Craig Interpolation for a Sequent Calculus for Combining Intuitionistic and Classical Propositional Logic

Masanobu Toyooka[1] and Katsuhiko Sano[2]

[1] Graduate School of Letters, Hokkaido University, Sapporo, Japan
toyooka.masanobu.t1@elms.hokudai.ac.jp
[2] Faculty of Humanities and Human Sciences, Hokkaido University, Sapporo, Japan
v-sano@let.hokudai.ac.jp

1 Introduction

This paper establishes the Craig interpolation for a multi-succedent sequent calculus for a combination of intuitionistic and classical propositional logic, denoted by $\mathsf{G}(\mathbf{C}+\mathbf{J})$. The calculus was provided in [16] and is based on the semantics offered in [4, 5]. The logic, called $\mathbf{C}+\mathbf{J}$, has two implications: intuitionistic and classical one[1]. They are interpreted in the Kripke semantics as follows (cf. [4, 5]):

$$\begin{array}{lll} w \models_M A \to_{\mathtt{i}} B & \text{iff} & \text{for all } v \in W, (wRv \text{ and } v \models_M A \text{ jointly imply } v \models_M B), \\ w \models_M A \to_{\mathtt{c}} B & \text{iff} & w \models_M A \text{ implies } w \models_M B, \end{array}$$

where M is an intuitionistic Kripke model, w is a possible world in M, and R is a preorder equipped in M. However this semantic treatment breaks one feature of intuitionistic logic called *heredity*, which is defined as: $w \models A$ and wRv jointly imply $v \models A$ for all Kripke models M and all states w and v in M. It is a well-known fact that this feature corresponds to an intuitionistically valid formula $A \to_{\mathtt{i}} (B \to_{\mathtt{i}} A)$. Therefore, the formula is not valid in the Kripke semantics of $\mathbf{C}+\mathbf{J}$. In order to avoid the formula being derivable in $\mathsf{G}(\mathbf{C}+\mathbf{J})$, the right rule for the intuitionistic implication should be restricted as follows:

$$\frac{A, C_1 \to_{\mathtt{i}} D_1, \ldots, C_m \to_{\mathtt{i}} D_m, p_1, \ldots, p_n \Rightarrow B}{C_1 \to_{\mathtt{i}} D_1, \ldots, C_m \to_{\mathtt{i}} D_m, p_1, \ldots, p_n \Rightarrow A \to_{\mathtt{i}} B} \; (\Rightarrow \to_{\mathtt{i}}).$$

The resulting calculus is sound and complete and a conservative extension of both an intuitionistic and a classical propositional sequent calculus (see [16]).

It is well-known that classical propositional logic and intuitionistic propositional logic enjoy the Craig interpolation theorem:

> If $A \to B$ is derivable, then there exists a formula C such that both $\Rightarrow A \to C$ and $\Rightarrow C \to B$ are also derivable and that $\mathsf{Prop}(C) \subseteq \mathsf{Prop}(A) \cap \mathsf{Prop}(B)$,

where $\mathsf{Prop}(D)$ denotes the set of all propositional variables in a formula D. The theorem can be shown in terms of a classical sequent calculus $\mathbf{LK}$ by Maehara's method in [9]. In multi-succedent intuitionistic sequent calculus $\mathbf{mLJ}$, the theorem can also be shown, though some modification of the ways is needed, as is noted in [10]. Since $\mathbf{C}+\mathbf{J}$ contains the two kinds of implication, the two types of Craig interpolation theorem can be considered in $\mathsf{G}(\mathbf{C}+\mathbf{J})$.

[1] In addition to $\mathbf{C}+\mathbf{J}$, other attempts to combine intuitionistic and classical logic are displayed in [1, 2, 3, 6, 7, 11, 12, 13, 14].

2 Syntax, Kripke Semantics and Sequent Calculus

2.1 Syntax and Kripke Semantics

This section reviews the syntax and the Kripke semantics of $\mathbf{C}+\mathbf{J}$. The syntax is defined in [16], and the Kripke semantics is based on the ones in [4, 5]. The syntax $\mathcal{L}$ consists of a countably infinite set Prop of propositional variables and the following logical connectives: falsum $\bot$, disjunction $\vee$, conjunction $\wedge$, intuitionistic implication $\rightarrow_{\mathtt{i}}$, and classical implication $\rightarrow_{\mathtt{c}}$. The set Form of all formulas in our syntax is defined inductively as follows:

$$A ::= p \mid \ \bot \ \mid A \vee A \mid A \wedge A \mid A \rightarrow_{\mathtt{i}} A \mid A \rightarrow_{\mathtt{c}} A,$$

where $p \in \mathsf{Prop}$. We define $\top := \bot \rightarrow_{\mathtt{i}} \bot$, $\neg_{\mathtt{c}} A := A \rightarrow_{\mathtt{c}} \bot$ and $\neg_{\mathtt{i}} A := A \rightarrow_{\mathtt{i}} \bot$.

Let us move to the semantics for the syntax $\mathcal{L}$.

Definition 1. A *model* is a tuple $M = (W, R, V)$ where

- W is a non-empty set of possible worlds,
- R is a preorder on W, i.e., R satisfies reflexivity and transitivity,
- $V : \mathsf{Prop} \rightarrow \mathcal{P}(W)$ is a valuation function satisfying the following *heredity* condition: $w \in V(p)$ and wRv jointly imply $v \in V(p)$ for all worlds $w, v \in W$.

Definition 2. Given a model $M = (W, R, V)$, a world $w \in W$ and a formula A, the *satisfaction relation* $w \models_M A$ is inductively defined as follows:

$$\begin{array}{lll}
w \models_M p & \text{iff} & w \in V(p), \\
w \not\models_M \bot, & & \\
w \models_M A \wedge B & \text{iff} & w \models_M A \text{ and } w \models_M B, \\
w \models_M A \vee B & \text{iff} & w \models_M A \text{ or } w \models_M B, \\
w \models_M A \rightarrow_{\mathtt{i}} B & \text{iff} & \text{for all } v \in W, (wRv \text{ and } v \models_M A \text{ jointly imply } v \models_M B). \\
w \models_M A \rightarrow_{\mathtt{c}} B & \text{iff} & w \models_M A \text{ implies } w \models_M B.
\end{array}$$

Let us say that a formula A is a *semantic consequence* of a set of formulas Γ, represented as $\Gamma \models A$, if $w \models_M C$ for any formula $C \in \Gamma$, then $w \models_M A$ for all models $M = (W, R, V)$ and all worlds $w \in W$. We use $\Gamma \models \Delta$ if $\Gamma \models A$ for some formula $A \in \Delta$. We say that A is *valid* if $\emptyset \models A$ holds. We say a formula A satisfies *heredity* if the following holds: $w \models A$ and wRv jointly imply $v \models A$ for all Kripke models M and all states w and v in M.

Proposition 1. A formula $\neg_{\mathtt{c}} p$ does not satisfy heredity.

Proposition 2. Neither $\neg_{\mathtt{c}} p \rightarrow_{\mathtt{i}} (\top \rightarrow_{\mathtt{i}} \neg_{\mathtt{c}} p)$ nor $\neg_{\mathtt{c}} p \rightarrow_{\mathtt{c}} (\top \rightarrow_{\mathtt{i}} \neg_{\mathtt{c}} p)$ is valid.

Proposition 2 implies that an intuitionistic tautology $A \rightarrow_{\mathtt{i}} (B \rightarrow_{\mathtt{i}} A)$, which is known for the correspondence to heredity in intuitionistic logic, is no longer valid.

2.2 Multi-succedent sequent calculus $\mathsf{G}(\mathbf{C}+\mathbf{J})$

This section reviews the sequent calculus $\mathsf{G}(\mathbf{C}+\mathbf{J})$ provided in [16]. In what follows, we use the ordinary notion of multi-succedent sequent. A *sequent* is a pair of finite multisets denoted by $\Gamma \Rightarrow \Delta$, which is read as "if all formulas in Γ hold then some formulas in Δ hold." Table 1 provides our multi-succedent sequent calculus $\mathsf{G}(\mathbf{C}+\mathbf{J})$, where the notion of derivability is defined as an existence of a

Table 1: Sequent Calculus $\mathsf{G}(\mathbf{C}+\mathbf{J})$

Axioms

$$\frac{}{A \Rightarrow A}\ (Id) \quad \frac{}{\bot \Rightarrow}\ (\bot)$$

Structural Rules

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ (\Rightarrow w) \quad \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ (w \Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ (\Rightarrow c) \quad \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ (c \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ (Cut)$$

Propositional Logical Rules

$$\frac{A, C_1 \to_{\mathtt{i}} D_1, \ldots, C_m \to_{\mathtt{i}} D_m, p_1, \ldots, p_n \Rightarrow B}{C_1 \to_{\mathtt{i}} D_1, \ldots, C_m \to_{\mathtt{i}} D_m, p_1, \ldots, p_n \Rightarrow A \to_{\mathtt{i}} B}\ (\Rightarrow \to_{\mathtt{i}}) \quad \frac{\Gamma_1 \Rightarrow \Delta_1, A \quad B, \Gamma_2 \Rightarrow \Delta_2}{A \to_i B, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (\to_i \Rightarrow)$$

$$\frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to_{\mathtt{c}} B}\ (\Rightarrow \to_{\mathtt{c}}) \quad \frac{\Gamma_1 \Rightarrow \Delta_1, A \quad B, \Gamma_2 \Rightarrow \Delta_2}{A \to_{\mathtt{c}} B, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (\to_{\mathtt{c}} \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ (\Rightarrow \wedge) \quad \frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge \Rightarrow_1) \quad \frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge \Rightarrow_2)$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B}\ (\Rightarrow \vee_1) \quad \frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \vee B}\ (\Rightarrow \vee_2) \quad \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ (\vee \Rightarrow)$$

finite tree, which is called a *derivation*, generated by inference rules of Table 1 from initial sequents (Id) and $(\bot)$ of Table 1.

Our basic strategy of constructing $\mathsf{G}(\mathbf{C}+\mathbf{J})$ is to add classical implication to the propositional fragment of multi-succedent sequent calculus $\mathbf{mLJ}$ of intuitionistic propositional logic, proposed by Maehara [8]. However, if the ordinary left and right rules of classical implication were added, the soundness of the resulting calculus would fail, because a formula $\neg_{\mathtt{c}} p \to_{\mathtt{c}} (\top \to_{\mathtt{i}} \neg_{\mathtt{c}} p)$, which is not valid by Proposition 2, would be derivable. This is the reason why the original right rule

$$\frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to_{\mathtt{i}} B}$$

of intuitionistic implication of $\mathbf{mLJ}$ is restricted to the right rule given in Table 1. Based on the abbreviation defined in Section 2.1, the following rules for negations are obtained respectively:

$$\frac{A, C_1 \to_{\mathtt{i}} D_1, \ldots, C_m \to_{\mathtt{i}} D_m, p_1, \ldots, p_n \Rightarrow}{C_1 \to_{\mathtt{i}} D_1, \ldots, C_m \to_{\mathtt{i}} D_m, p_1, \ldots, p_n \Rightarrow \neg_{\mathtt{i}} A}\ (\Rightarrow \neg_{\mathtt{i}}) \quad \frac{\Gamma \Rightarrow \Delta, A}{\neg_{\mathtt{i}} A, \Gamma \Rightarrow \Delta}\ (\neg_i \Rightarrow)$$

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg_{\mathtt{c}} A, \Delta}\ (\Rightarrow \neg_{\mathtt{c}}) \quad \frac{\Gamma \Rightarrow \Delta, A}{\neg_{\mathtt{c}} A, \Gamma \Rightarrow \Delta}\ (\neg_{\mathtt{c}} \Rightarrow).$$

Proposition 3. For any $\Gamma \cup \Delta \subseteq \mathsf{Form}$, $\Gamma \Rightarrow \Delta$ is derivable in $\mathsf{G}(\mathbf{C}+\mathbf{J})$ iff $\Gamma \models \Delta$ holds.

Proposition 4. If $\Gamma \Rightarrow \Delta$ is derivable in $\mathsf{G}(\mathbf{C}+\mathbf{J})$, then $\Gamma \Rightarrow \Delta$ is derivable in $\mathsf{G}^{-}(\mathbf{C}+\mathbf{J})$, where $\mathsf{G}^{-}(\mathbf{C}+\mathbf{J})$ is the calculus obtained by removing the rule (Cut) from $\mathsf{G}(\mathbf{C}+\mathbf{J})$.

By Proposition 4, the subformula property is obtained, which ensures the calculus is a conservative extension of both intuitionistic and classical propositional logic.

3 Craig Interpolation

In this section, we establishes two types of Craig interpolation theorem for $\mathsf{G}(\mathbf{C}+\mathbf{J})$, based on Maehara's partition argument in [9]. This argument is originally for classical sequent calculus $\mathbf{LK}$, and is dependent on the fact that the cut elimination holds in the calculus. Since cut elimination holds also in $\mathsf{G}(\mathbf{C}+\mathbf{J})$, as is guaranteed by Proposition 4, this method can be employed. In the following part of this section, $\mathsf{Prop}(D)$ denotes the set of all propositional variables in a formula D. And if Γ is a finite multiset of formulas, we define $\mathsf{Prop}(\Gamma) = \bigcup\{\mathsf{Prop}(D) \mid D \in \Gamma\}$. Especially, we have $\mathsf{Prop}(\perp) = \emptyset$. We call $\langle(\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2)\rangle$ a *partition* of a sequent $\Gamma \Rightarrow \Delta$, if Γ is Γ_1, Γ_2 and Δ is Δ_1, Δ_2. Let us say that C is an *interpolant* of $\langle(\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2)\rangle$ if $\Gamma_1 \Rightarrow \Delta_1, C$ and $C, \Gamma_2 \Rightarrow \Delta_2$ are derivable and $\mathsf{Prop}(C) \subseteq \mathsf{Prop}(\Gamma_1, \Delta_1) \cap \mathsf{Prop}(\Gamma_2, \Delta_2)$.

Although the main idea of giving $\mathsf{G}(\mathbf{C}+\mathbf{J})$ is adding classical implication to intuitionistic logic, our proof is similar to that in classical logic. For establishing the Craig interpolation theorem for $\mathbf{mLJ}$, we cannot employ the notion of partition of the form $\langle(\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2)\rangle$. This is because we cannot find an interpolant for $\langle(\emptyset : A); (A : \emptyset)\rangle$ as noted in [10]. Therefore, in order to show the theorem for $\mathbf{mLJ}$, the form of a partition should be restricted to $\langle(\Gamma_1 : \emptyset); (\Gamma_2 : \Delta)\rangle$. However, this restriction makes it possible to show neither of the two types of theorem in $\mathsf{G}(\mathbf{C}+\mathbf{J})$. Considering this situation, it seems difficult to establish the theorem for $\mathsf{G}(\mathbf{C}+\mathbf{J})$. However, the classical negation (or implication) enables us to use partitions of the form $\langle(\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2)\rangle$ without any restriction to calculate an interpolant by Maehara method. This fact about the way of showing Craig interpolation theorem implies that $\mathbf{C}+\mathbf{J}$ can be regarded as the logic obtained by adding the special (intuitionistic) implication to classical logic[2].

Lemma 1. Suppose that $\Gamma \Rightarrow \Delta$ is derivable in $\mathsf{G}(\mathbf{C}+\mathbf{J})$. Then for any partition $\langle(\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2)\rangle$ of the sequent, there exits an interpolant C in $\mathsf{G}(\mathbf{C}+\mathbf{J})$, i.e., such that both $\Gamma_1 \Rightarrow \Delta_1, C$ and $C, \Gamma_2 \Rightarrow \Delta_2$ are also derivable in $\mathsf{G}(\mathbf{C}+\mathbf{J})$, and $\mathsf{Prop}(C) \subseteq \mathsf{Prop}(\Gamma_1, \Delta_1) \cap \mathsf{Prop}(\Gamma_2, \Delta_2)$.

With Lemma 1, which is the core of the proof, we can easily show the following two types of Craig interpolation theorem.

Theorem 1. (Intuitionistic Craig Interpolation Theorem of $\mathsf{G}(\mathbf{C}+\mathbf{J})$). If $\Rightarrow A \to_{\mathtt{i}} B$ is derivable in $\mathsf{G}(\mathbf{C}+\mathbf{J})$, then there exists a formula C such that $\Rightarrow A \to_{\mathtt{i}} C$ and $\Rightarrow C \to_{\mathtt{i}} B$ are also derivable in $\mathsf{G}(\mathbf{C}+\mathbf{J})$ and that $\mathsf{Prop}(C) \subseteq \mathsf{Prop}(A) \cap \mathsf{Prop}(B)$.

Theorem 2. (Classical Craig Interpolation Theorem of $\mathsf{G}(\mathbf{C}+\mathbf{J})$). If $\Rightarrow A \to_{\mathtt{c}} B$ is derivable in $\mathsf{G}(\mathbf{C}+\mathbf{J})$, then there exists a formula C such that $\Rightarrow A \to_{\mathtt{c}} C$ and $\Rightarrow C \to_{\mathtt{c}} B$ are also derivable in $\mathsf{G}(\mathbf{C}+\mathbf{J})$ and that $\mathsf{Prop}(C) \subseteq \mathsf{Prop}(A) \cap \mathsf{Prop}(B)$.

4 Further Direction

In [15], the first-order expansion $\mathsf{G}(\mathbf{FOC}+\mathbf{J})$ of $\mathsf{G}(\mathbf{C}+\mathbf{J})$ can be given by adding classical universal quantifier to first-order multi-succedent intuitionistic sequent calculus $\mathbf{mLJ}$, although the similar restriction on the right rule for the intuitionistic universal quantifier is needed. Whether Craig interpolation holds in this expansion is an open question, which deserves being inquired.

References

[1] Carlos Caleiro and Jaime Ramos. Combining classical and intuitionistic implications. In Boris Konev and Frank Wolter, editors, *Frontiers of Combining Systems. FroCoS 2007*, Lecture Notes in Computer Science, pages 118–132. Springer, 2007.

[2] This interpretation of $\mathbf{C}+\mathbf{J}$ was already noted in [4].

[2] Michael De. Empirical negation. *Acta Analytica: International Periodical for Philosophy in the Analytical Tradition*, 28(1):49–69, March 2013.

[3] Michael De and Hitoshi Omori. More on empirical negation. In Rajeev Goré, Bateld Kooi, and Agi Kurucz, editors, *Advances in Modal Logic*, volume 10, pages 114–133. College Publications, 2014.

[4] Luis Fariñas del Cerro and Andreas Herzig. Combining classical and intuitionistic logic or: Intuitionistic implication as a conditional. In Franz Badder and Klaus U Schulz, editors, *Frontiers of Combining Systems. FroCoS 1996*, pages 93–102. Springer, March 1996.

[5] Llyod Humberstone. Interval semantics for tense logic: some remarks. *Journal of Philosophical Logic*, 8:171–196, 1979.

[6] Chuck Liang and Dale Miller. A focused approach to combining logics. *Annals of Pure and Applied Logic*, 162:679–697, 2011.

[7] Paqui Lucio. Structured sequent calculi for combining intuitionistic and classical first-order logic. In Hélène Kirchner and Christophe Ringeissen, editors, *Frontiers of Combining Systems. FroCoS 2000*, pages 88–104. Springer, 2000.

[8] Shôji Maehara. Eine Darstellung der intuitionistischen Logik in der klassischen. *Nagoya Mathematical Journal*, 7:45–64, 1954.

[9] Shôji Maehara. Craig's interpolation theorem (in japanese). *Sûgaku*, 12(4):235–237, 1961.

[10] Giorgi Mints. Interpolation theorems for intuitionistic predicate logic. *Annals of Pure and Applied Logic*, 113(1-3):225–242, December 2001.

[11] Luiz Carlos Pereira and Ricardo Oscar Rodriguez. Normalization, soundness and completeness for the propositional fragment of Prawitz' ecumenical system. *Revista Portuguesa de Filosofia*, 73(3/4):1153–1168, 2017.

[12] Elaine Pimentel, Luiz Carlos Pereira, and Valeria de Paiva. A proof theoretical view of ecumenical systems. In *Proceedings of the 13th Workshop on Logical and Semantic Framework with Applications*, pages 109–114. 2018.

[13] Elaine Pimentel, Luiz Carlos Pereira, and Valeria de Paiva. An ecumenical notion of entailment. *Synthese*, May 2019.

[14] Dag Prawitz. Classical versus intuitionistic logic. In Edward Herman Haeusler, Wagner de Campos Sanz, and Bruno Lopes, editors, *Why is this a Proof?*, pages 15–32. College Publications, June 2015.

[15] Masanobu Toyooka and Katsuhiko Sano. Combining first-order classical and intuitionistic logic. In *Proceedings of NCL' 22: Non-Classical Logics. Theory and Applications 2022, forthcoming*.

[16] Masanobu Toyooka and Katsuhiko Sano. Analytic multi-succedent sequent calculus for combining intuitionistic and classical propositional logic. In Sujata Ghosh and R Ramanujam, editors, *ICLA 2021 Proceedings: 9th Indian Conference on Logic and its Applications*, pages 128–133. March 2021.