# Simple Model and the Deduction System for Dynamic Epistemic Quantum Logic

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*Quantum logic* (QL) has been studied to handle strange propositions of quantum physics. In particular, logic based on *orthomodular lattices*, namely, *orthomodular logic* (OML), has been studied since 1936, proposed by Birkhoff and Von Neumann [10]. An orthomodular lattice is related to the closed subspaces of a Hilbert space, which is a state space of a particle in quantum physics. Instead of these lattices, the *Kripke model* (possible world model) of OML can be used, which is called the *orthomodular-model* (OM-model) [11] [12]. Intuitively, each possible world of an OM-model expresses a one-dimensional subspace of a Hilbert space, corresponding to a *quantum state*.

In quantum mechanics, due to the *uncertainty principle*, exact values cannot be simultaneously obtained for a specific set of physical quantities (for example, momentum and position along an axis). This statistical property is the nature of the states of the object and exist independently of an experimenter's knowledge. OML handles the most basic part of this strange nature of states.

To treat an agent's knowledge in quantum mechanics, some studies combine *epistemic logic* (EL) with QL. EL is a field of modal logic that treats the proposition of an agent's knowledge. In the Kripke model of EL, the *indistinguishability of states* is used to express knowledges. That is, if a formula $\phi$ is true at all states that are indistinguishable from the current state for agent $i$, then agent $i$ knows that $\phi$ is true. Furthermore, *dynamic EL* (DEL) has been studied to handle the transitions of knowledge [15]. In general, *public announcement logic* (PAL) is treated as the most basic and simple logic in DEL. Basic PAL includes only two types of modal symbols: the symbols for knowledge $\mathsf{K}_i$ of individual agents and the symbol [ ] for public announcements. $[\phi]\psi$ can be read as " After a public announcement $\phi$, $\psi$ is true. "

Ref [8] and [9] can be cited as one of the studies of logic that deal with the concept of knowledge with quantum physics. In these studies, the models which incorporate specific *quantum information* concepts were used. Ref [2] and [3] can be cited as the studies of knowledge with more general concepts of quantum physics. In these studies, similar to EL, knowledge was expressed using the indistinguishability of states.

To discuss the general *change* of knowledges due to the procurement of informations, other concepts have to be introduced and the field of *dynamic epistemic QL* (DEQL) has to be developed. In [4], *quantum test frame* is introduced as a part of the study of the *dynamic logic of test* (DLT). DLT is a logic for dealing with general changes in knowledge due to information obtained by testing. Quantum test frame is based on the frame for DLT and the frame for *dynamic QL* (DQL) [5] [6] [7]. DQL uses modal symbols for several types of transitions of quantum states, such as *unitary evolutions* and *projections*. An important aspect of quantum physics is the change of state due to measurement. In quantum physics, when a physical quantity is observed, the state is projected to an eigenstate of the physical quantity. That is, the state of the particle itself changes depending on the obtained information. In (classical) EL, if $x(\phi)y$, then $x = y$; where $x$ and $y$ denote states and $(\phi)$ is the relation for information $\phi$. Reflecting the nature of quantum physics, in quantum test frame, this property is not true [5] [6] [7].

As mentioned above, the transition of knowledge in quantum mechanics has been analyzed in some directions. However, some problems remain.

1. These models in previous studies are little complicated because these models introduce almost every modal element related to quantum mechanics. Such a model is also needed, but a somewhat simple and abstracted model that leaves only the important notions is also useful to analyze specific feature of knowledge in quantum mechanics.

2. As the models and symbols are complicated, constructing a *deduction system* for this types of logic is somewhat complicated task because we have to deal with the mutual consistency of many conditions. Actually, deduction systems for DEQL have not been analyzed much.

Therefore, in this study, as a basis for solving these problems, we construct new logic and models for the transition of knowledge in quantum mechanics that is simpler than previous studies, while retaining the essence of these studies. Furthermore, we construct a deduction system that holds soundness and completeness for those new models. Because of these purposes, herein, we mainly focus on mathematical and logical analysis, rather than quantum mechanical analysis.

We construct *dynamic epistemic orthomodular logic* (DEOML) by combining the frames and models of OML and PAL, and we simply use a combination of logical symbols for OML and PAL. The meaning of $[\phi]$ in DEOML is different from that in PAL. In DEOML( and in quantum test frames), $[\phi]$ denotes the action that the agent obtains the information $\phi$ by observing a state of the particle. However, they are the same in terms of "obtaining the information that $\phi$ is true." Therefore, in fact, the logical nature for this symbol are almost the same in each logic. Moreover, due to the simplicity of DEOML, this similarity is used to prove useful theorem (which is described in last paragraph) similar to PAL, which is difficult to established in the models in previous studies.

OML is adopted instead of DQL for the foundation of logic because of the following advantages.

1. Although OML is not a modal logic, OM-models *implicitly* include the concept of the modality of projection as binary relations that satisfies some important conditions [17]. Therefore, OML can handle the concept of projection while being a simpler model than DQL, which include the notion of of projections explicitly.

2. OML does not include the other dynamic concepts of quantum mechanics, such as unitary evolutions. However, the most important strange properties of the agent's knowledge that appear in quantum mechanics are related to projective observations. Therefore, the important properties can be analyzed as long as the concept of projection is included in the logic.

3. Different from DQL, deduction systems for OML are well argued in previous studies [13] [14] [16] [18] [19], and we can use them directly to construct a deduction system for DEOML.

We construct a *sequent calculus* type deduction system for DEOML and prove the soundness and completeness theorem with respect to new models.

Sequent calculus is suitable for this study because it is compatible with OML and modal logics [13] [18] [19]. Hilbert-style systems for OML have also been studied [14] [16]. However, they contain unique symbols for creating the Hilbert-style system, which are not suitable for combination with other (modal) symbols.

In this new logic, two types of formulae are used: a *quantum formula* (q-formula), and a *general formula* (g-formula).

q-formula $A ::= p \mid \bot \mid\sim A \mid A {\wedge} A$

g-formula $\phi ::= A \mid \neg\phi \mid \phi{\wedge}\phi \mid \mathsf{K}\phi \mid [A]\phi$

The q-formulae are included in g-formulae. The q-formulae are correspond to the propositions in OML. That is, q-formulae are used to express the propositions of quantum mechanics. g-formulae are used to express modal notions including knowledge and change of informations. We use the definition that only q-formulae can be placed in the modal symbol [ ] because we deal with the situation where the agent gets information about the particles in an experiment. By using this condition, the same concept of projections [ ] defined in advanced OM-model [17] can be used.

In this study, similar to [1] [4], we focus on the situations where only one agent is present. The main reason for this restriction is that models for QL which are currently configured are not very suitable for dealing with *product* Hilbert spaces, which represent state spaces of multiple particles and agents. Therefore, a study of logic that includes more than one agent or more than one particle in binary relational model is somehow different from this study.

It is shown that even with these restricted definition, important parts of knowledge in quantum mechanics still can be expressed. For example, $\mathsf{K}p \to [A]\mathsf{K}p$ is valid in PAL but not always valid in models of DEOML. Intuitively, this is because an announcement may change an agent's knowledge but not change the environment in PAL. In contrast, as mentioned earlier, in quantum mechanics, when we obtain information from the environment (particles), the state of the environment may change because of projections.

The main contributions of this study are as follows.

A novel model for DEQL that can analyze the transitions of knowledge is constructed, and it is simpler than the models in previous studies. The method of configuration of the model is also completely different from the previous studies.

Using the new model, we construct a new logic DEOML.

Some similarity and differences between PAL and DEOML from the mathematical logic perspective are analyzed. That is, following formulae are valid in DEOML.

$[A]B \leftrightarrow \sim A \sqcup (A \wedge B))$

$[A](\phi \wedge \psi) \leftrightarrow ([A]\phi \wedge [A]\psi)$

$[A]\neg\phi \leftrightarrow (\neg \sim A \rightarrow \neg[A]\phi)$

$[A]\mathsf{K}\phi \leftrightarrow (\neg \sim A \rightarrow \mathsf{K}[A]\phi)$

Deduction system for DEOML, which is sound and complete with respect to these new models are established. This results of deduction system for DEQL is completely new.

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