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The Dynamics of Takeovers through Exchange Offers in the Presence of Competition

Ryo Miyata
Teruyoshi Suzuki
Kyoko Yagi

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Faculty of Economics and Business
Hokkaido University
Kita 9 Nishi 7, Kita-Ku, Sapporo 060-0809, JAPAN

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Ryo Miyata*, Teruyoshi Suzuki†, Kyoko Yagi‡

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Abstract

This study examines the characteristics of takeovers in the presence of competition using a three-stage model. We first investigate the property of the equilibrium of mergers and takeovers in a frictionless market, and then analyze the effect that the existence of the competitors had on this process. We apply a general surplus function, rather than a specific one, eliminating the modifying effects of this factor from our study. Our model predicts that the existence of heavy competition in the takeover increases the number of unsuccessful or incomplete deals. Furthermore, we find that the shareholders of the target in a competitive market can choose the timing of accepting an offer, without the need to observe the surplus benefit of the deal. Our model shows that the presence of competition does not always provide the target shareholders with an advantage.

Keywords: Merger and acquisitions; real options; competitions

1 Introduction

During takeovers, if competition exists, it has a substantial influence on the timing and terms of the equilibrium. For example, the presence of competition will modify exchanges between a bidder and target shareholders, and also change the time frame. Accordingly, studies have attempted to develop a framework for the joint determination of the timing and terms of takeovers that occur in the face of competitors. Yet, little is known about the effects of competition on the time span of takeovers from initiation to completion. It is important to understand how their presence affects this timing, as delays have a significant effect on their failure. Prior studies have not analyzed the period of the deal as they have only constructed one-stage frameworks. In the one-stage model, the time taken to start the deal is consistent with the time taken to complete it. Therefore, we proposed a three-stage model for takeovers, in order to expand the analysis of the period of takeovers.

The model in this study has another feature: the general surplus function is considered. A prime motivation of those participating in takeovers is surplus benefit. Existing studies have assumed the specific form of the surplus function to meet their purpose. Current analyses of the effect of competition on takeovers have possibly relied on these assumptions. To avoid this drawback, we developed our model by supposing a general surplus function rather than a specific one, eliminating the modifying effects of this factor from our study. Thus, our model can examine more specifically the effect of competition on the period of takeovers.

*Faculty of Global and Regional Studies, University of the Ryukyus, 1, Sembaru, Nakagami-gun Nishihara-cho, Okinawa, 903-0129, Japan. ryo@eve.u-ryukyu.ac.jp

†Corresponding author, Faculty of Economics and Business, Hokkaido University, Kita 9, Nishi 7, Kita-ku, Sapporo 060-0809, Japan. suzuki@econ.hokudai.ac.jp

‡Graduate School of Management, Tokyo Metropolitan University, 1-4-1 Marunouchi, Chiyoda-ku, Tokyo 100-0005, Japan. kyagi@tmu.ac.jp

The main components in a construction of the three-stage takeover are the option to obtain the surplus benefit and a share-exchange option. We can show that a merger is expressed in this way. Hence, we first extend the previous results of mergers to those with the general surplus function. We then propose the takeover model and finally introduce a game between bidders and target shareholders.

This study's main findings are as follows. First, in the merger model, we provide sufficient conditions to attain an equilibrium under the general surplus function. One of sufficient conditions is that a central planner has a certain marginal profit in the merge surplus. This stems from the logic that an acquirer needs to provide the share-exchange option to the target firm.

Second, in the takeover model without competition, we prove that a takeover minimizing the expected time to propose an exchange offer, is equivalent to a globally optimal takeover maximizing the joint value of the bidding and target firms. This outcome proves beneficial for market participants. A shorter expected duration of the deal not only circumvents the entry of potential competitors, but also ensures global efficiency. Furthermore, we verify that under certain conditions, these two deals are identical to the globally efficient *merger*. Moreover, the model highlights that a decrease in the shareholder approval threshold causes a transfer of wealth from the target shareholders to the shareholders of the bidding firm. This finding is consistent with the study of Boone, et. al.(2018) wherein a bidder in Delaware is more likely to prefer takeovers after the enactment of a new law reducing the approval threshold.[1]

Last, after taking into consideration the presence of competitors in takeover markets, we find that competition increases the likelihood of the deal collapsing. Interestingly, if the market is relatively competitive, the stockholders of the target firm can determine when to accept the takeover, without observing the amount of surplus benefit in the deal. This means that even if the target shareholders suffer from information asymmetry, they can - once the options are provided by the bidder - implement the share-exchange option. Additionally, we show that the winning bidder can maximize their benefit from the deal as if the competitor does not exist, provided the losing bidder is not at all competitive. Hence, the presence of competition does not always offer an advantage to the target shareholders.

Our merger model is based on the seminal work of Lambrecht (2004). A number of earlier studies have targeted the endogenous timing and terms of mergers and acquisitions. Lambrecht (2004) assesses friendly mergers motivated by economies of scale in a one-factor model. He presents a model that provides the optimal timing of a merger, and indicates how the merger surplus could be divided. To analyze various aspects of mergers, previous papers present models specifying appropriate surplus benefit functions. For example, Hackbarth & Miao(2012) develop the timing and the terms of a takeover using the relation between oligopolistic industry equilibria. Hackbarth & Morellec (2008) derive the timing and the terms of a takeover that incorporates a follow-up operating option to analyze the behavior of firm-level betas in a takeover deal. Thijssen (2008) utilizes a double-boundary model to appraise mergers and takeovers that integrate both synergy effects and diversification as an investment. We follow the mechanism analyzed by these studies. However, we generalize the surplus function to investigate the effect of competition itself on the mergers and takeovers.

Concerning the exchange offer, Margrabe (1978) was the first to model it as an option-valuing problem. Moreover, Lambrecht (2004) examines takeovers by extending his merger model. Morellec & Zhdanov (2005) extend Lambrecht's endeavor by presenting competition and imperfect information. Morellec & Zhdanov (2005) study the financing strategy considering a simple takeover. A noteworthy difference between the present study and previous studies is that we model the takeover according to three-stage dynamics, which is based on appropriate business practices. Another point of contrast is the consideration of the general surplus function.

Looking at factors between players to introduce our game, we followed Hackbarth & Morellec (2008), in which equilibria depended on the competitiveness of the losing bidder. This study is different in that we introduce a leader-follower problem. This is where the winning bidder maximizes their firm value. From this feature, we conclude that the existence of competition at times works against the target shareholders unlike the previous study.

[1] In particular, their research compared the merger with the cash tender offer.

The remainder of this article is organized as follows. Section 2 scrutinizes the global efficiency of a friendly merger extending the previous studies to the general surplus function. Section 3 outlines the three-stage model for takeovers and investigates the timing and terms of takeovers in the equilibrium. Section 4 extends the three-stage model to examine its relationship to competition. Section 5 presents the concluding remarks. Unless otherwise noted, all the proofs of the results are provided in Section 6.

2 The Timing and Terms of Friendly Mergers

In this section, we follow Hackbarth & Morellec (2008) and generalize their surplus functions. We then show the general property of the equilibrium merger.

2.1 The Setup

The acquiring and target firms' unit values of capital are denoted by X_t and Y_t, respectively. Suppose that X_t and Y_t follow geometric Brownian motions:

$$\mathrm{d}X_t = \mu_X X_t \mathrm{d}t + \sigma_X X_t \mathrm{d}W_t^X, X_0 = x, \quad \mathrm{d}Y_t = \mu_Y Y_t \mathrm{d}t + \sigma_Y Y_t \mathrm{d}W_t^Y, Y_0 = y.$$

Note that the risk-adjusted drifts[2] and volatilities are constant and satisfy $\mu_X < r, \mu_Y < r, \sigma_X > 0$, and $\sigma_Y > 0$, where r denotes the risk-free interest rate. Furthermore, let W_t^X and W_t^Y be standard Brownian motions under the risk-neutral measure Q, and let the correlation coefficient between W_t^X and W_t^Y be constant and equal to $\rho \in (-1, 1)$. In Section 2–3, we assume a frictionless market.

Our study focuses on pure equity firms, where the merger and takeovers are supposed to be irreversible. The acquiring and target firms have capital stocks k_X and k_Y, respectively. Hence, each firm's equity value at time 0 equals $k_X x$ or $k_Y y$. Additionally, the processes of X_t and Y_t are independent of the deals. We assume that both the investors and the firms have complete information concerning all the parameters and functions of the model. Furthermore, both firms should have an option to merge the firms, wherein they can exchange their shares (i.e., $k_X x$ or $k_Y y$) for a fraction of the shares of the combined firm. The value of the combined firm is $G_M(x, y) + k_X x + k_Y y$ where $G_M(x, y)$ symbolizes a merger surplus from the deal, and satisfies:

Assumption 1 *$G_M(x, y)$ is homogeneous of degree 1 with respect to (w.r.t.) x and y, which follows that we can introduce $g_M(z)$ where $G_M(x, y)/y = g_M(z)$ for $z = x/y, y \neq 0$.*

We label the firm holding the asset as having the role of X_t as an acquirer in mergers or a bidder in takeovers. We also identify the firm whose asset is denoted by Y_t as the target in both mergers and takeovers.

We now define *call option characteristics* for the surplus functions on Z_t. Note that Z_t follows:

$$\mathrm{d}Z_t/Z_t = \left(\mu_X - \mu_Y - \sigma_Y^2\right)\mathrm{d}t + \sigma_X \mathrm{d}W_t^X - \sigma_Y \mathrm{d}W_t^Y, \quad Z_0 = z,$$

under Q where $\mu_X < r$ implies:

$$\mu_X - \mu_Y - \sigma_Y^2 < r. \tag{1}$$

Moreover, the characteristic function of Z_t is given by:

$$\left(\frac{1}{2}\sigma_X^2 + \frac{1}{2}\sigma_Y^2 - \rho\sigma_X\sigma_Y\right)\theta(\theta - 1) + (\mu_X - \mu_Y)\theta - (r - \mu_Y) = 0. \tag{2}$$

Definition 1 (Call Option Characteristics) *Suppose a call option with the payoff function* $\max\{f(x), 0\}$ *where $f(x)$ is called a benefit function. We say that $f(x)$ has call option characteristics, if*

$$f'(x) > 0, \quad f''(x) \leq 0,$$

[2]See Dixit & Pindyck(1993) p.178 for risk-adjusted drift.

there exists $x_0 > 0$ such that $f(x_0) = 0$, and $f(x)/f'(x) > w/\theta$ holds for sufficiently large x where θ is the positive root of (2) satisfying $\theta > 1$.

Throughout the study, we suppose that:

Assumption 2 *The surplus function $g_M(z)$ has the call option characteristics.*

To employ the findings of this section to analyze *takeovers*, we introduce the shareholder approval threshold $0 < \kappa \leq 1$ in the mergers. For takeovers, provided that shareholders of a bidding firm collect κ of target shares, the bidding firm can control the target firm. Hence, we assume that if κ of a target shareholder concurs with a merger, the decision of the target is finalized. Nevertheless, this scenario is a technical assumption for takeovers. Hence, we should understand the results in this section by setting $\kappa = 1$.

Consider ξ, which represents the share of the combined firm's asset where

$$0 < \xi < 1.$$

Then, the acquiring firm has an option to benefit

$$G_M^X(x,y) = \xi G_M(x,y) + \xi\{k_X x + \kappa k_Y y\} - k_X x,$$

from the merging. Furthermore, the approving target shareholders (κ of all shareholders) can receive

$$G_M^Y(x,y) = (1-\xi)G_M(x,y) + (1-\xi)\{k_X x + \kappa k_Y y\} - \kappa k_Y y,$$

from the deal. Note that the combined firm raises its capital by $(1-\kappa)k_Y y$, marginalizing the disapproving shareholders of the target firm.

To develop the takeover model in a later section, we introduce the benefit function regarding *a share-exchange option*, $G_E(x,y,\xi)$, by rewriting the benefit functions of mergers as:

$$\begin{array}{rclcl} G_M^X(x,y) & = & \xi G_M(x,y) & - & G_E(x,y,\xi), \\ G_M^Y(x,y) & = & (1-\xi)G_M(x,y) & + & G_E(x,y,\xi), \end{array} \tag{3}$$

where:

$$G_E(x,y,\xi) \quad = \quad (1-\xi)(k_X x + \kappa k_Y y) - \kappa k_Y y. \tag{4}$$

From (3), the options of a merger given to both firms are reinterpreted as follows. The acquirer has an option to receive a part of the merger surplus and writes the share-exchange option. The target shareholders have the option to secure the remaining part of the merger surplus and hold the share-exchange option documented by the acquirer.

Apparently, the function $G_E(x,y,\xi)$ is linearly homogeneous. Thus, we define $g_E(z,\xi) = G_E(x,y,\xi)/y$ where

$$g_E(z,\xi) = (1-\xi)k_X z - \xi\kappa k_Y. \tag{5}$$

and $z = x/y$. Therefore, the benefit functions of the acquirer and the target shareholders are also linearly homogeneous and can be rewritten as: $G_M^X(x,y) = y g_M^X(z,\xi)$ where

$$g_M^X(z,\xi) = \xi g_M(z) - g_E(z,\xi); \tag{6}$$

and $G_M^Y(x,y) = y g_M^Y(z,\xi)$ where

$$g_M^Y(z,\xi) = (1-\xi)g_M(z) + g_E(z,\xi). \tag{7}$$

It is easy to check $g_E(z,\xi)$ and $g_M^Y(z,\xi)$ has call option characteristics under Assumption 2. However, $g_M^X(z,\xi)$ does not necessarily have call option characteristics because the acquirer sells the share-exchange option. Thus, we impose an additional condition on $g_M^X(z,\xi)$:

Assumption 3 *Let $\mathcal{A} = \{\xi | g_M^X(z,\xi)$ has call option characteristics$\} \neq \emptyset$.*

2.2 The Equilibrium of the Merger

Initially, we present a globally efficient timing of the merger followed by an analysis of the exercise timing of the share-exchange option. Subsequently, we provide each optimal timing of the merger for the acquirer and target shareholders, and lastly, we present the main results of this section.

First, we consider the central planner who owns all the shares of the acquiring firm plus the sufficient shares of the target, $\kappa k_Y y$. We suppose that the combined firm can squeeze out the remaining shareholders by increasing their capital. Hence, the value of the merged firm is given by $k_X x + k_Y y + G_M(x, y)$. Therefore, the benefit of the central planner from the deal is given by:

$$k_X x + k_Y y + G_M(x, y) - (k_X x + \kappa k_Y y) - (1-\kappa) k_Y y = G_M(x, y).$$

Thus, we ensure that the central planner receives the total surplus $G_M(x, y)$ from the deal. Lemma 10 in appendix shows the valuation formula of the call option whose benefit function is characterized by Assumption 1–2. The next lemma is a direct consequence of the formula.

Lemma 1 (Globally Efficient Timing of the Merger) *Consider Assumptions 1–2 are valid. Then, the value of the option to merge the firms as a central planner is given by $V_M(x, y) = y v_M(z)$ with:*

$$v_M(z) = \begin{cases} g_M(z_M)\left(\dfrac{z}{z_M}\right)^{\theta}, & z < z_M, \\ g_M(z), & z \geq z_M; \end{cases} \tag{8}$$

and where z_M is a unique root of:

$$h_M(w) = w g_M'(w) - \theta g_M(w) = 0. \tag{9}$$

If $z < z_M$, the central planner must keep the option alive, and if $z \geq z_M$, the central planner must merge the firms. We call z_M the globally efficient timing of the merger.

Additionally, the value of the share-exchange option is directly given from Lemma 10 while omitting the proof.

Lemma 2 (The Share-Exchange Option) *Suppose that a bidder sells the share-exchange option to the target, for which the payoff function is given by $\max\{G_E(x, y), 0\}$ where $G_E(x, y)$ is defined by (4). Then, the value of the option is $V_E(x, y) = y v_E(z, \xi)$ with:*

$$v_E(z, \xi) = \begin{cases} g_E(z_E(\xi), \xi)\left(\dfrac{z}{z_E(\xi)}\right)^{\theta}, & z < z_E(\xi), \\ g_E(z, \xi), & z \geq z_E(\xi); \end{cases}$$

where:

$$z_E(\xi) = \frac{\theta}{\theta - 1} \frac{\xi \kappa k_Y}{(1-\xi) k_X}, \tag{10}$$

which is the root of equation:

$$h_E(w, \xi) = (1-\xi)(1-\theta) k_X w + \xi \theta \kappa k_Y = 0, \tag{11}$$

for given ξ. If $z < z_E(\xi)$, this option must be kept alive, and if $z \geq z_E(\xi)$, it must be exercised.

Third, we analyze the merger options held by the acquirer and the target shareholders. Under Assumption 1–3 their benefit functions have call option characteristics, respectively. Hence, Lemma 10 also derives the valuation formulae of the options.

Lemma 3 (Merger Options of the Acquirer and the Target) *Let Assumptions 1–3 hold. Consider firm i as the acquirer when $i = X$, or the target firm when $i = Y$. Then, the exercising strategy of firm i in terms of the merger option and its value are given as follows. If $z < z_M^i(\xi)$, firm i must keep the option alive, and if $z \geq z_M^i(\xi)$, firm i must immediately agree to the merger, where $z_M^i(\xi)$ is given by a root of:*

$$h_M^X(w,\xi) = \xi h_M(w) - h_E(w,\xi) = 0, \tag{12}$$

or

$$h_M^Y(w,\xi) = (1-\xi)h_M(w) + h_E(w,\xi) = 0, \tag{13}$$

for given ξ. Moreover, the value of the merger options for firm i are given by $V_M^i(x,y) = yv_M^i(z, z_M^i(\xi), \xi)$ where:

$$v_M^X(z, z_M^X(\xi), \xi) = \begin{cases} \left(\xi g_M(z_M^X(\xi)) - g_E(z_M^X(\xi),\xi)\right)\left(\dfrac{z}{z_M^X(\xi)}\right)^\theta, & z < z_M^X(\xi), \\ \xi g_M(z) - g_E(z,\xi), & z \geq z_M^X(\xi), \end{cases} \tag{14}$$

or

$$v_M^Y(z, z_M^Y(\xi), \xi) = \begin{cases} \left((1-\xi) g_M(z_M^Y(\xi)) + g_E(z_M^Y(\xi),\xi)\right)\left(\dfrac{z}{z_M^Y(\xi)}\right)^\theta, & z < z_M^Y(\xi), \\ (1-\xi) g_M(z) + g_E(z,\xi), & z \geq z_M^Y(\xi). \end{cases} \tag{15}$$

Lemmas 1–3 clarify the equilibrium timing such that $z_M^X(\xi) = z_M^Y(\xi)$ holds, in which the acquiring and target firms simultaneously want to enter the deal.

Therem 1 (The Uniqueness of the Equilibrium Merger) *Let Assumption 1–3 hold. Define*

$$\tilde{\xi} = \frac{(\theta-1)k_X z_M}{(\theta-1)k_X z_M + \theta\kappa k_Y}, \tag{16}$$

which is uniquely determined from $z_E(\tilde{\xi}) = z_M$ where z_M is the globally efficient timing of the merger. Then, $\tilde{\xi}$ gives the equilibrium such that both firms enter the deal simultaneously when Z_t reaches z_M where $z_M^X(\tilde{\xi}) = z_M^Y(\tilde{\xi}) = z_M$.

The theorem expresses the following implications.
(i) The merger timing is given by the globally efficient timing z_M as shown in previous studies.[3] A key contribution of this study is that we show this with the general surplus function.
(ii) Another contrast is that we derive the equilibrium term $\tilde{\xi}$ from $z_E(\xi) = z_M$. Surprisingly, the result suggests that how the surplus benefit is divided does not affect the equilibrium. For example, suppose that the benefit functions are extended to:

$$g_M^X(z,\xi;\delta) = \delta g_M(z) - g_E(z,\xi), \tag{17}$$
$$g_M^Y(z,\xi;\delta) = (1-\delta) g_M(z) + g_E(z,\xi). \tag{18}$$

Introducing δ, we can show $(z_M, \tilde{\xi})$ is still the equilibrium of the merger for any δ satisfying $0 \leq \delta \leq 1$. Hence, even in a merger where the acquirer or the target receive the total surplus ($\delta = 1$ or $\delta = 0$), the

[3]Hackbarth & Morellec (2008) prove that there is a division of share whereby $z_M^Y = z_M^X$ is verified, supposing a specific surplus function. Subsequently, $z_M^Y = z_M^X = z_M$ is presented as an interesting feature. Lambrecht (2004) reveals that there exist divisions of the share ξ_M^X such that $z_M^X = z_M$ holds and ξ_M^Y such that $z_M^Y = z_M$ holds for the Cobb–Douglas-type surplus function. He then verifies that $\xi_M^X + \xi_M^Y = 1$ is also valid. Lambrecht (2004) mentions that the Coase theorem achieves this optimality if the market is frictionless.

merger is still globally efficient.
(iii) From the direct relation (16) between $\tilde{\xi}$ and z_M, we derive a general property, $\partial\tilde{\xi}/\partial z_M > 0$, or equivalently $\partial z_M/\partial\tilde{\xi} > 0$ under Assumption 1–3. Thus, in the equilibrium, an increase of the share of the acquirer always delays the deal. This is because if the share of the acquirer increases, namely the target's share decreases, the timing of the merger is delayed since the target waits until the benefit from the exchange is sufficiently large.
(iv) For the call option characteristics of the acquirer's option, we imposed Assumption 3. Hence, a sufficient condition for the existence and uniqueness of the equilibrium is $\tilde{\xi} \in \mathcal{A}$. It follows that

$$\frac{\partial g_M^X(z,\tilde{\xi})}{\partial z} > 0 \Leftrightarrow g_M' > \frac{\theta\kappa k_Y}{z_M}. \tag{19}$$

Therefore, the merger occurs if the merger surplus has a certain marginal profit. The reason is that the acquirer provides the share-exchange option. Interestingly, decreasing the shareholder approval threshold κ creates the chance of the takeover because we will show that $\tilde{\xi}$ also provides a globally optimal takeover in Theorem 7. Furthermore, condition (19) is useful for the firms that are searching for a counter party to the merger.

2.3 General Properties of the Equilibrium

We discuss the general properties of the equilibrium from Theorem 1.

Corollary 1 (Global and Pareto Optimality) *Consider Assumptions 1–3 as true. Then,*
(i) The total benefit in the equilibrium is globally optimal, namely

$$v_M^X(z,z_M,\tilde{\xi}) + v_M^Y(z,z_M,\tilde{\xi}) = v_M(z). \tag{20}$$

We call $\tilde{\xi}$ the globally optimal division of shares.
(ii) The equilibrium merger is Pareto optimal for the choice of ξ because:

$$\left.\frac{\mathrm{d}v_M^X(z,z_M^X(\xi),\xi)}{\mathrm{d}\xi}\right|_{\xi=\tilde{\xi}} > 0, \quad \left.\frac{\mathrm{d}v_M^Y(z,z_M^Y(\xi),\xi))}{\mathrm{d}\xi}\right|_{\xi=\tilde{\xi}} < 0. \tag{21}$$

The properties are mentioned in previous studies. We present those generality. We also illustrate the comparative statistics of the thresholds, and their interrelationship. The results are necessary for the formulation of our takeover model.

Lemma 4 (Properties of Thresholds) *Let Assumptions 1–3 hold. Then,*

$$\frac{\mathrm{d}z_M^X(\xi)}{\mathrm{d}\xi} < 0, \quad \frac{\mathrm{d}z_M^Y(\xi)}{\mathrm{d}\xi} > 0, \quad \frac{\mathrm{d}z_E(\xi)}{\mathrm{d}\xi} > 0. \tag{22}$$

Furthermore, if $\xi \le \tilde{\xi}$,

$$z_E(\xi) \le z_M^Y(\xi) \le z_M \le z_M^X(\xi); \tag{23}$$

and if $\xi > \tilde{\xi}$,

$$z_M^X(\xi) < z_M < z_M^Y(\xi) < z_E(\xi). \tag{24}$$

The equalities in (23) are true if $\xi = \tilde{\xi}$.

Figure 1 illustrates four boundaries, namely, z_M, $z_M^X(\xi)$, $z_M^Y(\xi)$, and $z_E(\xi)$, which change ξ when the surplus function is given by

$$G_M(x,y) = k_Y\alpha(x-y) - k_Y\omega y, \tag{25}$$

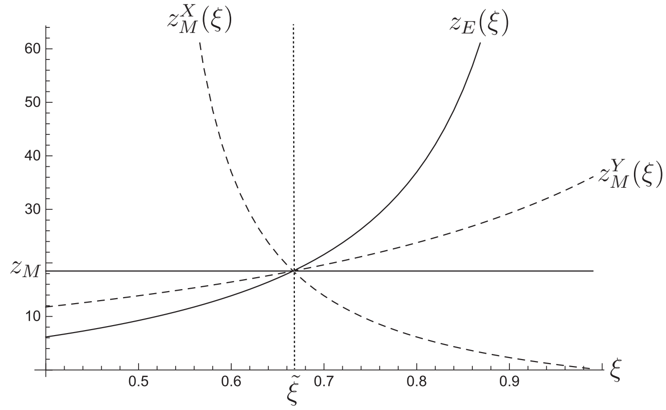

Figure 1: Optimal thresholds for mergers when the surplus function is given by (25) where players' benefit functions are extended to (17) and (18). The parameters are provided by $\alpha = \omega = \kappa = 1.0$, $r = 0.06$, $\mu_X = 0.055$, $\mu_Y = 0.025$, $\sigma_X = \sigma_Y = 0.2$, $\rho = 0.75$, and $k_X = k_Y = 1$. We set $\delta = 0.5$.

following Hackbarth & Morellec (2008) where $\alpha > 0$ and $\omega > 0$, which satisfies Assumptions 1–2. Further, we can show $\mathcal{A} \neq \emptyset$ if

$$\alpha + \omega > \kappa, \quad \xi > \frac{k_X}{k_Y + \alpha k_X}.$$

From Figure 1, we see that $z_M^X(\xi)$ is decreasing against ξ. The reason is that the acquirer needs to delay the merging in order to earn an adequate profit for smaller ξ. We also observe that z_M^Y is increasing with ξ. This observation can be understood by a reason similar to the first observation. Likewise, we find that when $\xi = \tilde{\xi}$, both the firms enter the deal at the same time as stated by Theorem 1. Furthermore, Figure 1 displays the result that $z_E(\xi) = z_M$ gives equilibrium share $\tilde{\xi}$ so that both players enter the deal at the same time, or $z_M^X(\tilde{\xi}) = z_M^Y(\tilde{\xi})$.

We now present the uniqueness of a leader-follower problem from Lemma 3–4. Our contribution is to show a sufficient condition for the uniqueness of the equilibrium in the leader-follower problem under the general surplus function. The condition is applied when competition is introduced in the takeover. Lambrecht (2004) investigates a similar problem as a hostile takeover model. eung & Kwok (2018) analyzed a similar setting under information asymmetry with a specific surplus function.

Proposition 1 (A Leader-Follower Problem in the Merger) *Letting Assumptions 1–3 hold, consider:*

$$z g_M'''(z) + (2 - \theta) g_M''(z) \geq 0. \tag{26}$$

Furthermore, suppose that the acquirer is the leader who controls the division of shares ξ, and the target is a follower who governs the timing of the merger: $z_M^Y(\xi)$ for given ξ. Then, the acquirer offers:

$$\xi^* = \arg\max_{0<\xi<1} v_M^X(z, z_M^Y(\xi), \xi),$$

which is unique and satisfies $\tilde{\xi} \leq \xi^* < 1$.

3 The Model for Takeovers

We first explain the process of the exchange offer and then present the three-stage model for takeovers. Finally, we state the property of the takeover in the frictionless market.

3.1 Exchange Offer Process

An outline of the three-stage model for a takeover is as follows: We assume that when a bidding firm holds a certain proportion of a target firm's shares, it obtains the management rights for the target firm. Moreover, these management rights generate an option to complete the *back-end merger*, which does not rely on the agreement of minority shareholders. Further, we suppose that the bidding firm has an option to propose an exchange offer by providing an exchange option directly to the shareholders of the target. If the exchange offer is proposed, the target shareholders have an option to exchange their shares for the share of the bidding firm. Provided the shareholders of the target accept the exchange offer, or they execute the exchange option equivalently, the shareholders of the bidding firm become the main shareholder of the target. They then have the right to complete the back-end merger. Under these circumstances, the bidder can ascertain the optimal timing of the back-end merger as a central planner. Thus, our model comprises three steps:

1. The bidder decides to propose an exchange offer;

2. The target shareholders accept the exchange offer; and

3. The bidder decides to complete the back-end merger.

Note that the three decisions above are supposed to be irreversible in this study.

We evaluate the aforementioned three decisions using the dynamic programming approach. We first show the value of the option in order to complete the back-end merger. Second, we deduce the value of the option to accept the exchange offer. Lastly, we suggest the value of the option to propose the exchange offer.

3.2 Value of Options

First, we discuss the value of the option to complete the back-end merger as follows: Assume that κ of stockholders in the target firm have exercised the exchange option. Then, the bidding firm holds shares $k_X x + \kappa k_Y y$. We also consider that the bidder can eject the target firm's remaining shareholders by increasing its stock. Therefore, we can regard the shareholders of the bidder as the central planner. Necessarily, the value of the option to conclude the back-end merger is given by Lemma 1.

The next result presents the value of the option to accept the exchange offer. We signify the optimal threshold w.r.t. z to *accept* the exchange offer by $z_A(\xi)$. The target shareholders valuate the option while taking into consideration that they will become the central planner at the time of acceptance.

Proposition 2 (The Value of the Option to Accept the Exchange Offer) *Consider that the bidder has proposed the exchange offer, and let Assumptions 1–3 hold. The value of the option to accept the offer is then given by $V_A(x,y) = yv_A(z,\xi)$, with:*
If $\xi \leq \tilde{\xi}$,

$$v_A(z,\xi) = \begin{cases} (1-\xi)g_M(z_M)\left(\dfrac{z}{z_M}\right)^{\theta} + g_E(z_A(\xi),\xi)\left(\dfrac{z}{z_A(\xi)}\right)^{\theta}, & z < z_E(\xi), \\ (1-\xi)g_M(z_M)\left(\dfrac{z}{z_M}\right)^{\theta} + g_E(z,\xi), & z_E(\xi) \leq z < z_M, \\ (1-\xi)g_M(z) + g_E(z,\xi), & z \geq z_M, \end{cases} \tag{27}$$

and if $\xi > \tilde{\xi}$,

$$v_A(z,\xi) = v_M^Y(z, z_A(\xi), \xi), \tag{28}$$

where

$$z_A(\xi) = \begin{cases} z_E(\xi), & \xi \leq \tilde{\xi}, \\ z_M^Y(\xi), & \xi > \tilde{\xi}. \end{cases} \tag{29}$$

The target shareholders should not accept the exchange offer when $z < z_A(\xi)$ and should accept the offer when $z \geq z_A(\xi)$.

Of central importance is the first equation of (27), which expresses that if $\xi \leq \tilde{\xi}$ the target shareholders accept the deal even though the back-end merger is uncertain. The target shareholders first accept the exchange offer, then they will complete the back-end merger as the central planner. In other words, the deal allows a certain period to complete the back-end merger. Remarkably, this equation implies that the value of the target's option equals the sum of the value of the share-exchange option, and the present value of the central planner's option. We can confirm that the value of the central planner's option does not depend on the time of acceptance. Hence, the timing of acceptance ignores the value of the surplus benefit and instead the timing is given by the optimal timing of the share-exchange option as in (29) for $\xi \leq \tilde{\xi}$.

In contrast, if $\xi > \tilde{\xi}$, (28) indicates that the target's option is regarded as a merger option, wherein the acceptance of the offer immediately finalizes the deal. The target shareholders delay the deal until the back-end merger becomes sure.

Finally, we display the value of the bidder's option. The bidder proposes the deal taking into consideration the target shareholders' optimal strategy, which we have specified by Proposition 2.

Therem 2 (The Value of the Option to Propose the Exchange Offer) *Let Assumptions 1–3 hold. Then, the value of the option to propose the exchange offer is given by $V_P(x, y) = yv_P(z, \xi)$, with:*

$$v_P(z, \xi) = \begin{cases} g_M^X(z_P(\xi), \xi) \left(\dfrac{z}{z_P(\xi)} \right)^\theta, & z < z_P(\xi), \\ g_M^X(z, \xi), & z \geq z_P(\xi), \end{cases} \tag{30}$$

where

$$z_P(\xi) = \begin{cases} z_M^X(\xi), & \xi \leq \tilde{\xi}, \\ z_M^Y(\xi), & \xi > \tilde{\xi}, \end{cases} \tag{31}$$

and where $z_P(\xi) \geq z_M$ holds. If $z < z_P(\xi)$, the bidder must keep this option alive, and if $z \geq z_P(\xi)$, the bidder must propose the exchange offer.

3.3 The Property of the Takeover in the Frictionless Market

We present several implication from Proposition 2 and Theorem 2. First, we see that the benefit at proposing in (30) is similar to the benefit from the merger. This means that the takeover proposed at the optimal timing is completed immediately upon initiation. The contrast to the merger is the timing $z_P(\xi) = z_M^Y(\xi)$ when $\xi > \tilde{\xi}$. We encapsulate the result as:

Corollary 2 (Takeover Dynamics under a Frictionless Market) *Letting Assumptions 1–3 hold, the proposition of the exchange offer immediately causes acceptance of the offer, which follows the immediate completion of the back-end merger.*

The result is understood as follows. Provided $\xi \leq \tilde{\xi}$, the target shareholders are ready to accept the offer before Z_t reaches z_M. However, the bidder delays the offer until it gains adequate benefits from the deal. This is similar to the merger. On the contrary, if $\xi > \tilde{\xi}$, the bidder delays the offer until the target shareholders are ready to accept the offer. This is because an early offer requires an additional cost in

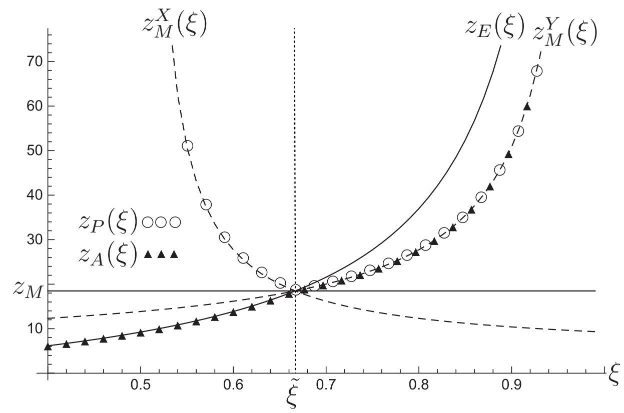

Figure 2: Optimal thresholds for the takeover when a surplus function is given by function (25). We utilize parameters similar to those in Figure 1.

our framework (See Remark 1 in Section 6.7). Figure 2 illustrates optimal thresholds for the option to accept the offer and the option to propose the exchange offer when a surplus function is given by (25). We will show Corollary 2 is not true if we consider the competitors in the market.

Next, we discusses the time to conclude the deal:

Corollary 3 (Minimization of the Time to Propose the Exchange Offer) *The bidding firm can minimize the expected time to propose the exchange offer by choosing $\tilde{\xi}$. In summary, $\tilde{\xi} = \arg\min_\xi z_P(\xi)$.*

Combining Lemma 4 and (31) directly proves the corollary. In practice, it is imperative to complete takeovers as quickly as possible to eliminate potential competitors who can provide a higher surplus function than the existing bidders. Hence, even though there appear to be no competitors in the market, the better strategy is to propose via $\tilde{\xi}$, which completes the deal at the earliest timing.

Next, we state the equivalence between the merger and takeovers. To this end, we develop three lemmas. We first discuss the acquisition premium of the target. Next, the present value of the merged firm is given. Third, the timing and the term maximizing the value of merged firm are shown. Finally, the result for the equivalence is shown.

Lemma 5 (Acquisition Premium of the Target's Equity) *An acquisition premium of the target firm is given by $y\bar{v}_A(z,\xi)$ where:*

$$\bar{v}_A(z,\xi) = \begin{cases} g_M^Y(z_P(\xi),\xi)\left(\dfrac{z}{z_P(\xi)}\right)^\theta, & z < z_P(\xi), \\ g_M^Y(z,\xi), & z \geq z_P(\xi). \end{cases}$$

Lemma 5 and Theorem 2 allow the immediate deduction of the value of a merged firm when the bidder considers the takeover.

Lemma 6 (Present Value of the Merged Firm) *The total equity value of the bidder and target when the bidder decides to plan the takeover is given as follows:*

$$k_X z + v_P(z,\xi) + k_Y + \bar{v}_A(z,\xi) = \begin{cases} k_X z + k_Y + g_M(z_P(\xi))\left(\dfrac{z}{z_P(\xi)}\right)^\theta, & z < z_P(\xi), \\ k_X z + k_Y + g_M(z), & z \geq z_P(\xi). \end{cases}$$

The expected value of the total surplus benefit from the takeover is equivalent to the value of the state price security, which pays the total merger surplus when the bidding firm proposes the exchange offer. The difference between the merger and the takeover comes from the timing of the completion of the deal. The upper bound of the $g_M(w)(z/w)^\theta$ is given by $g_M(z_M)(z/z_M)^\theta$ from Lemma 1. Additionally, (31) states $z_P(\tilde{\xi}) = z_M$. Therefore, we conclude the following scenario:

Lemma 7 (The Globally Optimal Takeover) *Division $\tilde{\xi}$ maximizes the present value of the bidder and target firm such that*

$$\tilde{\xi} = \arg\max_{\xi} \left\{ v_P(z,\xi) + \bar{v}_A(z,\xi) \right\} = \arg\max_{\xi} g_M(z_P(\xi)) \left(\frac{z}{z_P(\xi)} \right)^\theta .$$

Furthermore,

$$v_P(z,\tilde{\xi}) + \bar{v}_A(z,\tilde{\xi}) = v_M(z), \tag{32}$$

where

$$z_P(\tilde{\xi}) = z_M, \quad v_P(z,\tilde{\xi}) = v_M^X(z,\tilde{\xi}), \quad \bar{v}_A(z,\tilde{\xi}) = v_M^Y(z,\tilde{\xi}). \tag{33}$$

Corollary 3, Lemma 7 and Theorem 1 directly derive the next consequence.

Therem 3 (The Relationship between the Merger and Takeovers) *Suppose that the same surplus function and the shareholder approval threshold (κ) are given in the three deals mentioned below. Then, these deals are identical in the sense that they take place simultaneously and the divisions of their shares are exactly the same.*
(i) The takeover minimizing the expected time to propose the exchange offer.
(ii) The globally optimal takeover maximizing the joint firm-value of the bidder and the target.
(iii) The globally optimal merger that maximizes the joint firm-value of the acquirer and the target.

Theorem 3 states that minimizing the time to propose the offer results in the globally efficient takeover. Notably, $\kappa = 1$ should be determined in mergers but $\kappa < 1$ should be generalized in takeovers. Therefore, the result that (iii) is identical to (i) or (ii) is theoretical. Conversely, if the approval threshold is strictly given such that $\kappa = 1$, the globally optimal takeover is indistinguishable from the globally optimal merger.

Next, we present the analysis on shareholder approval threshold. As in Boone, et. al.(2018), a decrease in the shareholder approval threshold, κ, has been brought to public attention. We discuss below how the parameter κ affects the condition of the globally optimal takeover.

Corollary 4 (The Effect of the Shareholder Approval Threshold) *Considering $\xi = \tilde{\xi}$, the globally optimal takeovers satisfy:*

$$\frac{\partial \tilde{\xi}}{\partial \kappa} < 0, \quad \frac{\mathrm{d} v_P}{\mathrm{d}\kappa} < 0, \quad \frac{\mathrm{d} \bar{v}_A}{\mathrm{d}\kappa} > 0. \tag{34}$$

As expected, the first inequality in (34) implies that a decrease in the shareholders' approval threshold is beneficial for the acquirer. From the second and third inequalities in (34), we see that a decrease in κ increases the option value of the bidder and decreases that of the target shareholders. From these evaluations combining with (32), we conclude that a decrease in the shareholders' approval threshold enables the transfer of wealth from the target shareholders to the shareholders of the bidding firm. This outcome is consistent with the empirical finding by Boone, et. al.(2018) as mentioned in Section 1.

Finally, employing Theorem 2, we provide the study with a leader-follower problem, which develops the takeover dynamics in the presence of competition.

Lemma 8 (A Leader-Follower Problem in the Takeover) *Let Assumptions 1–3 and (26) hold. Suppose that the bidder is a leader who controls ξ and the timing of the offer, and the target is a follower who controls the timing of the offer's acceptance. Then, the problem:*

$$\max_{\xi} v_P(z,\xi) \tag{35}$$

solves the leader-follower problem, wherein the solution is given by ξ^* that appears in Proposition 1.

In this problem, the bidder proposes the offer when Z_t reaches $z_P(\xi^*) = z_M^Y(\xi^*)$, which immediately causes the acceptance of the offer and completion of the back-end merger. Interestingly, the leader-follower problem in the takeover occurs with a similar term and timing to the leader-follower problem in the merger provided that the market is frictionless.

4 Takeover under Competition among Multiple Bidders

We now introduce competition between multiple bidders. Without loss of generality, we can assume that there are two bidders, namely a winner and a loser, denoted by $j = w$ and ℓ, respectively. Even though there are several competitors, the equilibrium is determined by the first- and second-place bidders. The corresponding functions and parameters are superscripted or subscripted by $j = w, \ell$, which allows the notations: $z_M^{Y,j}$, z_M^j, g_M^j, g_P^j, v_A^j, z_A^j and $\tilde{\xi}_j$. Following Hackbarth & Morellec (2008), we assume that the firm sizes of the competitors are the same such that $k_X = k_X^j$ and $j = w, \ell$. Hence, $z_E(\cdot) = z_E^j(\cdot)$ where $j = w, \ell$ holds from (10). We also follow the previous study, in which the unit processes of capital are governed by the same process X_t under the supposition that the bidders belong to the same industry. As before, we also impose Assumption 1–3 on the surplus functions. The next assumption makes a difference between the winner and the loser.

Assumption 4 *The present value of Z_t is denoted by z_0, which is sufficiently low. Let the surplus functions satisfy:*

$$g_M^\ell(z_M^\ell)\left(\frac{z_0}{z_M^\ell}\right)^\theta \leq g_M^w(z_M^w)\left(\frac{z_0}{z_M^w}\right)^\theta. \tag{36}$$

Finally, we suppose (26) for the existence of the leader-follower problem in which the leader is the winner.

In this section, we employ a game to assess the effect of the strategic interaction of multiple bidders on the condition of the takeover. The players' utility functions are defined by the values of options, meaning that we use backward induction to study the players' strategies. The strategy space of the bidders is the set of proposing time $Z_t = z$ and division of shares ξ under the non-negative constraint of the bidder's utility value. The bidders govern the proposing time depending on Z_t and the public information defining whether the competitor has proposed the offer or not, and the offered share if the takeover has been proposed. When a bidder offers a deal, the remaining bidders should recommend an offer before the target shareholders accept the offer from the first bidder. The strategy space of the target is to specify the timing of the acceptance of the offer from the bidder that provides a higher benefit. The time to accept the offer is given by $z_A^j(\xi)$ in Proposition 2. It is noteworthy that the target can reject the offer and cannot accept offers from multiple bidders.

4.1 The sequence of the takeover decisions

Under this environment, we first consider the loser's breakeven offer and the winner's first-best strategy to specify which bidder proposes first. Next, we define the loser's competitiveness (*strong, weak, or extremely weak*) to classify the equilibria.

4.1.1 Which bidder proposes first?

The winner's first-best strategy should be given as the leader's strategy of the leader-follower problem in Lemma 8 because the problem maximizes the leader's value (assuming there are no competitors). Recall that the optimal proposing time of the problem is given by $z_M^{Y,w}(\xi_w^*)$ where the follower (the target) immediately accepts its offer. Thus, the winning bidder's first-best strategy is to offer ξ_w^* when Z_t reaches $z_M^{Y,w}(\xi_w^*)$.

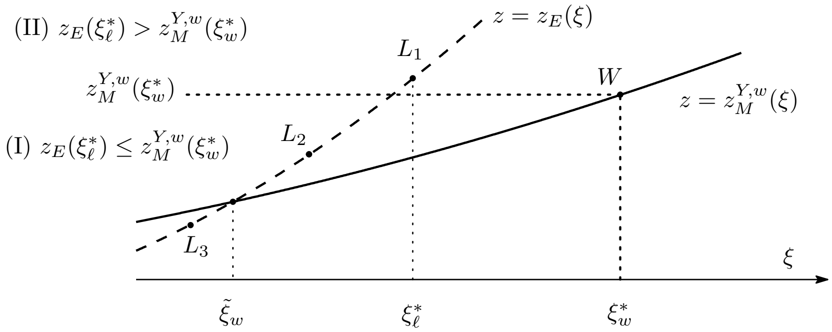

Figure 3: The winner's first best strategy is represented by $W = (\xi_w^*, z_M^{Y,w}(\xi_w^*))$. The loser's breakeven policies are illustrated by $L_i = (\xi_\ell^*, z_E(\xi_\ell^*))$ where $i = 1, 2, 3$. When the loser offers L_2 or L_3, the loser proposes first since $z_E(\xi_\ell^*) \le z_M^{Y,w}(\xi_w^*)$ holds. When the loser proposes L_1, the winner offers first since $z_E(\xi_\ell^*) > z_M^{Y,w}(\xi_w^*)$.

The loser's *breakeven division*, ξ_ℓ^* makes the loser's benefit zero. We now determine ξ_ℓ^* and its optimal proposing time. We can verify $\xi_\ell^* < \tilde{\xi}_\ell$ from Theorem 2 and (50). Hence (70) yields

$$v_P^\ell(z, \xi_\ell^*) = \xi_\ell^* g_M^\ell(z_M^\ell)\left(\frac{z}{z_M^\ell}\right)^\theta - g_E(z_E(\xi_\ell^*), \xi_\ell^*)\left(\frac{z}{z_E(\xi_\ell^*)}\right)^\theta - g_C(z, z_E(\xi_\ell^*)) = 0, \tag{37}$$

when the breakeven division is proposed at $Z_t = z \le z_E(\xi)$. The optimal timing of offer z is derived as follows. Eq. (37) and the value of the option to accept the offer (27) when $z \le z_E(\xi)$ imply that the value of the option to accept the offer at z is given by

$$v_A^\ell(z, \xi_\ell^*) = g_M^\ell(z_M^\ell)(z/z_M^\ell)^\theta - g_C(z, z_E(\xi_\ell^*)),$$

where (72) yields that

$$\begin{cases} g_C(z, z_E(\xi_\ell^*)) = 0, & z = z_E(\xi_\ell^*), \\ g_C(z, z_E(\xi_\ell^*)) > 0, & z < z_E(\xi_\ell^*), \end{cases}$$

Hence, to maximize the value of the option to accept the exchange offer, the bidders advise the offer when $z = z_E(\xi_\ell^*)$. Consequently, the value of the option to accept the offer becomes

$$v_A^\ell(z, \xi_\ell^*) = g_M^\ell(z_M^\ell)\left(\frac{z}{z_M^\ell}\right)^\theta, \quad z = z_E(\xi_\ell^*), \tag{38}$$

which is the maximum of the surplus benefit. Thus, the losing bidder's breakeven strategy is to offer ξ_ℓ^* determined by (37) when $z = z_E(\xi_\ell^*)$.

By comparing the winner's first best plan and the loser's breakeven policy, we understand that

(I) if $z_E(\xi_\ell^*) \le z_M^{Y,w}(\xi_w^*)$, the loser proposes first, else

(II) if $z_E(\xi_\ell^*) > z_M^{Y,w}(\xi_w^*)$, the winner proposes first.

Figure 3 illustrates the typical relationship between the loser's and winner's best timing $z_E(\xi_\ell^*)$ and $z_M^{Y,w}(\xi_w^*)$ changing ξ_ℓ^*.

4.1.2 The loser's competitiveness

Depending on which bidder proposes first, we introduce the competitiveness (*strong, weak, or extremely weak*) of losers.

Suppose (I) or that the loser proposes the breakeven division, ξ_ℓ^* before the winner proposes, when Z_t reaches $z_E(\xi_\ell^*)$. Then, the winner should deliberate whether to propose the first best strategy or to bid the offer that provides the same benefits to the target acquired by the loser. Therefore, we define the winner's second-best strategy ξ_w^{**} that satisfies:

$$v_A^w(z, \xi_w^{**}) = v_A^\ell(z, \xi_\ell^*), \quad v_P^\ell(z, \xi_\ell^*) = 0, \quad z = z_E(\xi_\ell^*). \tag{39}$$

Suppose (II) or that the winner proposes first and offers its first-best plan when Z_t reaches $z_M^{Y,w}(\xi_w^*)$. The loser immediately plans to offer the breakeven offer, ξ_ℓ^*, that satisfies (37) for $z = z_M^{Y,w}(\xi_w^*)$, and aims to supply $v_A^\ell(z, \xi_\ell^*)$ to the target. In response to the loser's offer, the winner considers whether or not to offer the second-best policy, ξ_w^{**}, which satisfies

$$v_A^w(z, \xi_w^{**}) = v_A^\ell(z, \xi_\ell^*), \quad v_P^\ell(z, \xi_\ell^*) = 0, \quad z = z_M^{Y,w}(\xi_w^*). \tag{40}$$

To define the loser's competitiveness, we employ the target's value given by the winner's second-best policy:

$$V^{**} = v_A^w(z, \xi_w^{**}),$$

where (z, ξ_w^{**}) satisfies (39) or (40). Furthermore, we define

$$\tilde{V} = v_A^w(z, \tilde{\xi}_w), \quad V^* = v_A^w(z, \xi_w^*).$$

Contingent on V^{**}, we outline the loser's situation as follows:

i) *The loser is strong* when $\tilde{V} < V^{**}$.

ii) *The loser is weak* when $V^* < V^{**} \leq \tilde{V}$.

iii) *The loser is extremely weak* when $V^{**} \leq V^*$.

For example, if the loser is weak, the target's value given by the winner's second-best policy is greater than that given by the winner's first best plan, but less than the one given by the winner's globally optimal policy.

As to the competitiveness of the loser and the second-best policy of the winner, Assumption 4 and Lemma 8 derive the following property:

Lemma 9 *1. The loser's breakeven policy ξ_ℓ^* given when $z = \min\{z_E(\xi_\ell^*), z_M^Y(\xi_\ell^*)\}$ satisfies: $\xi_\ell^* < \xi_w^{**}$.*

2. The second-best policy of the winner satisfies:

i) *if the loser is strong, $\xi_w^{**} < \tilde{\xi}_w$,*

ii) *if the loser is weak, $\tilde{\xi}_w \leq \xi_w^{**} < \xi_w^*$, or*

iii) *if the loser is extremely weak, $\xi_w^* \leq \xi_w^{**}$.*

3. When (II) *takes place, or $z_E(\xi_\ell^*) > z_M^{Y,w}(\xi_w^*)$, the loser cannot be strong.*

We illustrate Figure 4, which depicts the value functions of bidders and the target when the loser is weak. We can convince Lemma 9 2.-ii).

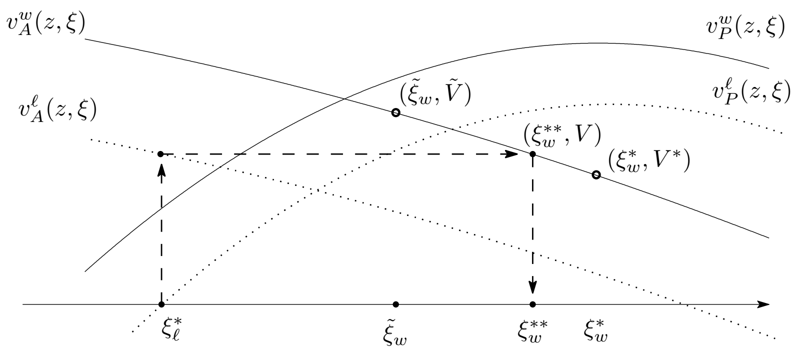

Figure 4: Function $v_P^j(z,\xi)$ draws winner's ($j = w$) or loser's ($j = \ell$) option value. Furthermore, $v_A^j(z,\xi)$ illustrates the target's option value offered by j. The loser's breakeven offer, ξ_ℓ^* is determined as $v_P^\ell(z,\xi_\ell^*) = 0$ where $z = \min\{z_E(\xi_\ell^*), z_M^Y(\xi_w^*)\}$. The winner's second-best offer, ξ_w^{**} is specified to hold $v_A^\ell(z,\xi_\ell^*) = v_A^w(z,\xi_w^{**})$. We observe $\tilde{\xi}_w \leq \xi_w^{**} < \xi_w^*$ when the loser is weak.

4.1.3 Equilibria in the market with competitors

We now present the equilibria of the takeover employing the bidders' offers introduced so far;
ξ_w^*: the winner's first best plan,
ξ_w^{**}: the winner's second-best strategy satisfying (39) or (40),
ξ_ℓ^*: the loser's breakeven policy determined by (39) or (40).

Therem 4 *Suppose that ξ_w^{**} and ξ_ℓ^* are defined by (39) until we redefine them. In the equilibria, the takeovers under competition follow the sequences shown below. Figure 5-9 explains the offers and timings of the takeovers for (I)-i),ii),iii) and (II)-ii),iii), respectively.*

(I) When $z_E(\xi_\ell^*) \leq z_M^{Y,w}(\xi_w^*)$, the loser proposes first and offers the breakeven division when Z_t reaches $z_E(\xi_\ell^*)$. The subsequent processes are specified as follows.

- *i) If the loser is strong, the winner immediately proposes the second-best plan, ξ_w^{**}. Thereafter, when Z_t exceeds $z_E(\xi_w^{**})$, the target shareholders accept the offer from the winner, and the back-end merger is concluded when Z_t reaches z_M^w.*
- *ii) If the loser is weak, the winner immediately proposes the second-best policy ξ_w^{**}. The target accepts the winner's offer simultaneously, or when Z_t exceeds $z_M^{Y,w}(\xi_w^{**})$. Upon acceptance, the completion of the back-end merger occurs.*
- *iii) If the loser is extremely weak, the winner immediately proposes the first-best offer, ξ_w^*. The target accepts the winner's offer when Z_t becomes $z_M^{Y,w}(\xi_w^*)$, which concludes the back-end merger.*

(II) When $z_E(\xi_\ell^*) > z_M^{Y,w}(\xi_w^*)$, the winner proposes first and offers the first-best plan when Z_t reaches $z_M^{Y,w}(\xi_w^*)$. The loser immediately proposes the breakeven division, ξ_ℓ^{**} determined by (40). The subsequent processes are stated below:

- *ii) If the loser is weak, the winner instantly proposes the second-best plan, ξ_w^{**}, which satisfies (40). The target immediately accepts the winner's offer, which follows the back-end merger.*
- *iii) If the loser is extremely weak, the target promptly accepts the winner's first-best offer, which immediately follows the completion of the back-end merger.*

Table 1: Property of the takeover under competition compared with the friendly merger.

first proposer	loser is	target's benefit		start	complete
loser (Figure 5)	strong	loser's total surplus*	greater than merger's**	early†	at equilibrium timing of merger
loser (Figure 6) or winner (Figure 7)	weak	loser's total surplus	less than merger's	early or late	late
loser (Figure 8) or winner (Figure 9)	extremely weak	minimized‡	less than merger's	early or late	late

* The winner presents the same benefit as the total surplus of the loser.
** The target's benefit is greater than that obtained through the merger with the winner.
† The deal starts earlier than the equilibrium timing of the merger.
‡ The winner maximizes his/her benefit as the leader of the leader-follower problem.

Proof. Lemma 4 and Lemma 9 derive the sequence of takeovers directly. Moreover, we see that no player has an incentive to change the strategies given that the agendas of the other player remain unchanged, as the deviation of the strategies decreases the gain or violates the constraints. Hence, the set of strategies given above is a Nash equilibrium of the takeover game. □

4.2 The effects of the competition on the takeover decisions

The implications of Theorem 4 are the following:
1. The most interesting result is that the presence of the competition increases the number of collapsing deals. Figure 5 can explain this result. We understand that as the losing bidder's competitiveness is stronger or the market is more competitive, the takeover starts earlier (ξ_ℓ^* becomes smaller). In the equilibrium, the target accepts the offer after an interval and the deal completes at the equilibrium timing of the merger. Hence, in the competitive market, the takeover takes much longer from the offer to completion. It follows that the deal cannot be completed because of the insufficient surplus benefit in some cases. Thus, the competition increases the collapsing or incomplete deals.

The reasoning of the result is more clear if we suppose the limit of the loser's strength such that $g_M^\ell(\cdot) \to g_M^w(\cdot)$. At the limit, the winner's benefit becomes zero because the winner provides the share-exchange option by spending all the benefit from the deal as the loser does. The deal starts the winner's breakeven timing and is accepted immediately because the offers are indistinguishable. But the deal is also completed at the equilibrium timing of the merger. As a result, the period from the offer to the completion is maximized. Thus, we obtain the result that the presence of competition increases the risk of the deal being incomplete.

2. More interesting is that the target's stockholders can determine the timing of the acceptance only from the market parameters, provided that the loser is strong. Recall that the time to accept the deal is given as the optimal timing of the share-exchange option, $z_E(\xi)$ when the loser is strong (See Figure 2 and 5). This implies that once the deal is offered, the timing of acceptance does not depend on the surplus function. It follows that the target's stockholders need not observe the surplus function to decide when to accept the deal. From (10), the functional form of $z_E(\xi)$ allows the stockholder to decide the time to accept the deal only based on the market parameters. The result becomes significant when an information asymmetry exists between the stockholders and the managers of the participating firms. Note that (10) suggests that decreasing κ accelerates the acceptance of the deal as Delaware's law expects.

3. It is remarkable that the competition does not always give the target's shareholders an advantage. Table 1 summarizes the property of takeovers under competition and shows that the target's shareholders achieve less benefit than the equilibrium merger unless the loser is strong. Especially, if the second-place bidder is extremely weak, the competition ends up allowing the first-place bidder to maximize its benefit.

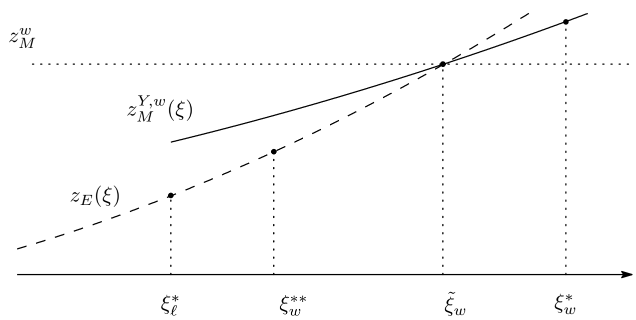

Figure 5: When $z_E(\xi_\ell^*) \leq z_M^{Y,w}(\xi_w^*)$ and the loser is strong ($\xi_w^{**} < \tilde{\xi}_w$), the loser proposes breakeven division ξ_ℓ^* when $Z_t = z_E(\xi_\ell^*)$. Simultaneously, the winner offers the second-best plan ξ_w^{**}. After that the target accepts the offer from the winner when $Z_t = z_E(\xi_w^{**})$. Finally, the winner completes the back-end merger at $Z_t = z_M^w$.

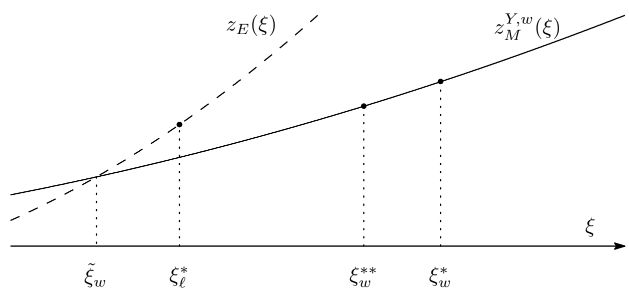

Figure 6: When $z_E(\xi_\ell^*) \leq z_M^{Y,w}(\xi_w^*)$ and the loser is weak ($\tilde{\xi}_w \leq \xi_w^{**} < \xi_w^*$), the loser proposes breakeven division ξ_ℓ^* when $Z_t = z_E(\xi_\ell^*)$. Simultaneously, the winner proposes the second-best plan ξ_w^{**}. The target shareholders accept the winner's offer when Z_t gains $\min\{z_E(\xi_\ell^*), z_M^{Y,w}(\xi_w^{**})\}$, which immediately completes the deal.

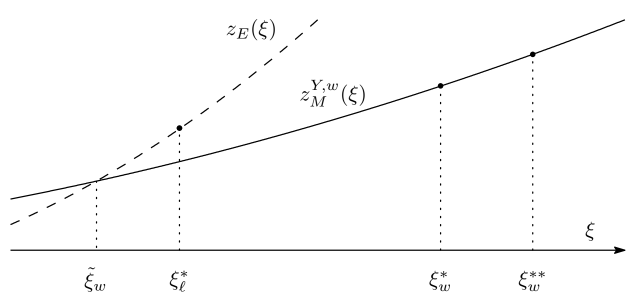

Figure 7: When $z_E(\xi_\ell^*) \leq z_M^{Y,w}(\xi_w^*)$ and the loser is extremely weak ($\xi_w^* \leq \xi_w^{**}$), the loser proposes breakeven division ξ_ℓ^* when $Z_t = z_E(\xi_\ell^*)$. Simultaneously, the winner proposes the first-best plan ξ_w^*. The target accepts the winner's offer when Z_t reaches $z_M^{Y,w}(\xi_w^*)$. Simultaneously, the backend merger is completed.

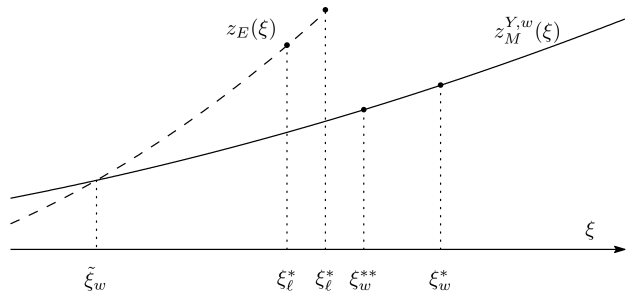

Figure 8: When $z_E(\xi_\ell^*) > z_M^{Y,w}(\xi_w^*)$ and the loser is weak ($\tilde{\xi}_w \leq \xi_w^{**} < \xi_w^*$), the winner proposes its first-best plan, ξ_w^* when $Z_t = z_M^{Y,w}(\xi_w^*)$. Simultaneously, the loser proposes breakeven offer ξ_ℓ^* defined by (40), which immediately follows the winner's second offer, ξ_w^{**} satisfying (40). The target instantaneously accepts the winner's second offer, which completes the deal. Note that the left and right ξ_ℓ^* satisfy (39) and (40), respectively.

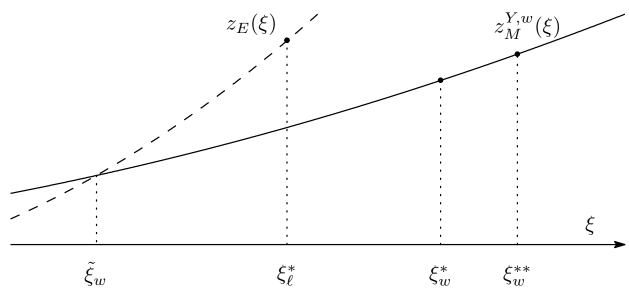

Figure 9: When $z_E(\xi_\ell^*) > z_M^{Y,w}(\xi_w^*)$ and the loser is extremely weak ($\xi_w^* \leq \xi_w^{**}$), the winner first proposes its first-best plan, ξ_w^* when $Z_t = z_M^{Y,w}(\xi_w^*)$. Simultaneously, the loser proposes the breakeven offer ξ_ℓ^* satisfying (40). The target instantaneously accept the winner's offer, which completes the deal.

4. As expected, the takeovers under competition are not globally optimal. As we can easily prove:

$$v_P^w(z, \xi_w^{**}) + \bar{v}_A^w(z, \xi_w^{**}) = v_M^w(z) - g_C^w(z_E(\xi_\ell^*), z_E(\xi_w^{**}))(z_E(\xi_\ell^*)/z_E(\xi_w^{**}))^\theta < v_M^w(z),$$

when the target delays in accepting the offer or when the loser is strong. When the loser is weak or extremely weak, the target immediately accepts the offer in many cases. But the completion of the deal is inefficiently delayed. This is because the weakness of the loser decreases the share to the target, which results in the target waiting until the surplus benefit is sufficiently high to accept the offer.

5 Concluding Remarks

This study presented a real-options model to study the equilibria of mergers and takeovers through an exchange offer. Since we considered the general surplus function, this study's results did not depend on the specific form of the surplus function. Specifically, the results - with respect to competition in takeover markets - can be developed by only considering the existence of competitors. Further, we established a three-stage model for the takeover. Therefore, we could discuss the period of the takeover from initiation to the completion, which is a difficult exercise in the previous frameworks. This extension is essential because the period of the takeover strongly affects the possibility of the takeover's success.

As regards the shareholder approval threshold, κ, we showed that decreasing κ creates the chance of takeovers since the sufficient marginal profit of the surplus is lowered. This result is suitable to promote takeovers.

Having discussed our results, we would now like to acknowledge one limitation to this study. Generally, a takeover is proposed by one of two ways: a tender offer or an exchange offer. Due to model constraints, only an exchange offer was analyzed in this study. In the tender offer, target shareholders can sell their shares and receive the offer price in cash. However, our model cannot treat the tender offer because the benefit function in the tender offer does not satisfy Assumption 1.

The model can be extended in several dimensions. It is interesting to incorporate asymmetric information (orellec & Schurhoff, 2011) in takeovers. Moreover, to consider the reciprocal chance to propose the exchange offer is interesting. More difficult extensions include earn outs (Lukas et. al, 2012), and debt financing (Morellec & Zhdanov, 2008).

6 Appendix

6.1 The Value of the Call Option where the Benefit Function has Call Option Characteristics

Lemma 1–3 are the direct consequence of the next lemma:

Lemma 10 *Suppose a perpetual call option with a payoff function given by* $\max\{F(x,y),0\}$ *where* $F(x,y) = yf(z), z = x/y$. *Suppose further* $f(z)$ *has the call option characteristics, which follows that there exists* $w_0 > 0$ *such that* $f(w_0) = 0$. *Then,*

$$h(w) = wf'(w) - \theta f(w) = 0, \tag{41}$$

has a unique root $w = z^* > w_0$, *and the value of the call option is given by* $V(x,y) = yv(z)$, *where*

$$v(z) = \begin{cases} f(z^*)\left(\frac{z}{z^*}\right)^{\theta}, & z < z^*, \\ f(z), & z \geq z^*; \end{cases}$$

and where θ *is the positive root of (2). If* $z < z^*$, *the option must be active, and if* $z \geq z^*$, *the option must be exercised.*

Proof. The value of the call option $V = V(x,y)$, when the option is alive, satisfies:

$$rV = \mu_X x\frac{\partial V}{\partial x} + \mu_Y y\frac{\partial V}{\partial y} + \frac{1}{2}\sigma_X^2 x^2\frac{\partial^2 V}{\partial x^2} + \frac{1}{2}\sigma_Y^2 y^2\frac{\partial^2 V}{\partial y^2} + \rho\sigma_X\sigma_Y xy\frac{\partial^2 V}{\partial x\partial y}. \tag{42}$$

Since boundary conditions for (42) are given by the linearly homogeneous function $F(x,y)$, we choose a candidate of $V(x,y)$ as a linearly homogeneous function in x and y. Hence, we can reduce a dimension by $(1/y)V(x,y) = V(z,1) \equiv v(z)$ where $z = x/y$, obtaining:

$$\left(\frac{1}{2}\sigma_X^2 + \frac{1}{2}\sigma_Y^2 - \rho\sigma_X\sigma_Y\right)z^2\frac{\mathrm{d}^2 v(z)}{\mathrm{d}z^2} + (\mu_X - \mu_Y)z\frac{\mathrm{d}v(z)}{\mathrm{d}z} - (r - \mu_Y)v(z) = 0. \tag{43}$$

The general solution of (43) is represented by $v(z) = A_1 z^{\theta} + A_2 z^{\theta_0}$, where A_1 and A_2 are undetermined multipliers, and $\theta > 1$ and $\theta_0 < 0$ are roots of the quadratic equation (2). Further, boundary condition $\lim_{z\to 0} v(z) = 0$ implies

$$v(z) = A_1 z^{\theta}, \quad \theta > 1. \tag{44}$$

Similarly, the value-matching condition $f(z^*) = A_1(z^*)^{\theta}$, where z^* means the optimal threshold, leads to $A_1 = f(z^*)(z^*)^{-\theta}$ with $z^* > w_0$. It follows that $v(z) = f(z^*)\left(z/z^*\right)^{\theta}$ where z^* is set to maximize $v(z)$ under $z^* > w_0$. Thus, z^* is obtained by solving:

$$v(z) = \max_{w > w_0} f(w)\left(\frac{z}{w}\right)^{\theta}. \tag{45}$$

To solve (45), we let $\tilde{f}(w) = f(w)/w^\theta$. We then have

$$\tilde{f}'(w) = h(w)/w^{\theta+1},$$

where

$$h(w) = wf'(w) - \theta f(w), \quad h'(w) = wf''(w) - (\theta - 1)f'(w).$$

As $f(x)$ has the call option characteristics, $f'(w) > 0$ and $f(w_0) = 0$ for $w_0 > 0$. Hence, $h(w_0) = w_0 f'(w_0) > 0$. Further, the call option characteristics follow that $h(w) < 0$ for sufficiently large w, and $h'(w) < 0$. Therefore, $h(w) = 0$ has a unique solution $w = z^* > w_0 > 0$ such that

$$h(w) = \begin{cases} (+) & w \leq z^*, \\ (-) & w > z^*. \end{cases} \tag{46}$$

Thus, z^* is unique and defines $v(z)$. □

Corollary 5 *If $f(z)$ has the call option characteristics, $\tilde{f}(w) = f(w)/w^\theta$ is convex for $w > w_0$ where $f(w_0) = 0$.*

Proof. We proved (45) has a unique solution $w = z^* > w_0$, which proves the corollary.

6.2 Proof of Theorem 1 (The Uniqueness of the Equilibrium Merger)

Let w be the threshold for z, at which both the firms enter the deal together. Then, Lemma 3 infers that the condition to agree with the deal through both the same timing and the term is given by

$$h_M^X(w, \xi) = \xi h_M(w) - h_E(w, \xi) = 0, \tag{47}$$

$$h_M^Y(w, \xi) = (1 - \xi)h_M(w) + h_E(w, \xi) = 0, \tag{48}$$

which are equivalent to the following simultaneous equations:

$$h_M(w) = 0, \quad h_E(w, \xi) = 0. \tag{49}$$

Thus, Lemma 1–2 lead to $w = z_M$ and $\xi = \tilde{\xi}$ being a solution for the simultaneous equations (47) and (48). Lemma 1 provides the uniqueness of z_M. Moreover, $\tilde{\xi}$ is uniquely determined by (16) for given z_M. Thus, the uniqueness of $(z_M, \tilde{\xi})$ is confirmed. □

6.3 Proof of Corollary 1 (Global and Pareto Optimality)

(i) Adding (14) to (15), we acquire (20) from (8).
(ii) From the optimality of $v_M^X(z, w, \xi)$ w.r.t $w = z_M^X(\xi)$, we obtain

$$\frac{\mathrm{d}v_M^X(z, w, \xi)}{\mathrm{d}\xi} = \frac{\partial v_M^X(z, w, \xi)}{\partial w}\frac{\mathrm{d}w}{\mathrm{d}\xi} + \frac{\partial v_M^X(w, \xi)}{\partial \xi} = -\frac{\partial g_E(z, \xi)}{\partial \xi}\left(\frac{z}{w}\right)^\theta > 0. \tag{50}$$

In a similar method, we can show

$$\frac{\mathrm{d}v_M^Y}{\mathrm{d}\xi}(z, w, \xi) < 0, \quad w = z_M^Y(\xi).$$

Thus, we conclude (22) since Theorem 1 implies $w = z_M^X(\tilde{\xi}) = z_M^Y(\tilde{\xi})$. □

6.4 Proof of Lemma 4 (Properties of the Equilibrium)

We first prepare several inequalities. We can show:

$$h'_M(w) < 0, \quad \frac{\partial h^X_M(w,\xi)}{\partial w} < 0, \quad \frac{\partial h^Y_M(w,\xi)}{\partial w} < 0, \tag{51}$$

similarly to provide $h'(w) < 0$ in Section 6.1. We attain

$$\frac{\partial h_E(w,\xi)}{\partial w} = (1-\xi)(1-\theta)k_X < 0, \quad \frac{\partial h_E(w,\xi)}{\partial \xi} = (\theta-1)k_X w + \theta\kappa k_Y > 0, \tag{52}$$

from (11). In addition, (11), (12) and (52) provide

$$\frac{\partial h^X_M(w,\xi)}{\partial \xi} = h_M(w) - \frac{\partial h_E(w,\xi)}{\partial \xi} = \frac{h_E(w,\xi)}{\xi} - \frac{\partial h_E(w,\xi)}{\partial \xi} = \frac{(1-\theta)k_X w}{\xi} < 0. \tag{53}$$

Similarly, we obtain $\partial h^Y_M(w,\xi)/\partial\xi = \theta\kappa k_Y/(1-\xi) > 0$.

To prove (22), we apply the implicit function theorem to (12), which follows

$$\frac{\mathrm{d}h^X_M(w,\xi)}{\mathrm{d}\xi} = \frac{\partial h^X_M(w,\xi)}{\partial w}\frac{\mathrm{d}w}{\mathrm{d}\xi} + \frac{\partial h^X_M(w,\xi)}{\partial \xi} = 0, \quad w = z^X_M(\xi). \tag{54}$$

Therefore, (51), and (53) yield $\mathrm{d}z^X_M(\xi)/\mathrm{d}\xi < 0$. Using a similar method, we obtain $\mathrm{d}z^Y_M(\xi)/\mathrm{d}\xi > 0$. Moreover, the implicit function theorem applied to (11) derives $\mathrm{d}z_E(\xi)/\mathrm{d}\xi > 0$. Thus, (22) is substantiated.

To prove (23)–(24), we first show

$$\begin{cases} z_E(\xi) \le z^Y_M(\xi), & \xi \le \tilde{\xi}, \\ z_E(\xi) > z^Y_M(\xi), & \xi > \tilde{\xi}. \end{cases} \tag{55}$$

This is given from

$$\begin{aligned} h^Y_M(z_E(\xi),\xi) &= (1-\xi)h_M(z_E(\xi)) + h_E(z_E(\xi),\xi) \\ &= (1-\xi)h_M(z_E(\xi)) \\ &= \begin{cases} (+), & z_E(\xi) \le z_M \Leftrightarrow \xi \le \tilde{\xi}, \\ (-), & z_E(\xi) > z_M \Leftrightarrow \xi > \tilde{\xi}. \end{cases} \end{aligned} \tag{56}$$

Note that the third equality holds since $h_M(w)$ is decreasing where $h_M(z_M) = 0$. Note also that since $h^Y_M(w,\xi)$ is decreasing w.r.t. w where $h^Y_M(z^Y_M(\xi),\xi) = 0$, (56) leads to (55).

Second, $z^X_M(\tilde{\xi}) = z^Y_M(\tilde{\xi}) = z_M$ from Theorem 1, $\mathrm{d}z^X_M/\mathrm{d}\xi < 0$, and $\mathrm{d}z^Y_M/\mathrm{d}\xi > 0$ infer:

$$\begin{cases} z^Y_M(\xi) \le z_M \le z^X_M(\xi), & \xi \le \tilde{\xi}, \\ z^X_M(\xi) < z_M < z^Y_M(\xi), & \xi > \tilde{\xi}. \end{cases} \tag{57}$$

Thus, (55) and (57) complete the proof. □

6.5 Proof of Proposition 1 (The Leader-Follower Problem in the Merger)

We first prepare a lemma to prove the proposition.

Lemma 11 *Let Assumption 1–3 hold. Then,*

$$\frac{\mathrm{d}^2 z_E(\xi)}{\mathrm{d}\xi^2} > 0; \quad \textit{and if (26) holds,} \quad \frac{\mathrm{d}^2 z^X_M(\xi)}{\mathrm{d}\xi^2} > 0, \frac{\mathrm{d}^2 z^Y_M(\xi)}{\mathrm{d}\xi^2} > 0. \tag{58}$$

Proof. It is easy to show the first inequality from (10). To prove the second inequality, we employ the implicit function theorem. From (54), we obtain

$$\begin{aligned}\frac{\mathrm{d}^2 h_M^X(w,\xi)}{\mathrm{d}\xi^2} = \frac{\partial^2 h_M^X(w,\xi)}{\partial w^2}\left(\frac{\mathrm{d}w}{\mathrm{d}\xi}\right)^2 + \frac{\partial h_M^X(w,\xi)}{\partial w}\frac{\mathrm{d}^2 w}{\mathrm{d}\xi^2} + 2\frac{\partial^2 h_M^X(w,\xi)}{\partial w \partial \xi}\frac{\mathrm{d}w}{\mathrm{d}\xi} \\ + \frac{\partial^2 h_M^X(w,\xi)}{\partial \xi^2} = 0, \quad w = z_M^X(\xi).\end{aligned}$$

This gives $\mathrm{d}^2 z_M^X(\xi)/\mathrm{d}\xi^2 > 0$ because (26) implies

$$\frac{\partial^2 h_M^X(w,\xi)}{\partial w^2} = \xi h_M''(w) = w g_M'''(w) + (2-\theta) g_M'' \geq 0;$$

(51) shows $\partial h_M^X(w,\xi)/\partial w < 0$; (53) leads to

$$\frac{\partial^2 h_M^X(w,\xi)}{\partial w \partial \xi} < 0, \quad \frac{\partial^2 h_M^X(w,\xi)}{\partial \xi^2} > 0;$$

and $\mathrm{d}w/\mathrm{d}\xi < 0$ from (22). Similarly, we can show $\mathrm{d}^2 z_M^Y(\xi)/\mathrm{d}\xi^2 > 0$. □

Proof of proposition. First, we assert that $v_M^X(z, z_M^Y(\xi), \xi)$ is increasing when $\xi < \tilde{\xi}$, or

$$\frac{\mathrm{d}v_M^X(z,w,\xi)}{\mathrm{d}\xi} = \frac{\partial v_M^X(z,w,\xi)}{\partial w}\frac{\mathrm{d}w}{\mathrm{d}\xi} + \frac{\partial v_M^X(w,\xi)}{\partial \xi} > 0, \quad w = z_M^Y(\xi), \quad \xi < \tilde{\xi}.$$

This is true because $\partial w/\partial \xi > 0$ is given by Lemma 4; Lemma 3 derives

$$\frac{\partial v_M^X(z,w,\xi)}{\partial \xi} = (g_M(w) + k_X w + \kappa k_Y)\left(\frac{z}{w}\right)^\theta > 0; \tag{59}$$

and we can show

$$\frac{\partial v_M^X(z,w,\xi)}{\partial w} = \frac{h_M^X(w,\xi)}{w^{\theta+1}} = \frac{h_M(w,\xi)}{w^{\theta+1}} = \begin{cases} (+) & , \quad w \leq z_M \Leftrightarrow \xi < \tilde{\xi}, \\ (-) & , \quad w > z_M \Leftrightarrow \xi \geq \tilde{\xi}. \end{cases} \tag{60}$$

We derive (60) from Lemma 4 and (46) considering that timing $w = z_M^Y(\xi)$ is optimal to the target for given ξ, which means $h_M^Y(w,\xi) = (1-\xi)h_M + h_E(w,\xi) = 0$, following $h_M^X(w,\xi) = \xi h_M(w) - h_E(w,\xi) = h_M(w)$.

Second, we verify that $v_M^X(z, z_M^Y(\xi), \xi)$ is concave when $\tilde{\xi} \leq \xi < 1$, that is,

$$\begin{aligned}\frac{\mathrm{d}^2 v_M^X(z,w,\xi)}{\mathrm{d}\xi^2} = \frac{\partial^2 v_M^X(z,w,\xi)}{\partial w^2}\left(\frac{\mathrm{d}w}{\mathrm{d}\xi}\right)^2 + \frac{\partial v_M^X(z,w,\xi)}{\partial w}\frac{\partial^2 w}{\partial \xi^2} + 2\frac{\partial^2 v_M^X(w,\xi)}{\partial \xi \partial w}\frac{\mathrm{d}w}{\mathrm{d}\xi} \\ + \frac{\partial^2 v_M^X(z,w,\xi)}{\partial \xi^2} < 0, \quad w = z_M^Y(\xi), \quad \xi \geq \tilde{\xi}.\end{aligned}$$

This is shown because Corollary 5 states $\partial^2 v_M^X(z,w,\xi)/\partial w^2 < 0$; (60) when $\xi \geq \tilde{\xi}$ shows $\partial v_M^X(z,w,\xi)/\partial w < 0$ and

$$\frac{\partial^2 v_M^X(w,\xi)}{\partial \xi \partial w} = \frac{h_M(w) - \partial h_E(w,\xi)/\partial \xi}{w^{\theta+1}} < 0,$$

from (52); (58) shows $\partial^2 w/\partial \xi^2 > 0$; Lemma 4 shows $\mathrm{d}w/\mathrm{d}\xi > 0$; and (59) follows $\partial^2 v_M^X(z,w,\xi)/\partial \xi^2 = 0$.

Finally, suppose $\xi \to 1$. Then, we have $h_M^Y(w,\xi) = h_E(w,\xi)$, which means $z_M^Y(\xi) \to z_E(\xi)$. Furthermore, (10) deduces $z_E(\xi) \to \infty$. Hence, we assure

$$\lim_{\xi \to 1} v_M^X(z, z_M^Y(\xi), \xi) = 0.$$

Thus, ξ^* is unique where $\tilde{\xi} \leq \xi^* < 1$. □

6.6 Proof of Proposition 2 (Option to Accept the Exchange Offer)

Suppose that the bidding firm has proposed an exchange offer. If the target shareholders accept the exchange offer, the value of the bidding firm's equity is given by[4]

$$S_X^{(2)}(x,y) = V_M(x,y) + k_X x + \kappa k_Y y. \tag{61}$$

We define $s_X^{(2)}(z) = S_X^{(2)}(x,y)/y$. Then, Lemma 1 yields:

$$s_X^{(2)}(z) = \begin{cases} g_M(z_M)\left(\dfrac{z}{z_M}\right)^{\theta} + k_X z + \kappa k_Y, & z < z_M, \\ g_M(z) + k_X z + \kappa k_Y, & z \geq z_M. \end{cases} \tag{62}$$

From (61) and (62), the benefit function of the option to *accept* the exchange offer is given as follows.

$$G_A(x,y) = (1-\xi) S_X^{(2)}(x,y) - \kappa k_Y y = y g_A(z,\xi), \tag{63}$$

where:

$$g_A(z,\xi) = (1-\xi) s_X^{(2)}(z) - \kappa k_Y = \begin{cases} (1-\xi) g_M(z_M)\left(\dfrac{z}{z_M}\right)^{\theta} + g_E(z,\xi), & z < z_M, \\ \\ (1-\xi) g_M(z) + g_E(z,\xi), & z \geq z_M. \end{cases} \tag{64}$$

Now, parallel to the proof of Lemma 10, $v_A(z,\xi)$ satisfies (43). Moreover, the general solution is outlined by (44). Hence, the value matching condition with (64) and (7) indicates:

$$v_A(z,\xi) = \max_w g_A(w,\xi)\left(\frac{z}{w}\right)^{\theta} \tag{65}$$

$$= \max_w \begin{cases} (1-\xi) g_M(z_M)\left(\dfrac{z}{z_M}\right)^{\theta} + g_E(w,\xi)\left(\dfrac{z}{w}\right)^{\theta}, & w < z_M, \\ \\ g_M^Y(w,\xi)\left(\dfrac{z}{w}\right)^{\theta}, & w \geq z_M. \end{cases} \tag{66}$$

Hereafter, we unravel (66). First, consider $\xi \leq \tilde{\xi}$. Then, Lemma 4 hints that $z_E(\xi) \leq z_M^Y(\xi) \leq z_M$. Therefore, Lemma 2 articulates that $g_E(w,\xi)(z/w)^{\theta}$ takes its maximum at $z_E(\xi)$ against w when $w < z_M$. Notably, $(1-\xi) g_M(z_M)(z/z_M)^{\theta}$ is constant against w. Furthermore, Lemma 3 and 4 say that $g_M^Y(w,\xi)(z/w)^{\theta}$ is decreasing in w when $w \geq z_M$ if $\xi \leq \tilde{\xi}$. Thus, $w = z_E(\xi)$ gives the solution of (65) provided $\xi \leq \tilde{\xi}$.

Second, assume $\xi > \tilde{\xi}$. Then, Lemma 4 specifies that $z_E(\xi) \geq z_M^Y(\xi) > z_M$. Accordingly, Lemma 2 indicates that $g_E(w,\xi)(z/w)^{\theta}$ is increasing w.r.t w when $w < z_M$, and Lemma 3 mentions that $g_M^Y(w,\xi)(z/w)^{\theta}$ takes its maximum at $z_M^Y(\xi)$ against w when $w \geq z_M$. Consequently, $w = z_M^Y(\xi)$ gives the solution of (66) if $\xi > \tilde{\xi}$. □

6.7 Proof of Theorem 2 (Option to Propose the Exchange Offer)

We first demonstrate the equity value of the bidding firm when the exchange offer is proposed (Lemma 12). Subsequently, the benefit function of the exchange offer is introduced (Lemma 13). Eventually, we show the proof of Theorem 2.

[4]A variable $S_X^{(1)}(x,y)$ will be defined as the equity value of the bidding firm when the bidding firm proposes the exchange offer, as the option to propose the exchange offer is the "first" to be executed in the deal. Correspondingly, we write (2) on a superscript of $S_X^{(2)}(x,y)$ because the option to accept the exchange offer is the "second" one to be executed.

Lemma 12 (Equity Value of the Bidding Firm) *The equity value of the bidding firm when the exchange offer is proposed is denoted by $S_X^{(1)}(x,y)$. Assume that the bidding firm proposes the exchange offer. Then, $S_X^{(1)}(x,y) = ys_X^{(1)}(z)$ and $z = x/y$ where:*

$$s_X^{(1)}(z) = \begin{cases} \xi s_X^{(2)}(z_A(\xi)) \left(\dfrac{z}{z_A(\xi)}\right)^{\theta}, & z < z_A(\xi), \\ \xi s_X^{(2)}(z), & z \geq z_A(\xi), \end{cases}$$

and $z_A(\xi)$ is given by (29).

Proof. Consider that the target shareholders accept the offer at time $\tau < \infty$. Concurrently, the equity value of the bidding firm $S_X^{(1)}(X_\tau, Y_\tau)$ will be changed to $\xi S_X^{(2)}(X_\tau, Y_\tau)$ because:

$$\begin{aligned} S_X^{(1)}(X_\tau, Y_\tau) &= V_M(X_\tau, Y_\tau) - G_A(X_\tau, Y_\tau) + k_X X_\tau && (67) \\ &= V_M(X_\tau, Y_\tau) - \left\{(1-\xi)S_X^{(2)}(X_\tau, Y_\tau) - \kappa k_Y Y_\tau\right\} + k_X X_\tau && (68) \\ &= \xi S_X^{(2)}(X_\tau, Y_\tau). && (69) \end{aligned}$$

Note that the consistency of the assets and liabilities implies (67); inserting (63) into (67) leads to (68); and inserting (61) into (68) yields (69). Therefore, $S_X^{(1)}(x,y)$ is specified as follows: As in (42)–(44), the general solution of $s_X^{(1)}(z)$ is given by $s_X^{(1)}(z) = A_S^{(1)} z^\theta$ where $A_S^{(1)}$ is an undetermined multiplier. From the value-matching condition at $z_A(\xi)$, we acquire $A_S^{(1)} z_A(\xi)^\theta = \xi s_X^{(2)}(z_A(\xi))$ where $z_A(\xi)$ is specified by (29), thus, validating the lemma. □

Lemma 13 (Benefit Function of the Acquirer) *Denote by $G_P(x,y)$ the benefit function of the bidding firm when it proposes the exchange offer where $g_P(z,\xi) = G_P(x,y)/y$. Then,*

If $\xi \leq \tilde{\xi}$,

$$g_P(z,\xi) = \begin{cases} \xi g_M(z_M)\left(\dfrac{z}{z_M}\right)^\theta - g_E(z_E(\xi),\xi)\left(\dfrac{z}{z_E(\xi)}\right)^\theta & \\ -g_C(z, z_E(\xi)) & z \leq z_E(\xi), \\ \xi g_M(z_M)\left(\dfrac{z}{z_M}\right)^\theta - g_E(z,\xi), & z_E(\xi) < z \leq z_M, \\ g_M^X(z,\xi) & z > z_M, \end{cases} \tag{70}$$

and if $\xi > \tilde{\xi}$,

$$g_P(z,\xi) = \begin{cases} g_M^X(z_M^Y(\xi),\xi)\left(\dfrac{z}{z_M^Y(\xi)}\right)^\theta - g_C(z, z_M^Y(\xi)), & z \leq z_M^Y(\xi), \\ g_M^X(z,\xi) & z > z_M^Y(\xi), \end{cases} \tag{71}$$

where

$$g_C(z,w) = k_X\left\{z - w\left(\frac{z}{w}\right)^\theta\right\} > 0, \ \textit{for } z \leq w. \tag{72}$$

Proof. The benefit of the acquirer from the deal is given by $G_P(x,y) = S_X^{(1)}(x,y) - k_X x$. It follows that

$$g_P(z,\xi) = s_X^{(1)} - k_X z. \tag{73}$$

Therefore, the proof is given from Lemma 12 and (62) as follows. Lemma 4 and Proposition 2 suggest that if $\xi \le \tilde{\xi}$, $z_E(\xi) = z_A(\xi) \le z_M \le z_M^X(\xi)$, and if $\xi > \tilde{\xi}$, $z_M^X(\xi) < z_M < z_M^Y(\xi) = z_A(\xi) \le z_E(\xi)$. Accordingly, Lemma 12 results in:
If $\xi \le \tilde{\xi}$,

$$s_X^{(1)}(z) = \begin{cases} \xi g_M(z_M)\left(\dfrac{z}{z_M}\right)^\theta + \xi\left(k_X z_E(\xi) + \kappa k_Y\right)\left(\dfrac{z}{z_E(\xi)}\right)^\theta, & z \le z_E(\xi), \\ \xi g_M(z_M)\left(\dfrac{z}{z_M}\right)^\theta + \xi\left(k_X z_E(\xi) + \kappa k_Y\right), & z_E(\xi) < z \le z_M, \\ \xi\big(g_M(z) + k_X z + \kappa k_Y\big), & z > z_M, \end{cases}$$

and if $\xi > \tilde{\xi}$,

$$s_X^{(1)}(z) = \begin{cases} \xi\Big(g_M(z_M^Y(\xi)) + k_X z_M^Y(\xi) + \kappa k_Y\Big)\left(\dfrac{z}{z_M^Y(\xi)}\right)^\theta, & z \le z_M^Y(\xi), \\ \xi\Big(g_M(z) + k_X z + \kappa k_Y\Big), & z > z_M^Y(\xi). \end{cases} \tag{74}$$

Since $\xi(k_X z + \kappa k_Y) - k_X z = -g_E(z, \xi)$, we can express (73) by (70) and (71). Notably, we use the equivalence

$$\xi(k_X w + \kappa k_Y)\,(z/w)^\theta - k_X z = -g_E(w, \xi)\,(z/w)^\theta - g_C(z, w),$$

where $g_C(z, w)$ is signified by (72). Finally, by letting $f(z, t) = \mathrm{e}^{-rt} z$, (1) and Dynkin's formula lead to

$$g_C(z, w) = k_X\Big\{r - \left(\mu_X - \mu_Y - \sigma_Y^2\right)\Big\}E^Q\left[\int_0^\tau \mathrm{e}^{-rt} Z_t \mathrm{d}t\right] > 0, \tag{75}$$

where τ is the first hitting time when Z_t reaches $w > z$ where $Z_0 = z$. □

Proof of theorem. Analogous to the proof of Lemma 10, $V_P(x, y)$ satisfies (42) and $v_P(z, \xi)$ satisfies (43), respectively. Furthermore, the general solution is given by (44). Therefore, the value matching condition indicates $v_P(z, \xi) = \max_w g_P(w, \xi)\,(z/w)^\theta$. Consequently, Lemma 13 implies that:
If $\xi \le \tilde{\xi}$,

$$v_P(z, \xi) = \max_w \begin{cases} \xi g_M(z_M)\left(\dfrac{z}{z_M}\right)^\theta - g_E(w, \xi)\left(\dfrac{z}{z_E(\xi)}\right)^\theta - g_C(w, z_E(\xi))\left(\dfrac{z}{w}\right)^\theta, & w \le z_E(\xi) \\ \xi g_M(z_M)\left(\dfrac{z}{z_M}\right)^\theta - g_E(w, \xi)\left(\dfrac{z}{w}\right)^\theta, & z_E(\xi) < w \le z_M, \\ g_M^X(w, \xi)\left(\dfrac{z}{w}\right)^\theta, & w > z_M, \end{cases} \tag{76}$$

and if $\xi > \tilde{\xi}$,

$$v_P(z, \xi) = \max_w \begin{cases} g_M^X(z_M^Y(\xi), \xi)\left(\dfrac{z}{z_M^Y(\xi)}\right)^\theta - g_C(w, z_M^Y(\xi))\left(\dfrac{z}{w}\right)^\theta & w \le z_M^Y(\xi), \\ g_M^X(w, \xi)\left(\dfrac{z}{w}\right)^\theta, & w > z_M^Y(\xi). \end{cases} \tag{77}$$

Suppose $\xi \le \tilde{\xi}$. We assert that the objective function in (76) is increasing in w when $w \le z_E(\xi)$ and $z_E(\xi) < w \le z_M$ because it is easy to show that $g_C(w, z_E(\xi))(z/w)^\theta$ is decreasing in w, and Lemma 2 implies that $g_E(w,\xi)(z/w)^\theta$ takes maximum when $w = z_E(\xi)$. Furthermore, Lemma 3 says that $\arg\max_w g_M^X(w,\xi)(z/w)^\theta = z_M^X(\xi)$, which satisfies $z_M^X(\xi) \ge z_M$ when $\xi \le \tilde{\xi}$. Thus, $w = z_M^X(\xi)$ resolves (76).

Suppose $\xi > \tilde{\xi}$. Then, we can show that the objective function in (77) is increasing in w when $z < z_M^Y(\xi)$ and decreasing in w when $z \le z_M^Y(\xi)$, which follows that $w = z_M^Y(\xi)$ gives the solution of (77). Because, $g_C(w, z_M^Y(\xi))\,(z/w)^\theta$ is decreasing in w, and Lemma 3 implies that $g_M^X(w,\xi)\,(z/w)^\theta$ takes its maximum at $z_M^X(\xi)$ where $z_M^X(\xi) < z_M^Y(\xi)$ holds when $\xi > \tilde{\xi}$. Accordingly, (30) and (31) are verified. Furthermore, Lemma 4 and (31) implies $z_P(\xi) \le z_M$. □

Remark 1 *From (75), we understand that $g_C(z,w)$ means the continuously paid cost until Z_t reaches w from $Z_t = z$. Consequently, $g_C(w, z_E(\xi))\,(z/w)^\theta$ in (76) suggest the present value of the additional cost for writing the exchange option, which is in the money.*

6.8 Proof of Lemma 5(Acquisition Premium of the Target)

As shown by Corollary 2, the share-exchange option is expired when it is provided. Further, the back-end merger completes immediately. Hence, the acquisition premium is the present value of the benefit from the merger while the timing is given by $z_P(\xi)$. Thus, the value of the acquisition premium can be derived in the same way to show Lemma 12. □

6.9 Proof of Corollary 4

The proof is shown when z is sufficiently low. Otherwise, the proof is straightforward. Now, we assert

$$\begin{aligned}\frac{\mathrm{d}v_M^X(z,z_M,\tilde{\xi})}{\mathrm{d}\kappa} &= \frac{\partial v_M^X(z,z_M,\tilde{\xi})}{\partial \tilde{\xi}}\frac{\partial \tilde{\xi}}{\partial \kappa} + \frac{\partial v_M^X(z,z_M,\tilde{\xi})}{\partial \kappa} \\ &= \left(g_M(z_M)\frac{\partial \tilde{\xi}}{\partial \kappa} - \frac{\partial g_E(z_M,\tilde{\xi})}{\partial \tilde{\xi}}\frac{\partial \tilde{\xi}}{\partial \kappa} - \frac{\partial g_E(z_M,\tilde{\xi})}{\partial \kappa}\right)\left(\frac{z}{z_M}\right)^\theta < 0, \qquad (78)\end{aligned}$$

regarding $\tilde{\xi}$ is a function of κ. It can be shown that (16) leads to

$$\frac{\partial \tilde{\xi}}{\partial \kappa} = -\frac{\tilde{\xi}\theta k_Y}{(\theta-1)k_X z_M + \theta\kappa k_Y} < 0, \qquad (79)$$

, and this implies

$$\frac{\mathrm{d}g_E(z_M,\tilde{\xi})}{\mathrm{d}\kappa} = \frac{\partial g_E(z_M,\tilde{\xi})}{\partial \tilde{\xi}}\frac{\partial \tilde{\xi}}{\partial \kappa} + \frac{\partial g_E(z_M,\tilde{\xi})}{\partial \kappa} = \frac{k_Y\tilde{\xi}^2}{\theta-1} > 0.$$

Similarly, we can state $\mathrm{d}v_M^Y(z,z_M,\tilde{\xi})/\mathrm{d}\kappa > 0$, which concludes (34) from (33) with (78), and (79). □

6.10 Proof of Lemma 8 (A Leader-Follower Problem in the Takeover)

Theorem 2 provides $v_P(z,\xi) = v_M^X(z, z_M^X(\xi), \xi)$ when $\xi \le \tilde{\xi}$. Hence, $v_P(z,\xi)$ is increasing in ξ when $\xi \le \tilde{\xi}$ from (50). Moreover, Theorem 2 implies $v_P(z,\xi) = v_M^X(z, z_M^Y(\xi), \xi)$ when $\xi > \tilde{\xi}$. This is the value function that appears in the leader-follower problem in the merger. Thus, ξ^* solves (35). □

6.11 Proof of Lemma 9

1) Targets value is monotonically decreasing in ξ. Because, Proposition 2 implies that if $\xi \le \tilde{\xi}$,

$$\frac{\mathrm{d}v_A(z,\xi)}{\mathrm{d}\xi} = \begin{cases} -g_M(z_M)\left(\dfrac{z}{z_M}\right)^{\theta} + \dfrac{\partial v_E(z,\xi)}{\partial \xi}, & z < z_E(\xi), \\[2ex] -g_M(z_M)\left(\dfrac{z}{z_M}\right)^{\theta} + \dfrac{\partial g_E(z,\xi)}{\partial \xi}, & z_E(\xi) \le z < z_M, \\[2ex] -g_M(z_M) + \dfrac{\partial g_E(z,\xi)}{\partial \xi}, & z \ge z_M, \end{cases}$$

and if $\xi > \tilde{\xi}$, $\mathrm{d}v_A(z,\xi)/\mathrm{d}\xi = \mathrm{d}v_M^Y(z, z_M^Y`(\xi), \xi)/\mathrm{d}\xi$, where we can show $\partial v_E(z,\xi)/\partial\xi < 0$ and $\partial g_E(z,\xi)/\partial\xi < 0$ from Lemma 2; and $\mathrm{d}v_M^Y(z, z_A(\xi), \xi)/\mathrm{d}\xi < 0$ from Lemma 3. Hence, Assumption 4 leads to 1). See Figure 4.

2) Recall that the uniqueness of ξ_w^* is given in Lemma 8. Further, the concavity of the bidder's value is shown in the proof of Lemma 8. Furthermore, we have shown that $v_A(z,\xi)$ is monotonically decreasing in ξ. Therefore, the definition of V^{**} implies the result directly. See also Figure 4.

3) If $z_E(\xi_\ell^*) > z_M^{Y,w}(\xi_w^*)$, we have $\xi_\ell^* > \tilde{\xi}_w$ from Lemma 4 (See Figure 3). As stated above, $\xi_\ell^* < \xi_w^{**}$ holds. Hence, we can assure that $V^{**} < v_A^w(z, \tilde{\xi}_w)$ since $v_A^j(z,\xi)$ is decreasing in ξ (See also Figure 4). It follows that the loser will never be strong when the winner proposes first. □

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