is provable from Γ in $\mathsf{H}(\mathbf{tKBF}_n\Sigma_n^{=}\restriction \mathsf{L}_n^{=})$.

Proof. Let $\Lambda = \mathsf{H}(\mathbf{tKBF}_n\Sigma_n^{=} \restriction \mathsf{L}_n^{=})$ for short. Suppose φ is not provable from Γ in Λ. We show φ is not a consequence from Γ in $\mathbb{F}_{\Sigma_n} \cap \mathbb{LCD}$. Note first that $\Gamma \cup \{ \neg\varphi \}$ is Λ-consistent in L. We claim $\Gamma \cup \{ \neg\varphi \}$ is Λ-consistent in $\mathsf{L}(\mathsf{Var}^+)$. Similarly as in the proof of Theorem 104, we obtain by item 2 of Lindenbaum Lemma (Lemma 131) a witnessed Λ-MCS Γ^+ in $\mathsf{L}(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma^+) = \mathsf{Var}^+$ and $\Gamma \cup \{ \neg\varphi \} \subseteq \Gamma^+$. It follows from Truth Lemma (Lemma 137) that

$$\mathfrak{M}^\Lambda, \Gamma^+, \iota \models \Gamma \quad \text{and} \quad \mathfrak{M}^\Lambda, \Gamma^+, \iota \not\models \varphi,$$

where $\mathfrak{M}^\Lambda = (\mathfrak{F}^\Lambda, I^\Lambda)$ is the canonical Λ-model and ι is the canonical assignment. We also need to establish that $\mathfrak{F}^\Lambda \in \mathbb{F}_{\Sigma_n} \cap \mathbb{LCD}$ holds. This is established since $\mathfrak{F}^\Lambda \in \mathbb{F}_{\Sigma_n} \cap \mathbb{CD}$ by Proposition 138 and $\mathbb{F}_{\Sigma_n} \cap \mathbb{CD} \subseteq \mathbb{F}_{\Sigma_n} \cap \mathbb{LCD}$. Therefore, in $\mathsf{L}(\mathsf{Var}^+)$, φ is not a consequence from Γ in $\mathbb{F}_{\Sigma_n} \cap \mathbb{LCD}$. By restricting $\mathsf{L}(\mathsf{Var}^+)$ to L, we conclude in L that φ is not a consequence from Γ in $\mathbb{F}_{\Sigma_n} \cap \mathbb{LCD}$. ■

Corollary 140. (Strong completeness of $\mathsf{H}(\mathbf{tS5}_1^{=}\restriction \mathsf{L}_1^{=})$) Let $\mathbb{F}$ be the class of all frames to which $\{ \mathrm{T}_1, 4_1, \mathrm{B}_1 \}$ corresponds. For all formulas φ and sets Γ of formulas in $\mathsf{L}_1^{=}$, if φ is a consequence from Γ in $\mathbb{F}$, then φ is provable from Γ in $\mathsf{H}(\mathbf{tS5}_1^{=}\restriction \mathsf{L}_1^{=})$.

Chapter 4
Term-Sequence-Dyadic Deontic Logic

In Chapter 3 we have developed an expansion of term-modal logic, term-sequence-modal logic. In this chapter, we will see how we can apply term-sequence-modal logic in order to develop a deontic logic that accommodates normative conflicts in a way compatible with the thesis that norms are neither true nor false.

In Chapter 4, we will see how we can apply TSML in order to develop a deontic logic that accommodates normative conflicts in a way compatible with the thesis that norms are neither true nor false. Towards the development of such a deontic logic, we will first present TDDL in Section 4.1. Our TDDL is a combination of TSML and the minimal normal conditional logic **CK** introduced in Chellas [10, p. 269]. This logic has, for example, $\mathbf{O}_{ts}\varphi$ and $\mathbf{O}_{ts}(\varphi|\psi)$ as well-formed formulas. We will further provide a cut-free ordinary sequent calculus for TDDL in Section 4.2. In Section 4.3, we will use these results to argue that TDDL can accommodate two kinds of normative conflicts. We then claim that TDDL is consistent without the truth-ascription to (deontic) formulas of TDDL, as well as that TDDL can accommodate normative conflicts of the above kinds without the truth-ascription.

This chapter is based on Sawasaki and Sano [84] "Term-Sequence-Dyadic Deontic Logic".

4.1 Conditionalizing Term-Sequence-Modality

As before, this section first presents a syntax and Kripke semantics for term-sequence-dyadic deontic logic (4.1.1). Soundness is straightforward and strong completeness is proved by a suitable combination of the canonical models for TSML and conditional logic (4.1.2).

4.1.1 Syntax and Kripke Semantics

Definition 141. The *language* $\mathsf{L}_{\mathsf{TDDL}}$ of *term-sequence-dyadic deontic logic* (TDDL) consists of a countably infinite set $\mathrm{Var} = \{x, y, \dots\}$ of variables, a countable set $\mathrm{Con} = \{c, d, \dots\}$ of constant symbols, a countable set $\mathsf{Func} = \{f, g, \dots\}$ of function symbols each of which has a fixed finite arity more than zero, the union $\mathsf{Pred} = \bigcup_{n\in\mathbb{N}} \mathsf{Pred}_n$ of countably infinite sets $\mathsf{Pred}_n = \{P, Q, \dots\}$ of predicate symbols with arity n, the set $\{\bot, \rightarrow, \forall\}$ of logical constants, and the set $\mathsf{Mod} = \{\mathbf{O}_{(\cdot)}(\cdot|\cdot)\}$ of the *ternary modal operator* $\mathbf{O}_{(\cdot)}(\cdot|\cdot)$. A *term* t *in* $\mathsf{L}_{\mathsf{TDDL}}$ is recursively defined by

$$t ::= x \mid c \mid f(t_1, \dots, t_n)$$

where x is a variable, c is a constant symbol and f is a function symbol with arity n. A *formula* φ *in* $\mathsf{L}_{\mathsf{TDDL}}$ is recursively defined by

$$\varphi ::= Pt_1 \dots t_n \mid \bot \mid (\varphi \rightarrow \varphi) \mid \forall x\varphi \mid \mathbf{O}_{ts}(\varphi|\varphi)$$

where P is a predicate symbol with arity n, x is a variable, $t_1, \dots, t_n, t, s$ are terms. Instead of $\mathsf{L}_{\mathsf{TDDL}}$, we often write L when it is clear from the context. We also write $\mathsf{L}_{\mathsf{TDDL}}(V)$ to denote $\mathsf{L}_{\mathsf{TDDL}}$ in which the set of variables is V.

The other Boolean connectives and the existential quantifier are defined as in QML. We also define $\mathbf{P}_{ts}(\varphi|\psi)$, $\mathbf{O}_{ts}\varphi$ and $\mathbf{P}_{ts}\varphi$ as $\neg\mathbf{O}_{ts}(\neg\varphi|\psi)$, $\mathbf{O}_{ts}(\varphi|\top)$ and $\mathbf{P}_{ts}(\varphi|\top)$, respectively. The modal operators $\mathbf{O}_{ts}(\cdot|\psi)$ and $\mathbf{P}_{ts}(\cdot|\psi)$ represent obligation and permission relative to agents t, s and a condition ψ. The modal operators $\mathbf{O}_{ts}$ and $\mathbf{P}_{ts}\varphi$ represent obligation and permission relative only to agents t, s. The former operators are sometimes called *dyadic* and the latter operators *monadic*. On the other hand, modal operators like "$\mathbf{O}_t(\cdot|\psi)$" and "$\mathbf{O}_t$" are not defined here.

The intended readings of formulas $\mathbf{O}_{ts}(\varphi|\psi)$, $\mathbf{P}_{ts}(\varphi|\psi)$, $\mathbf{O}_{ts}$ and $\mathbf{P}_{ts}\varphi$ are presented in Table 4.1.

Formulas	Intended Readings
$\mathbf{O}_{ts}\varphi$	agent t has an obligation towards agent s to see to it that φ
$\mathbf{O}_{ts}(\varphi|\psi)$	agent t has an obligation towards agent s to see to it that φ given that ψ
$\mathbf{P}_{ts}\varphi$	agent t is permitted towards agent s to see to it that φ
$\mathbf{P}_{ts}(\varphi|\psi)$	agent t is permitted towards agent s to see to it that φ given that ψ

Table 4.1: The Intended Readings of Deontic Operators

The Kripke semantics for $\mathsf{L}_{\mathsf{TDDL}}$ is obtained by combining the semantics for $\mathsf{L}_{\mathsf{TSML}}$ with the semantics for a language of the conditional logic **CK** introduced in Chellas [9, 10].

Definition 142. A *frame* (*for* $\mathsf{L}_{\mathsf{TDDL}}$) is a tuple $\mathfrak{F} = (W, f, (D_w)_{w \in W})$, where W is a non-empty set of worlds; D_w is a non-empty domain of a world w[1]; f is an *indexed selection function* that maps each pair $(d, e) \in \bigcup_{w \in W}(D_w^2)$ to a *selection function*, i.e., $f_{de} : W \times 2^W \to 2^W$, where 2^W is the power set $\{ X \mid X \subseteq W \}$ of W; $\mathfrak{F}$ has *increasing domains* (*in TDDL*):

$$\text{for all } w, v \in W, X \subseteq W \text{ and } (d, e) \in D_w^2, \text{ if } v \in f_{de}(w, X) \text{ then } D_w \subseteq D_v.$$

Given a frame $\mathfrak{F}$, a *model* (*for* $\mathsf{L}_{\mathsf{TDDL}}$) is a tuple $\mathfrak{M} = (\mathfrak{F}, I)$, where I is an interpretation that maps each predicate symbol P with arity n and world w to a subset $I(P, w)$ of D_w^n; each constant symbol to an agent $I(c) \in \bigcap_{w \in W} D_w$; each function symbol with arity n to an n-place function $I(f) : (\bigcup_w D_w)^n \to \bigcup_w D_w$ such that $I(f)(d_1, \ldots, d_n) \in D_w$ for all $(d_1, \ldots, d_n) \in D_w^n$. We define D as $\bigcup_w D_w$. We sometimes write `True` and `False` instead of D_w^0 and $\emptyset$, respectively.

Let us illustrate what a selection function intuitively does. As well known in the literature, a set of worlds can be understood as a *proposition*. (see e.g. a footnote of Lewis [52, p. 46].) Thus a selection function f_{de} can be seen as "picking out" a proposition $f_{de}(w, X)$ for each world w and proposition X. Moreover, it can be treated as a binary relation $R_{(d,e),X}$ on W relative to (d, e) and X, because we can define it by

$$v \in f_{de}(w, X) \quad \text{iff} \quad w R_{(d,e),X} v.$$

Thus, we could develop a Kripke semantics for TDDL using $R_{(d,e),X}$. However, we adopted the current form since conditional logic has been traditionally developed using selection functions. (e.g. Chellas [9, 10].)

The notion of assignment is defined in the same way as in TSML. The satisfaction relation for $\mathsf{L}_{\mathbf{TDDL}}$ is also the same as for $\mathsf{L}_{\mathbf{TDDL}}$, except for the clause of $\mathbf{O}_{ts}(\varphi | \psi)$.

Definition 143. Let $\mathfrak{M} = (W, f, (D_w)_{w \in W}, I)$ be a model, w a world, α an assignment, and φ a formula such that φ is an α_w-formula. The *satisfaction relation* $\mathfrak{M}, w, \alpha \models \varphi$ *between* $\mathfrak{M}, w, \alpha$ *and* φ (*for* $\mathsf{L}_{\mathsf{TDDL}}$) is defined as follows.

$$\begin{array}{lll}
\mathfrak{M}, w, \alpha \models P t_1 \ldots t_n & \text{iff} & \alpha(t_1, \ldots, t_n) \in I(P, w) \\
\mathfrak{M}, w, \alpha \not\models \bot & & \\
\mathfrak{M}, w, \alpha \models \varphi \to \psi & \text{iff} & \mathfrak{M}, w, \alpha \models \varphi \text{ implies } \mathfrak{M}, w, \alpha \models \psi \\
\mathfrak{M}, w, \alpha \models \forall x \varphi & \text{iff} & \text{for all agents } d \in D_w,\ \ \mathfrak{M}, w, \alpha(x|d) \models \varphi \\
\mathfrak{M}, w, \alpha \models \mathbf{O}_{ts}(\varphi | \psi) & \text{iff} & \text{for all worlds } v \in W, \\
& & v \in f_{\alpha(ts)}(w, [\![\psi]\!]_\alpha^M) \text{ implies } M, \alpha, v \models \varphi
\end{array}$$

where $[\![\psi]\!]_\alpha^{\mathfrak{M}} = \{ v \in W \mid \mathfrak{M}, \alpha, v \models \psi \}$. For a set Γ of formulas, $\mathfrak{M}, w, \alpha \models \Gamma$ means that $\mathfrak{M}, w, \alpha \models \psi$ for all formulas $\psi \in \Gamma$.

[1]When Con $\neq \emptyset$, it is also required that $\bigcap_{w \in W} D_w \neq \emptyset$.

Selection functions make it possible to express *non-monotonicity* of conditional obligations relative to agents. For example, consider a sentence "Adam has an obligation towards Barbara to be with her given that she is old." Moreover, consider another sentence "Adam has an obligation towards Barbara to be with her given that she is old but not lonely." Intuitively, the former does not imply the latter since they are under different conditions. Selection functions can capture the intuition as follows. Let Wab, Ob and Lb represent "Adam is with Barbara," "Barbara is old" and "Barbara is lonely," respectively. Let also $\mathfrak{M}$, α and w be a model, an assignment and a world, respectively. Then, $\mathfrak{M}, w, \alpha \models \mathbf{O}_{ab}(Wab|Ob)$ does not imply $\mathfrak{M}, w, \alpha \models \mathbf{O}_{ts}(Wab|Ob \wedge \neg Lb)$ since $f_{\alpha(ab)}(w, [\![Ob]\!]^M_\alpha)$ and $f_{\alpha(ab)}(w, [\![Ob \wedge \neg Lb]\!]^M_\alpha)$ may be thoroughly different.

Definition 144. Let φ be a formula and Γ a set of formulas.

1. φ is *valid in the class of all frames* (*for* $\mathsf{L}_{\mathsf{TDDL}}$), denoted by $\models \varphi$, if $(\mathfrak{F}, I), w, \alpha \models \varphi$ for for all frames $\mathfrak{F}$, interpretations I, worlds w, assignments α such that φ is an α_w-formula.
2. φ is a *consequence from* Γ *in the class of all frames* (*for* $\mathsf{L}_{\mathsf{TDDL}}$) if $(\mathfrak{F}, I), w, \alpha \models \Gamma$ implies $(\mathfrak{F}, I), w, \alpha \models \varphi$ for all frames $\mathfrak{F}$, interpretations I, worlds w, assignments α such that ψ is an α_w-formula for all formulas ψ in $\Gamma \cup \{ \varphi \}$.

4.1.2 Hilbert System $\mathsf{H}(\mathbf{TDDL})$

A Hilbert system $\mathsf{H}(\mathbf{TDDL})$ consists of axioms and inference rules in Table 4.2. The axioms and inference rules other than (RCEA) and (RCK) are from first order logic, and (RCEA) and (RCK) are term-sequence-modal expansions of the corresponding rules of the conditional logic **CK**. When $n = 0$, (RCK) essentially means that we may infer $\mathbf{O}_{ts}(\varphi|\psi)$ from φ. The notion of proof in $\mathsf{H}(\mathbf{TDDL})$ is defined as in $\mathsf{H}(\mathbf{K\Sigma})$.

(Taut1)	$\varphi \to (\psi \to \varphi)$
(Taut2)	$(\varphi \to (\psi \to \gamma)) \to ((\varphi \to \psi) \to (\varphi \to \gamma))$
(Taut3)	$(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$
(U)	$\forall x\varphi \to \varphi(y/x)$
(MP)	From $\varphi \to \psi$ and φ, we may infer ψ
(Gen)	From $\varphi \to \psi(y/x)$, we may infer $\varphi \to \forall x\psi$ if $y \notin \mathsf{FV}(\varphi, \forall x\psi)$
(RCEA)	From $\psi \leftrightarrow \psi'$, we may infer $\mathbf{O}_{ts}(\varphi|\psi) \leftrightarrow \mathbf{O}_{ts}(\varphi|\psi')$
(RCK)	From $(\varphi_1 \wedge \cdots \wedge \varphi_n) \to \varphi$, we may infer $(\mathbf{O}_{ts}(\varphi_1|\psi) \wedge \cdots \wedge \mathbf{O}_{ts}(\varphi_n|\psi)) \to \mathbf{O}_{ts}(\varphi|\psi)$

Table 4.2: Hilbert system $\mathsf{H}(\mathbf{TDDL})$

Theorem 145. (Soundeness of $\mathsf{H}(\mathbf{TDDL})$) For all formulas φ, if a formula φ is provable in $\mathsf{H}(\mathbf{TDDL})$, then φ is valid.

Proof. By induction on the length of a proof of φ. No difficulty arises in first order cases. One can also easily check that (RCEA) and (RCK) preserve the validity. ■

As before, we prove the strong completeness of $\mathsf{H}(\mathbf{TDDL})$ via construction of the canonical model defined below. The canonical model for TDDL is a suitable combination of the canonical models for TSML and conditional logic. The former model was defined in Section 3.1. The latter models for propositional and first order conditional logics are introduced in [52, 9] and [13, 14], respectively.

Recall that we write $\mathsf{L}_{\mathsf{TDDL}}(V)$ to denote $\mathsf{L}_{\mathsf{TDDL}}$ in which the set of variables is V.

Definition 146. We define Var^+ as $\mathsf{Var} \cup \mathsf{Var}'$, where Var' is a *fresh* countably infinite set of variables disjoint from Var. Given $V \subseteq \mathsf{Var}^+$, the set $\mathsf{Term}(V)$ refers to the set of all terms in $\mathsf{L}_{\mathsf{TDDL}}(V)$. Given a set Γ of formulas, $\mathsf{L}_{\mathsf{TDDL}}(\Gamma)$ and $\mathsf{Term}(\Gamma)$ denote $\mathsf{L}_{\mathsf{TDDL}}(\mathsf{FV}(\Gamma))$ and $\mathsf{Term}(\mathsf{FV}(\Gamma))$, respectively. Given $V, V' \subseteq \mathsf{Var}^+$, by a notation $V \sqsubset V'$ we mean that $V \subseteq V'$ and $V' \backslash V$ is countably infinite.

Proposition 147. Let Γ be an MCS in $\mathsf{L}(\mathsf{Var}^+)$ and φ, ψ formulas in $\mathsf{L}(\Gamma)$.

1. $\Gamma \vdash_{\mathsf{H}(\mathbf{TDDL})} \varphi$ iff $\varphi \in \Gamma$.
2. If $\varphi \in \Gamma$ and $\vdash_{\mathsf{H}(\mathbf{TDDL})} \varphi \to \psi$, then $\psi \in \Gamma$.
3. $\bot \notin \Gamma$.
4. $\varphi \to \psi \in \Gamma$ iff $\varphi \notin \Gamma$ or $\psi \in \Gamma$.

Lemma 148. (Lindenbaum Lemma) Let Γ be a consistent set in $\mathsf{L}(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma) \sqsubset \mathsf{Var}^+$. There exists a witnessed MCS Γ^+ in $\mathsf{L}(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma^+) \sqsubset \mathsf{Var}^+$ and $\Gamma \subseteq \Gamma^+$.

Definition 149. Define

$$W^{\mathrm{c}} := \left\{ \Gamma \mid \Gamma \text{ is a witnessed MCS in } \mathsf{L}(\mathsf{Var}^+) \text{ such that } \mathsf{FV}(\Gamma) \sqsubset \mathsf{Var}^+ \right\};$$
$$|\varphi| := \{ \Gamma \in W^{\mathrm{c}} \mid \varphi \in \Gamma \}.$$

The *canonical model* $\mathfrak{M}^{\mathrm{c}}$ is the tuple $(W^{\mathrm{c}}, f^{\mathrm{c}}, (D^{\mathrm{c}}_{\Gamma})_{\Gamma \in W^{\mathrm{c}}}, I^{\mathrm{c}})$ where

- $D^{\mathrm{c}}_{\Gamma} = \mathsf{Term}(\Gamma)$;
- f^{c}_{ts} is defined as follows:
  - if $X = |\psi|$ for some formula ψ in $\mathsf{L}(\Gamma)$,

$$\Delta \in f^{\mathrm{c}}_{ts}(\Gamma, |\psi|) \quad \text{iff} \quad \{ \varphi \mid \mathbf{O}_{ts}(\varphi|\psi) \in \Gamma \} \subseteq \Delta;$$

  - otherwise, $f^{c}_{ts}(\Gamma, X) := \varnothing$;
- $(t_1, \ldots, t_n) \in I^{c}(P, \Gamma)$ iff $Pt_1 \ldots t_n \in \Gamma$;
- $I^{c}(c) = c$.
- $I^{c}(f)(t_1, \ldots, t_n) := f(t_1, \ldots, t_n)$.

The *canonical assignment* is the assignment $\iota : \mathrm{Var}^{+} \to D^{\Lambda}$ defined by $\iota(x) := x$.

Proposition 150. The canonical model's f^{c} is well-defined.

Proof. Take any $\Gamma \in W^{c}$ and $(t, s) \in (D^{c}_{\Gamma})^2$, and suppose $|\psi| = |\psi'|$. We show the following equivalence

$$\mathbf{O}_{ts}(\gamma|\psi) \in \Gamma \quad \text{iff} \quad \mathbf{O}_{ts}(\gamma|\psi') \in \Gamma$$

holds for any formula ψ, ψ' in $\mathsf{L}(\Gamma)$. We first establish $\vdash_{\mathsf{H}(\mathbf{TDDL})} \psi \to \psi'$. Suppose not. Since the set $\{\psi, \neg\psi'\}$ is then consistent, by Lindenbaum Lemma (Lemma 148) we have an MCS $\Delta \in W^{c}$ such that $\{\psi, \neg\psi'\} \subseteq \Delta$. Thus, we can deduce by $|\psi| = |\psi'|$ that $\psi', \neg\psi' \in \Delta$, which contradicts the consistency of Δ. Hence $\vdash_{\mathsf{H}(\mathbf{TDDL})} \psi \to \psi'$. Similarly, $\vdash_{\mathsf{H}(\mathbf{TDDL})} \psi' \to \psi$. As $\vdash_{\mathsf{H}(\mathbf{TDDL})} \psi \leftrightarrow \psi'$ holds, it follows from (RCEA) that $\vdash_{\mathsf{H}(\mathbf{TDDL})} \mathbf{O}_{ts}(\gamma|\psi) \leftrightarrow \mathbf{O}_{ts}(\gamma|\psi')$. This gives the above equivalence. ■

Proposition 151. The canonical model *is* a model.

Proof. We confirm that the frame $\mathfrak{F}^{c} = (W^{c}, f^{c}, (D^{c}_{\Gamma})_{\Gamma \in W^{c}})$ of the canonical model has increasing domains, i.e., for all $\Gamma, \Delta \in W^{c}$, $X \subseteq W^{c}$ and $(t, s) \in (D^{c}_{\Gamma})^2$, if $\Delta \in f_{ts}(\Gamma, X)$ then $D_{\Gamma} \subseteq D_{\Delta}$. Take any $\Gamma, \Delta \in W^{c}$, $X \subseteq W^{c}$ and $(t, s) \in (D^{c}_{\Gamma})^2$. Suppose $\Delta \in f^{c}_{ts}(\Gamma, X)$ and $u \in D^{c}_{\Gamma}$. By $f^{c}_{ts}(\Gamma, X) \neq \varnothing$, it holds that $\{\gamma \mid \mathbf{O}_{ts}(\gamma|\psi) \in \Gamma\} \subseteq \Delta$ for some formula ψ in $\mathsf{L}(\Gamma)$ such that $X = |\psi|$. Also, $\mathbf{O}_{ts}(Pu \to Pu|\psi) \in \Gamma$ holds. Therefore $Pu \to Pu \in \Delta$ hence $u \in D^{c}_{\Delta}$, as required. ■

Lemma 152. (Existence Lemma) If $\neg\mathbf{O}_{ts}(\varphi|\psi) \in \Gamma \in W^{c}$, then there exists some $\Delta \in W^{c}$ such that $\neg\varphi \in \Delta$ and $\Delta \in f^{c}_{ts}(\Gamma, |\psi|)$.

Proof. The proof is analogous to the proof of Existence Lemma for TSML (Lemma 92). Suppose $\neg\mathbf{O}_{ts}(\varphi|\psi) \in \Gamma \in W^{c}$. To establish that $\Delta_0 := \{\neg\varphi\} \cup \{\gamma \mid \mathbf{O}_{ts}(\gamma|\psi) \in \Gamma\}$ is consistent, suppose not. Then, (1) $\vdash_{\mathsf{H}(\mathbf{TDDL})} (\gamma_1 \wedge \cdots \wedge \gamma_n) \to \varphi$ for some $\gamma_1, \ldots, \gamma_n$ such that (2) $\mathbf{O}_{ts}(\gamma_i|\psi) \in \Gamma$. We obtain $\Gamma \vdash_{\mathsf{H}(\mathbf{TDDL})} \bot$ as follows.

1.	$\vdash (\gamma_1 \wedge \cdots \wedge \gamma_n) \to \varphi$	(1)
2.	$\vdash (\mathbf{O}_{ts}(\gamma_1|\psi) \wedge \cdots \wedge \mathbf{O}_{ts}(\gamma_n|\psi)) \to \mathbf{O}_{ts}(\varphi|\psi)$	1, (RCK)

3. $\Gamma \vdash \mathbf{O}_{ts}(\gamma_1|\psi) \wedge \cdots \wedge \mathbf{O}_{ts}(\gamma_n|\psi)$ (2)
4. $\Gamma \vdash \mathbf{O}_{ts}(\varphi|\psi)$ 2, 3, MP
5. $\Gamma \vdash \neg\mathbf{O}_{ts}(\varphi|\psi)$ $\neg\mathbf{O}_{ts}(\varphi|\psi) \in \Gamma$
6. $\Gamma \vdash \bot$ 4, 5, PC

However, Γ should be consistent so a contradiction. Thus Δ_0 is consistent. By applying Lindenbaum Lemma (Lemma 148) to Δ_0, we obtain some $\Delta \in W^{\mathrm{c}}$ such that $\neg\varphi$ and $\Delta \in f^{\mathrm{c}}_{ts}(\Gamma, |\psi|)$. ■

Lemma 153. (Truth Lemma) Let $\mathfrak{M}^{\mathrm{c}}$ be the canonical model and ι the canonical assignment. For all MCSs $\Gamma \in W^{\Lambda}$ and formulas φ in $\mathsf{L}(\Gamma)$,

$$\mathfrak{M}^{\mathrm{c}}, \Gamma, \iota \models \varphi \quad \text{iff} \quad \varphi \in \Gamma.$$

Proof. By induction on the length of φ. We skip cases involving first order logic and prove a case in which φ is $\mathbf{O}_{ts}(\gamma|\psi)$. Note that $\iota(ts) = ts$ by definition.

For the right-to-left direction, suppose $\mathbf{O}_{ts}(\gamma|\psi) \in \Gamma$. To show $\mathfrak{M}^{\mathrm{c}}, \Gamma, \iota \models \mathbf{O}_{ts}(\gamma|\psi)$, consider any world $\Delta \in W^{\mathrm{c}}$ such that $\Delta \in f^{\mathrm{c}}_{ts}(\Gamma, [\![\psi]\!]^{\mathfrak{M}^{\mathrm{c}}}_{\iota})$. What should be established is $\mathfrak{M}, \Delta, \iota \models \gamma$. Recall $[\![\psi]\!]^{\mathfrak{M}^{\mathrm{c}}}_{\iota} = \{\, \Theta \mid \mathfrak{M}^{\mathrm{c}}, \Theta, \iota \models \psi \,\}$ and $|\psi| = \{\, \Theta \in W^{\mathrm{c}} \mid \psi \in \Theta \,\}$. By $\Delta \in f^{\mathrm{c}}_{ts}(\Gamma, [\![\psi]\!]^{\mathfrak{M}^{\mathrm{c}}}_{\iota})$ and inductive hypothesis, we have $\Delta \in f^{\mathrm{c}}_{ts}(\Gamma, |\psi|)$. Then $\{\, \gamma' \mid \mathbf{O}_{ts}(\gamma'|\psi) \in \Gamma \,\} \subseteq \Delta$. As we have supposed $\mathbf{O}_{ts}(\gamma|\psi) \in \Gamma$, it follows that $\gamma \in \Delta$. Therefore, by inductive hypothesis, $\mathfrak{M}^{\mathrm{c}}, \iota, \Delta \models \gamma$ holds.

For the left-to-right direction, we show the contraposition. Suppose $\mathbf{O}_{ts}(\gamma|\psi) \notin \Gamma$. What should be established is $\mathfrak{M}^{\mathrm{c}}, \iota, \Gamma \not\models \mathbf{O}_{ts}(\gamma|\psi)$. Then $\neg\mathbf{O}_{ts}(\gamma|\psi) \in \Gamma$ since $\mathbf{O}_{ts}(\gamma|\psi)$ is a formula in $\mathsf{L}(\Gamma)$. By Existence Lemma (Lemma 152), we obtain some MCS $\Delta \in W^{\mathrm{c}}$ such that $\neg\gamma \in \Delta$ and $\Delta \in f^{\mathrm{c}}_{ts}(\Gamma, |\psi|)$. As $\neg\gamma \in \Delta$ implies $\gamma \notin \Delta$, inductive hypothesis provides that $\Delta \in f^{\mathrm{c}}_{ts}(\Gamma, [\![\psi]\!]^{\mathfrak{M}^{\mathrm{c}}}_{\iota})$ but $\mathfrak{M}^{\mathrm{c}}, \iota, \Delta \not\models \gamma$. This implies $\mathfrak{M}^{\mathrm{c}}, \iota, \Gamma \not\models \mathbf{O}_{ts}(\gamma|\psi)$. ■

Theorem 154. (Strong completeness of $\mathsf{H}(\mathbf{TDDL})$) For all formulas φ and sets Γ of formulas, if φ is a consequence from Γ in $\mathbb{F}_{\Sigma}$, then φ is provable from Γ in $\mathsf{H}(\mathbf{TDDL})$.

Proof. Similarly done as in the proof of Theorem 95. ■

4.2 Proof-Theoretic Analysis

In this section we first present a sequent calculus equipollent to $\mathsf{H}(\mathbf{TDDL})$ (4.2.1). We then prove by a similar argument as in TSML that it admits the cut elimination theorem and the Craig interpolation theorem (4.2.2).

4.2.1 Sequent Calculus G(TDDL)

A sequent calculus G(**TDDL**) is presented by Table 4.3, where (**O**) is a logical rule. The notion of derivation in G(**TDDL**) is defined as in G(**K**Σ).

$$\frac{}{\varphi \Rightarrow \varphi}\ (id) \qquad\qquad \frac{}{\bot \Rightarrow}\ (\bot)$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}\ (\Rightarrow w) \qquad\qquad \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (w \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}\ (\Rightarrow c) \qquad\qquad \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (c \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Theta \Rightarrow \Sigma}{\Gamma, \Theta \Rightarrow \Delta, \Sigma}\ (Cut)$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}\ (\Rightarrow \to) \qquad\qquad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Theta \Rightarrow \Sigma}{\varphi \to \psi, \Gamma, \Theta \Rightarrow \Delta, \Sigma}\ (\to \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi(y/x)}{\Gamma \Rightarrow \Delta, \forall x \varphi}\ (\Rightarrow \forall)^{\dagger} \qquad\qquad \frac{\varphi(t/x), \Gamma \Rightarrow \Delta}{\forall x \varphi, \Gamma \Rightarrow \Delta}\ (\forall \Rightarrow)$$

$$\frac{\psi_1 \Rightarrow \psi \quad \psi \Rightarrow \psi_1 \qquad \cdots \qquad \psi_n \Rightarrow \psi \quad \psi \Rightarrow \psi_n \qquad \varphi_1, \ldots, \varphi_n \Rightarrow \varphi}{\mathbf{O}_{ts}(\varphi_1|\psi_1), \ldots, \mathbf{O}_{ts}(\varphi_n|\psi_n) \Rightarrow \mathbf{O}_{ts}(\varphi|\psi)}\ (\mathbf{O})$$

where †: y is not a free variable in $\Gamma, \Delta, \forall x \varphi$.

Table 4.3: Sequent calculus G(**TDDL**)

For short, we write the rule (**O**) as

$$\frac{(\psi_i \Leftrightarrow \psi)_{1 \leqslant i \leqslant n} \qquad \varphi_1, \ldots, \varphi_n \Rightarrow \varphi}{\mathbf{O}_{ts}(\varphi_1|\psi_1), \ldots, \mathbf{O}_{ts}(\varphi_n|\psi_n) \Rightarrow \mathbf{O}_{ts}(\varphi|\psi)}\ (\mathbf{O})$$

where $\psi_i \Leftrightarrow \psi$ is a pair of two sequents $\psi_i \Rightarrow \psi$ and $\psi \Rightarrow \psi_i$. The rule (**O**) is imported in the form of the two-sided rule from the one-sided rule (CK_g) introduced in Pattinson and Lutz [77, p. 14]. One may regard (**O**) as a rule "applying (`RCK`) together with (`RCEA`)."

Example 155. A sequent

$$\forall x \forall y (Mxy \to \mathbf{O}_{xy}(Wxy \land Txy|Oy \land Ly)), Mab \Rightarrow \mathbf{O}_{ab}(Wab|Ob \land Lb)$$

is derivable in G(**TDDL**) as follows, where $(\land \Rightarrow)$ is a derived rule in G(**TDDL**).

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(id)}
\UnaryInfC{$Mab \Rightarrow Mab$}
\AxiomC{$Ob \land Lb$}
\AxiomC{}
\RightLabel{(id)}
\UnaryInfC{$Wab \Rightarrow Wab$}
\RightLabel{$(\land \Rightarrow)$}
\UnaryInfC{$Wab \land Tab \Rightarrow Wab$}
\RightLabel{$(\mathbf{O})$}
\BinaryInfC{$\mathbf{O}_{ab}(Wab \land Tab|Ob \land Lb) \Rightarrow \mathbf{O}_{ab}(Wab|Ob \land Lb)$}
\RightLabel{$(\rightarrow \Rightarrow)$}
\BinaryInfC{$Mab \rightarrow \mathbf{O}_{ab}(Wab \land Tab|Ob \land Lb), Mab \Rightarrow \mathbf{O}_{ab}(Wab|Ob \land Lb)$}
\RightLabel{$(\forall \Rightarrow)$}
\doubleLine
\UnaryInfC{$\forall x \forall y(Mxy \rightarrow \mathbf{O}_{xy}(Bxy \land Txy|Ox \land Lx)), Mab \Rightarrow \mathbf{O}_{ab}(Wab|Ob \land Lb)$}
\end{prooftree}
$$

Similarly to Proposition 23, we have the equipollence of H(**TDDL**) and G(**TDDL**). Note that the equipollence does not imply that we have to ascribe truth values to formulas in TDDL.

Proposition 156. (Equipollence of H(**TDDL**) and G(**TDDL**)) A formula φ is provable in H(**TDDL**) iff a sequent $\Rightarrow \varphi$ is derivable in G(**TDDL**).

Proof. The proof is analogous to the proof of Proposition 109. The left-to-right direction is established by induction on the length of a proof of φ. We check only a case in which a proved formula is obtained from a preceding formula by (`RCEA`) or (`RCK`).

Case of (`RCEA`)**.** The proved formula has the form $\mathbf{O}_{ts}(\varphi|\psi) \leftrightarrow \mathbf{O}_{ts}(\varphi'|\psi)$ and is obtained from a preceding formula of the form $\psi \leftrightarrow \psi'$. A derivation of it in G(**TDDL**) is as follows.

$$
\begin{prooftree}
\AxiomC{$\psi \Leftrightarrow \psi'$}
\AxiomC{}
\RightLabel{(id)}
\UnaryInfC{$\gamma \Rightarrow \gamma$}
\RightLabel{$(\mathbf{O})$}
\BinaryInfC{$\mathbf{O}_{ts}(\gamma|\psi) \Rightarrow \mathbf{O}_{ts}(\gamma|\psi')$}
\RightLabel{$(\Rightarrow \rightarrow)$}
\UnaryInfC{$\Rightarrow \mathbf{O}_{ts}(\gamma|\psi) \rightarrow \mathbf{O}_{ts}(\gamma|\psi')$}
\AxiomC{$\psi' \Leftrightarrow \psi$}
\AxiomC{}
\RightLabel{(id)}
\UnaryInfC{$\gamma \Rightarrow \gamma$}
\RightLabel{$(\mathbf{O})$}
\BinaryInfC{$\mathbf{O}_{ts}(\gamma|\psi') \Rightarrow \mathbf{O}_{ts}(\gamma|\psi)$}
\RightLabel{$(\Rightarrow \rightarrow)$}
\UnaryInfC{$\Rightarrow \mathbf{O}_{ts}(\gamma|\psi') \rightarrow \mathbf{O}_{ts}(\gamma|\psi)$}
\RightLabel{$(\Rightarrow \land)$}
\BinaryInfC{$\Rightarrow \mathbf{O}_{ts}(\gamma|\psi) \leftrightarrow \mathbf{O}_{ts}(\gamma|\psi')$}
\end{prooftree}
$$

where $(\Rightarrow \land)$ is a derived rule in G(**TDDL**), and $\psi \Rightarrow \psi'$ and $\psi' \Rightarrow \psi$ are derivable by inductive hypothesis and (Cut).

Case of (`RCK`)**.** The proved formula has the form $(\mathbf{O}_{ts}(\varphi_1|\psi) \land \cdots \land \mathbf{O}_{ts}(\varphi_n|\psi)) \rightarrow \mathbf{O}_{ts}(\varphi|\psi)$ and is obtained from a preceding formula of the form $(\varphi_1 \land \cdots \land \varphi_n) \rightarrow \varphi$. A derivation of it in G(**TDDL**) is as follows.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(id)}
\UnaryInfC{$\psi \Leftrightarrow \psi$}
\AxiomC{$\varphi_1, \ldots, \varphi_n \Rightarrow \varphi$}
\RightLabel{$(\mathbf{O})$}
\BinaryInfC{$\mathbf{O}_{ts}(\varphi_1|\psi), \ldots, \mathbf{O}_{ts}(\varphi_n|\psi) \Rightarrow \mathbf{O}_{ts}(\varphi|\psi)$}
\RightLabel{$(\land \Rightarrow)$}
\UnaryInfC{$\mathbf{O}_{ts}(\varphi_1|\psi) \land \cdots \land \mathbf{O}_{ts}(\varphi_n|\psi) \Rightarrow \mathbf{O}_{ts}(\varphi|\psi)$}
\RightLabel{$(\Rightarrow \rightarrow)$}
\UnaryInfC{$\Rightarrow (\mathbf{O}_{ts}(\varphi_1|\psi) \land \cdots \land \mathbf{O}_{ts}(\varphi_n|\psi)) \rightarrow \mathbf{O}_{ts}(\varphi|\psi)$}
\end{prooftree}
$$

where $(\land \Rightarrow)$ is a derived rule in G(**TDDL**) and $\varphi_1, \ldots, \varphi_n \Rightarrow \varphi$ is derivable by inductive hypothesis and (Cut).

The right-to-left direction immediately follows from the claim that $\vdash_{\mathsf{G}(\mathbf{TDDL})} \Gamma \Rightarrow \Delta$ implies $\vdash_{\mathsf{H}(\mathbf{TDDL})} \bigwedge \Gamma \rightarrow \bigvee \Delta$. The claim is shown by induction on the height of a derivation of $\Gamma \Rightarrow \Delta$. In the case that the derivation ends with an application of $(\mathbf{O})$, we obtain a proof in $\mathsf{H}(\mathbf{TDDL})$ of the formula corresponding to $\Gamma \Rightarrow \Delta$ by applying (RCK) together with (RCEA). ■

4.2.2 Cut Elimination

The cut elimination theorem for $\mathsf{G}(\mathbf{TDDL})$ is proved along the same lines as the proof of the cut elimination theorem of TSML (Theorem 115). Recall the extended rule (Cut^*) of (Cut)

$$\frac{\Gamma \Rightarrow \Delta, \varphi^l \qquad \varphi^m, \Xi \Rightarrow \Pi}{\Gamma, \Xi \Rightarrow \Delta, \Pi}\ (Cut^*)$$

where l, m can be zero. Let $\mathsf{G}^-(\mathbf{TDDL})$ be the calculus obtained from $\mathsf{G}(\mathbf{TDDL})$ by removing (Cut), and $\mathsf{G}^*(\mathbf{TDDL})$ the calculus obtained from $\mathsf{G}(\mathbf{TDDL})$ by replacing (Cut) with (Cut^*). In a similar vein as in $\mathsf{G}(\mathbf{tK\Sigma})$, we define notions of principal formulas, the (Cut^*)-bottom form, and the grade and the weight of an application of (Cut^*) in $\mathsf{G}(\mathbf{TDDL})$, except that in an application of $(\mathbf{O})$ of $\mathsf{G}(\mathbf{TDDL})$:

$$\frac{(\psi_i \Leftrightarrow \psi)_{1 \leqslant i \leqslant n} \qquad \varphi_1, \ldots, \varphi_n \Rightarrow \varphi}{\mathbf{O}_{ts}(\varphi_1|\psi_1), \ldots, \mathbf{O}_{ts}(\varphi_n|\psi_n) \Rightarrow \mathbf{O}_{ts}(\varphi|\psi)}\ (\mathbf{O})$$

the formulas $\mathbf{O}_{ts}(\varphi_i|\psi_i)$ and $\mathbf{O}_{ts}(\varphi|\psi)$ are principal. In what follows, we assume that free variables and bound variables in derivations are thoroughly separated.

Theorem 157. (Cut Elimination) If $\Gamma \Rightarrow \Delta$ is derivable in $\mathsf{G}(\mathbf{TDDL})$ then it is also derivable in $\mathsf{G}^-(\mathbf{TDDL})$.

Proof. It suffices to show that $\vdash_{\mathsf{G}^*} \Gamma \Rightarrow \Delta$ implies $\vdash_{\mathsf{G}^-} \Gamma \Rightarrow \Delta$, which is immediately obtained from the following claim: if there is a derivation $\mathfrak{D}$ of the (Cut^*)-bottom form of a sequent $\Gamma \Rightarrow \Delta$ in $\mathsf{G}^*(\mathbf{TDDL})$, there is also a derivation of $\Gamma \Rightarrow \Delta$ in $\mathsf{G}^-(\mathbf{TDDL})$. The claim is established by double induction on a pair $(g(\sigma), w(\sigma))$ of the grade $g(\sigma)$ and the weight $w(\sigma)$ of the only application σ of (Cut^*) in a derivation $\mathfrak{D}$ of the (Cut^*)-bottom form. We skip cases involving first order logic and confine ourselves to a case in which both of the left and right upper sequents of the only one application σ of (Cut^*) in $\mathfrak{D}$ are obtained by $(\mathbf{O})$. Thus, $\mathfrak{D}$ is now of the form

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ (\psi_i \Leftrightarrow \psi)_{1 \leqslant i \leqslant h}\end{matrix} \qquad \begin{matrix}\vdots \\ \overrightarrow{\varphi_h} \Rightarrow \varphi\end{matrix}}{\overrightarrow{\mathbf{O}_{ts}(\varphi_h|\psi_h)} \Rightarrow \mathbf{O}_{ts}(\varphi|\psi)}\ (\mathbf{O}) \quad \dfrac{\begin{matrix}\vdots \\ (\psi \Leftrightarrow \psi')^m\end{matrix} \qquad \begin{matrix}\vdots \\ (\psi'_j \Leftrightarrow \psi')_{1 \leqslant j \leqslant k}\end{matrix} \qquad \begin{matrix}\vdots \\ \varphi^m, \overrightarrow{\varphi'_k} \Rightarrow \varphi'\end{matrix}}{(\mathbf{O}_{ts}(\varphi|\psi))^m, \overrightarrow{\mathbf{O}_{ts}(\varphi'_k|\psi'_k)} \Rightarrow \mathbf{O}_{ts}(\varphi'|\psi')}\ (\mathbf{O})}{\overrightarrow{\mathbf{O}_{ts}(\varphi_h|\psi_h)}, \overrightarrow{\mathbf{O}_{ts}(\varphi'_k|\psi'_k)}, \Rightarrow \mathbf{O}_{ts}(\varphi'|\psi')}\ (Cut)^*$$

where $\overrightarrow{\mathbf{O}_{ts}(\varphi_h|\psi_h)} = \mathbf{O}_{ts}(\varphi_1|\psi_1), \ldots, \mathbf{O}_{ts}(\varphi_h|\psi_h)$, $\overrightarrow{\mathbf{O}_{ts}(\varphi'_k|\psi'_k)} = \mathbf{O}_{ts}(\varphi'_1|\psi'_1), \ldots, \mathbf{O}_{ts}(\varphi'_k|\psi'_k)$, $\overrightarrow{\varphi_h} = \varphi_1, \ldots, \varphi_h$, and $\overrightarrow{\varphi'_k} = \varphi'_1, \ldots, \varphi'_k$. Put derivations $\mathfrak{D}^i_1$, $\mathfrak{D}^i_2$ and $\mathfrak{D}_3$ as

$$\mathfrak{D}^i_1 = \dfrac{\begin{matrix}\vdots\\ \psi_i \Rightarrow \psi\end{matrix} \quad \begin{matrix}\vdots\\ \psi \Rightarrow \psi'\end{matrix}}{\psi_i \Rightarrow \psi'}\ (Cut^*) \qquad \mathfrak{D}^i_2 = \dfrac{\begin{matrix}\vdots\\ \psi' \Rightarrow \psi\end{matrix} \quad \begin{matrix}\vdots\\ \psi \Rightarrow \psi_i\end{matrix}}{\psi' \Rightarrow \psi_i}\ (Cut^*)$$

$$\mathfrak{D}_3 = \dfrac{\begin{matrix}\vdots\\ \overline{\varphi_h} \Rightarrow \varphi\end{matrix} \quad \begin{matrix}\vdots\\ \varphi^n, \overline{\varphi'_k} \Rightarrow \varphi'\end{matrix}}{\overline{\varphi_h}, \overline{\varphi'_k} \Rightarrow \varphi'}\ (Cut^*)$$

respectively. Let also σ^i_1, σ^i_2 and σ_3 be the only applications of (Cut^*) in $\mathfrak{D}^i_1$, $\mathfrak{D}^i_2$ and $\mathfrak{D}_3$, respectively. They are all eliminable by inductive hypothesis, since $g(\sigma^i_1) < g(\sigma)$, $g(\sigma^i_2) < g(\sigma)$ and $g(\sigma_3) < g(\sigma)$. With $\mathfrak{D}^i_1$, $\mathfrak{D}^i_2$ and $\mathfrak{D}_3$, we can construct a derivation of $\overrightarrow{\mathbf{O}_{ts}(\varphi_h|\psi_h)}, \overrightarrow{\mathbf{O}_{ts}(\varphi'_k|\psi'_k)}, \Rightarrow \mathbf{O}_{ts}(\varphi'|\psi')$ in $\mathrm{G}^-(\mathbf{TDDL})$, as below.

$$\dfrac{\begin{matrix}\mathfrak{D}^i_1, \mathfrak{D}^i_2\\ (\psi_i \Leftrightarrow \psi')_{1\leqslant i\leqslant k}\end{matrix} \quad \begin{matrix}\vdots\\ (\psi'_j \Leftrightarrow \psi')_{1\leqslant i\leqslant l}\end{matrix} \quad \begin{matrix}\mathfrak{D}_3\\ \overrightarrow{\varphi_k}, \overrightarrow{\varphi'_l} \Rightarrow \varphi'\end{matrix}}{\overrightarrow{\mathbf{O}_{ts}(\varphi_k|\psi_k)}, \overrightarrow{\mathbf{O}_{ts}(\varphi'_l|\psi'_l)} \Rightarrow \mathbf{O}_{ts}(\varphi'|\psi')}\ (\mathbf{O})$$

Therefore, the claim was established. ∎

Similarly as in TSML, the consistency of $\mathrm{G}(\mathbf{TDDL})$ follows from the cut elimination theorem as a corollary. The proof is the same as for Corollary 116, but we spell out it since it is of importance in Section 4.3. Note that the following proof proceeds without invoking any semantics notions from the Kripke semantics given to TDDL.

Corollary 158. A sequent $\Rightarrow \bot$ is not derivable in $\mathrm{G}(\mathbf{TDDL})$.

Proof. Suppose for contradiction that $\vdash_{\mathrm{G}(\mathbf{TDDL})} \Rightarrow \bot$. Then it follows from $\vdash_{\mathrm{G}(\mathbf{TDDL})} \bot \Rightarrow$ and (Cut) that $\vdash_{\mathrm{G}(\mathbf{TDDL})} \Rightarrow$. Thus the cut elimination theorem (Theorem 115) tells us that $\vdash_{\mathrm{G}^-(\mathbf{TDDL})} \Rightarrow$, which cannot be the case from all the rules of $\mathrm{G}^-(\mathbf{TDDL})$. ∎

As before, the cut elimination theorem of $\mathrm{G}(\mathbf{TDDL})$ enables us to prove the Craig interpolation theorem of $\mathrm{G}(\mathbf{TDDL})$ in a proof-theoretic way. Lemma 159 and Theorem 160 are contributions of K. Sano in [84]. Similarly to Lemma 118 and Theorem 119, Lemma 159 and Theorem 160 might be generalized so that $\mathsf{Func}(\varphi) \subseteq \mathsf{Func}(\Gamma_1, \Delta_1) \cap \mathsf{Func}(\Gamma_2, \Delta_2)$ and $\mathsf{Func}(\chi) \subseteq \mathsf{Func}(\varphi) \cap \mathsf{Func}(\psi)$, respectively, where $\mathsf{Func}(\Gamma)$ means the set of all function symbols in a set Γ of formulas. However, such a generalization is not done in this thesis. See [62] for such a possible generalization. Below, $\mathsf{Pred}(\Gamma)$ and $\mathsf{Con}(\Gamma)$ mean the sets of all predicate symbols and all constant symbols in a set Γ of formulas, respectively. We also use the notion of partition carried over from $\mathrm{G}(\mathbf{tK\Sigma})$.

Lemma 159. Let $\Gamma \Rightarrow \Delta$ be a sequent derivable in G(**TDDL**). If $(\Gamma_1 : \Delta_1)$, $(\Gamma_2 : \Delta_2)$ is a partition of $\Gamma \Rightarrow \Delta$, there is an interpolant of it, i.e., a formula φ such that

1. $\vdash_{\mathrm{G}(\mathbf{tK\Sigma})} \Gamma_1 \Rightarrow \Delta_1, \varphi$ and $\vdash_{\mathrm{G}(\mathbf{tK\Sigma})} \varphi, \Gamma_2 \Rightarrow \Delta_2$;
2. $\mathsf{Pred}(\varphi) \subseteq \mathsf{Pred}(\Gamma_1, \Delta_1) \cap \mathsf{Pred}(\Gamma_2, \Delta_2)$;
3. $\mathsf{FV}(\varphi) \subseteq \mathsf{FV}(\Gamma_1, \Delta_1) \cap \mathsf{FV}(\Gamma_2, \Delta_2)$;
4. $\mathsf{Con}(\varphi) \subseteq \mathsf{Con}(\Gamma_1, \Delta_1) \cap \mathsf{Con}(\Gamma_2, \Delta_2)$.

Proof. The proof is analogous to the proof of Lemma 118. By the cut elimination theorem (Theorem 115), we have a derivation of $\Gamma \Rightarrow \Delta$ in $\mathrm{G}^-(\mathbf{TDDL})$. Our proof is done by induction on the height of the derivation of $\Gamma \Rightarrow \Delta$ in $\mathrm{G}^-(\mathbf{TDDL})$. As before, we can exclude partitions of the form $(\emptyset : \emptyset), (\Gamma : \Delta)$ or $(\Gamma : \Delta), (\emptyset : \emptyset)$. Here we only show a case in which $\Gamma \Rightarrow \Delta$ is obtained by (**O**). In this case, the derivation ends with

$$\frac{(\psi_i \Leftrightarrow \psi)_{1\leqslant i\leqslant n} \qquad (\Sigma_{i1}, \Sigma_{i2})_{1\leqslant i\leqslant n} \Rightarrow \varphi}{(\mathbf{O}_{ts}(\Sigma_{i1}|\psi_i), \mathbf{O}_{ts}(\Sigma_{i2}|\psi_i))_{1\leqslant i\leqslant n} \Rightarrow \mathbf{O}_{ts}(\varphi|\psi)}\ (\mathbf{O})$$

where $\Sigma_i = \Sigma_{i1}, \Sigma_{i2}$ and $\mathbf{O}_{ts}(\Sigma|\psi)$ means $\{\, \mathbf{O}_{ts}(\gamma|\psi) \mid \gamma \in \Sigma \,\}$. There are two cases depending on which side of a partition of the lower sequent contains $\mathbf{O}_{ts}(\varphi|\psi)$. In what follows, we prove only a case in which the right side of the partition contains $\mathbf{O}_{ts}(\varphi|\psi)$, i.e., the partition is $((\mathbf{O}_{ts}(\Sigma_{i1}|\psi_i))_{1\leqslant i\leqslant n} : \varnothing), (\mathbf{O}_{ts}(\Sigma_{i2}|\psi_i))_{1\leqslant i\leqslant n} : \mathbf{O}_{ts}(\varphi|\psi))$.

It is remarked that $(\mathbf{O}_{ts}(\Sigma_{i1}|\psi_i))_{1\leqslant i\leqslant n}$ is not empty. Fix such k. Since $\psi \Rightarrow \psi_k$ is derivable by assumption, our inductive hypothesis implies there is an interpolant ρ_k for $(\psi : \varnothing), (\varnothing : \psi_k)$, i.e., both of $\psi \Rightarrow \rho_k$ and $\rho_k \Rightarrow \psi_k$ are derivable in G(**TDDL**) and $\mathsf{X}(\rho_k) \subseteq \mathsf{X}(\psi) \cap \mathsf{X}(\psi_k)$ where X is FV, Con or Pred. Together with an assumption that $\psi_k \Rightarrow \psi$ is derivable in G(**TDDL**), we can deduce that $\rho_k \Leftrightarrow \psi$ is derivable in G(**TDDL**). Moreover, for each i, $\psi_i \Leftrightarrow \psi$ is derivable in G(**TDDL**) by assumption. Thus $\psi_i \Leftrightarrow \rho_k$ is derivable in G(**TDDL**).

By inductive hypothesis to $(\Sigma_{i1}, \Sigma_{i2})_{1\leqslant i\leqslant n} \Rightarrow \varphi$, we can also obtain

- $\vdash (\Sigma_{i1})_{1\leqslant i\leqslant n} \Rightarrow \chi$ and $\vdash \chi, (\Sigma_{i2})_{1\leqslant i\leqslant n} \Rightarrow \varphi$;
- $\mathsf{X}(\chi) \subseteq \mathsf{X}((\Sigma_{i1})_{1\leqslant i\leqslant n}) \cap \mathsf{X}((\Sigma_{i2})_{1\leqslant i\leqslant n}, \varphi)$ where X is FV, Con or Pred.

Then, we have

$$\frac{(\psi_i \Leftrightarrow \rho_k)_{1\leqslant i\leqslant n} \qquad (\Sigma_{i1})_{1\leqslant i\leqslant n} \Rightarrow \chi}{(\mathbf{O}_{ts}(\Sigma_{i1}|\psi_i))_{1\leqslant i\leqslant n} \Rightarrow \mathbf{O}_{ts}(\chi|\rho_k)}\ (\mathbf{O})$$

In addition, we have

$$\frac{(\psi_i \Leftrightarrow \psi)_{1\leqslant i\leqslant n} \qquad \rho_k \Leftrightarrow \psi \qquad (\Sigma_{i2})_{1\leqslant i\leqslant n}, \chi \Rightarrow \varphi}{(\mathbf{O}_{ts}(\Sigma_{i2}|\psi_i))_{1\leqslant i\leqslant n}, \mathbf{O}_{ts}(\chi|\rho_k) \Rightarrow \mathbf{O}_{ts}(\varphi|\psi)}\ (\mathbf{O})$$

Moreover, it is easy to verify the following conditions:

$$\mathsf{FV}(\mathbf{O}_{ts}(\chi|\rho_k)) \subseteq \mathsf{FV}((\mathbf{O}_{ts}(\Sigma_{i1}|\psi_i))_{1\leqslant i\leqslant n}) \cap \mathsf{FV}((\mathbf{O}_{ts}(\Sigma_{i2}|\psi_i))_{1\leqslant i\leqslant n}, \mathbf{O}_{ts}(\varphi|\psi));$$
$$\mathsf{Con}(\mathbf{O}_{ts}(\chi|\rho_k)) \subseteq \mathsf{Con}((\mathbf{O}_{ts}(\Sigma_{i1}|\psi_i))_{1\leqslant i\leqslant n}) \cap \mathsf{Con}((\mathbf{O}_{ts}(\Sigma_{i2}|\psi_i))_{1\leqslant i\leqslant n}, \mathbf{O}_{ts}(\varphi|\psi));$$
$$\mathsf{Pred}(\mathbf{O}_{ts}(\chi|\rho_k)) \subseteq \mathsf{Pred}((\mathbf{O}_{ts}(\Sigma_{i1}|\psi_i))_{1\leqslant i\leqslant n}) \cap \mathsf{Pred}((\mathbf{O}_{ts}(\Sigma_{i2}|\psi_i))_{1\leqslant i\leqslant n}, \mathbf{O}_{ts}(\varphi|\psi)).$$

Therefore $\mathbf{O}_{ts}(\chi|\rho_k)$ is an interpolant of the current partition. ■

Theorem 160. (Craig interpolation theorem) If $\Rightarrow \varphi \to \psi$ is derivable in G(**TDDL**), then there is a formula χ such that

- $\vdash_{\mathrm{G}(\mathbf{TDDL})} \Rightarrow \varphi \to \chi$ and $\vdash_{\mathrm{G}(\mathbf{TDDL})} \Rightarrow \chi \to \psi$;
- $\mathsf{Pred}(\chi) \subseteq \mathsf{Pred}(\varphi) \cap \mathsf{Pred}(\psi)$;
- $\mathsf{FV}(\chi) \subseteq \mathsf{FV}(\varphi) \cap \mathsf{FV}(\psi)$;
- $\mathsf{Con}(\chi) \subseteq \mathsf{Con}(\varphi) \cap \mathsf{Con}(\psi)$.

4.3 Accommodating Normative Conflicts

In this section, we will first argue that TDDL can accommodate two kinds of normative conflicts (Section 4.3.1). This suggests that TDDL works well for accommodating normative conflicts.

We will then claim that TDDL is consistent without the truth-ascription (Section 4.3.2). This means that TDDL is a desired deontic logic. It is because, as we pointed out in Chapter 1.1.2, proving the consistency of a presented deontic logic without the truth-ascription is a first step towards the development of deontic logics compatible with the thesis that norms are neither true nor false. Moreover, we will claim here that TDDL can accommodate normative conflicts of the above kinds without the truth-ascription.

4.3.1 The Problem of Normative Conflict

It is widely accepted in the literature [32, 26, 60] that the standard deontic logic **SDL** cannot properly formalize situations called *normative conflicts* in which an agent has obligations to do incompatible things. Recall that **SDL** is the normal modal propositional logic **KD**. In **SDL**, the only natural formalization of such situations is $\mathbf{O}P \wedge \mathbf{O}\neg P$, which contradicts modal axiom (D) in **SDL**. As often pointed out, the logic obtained from **SDL** by removing (D), i.e., the smallest normal modal propositional logic **K**, still fails to formalize normative situations properly. It is because arbitrary obligations $\mathbf{O}\psi$ follow from a normative conflict $\mathbf{O}P \wedge \mathbf{O}\neg P$ via a formula $(\mathbf{O}P \wedge \mathbf{O}\neg P) \to \mathbf{O}\psi$ called *deontic explosion* (e.g. [26, pp. 297–8]). Therefore, it has been explored by many deontic logicians how normative conflicts can be accommodated, i.e., how they can be formalized giving rise to neither contradictions nor arbitrary obligations.

Our approach to accommodate normative conflicts is to relativize obligations to a pair of agents and a condition. Thus our approach falls into the first approach on normative conflicts explained in Chapter 1, i.e., an approach to relativize obligations to agents or the like. It is a familiar strategy in the literature and in fact found e.g. in [44, 45, 25, 107]. However, TDDL can accommodate a larger class of normative conflicts than their logics can.

Normative conflicts of the first kind that TDDL can accommodate are situations in which incompatible obligations are directed towards different agents. Consider a situation in which Adam (a) has obligations towards Barbara (b) and Charles (c) to be with her and him, respectively, but cannot be with both. Let Wab and Wac represent "Adam is with Barbara" and "Adam is with Charles". Since we may identify Wac with $\neg Wab$ in this specific example, the situation is simply formalized in TDDL by

$$\mathbf{O}_{ab}Wab \wedge \mathbf{O}_{ac}\neg Wab.$$

It does not give rise to contradictions $\bot$ or arbitrary obligations $\mathbf{O}_{ts}\varphi$ because Barbara and Charles are assumed to be different persons. This can be proved as follows. Note first that the fact that it is not the case that $\vdash_{\mathsf{H}(\mathbf{TDDL})} (\mathbf{O}_{ab}Wab \wedge \mathbf{O}_{ac}\neg Wab) \to \mathbf{O}_{ts}\varphi$ for any term t, s and formula φ follows from the fact that $\nvdash_{\mathsf{H}(\mathbf{TDDL})} (\mathbf{O}_{ab}Wab \wedge \mathbf{O}_{ac}\neg Wab) \to \mathbf{O}_{xy}P$. Thus, our goal is to show

(1) $\nvdash_{\mathsf{H}(\mathbf{TDDL})} (\mathbf{O}_{ab}Wab \wedge \mathbf{O}_{ac}\neg Wab) \to \bot$;

(2) $\nvdash_{\mathsf{H}(\mathbf{TDDL})} (\mathbf{O}_{ab}Wab \wedge \mathbf{O}_{ac}\neg Wab) \to \mathbf{O}_{xy}P$.

It is not easy to show the facts (1) and (2) by examining all possible proofs in $\mathsf{H}(\mathbf{TDDL})$. However, the soundness of $\mathsf{H}(\mathbf{TDDL})$ (Theorem 145) tells us that it is sufficient to show

(3) $\not\models (\mathbf{O}_{ab}Wab \wedge \mathbf{O}_{ac}\neg Wab) \to \bot$;

(4) $\not\models (\mathbf{O}_{ab}Wab \wedge \mathbf{O}_{ac}\neg Wab) \to \mathbf{O}_{xy}P$.

And we can easily show the facts (3) and (4) by our assumption that Barbara and Charles are different persons, i.e., $I(b) \neq I(c)$ for all interpretations I. Therefore the facts (1) and (2) also hold, which mean that the formula $\mathbf{O}_{ab}Wab \wedge \mathbf{O}_{ac}\neg Wab$ does not give rise to contradictions $\bot$ or arbitrary obligations $\mathbf{O}_{ts}\varphi$. It should be noted here that deontic logics with no indices are difficult to accommodate normative conflicts of the first kind with a succinct formulation.

Normative conflicts of the second kind that TDDL can accommodate are situations in which incompatible obligations are directed towards the same agent under different conditions. For example, consider a situation in which Adam has an obligation towards

Barbara to be with her given that she is old and in which he has another obligation towards her not to be with her given that COVID-19 is still spreading. Let Wab, Ob and C represent "Adam is with Barbara", "Barbara is old" and "COVID-19 is spreading", respectively. This situation is formalized in TDDL by

$$\mathbf{O}_{ab}(Wab|Ob) \wedge \mathbf{O}_{ab}(\neg Wab|C).$$

It does not also give rise to contradictions $\bot$ or arbitrary obligations $\mathbf{O}_{ts}(\varphi|\psi)$, because the conditions Ob and C are not equivalent. Similarly to normative conflicts of the first kind, this can be proved via the soundness of $\mathsf{H}(\mathbf{TDDL})$ (Theorem 145). It should be noted here that deontic logics with no conditionals face difficulties in accommodating normative conflicts of the second kind. The most well-known difficulty is that such logics cannot express non-monotonicity of obligations. For example, consider two situations in which Adam has an obligation towards Barbara to be with her given that she is old and in which Adam has an obligation towards Barbara to be with her given that she is old but not lonely. Let also Lb represent "b is lonely". Then, the most natural formalizations in the deontic logics with no conditionals would be $Ob \rightarrow \mathbf{O}_{ab}Wab$ and $Ob \wedge \neg Lb \rightarrow \mathbf{O}_{ab}Wab$. However, since the former implies the latter, these formalizations do not reflect our intuition that the latter may fail even if the former holds.

Normative conflicts of the second kind need finer-grained formalizations when they follow from another situations involved in quantification. Consider a situation in which Adam has an obligation towards Barbara not to be with her given that COVID-19 is still spreading but in which she is his mother. Suppose also that we accept that anyone has an obligation towards one's mother to be with her given that she is old. Then, from the current situation, a normative conflict of the second kind follows in which Adam has an obligation towards Barbara not to be with her given that COVID-19 is still spreading, as well as another obligation towards her to be with her given that she is old. We shall call such a normative conflict *derived normative conflict*.

Our TDDL can formalize derived normative conflicts as well. Let Wab, Ob, C and Mba represent "Adam is with Barbara", "Barbara is old", "COVID-19 is spreading" and "Barbara is Adam's mother", respectively. Then, the first situation is formalized in TDDL by

$$\mathbf{O}_{ab}(\neg Wab|C) \wedge Mba \wedge \forall x \forall y(Myx \rightarrow \mathbf{O}_{xy}(Wxy|Oy)),$$

from which the derived normative conflict above is obtained in the following form:

$$\mathbf{O}_{ab}(\neg Wab|C) \wedge \mathbf{O}_{ab}(Wab|Ob).$$

As before, it does not give rise to contradictions $\bot$ or arbitrary obligations $\mathbf{O}_{ts}(\varphi|\psi)$, which can be proved via the soundness of $\mathsf{H}(\mathbf{TDDL})$ (Theorem 145).

Against our claim requiring conditionals for the second kind, there might be the following objection: no additional machineries are necessary to accommodate the second kind, since we can formalize even such conditions just by adding the third index to modal operators. For example, consider the aforementioned situation in which Adam has an obligation towards Barbara to be with her given that she is old and in which he has another obligation towards her not to be with her given that COVID-19 is still spreading. According to the objection, it can be formalized as $\mathbf{O}_{abo}Wab \wedge \mathbf{O}_{abc}Wab$, where o and c denotes the conditions that Barbara is old and that COVID-19 is spreading. Clearly, it does not give rise to contradictions or arbitrary obligations. For ease of reference, we shall refer to an approach formalizing conditions in such a way as the indexing approach.

However, the use of indices representing conditions should be avoided to keep the same interpretation of logical constants in a given context. Consider a situation in which Adam has an obligation to be *and* talk with Barbara given that she is old *and* lonely. Whatever formalization is given to it, we would require that two logical constants “and” in it be formalized in the same way. The indexing approach basically face a difficulty when formalizing logical constants appearing in both of the antecedent and the consequent of a conditional obligation. Amongst those which might fall into the indexing approach, Gabbay [21] and Tamminga [91] can avoid the difficulty. It is because the former allows indices to be formulas and the latter represents conditionals by actions of some group which have no logical forms. However, they are based on propositional logic, so difficult to accommodate derived normative conflicts requiring quantifiers to formalize.

Against our claim that derived normative conflicts require quantifiers for formalization, one might think that finite conjunctions or disjunctions are enough and that quantifiers are not necessary in deontic logic, since we may assume the number of agents to be finite. We can answer this criticism by putting forth two points.

First, as pointed out e.g. in Hilpinen and McNamara [35, p. 53] and Frijters [18, p. 70], there are deontic sentences in which the *de re*/*de dicto* distinction should be made. For example, it may be the case that Adam has an obligation towards Barbara to take someone’s class in the university (since she is sending him money for class), but not that there is someone such that he has an obligation towards her to take the person’s class in the university. Given a formula Cax representing “Adam takes x’s class in the university”, the intuition can be captured in TDDL with formulas $\mathbf{O}_{ab}\exists xCax$ and $\exists x\mathbf{O}_{ab}Cax$ since the former does not imply the latter. It seems difficult to capture this intuition in the framework of propositional deontic logic. See Frijters [18, p. 70] for a similar example.

Second, even when the domain of quantification is a finite set $\{ a_1, \dots, a_n \}$ of agents, it is not a decisive proposition in deontic logic that a finite conjunction $Pa_1 \wedge \dots \wedge Pa_n$ and a universally quantified sentence $\forall xPx$ have the same truth value. It is because we sometimes consider a universally quantified sentence like “everyone should be honest” to be true irrespective of how the domain of quantification is like. We can find this interesting idea in F. Ramsey’s *Philosophical Papers* [79, p. 145], in which he calls such universal sentences *variable hypotheticals*. Of course our current semantics is not suitable for this idea. However, with this idea we can claim the need of quantifiers for deontic logic.

To sum up, TDDL can accommodate the class of normative conflicts of the first and second kinds. **SDL** and **K** cannot accommodate the first kind, nor the second kind. Kooi and Tamminga [44, 45], Glavanicova[25] and Yamada [107] can accommodate the first kind, but not the second kind due to the lack of conditionals. Gabbay [21] and Tamminga [91] can be applied to accommodate some of the second kind, but not all. They cannot be applied to accommodate derived normative conflicts due to the lack of quantifiers.

4.3.2 The Problem of Truth-Ascription

We will then claim that TDDL is consistent without the truth-ascription, as well as that TDDL can accommodate normative conflicts of the above kinds without the truth-ascription.

To begin, we shall again spell out Corollary 158 with the proof just for a reminder.

Corollary 158. A sequent $\Rightarrow \bot$ is not derivable in $\mathsf{G}(\mathbf{TDDL})$.

Proof. Suppose for contradiction that $\vdash_{\mathsf{G}(\mathbf{TDDL})} \Rightarrow \bot$. Then it follows from $\vdash_{\mathsf{G}(\mathbf{TDDL})} \bot \Rightarrow$ and (Cut) that $\vdash_{\mathsf{G}(\mathbf{TDDL})} \Rightarrow$. Thus the cut elimination theorem (Theorem 115) tells us that $\vdash_{\mathsf{G}^-(\mathbf{TDDL})} \Rightarrow$, which cannot be the case from all the rules of $\mathsf{G}^-(\mathbf{TDDL})$. ■

This corollary straightforwardly tells us that TDDL is consistent. What we have to further confirm is that the proof proceeds without the truth-ascription to formulas of TDDL. We can easily observe this. Note first that $\vdash_{\mathsf{G}(\mathbf{TDDL})} \Rightarrow \bot$ means by definition that there is a finite tree generated from initial sequents by applying structural or logical rules of $\mathsf{G}(\mathbf{TDDL})$ such that the tree's root is $\Rightarrow \bot$. Thus our first supposition in the proof does not use any semantic notions from the Kripke semantics given to TDDL. Also, the transition from $\vdash_{\mathsf{G}(\mathbf{TDDL})} \bot \Rightarrow$ to $\vdash_{\mathsf{G}(\mathbf{TDDL})} \Rightarrow$ is just a syntactic manipulation by an application of (Cut). And importantly, as we have proved the cut elimination theorem of $\mathsf{G}(\mathbf{TDDL})$ in a *purely syntactic way*, our inference of $\vdash_{\mathsf{G}^-(\mathbf{TDDL})} \Rightarrow$ from $\vdash_{\mathsf{G}(\mathbf{TDDL})} \Rightarrow$ in the proof is also just a syntactic manipulation. The final step is obtained from a simple inspection of all the rules of $\mathsf{G}^-)\mathbf{TDDL}$. Thus, we could obtain the consistency of TDDL without any semantics, as well as without ascribing truth and falsity to formulas of TDDL.

Then, we establish without the truth-ascription the fact that TDDL can accommodate normative conflicts of the first and second kinds, i.e., the formulations of these normative conflicts do not give rise to contradictions nor arbitrary obligations. Recall that in Section 4.3.1 we used the soundness of $\mathsf{H}(\mathbf{TDDL})$ to prove the same fact.

This means that our proofs in Section 4.3.1 depend on the truth-ascription to deontic formulas in **TDDL**. In contrast, Proposition 161 is proved without the truth-ascription.

Proposition 161. Let x, y, z, w be pairwise distinct variables and P, Q, R, S, T pairwise distinct nullary predicate symbols.

1. A sequent $\mathbf{O}_{xy}P \wedge \mathbf{O}_{xz}\neg P \Rightarrow \bot$ is not derivable in G(**TDDL**).
2. A sequent $\mathbf{O}_{xy}P \wedge \mathbf{O}_{xz}\neg P \Rightarrow \mathbf{O}_{xw}S$ is not derivable in G(**TDDL**).
3. A sequent $\mathbf{O}_{xy}(P|Q) \wedge \mathbf{O}_{xy}(\neg P|R) \Rightarrow \bot$ is not derivable in G(**TDDL**).
4. A sequent $\mathbf{O}_{xy}(P|Q) \wedge \mathbf{O}_{xy}(\neg P|R) \Rightarrow \mathbf{O}_{xw}(S|T)$ is not derivable in G(**TDDL**).

Proof. We first give proofs of items 1, 2. Recall that $\neg\varphi$, $\varphi\wedge\psi$ and $\mathbf{O}_{ts}\varphi$ are abbreviations of $\varphi \to \bot$, $\neg(\varphi \to \neg\psi)$ and $\mathbf{O}_{ts}(\varphi|\top)$, respectively. Recall also that G^-(**TDDL**) is the sequent calculus obtained from G(**TDDL**) by removing (Cut). By induction on $n \in \mathbb{N}$, we can establish the fact that a sequent

$$\mathbf{O}_{xy}P^j, \mathbf{O}_{xz}\neg P^k, \mathbf{O}_{xy}P \wedge \mathbf{O}_{xz}\neg P^l \Rightarrow \mathbf{O}_{xy}P \to \neg\mathbf{O}_{xz}\neg P^m, \neg\mathbf{O}_{xz}\neg P^n, \bot^o, \mathbf{O}_{xw}S^p, S^q$$

is not derivable in G^-(**TDDL**) with at most height i for any $j, k, l, m, n, o, p, q \geqslant 0$, where each superscript denotes the number of occurrences of each formula. We can use the fact to show Items 1, 2. We first show Item 1, i.e., that $\mathbf{O}_{xy}P \wedge \mathbf{O}_{xy}\neg P \Rightarrow \bot$ is not derivable in G(**TDDL**). Suppose not. By the cut elimination theorem of G(**TDDL**) (Theorem 157), $\mathbf{O}_{xy}P \wedge \mathbf{O}_{xy}\neg P \Rightarrow \bot$ is derivable in G^-(**TDDL**). On the other hand, the above fact implies that it is not derivable in G^-(**TDDL**). This is a contradiction so Item 1 holds. By the same argument, Item 2 also holds.

We then give proofs of Items 3, 4. By induction on $n \in \mathbb{N}$, we can establish the fact that a sequent

$$\begin{aligned}\mathbf{O}_{xy}(P|Q)^j, \mathbf{O}_{xy}(\neg P|R)^k, \mathbf{O}_{xy}(P|Q) \wedge \mathbf{O}_{xy}(\neg P|R)^l \Rightarrow& \\ \mathbf{O}_{xy}(P|Q) \to \neg\mathbf{O}_{xy}(\neg P|R)^m, \neg\mathbf{O}_{xy}(\neg P|R)^n, \bot^o, \mathbf{O}_{xw}(S|T)^p, S^q&\end{aligned}$$

is not derivable in G^-(**TDDL**) with at most height i for any $j, k, l, m, n, o, p, q \geqslant 0$, where each superscript denotes the number of occurrences of each formula. Thus, by the same argument as for Items 1, 2, we can prove Items 3, 4. ■

Chapter 5
Common Sense Modal Predicate Logic

In Chapter 3 we have developed term-sequence-modal logic and in Chapter 4 considered as an application of it term-sequence-dyadic deontic logic. In this Chapter, we will examine another modal predicate logic that is quite different from them, *common sense modal predicate logic*. We will begin this by considering a family of common sense modal predicate logics other than the original one (5.1). Then, we will examine proof-theoretic properties of these logics by presenting cut-free ordinary sequent calculi (5.2).

This chapter is based on Sawasaki and Sano [83] "Frame Definability, Canonicity and Cut Elimination in Common Sense Modal Predicate Logics".

5.1 Common Sense Modalities other than S5

We first describe a syntax and a Kripke semantics for common sense modal predicate logic (5.1.1). In the description, we examine several characteristic frame properties in this logic. The results given there can be seen as a supplement of van Benthem [95, 94, 93]. We then put forth Hilbert systems for common sense modal predicate logics, proving soundness and strong completeness of some class of them (5.1.2).

5.1.1 Syntax and Kripke Semantics

Definition 162. The *language* $\mathsf{L}_{\mathsf{CMPC}}$ of *common sense modal predicate logic* (CMPC) consists of a countably infinite set $\mathsf{Var} = \{ x, y, \dots \}$ of variables, the union $\mathsf{Pred} = \bigcup_{n \in \mathbb{N}} \mathsf{Pred}_n$ of countably infinite sets $\mathsf{Pred}_n = \{ P, Q, \dots \}$ of predicate symbols with arity n, the set $\{ \bot, \to, \forall \}$ of logical symbols, and the set $\mathsf{Mod} = \{ \Box \}$ of the unary modal operator $\Box$. A *formula φ in* $\mathsf{L}_{\mathsf{CMPC}}$ is recursively defined by

$$\varphi ::= Px_1 \dots x_n \mid \bot \mid (\varphi \to \varphi) \mid \forall x \varphi \mid \Box \varphi,$$

where P is a predicate symbol with arity n and $x, x_1, \ldots, x_n$ are variables. Instead of $\mathsf{L}_{\mathsf{CMPC}}$, we often write L when it is clear from the context. We also write $\mathsf{L}_{\mathsf{CMPC}}(V)$ to denote $\mathsf{L}_{\mathsf{CMPC}}$ in which the set of variables is V.

The length of a formula, the sets of free variables in a formula and a set of formulas, and the substitution of a variable for a variable are defined along the same lines as $\mathsf{L}_{\mathsf{QML}}$. In addition to these notions, we use the notation $\#X$ to mean the size of a set X.

We define frame and model for $\mathsf{L}_{\mathsf{CMPC}}$ similarly as before, except for one thing. *They have varying domains.*

Definition 163. A *frame* (*for* $\mathsf{L}_{\mathsf{CMPC}}$) is a tuple $\mathfrak{F} = (W, R, (D_w)_{w \in W})$, where W is a nonempty set of worlds; R is a binary relation on W; D_w is a non-empty domain of a world w. Given a frame $\mathfrak{F}$, a *model* (*for* $\mathsf{L}_{\mathsf{CMPC}}$) is a tuple $\mathfrak{M} = (\mathfrak{F}, I)$, where I is an interpretation that maps each predicate symbol P with arity n and world w to a subset $I(P, w)$ of D_w^n. We define D as $\bigcup_{w \in W} D_w$. We sometimes write `True` and `False` instead of D_w^0 and $\emptyset$, respectively.

Thus, in this semantics, what exists in each world may be thoroughly different across the worlds. For frame's examples, see Figure 5.1.

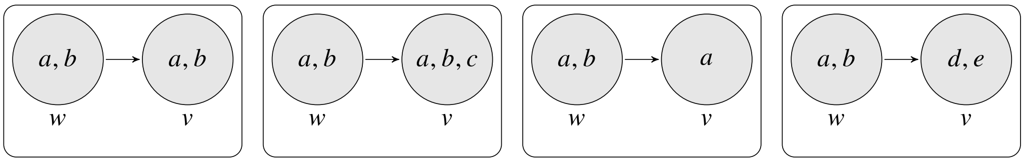

Figure 5.1: Frames for $\mathsf{L}_{\mathsf{CMPC}}$

The notion of assignment is straightforwardly carried over from the semantics for $\mathsf{L}_{\mathsf{QML}}$. Recall that a formula φ is said to be an α_w-formula if $\alpha(x) \in D_w$ for all variables $x \in \mathsf{FV}(\varphi)$.

Definition 164. (Satisfaction relation) Let $\mathfrak{M} = (W, R, (D_w)_{w \in W}, I)$ be a model, w a world, α an assignment, and φ a formula such that φ is an α_w-formula. The *satisfaction relation* $\mathfrak{M}, w, \alpha \models \varphi$ *between* $\mathfrak{M}, w, \alpha$ *and* φ is defined as follows.

$$
\begin{array}{lll}
\mathfrak{M}, w, \alpha \models Px_1 \ldots x_n & \text{iff} & \alpha(x_1, \ldots, x_n) \in I(P, w) \\
\mathfrak{M}, w, \alpha \not\models \bot & & \\
\mathfrak{M}, w, \alpha \models \varphi \to \psi & \text{iff} & \mathfrak{M}, w, \alpha \models \varphi \text{ implies } \mathfrak{M}, w, \alpha \models \psi \\
\mathfrak{M}, w, \alpha \models \forall x \varphi & \text{iff} & \text{for all objects } d \in D_w,\ \ \mathfrak{M}, w, \alpha(x|d) \models \varphi \\
\mathfrak{M}, w, \alpha \models \Box \varphi & \text{iff} & \text{for all worlds } v \in W \text{ such that } \varphi \text{ is an } \alpha_v\text{-formula}, \\
 & & wRv \text{ implies } \mathfrak{M}, v, \alpha \models \varphi
\end{array}
$$

For a set Γ of formulas, $\mathfrak{M}, w, \alpha \models \Gamma$ means that $\mathfrak{M}, w, \alpha \models \psi$ for all formulas $\psi \in \Gamma$.

Remark 165. In the increasing domain semantics for $\mathsf{L}_{\mathsf{QML}}$, we have defined the satisfaction relation of $\Box\varphi$, as follows.

$$\mathfrak{M}, w, \alpha \models \Box\varphi \quad \text{iff} \quad \text{for all worlds } v \in W,\ \ wRv \text{ implies } \mathfrak{M}, v, \alpha \models \varphi.$$

As introduced in Fitting and Mendelsohn [16], the varying domain semantics for $\mathsf{L}_{\mathsf{QML}}$ also takes the same definition. Contrary to this standard definition, the varying domain semantics for $\mathsf{L}_{\mathsf{CMPC}}$ requires the condition that φ be an α_v-formula. Unfolding the definition of α_v-formula, we can rephrase the satisfaction relation of $\Box\varphi$ in the varying domain semantics for CMPC, as below.

$$\begin{aligned} \mathfrak{M}, w, \alpha \models \Box\varphi \quad \text{iff} \quad & \text{for all worlds } v \in W \text{ such that } \alpha(x) \in D_v \text{ for all } x \in \mathsf{FV}(\varphi), \\ & wRv \text{ implies } \mathfrak{M}, v, \alpha \models \varphi. \end{aligned}$$

The additional condition intuitively means that φ is "well-defined" at a world v.

It should also be noted that the following equivalence follows by definition.

$$\begin{aligned} \mathfrak{M}, w, \alpha \models \Diamond\varphi \quad \text{iff} \quad & \text{for some worlds } v \in W \text{ such that } \varphi \text{ is an } \alpha_v\text{-formula}, \\ & wRv \text{ and } \mathfrak{M}, v, \alpha \models \varphi. \end{aligned}$$

The unfolded form of it is van Benthem's satisfaction relation of $\Diamond\varphi$ in [95, p. 121].

$$\begin{aligned} \mathfrak{M}, w, \alpha \models \Diamond\varphi \quad \text{iff} \quad & \text{for some worlds } v \in W \text{ such that } \alpha(x) \in D_v \text{ for all } x \in \mathsf{FV}(\varphi), \\ & wRv \text{ and } \mathfrak{M}, v, \alpha \models \varphi. \end{aligned}$$

Finally, we note that our definition of frame is more general than that of Seligman in his drafts [87, 89] since he has defined frame without accessibility relation. In short, Seligman [87, 89]'s satisfaction relations of $\Box\varphi$ and $\Diamond\varphi$ are given as follows.

$$\begin{aligned} \mathfrak{M}, w, \alpha \models \Box\varphi \quad \text{iff} \quad & \text{for all worlds } v \in W \text{ such that } \varphi \text{ is an } \alpha_v\text{-formula}, \\ & \mathfrak{M}, v, \alpha \models \varphi. \\ \mathfrak{M}, w, \alpha \models \Diamond\varphi \quad \text{iff} \quad & \text{for some worlds } v \in W \text{ such that } \varphi \text{ is an } \alpha_v\text{-formula}, \\ & \mathfrak{M}, v, \alpha \models \varphi. \end{aligned}$$

These satisfaction relations are obtained from ours when R is the total relation $W \times W$.

Contrary to the satisfaction relation, validity is defined in the same way as for $\mathsf{L}_{\mathsf{QML}}$, i.e., as below.

Definition 166. Let φ be a formula and Γ a set of formulas.

- φ is *valid in a frame* $\mathfrak{F}$ *(for* $\mathsf{L}_{\mathsf{CMPC}}$*)*, denoted by $\mathfrak{F} \models \varphi$, if $(\mathfrak{F}, I), w, \alpha \models \varphi$ for all interpretations I, worlds w and assignments α such that φ is an α_w-formula.

- Γ is *valid in a frame* $\mathfrak{F}$ (*for* $\mathsf{L}_{\mathsf{CMPC}}$), denoted by $\mathfrak{F} \models \Gamma$, if $\mathfrak{F} \models \gamma$ for all formulas $\gamma \in \Gamma$.
- φ is *valid in a class* $\mathbb{F}$ *of frames* (*for* $\mathsf{L}_{\mathsf{CMPC}}$), denoted by $\mathbb{F} \models \varphi$, if $\mathfrak{F} \models \varphi$ for all frames $\mathfrak{F} \in \mathbb{F}$.
- φ is a *consequence from* Γ *in a class* $\mathbb{F}$ *of frames* (*for* $\mathsf{L}_{\mathsf{CMPC}}$) if $(\mathfrak{F}, I), w, \alpha \models \Gamma$ implies $(\mathfrak{F}, I), w, \alpha \models \varphi$ for all frames $\mathfrak{F} \in \mathbb{F}$, interpretations I, worlds w, assignments α such that φ is an α_w-formula.

Proposition 167. Let $\mathfrak{M}$ be a model, w a world in $\mathfrak{M}$, α an assignment. It holds that $\mathfrak{M}, w, \alpha \models \varphi(y/x)$ iff $\mathfrak{M}, w, \alpha(x|\alpha(y)) \models \varphi$.

Here we will import Seligman [87]'s propositions involving validity into our generalized setting. Item 1 of the following proposition explains the reason why CMPC must give up axiom (K).

Proposition 168.

1. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ is not valid in the class of all frames.
2. $\Box(\varphi \land \psi) \to (\Box\varphi \land \Box\psi)$ is not valid in the class of all frames.
3. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ is valid in the class of all frames if $\mathsf{FV}(\varphi) \subseteq \mathsf{FV}(\psi)$.
4. $(\Box\varphi \land \Box\psi) \to \Box(\varphi \land \psi)$ is valid in the class of all frames.

Proof.

1. Consider a formula $\Box(Px \to \exists xPx) \to (\Box Px \to \Box\exists xPx)$. Let $\mathfrak{M}$ be a model (W, R, D, I) and α an assignment such that $W = \{ w, v \}$, $D_w = \{ a \}$, $D_v = \{ b \}$, $R = \{ (w, v) \}$, $I(P, w) = I(P, v) = \emptyset$ and $\alpha(x) = a$. Then $\mathfrak{M}, w, \alpha \models \Box(Px \to \exists xPx)$ and $\mathfrak{M}, w, \alpha \models \Box Px$, but $\mathfrak{M}, w, \alpha \not\models \Box\exists xPx$.
2. Consider a formula $\Box(Px \land \exists xPx) \to (\Box Px \land \Box\exists xPx)$ and take the same model $\mathfrak{M}$ and assignment α as above. It is easy to see $\mathfrak{M}, w, \alpha \models \Box(Px \land \exists xPx)$ but $\mathfrak{M}, w, \alpha \not\models \Box\exists xPx$.
3. Suppose $\mathsf{FV}(\varphi) \subseteq \mathsf{FV}(\psi)$ and take any frame $\mathfrak{F}$, interpretation I, world w, assignment α such that $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ is an α_w-formula. Assume $(\mathfrak{F}, I), w, \alpha \models \Box(\varphi \to \psi)$ and $(\mathfrak{F}, I), w, \alpha \models \Box\varphi$. We show $(\mathfrak{F}, I), w, \alpha \models \Box\psi$. Take any world v such that wRv and ψ is an α_v-formula. Since φ is an α_v-formula by $\mathsf{FV}(\varphi) \subseteq \mathsf{FV}(\psi)$, $(\mathfrak{F}, I), v, \alpha \models \varphi \to \psi$ and $(\mathfrak{F}, I), v, \alpha \models \varphi$. Thus $(\mathfrak{F}, I), v, \alpha \models \psi$, as required.

4. Take any frame $\mathfrak{F}$, interpretation I, world w, assignment α such that $(\Box\varphi \land \Box\psi) \to \Box(\varphi \land \psi)$ is an α_w-formula. Assume $(\mathfrak{F}, I), w, \alpha \models \Box\varphi \land \Box\psi$. We show $(\mathfrak{F}, I), w, \alpha \models \Box(\varphi \land \psi)$. Take any world v such that wRv and φ, ψ are α_v-formulas. Then $(\mathfrak{F}, I), v, \alpha \models \varphi$ and $(\mathfrak{F}, I), v, \alpha \models \psi$. Thus $(\mathfrak{F}, I), v, \alpha \models \varphi \land \psi$, as required.

■

As we have provided for QML and TSML, we will also provide frame definability results for CMPC. Recall that, for example, D corresponds to seriality in QML. In CMPC, the same correspondence relation does not hold due to the semantics. For example, similarly as in QML, an instance $\Box P \to \Diamond P$ of D corresponds to seriality, i.e., a property that for any world w there is a world v such that wRv. On the other hand, unlike QML, an instance $\Box Px_1x_2 \to \Diamond Px_1x_2$ of D does not correspond to seriality, instead a property that for any world w and object $d_1, d_2 \in D_w$, there is a world v such that wRv and $d_1, d_2 \in D_v$. This is because the satisfaction relations of $\Box Px_1x_2$ and $\Diamond Px_1x_2$ in $\mathsf{L}_{\mathsf{CMPC}}$ require that the values of x_1, x_2 still exist at any (some) accessible world. This semantic characteristics of CMPC also affects the frame correspondences of other formulas. For example, axiom K without any restriction on variables corresponds to increasing domains in CMPC, which is known as the frame property defined by a formula $\exists x \Box Px \to \Box \exists x Px$ in CMPC [93, p. 368], [94, p. 4]. This frame property is also commonly known as the property defined by converse Barcan formula $\Box \forall x Px \to \forall x \Box Px$ in modal predicate logic with the ordinary varying domain introduced in Fitting and Mendelsohn [16, p. 111].

Definition 169. A set Γ of formulas in $\mathsf{L}_{\mathsf{CMPC}}$ *corresponds* to a class $\mathbb{F}$ of frames for $\mathsf{L}_{\mathsf{CMPC}}$ if the equivalence

$$\mathfrak{F} \models \Gamma \quad \text{iff} \quad \mathfrak{F} \in \mathbb{F}$$

holds for all frames $\mathfrak{F}$ for $\mathsf{L}_{\mathsf{CMPC}}$. If $\Gamma = \{ \varphi \}$, we just say that φ corresponds to $\mathbb{F}$.

Some of the following frame properties in CMPC are exactly the same as those in QML, but we will spell out them for a reminder.

Definition 170. (Frame properties) Let $\mathfrak{F} = (W, R, (D_w)_{w \in W})$ be a frame.

1. $\mathfrak{F}$ is *serial with n objects* if, for all $w \in W$ and $d_1, \dots, d_n \in D_w$, there exists some $v \in W$ such that wRv and $d_1, \dots, d_n \in D_v$.
2. $\mathfrak{F}$ is *reflexive* if, for any world $w \in W$, wRw.
3. $\mathfrak{F}$ is *symmetric* if, for any world $w, v \in W$, wRv implies vRw.
4. $\mathfrak{F}$ is *weakly symmetric* if, for all $w, v \in W$, wRv and $D_w \cap D_v \neq \emptyset$ jointly imply vRw.

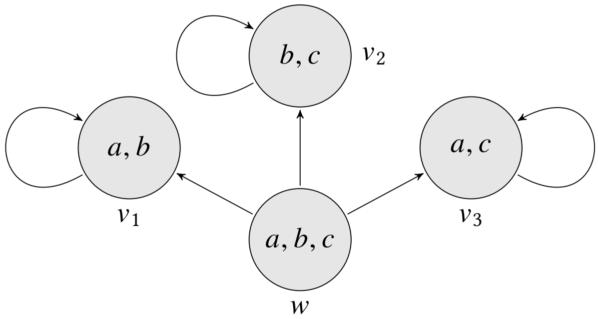

Figure 5.2: A scenario guaranteed by seriality with 2 objects

5. $\mathfrak{F}$ is *transitive* if, for any world w, v, u, wRv and vRu jointly imply wRu.
6. $\mathfrak{F}$ is *weakly transitive* if, for all $w, v, u \in W$, wRv and vRu and $D_w \cap D_v \cap D_u \neq \emptyset$ jointly imply wRu.
7. $\mathfrak{F}$ is *euclidean* if, for any world w, v, u, wRv and wRu jointly imply vRu.
8. $\mathfrak{F}$ is *weakly euclidean* if, for all $w, v, u \in W$, wRv and wRu and $D_w \cap D_v \cap D_u \neq \emptyset$ jointly imply vRu.

Proposition 171. Given a reflexive frame $\mathfrak{F}$, $\mathfrak{F}$ is weakly euclidean iff $\mathfrak{F}$ is weakly symmetric and weakly transitive.

Proof. This is proved similarly as in modal propositional logic. ∎

Remark 172. Clearly, seriality with 0 object is the ordinary seriality. One can also observe that seriality with n objects implies seriality with m objects for $m \leqslant n$ since the n objects are not required to be pairwise distinct.

Example 173. Since seriality with n objects might be somewhat complicated, we will explain this with a simple example. Consider a world w in which agents a, b, c are drowning in a river. In this case, seriality with 2 objects guarantees that w has one or more worlds such that *up to two arbitrary agents* are still alive. Figure 5.2 presents such a scenario.

Proposition 174. (Frame definability) Let $x_1, \ldots, x_n$ be pairwise distinct variables, $\vec{x} = (x_1, \ldots, x_n)$, and P, Q, S predicate symbols with arities 0, 1, n, respectively.

1. $\mathrm{K} := \Box(Qx_1 \rightarrow P) \rightarrow (\Box Qx_1 \rightarrow \Box P)$ corresponds to the class of all the increasing domain frames.

2. $\mathrm{D}_{\leqslant n} := \Box S\vec{x} \to \Diamond S\vec{x}$ corresponds to the class of all the serial frames with n objects.
3. $\mathrm{T} = \Box P \to P$ corresponds to the class of all the reflexive frames.
4. $\mathrm{B} = P \to \Box\Diamond P$ corresponds to the class of all the symmetric frames.
5. $\mathrm{B}^- := Qx_1 \to \Box\Diamond Qx_1$ corresponds to the class of all the weakly symmetric frames.
6. $4 = \Box P \to \Box\Box P$ corresponds to the class of all the transitive frames.
7. $4^- := \Box Qx_1 \to \Box\Box Qx_1$ corresponds to the class of all the weakly transitive frames.
8. $5 = \Diamond P \to \Box\Diamond P$ corresponds to the class of all the euclidean frames.
9. $5^- := \Diamond Qx_1 \to \Box\Diamond Qx_1$ corresponds to the class of all the weakly euclidean frames.

Proof. Proofs for 3, 4, 6 and 8 are done similarly as in QML, because they do not contain any variables and thus one can ignore how domains are like. We show the other cases.

1. For one direction, suppose $\mathfrak{F}$ has increasing domains, i.e., for all $w, v \in W$, wRv implies $D_w \subseteq D_v$. We show $\mathfrak{F} \models \Box(Qx \to P) \to (\Box Qx \to \Box P)$. Take any interpretation I, world w, assignment α such that $\Box(Qx \to P) \to (\Box Qx \to \Box P)$ is an α_w-formula. Assume $(\mathfrak{F}, I), w, \alpha \models \Box(Qx \to P)$ and $(\mathfrak{F}, I), w, \alpha \models \Box Qx$. To show $(\mathfrak{F}, I), w, \alpha \models \Box Px$, take also any world v such that wRv. Since $\mathfrak{F}$ has increasing domains, $Qx \to P$ and Qx are also α_v-formulas. Hence $(\mathfrak{F}, I), w, \alpha \models P$ follows from our assumption and wRv, as required.

   For the other direction, suppose $\mathfrak{F} \models \Box(Qx \to P) \to (\Box Qx \to \Box P)$. Take any worlds w, v such that wRv, and suppose $d \in D_w$. We show $d \in D_v$. Let I and α be an interpretation and an assignment such that, for all $u \in W$, $I(Q, u) = D_u$; for all $u \in W$, if $d \in D_u$ then $I(P, u) = \mathsf{True}$, otherwise $I(P, u) = \mathsf{False}$; $\alpha(x) = d$. We claim $(\mathfrak{F}, I), w, \alpha \models \Box Qx$ and $(\mathfrak{F}, I), w, \alpha \models \Box(Qx \to P)$. The former follows from $I(Q, u) = D_u$ for all $u \in W$. The latter is established as follows. Consider any world u such that $Qx \to P$ is an α_u-formula and wRu. Then $(\mathfrak{F}, I), u, \alpha \models Qx \to P$ holds, because $\alpha(x) = d \in D_u$ and thus $(\mathfrak{F}, I), u, \alpha \models P$. Now, we also have $(\mathfrak{F}, I), w, \alpha \models \Box(Qx \to P) \to (\Box Qx \to \Box P)$ by our initial supposition and $\alpha(x) = d \in D_w$. Hence it follows from the claim above that $(\mathfrak{F}, I), w, \alpha \models \Box P$. By wRv this provides $(\mathfrak{F}, I), v, \alpha \models P$, which implies $d \in D_v$, as required.

2. For one direction, suppose $\mathfrak{F}$ is serial with n objects, i.e., for all $w \in W$ and $d_1, \dots, d_n \in D_w$, there exists some $v \in W$ such that wRv and $d_1, \dots, d_n \in D_v$. We show $\mathfrak{F} \models \Box S\vec{x} \to \Diamond S\vec{x}$, where $\vec{x} = (x_1, \dots, x_n)$. Take any interpretation I, world w, assignment α such that $\Box S\vec{x} \to \Diamond S\vec{x}$ is an α_w-formula. Assume $(\mathfrak{F}, I), w, \alpha \models \Box S\vec{x}$. By seriality with n objects of $\mathfrak{F}$, we have some world v such that wRv and $\alpha(x_1), \dots, \alpha(x_n) \in D_v$. Then $S\vec{x}$ is an α_v-formula. Hence $(\mathfrak{F}, I), w, \alpha \models \Diamond S\vec{x}$ follows from our assumption and wRv, as required.

For the other direction, suppose $\mathfrak{F} \models \Box S\vec{x} \to \Diamond S\vec{x}$. Take any world w and any objects $d_1, \ldots, d_n \in D_w$. We show there exists some world v such that wRv and $d_1, \ldots, d_n \in D_v$. Let I and α be an interpretation and an assignment such that $I(S, u) = D_u^n$ for all $u \in W$; $\alpha(x_i) = d_i$ for all i ($1 \leqslant i \leqslant n$). We claim $(\mathfrak{F}, I), w, \alpha \models \Box S\vec{x}$. We also have $(\mathfrak{F}, I), w, \alpha \models \Box S\vec{x} \to \Diamond S\vec{x}$ by our initial supposition and $\alpha(x_i) = d_i \in D_w$. Thus $(\mathfrak{F}, I), w, \alpha \models \Diamond S\vec{x}$ follows from the claim above. This implies the existence of some world v such that wRv and $d_1, \ldots, d_n \in D_v$, as required.

5. For one direction, suppose $\mathfrak{F}$ is weakly symmetric, i.e., for all $w, v \in W$, wRv and $D_w \cap D_v \neq \emptyset$ jointly imply vRw. We show $\mathfrak{F} \models Qx \to \Box\Diamond Qx$. Take any interpretation I, world w, assignment α such that $Qx \to \Box\Diamond Qx$ is an α_w-formula. Assume $(\mathfrak{F}, I), w, \alpha \models Qx$. To show $(\mathfrak{F}, I), w, \alpha \models \Box\Diamond Qx$, take also any world v such that $\Diamond Qx$ is an α_v-formula and wRv. Then $\alpha(x) \in D_w \cap D_v$. By weak symmetry of $\mathfrak{F}$, we have vRw. Hence $(\mathfrak{F}, I), v, \alpha \models \Diamond Qx$ follows from our assumption.

   For the other direction, suppose $\mathfrak{F} \models Qx \to \Box\Diamond Qx$. Take any worlds w, v such that wRv and $D_w \cap D_v \neq \emptyset$. We show vRw. Consider some $d \in D_w \cap D_v$ and define an interpretation I and an assignment α such that, for all $u \in W$, if $u = w$ then $I(Q, u) = D_u$, otherwise $I(Q, u) = \emptyset$; $\alpha(x) = d$. We claim $(\mathfrak{F}, I), w, \alpha \models Qx$. We also have $(\mathfrak{F}, I), w, \alpha \models Qx \to \Box\Diamond Q$ by our initial supposition and $\alpha(x) = d \in D_w$. Thus $(\mathfrak{F}, I), w, \alpha \models \Box\Diamond Qx$ follows from the claim above. Since $\Diamond Qx$ is an α_v-formula and wRv, $(\mathfrak{F}, I), v, \alpha \models \Diamond Qx$ holds. This implies vRw, as required.

7. For one direction, suppose $\mathfrak{F}$ is weakly transitive, i.e., for all $w, v, u \in W$, wRv and vRu and $D_w \cap D_v \cap D_u \neq \emptyset$ jointly imply wRu. We show $\mathfrak{F} \models \Box Qx \to \Box\Box Qx$. Take any interpretation I, world w, assignment α such that $\Box Qx \to \Box\Box Qx$ is an α_w-formula. Assume $(\mathfrak{F}, I), w, \alpha \models \Box Qx$. To show $(\mathfrak{F}, I), w, \alpha \models \Box\Box Qx$, take also any world v such that $\Box Qx$ is an α_v-formula and wRv, and any world u such that Qx is an α_u-formula and vRu. Then $\alpha(x) \in D_w \cap D_v \cap D_u$. By weak transitivity of $\mathfrak{F}$ we have wRu. Hence $(\mathfrak{F}, I), u, \alpha \models Qx$ follows from our assumption, as required.

   For the other direction, suppose $\mathfrak{F} \models \Box Qx \to \Box\Box Qx$. Take any worlds w, v, u such that wRv, vRu and $D_w \cap D_v \cap D_u \neq \emptyset$. We show wRu. Consider some $d \in D_w \cap D_v \cap D_u$ and define an interpretation I and an assignment α such that, for all $u' \in W$, if wRu' then $I(Q, u') = D_{u'}$, otherwise $I(Q, u') = \emptyset$; $\alpha(x) = d$. We claim $(\mathfrak{F}, I), w, \alpha \models \Box Qx$. We also have $(\mathfrak{F}, I), w, \alpha \models \Box Qx \to \Box\Box Qx$ by our initial supposition and $\alpha(x) = d \in D_w$. Hence $(\mathfrak{F}, I), w, \alpha \models \Box\Box Qx$ follows from the claim above. Since Qx is an α_v-formula and an α_u-formula, it also follows from wRv and vRu that $(\mathfrak{F}, I), u, \alpha \models Qx$. This implies $I(Q, u) \neq \emptyset$ so wRu, as required.

9. For one direction, suppose $\mathfrak{F}$ is weakly euclidean, i.e., for all $w, v, u \in W$, wRv and wRu and $D_w \cap D_v \cap D_u \neq \emptyset$ jointly imply vRu. We show $\mathfrak{F} \models \Diamond Qx \to \Box\Diamond Qx$. Take

any interpretation I, world w, assignment α such that $\Diamond Qx \to \Box\Diamond Qx$ is an α_w-formula. Assume $(\mathfrak{F}, I), w, \alpha \models \Diamond Qx$. To show that $(\mathfrak{F}, I), w, \alpha \models \Box\Diamond Qx$, take also any world v such that $\Diamond Qx$ is an α_v-formula and wRv. By our assumption, there exists some world u such that Qx is an α_u-formula, wRu, and $(\mathfrak{F}, I), u, \alpha \models Qx$. Therefore, it holds that wRv, wRu and $\alpha(x) \in D_w \cap D_v \cap D_u$. By weak euclideaness of $\mathfrak{F}$ we have vRu. Hence, as $(\mathfrak{F}, I), u, \alpha \models Qx$, $(\mathfrak{F}, I), u, \alpha \models \Diamond Qx$ holds, as required.

For the other direction, suppose $\mathfrak{F} \models \Diamond Qx \to \Box\Diamond Qx$. Take any worlds w, v, u such that wRv, wRu and $D_w \cap D_v \cap D_u \neq \emptyset$. We show vRu. Consider some $d \in D_w \cap D_v \cap D_u$ and define an interpretation I and an assignment α such that, for all $u' \in W$, if $u' = u$ then $I(Q, u') = D_{u'}$, otherwise $I(Q, u') = \emptyset$; $\alpha(x) = d$. We claim $(\mathfrak{F}, I), w, \alpha \models \Diamond Qx$. We also have $(\mathfrak{F}, I), w, \alpha \models \Diamond Qx \to \Box\Diamond Qx$ by our initial supposition and $\alpha(x) = d \in D_w$. It then follows from the claim above that $(\mathfrak{F}, I), w, \alpha \models \Box\Diamond Qx$. Since $\Diamond Qx$ is an α_v-formula, it also follows from wRv that $(\mathfrak{F}, I), v, \alpha \models \Diamond Qx$. This implies vRu, as required.

■

Let ω be the cardinality of $\mathbb{N}$. As a corollary, it follows that $\{ \mathrm{D}_{\leqslant n} \mid n \leqslant \omega \}$ corresponds to the class of all the serial frames with n objects for all $n \in \mathbb{N}$.

5.1.2 Hilbert System $\mathsf{H}(\mathbf{cK}\Sigma)$

A Hilbert system $\mathsf{H}(\mathbf{cK})$ for CMPC consists of axioms and inference rules in Table 5.1, where all axioms and inference rules are presented as schemas.

(Taut1)	$\varphi \to (\psi \to \varphi)$	
(Taut2)	$(\varphi \to (\psi \to \gamma)) \to ((\varphi \to \psi) \to (\varphi \to \gamma))$	
(Taut3)	$(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$	
(U)	$\forall x\varphi \to \varphi(y/x)$	
($\mathtt{K_{inv}}$)	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	if $\mathsf{FV}(\varphi) \subseteq \mathsf{FV}(\psi)$
(MP)	From $\varphi \to \psi$ and φ, we may infer ψ	
(Gen)	From $\varphi \to \psi(y/x)$, we may infer $\varphi \to \forall x\psi$	if $y \notin \mathsf{FV}(\varphi, \forall x\psi)$
(Nec)	From φ, we may infer $\Box\varphi$	

Table 5.1: Hilbert system $\mathsf{H}(\mathbf{cK})$

Expansions of $\mathsf{H}(\mathbf{cK})$ for CMPC are obtained as follows. Put

$$\mathsf{Axiom}_{\mathsf{CMPC}} := \{ \mathrm{K}, \mathrm{T}, \mathrm{B}, \mathrm{B}^-, 4, 4^-, 5, 5^- \} \cup \{ \mathrm{D}_{\leqslant n} \mid n < \omega \}$$

where ω is the cardinality of $\mathbb{N}$. For a set $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{CMPC}}$, we mean by $\mathsf{Inst}(\Sigma)$ the set of all instances of the schema corresponding to a formula of Σ which is listed in Table 5.2.

Formulas	Schemas
$\mathrm{K} = \Box(Qx_1 \to P) \to (\Box Qx_1 \to \Box P)$	$(\mathtt{K}) = \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
$\mathrm{D}_{\leqslant n} = \Box S\vec{x} \to \Diamond S\vec{x}$	$(\mathtt{D}_{\leqslant n})^{\dagger} := \Box\varphi \to \Diamond\varphi$
$\mathrm{T} = \Box P \to P$	$(\mathtt{T}) = \Box\varphi \to \varphi$
$\mathrm{B} = P \to \Box\Diamond P$	$(\mathtt{B}) = \varphi \to \Box\Diamond\varphi$
$\mathrm{B}^{-} = Qx_1 \to \Box\Diamond Qx_1$	$(\mathtt{B}^{-})^{\ddagger} := \varphi \to \Box\Diamond\varphi$
$4 = \Box P \to \Box\Box P$	$(\mathtt{4}) = \Box\varphi \to \Box\Box\varphi$
$4^{-} = \Box Qx_1 \to \Box\Box Qx_1$	$(\mathtt{4}^{-})^{\ddagger} := \Box\varphi \to \Box\Box\varphi$
$5 = \Diamond P \to \Box\Diamond P$	$(\mathtt{5}) = \Diamond\varphi \to \Box\Diamond\varphi$
$5^{-} = \Diamond Qx_1 \to \Box\Diamond Qx_1$	$(\mathtt{5}^{-})^{\ddagger} := \Diamond\varphi \to \Box\Diamond\varphi$
$\dagger$: $\#\mathsf{FV}(\varphi) \leqslant n$ and $\ddagger$: $\mathsf{FV}(\varphi) \neq \emptyset$.	

Table 5.2: The schemas corresponding to formulas of Σ

Definition 175. Given a set Φ of formulas, the Hilbert system $\mathsf{H}(\mathbf{cK}\oplus\Phi)$ is the system obtained from $\mathsf{H}(\mathbf{cK})$ by adding all formulas of Φ as axioms. Given $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{CMPC}}$, $\mathsf{H}(\mathbf{cK}\Sigma)$ denotes the system $\mathsf{H}(\mathbf{cK} \oplus \mathsf{Inst}(\Sigma))$. We sometimes write $\mathsf{H}(\mathbf{cK}\{\, \mathrm{T}, 4, \mathrm{B} \,\})$ as $\mathsf{H}(\mathbf{cS5})$.

We note here that $\mathsf{H}(\mathbf{cS5})$ is Seligman's original CMPC [87, 89]. The notion of proof in $\mathsf{H}(\mathbf{cK}\Sigma)$ is defined as in $\mathsf{H}(\mathbf{K}\Sigma)$.

Example 176. In the same way that seriality with n objects is somewhat complicated, it might take time to see what $(\mathtt{D}_{\leqslant n})$ is saying. It is relatively easy to understand if we read $\Box\varphi$ as "it is obligatory that φ." Then, we could understand $(\mathtt{D}_{\leqslant n})$ as a claim that there are no moral dilemmas *for at most n agents*. Consider again the world w in which agents a, b, c are drowning in the river. In this case, $(\mathtt{D}_{\leqslant 2})$ guarantees that there are no moral dilemmas for at most 2 agents. However, it does not guarantee that there are no moral dilemmas for 3 agents.

For convenience, we list some formulas provable in $\mathsf{H}(\mathbf{cK}\Sigma)$.

Proposition 177. Let $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{CMPC}}$.

1. $\vdash_{\mathsf{H}(\mathbf{cK}\Sigma)} (\Box\varphi_1 \wedge \cdots \wedge \Box\varphi_n) \to \Box(\varphi_1 \wedge \cdots \wedge \varphi_n)$.
2. $\vdash_{\mathsf{H}(\mathbf{cK}\Sigma)} \Diamond\neg(\varphi \to \varphi) \leftrightarrow \bot$.

Theorem 178. (Soundness of $\mathsf{H}(\mathbf{tK}\Sigma)$) Let $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{CMPC}}$ and $\mathbb{F}_\Sigma$ be the class of all the frames to which Σ corresponds. For all formulas φ, if φ is provable in $\mathsf{H}(\mathbf{cK}\Sigma)$, then φ is valid in $\mathbb{F}_\Sigma$.

Proof. By induction on the length of a proof of φ. The validity of $(\mathtt{K_{inv}})$ is established in item 3 of Proposition 168. One can also easily establish that the other axioms of $\mathsf{H}(\mathbf{cK})$ are valid in $\mathbb{F}_\Sigma$ and each inference rule of $\mathsf{H}(\mathbf{cK})$ preserves the validity in $\mathbb{F}_\Sigma$. The validity of additional axioms from Σ are shown similarly as in Proposition 174. ■

For strong completeness of CMPCs, Seligman [87] first used a canonical model construction to attempt to prove the strong completeness of his original CMPC $\mathsf{H}(\mathbf{cS5})$. Unfortunately, it turned out by K. Sano's contribution in [83] that Seligman's canonical model did not satisfy transitivity (Theorem 193). This means that Seligman's first attempt failed. For this reason, Seligman [89] proved the strong completeness by the step-by-step method introduced in Blackburn et al. [6]. In this section, we use Seligman's canonical model with accessibility relation to prove the strong completeness of $\mathsf{H}(\mathbf{cK})$ and its extensions with $\mathrm{D}_{\leqslant n}$, T, B and B^- (Theorem 189). Furthermore, we prove in the end of this section that K, 4, 4^-, 5 and 5^- are *not* canonical (Theorems 192, 193, 194). Thus, our canonical model does not provide us the strong completeness of the other systems like $\mathsf{H}(\mathbf{cK}4)$ and $\mathsf{H}(\mathbf{cK}5)$.

Recall that we write $\mathsf{L}_{\mathsf{CMPC}}(V)$ to denote $\mathsf{L}_{\mathsf{CMPC}}$ in which the set of variables is V.

Definition 179. We define Var^+ as $\mathsf{Var} \cup \mathsf{Var}'$, where Var' is a *fresh* countably infinite set of variables disjoint from Var. Given $V \subseteq \mathsf{Var}^+$, the set $\mathsf{Term}(V)$ refers to the set of all terms in $\mathsf{L}_{\mathsf{CMPC}}(V)$. Given a set Γ of formulas, $\mathsf{L}_{\mathsf{CMPC}}(\Gamma)$ and $\mathsf{Term}(\Gamma)$ denote $\mathsf{L}_{\mathsf{CMPC}}(\mathsf{FV}(\Gamma))$ and $\mathsf{Term}(\mathsf{FV}(\Gamma))$, respectively. Given $V, V' \subseteq \mathsf{Var}^+$, by a notation $V \sqsubset V'$ we mean that $V \subseteq V'$ and $V' \backslash V$ is countably infinite.

Recall $\mathsf{Axiom}_{\mathsf{CMPC}} = \{\mathrm{K}, \mathrm{T}, \mathrm{B}, \mathrm{B}^-, 4, 4^-, 5, 5^-\} \cup \{\mathrm{D}_{\leqslant n} \mid n < \omega\}$.

Definition 180. Let $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{CMPC}}$. Given a set Γ of formulas, Γ is $\mathsf{H}(\mathbf{cK}\Sigma)$-*inconsistent* if $\Gamma \vdash_\Lambda \bot$; Γ is $\mathsf{H}(\mathbf{cK}\Sigma)$-*consistent* if Γ is not $\mathsf{H}(\mathbf{cK}\Sigma)$-inconsistent; Γ is *maximal* if $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$ for all formulas φ in $\mathsf{L}(\Gamma)$; Γ is a *maximal* $\mathsf{H}(\mathbf{cK}\Sigma)$-*consistent set* ($\mathsf{H}(\mathbf{cK}\Sigma)$-*MCS*) if Γ is $\mathsf{H}(\mathbf{cK}\Sigma)$-consistent and maximal; Γ is *witnessed* if, for all formulas of the form $\forall x\varphi$ in $\mathsf{L}(\Gamma)$, there exists some $y \in \mathsf{FV}(\Gamma)$ such that $\varphi(y/x) \to \forall x\varphi \in \Gamma$.

Proposition 181. Let $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{CMPC}}$, Γ a $\mathsf{H}(\mathbf{cK}\Sigma)$-MCS in $\mathsf{L}(\mathsf{Var}^+)$ and φ, ψ formulas in $\mathsf{L}(\Gamma)$.

1. $\Gamma \vdash_{\mathsf{H}(\mathbf{cK}\Sigma)} \varphi$ iff $\varphi \in \Gamma$.
2. If $\varphi \in \Gamma$ and $\vdash_{\mathsf{H}(\mathbf{cK}\Sigma)} \varphi \to \psi$, then $\psi \in \Gamma$.

3. $\bot \notin \Gamma$.
4. $\varphi \to \psi \in \Gamma \quad$ iff $\quad \varphi \notin \Gamma$ or $\psi \in \Gamma$.

Until Theorem 189, we abbreviate $\mathsf{H}(\mathbf{cK}\Sigma)$ as Λ for some fixed $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{CMPC}}$.

Lemma 182. (Lindenbaum Lemma) Let Γ be a Λ-consistent set of formulas in $\mathsf{L}(\mathsf{Var}^+)$ and $X \subseteq \mathsf{Var}^+$ such that $X \backslash \mathsf{FV}(\Gamma)$ is countably infinite. There is a witnessed Λ-MCS Γ^+ in $\mathsf{L}(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma^+) \sqsubset \mathsf{Var}^+$, $\Gamma \subseteq \Gamma^+$, and $\mathsf{FV}(\Gamma^+) \subseteq \mathsf{FV}(\Gamma) \cup X$.

Definition 183. (Seligman [87]) The *canonical* Λ*-frame* is a tuple

$$\mathfrak{F}^\Lambda = (W^\Lambda, R^\Lambda, (D^\Lambda_\Gamma)_{\Gamma \in W^\Lambda}),$$

where

- $W^\Lambda := \{ \Gamma \mid \Gamma$ is a witnessed Λ-MCS in $\mathsf{L}(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma) \sqsubset \mathsf{Var}^+ \}$;
- $\Gamma R^\Lambda \Delta \quad$ iff $\quad \Box\varphi \in \Gamma$ implies $\varphi \in \Delta \quad$ for all formulas φ in $\mathsf{L}(\Delta)$;
- $D_\Gamma = \mathsf{FV}(\Gamma)$.

The *canonical* Λ*-model* is a tuple $\mathfrak{M}^\Lambda = (\mathfrak{F}^\Lambda, I^\Lambda)$, where

- $\mathfrak{F}^\Lambda$ is the canonical Λ-frame;
- $(x_1, \ldots, x_n) \in I^\Lambda(P, \Gamma) \quad$ iff $\quad Px_1 \ldots x_n \in \Gamma$;

The *canonical assignment* is the assignment $\iota : \mathsf{Var}^+ \to D^\Lambda$ defined by $\iota(x) := x$.

Remark 184. The canonical model for CMPC should be compared to the canonical model for QML having increasing domains (Definition 49). The only difference lies in the definition of accessibility relation R^Λ. The canonical model for CMPC requires a condition that formulas φ are in $\mathsf{L}(\Delta)$. The condition mirrors the condition of α_Δ-formula in the satisfaction relation of $\Box\varphi$, i.e., a condition that φ is an α_Δ-formula.

Proposition 185. The canonical Λ-model *is* a model.

Lemma 186. (Existence Lemma, Seligman [87]) If $\neg\Box\varphi \in \Gamma \in W^\Lambda$, then

$$\{ \neg\varphi \} \cup \{ \gamma \mid \Box\gamma \in \Gamma \text{ and } \mathsf{FV}(\gamma) \subseteq \mathsf{FV}(\varphi) \} \nvdash_\Lambda \bot$$

and there exists some $\Delta \in W^\Lambda$ such that $\neg\varphi \in \Delta$, $\Gamma R^\Lambda \Delta$ and $\mathsf{FV}(\Gamma) \cap \mathsf{FV}(\Delta) = \mathsf{FV}(\varphi)$.

Proof. Suppose $\neg\Box\varphi \in \Gamma \in W^\Lambda$. We first establish

$$\Delta_0 := \{ \neg\varphi \} \cup \{ \gamma \mid \Box\gamma \in \Gamma \text{ and } \mathsf{FV}(\gamma) \subseteq \mathsf{FV}(\varphi) \} \nvdash_\Lambda \bot.$$

Suppose not. Then (1) $\vdash \bigwedge_{i \leqslant m} \gamma_i \to \varphi$ for some $\gamma_1, \ldots, \gamma_m$ such that (2) $\Box\gamma_i \in \Gamma$ and (3) $\mathsf{FV}(\gamma_i) \subseteq \mathsf{FV}(\varphi)$. We use (1) to obtain $\Gamma \vdash \bot$ as follows.

1. $\vdash \bigwedge_{i\leqslant m}\gamma_i \to \varphi$ (1)
2. $\vdash \Box(\bigwedge_{i\leqslant m}\gamma_i \to \varphi)$ 1, $(\texttt{Nec})$
3. $\vdash \Box(\bigwedge_{i\leqslant m}\gamma_i \to \varphi) \to (\Box\bigwedge_{i\leqslant m}\gamma_i \to \Box\varphi)$ (3), $(\texttt{K}_{\texttt{inv}})$
4. $\vdash \Box\bigwedge_{i\leqslant m}\gamma_i \to \Box\varphi$ 2, 3, MP
5. $\vdash \bigwedge_{i\leqslant m}\Box\gamma_i \to \Box\varphi$ 4, item 1 of Proposition 177
6. $\Gamma \vdash \bigwedge_{i\leqslant m}\Box\gamma_i$ (2)
7. $\Gamma \vdash \Box\varphi$ 5, 6, MP
8. $\Gamma \vdash \neg\Box\varphi$ $\neg\Box\varphi \in \Gamma$
9. $\Gamma \vdash \bot$ 7, 8, PC

However, Γ should be Λ-consistent, so a contradiction. Thus Δ_0 is Λ-consistent.

We use Δ_0 to construct a desired Λ-MCS Δ. Put $X = \mathsf{Var}^+ \backslash \mathsf{FV}(\Gamma)$. From $\mathsf{FV}(\Gamma) \sqsubset \mathsf{Var}^+$ and $\mathsf{FV}(\Delta_0) \subseteq \mathsf{FV}(\Gamma)$, one can observe that $X\backslash\mathsf{FV}(\Delta_0)$ is countably infinite. By Lemma 182, we obtain some $\Delta \in W^\Lambda$ such that $\Delta_0 \subseteq \Delta$ and $\mathsf{FV}(\Delta) \subseteq \mathsf{FV}(\Delta_0) \cup X$. What must be established now is (a) $\neg\varphi \in \Delta$, (b) $\Gamma R^\Lambda \Delta$, and (c) $\mathsf{FV}(\Gamma) \cap \mathsf{FV}(\Delta) = \mathsf{FV}(\varphi)$. Since (a) is easy, we confirm only (b) and (c).

For (c), as $\mathsf{FV}(\varphi) \subseteq \mathsf{FV}(\Gamma) \cap \mathsf{FV}(\Delta)$ is immediate from $\neg\Box\varphi \in \Gamma$ and (a), we show $\mathsf{FV}(\Gamma) \cap \mathsf{FV}(\Delta) \subseteq \mathsf{FV}(\varphi)$. Suppose $x \in \mathsf{FV}(\Gamma) \cap \mathsf{FV}(\Delta)$. Then $x \notin X$ from $x \in \mathsf{FV}(\Gamma)$, and also $x \in \mathsf{FV}(\Delta_0) \cup X$ from $x \in \mathsf{FV}(\Delta)$ and $\mathsf{FV}(\Delta) \subseteq \mathsf{FV}(\Delta_0) \cup X$. Thus $x \in \mathsf{FV}(\Delta_0) = \mathsf{FV}(\varphi)$, as required.

For (b), we show $\Box\psi \in \Gamma$ implies $\psi \in \Delta$ for all formulas ψ in $\mathsf{L}(\Delta)$. Take any formula ψ in $\mathsf{L}(\Delta)$ and suppose $\Box\psi \in \Gamma$. Then $\mathsf{FV}(\psi) \subseteq \mathsf{FV}(\Gamma) \cap \mathsf{FV}(\Delta)$. By (c), $\mathsf{FV}(\psi) \subseteq \mathsf{FV}(\varphi)$. Since $\Box\psi \in \Gamma$, we can deduce $\psi \in \Delta_0 \subseteq \Delta$, as required. ■

Lemma 187. (Truth Lemma) Let $\mathfrak{M}^\Lambda$ be the canonical Λ-model and ι be the canonical assignment. For all Λ-MCS $\Gamma \in W^\Lambda$ and formulas φ in $\mathsf{L}(\Gamma)$,

$$\mathfrak{M}^\Lambda, \Gamma, \iota \models \varphi \quad \text{iff} \quad \varphi \in \Gamma$$

Proof. By induction on the length of φ. First order cases are proved in the same way as the standard proof of Truth Lemma for QML (Lemma 52), but we spell out a case in which φ is $\forall x\psi$. In addition to the case, we also confirm a case in which φ is $\Box\psi$.

We first show the case in which φ is $\forall x\psi$. For the right-to-left direction, suppose $\forall x\psi \in \Gamma$ and consider any $y \in D_\Gamma$. What should be established is $\mathfrak{M}^\Lambda, \Gamma, \iota(x|y) \models \psi$. As $\psi(y/x) \in \Gamma$, it follows from inductive hypothesis that $\mathfrak{M}^\Lambda, \Gamma, \iota \models \psi(y/x)$. By Proposition 167 and $\iota(y) = y$, it holds that $\mathfrak{M}^\Lambda, \Gamma, \iota(x|y) \models \psi$. For the left-to-right direction, we show the contraposition. Suppose $\forall x\psi \notin \Gamma$. Since $\forall x\psi$ is a formula in $\mathsf{L}(\Gamma)$ and Γ is witnessed, there exists some $y \in \mathsf{FV}(\Gamma)$ such that $\psi(y/x) \to \forall x\psi \in \Gamma$. As

$\psi(y/x) \notin \Gamma$ by our supposition, $\mathfrak{M}, \Gamma, \alpha \not\models \psi(y/x)$ follows from inductive hypothesis. By Proposition 167 and $\iota(y) = y$, it holds that $\mathfrak{M}, \Gamma, \alpha \not\models \forall x\psi$, as required.

We then prove the case in which φ is $\Box\psi$. For the right-to-left direction, suppose $\Box\psi \in \Gamma$ and consider any $\Delta \in W$ such that ψ is an ι_Δ-formula and $\Gamma R^\Lambda \Delta$. What should be established is $\mathfrak{M}, \Delta, \iota \models \psi$. Since ψ is an ι_Δ-formula, ψ is a formula in $\mathsf{L}(\Delta)$. This implies $\psi \in \Delta$ by $\Box\psi \in \Gamma$ and $\Gamma R^\Lambda \Delta$, so $\mathfrak{M}, \Delta, \iota \models \psi$ follows from inductive hypothesis. For the left-to-right direction, we show the contraposition. Suppose $\Box\psi \notin \Gamma$. Then $\neg\Box\psi \in \Gamma$ since $\Box\psi$ is a formula in $\mathsf{L}(\Gamma)$. By Lemma 186, we obtain some $\Delta \in W^\Lambda$ such that $\neg\psi \in \Delta$ and $\Gamma R^\Lambda \Delta$. As $\neg\psi \in \Delta$ implies $\psi \notin \Delta$, inductive hypothesis provides $\mathfrak{M}, \Delta, \iota \not\models \psi$. Thus $\mathfrak{M}, \Gamma, \iota \not\models \Box\psi$ follows from $\Gamma R^\Lambda \Delta$, as required. ■

Proposition 188. Suppose $\Sigma \subseteq \{\mathrm{T}, \mathrm{B}, \mathrm{B}^-\} \cup \{\mathrm{D}_{\leqslant n} \mid n < \omega\}$. Let Φ be a set of formulas such that $\mathsf{Inst}(\Sigma) \subseteq \Phi$, $\Lambda_\Phi = \mathsf{H}(\mathbf{cK} \oplus \Phi)$, and $\mathfrak{F}^{\Lambda_\Phi}$ the canonical Λ_Φ-frame. Then the following hold.

1. If $\mathrm{T} \in \Sigma$ then $\mathfrak{F}^{\Lambda_\Phi}$ is reflexive.
2. If $\mathrm{D}_{\leqslant n} \in \Sigma$ then $\mathfrak{F}^{\Lambda_\Phi}$ is serial with n objects.
3. If $\mathrm{B} \in \Sigma$ then $\mathfrak{F}^{\Lambda_\Phi}$ is symmetric.
4. If $\mathrm{B}^- \in \Sigma$ then $\mathfrak{F}^{\Lambda_\Phi}$ is weakly symmetric.

Therefore, the canonical Λ_Φ-frame is contained in the class of all the frames to which Σ corresponds.

Proof.

1. Suppose $\mathrm{T} \in \Sigma$. We show $\mathfrak{F}^{\Lambda_\Phi}$ is reflexive, i.e., for all $\Gamma \in W^{\Lambda_\Phi}$, $\Gamma R^{\Lambda_\Phi} \Gamma$. Take any formula φ in $\mathsf{L}(\Gamma)$ such that $\Box\varphi \in \Gamma$. It holds that $\varphi \in \Gamma$ by $(\mathtt{T})$.
2. Suppose $\mathrm{D}_{\leqslant n} \in \Sigma$. We show $\mathfrak{F}^{\Lambda_\Phi}$ is serial with n objects, i.e., for all $\Gamma \in W^{\Lambda_\Phi}$ and $x_1, \ldots, x_n \in D_\Gamma^{\Lambda_\Phi}$, there exists some $\Delta \in W^{\Lambda_\Phi}$ such that $\Gamma R^{\Lambda_\Phi} \Delta$ and $x_1, \ldots, x_n \in D_\Delta$. Take any $\Gamma \in W^{\Lambda_\Phi}$ and $x_1, \ldots, x_n \in D_\Gamma^{\Lambda_\Phi}$. Let $\vec{x} = (x_1, \ldots, x_n)$ and $\bot(\vec{x}) := \neg(P\vec{x} \to P\vec{x})$. We claim that $\neg\Box\bot(\vec{x}) \in \Gamma$ since $\Box\bot(\vec{x}) \to \bot$ is provable in $\Lambda_\Phi = \mathsf{H}(\mathbf{cK} \oplus \Phi)$ by $(\mathtt{D}_{\leqslant n})$ and item 2 of Proposition 177 and thus $\Box\bot(\vec{x}) \to \bot \in \Gamma$ holds. By Existence Lemma (Lemma 186), there exists some $\Delta \in W^\Lambda$ such that $\Gamma R^\Lambda \Delta$ and $\mathsf{FV}(\Gamma) \cap \mathsf{FV}(\Delta) = \{x_1, \ldots, x_n\}$. This implies $x_1, \ldots, x_n \in D_\Delta^{\Lambda_\Phi}$.
3. Suppose $\mathrm{B} \in \Sigma$. We show $\mathfrak{F}^{\Lambda_\Phi}$ is symmetric, i.e., for all $\Gamma, \Delta \in W^{\Lambda_\Phi}$, $\Gamma R^{\Lambda_\Phi} \Delta$ implies $\Delta R^{\Lambda_\Phi} \Gamma$. Assume $\Gamma R^{\Lambda_\Phi} \Delta$ and take any formula φ in $\mathsf{L}(\Gamma)$ such that $\Box\varphi \in \Delta$. We claim $\Diamond\Box\varphi \in \Gamma$, because otherwise $\Box\neg\Box\varphi \in \Gamma$ holds and thus $\neg\Box\varphi \in \Delta$ follows from $\Gamma R^{\Lambda_\Phi} \Delta$, which gives a contradiction together with $\Box\varphi \in \Delta$. Then it follows from the claim above and $(\mathtt{B})$ that $\varphi \in \Gamma$, as required.

4. Suppose $\mathrm{B}^- \in \Sigma$. We show $\mathfrak{F}^{\Lambda_\Phi}$ is weakly symmetric, i.e., for all $\Gamma, \Delta \in W^{\Lambda_\Phi}$, $\Gamma R^{\Lambda_\Phi} \Delta$ and $D_\Gamma \cap D_\Delta \neq \emptyset$ jointly imply $\Delta R^{\Lambda_\Phi} \Gamma$. Assume $\Gamma R^{\Lambda_\Phi} \Delta$ and $D_\Gamma \cap D_\Delta \neq \emptyset$. Take any formula φ in $\mathsf{L}(\Gamma)$ such that $\Box\varphi \in \Delta$. Consider some variable $x \in D_\Gamma \cap D_\Delta$ and a formula $Px \to Px$. Since $\Box\varphi \in \Delta$ and $\Box(Px \to Px) \in \Delta$, we obtain $\Box(\varphi \land (Px \to Px)) \in \Delta$ by item 1 of Proposition 177. This enables us to claim $\Diamond\Box(\varphi \land (Px \to Px)) \in \Gamma$, because otherwise $\Box\neg\Box(\varphi \land (Px \to Px)) \in \Gamma$ holds and thus $\neg\Box(\varphi \land (Px \to Px)) \in \Delta$ follows from $\Gamma R^{\Lambda_\Phi} \Delta$, which gives a contradiction together with $\Box(\varphi \land (Px \to Px)) \in \Gamma$. Then it follows from the claim above and (B^-) that $\varphi \land (Px \to Px) \in \Gamma$. This implies $\varphi \in \Gamma$, as required.

■

Theorem 189. (Strong completeness of $\mathsf{H}(\mathbf{cK}\Sigma)$) Let $\Sigma \subseteq \{\mathrm{T}, \mathrm{B}, \mathrm{B}^-\} \cup \{\mathrm{D}_{\leqslant n} \mid n < \omega\}$ and $\mathbb{F}_\Sigma$ be the class of all the frames to which Σ corresponds. For all formulas φ and sets Γ of formulas, if φ is a consequence from Γ in $\mathbb{F}_\Sigma$, then φ is provable from Γ in $\mathsf{H}(\mathbf{cK}\Sigma)$.

Proof. Let $\Lambda = \mathsf{H}(\mathbf{cK}\Sigma)$ for short. Suppose φ is not provable from Γ in Λ. We show φ is not a consequence from Γ in $\mathbb{F}_\Sigma$. Note first that $\Gamma \cup \{\neg\varphi\}$ is Λ-consistent in L. We claim $\Gamma \cup \{\neg\varphi\}$ is Λ-consistent in $\mathsf{L}(\mathsf{Var}^+)$. By Lindenbaum Lemma (Lemma 182), we obtain a witnessed Λ-MCS Γ^+ in $\mathsf{L}(\mathsf{Var}^+)$ such that $\mathsf{FV}(\Gamma^+) \sqsubset \mathsf{Var}^+$ and $\Gamma \cup \{\neg\varphi\} \subseteq \Gamma^+$. It then follows from Truth Lemma (Lemma 187) that

$$\mathfrak{M}^\Lambda, \Gamma^+, \iota \models \Gamma \quad \text{and} \quad \mathfrak{M}^\Lambda, \Gamma^+, \iota \not\models \varphi,$$

where $\mathfrak{M}^\Lambda = (\mathfrak{F}^\Lambda, I^\Lambda)$ is the canonical Λ-model and ι is the canonical assignment. We must further show that $\mathfrak{F}^\Lambda \in \mathbb{F}_\Sigma$, which is established by Proposition 188. Hence, in $\mathsf{L}(\mathsf{Var}^+)$, φ is not a consequence from Γ in $\mathbb{F}_\Sigma$. By restricting $\mathsf{L}(\mathsf{Var}^+)$ to L, we conclude in L that φ is not a consequence from Γ in $\mathbb{F}_\Sigma$. ■

Theorem 189 tells us that $\mathsf{H}(\mathbf{cK})$ and its extensions with $\mathrm{D}_{\leqslant n}, \mathrm{T}, \mathrm{B}, \mathrm{B}^-$ are strongly complete with respect to the corresponding classes of frames. Unfortunately, this result is not carried over to the other extensions such as $\mathsf{H}(\mathbf{cK4})$ and $\mathsf{H}(\mathbf{cK5})$. This is because $\mathrm{K}, 4, 4^-, 5, 5^-$ are *not* canonical in the following sense.[1] Recall

$$\mathsf{Axiom}_{\mathsf{CMPC}} = \{\mathrm{K}, \mathrm{T}, \mathrm{B}, \mathrm{B}^-, 4, 4^-, 5, 5^-\} \cup \{\mathrm{D}_{\leqslant n} \mid n < \omega\}$$

and $\mathsf{Inst}(\Sigma)$ is the set of all instances of the schema corresponding to a formula of Σ. We note that Theorems 193 and 194 are the contributions of K. Sano in [83].

[1]Definitions of canonicity given in the literature usually require the notion of a logic that is closed under uniform substitution (e.g. [6, p. 203]). However, CMPCs are not closed under uniform substitution. For this reason, we employ the notion of canonicity as in Definition 190, adding to the Hilbert system $\mathsf{H}(\mathbf{cK})$ all instances of additional axioms instead of the axioms as schemas.

Definition 190. (Canonicity) A *formula* $\varphi \in \mathsf{Axiom}_{\mathsf{CMPC}}$ *is canonical* if for any set Φ of formulas such that $\mathsf{Inst}(\varphi) \subseteq \Phi$, $\mathsf{Inst}(\varphi)$ is valid in the canonical $\mathsf{H}(\mathbf{cK} \oplus \Phi)$-frame.

By Propositions 174 and 188, it follows that D_n, T, B and B^- are canonical.

Corollary 191. D_n, T, B and B^- are canonical.

On the other hand, as shown below, K, 4, 4^-, 5 and 5^- are not canonical. In what follows, we say that a set Γ of formulas is *satisfiable on a class of frames* if there exist a frame $\mathfrak{F}$ in the class, an interpretation I, a world w and an assignment α such that φ is an α_w-formula and $\mathfrak{M}, w, \alpha \models \varphi$ for all $\varphi \in \Gamma$.

Theorem 192. (Non-canonicity of K) The canonical $\mathsf{H}(\mathbf{cK} \oplus \mathsf{Inst}(\mathrm{K}))$-frame does not have increasing domains. Thus, K is not canonical.

Proof. Let Λ be $\mathsf{H}(\mathbf{cK} \oplus \mathsf{Inst}(\mathrm{K}))$ and $\mathfrak{F}^\Lambda = (W^\Lambda, R^\Lambda, (D_\Gamma^\Lambda)_{\Gamma \in W^\Lambda})$ the canonical Λ-frame. We show there exist some $\Gamma, \Delta \in W^\Lambda$ such that $\Gamma R^\Lambda \Delta$ but $D_\Gamma \nsubseteq D_\Delta$. Let P be a nullary predicate symbol in L. Put

$$\Gamma_0 \coloneqq \{\, \neg\Box\neg(P \to P) \,\}.$$

It is easy to see that Γ_0 is Λ-consistent since Γ_0 is satisfiable on the class of all increasing domain frames. Moreover, $\mathsf{Var}^+ \backslash \mathsf{FV}(\Gamma_0)$ is countably infinite. Thus, by Lindenbaum Lemma (Lemma 182), we have some $\Gamma \in W^\Lambda$ such that $\Gamma_0 \subseteq \Gamma$. Since $\neg\Box\neg(P \to P) \in \Gamma$, Existence Lemma (Lemma 186) provides us some $\Delta \in W^\Lambda$ such that $\neg\neg(P \to P) \in \Delta$, $\Gamma R^\Lambda \Delta$ and $\mathsf{FV}(\Gamma) \cap \mathsf{FV}(\Delta) = \mathsf{FV}(\neg(P \to P)) = \emptyset$.

Now that we have obtained $\Gamma, \Delta \in W^\Lambda$ such that $\Gamma R^\Lambda \Delta$, it is sufficient to establish $D_\Gamma \nsubseteq D_\Delta$. Since $\forall x P$ is a formula in $\mathsf{L}(\Gamma)$ and Γ is witnessed, there exists some variable $y \in \mathsf{FV}(\Gamma) = D_\Gamma$. Then $D_\Gamma \nsubseteq D_\Delta$ holds from $\mathsf{FV}(\Gamma) \cap \mathsf{FV}(\Delta) = \emptyset$. ∎

Theorem 193. (Non-canonicity of 4 and 4^-)

1. The canonical $\mathsf{H}(\mathbf{cK} \oplus \mathsf{Inst}(\{\mathrm{T}, \mathrm{B}, 4, 5\}))$-frame is not weakly transitive. Thus, 4^- is not canonical.
2. The canonical $\mathsf{H}(\mathbf{cK} \oplus \mathsf{Inst}(\{\mathrm{T}, \mathrm{B}, 4, 5\}))$-frame is not transitive. Thus, 4 is not canonical.

Proof. It suffices to show item 1, since any frame that is not weakly transitive is not transitive. Let Λ be $\mathsf{H}(\mathbf{cK} \oplus \mathsf{Inst}(\{\mathrm{T}, \mathrm{B}, 4, 5\}))$ and $\mathfrak{F}^\Lambda = (W^\Lambda, R^\Lambda, (D_\Gamma^\Lambda)_{\Gamma \in W^\Lambda})$ the canonical Λ-frame. We show there exist some $\Gamma, \Delta, \Theta \in W^\Lambda$ such that $\Gamma R^\Lambda \Delta$ and

$\Delta R^\Lambda \Theta$ hold and also $D_\Gamma \cap D_\Delta \cap D_\Theta \neq \emptyset$, but $\Gamma R^\Lambda \Theta$ fails. Let P be a unary predicate symbol in L and fix a variable x_0. Put

$$\Gamma_0 \coloneqq \{ \Box P x_0, \Box\neg\forall y P y \}.$$

We first establish Γ_0 is Λ-consistent by showing that Γ_0 is satisfiable on the class of all reflexive, transitive and symmetric frames. Consider a model $\mathfrak{M} = (W, R, D, I)$ such that $W = \{ w, v \}$, $R = W \times W$, $D_w = \{ d, e \}$, $D_v = \{ e \}$, $I(P, w) = \{ d \}$ and $I(P, v) = \emptyset$. Let also α be an assignment such that $\alpha(x_0) = d$. Then, $\mathfrak{M}, w, \alpha \models \Box P x_0$ and $\mathfrak{M}, w, \alpha \models \Box\neg\forall y P y$. Hence Γ_0 is satisfiable on the class of all reflexive, transitive and symmetric frames.

Moreover, $\mathsf{Var}^+ \backslash \mathsf{FV}(\Gamma_0)$ is infinite. Thus, by Lindenbaum Lemma (Lemma 182), we obtain some Λ-MCS $\Gamma \in W^\Lambda$ such that $\Gamma_0 \subseteq \Gamma$. Fix some variable $x_1 \in \mathsf{FV}(\Gamma)$ which is distinct from x_0. It is not difficult to see that such a variable necessarily exists. Then $\neg\Box\neg(Px_1 \to Px_1) \in \Gamma$ since $\vdash_\Lambda (Px_1 \to Px_1) \to \neg\Box\neg(Px_1 \to Px_1)$. Thus Existence Lemma (Lemma 186) provides us some $\Delta \in W^\Lambda$ such that $\neg\neg(Px_1 \to Px_1) \in \Delta$, $\Gamma R^\Lambda \Delta$ and $\mathsf{FV}(\Gamma) \cap \mathsf{FV}(\Delta) = \mathsf{FV}(\neg(Px_1 \to Px_1)) = \{ x_1 \}$.

We now have $\Box\Box\neg\forall y P y \in \Gamma$ since $\vdash_\Lambda \Box\neg\forall y P y \to \Box\Box\neg\forall y P y$ and $\Box\neg\forall y P y \in \Gamma$. Thus, we have $\Box\neg\forall y P y \in \Delta$ because $\Gamma R^\Lambda \Delta$ and $\Box\neg\forall y P y$ is a formula in $\mathsf{L}(\Delta)$. Similarly as before, we have $\neg\Box\neg(Px_1 \to Px_1) \in \Delta$ since $\vdash_\Lambda (Px_1 \to Px_1) \to \neg\Box\neg(Px_1 \to Px_1)$. By Existence Lemma (Lemma 186), we know that

$$\Theta_0^- \coloneqq \{ \neg\neg(Px_1 \to Px_1) \} \cup \{ \delta \mid \Box\delta \in \Delta \text{ and } \mathsf{FV}(\delta) \subseteq \{ x_1 \} \}.$$

is Λ-consistent. Now we claim that

$$\Theta_0 \coloneqq \Theta_0^- \cup \{ \neg P x_0 \}$$

is also Λ-consistent. Suppose not. This means $\Theta_0^- \cup \{ \neg P x_0 \} \vdash_\Lambda \bot$ hence $\Theta_0^- \vdash_\Lambda P x_0$. Since x_0 does not belong to $\mathsf{FV}(\Theta_0^-)$ (if so $x_0 \in \mathsf{FV}(\Gamma) \cap \mathsf{FV}(\Delta) = \{ x_1 \}$, which implies $x_1 = x_0$, a contradiction), we obtain $\Theta_0^- \vdash_\Lambda \forall y P y$. Since $\Box\neg\forall y P y \in \Delta$ and $\mathsf{FV}(\Box\neg\forall y P y) = \emptyset \subseteq \{ x_1 \}$, we have $\neg\forall y P y \in \Theta_0^-$ by definition. Together with $\Theta_0^- \vdash_\Lambda \forall y P y$, we obtain the Λ-inconsistency of Θ_0^-, which is a contradiction. Therefore Θ_0 is Λ-consistent.

Put $X \coloneqq \mathsf{Var}^+ \backslash \mathsf{FV}(\Delta)$, which is countably infinite. Since $\mathsf{FV}(\Theta_0) = \{ x_0, x_1 \}$, we know $X \backslash \mathsf{FV}(\Theta_0)$ is still countably infinite. By Lindenbaum Lemma (Lemma 182), there exists a Λ-MCS $\Theta \in W^\Lambda$ such that $\Theta_0 \subseteq \Theta$ and $\mathsf{FV}(\Theta) \subseteq \mathsf{FV}(\Theta_0) \cup X = \{ x_0, x_1 \} \cup X$. Then we also have $x_1 \in D_\Gamma \cap D_\Delta \cap D_\Theta$.

Now we establish that $\mathsf{FV}(\Delta) \cap \mathsf{FV}(\Theta) = \{ x_1 \}$. It is easy to see that $\{ x_1 \} \subseteq \mathsf{FV}(\Delta) \cap \mathsf{FV}(\Theta)$. Conversely, fix any $y \in \mathsf{FV}(\Delta) \cap \mathsf{FV}(\Theta)$. Our goal is to show $y = x_1$. We have $y \in \mathsf{FV}(\Delta)$ and $y \in \mathsf{FV}(\Theta)$. Since $\mathsf{FV}(\Theta) \subseteq \mathsf{FV}(\Theta_0) \cup X$, $y \in \{ x_0, x_1 \} \cup X$. Since $y \in \mathsf{FV}(\Delta)$, $y \notin X = \mathsf{Var}^+ \backslash \mathsf{FV}(\Delta)$. Thus $y \in \{ x_0, x_1 \}$. Suppose $y = x_0$. Since $x_0 \in \mathsf{FV}(\Gamma)$ and $x_0 = y \in \mathsf{FV}(\Delta)$, we have $x_0 \in \mathsf{FV}(\Gamma) \cap \mathsf{FV}(\Delta) = \{ x_1 \}$. This is a contradiction with the fact that x_0 is distinct from x_1. So $y = x_1$ hence $y \in \{ x_1 \}$. Therefore, $\mathsf{FV}(\Delta) \cap \mathsf{FV}(\Theta) \subseteq \{ x_1 \}$. This finishes establishing $\mathsf{FV}(\Delta) \cap \mathsf{FV}(\Theta) = \{ x_1 \}$.

Let us establish $\Delta R^{\Lambda}\Theta$. Suppose that δ is a formula in $\mathsf{L}(\Theta)$ and $\Box\delta \in \Delta$. This implies $\mathsf{FV}(\delta) \subseteq \mathsf{FV}(\Delta) \cap \mathsf{FV}(\Theta) = \{ x_1 \}$. Then we have $\delta \in \Theta_0^-$ by definition. This allows us to conclude that $\delta \in \Theta$ by $\Theta_0^- \subseteq \Theta_0 \subseteq \Theta$.

We have shown that $\Gamma R^{\Lambda}\Delta$, $\Delta R^{\Lambda}\Theta$ and $D_\Gamma \cap D_\Delta \cap D_\Theta \neq \emptyset$. Finally we show that $\Gamma R^{\Lambda}\Theta$ fails. While $\Box P x_0 \in \Gamma$, $\neg P x_0 \in \Theta$ hence $P x_0 \notin \Theta$. It is trivial to see that $P x_0 \in \mathsf{Form}(\Theta)$. This finishes showing that $\Gamma R^{\Lambda}\Theta$ fails. ∎

Theorem 194. (Non-canonicity of 5 and 5^-)

1. The canonical $\mathsf{H}(\mathbf{cK} \oplus \mathsf{Inst}(\{ \mathrm{T}, \mathrm{B}, 4, 5 \}))$-frame is not weakly euclidean. Thus, 5^- is not canonical.
2. The canonical $\mathsf{H}(\mathbf{cK} \oplus \mathsf{Inst}(\{ \mathrm{T}, \mathrm{B}, 4, 5 \}))$-frame is not euclidean. Thus, 5 is not canonical.

Proof. It suffices to show item 1, since any frame that is not weakly euclidean is not euclidean. Let $\Lambda = \mathsf{H}(\mathbf{cK} \oplus \mathsf{Inst}(\{ \mathrm{T}, \mathrm{B}, 4, 5 \}))$ and $\mathfrak{F}^{\Lambda}$ the canonical Λ-frame. Suppose for contradiction that $\mathfrak{F}^{\Lambda}$ is weakly euclidean. By Proposition 188, we know $\mathfrak{F}^{\Lambda}$ is also reflexive. Then, it follows from Proposition 171 that $\mathfrak{F}^{\Lambda}$ is weakly transitive, which contradicts item 1 of Theorem 193. ∎

Therefore, the strong completeness of Hilbert systems such as $\mathsf{H}(\mathbf{cK4})$, $\mathsf{H}(\mathbf{cKT4})$ and $\mathsf{H}(\mathbf{cK5})$ are left open. As we mentioned above, the strong completeness of $\mathsf{H}(\mathbf{cS5})$ is proved by Seligman [89] via the step-by-step method.

5.2 Proof-Theoretic Analysis

In this section we first present sequent calculi for CMPCs which are equipollent to the Hilbert system $\mathsf{H}(\mathbf{cK})$ and its extensions with axiom schemata D_n, T, 4 (5.2.1). We note that their modal rules, except for ($\Box$T), have some restrictions on variables. We then prove by a similar argument as in TSML that they admit the cut elimination theorems (5.2.2).

5.2.1 Sequent Calculus $\mathsf{G}(\mathbf{cK}\Sigma)$

Similarly to $\mathsf{G}(\mathbf{tK}\Sigma)$, a sequent calculus $\mathsf{G}(\mathbf{cK})$ and its expansions for CMPC are obtained from $\mathsf{G}(\mathbf{QK})$ as follows. Put

$$\mathsf{Axiom}^-_{\mathsf{CMPC}} := \{ \mathrm{T}, 4 \} \cup \{ \mathrm{D}_{\leqslant n} \mid n < \omega \}.$$

Definition 195. Given $\Sigma \subseteq \mathsf{Axiom}^{-}_{\mathsf{CMPC}}$, the sequent calculus $\mathrm{G}(\mathbf{cK}\Sigma)$ is the calculus obtained from $\mathrm{G}(\mathbf{QK})$ by adding all logical rules for Σ in Table 5.3, where

- we define logical rules for $\{ \mathrm{D}_{\leqslant n} \mid n < k \} \cup \{ \mathrm{T} \}$ and $\{ \mathrm{D}_{\leqslant n} \mid n < k \} \cup \{ \mathrm{T}, 4 \}$ by those for $\{ \mathrm{T} \}$ and $\{ \mathrm{T}, 4 \}$, respectively, where $k \leqslant \omega$;
- we define logical rules for $\left\{ \mathrm{D}_{\leqslant n_i} \mid i \in I \right\}$ and $\left\{ \mathrm{D}_{\leqslant n_i} \mid i \in I \right\} \cup \{ 4 \}$ by
  - if there is the largest number k in $\{ n_i \mid i \in I \}$, the logical rules for $\{ \mathrm{D}_{\leqslant n} \mid n < k \}$ and $\{ \mathrm{D}_{\leqslant n} \mid n < k \} \cup \{ 4 \}$, respectively;
  - if there is not the largest number k in $\{ n_i \mid i \in I \}$, the logical rules for $\{ \mathrm{D}_{\leqslant n} \mid n < \omega \}$ and $\{ \mathrm{D}_{\leqslant n} \mid n < \omega \} \cup \{ 4 \}$, respectively.

We follow Seligman's drafts [87, 89] to call the side conditions of $(\Box\mathrm{K}_{\mathrm{inv}})$, $(\Box 4_{\mathrm{inv}})$ and $(\Box\mathrm{S4}_{\mathrm{inv}})$ *involvement conditions*. We also call the side condition of $(\Box\mathrm{D}_{\leqslant n})$ *n-variable condition*.

We note that the formulation of $(\Box\mathrm{K}_{\mathrm{inv}})$ is by K. Sano in [83]. The notion of derivation in $\mathrm{G}(\mathbf{cK}\Sigma)$ is defined as in $\mathrm{G}(\mathbf{K})$. As before, we use the notation $\vdash_{\mathrm{G}(\mathbf{cK}\Sigma)} \Gamma \Rightarrow \Delta$ to mean that $\Gamma \Rightarrow \Delta$ is derivable in $\mathrm{G}(\mathbf{cK}\Sigma)$.

In the remaining of this subsection, we establish the equipollence of $\mathsf{H}(\mathbf{cK}\Sigma)$ and $\mathrm{G}(\mathbf{cK}\Sigma)$ for any $\Sigma \subseteq \mathsf{Axiom}^{-}_{\mathsf{CMPC}}$. The proof of this equipollence proceeds along the same lines as the proof of the equipollences of TSML. Let Σ be some fixed subset of $\mathsf{Axiom}^{-}_{\mathsf{CMPC}}$ throughout this section.

Proposition 196. Suppose $\mathsf{FV}(\varphi) \subseteq \mathsf{FV}(\psi)$. A sequent $\Rightarrow \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ is derivable in $\mathrm{G}(\mathbf{cK}\Sigma)$.

Proof. The proof is done depending on what Σ is like. Note that the involvement conditions of $(\Box\mathrm{K}_{\mathrm{inv}})$, $(\Box 4_{\mathrm{inv}})$ and $(\Box\mathrm{S4}_{\mathrm{inv}})$ are satisfied by our assumption $\mathsf{FV}(\varphi) \subseteq \mathsf{FV}(\psi)$.

Case in which either $\Sigma = \emptyset$, $\Sigma = \{ \mathrm{D}_{\leqslant n} \mid n < k \}$ or $\Sigma = \{ \mathrm{T} \}$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(id) \qquad \dfrac{}{\psi \Rightarrow \psi}\,(id)}{\varphi \to \psi, \varphi \Rightarrow \psi}\,(\to\Rightarrow)}{\Box(\varphi \to \psi), \Box\varphi \Rightarrow \Box\psi}\,(\Box\mathrm{K}_{\mathrm{inv}})}{\Rightarrow \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)}\,(\Rightarrow\to)
$$

Case in which either $\Sigma = \{ 4 \}$ or $\Sigma = \{ \mathrm{D}_{\leqslant n} \mid n < k \} \cup \{ 4 \}$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(id) \qquad \dfrac{}{\psi \Rightarrow \psi}\,(id)}{\varphi \to \psi, \varphi \Rightarrow \psi}\,(\to\Rightarrow)}{\varphi \to \psi, \varphi, \Box(\varphi \to \psi), \Box\varphi \Rightarrow \psi}\,(w\Rightarrow)}{\Box(\varphi \to \psi), \Box\varphi \Rightarrow \Box\psi}\,(\Box 4_{\mathrm{inv}})}{\Rightarrow \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)}\,(\Rightarrow\to)
$$

Sequent Calculus G	
$\dfrac{}{\varphi \Rightarrow \varphi}\ (id)$	$\dfrac{}{\bot \Rightarrow}\ (\bot)$
$\dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}\ (\Rightarrow w)$	$\dfrac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (w \Rightarrow)$
$\dfrac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}\ (\Rightarrow c)$	$\dfrac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (c \Rightarrow)$
$\dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Xi \Rightarrow \Sigma}{\Gamma, \Xi \Rightarrow \Delta, \Sigma}\ (Cut)$	
$\dfrac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}\ (\Rightarrow \to)$	$\dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Xi \Rightarrow \Sigma}{\varphi \to \psi, \Gamma, \Xi \Rightarrow \Delta, \Sigma}\ (\to \Rightarrow)$
$\dfrac{\Gamma \Rightarrow \Delta, \varphi(y/x)}{\Gamma \Rightarrow \Delta, \forall x \varphi}\ (\Rightarrow \forall)^{\dagger}$	$\dfrac{\varphi(t/x), \Gamma \Rightarrow \Delta}{\forall x \varphi, \Gamma \Rightarrow \Delta}\ (\forall \Rightarrow)$

where †: y is not a free variable in $\Gamma, \Delta, \forall x \varphi$.

Σ	Logical Rules for Σ	
$\emptyset$	$\dfrac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}\ (\Box\mathrm{K}_{\mathrm{inv}})^{\dagger}$	
$\{ \mathrm{D}_{\leqslant n} \mid n < k \}^{\star}$	$\dfrac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}\ (\Box\mathrm{K}_{\mathrm{inv}})^{\dagger}$	$\dfrac{\Gamma \Rightarrow}{\Box\Gamma \Rightarrow}\ (\Box\mathrm{D}_{\leqslant n})^{\ddagger}$
$\{ \mathrm{T} \}$	$\dfrac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}\ (\Box\mathrm{K}_{\mathrm{inv}})^{\dagger}$	$\dfrac{\varphi, \Gamma \Rightarrow \Delta}{\Box\varphi, \Gamma \Rightarrow \Delta}\ (\Box\mathrm{T})$
$\{ 4 \}$	$\dfrac{\Gamma, \Box\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}\ (\Box 4_{\mathrm{inv}})^{\dagger}$	
$\{ \mathrm{D}_{\leqslant n} \mid n < k \} \cup \{ 4 \}^{\star}$	$\dfrac{\Gamma, \Box\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}\ (\Box 4_{\mathrm{inv}})^{\dagger}$	$\dfrac{\Gamma, \Box\Gamma \Rightarrow}{\Box\Gamma \Rightarrow}\ (\Box\mathrm{D}4_{\leqslant n})^{\ddagger}$
$\{ \mathrm{T}, 4 \}$	$\dfrac{\Box\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}\ (\Box\mathrm{S}4_{\mathrm{inv}})^{\dagger}$	$\dfrac{\varphi, \Gamma \Rightarrow \Delta}{\Box\varphi, \Gamma \Rightarrow \Delta}\ (\Box\mathrm{T})$

where $\Box\Gamma \coloneqq \{ \Box\varphi \mid \varphi \in \Gamma \}$, ★: $k \leqslant \omega$, †: $\mathsf{FV}(\Gamma) \subseteq \mathsf{FV}(\varphi)$, and ‡: $\#\mathsf{FV}(\Gamma) \leqslant n < k$.

Table 5.3: Sequent Calculus G($\mathbf{cK\Sigma}$)

Case in which $\Sigma = \{ \mathrm{T}, 4 \}$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(id)}
\UnaryInfC{$\varphi \Rightarrow \varphi$}
\AxiomC{}
\RightLabel{(id)}
\UnaryInfC{$\psi \Rightarrow \psi$}
\RightLabel{$(\to\Rightarrow)$}
\BinaryInfC{$\varphi \to \psi, \varphi \Rightarrow \psi$}
\doubleLine
\RightLabel{$(\Box\mathrm{T})$}
\UnaryInfC{$\Box(\varphi \to \psi), \Box\varphi \Rightarrow \psi$}
\RightLabel{$(\Box\mathrm{S4}_{\mathrm{inv}})$}
\UnaryInfC{$\Box(\varphi \to \psi), \Box\varphi \Rightarrow \Box\psi$}
\doubleLine
\RightLabel{$(\Rightarrow\to)$}
\UnaryInfC{$\Rightarrow \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$}
\end{prooftree}
$$

■

Proposition 197. (Equipollence of $\mathsf{H}(\mathbf{cK}\Sigma)$ and $\mathrm{G}(\mathbf{cK}\Sigma)$) A formula φ is provable in $\mathsf{H}(\mathbf{cK}\Sigma)$ iff a sequent $\Rightarrow \varphi$ is derivable in $\mathrm{G}(\mathbf{cK}\Sigma)$.

Proof. The proof proceeds similarly to the proof of Proposition 109. The left-to-right direction is by induction on the length of a proof of φ. We skip cases involving first order logic and a case in which $(\mathtt{Nec})$ is lastly applied. A case in which a proved formula is an instance of $(\mathtt{K}_{\mathtt{inv}})$ is shown by Proposition 196. Thus it suffices to check only cases in which a proved formula belongs to $\mathsf{Inst}(\Sigma)$.

Case in which $\Sigma = \{ \mathrm{D}_{\leqslant n} \mid n < k \}$. The proved formula belongs to $\mathsf{Inst}(\{ \mathrm{D}_{\leqslant n} \mid n < k \})$. This means it belongs to $\mathsf{Inst}(\mathrm{D}_{\leqslant n})$ for some $n < k$. Then, it has the form $\Box\varphi \to \Diamond\varphi$ such that $\#\mathsf{FV}(\Box\varphi \to \Diamond\varphi) \leqslant n$, so a set $\{ \varphi, \neg\varphi \}$ satisfies the n-variable condition of $(\Box\mathrm{D}_{\leqslant n})$. Thus we can obtain a derivation of $\Box\varphi \to \Diamond\varphi$ in $\mathrm{G}(\mathbf{cK}\Sigma)$ as follows.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(id)}
\UnaryInfC{$\varphi \Rightarrow \varphi$}
\AxiomC{}
\RightLabel{$(\bot)$}
\UnaryInfC{$\bot \Rightarrow$}
\RightLabel{$(\to\Rightarrow)$}
\BinaryInfC{$\varphi, \neg\varphi \Rightarrow$}
\RightLabel{$(\Box\mathrm{D}_{\leqslant n})$}
\UnaryInfC{$\Box\varphi, \Box\neg\varphi \Rightarrow$}
\RightLabel{$(\Rightarrow w)$}
\UnaryInfC{$\Box\varphi, \Box\neg\varphi \Rightarrow \bot$}
\doubleLine
\RightLabel{$(\Rightarrow\to)$}
\UnaryInfC{$\Rightarrow \Box\varphi \to \Diamond\varphi$}
\end{prooftree}
$$

Case in which $\Sigma = \{ \mathrm{T} \}$. The proved formula belongs to $\mathsf{Inst}(\mathrm{T})$, so has the form $\Box\varphi \to \varphi$. A derivation of it in $\mathrm{G}(\mathbf{cK}\Sigma)$ is as follows.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(id)}
\UnaryInfC{$\varphi \Rightarrow \varphi$}
\RightLabel{$(\Box\mathrm{T})$}
\UnaryInfC{$\Box\varphi \Rightarrow \varphi$}
\RightLabel{$(\Rightarrow\to)$}
\UnaryInfC{$\Rightarrow \Box\varphi \to \varphi$}
\end{prooftree}
$$

Case in which $\Sigma = \{ 4 \}$. The proved formula belongs to $\mathsf{Inst}(4)$, so has the form $\Box\varphi \to \Box\Box\varphi$. A derivation of it in $\mathrm{G}(\mathbf{cK}\Sigma)$ is as follows.

$$\dfrac{\dfrac{\dfrac{\overline{\Box\varphi \Rightarrow \Box\varphi}\ (id)}{\varphi, \Box\varphi \Rightarrow \Box\varphi}\ (w \Rightarrow)}{\Box\varphi \Rightarrow \Box\Box\varphi}\ (\Box 4_{\mathrm{inv}})}{\Rightarrow \Box\varphi \to \Box\Box\varphi}\ (\Rightarrow \to)$$

Case in which $\Sigma = \{ \mathrm{D}_{\leqslant n} \mid n < k \} \cup \{ 4 \}$. The proved formula belongs to $\mathsf{Inst}(\{ \mathrm{D}_{\leqslant n} \mid n < k \} \cup \{ 4 \})$.

Case belonging to $\mathsf{Inst}(\{ \mathrm{D}_{\leqslant n} \mid n < k \})$**.** The proved formula belongs to $\mathsf{Inst}(\mathrm{D}_{\leqslant n})$ for some $n < k$. Then, it has the form $\Box\varphi \to \Diamond\varphi$ such that $\#\mathsf{FV}(\Box\varphi \to \Diamond\varphi) \leqslant n$, so a set $\{ \varphi, \neg\varphi, \Box\varphi, \Box\neg\varphi \}$ satisfies the n-variable condition of $(\Box\mathrm{D}4_{\leqslant n})$. Thus we can construct a derivation of $\Box\varphi \to \Diamond\varphi$ in $\mathrm{G}(\mathbf{cK}\Sigma)$ as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\varphi \Rightarrow \varphi}\ (id) \quad \overline{\bot \Rightarrow}\ (\bot)}{\varphi, \neg\varphi \Rightarrow}\ (\to\Rightarrow)}{\varphi, \neg\varphi, \Box\varphi, \Box\neg\varphi \Rightarrow}\ (w \Rightarrow)}{\Box\varphi, \Box\neg\varphi \Rightarrow}\ (\Box\mathrm{D}4_{\leqslant n})}{\Box\varphi, \Box\neg\varphi \Rightarrow \bot}\ (\Rightarrow w)}{\Rightarrow \Box\varphi \to \Diamond\varphi}\ (\Rightarrow \to)$$

Case belonging to $\mathsf{Inst}(4)$**.** Same as the case in which $\Sigma = \{ 4 \}$.

Case in which $\Sigma = \{ \mathrm{T}, 4 \}$. The proved formula belongs to $\mathsf{Inst}(\mathrm{T}, 4)$.

Case belonging to $\mathsf{Inst}(\mathrm{T})$**.** Same as the case in which $\Sigma = \{ \mathrm{T} \}$.

Case belonging to $\mathsf{Inst}(4)$**.** It has the form $\Box\varphi \to \Box\Box\varphi$. A derivation of it in $\mathrm{G}(\mathbf{cK}\Sigma)$ is as follows.

$$\dfrac{\dfrac{\overline{\Box\varphi \Rightarrow \Box\varphi}\ (id)}{\Box\varphi \Rightarrow \Box\Box\varphi}\ (\Box\mathrm{S}4)}{\Rightarrow \Box\varphi \to \Box\Box\varphi}\ (\Rightarrow \to)$$

Therefore, the proof of the left-to-right direction was done.

The right-to-left direction is obtained from the following two claims:

- $\vdash_{\mathsf{H}(\mathbf{tK}\Sigma)}$ ... $\vdash_{\mathsf{G}(\mathbf{tK}\Sigma)} \Gamma \Rightarrow \Delta$ implies $\vdash_{\mathsf{H}(\mathbf{tK}\Sigma)} \bigwedge \Gamma \to \bigvee \Delta$;
- $\vdash_{\mathsf{H}(\mathbf{tK}\Sigma)} (\bigwedge \emptyset \to \bigvee \{ \varphi \}) \to \varphi$.

The latter is easy to establish, so we show the former by induction on the height of a derivation of $\Gamma \Rightarrow \Delta$ in $\mathrm{G}(\mathbf{tK}\Sigma)$. We skip cases in which the last applied rule is a rule from G, and show only cases in which it is a logical rule from Σ.

Case in which $\Sigma = \emptyset$. The last applied rule is $(\Box\mathrm{K}_{\mathrm{inv}})$ so the derivation is of the form

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \varphi\end{array}}{\Box\Gamma \Rightarrow \Box\varphi}\ (\Box\mathrm{K}_{\mathrm{inv}})$$

where (1) $\mathsf{FV}(\Gamma) \subseteq \mathsf{FV}(\varphi)$. We can obtain $\vdash \bigwedge \Box\Gamma \to \Box\varphi$ as follows.

1. $\vdash \bigwedge \Gamma \to \varphi$	Inductive hypothesis
2. $\vdash \Box(\bigwedge \Gamma \to \varphi)$	1, $(\mathtt{Nec})$
3. $\vdash \Box(\bigwedge \Gamma \to \varphi) \to (\Box \bigwedge \Gamma \to \Box\varphi)$	(1), $(\mathtt{K_{inv}})$
4. $\vdash \Box \bigwedge \Gamma \to \Box\varphi$	2, 3, MP
5. $\vdash \bigwedge \Box\Gamma \to \Box\varphi$	4, Item 1 of Proposition 83

Case in which $\Sigma = \{ \mathrm{D}_{\leqslant n} \mid n < k \}$. The last applied rule is $(\Box\mathrm{K}_{\mathrm{inv}})$ or $(\Box\mathrm{D}_{\leqslant n})$.

Case of $(\Box\mathrm{K}_{\mathrm{inv}})$. Same as the case in which $\Sigma = \emptyset$.

Case of $(\Box\mathrm{D}_{\leqslant n})$. The derivation is of the form

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow\end{array}}{\Box\Gamma \Rightarrow}\ (\Box\mathrm{D}_{\leqslant n})$$

where $\#\mathsf{FV}(\Gamma) \leqslant n$. Then (1) $\mathsf{FV}(\Gamma) = \{ x_1, \ldots, x_{n'} \}$. Let $\vec{x} = (x_1, \ldots, x_{n'})$ and $\bot(\vec{x}) := \neg(P\vec{x} \to P\vec{x})$. We can obtain $\vdash \bigwedge \Box\Gamma \to \bot$ as follows.

1. $\vdash \bigwedge \Gamma \to \bot$	Inductive hypothesis
2. $\vdash \bigwedge \Gamma \to \bot(\vec{x})$	1, PC
3. $\vdash \Box(\bigwedge \Gamma \to \bot(\vec{x}))$	2, $(\mathtt{Nec})$
4. $\vdash \Box(\bigwedge \Gamma \to \bot(\vec{x})) \to (\Box \bigwedge \Gamma \to \Box\bot(\vec{x}))$	(1), $(\mathtt{K_{inv}})$
5. $\vdash \Box \bigwedge \Gamma \to \Box\bot(\vec{x})$	3, 4, MP
6. $\vdash \bigwedge \Box\Gamma \to \Box\bot(\vec{x})$	5, Item 5 of Proposition 177
7. $\vdash \Box\bot(\vec{x}) \to \Diamond\bot(\vec{x})$	$(\mathtt{D}_{\leqslant n})$
8. $\vdash \bigwedge \Box\Gamma \to \Diamond\bot(\vec{x})$	6, 7, PC
9. $\vdash \bigwedge \Box\Gamma \to \bot$	8, Item 2 of Proposition 177

Case in which $\Sigma = \{ \mathrm{T} \}$. The last applied rule is $(\Box\mathrm{K}_{\mathrm{inv}})$ or $(\Box\mathrm{T})$.

Case of $(\Box\mathrm{K}_{\mathrm{inv}})$. Same as the case in which $\Sigma = \emptyset$.

Case of $(\Box\mathrm{T})$. The derivation is of the form

$$\dfrac{\begin{array}{c}\vdots\\ \varphi, \Gamma \Rightarrow \Delta\end{array}}{\Box\varphi, \Gamma \Rightarrow \Delta}\ (\Box\mathrm{T})$$

We can obtain $\vdash (\Box\varphi \wedge \bigwedge \Gamma) \to \bigvee \Delta$ as follows.

1. $\vdash (\varphi \wedge \bigwedge \Gamma) \to \bigvee \Delta$ — Inductive hypothesis
2. $\vdash \Box\varphi \to \varphi$ — $(\mathtt{T})$
3. $\vdash (\Box\varphi \wedge \bigwedge \Gamma) \to \bigvee \Delta$ — 1, 2, PC

Case in which $\Sigma = \{ 4 \}$. The last applied rule is $(\Box 4_{\mathrm{inv}})$ so the derivation is of the form

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma, \Box\Gamma \Rightarrow \varphi\end{array}}{\Box\Gamma \Rightarrow \Box\varphi}\ (\Box 4_{\mathrm{inv}})$$

where (1) $\mathsf{FV}(\Gamma) \subseteq \mathsf{FV}(\varphi)$. We can obtain $\vdash \bigwedge \Box\Gamma \to \Box\varphi$ as follows.

1. $\vdash (\bigwedge \Gamma \wedge \bigwedge \Box\Gamma) \to \varphi$ — Inductive hypothesis
2. $\vdash \Box((\bigwedge \Gamma \wedge \bigwedge \Box\Gamma) \to \varphi)$ — 1, $(\mathtt{Nec})$
3. $\vdash \Box((\bigwedge \Gamma \wedge \bigwedge \Box\Gamma) \to \varphi) \to (\Box(\bigwedge \Gamma \wedge \bigwedge \Box\Gamma) \to \Box\varphi)$ — (1), $(\mathtt{K})$
4. $\vdash \Box(\bigwedge \Gamma \wedge \bigwedge \Box\Gamma) \to \Box\varphi$ — 2, 3, MP
5. $\vdash (\bigwedge \Box\Gamma \wedge \bigwedge \Box\Box\Gamma) \to \Box\varphi$ — 4, Item 1 of Proposition 83
6. $\vdash \Box\gamma \to \Box\Box\gamma$ — $\gamma \in \Gamma$, (4)
7. $\vdash \bigwedge \Box\Gamma \to \bigwedge \Box\Box\Gamma$ — 6, PC
8. $\vdash \bigwedge \Box\Gamma \to \Box\varphi$ — 5, 7, PC

Case in which $\Sigma = \{ \mathrm{D}_{\leqslant n} \mid n < k \} \cup \{ 4 \}$. The last applied rule is $(\Box 4_{\mathrm{inv}})$ or $(\Box \mathrm{D}4_{\leqslant n})$.

Case of $(\Box 4_{\mathrm{inv}})$. Same as the case in which $\Sigma = \{ 4 \}$.

Case of $(\Box \mathrm{D}4_{\leqslant n})$. The derivation is of the form

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma, \Box\Gamma \Rightarrow\end{array}}{\Box\Gamma \Rightarrow}\ (\Box \mathrm{D}4_{\leqslant n})$$

where $\#\mathsf{FV}(\Gamma) \leqslant n$. Then (1) $\mathsf{FV}(\Gamma) = \{ x_1, \ldots, x_{n'} \}$. Let $\vec{x} = (x_1, \ldots, x_{n'})$ and $\bot(\vec{x}) := \neg(P\vec{x} \to P\vec{x})$. We can obtain $\vdash \bigwedge \Box\Gamma \to \bot$ as follows.

1. $\vdash (\bigwedge \Gamma \wedge \bigwedge \Box\Gamma) \to \bot$ — Inductive hypothesis

2. $\vdash (\bigwedge\Gamma \land \bigwedge\Box\Gamma) \to \bot(\vec{x})$ 1, PC
3. $\vdash \Box((\bigwedge\Gamma \land \bigwedge\Box\Gamma) \to \bot(\vec{x}))$ 2, $(\mathtt{Nec})$
4. $\vdash \Box((\bigwedge\Gamma \land \bigwedge\Box\Gamma) \to \bot(\vec{x})) \to (\Box(\bigwedge\Gamma \land \bigwedge\Box\Gamma) \to \Box\bot(\vec{x}))$ (1), $(\mathtt{K}_{\mathtt{inv}})$
5. $\vdash \Box(\bigwedge\Gamma \land \bigwedge\Box\Gamma) \to \Box\bot(\vec{x})$ 3, 4, MP
6. $\vdash (\bigwedge\Box\Gamma \land \bigwedge\Box\Box\Gamma) \to \Box\bot(\vec{x})$ 5, Item 1 of Proposition 177
7. $\vdash \Box\gamma \to \Box\Box\gamma$ $\gamma \in \Gamma$, (4)
8. $\vdash \bigwedge\Box\Gamma \to \bigwedge\Box\Box\Gamma$ 7, PC
9. $\vdash \bigwedge\Box\Gamma \to \Box\bot(\vec{x})$ 6, 8, PC
10. $\vdash \Box\bot(\vec{x}) \to \Diamond\bot(\vec{x})$ $(\mathtt{D}_{\leqslant n})$
11. $\vdash \bigwedge\Box\Gamma \to \Diamond\bot(\vec{x})$ 9, 10, PC
12. $\vdash \bigwedge\Box\Gamma \to \bot$ 11, Item 2 of Proposition 177

Case in which $\Sigma = \{\mathrm{T}, 4\}$. The last applied rule is $(\Box\mathrm{S4}_{\mathrm{inv}})$ or $(\Box\mathrm{T})$.

Case of $(\Box\mathrm{S4}_{\mathrm{inv}})$. The derivation is of the form

$$\cfrac{\begin{array}{c}\vdots\\ \Box\Gamma \Rightarrow \varphi\end{array}}{\Box\Gamma \Rightarrow \Box\varphi}\ (\Box\mathrm{S4}_{\mathrm{inv}})$$

where (1) $\mathsf{FV}(\Gamma) \subseteq \mathsf{FV}(\varphi)$. We can obtain $\vdash \bigwedge\Box\Gamma \to \Box\varphi$ as follows.

1. $\vdash \bigwedge\Box\Gamma \to \varphi$ Inductive hypothesis
2. $\vdash \Box(\bigwedge\Box\Gamma \to \varphi)$ 1, $(\mathtt{Nec})$
3. $\vdash \Box(\bigwedge\Box\Gamma \to \varphi) \to (\Box\bigwedge\Box\Gamma \to \Box\varphi)$ (1), $(\mathtt{K}_{\mathtt{inv}})$
4. $\vdash \Box\bigwedge\Box\Gamma \to \Box\varphi$ 2, 3, MP
5. $\vdash \bigwedge\Box\Box\Gamma \to \Box\varphi$ 4, Item 1 of Proposition 83
6. $\vdash \Box\gamma \to \Box\Box\gamma$ $\gamma \in \Gamma$, (4)
7. $\vdash \bigwedge\Box\Gamma \to \bigwedge\Box\Box\Gamma$ 6, PC
8. $\vdash \bigwedge\Box\Gamma \to \Box\varphi$ 5, 7, PC

Case of $(\Box\mathrm{T})$. Same as the case in which $\Sigma = \{\mathrm{T}\}$.

Thus the proof of the right-to-left direction was also done. ■

5.2.2 Cut Elimination

In this subsection, we prove the cut elimination theorem of $\mathrm{G}(\mathbf{cK}\Sigma)$. The proof strategy we take here is the same as the strategy that we took for the cut elimination theorem of TSML. Recall the extended rule (Cut^*) of (Cut)

$$\frac{\Gamma \Rightarrow \Delta, \varphi^l \qquad \varphi^m, \Xi \Rightarrow \Pi}{\Gamma, \Xi \Rightarrow \Delta, \Pi} \ (Cut^*)$$

where l, m can be zero. Let $\mathrm{G}^-(\mathbf{cK}\Sigma)$ be the calculus obtained from $\mathrm{G}(\mathbf{cK}\Sigma)$ by removing (Cut), and $\mathrm{G}^*(\mathbf{cK}\Sigma)$ the calculus obtained from $\mathrm{G}(\mathbf{cK}\Sigma)$ by replacing (Cut) with (Cut^*). In a similar vein as in $\mathrm{G}(\mathbf{tK}\Sigma)$, we define notions of principal formula, the (Cut^*)-bottom form, and the grade and the weight of an application of (Cut^*) in $\mathrm{G}(\mathbf{cK}\Sigma)$. In what follows, we assume that free variables and bound variables in derivations are thoroughly separated.

Theorem 198. (Cut elimination) If a sequent $\Gamma \Rightarrow \Delta$ is derivable in $\mathrm{G}(\mathbf{cK}\Sigma)$, then $\Gamma \Rightarrow \Delta$ is also derivable in $\mathrm{G}^-(\mathbf{cK}\Sigma)$.

Proof. The proof proceeds along the same lines as the proof of the cut elimination theorem of TSML (Theorem 115). To eliminate (Cut), it suffices to show that $\vdash_{\mathrm{G}^*(\mathbf{cK}\Sigma)} \Gamma \Rightarrow \Delta$ implies $\vdash_{\mathrm{G}^-(\mathbf{cK}\Sigma)} \Gamma \Rightarrow \Delta$, which is obtained immediately from the following claim.

> If there is a derivation $\mathfrak{D}$ of the (Cut^*)-bottom form of a sequent $\Gamma \Rightarrow \Delta$ in $\mathrm{G}^*(\mathbf{cK}\Sigma)$, there is also a derivation of $\Gamma \Rightarrow \Delta$ in $\mathrm{G}^-(\mathbf{cK}\Sigma)$.

We show this claim by double induction on a pair $(g(\sigma), w(\sigma))$ of the grade $g(\sigma)$ and the weight $w(\sigma)$ of the only application σ of (Cut^*) in a derivation $\mathfrak{D}$ of the (Cut^*)-bottom form. Assume we are given a derivation $\mathfrak{D}$ of the (Cut^*)-bottom form of a sequent $\Gamma \Rightarrow \Delta$ in $\mathrm{G}^*(\mathbf{cK}\Sigma)$. Then $\mathfrak{D}$ is of the form

$$\frac{\dfrac{\mathfrak{D}_1}{\Gamma \Rightarrow \Delta, \varphi^l}\ \rho(L) \qquad \dfrac{\mathfrak{D}_2}{\varphi^m, \Xi \Rightarrow \Pi}\ \rho(R)}{\Gamma, \Xi \Rightarrow \Delta, \Pi} \ (Cut^*)$$

where $\rho(L)$ and $\rho(R)$ are the last applied rule to $\mathfrak{D}_1$ and $\mathfrak{D}_2$, respectively. We may also assume that both of the numbers l, m are more than zero, because if not we can obtain a derivation of $\Gamma \Rightarrow \Delta$ in $\mathrm{G}^-(\mathbf{cK}\Sigma)$ by applying $(\Rightarrow w)$ and $(w \Rightarrow)$ repeatedly to $\mathfrak{D}_1$ or $\mathfrak{D}_2$.

Regardless of our choice of Σ, we divide our proof into the following four cases, though our arguments for additional rules depend on our choice of Σ.

- $\rho(L)$ or $\rho(R)$ is an initial sequent.
- $\rho(L)$ or $\rho(R)$ is an application of a structural rule.
- $\rho(L)$ or $\rho(R)$ is an application of a logical rule but the cut-formula is not principal in $\rho(L)$ or $\rho(R)$, respectively.
- Both of $\rho(L)$ and $\rho(R)$ are applications of logical rules and the cut-formula is principal in $\rho(L)$ and $\rho(R)$.

Since the first and second cases are similarly done as in the proof of the cut elimination theorem of TSML (Theorem 115), we skip them.

For the third case, as before, we divide two subcases depending on whether $\mathrm{T} \in \Sigma$ or not. We note that both of subcases are checked similarly as before.

Consider first a subcase in which $\mathrm{T} \notin \Sigma$. Recall the definition of principality and our assumption that both of the number l, m of the cut-formula are more than zero. These tell us that it cannot be the case that $\rho(L)$ or $\rho(R)$ is a logical rule from Σ. Thus, in this subcase, it suffices to check when $\rho(L)$ or $\rho(R)$ is a logical rule from $\mathrm{G}(\mathbf{QK})$. This is shown by a usual argument found in [67, p. 29]. Therefore, this subcase is done.

Consider then the other subcase in which $\mathrm{T} \in \Sigma$. In this subcase, in addition to the cases in which $\rho(L)$ or $\rho(R)$ is a logical rule from $\mathrm{G}(\mathbf{QK})$, we must check cases in which $\rho(L)$ or $\rho(R)$ is $(\Box\mathrm{T})$. Since the former cases are proved by the usual argument found in [67, p. 29], we check the cases in which $\rho(L)$ or $\rho(R)$ is $(\Box\mathrm{T})$.

Case in which $\rho(L) = (\Box\mathrm{T})$. Then $\mathfrak{D} =$

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \psi, \Gamma \Rightarrow \Delta, \varphi^l\end{matrix}}{\Box\psi, \Gamma \Rightarrow \Delta, \varphi^l}\,(\Box\mathrm{T}) \qquad \begin{matrix}\mathfrak{D}_2\\ \varphi^m, \Xi \Rightarrow \Pi\end{matrix}}{\Box\psi, \Gamma, \Xi \Rightarrow \Delta, \Pi}\,(Cut^*)$$

which can be transformed into a derivation $\mathfrak{D}' =$

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \psi, \Gamma \Rightarrow \Delta, \varphi^l\end{matrix} \qquad \begin{matrix}\mathfrak{D}_2\\ \varphi^m, \Xi \Rightarrow \Pi\end{matrix}}{\psi, \Gamma, \Xi \Rightarrow \Delta, \Pi}\,(Cut^*)}{\Box\psi, \Gamma, \Xi \Rightarrow \Delta, \Pi}\,(\Box\mathrm{T})$$

The application σ_1 of (Cut^*) in $\mathfrak{D}'$ is eliminable by inductive hypothesis, because $g(\sigma_1) = g(\sigma)$ and $w(\sigma_1) < w(\sigma)$ hold. Thus, there is a derivation of $\Box\psi, \Gamma, \Xi \Rightarrow \Delta, \Pi$ in $\mathrm{G}^{-}(\mathbf{cK\Sigma})$.

Case in which $\rho(R) = (\Box\mathrm{T})$. If the cut-formula is principal in $\rho(R)$, as the third case is now under consideration, it has to be the case that $\rho(L)$ is an application of a logical rule but the cut-formula is not principal in $\rho(L)$. When it is a logical rule from $\mathrm{G}(\mathbf{QK})$, the current case is established by the usual argument found in [67, p. 29]. When it is a logical rule from Σ, the current case must be the case of $\rho(L) = (\Box\mathrm{T})$, which we just saw above. Therefore, we may assume the cut-formula is not principal in $\rho(R)$. Then, $\mathfrak{D} =$

$$\dfrac{\begin{matrix}\mathfrak{D}_1\\ \Gamma \Rightarrow \Delta, \varphi^l\end{matrix} \qquad \dfrac{\begin{matrix}\mathfrak{D}_2\\ \varphi^m, \psi, \Xi \Rightarrow \Pi\end{matrix}}{\varphi^m, \Box\psi, \Xi \Rightarrow \Pi}\ (\Box\mathrm{T})}{\Box\psi, \Gamma, \Xi \Rightarrow \Delta, \Pi}\ (Cut^*)$$

which can be transformed into a derivation $\mathfrak{D}' =$

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \Gamma \Rightarrow \Delta, \varphi^l\end{matrix} \qquad \begin{matrix}\mathfrak{D}_2\\ \varphi^m, \psi, \Xi \Rightarrow \Pi\end{matrix}}{\psi, \Gamma, \Xi \Rightarrow \Delta, \Pi}\ (Cut^*)}{\Box\psi, \Gamma, \Xi \Rightarrow \Delta, \Pi}\ (\Box\mathrm{T})$$

The application σ_1 of (Cut^*) in $\mathfrak{D}'$ is eliminable by inductive hypothesis, because $g(\sigma_1) = g(\sigma)$ and $w(\sigma_1) < w(\sigma)$ hold. Thus, there is a derivation of $\Box\psi, \Gamma, \Xi \Rightarrow \Delta, \Pi$ in $\mathrm{G}^-(\mathbf{cK\Sigma})$.

Now that we have done the first, second and third cases, the remaining case is the fourth case. Recall that the fourth case is the case in which both of $\rho(L)$ and $\rho(R)$ are applications of logical rules and the cut-formula is principal in $\rho(L)$ and $\rho(R)$. One can establish cases in which $\rho(L)$ and $\rho(R)$ are logical rules from $\mathrm{G}(\mathbf{QK})$ by a usual argument found in [67, pp. 29–30]. We show only cases in which $\rho(L)$ and $\rho(R)$ are logical rules from Σ.

Case in which $\rho(L) = \rho(R) = (\Box\mathrm{K}_{\mathrm{inv}})$**.** Then $\mathfrak{D} =$

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \Gamma \Rightarrow \varphi\end{matrix}}{\Box\Gamma \Rightarrow \Box\varphi}\ (\Box\mathrm{K}_{\mathrm{inv}}) \qquad \dfrac{\begin{matrix}\mathfrak{D}_2\\ \varphi^m, \Xi \Rightarrow \psi\end{matrix}}{(\Box\varphi)^m, \Box\Xi \Rightarrow \Box\psi}\ (\Box\mathrm{K}_{\mathrm{inv}})}{\Box\Gamma, \Box\Xi \Rightarrow \Box\psi}\ (Cut^*)$$

where $\mathsf{FV}(\Gamma) \subseteq \mathsf{FV}(\varphi)$ and $\mathsf{FV}(\{\varphi\} \cup \Xi) \subseteq \mathsf{FV}(\psi)$. Then $\mathsf{FV}(\Gamma \cup \Xi) \subseteq \mathsf{FV}(\psi)$. Therefore, $\mathfrak{D}$ can be transformed into a derivation $\mathfrak{D}' =$

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \Gamma \Rightarrow \varphi\end{matrix} \qquad \begin{matrix}\mathfrak{D}_2\\ \varphi^m, \Xi \Rightarrow \psi\end{matrix}}{\Gamma, \Xi \Rightarrow \psi}\ (Cut^*)}{\Box\Gamma, \Box\Xi \Rightarrow \Box\psi}\ (\Box\mathrm{K}_{\mathrm{inv}})$$

The application σ_1 of (Cut^*) in $\mathfrak{D}'$ is eliminable by inductive hypothesis, because $g(\sigma_1) < g(\sigma)$ holds by $l(\varphi) < l(\Box\varphi)$. Thus, there is a derivation of $\Box\Gamma, \Box\Xi \Rightarrow \Box\psi$ in $\mathrm{G}^-(\mathbf{cK\Sigma})$.

Case in which $\rho(L) = (\Box\mathrm{K_{inv}})$ **and** $\rho(R) = (\Box\mathrm{D}_{\leqslant n})$**.** Then $\mathfrak{D}$ =

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \Gamma \Rightarrow \varphi\end{matrix}}{\Box\Gamma \Rightarrow \Box\varphi}\,(\Box\mathrm{K_{inv}}) \qquad \dfrac{\begin{matrix}\mathfrak{D}_2\\ \varphi^m, \Xi \Rightarrow\end{matrix}}{(\Box\varphi)^m, \Box\Xi \Rightarrow}\,(\Box\mathrm{D}_{\leqslant n})}{\Box\Gamma, \Box\Xi \Rightarrow}\,(Cut^*)$$

where $\mathsf{FV}(\Gamma) \subseteq \mathsf{FV}(\varphi)$ and $\#\mathsf{FV}(\{\varphi\} \cup \Xi) \leqslant n$. Then $\#\mathsf{FV}(\Gamma \cup \Xi) \leqslant n$. Thus, $\mathfrak{D}$ can be transformed into a derivation $\mathfrak{D}'$ =

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \Gamma \Rightarrow \varphi\end{matrix} \qquad \begin{matrix}\mathfrak{D}_2\\ \varphi^m, \Xi \Rightarrow\end{matrix}}{\Gamma, \Xi \Rightarrow}\,(Cut^*)}{\Box\Gamma, \Box\Xi \Rightarrow}\,(\Box\mathrm{D}_{\leqslant n})$$

The application σ_1 of (Cut^*) in $\mathfrak{D}'$ is eliminable by inductive hypothesis, because $g(\sigma_1) < g(\sigma)$ holds by $l(\varphi) < l(\Box\varphi)$. Thus, there is a derivation of $\Box\Gamma, \Box\Xi \Rightarrow$ in $\mathrm{G}^-(\mathbf{cK\Sigma})$.

Case in which $\rho(L) = (\Box\mathrm{K_{inv}})$ **and** $\rho(R) = (\Box\mathrm{T})$**.** Then $\mathfrak{D}$ =

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \Gamma \Rightarrow \varphi\end{matrix}}{\Box\Gamma \Rightarrow \Box\varphi}\,(\Box\mathrm{K_{inv}}) \qquad \dfrac{\begin{matrix}\mathfrak{D}_2\\ \varphi, (\Box\varphi)^{m-1}, \Xi \Rightarrow \Delta\end{matrix}}{\Box\varphi, (\Box\varphi)^{m-1}, \Xi \Rightarrow \Delta}\,(\Box\mathrm{T})}{\Box\Gamma, \Box\Xi \Rightarrow \Delta}\,(Cut^*)$$

where $\mathsf{FV}(\Gamma) \subseteq \mathsf{FV}(\varphi)$. Therefore, $\mathfrak{D}$ can be transformed into a derivation $\mathfrak{D}'$ =

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \Gamma \Rightarrow \varphi\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \Gamma \Rightarrow \varphi\end{matrix}}{\Box\Gamma \Rightarrow \Box\varphi}\,(\Box\mathrm{K_{inv}}) \qquad \begin{matrix}\mathfrak{D}_2\\ \varphi, (\Box\varphi)^{m-1}, \Xi \Rightarrow \Delta\end{matrix}}{\varphi, \Box\Gamma, \Xi \Rightarrow \Delta}\,(Cut^*)}{\Gamma, \Box\Gamma, \Xi \Rightarrow \Delta}\,(Cut^*)}{\Box\Gamma, \Box\Gamma, \Box\Xi \Rightarrow \Delta}\,(\Box\mathrm{T})}{\Box\Gamma, \Box\Xi \Rightarrow \Delta}\,(c \Rightarrow)$$

The upper application σ_1 of (Cut^*) in $\mathfrak{D}'$ is eliminable by inductive hypothesis, because $g(\sigma_1) = g(\sigma)$ and $w(\sigma_1) < w(\sigma)$ hold. The lower application σ_2 of (Cut^*) in $\mathfrak{D}'$ is also eliminable by inductive hypothesis, because $g(\sigma_1) < g(\sigma)$ holds by $l(\varphi) < l(\Box\varphi)$. Thus, there is a derivation of $\Box\Gamma, \Box\Xi \Rightarrow \Delta$ in $\mathrm{G}^-(\mathbf{cK\Sigma})$.

Case in which $\rho(L) = \rho(R) = (\Box 4_{\mathrm{inv}})$**.** Then $\mathfrak{D}$ =

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \Gamma, \Box\Gamma \Rightarrow \varphi\end{matrix}}{\Box\Gamma \Rightarrow \Box\varphi}\,(\Box 4_{\mathrm{inv}}) \qquad \dfrac{\begin{matrix}\mathfrak{D}_2\\ \varphi^m, \Xi, (\Box\varphi)^m, \Box\Xi \Rightarrow \psi\end{matrix}}{(\Box\varphi)^m, \Box\Xi \Rightarrow \Box\psi}\,(\Box 4_{\mathrm{inv}})}{\Box\Gamma, \Box\Xi \Rightarrow \Box\psi}\,(Cut^*),$$

where $\mathsf{FV}(\Gamma) \subseteq \mathsf{FV}(\varphi)$ and $\mathsf{FV}(\{\varphi\} \cup \Xi) \subseteq \mathsf{FV}(\psi)$. Then $\mathsf{FV}(\Gamma \cup \Xi) \subseteq \mathsf{FV}(\psi)$. Therefore, $\mathfrak{D}$ can be transformed into a derivation $\mathfrak{D}' =$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \Gamma, \Box\Gamma \Rightarrow \varphi\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \Gamma, \Box\Gamma \Rightarrow \varphi\end{matrix}}{\Box\Gamma \Rightarrow \Box\varphi}\,(\Box 4_{\mathrm{inv}}) \qquad \begin{matrix}\mathfrak{D}_2\\ \varphi^m, \Xi, (\Box\varphi)^m, \Box\Xi \Rightarrow \psi\end{matrix}}{\Box\Gamma, \varphi^m, \Xi, \Box\Xi \Rightarrow \psi}\,(Cut^*)}{\Gamma, \Box\Gamma, \Box\Gamma, \Xi, \Box\Xi \Rightarrow \psi}\,(Cut^*)}{\Gamma, \Xi, \Box\Gamma, \Box\Xi \Rightarrow \psi}\,(c \Rightarrow)}{\Box\Gamma, \Box\Xi \Rightarrow \Box\psi}\,(\Box 4_{\mathrm{inv}})$$

The upper application σ_1 of (Cut^*) in $\mathfrak{D}'$ is eliminable by inductive hypothesis, because $g(\sigma_1) = g(\sigma)$ and $w(\sigma_1) < w(\sigma)$ hold. The lower application σ_2 of (Cut^*) in $\mathfrak{D}'$ is also eliminable by inductive hypothesis, because $g(\sigma_1) < g(\sigma)$ holds by $l(\varphi) < l(\Box\varphi)$. Thus, there is a derivation of $\Box\Gamma, \Box\Xi \Rightarrow \Box\psi$ in $\mathrm{G}^-(\mathbf{cK}\Sigma)$.

Case in which $\rho(L) = (\Box 4_{\mathrm{inv}})$ and $\rho(R) = (\Box\mathrm{D}4_{\leqslant n})$. Then $\mathfrak{D} =$

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \Gamma, \Box\Gamma \Rightarrow \varphi\end{matrix}}{\Box\Gamma \Rightarrow \Box\varphi}\,(\Box 4_{\mathrm{inv}}) \qquad \dfrac{\begin{matrix}\mathfrak{D}_2\\ \varphi^m, \Xi, (\Box\varphi)^m, \Box\Xi \Rightarrow\end{matrix}}{(\Box\varphi)^m, \Box\Xi \Rightarrow}\,(\Box\mathrm{D}4_{\leqslant n})}{\Box\Gamma, \Box\Xi \Rightarrow}\,(Cut^*)$$

where $\mathsf{FV}(\Gamma) \subseteq \mathsf{FV}(\varphi)$ and $\#\mathsf{FV}(\{\varphi\} \cup \Xi) \leqslant n$. Then $\#\mathsf{FV}(\Gamma \cup \Xi) \leqslant n$. Thus, $\mathfrak{D}$ can be transformed into a derivation $\mathfrak{D}' =$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \Gamma, \Box\Gamma \Rightarrow \varphi\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \Gamma, \Box\Gamma \Rightarrow \varphi\end{matrix}}{\Box\Gamma \Rightarrow \Box\varphi}\,(\Box 4_{\mathrm{inv}}) \qquad \begin{matrix}\mathfrak{D}_2\\ \varphi^m, \Xi, (\Box\varphi)^m, \Box\Xi \Rightarrow\end{matrix}}{\Box\Gamma, \varphi^m, \Xi, \Box\Xi \Rightarrow}\,(Cut^*)}{\Gamma, \Box\Gamma, \Box\Gamma, \Xi, \Box\Xi \Rightarrow}\,(Cut^*)}{\Gamma, \Xi, \Box\Gamma, \Box\Xi \Rightarrow}\,(c \Rightarrow)}{\Box\Gamma, \Box\Xi \Rightarrow}\,(\Box 4_{\mathrm{inv}})$$

The upper application σ_1 of (Cut^*) in $\mathfrak{D}'$ is eliminable by inductive hypothesis, because $g(\sigma_1) = g(\sigma)$ and $w(\sigma_1) < w(\sigma)$ hold. The lower application σ_2 of (Cut^*) in $\mathfrak{D}'$ is also eliminable by inductive hypothesis, because $g(\sigma_1) < g(\sigma)$ holds by $l(\varphi) < l(\Box\varphi)$. Thus, there is a derivation of $\Box\Gamma, \Box\Xi \Rightarrow$ in $\mathrm{G}^-(\mathbf{cK}\Sigma)$.

Case in which $\rho(L) = \rho(R) = (\Box\mathrm{S4_{inv}})$**.** Then $\mathfrak{D}$ =

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \Box\Gamma \Rightarrow \varphi\end{matrix}}{\Box\Gamma \Rightarrow \Box\varphi}\,(\Box\mathrm{S4_{inv}}) \qquad \dfrac{\begin{matrix}\mathfrak{D}_2\\ (\Box\varphi)^m, \Box\Xi \Rightarrow \psi\end{matrix}}{(\Box\varphi)^m, \Box\Xi \Rightarrow \Box\psi}\,(\Box\mathrm{S4_{inv}})}{\Box\Gamma, \Box\Xi \Rightarrow \Box\psi}\,(Cut^*)$$

where $\mathsf{FV}(\Gamma) \subseteq \mathsf{FV}(\varphi)$ and $\mathsf{FV}(\{\varphi\} \cup \Xi) \subseteq \mathsf{FV}(\psi)$. Then $\mathsf{FV}(\Gamma \cup \Xi) \subseteq \mathsf{FV}(\psi)$. Therefore, $\mathfrak{D}$ can be transformed into a derivation $\mathfrak{D}'$ =

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \Box\Gamma \Rightarrow \varphi\end{matrix}}{\Box\Gamma \Rightarrow \Box\varphi}\,(\Box\mathrm{S4_{inv}}) \qquad \begin{matrix}\mathfrak{D}_2\\ (\Box\varphi)^m, \Box\Xi \Rightarrow \psi\end{matrix}}{\Box\Gamma, \Box\Xi \Rightarrow \psi}\,(Cut^*)}{\Box\Gamma, \Box\Xi \Rightarrow \Box\psi}\,(\Box\mathrm{S4_{inv}})$$

The application σ_1 of (Cut^*) in $\mathfrak{D}'$ is eliminable by inductive hypothesis, because $g(\sigma_1) = g(\sigma)$ and $w(\sigma_1) < w(\sigma)$ hold. Thus, there is a derivation of $\Box\Gamma, \Box\Xi \Rightarrow \Box\psi$ in $\mathrm{G}^-(\mathbf{cK\Sigma})$.

Case in which $\rho(L) = (\Box\mathrm{S4_{inv}})$ **and** $\rho(R) = (\Box\mathrm{T})$**.** Then $\mathfrak{D}$ =

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \Box\Gamma \Rightarrow \varphi\end{matrix}}{\Box\Gamma \Rightarrow \Box\varphi}\,(\Box\mathrm{S4_{inv}}) \qquad \dfrac{\begin{matrix}\mathfrak{D}_2\\ \varphi, (\Box\varphi)^{m-1}, \Xi \Rightarrow \Delta\end{matrix}}{\Box\varphi, (\Box\varphi)^{m-1}, \Xi \Rightarrow \Delta}\,(\Box\mathrm{T})}{\Box\Gamma, \Xi \Rightarrow \Delta}\,(Cut^*)$$

where $\mathsf{FV}(\Gamma) \subseteq \mathsf{FV}(\varphi)$. Therefore, $\mathfrak{D}$ can be transformed into a derivation $\mathfrak{D}'$ =

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \Box\Gamma \Rightarrow \varphi\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \Box\Gamma \Rightarrow \varphi\end{matrix}}{\Box\Gamma \Rightarrow \Box\varphi}\,(\Box\mathrm{S4_{inv}}) \qquad \begin{matrix}\mathfrak{D}_2\\ \varphi, (\Box\varphi)^{m-1}, \Xi \Rightarrow \Delta\end{matrix}}{\varphi, \Box\Gamma, \Xi \Rightarrow \Delta}\,(Cut^*)}{\Box\Gamma, \Box\Gamma, \Xi \Rightarrow \Delta}\,(Cut^*)}{\Box\Gamma, \Xi \Rightarrow \Delta}\,(c \Rightarrow)$$

The upper application σ_1 of (Cut^*) in $\mathfrak{D}'$ is eliminable by inductive hypothesis, because $g(\sigma_1) = g(\sigma)$ and $w(\sigma_1) < w(\sigma)$ hold. The lower application σ_2 of (Cut^*) in $\mathfrak{D}'$ is also eliminable by inductive hypothesis, because $g(\sigma_1) < g(\sigma)$ holds by $l(\varphi) < l(\Box\varphi)$. Thus, there is a derivation of $\Box\Gamma, \Xi \Rightarrow \Delta$ in $\mathrm{G}^-(\mathbf{cK\Sigma})$.

■

As before, the cut elimination theorem of $\mathrm{G}(\mathbf{cK\Sigma})$ supplies us the consistency of $\mathrm{G}(\mathbf{cK\Sigma})$ in a purely proof-theoretic way.

Corollary 199. A sequent $\Rightarrow \bot$ is not derivable in $\mathrm{G}(\mathbf{cK\Sigma})$.

Remarkably, the Craig interpolation theorem of $\mathrm{G}(\mathbf{cK\Sigma})$ seems to need more tricks than $\mathrm{G}(\mathbf{tK\Sigma})$. One can find a difficulty just by considering a case of $(\mathtt{K}_{\mathtt{inv}})$. We leave it open the problem whether or not $\mathrm{G}(\mathbf{cK\Sigma})$ admits the Craig interpolation theorem.

Chapter 6
Conclusion and Open Questions

6.1 Conclusion

We will summarize our results to answer **Questions 1, 2, 3, 4** asked in Chapter 1.

In Chapter 3, *term-sequence-modal logics* (TSMLs) subsuming the original term-modal logics were provided. The soundness theorems were given for all the given Hilbert systems $\mathsf{H}(\mathbf{tK}\Sigma)$ (Theorem 85). The strong completeness theorems were given for two classes of their Hilbert systems. The first class is the class of all Hilbert systems $\mathsf{H}(\mathbf{tK}\Sigma)$ such that $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{TSML}} \setminus \{ \mathrm{B}_n, 5_n, \mathrm{BF}_n \mid n \in \mathbb{N} \}$ (Theorems 95). The second class is the class of all Hilbert systems $\mathsf{H}(\mathbf{tKBF}_n\Sigma_n \upharpoonright \mathsf{L}_n)$ such that $\Sigma \subseteq \mathsf{Axiom}_{\mathsf{TSML}}$, where $\Sigma_n := \{ \mathrm{X}_n \mid \mathrm{X}_n \in \Sigma \}$ (Theorem 104). Furthermore, for any $\Sigma \subseteq \{ \mathrm{D}_n, \mathrm{T}_n, 4_n \mid n \in \mathbb{N} \}$, ordinary sequent calculi $\mathsf{G}(\mathbf{tK}\Sigma)$ were presented, and it was proved that they admit the cut elimination theorems and the Craig interpolation theorems ((Theorems 115), 119). Finally, soundness and strong completeness of expansions of the first and second classes above with equality were proved (Theorems 121, 130, 139).

In Chapter 4, *term-sequence-dyadic deontic logic* (TDDL) based on a TSML and the conditional logic **CK** was given. Soundness and strong completeness were first proved for the given Hilbert system $\mathsf{H}(\mathbf{TDDL})$ (Theorem 145, 154), and then an ordinary sequent calculus $\mathsf{G}(\mathbf{TDDL})$ was presented for TDDL. The cut elimination was proved in a similar vein as in TSML, in terms of which the Craig interpolation theorem was proved (Theorem 160). After all the formal details, we first argued that TDDL can two kinds of accommodate normative conflicts, i.e., situations in which incompatible obligations are directed towards different agents, and situations in which incompatible obligations are directed towards the same agent under different conditions. Also, TDDL can use quantifiers to formalize even situations implying derived normative conflicts. (Section 4.3.1). We then claimed that TDDL is consistent without the truth-ascription to deontic formulas, as well as that TDDL can accommodate normative conflicts of the above kinds without the truth-ascription (4.3.2). Although it is still an open question whether or not TDDL is compatible with the thesis that norms are neither true nor false, the proof of the consistency of TDDL without the truth-ascription is a first step towards the development of deontic logic compatible with the thesis.

In Chapter 5, *common sense modal predicate logics* (CMPCs) with a varying domain semantics were studied. In this varying domain semantics, several remarkable frame correspondences were confirmed (Proposition 174). For example, on the one hand, $\mathrm{D}_{\leqslant 0} = \Box P \to \Diamond P$ corresponds to the class of all the ordinary serial frames, i.e., frames such that for all w there is some v such that wRv. On the other hand, $\mathrm{D}_{\leqslant 2} = \Box Px_1x_2 \to \Diamond Px_1x_2$ corresponds to the class of all the serial frames with 2 objects, i.e., frames such that for all w and all $d, e \in D_w$ there is some v such that wRv and $d, e \in D_v$. The soundness theorems were established as usual for all the given Hilbert systems $\mathsf{H}(\mathbf{cK}\Sigma)$. The strong completeness theorems were proved via construction of the canonical model (Theorems 178, 189). At the same time, it turned out that formulas $\mathrm{K} = \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$, $4 = \Box P \to \Box\Box P$, $4^- = \Box P \to \Box\Box P$, $5 = \Diamond P \to \Box\Diamond P$, $5^- = \Diamond Qx_1 \to \Box\Diamond Qx_1$ are *not* canonical (Theorems 192, 193, 194). Finally, for any $\Sigma \subseteq \{\mathrm{T}, 4\} \cup \{\mathrm{D}_{\leqslant n} \mid n < \omega\}$, ordinary sequent calculi $\mathsf{G}(\mathbf{cK}\Sigma)$ were also presented and the cut elimination theorems were proved (Theorem 198). Contrary to TSML and TDDL, the Craig interpolation theorem was left open since it seemed not so straightforward.

Therefore, each question is answered as follows.

Question 1. How can we provide well-behaved cut-free sequent calculi for TSMLs subsuming TMLs?

- We have provided cut-free *ordinary* sequent calculi for TSMLs as such sequent calculi (Section 3.2).

Question 2. How can we develop a deontic logic that accommodates normative conflicts in a compatible way with the thesis that norms are neither true nor false?

- We have developed TDDL towards such a deontic logic (Sections 4.1, 4.2, 4.3). Our TDDL can accommodate two kinds of normative conflicts, i.e., situations in which incompatible obligations are directed towards different agents and situations in which incompatible obligations are directed towards the same agent under different conditions (Section 4.3.1). The first kind can be accommodated by changing the second index s in a formula $\mathbf{O}_{ts}\varphi$. The second kind can be accommodated by changing the conditional ψ in a formula $\mathbf{O}_{ts}(\varphi|\psi)$. Also, TDDL can use quantifiers to formalize even situations implying derived normative conflicts. As for the thesis that norms are neither true nor false, it is still an open question whether or not TDDL is compatible with the thesis. However, we can prove the consistency of TDDL without the truth-ascription via the cut-free sequent calculus $\mathsf{G}(\mathbf{TDDL})$. As we claimed in Chapter 1.1.2, this would be a first step towards the development of deontic logics compatible with the thesis that norms are neither true nor false. Furthermore, we can prove that TDDL can accommodate normative conflicts of the above kinds without the truth-ascription.

Question 3. Are there any sound and strongly complete CMPCs other than the original one?

- We have provided a Hilbert system $\mathsf{H}(\mathbf{cK})$ and its extensions with $(\mathtt{T})$, $(\mathtt{B})$, $(\mathtt{B}^-)$ and $\mathrm{D}_{\leqslant n}$ (Section 5.1).

Question 4. How can we provide well-behaved cut-free sequent calculi for CMPCs?

- We have provided cut-free ordinary sequent calculi for CMPCs (Section 5.2).

6.2 Open Questions

Many questions should be still asked. We shall close this thesis by first listing technical questions, and then asking philosophical questions.

Technical Quesitions

- How can we expand TDDL?
  - By adding a condition on an indexed selection function f that $f_{de}(w, X) \subseteq X$, we can probably have the sound and strongly complete Hilbert system obtained from $\mathsf{H}(\mathbf{TDDL})$ by adding axiom $\mathbf{O}_{ts}(\varphi|\varphi)$ (cf. Chellas [10]). As for the sequent calculus $\mathsf{G}(\mathbf{TDDL})$, replace the rule

$$\frac{\psi_1 \Rightarrow \psi \quad \psi \Rightarrow \psi_1 \quad \cdots \quad \psi_n \Rightarrow \psi \quad \psi \Rightarrow \psi_n \quad \varphi_1, \ldots, \varphi_n \Rightarrow \varphi}{\mathbf{O}_{ts}(\varphi_1|\psi_1), \ldots, \mathbf{O}_{ts}(\varphi_n|\psi_n) \Rightarrow \mathbf{O}_{ts}(\varphi|\psi)} \ (\mathbf{O})$$

    with a new rule

$$\frac{\psi_1 \Rightarrow \psi \quad \psi \Rightarrow \psi_1 \quad \cdots \quad \psi_n \Rightarrow \psi \quad \psi \Rightarrow \psi_n \quad \psi, \varphi_1, \ldots, \varphi_n \Rightarrow \varphi}{\mathbf{O}_{ts}(\varphi_1|\psi_1), \ldots, \mathbf{O}_{ts}(\varphi_n|\psi_n) \Rightarrow \mathbf{O}_{ts}(\varphi|\psi)}$$

    Probably the resulting sequent calculus becomes equipollent to the new Hilbert system with $\mathbf{O}_{ts}(\varphi|\varphi)$, and satisfies the cut elimination and Craig interpolation theorems. (cf. Pattinson and Schröder [77]).
- How can we provide the canonical model for CMPC so that even 4, 4^-, 5 and 5^- as well as T, B, $(\mathtt{B}^-)$ and $\mathrm{D}_{\leqslant n}$ are canonical?
- How can (or should) we add equality, constant symbols, and functions symbols to CMPC?
- How can we develop TSML with a CMPC-style varying domain semantics?
- Can we provide cut-free ordinary sequent calculi for TSML with equality?
  - This might be attained as follows. Let $\mathsf{G}^{=}(\mathbf{tK})$ be the sequent calculus obtained from $\mathsf{G}(\mathbf{tK})$ by adding the following rules:

$$\overline{\Rightarrow t = t}$$

$$\frac{\Gamma[t/z] \Rightarrow \Delta[t/z]}{t = s, \Gamma[s/z] \Rightarrow \Delta[s/z]}$$

$$\frac{\Gamma[s/z] \Rightarrow \Delta[s/z]}{t = s, \Gamma[t/z] \Rightarrow \Delta[t/z]}$$

$$\frac{\Gamma \Rightarrow \varphi, t = s}{[\vec{u}]\Gamma \Rightarrow [\vec{u}]\varphi, t = s} \ (\Box \mathrm{K}_n^=)$$

where the number of $t = s$ in $(\Box \mathrm{K}_n^=)$ can be 0. Unfortunately, $\mathrm{G}^=(\mathbf{tK})$ does not satisfy the cut elimination theorem. However, it might be the case that we obtain the subformula property of $\mathrm{G}^=(\mathbf{tK})$ by restricting (Cut) with a condition that the cut formula is always a subformula of a formula in the lower sequent of it [90, 67]. The similar modifications might work well even for expansions $\mathrm{G}(\mathbf{tK}\Sigma)$ of $\mathrm{G}(\mathbf{tK})$.

- Can we provide cut-free ordinary sequent calculi for CMPC with equality?
  - The similar modifications to the above might work for $\mathrm{G}(\mathbf{cK})$ and its expansions $\mathrm{G}(\mathbf{cK}\Sigma)$
- Does CMPC admit the Craig interpolation theorem?

Philosophical Questions

- How should the domain of first order deontic logic be like?
  - The Kripke semantics that we have given to TDDL was an increasing domain semantics. Our choice was done to make it easy to build TDDL's cut-free ordinary sequent calculus. However, our naive intuition does not seem fit this semantics, because sometimes there would be an acceptable world from the actual world such that some agents exist at the actual world but do not exist at the acceptable world. For a similar reason, the domain of first order deontic logic should not be decreasing.
  - The question is thus whether the domain of first order deontic logic should be constant, or varying. Of course we cannot answer this here. For a fruitful discussion on this topic, see Calardo [7, pp. 85–89].
- How should we understand relations between obligations of someone towards someone ($\mathbf{O}_{ts}\varphi$) and obligations of someone ("$\mathbf{O}_t\varphi$")? And how should we understand relations between them and "impersonal obligations" or "ought-to-be obligations" ("$\mathbf{O}\varphi$")?
  - The notion of obligation allowed in TDDL is only the notion of obligation of someone towards someone. Thus some questions to be considered would

naturally arise. For example, when Adam has an obligation towards Barbara to talk with her, should we also consider that he has an obligation (towards no one) to talk with her? Moreover, when he has an obligation (towards no one) to talk with her, should we also consider that it ought to be the case (independently from agents) that he talks with her?

- As a recent attempt towards understanding, Frijters and De Coninck [19] argue that directed obligations ($\mathbf{O}_{ts}\varphi$) should imply undirected obligations ($\mathbf{O}_t\varphi$) except when normative conflicts are involved. They are motivated by this implication to offer a term-modal deontic logic with an axiom $\mathbf{O}_{ts}\varphi \to \mathbf{O}_t\varphi$ which can accommodate a normative conflict of undirected obligations $\mathbf{O}_t\varphi \wedge \mathbf{O}_t\neg\varphi$ which is derived from a normative conflict of directed obligations $\mathbf{O}_{ts}\varphi \wedge \mathbf{O}_{tu}\neg\varphi$.

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