On the Degrees of Ignorance: via Epistemic Logic and μ-Calculus

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Epistemic Logic normally discourses on knowledge, belief, and related concepts. We here study ignorance instead. With the help of the μ-calculus, we analyze the degrees of ignorance in which an agent doesn't know whether or not a given proposition is true. Building on the study by Stalnaker [14], we argue that logics "closer" to S4.2 allow greater degrees of ignorance, compared to logics "closer" to S5.

We consider Epistemic Logic with the modal operators K (for knowledge) and B (for belief). We will focus on the case where there is only one agent, following Hintikka [7], Lenzen [8], and Stalnaker [14]. We suppose that K satisfies (at least) S4 and B satisfies KD45. By belief, we mean strong belief, and suppose that the agent's beliefs has no contradiction and they have introspection about their own beliefs:

$$\begin{aligned}&K\varphi \to B\varphi,\\ &B\varphi \to KB\varphi, \text{ and}\\ &\neg B\varphi \to K\neg B\varphi.\end{aligned}$$

Lenzen [8] showed that the interaction axioms above imply that $B\varphi \leftrightarrow \hat{K}K\varphi$, that is, belief can be defined by knowledge. In practice, we consider B as a defined modality. Lenzen's proof also implies that K satisfies S4.2.

The concept of ignorance was also studied by van der Hoek and Lomuscio [6]. They defined a modal operator for ignorance by

$$I\varphi :\leftrightarrow \neg K\varphi \wedge \neg K\neg\varphi.$$

They also define a logic for ignorance Ig and prove that it is sound and complete over the class of all frames. Fine [5] studied the n^{th}-order ignorance $I^n\varphi$, where $I^{n+1}\varphi :\leftrightarrow I(I^n\varphi)$ and $I^0\varphi :\leftrightarrow \varphi$. In particular, it is shown that $I^2\varphi$ is equivalent to the so-called Rumsfeld ignorance "the unknown unknown", $I\varphi \wedge \neg KI\varphi$. Fine also showed that, for any φ, $\neg KI^2\varphi$ is valid on any frame of S4.2. Therefore, the knowledge of second-order ignorance is unobtainable. Note that the ignorance modality I is a particular case of the contingency modality ∇ (see Montgomery and Routley [10]). ∇ is defined by

$$\nabla\varphi :\leftrightarrow \Diamond\varphi \wedge \Diamond\neg\varphi.$$

We build on Stalnaker's [14] analysis and consider the logics S4.2, S4.3, S4.3.2, S4.4, and S5 for knowledge. They are defined using the axioms in Table 1 with the necessitation rule. Aucher [2] characterized these logics by axioms relating knowledge and belief/conditional belief similar to Lenzen's characterization of S4.2.

Axiom Name	Axiom	Frame conditions
K	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$	(no condition)
D	$K\varphi \to \hat{K}\varphi$	Serial
T	$K\varphi \to \varphi$	Reflexive
4	$K\varphi \to KK\varphi$	Transitive
5	$\hat{K}\varphi \to K\hat{K}\varphi$	Euclidean
.2	$\hat{K}K\varphi \to K\hat{K}\varphi$	Convergent
.3	$K(K\varphi \to \psi) \vee K(K\psi \to \varphi)$	Weakly Connected
.3.2	$(\hat{K}\varphi \wedge \hat{K}K\psi) \to K(\hat{K}\varphi \vee \psi)$	Semi-Euclidean
.4	$(\varphi \wedge \hat{K}K\varphi) \to K\varphi$	(no particular name)

Table 1: Modal axioms for K.

The (modal) μ-calculus is obtained by adding to modal logic the fixed-point operators μ and ν, for least and greatest fixed-points. The μ-formulas are generated by the grammar

$$\varphi := P \mid \neg P \mid X \mid \varphi \wedge \varphi \mid K\varphi \mid \mu X.\varphi \mid \nu X.\varphi.$$

We denote the dual operators of K and B by $\hat{K}$ and $\hat{B}$. For reasons that will become clear later, we consider only alternation-free formulas, that is, formulas with no nested alternation of μ and ν operators. More rigorously, a μ-formula is alternation-free if it has no subformula of the form $\mu X.\varphi$ (or $\nu X.\varphi$) such that φ has a subformula $\nu Y.\psi$ (or $\mu Y.\psi$) with a free occurrence of X in ψ.

The relational semantics for the μ-calculus is defined as follows. Given a model M and a μ-formula φ, we will define $\|\varphi\|^M$ to be the set of worlds w where φ holds. Propositional operators and modal operators are treated as usual. For fixed-point operators, letting $\Gamma_\varphi(X) = \|\varphi(X)\|^M$, we have

$$\|\mu X.\varphi(X)\|^M \text{ is the least fixed point of } \Gamma_\varphi, \text{ and}$$
$$\|\nu X.\varphi(X)\|^M \text{ is the greatest fixed point of } \Gamma_\varphi.$$

For an example of the use of fixed-point operators, suppose we have modality E for "everyone knows". Then a formula φ is common knowledge iff the following formula holds:

$$\nu X.(\varphi \wedge EX).$$

That is, φ is common knowledge iff it is true, everybody knows that φ is true, everybody knows that "everybody knows that φ is true", and so on.

The operators μ and ν induce a (syntactical) hierarchy of the μ-formulas, measuring the entanglement of least and greatest fixed-point operators. Bradfield [4] showed that, in general, the hierarchy is strict: for all natural number

n there is a formula with alternation depth $n+1$ which is not equivalent to any formula of alteration depth n. But this strictness may fail in a restricted class of models. In fact, Alberucci and Facchini [1] showed that on a frame satisfying $\mathsf{S4}$, the hierarchy collapses to its alternation-free fragment: every μ-formula is equivalent to an alternation-free μ-formula. This justifies our restriction to alternation-free formulas in the definition of our μ-calculus. They also showed that the hierarchy collapses to modal logic on frames of $\mathsf{S5}$: every μ-formula is equivalent to a modal formula. Also note that we can define a *weak* alternation hierarchy on the alternation-free fragment. The authors have shown the strictness of the weak alternation hierarchy on recursive frames [11].

In [12], the authors show that the alternation hierarchy collapses to its alternation-free fragment over frames of

$$\mathsf{S4.2} \text{ and } \mathsf{S4.3};$$

and collapses to modal logic over frames of

$$\mathsf{S4.3.2}, \mathsf{S4.4} \text{ and } \mathsf{KD45}.$$

Therefore there must be an (alternation-free) formula φ which is not equivalent to any modal formula over $\mathsf{S4.2}$ and $\mathsf{S4.3}$; but *is* equivalent to a modal formula over $\mathsf{S4.3.2}$, $\mathsf{S4.4}$ and $\mathsf{KD45}$. While this abstract uses only relational semantics, the collapses of the alternation hierarchy can be transferred to topological semantics by a result of Baltag *et al.* [3].

We analyze a formula which is not equivalent to any modal formula over $\mathsf{S4.2}$ and $\mathsf{S4.3}$. Let φ be any μ-formula, and define

$$\alpha_\varphi(X) := \hat{K}(\varphi \wedge X) \wedge \hat{K}(\neg\varphi \wedge X).$$

We study $\alpha_\varphi^\infty := \nu X.\alpha_\varphi$ and its approximants $\alpha_\varphi^1 := \alpha_\varphi(T)$, $\alpha_\varphi^{n+1} := \alpha_\varphi(\alpha_\varphi^n)$; they will be used to measure the agent's degree of ignorance with respect to φ. Each α_φ^i will represent a degree of ignorance. Over $\mathsf{S4.2}$, any degree implies the weaker degrees but the converse may not hold. That is, if $i, j \in \mathbb{N} \cup \{\infty\}$ and $i < j$, then α_φ^j implies α_φ^i; and the converse doesn't hold as $\alpha_\varphi^i \wedge \neg\alpha_\varphi^j$ is satisfiable. Therefore we have, in general, infinitely many degrees of ignorance. Our first degree of ignorance α_φ^1 is equivalent to $I\varphi$, and all the α_φ^i can be thought of as generalizations of $I\varphi$.

Van der Hoek and Lomuscio [6] state that the ignorance modality I is not intended to capture degrees of ignorance, while our α_φ^i's are intended to do so. Furthermore, $K\alpha_\varphi^i$ is satisfiable for any $i \in \mathbb{N} \cup \{\infty\}$. Therefore the α_φ^i are different from second-order ignorance $I^2\varphi$, and not obtainable by iterations of I.

If we change our settings, we may have finitely many degrees of ignorance. Consider $\mathsf{S4.4}$, the logic of knowledge as true belief. We can show here that $\alpha_\varphi^1 \wedge \neg\alpha_\varphi^2$ is equivalent to the agent having a false belief and that α_φ^2 is equivalent to α_φ^∞. That is, we have only two non-equivalent degrees of ignorance: α_φ^1, where the agent's belief is false; and α_φ^2, where the agent believes neither φ nor $\neg\varphi$.

The same analysis holds for S4.3.2, which has the same two degrees. This logic is used, for example, in [13].

In S5, the standard logic for multi-agent epistemic logic, we have only one degree of ignorance. In this setting, α^1_φ is equivalent to α^∞_φ. We also have that belief is equivalent to knowledge, $B\varphi \leftrightarrow K\varphi$, so the agent has no wrong beliefs, and being ignorant of φ also means that they believe neither φ nor $\neg\varphi$.

Now consider the interpretation of $\alpha^i_\varphi \wedge \neg\alpha^{i+1}_\varphi$ over S4.2 and S4.3. S4.2 is the logic of knowledge according to Lenzen [8] and Stalnaker [14]. S4.3 is Lehrer and Paxon's undefeated justified true belief [9]. Here, $\alpha^1_\varphi \wedge \neg\alpha^2_\varphi$ is equivalent to the agent's belief being false and the agent knowing whether φ holds in every world other than the real world. Likewise, $\alpha^2_\varphi \wedge \neg\alpha^3_\varphi$ holds exactly when the agent has a true belief but considers it possible that their belief is false. Symbolically,

$$\alpha^2_\varphi \wedge \neg\alpha^3_\varphi \equiv [\varphi \wedge B\varphi \wedge \hat{K}(\neg\varphi \wedge B\varphi)] \vee [\neg\varphi \wedge B\neg\varphi \wedge \hat{K}(\varphi \wedge B\neg\varphi)].$$

This analysis can be extended to other $\alpha^i_\varphi \wedge \neg\alpha^{i+1}_\varphi$ to show that each degree of ignorance expresses a higher level of the agent's self-doubt.

Still in S4.2 and S4.3, note that, for $n \in \mathbb{N}$, α^n_φ implies the agent has a belief (which may be true or false) and the agent not having a belief implies α^∞_φ. In other words, having no belief implies a high degree of ignorance, but a high degree of ignorance does not deny the agent having a belief.

From the point of view of our degrees of ignorance, S4.3 and S4.3.2 are very different: S4.3 has infinitely many degrees of ignorance; while S4.3.2 has only two degrees. This contrasts with Stalnaker's critics of S4.3 and S4.3.2, which argues that in both logics false belief can deny knowledge: in S4.3 a false belief can deny some knowledge the agent may be justified in having; and in S4.3.2 a false belief denies all non-trivial knowledge.

At last, we can do a similar analysis to belief, using

$$\delta_\varphi(X) := \hat{B}(\varphi \wedge X) \wedge \hat{B}(\neg\varphi \wedge X),$$

a belief variant of $\alpha_\varphi(X)$. Define $\delta^1_\varphi := \delta_\varphi(\top)$, $\delta^{n+1}_\varphi := \delta_\varphi(\delta^n_\varphi)$ and $\delta^\infty = \nu X.\delta(X)$. Then, for $i \in \mathbb{N} \cup \{\infty\}$, δ^1_φ is equivalent to δ^i_φ over KD45, similar to the case where knowledge satisfies S5. Therefore we can only define one degree of disbelief by our approach.

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