Measurement-Theoretic Remarks on Reducibility of Decision-Theoretic Values of Questions and Answers to Their Information Values (Extended Abstract)

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1 Motivation

The theory of questions and answers is one of the most popular topics in *speech act theory*. According to Cross and Roelofsen [4], *whether-questions* can be classified into at least two categories. The first category is an yes/no question like (1):

(1) Was there a quorum at the meeting?

(1) has the following two direct answers:

(1a) Yes. There was a quorum at the meeting.
(1b) No. There was not a quorum at the meeting.

(1) presupposes that the meeting took place. (1) also has a corrective answer:

(1c) The meeting did not take place.

Although (2) can be read as an yes/no question having two direct answers, it also has a reading on which it presents the following three direct answers:

(2) Does Jones live in Italy, in Spain, or in Germany?

(2a) Jones lives in Italy.
(2b) Jones lives in Spain.
(2c) Jones lives in Germany.

(2) falls under the second category of whether-questions. (2) presupposes that Jones lives in Italy, in Spain, or in Germany. (2) also has a corrective answer:

(2d) Jones does not live in Italy, in Spain, or in Germany.

Whether-questions have a finite number of direct answers, whereas *which-questions* like (3) and (4) may have an indefinite or infinite number of direct answers.

(3) Which Cardinal was elected Pope in 2013?
(4) Who shot J.R.?

Belnap and Steel [1] refer to wether- and which-questions like (3) and (4) as *elementary questions*. Hamblin [7] takes a question to denote, in a world w, the set of all propositions corresponding to a possible answer to the question. A fundamental problem is that Hamblin semantics does not specify what a possible answer is. Groenendijk and Stokhof [6] take a question to denote, in each world, a single proposition corresponding to the true exhaustive answer to the question in that world. What the true *exhaustive answer* to a question in a given world is is much clear than what *all the possible answers* to that question are. Then the meaning of a question can be identified with a set of *mutually exclusive and exhaustive* propositions (i.e., *partition*) of the logical space. In this paper, we would like to argue about the *crossroads* of the *theory of questions and answers*, *decision theory*, and *information theory* in terms of *measurement theory* (cf. Krantz et al. [8]). The aim of this paper is to remark, in terms of such measurement-theoretic concepts as scale types, on the reducibility of the decision-theoretic values of questions to the their information-theoretic values on the basis of Luce [9]'s theorems. The selling point of this paper is not giving a new linguistic (empirical) analysis of questions and answers but giving a new measurement-theoretic (conceptual) analysis of the decision-theoretic and information-theoretic sides of questions and answers.

2 Decision-Theoretic and Information-Theoretic Values of Questions and Answers

According to van Rooij [10, 11], the *relevance* of a question and its answers can be determined in terms of how much it contributes to solving a *decision problem* that can be modeled by a decision space $(\mathbf{W}, \mathscr{F}, P, U)$. When a partition R is given, *decision-theoretic value* $DV_R(B)$ of a proposition B with respect to R is defined as follows:

Definition 1 **(**$DV_R(B)$**).**

$$DV_R(B) := \max_U \sum_{A \in R} P(A|B)U(A \cap B) - \max_U \sum_{A \in R} P(A)U(A).$$

The expected decision-theoretic value $EDV_R(Q)$ of a question (partition) Q with respect to R is defined by $DV_R(B)$:

Definition 2 **(**$EDV_R(Q)$**).**

$$EDV_R(Q) := \sum_{B \in Q} P(B)DV_R(B).$$

On the other hand, the *relevance* of a question and its answers can be analyzed also in terms of information theory. The informational value $IV_R(A)$ of $A \in \mathscr{F}$ with respect to a partition R:

Definition 3 **($IV_R(B)$).**

$$IV_R(B) := H(R) - H_B(R) = \sum_{A \in R} P(A|B) \log P(A|B) - \sum_{A \in R} P(A) \log P(A),$$

where $H_B(R)$ is the entropy of R with respect to the probability function conditionalized on B.

The *expected information-theoretic* value $EIV_R(Q)$ of a question (partition) Q with respect to R that is defined by $IV_R(B)$:

Definition 4 **($EIV_R(Q)$).**

$$EIV_R(Q) := \sum_{B \in Q} P(B) IV_R(B) = \sum_{B \in Q} \sum_{A \in R} P(A \cap B) \log \frac{P(A \cap B)}{P(A)P(B)}.$$

3 Reducibility: Properness, Locality, and Underlying Context

In general, the decision-theoretic values of questions and answers do not agree with their information-theoretic values. Then when the decision-theoretic values of questions and answers can be reduced to their information-theoretic values? We would like to consider this problem. When this problem is considered, such properties of U as properness and locality are often focused. Properness is defined as follows:

Definition 5 **(Properness).** *U is a* proper *iff* $\sum_{A \in R} P(A) \cdot U(P, A) \geq \sum_{A \in R} P(A) \cdot U(P', A)$ *for any P and P'.*

Locality is defined as follows:

Definition 6 **(Locality).** *U is* local *iff U is defined only by $P(A)(P'(A))$ where $A \in R$ but not by $P(P')$.*

Fischer [5] proves the following theorem:

Fact 1 (Logarithmic Utility Function) *If U is differentiable, proper and local utility functions (scoring rules) for probability functions, and R has more than two cells, then $U(P(A)) = \alpha \log P(A) + \gamma$, where $\alpha > 0$.*

From Fact 1, van Rooij [10, p. 395] deduces the following proposition:

Fact 2 (Reducibility) *If U is differentiable, proper and local utility functions (scoring rules) for probability functions, and R has more than two cells, and moreover $\alpha = 1$ and $\gamma = 0$ in $U(P(A)) = \alpha \log P(A) + \gamma$, then both $DV_R(A) = IV_R(A)$ and $EDV_R(Q) = EIV_R(Q)$ hold, that is, $(E)DV_R$ can be reduced to $(E)IV_R$.*

Although deducing itself the logarithmic utility functions from properness and locality is clear, the statuses of these functions and conditions are not clear to us. So we would like to consider these statuses in terms of comparing the logarithmic utility functions with other proper utility functions. Besides the logarithmic utility functions, there are at least two kinds of frequently-used proper utility functions (scoring rules) for probability functions:

1. quadratic: $U(P(A)) := 2P(A) - \sum_{B \in R} P(B)^2$, and
2. spherical: $U(P(A)) := \dfrac{P(A)}{\sqrt{\sum_{B \in R} P(B)^2}}$.

Both the quadratic and spherical utility functions are not local. Among these three types of functions, the logarithmic utility functions only are both proper and local. Which of these three utility functions should be chosen? Bickel [2] criticizes the quadratic and spherical utility functions in the following two points:

1. The quadratic and spherical utility functions often result in extreme ranking differences when compared to the logarithmic utility functions.
2. Because of nonlocality, the quadratic and spherical utility functions allow for the undesirable possibility that one expert receives the highest utility (score) when assigning to the observed proposition a probability lower than the probabilities assigned by other experts.

On the other hand, Selten [12] criticizes the logarithmic utility functions in the following two points:

1. Their resulting utility (score) is too sensitive to small mistakes for small probabilities.
2. An expert's utility (score) is $-\infty$ when a proposition holds that she predicted to be impossible. So the logarithmic utility functions are unbounded and they need to be truncated, but it will be no longer be proper after such a truncation.

According to Carvalho [3, p. 4], "the choice of the most appropriate proper scoring rule is dependent on the desired properties, which in turn is dependent on the *underlying context.*" Properness and locality can be considered to be examples of "desired properties". Because the statuses of the logarithmic utility functions, properness and locality are not clear to us as we said before, we would like to change our viewpoint from the relation between these functions and conditions to the relation between these functions and the "underlying context" to determine when U is a logarithmic function. Then the following problem arises:

Problem 1 (Reducibility and Underlying Context) *What is an underlying context to determine when $(E)DV_R$ can be reduced to $(E)IV_R$, that is, when U is a logarithmic function and so both $DV_R(A) = IV_R(A)$ and $EDV_R(Q) = EIV_R(Q)$ hold?*

4 Luce's Theorems: Psychophysical Laws

Now we try to cope with Problem 1 in terms of such measurement-theoretic concepts as *scale types* based on the class of *admissible transformations*:

Definition 7 (Scale Types). *A scale is a triple $\langle \mathfrak{U}, \mathfrak{V}, f \rangle$ where $\mathfrak{U}$ is an observed relational structure that is qualitative, $\mathfrak{V}$ is a numerical relational structure that is quantitative, and f is a homomorphism from $\mathfrak{U}$ into $\mathfrak{V}$. A is the domain of $\mathfrak{U}$ and B is the domain of $\mathfrak{V}$. When the admissible transformations are all the functions $\varphi : f(A) \to B$, where $f(A)$ is the range of f, of the form $\varphi(x) := \alpha x; \alpha > 0$. φ is called a* similarity transformation*, and a scale with the similarity transformations as its class of admissible transformations is called a* ratio scale*. When the admissible transformations are all the functions $\varphi : f(A) \to B$ of the form $\varphi(x) := \alpha x + \beta; \alpha > 0$, φ is called a* positive affine transformation*, and a corresponding scale is called an* interval scale*.*

Remark 1 (Ratio and Interval Scales) *The indefinite integral of a ratio scale is an interval scale.*

Indeed the concept of (underlying) context is ambiguous. But when $U := \psi(P)$, ψ can be considered to be an *underlying context* to connect P to U and to determine when U is a logarithmic function and so both $DV_R(A) = IV_R(A)$ and $EDV_R(Q) = EIV_R(Q)$ hold. Luce [9] proves the theorems on the types of *psychophysical laws* that connect the *physical scales* to *psychological scales* in terms of *measurement theory*. First, Luce proves the following theorem that connects ratio scales as physical scales to ratio scales as psychological scales:

Fact 3 (From Ratio Scale to Ratio Scale) *Suppose that $f : A \to \mathbb{R}^+$ and $g : A \to \mathbb{R}^+$ are both ratio scales and that $g(a) = \psi(f(a))$ for any $a \in A$ and that ψ is continuous. Then $\psi(x) = \alpha x^\beta$, where $\alpha > 0$.*

Second, Luce [9] proves the following theorem that connects ratio scales as physical scales to interval scales as psychological scales:

Fact 4 (From Ratio Scale to Interval Scale) *Suppose that $f : A \to \mathbb{R}^+$ is a ratio scale and $g : A \to \mathbb{R}$ is an interval scale and that $g(a) = \psi(f(a))$ for any $a \in A$ and that ψ is continuous. Then $\psi(x) = \alpha x^\beta + \gamma$ or $\psi(x) = \alpha \log x + \gamma$.*

5 Reducibility and Scale Types

Luce proves Fact 4 independently of Fact 3. In addition, he proves Fact 4 as a corollary of Fact 3 on the assumption that ψ is not only continuous but also *differentiable* in such a way that since the *indefinite integral* of a *ratio scale* is an *interval scale*, if f is considered to be a ratio scale and g is an interval scale, then either $\psi(x) = \frac{\alpha}{\beta+1} x^{\beta+1} + \gamma$ if $\beta \neq -1$ or $\psi(x) = \alpha \log x + \gamma$ if $\beta = -1$. Facts 3 and 4 may be originally intended to determine the psychophysical laws that connect the physical scales to psychological scales. But we can regard Luce's theorems as the theorems which have *wider applicability* in the sense that these theorems can make clear connection between *scales in general*. Now we would like to use these theorems in order to furnish a solution to Problem 1:

Proposition 1 (Reducibility and Scale Types). *Suppose that a ratio scale P (in a wide sense) is given, and that an underlying context $\psi(x) := \alpha x^{\beta}; \alpha > 0$ is given connecting P to a ratio scale (stronger cardinal utility) U^*, and that R has more than two cells. Then if $\beta = -1$ and the integral constant of $\int U^*(P)dP$ equals 0, then such interval scale (weaker cardinal utility) U as $U(P) := \int U^*(P)dP$ is a* logarithmic function *and, when DV_R is defined by U, both $DV_R(A) = IV_R(A)$ and $EDV_R(Q) = EIV_R(Q)$ hold, that is, $(E)DV_R$ can be reduced to $(E)IV_R$, and if $\beta \neq -1$, then U is* no logarithmic function*—it may be quadratic or spherical function—and either $DV_R(A) = IV_R(A)$ or $EDV_R(Q) = EIV_R(Q)$ does not always hold, that is, $(E)DV_R$ cannot be reduced to $(E)IV_R$.*

Remark 2 (Solution to Problem 1) *Proposition 1 states that such conditions as especially the value of β (i.e., $\beta = -1$ or not) concerning the underlying context $\psi(x) := \alpha x^{\beta}$ connecting a ratio scale P to a ratio scale U^* determines when the* decision-theoretic *value of questions and answers can be reduced to their* information-theoretic *values, which furnishes a solution to Problem 1.*

References

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SOCREAL2022
Online
28 February 2022

Contents of This Talk

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Theory of Questions and Answers

The theory of questions and answers is one of the most popular topics in the philosophy of language (cf. Cross and Roelofsen (2020)).

C. Cross and F. Roelofsen.
Questions, 2020.
Stanford Encyclopedia of Philosophy.

Hamblin Semantics

- Hamblin (1973) takes a question to denote, in a world w, the set of all propositions corresponding to a possible answer to the question.

  C. L. Hamblin.
  Questions and Answers in Montague English.
  Foundations of Language, 10:41–53, 1973.

- A fundamental problem is that Hamblin semantics does not specify what a possible answer is.

Partition Semantics

- Groenendijk and Stokhof (1984) take a question to denote, in each world, a single proposition corresponding to the true exhaustive answer to the question in that world.

  J. Groenendijk and M. Stokhof.
  Studies on the Semantics of Questions and the Pragmatics of Answers.
  University of Amsterdam, 1984.

- The meaning of a question can be identified with a set of mutually exclusive and exhaustive propositions (i.e., partition) of the logical space.
- What the true exhaustive answer to a question in a given world is much clear than what all the possible answers to that question are.

Crossroads

In this talk, we would like to argue about the crossroads of the theory of questions and answers, decision theory, and information theory in terms of measurement theory (cf. Krantz et al. (1971)).

D. H. Krantz et al.
Foundations of Measurement, volume 1.
Academic Press, New York, 1971.

Aim of This Talk

- The aim of this talk is to remark, in terms of such measurement-theoretic concepts as scale types, on the reducibility of the decision-theoretic values of questions to the their information-theoretic values on the basis of Luce (1959)'s theorems.

  R. D. Luce.
  On the possible psychophysical laws.
  The Psychological Review, 66:81–95, 1959.

- The selling point of this talk is not giving a new linguistic (empirical) analysis of questions and answers but giving a new measurement-theoretic (conceptual) analysis of the decision-theoretic and information-theoretic sides of questions and answers.

Contents of This Talk

Relevance of Question to Decision Problem: Two Scales

According to van Rooij (2004, 2009), the relevance of a question to a decision problem can be scaled in terms of decision theory and information theory.

van Rooij, R.:
Utility, informativity and protocols.
Journal of Philosophical Logic, 33:389–419, 2004.

van Rooij, R.:
Comparing questions and answers: A bit of logic, a bit of language, and some bits of information.
In: Sommaruga, G. (ed.) Formal Theories of Information: from Shannon to Semantic Information Theory and General Concepts of Information, LNCS 5363, pp. 161–192. Springer, Heidelberg, 2009.

Decision-Theoretic Value

First, the relevance of a question to a decision problem can be scaled in terms of decision theory:

Definition (DV_R)

When a partition R is given, we define the decision-theoretic value $DV_R(B)$ of a proposition B with respect to R:

$$DV_R(B) := \max_{U\in\mathscr{U}} EU(A\cap B) - \max_{U\in\mathscr{U}} EU(A)$$

$$= \max_{U\in\mathscr{U}} \sum_{A\in R} P(A|B)U(A\cap B) - \max_{U\in\mathscr{U}} \sum_{A\in R} P(A)U(A),$$

where U is a variable and $\mathscr{U}$ is the class of all utility functions.

Definition (EDV_R)

The expected decision-theoretic value $EDV_R(Q)$ of a question (partition) Q with respect to R is defined by $DV_R(B)$:

$$EDV_R(Q) := \sum_{B\in Q} P(B)DV_R(B).$$

Information-Theoretic Value

Second, the relevance of a question to a decision problem can be scaled also in terms of information theory:

Definition (IV_R)

The information-theoretic value $IV_R(A)$ of $A \in \mathscr{F}$ with respect to a partition R:

$$IV_R(B) := H(R) - H_B(R) = \sum_{A\in R} P(A|B)\log P(A|B) - \sum_{A\in R} P(A)\log P(A),$$

where $H(R)$ is the entropy of R, and $H_B(R)$ is the entropy of R with respect to the probability function conditionalized on B.

Definition (EIV_R)

The expected information-theoretic value $EIV_R(Q)$ of a question (partition) Q with respect to R that is defined by $IV_R(B)$:

$$EIV_R(Q) := \sum_{B\in Q} P(B)IV_R(B) = \sum_{B\in Q}\sum_{A\in R} P(A\cap B)\log\frac{P(A\cap B)}{P(A)P(B)}.$$

Example

In general, the decision-theoretic values of questions and answers do not agree with their information-theoretic values. The following example by van Rooij (2009) illustrates this fact:

Example (Discrepancy)

- John considers the decision problem of whether he should go to the party tonight.
- This decision problem depends almost entirely on whether Mary will go, because he is secretly in love with Mary, and believes that going to the party is his only chance to meet her.
- He prefers meeting her tonight to not meeting her, but if Mary wont go, he prefers to stay home.
- But going to the party when Mary comes too obviously involves a risk: Mary might turn him down when he makes his advances.
- In this situation 4 different worlds are involved:
  - w_1: Mary goes to the party, John will go, too, he will try his luck, and is successful.
  - w_2: Mary goes, John goes, he tries his luck, and is unsuccessful.
  - w_3: Mary won't go to the party, and thus neither does John, and if John tried his luck, he would be successful.
  - w_4: similar to w_3 except that in this world if John tried his luck, he would be unsuccessful.
- Then the partition is $R := \{\{w_1, w_2\}, \{w_3, w_4\}\}$.

Example (Continued)

Example (Discrepancy (Continued))

- John thinks all worlds are equally likely to come out true.
- He has a negative attitude towards taking risks.
- He doesn't care about what Mary would do if they don't go to the party.
- His decision problem might be represented by the following table:

World	Probability	Utility
w_1	$\frac{1}{4}$	12
w_2	$\frac{1}{4}$	2
w_3	$\frac{1}{4}$	8
w_4	$\frac{1}{4}$	8

- Suppose that John considers the question of whether he will be successful if he tries the luck.
- Then the semantic value of the question is $Q := \{\{w_1, w_3\}, \{w_2, w_4\}\}$.
- When the semantic value of the positive answer (to the question) that he will be successful is $B := \{w_1, w_3\}$, that of the negative answer that he will not be successful is $B^C = \{w_2, w_4\}$.

Example (Continued)

Example (Discrepancy (Continued))

- $DVR(B) = \max_{U\in\mathscr{U}} EU(A\cap B) - \max_{U\in\mathscr{U}} EU(A) = EU(A\cap B) - EU(A) = \frac{5}{2}$.
- On the other hand, neither learning B nor learning B^C changes the entropy of the partition R, that is,

$$IV_R(B) = H(R) - H_B(R) = IV_R(B^C) = H(R) - H_{B^C}(R) = 0,$$

  because neither learning B nor learning B^C changes the probability distribution of the elements of R, that is, $H(R)$, $H_B(R)$ and $H_{B^C}(R)$ each have a value of 1.
-

$$EDV_R(Q) := P(B)DV_R(B) + P(B^C)DV_R(B^C)$$
$$= P(B)(EU(A\cap B) - EU(A)) + P(B^C)(EU(A\cap B^C) - EU(A)) = 0$$

- Because not only the positive answer B to the question, but also the negative answer B^C has no effect on the probability distribution of the elements of Q, $EIV_R(Q) = H(R) - H_Q(R) = 0$.

Contents of This Talk

- In general, the decision-theoretic values of questions and answers do not agree with their information-theoretic values.
- Then when the decision-theoretic values of questions and answers can be reduced to their information-theoretic values?
- We would like to consider this problem.
- When this problem is considered, such properties of U as properness and locality are often focused.

Properness and Locality

A score rule is defined as follows:

Definition (Scoring Rule)

We call a utility function U for probability functions P defined on a partition R a scoring rule for P. For any $A, B, \ldots \in R$, we abbreviate $U(P(A), P(B), \ldots)$ as $U(P)$.

Properness is defined as follows:

Definition (Properness)

A scoring rule U for P is proper iff $\sum_{A \in R} P(A) \cdot U(P) = \sup_{P' \in \mathscr{P}} \sum_{A \in R} P(A) \cdot U(P')$ for any $P \in \mathscr{P}$ that is the class of all probability functions defined on a partition R.

Locality is defined as follows:

Definition (Locality)

A scoring rule U for P is local iff, for any $P \in \mathscr{P}$, there exist such U' that $U(P) = U'(P(A))$ for any $A \in R$.

Fisher's Theorem

Fischer (1972) proves the following theorem:

Fact (Logarithmic Scoring Rule)

If U is differentiable, proper and local scoring rules for probability functions P, and R has more than two cells, then $U(P(A)) = \alpha \log P(A) + \gamma$, where $\alpha > 0$.

Fisher, P.:
On the inequality $\sum p_i f(p_i) \geq \sum p_i f(q_i)$. Metrika 18, 199–208 (1972)

From Fact (Logarithmic Scoring Rule), van Rooij (2004) deduces the following proposition:

Fact (Reducibility)

If U is differentiable, proper and local scoring rules for probability functions P, and R has more than two cells, and moreover $\alpha = 1$ and $\gamma = 0$ in $U(P(A)) = \alpha \log P(A) + \gamma$, then both $DV_R(A) = IV_R(A)$ and $EDV_R(Q) = EIV_R(Q)$ hold, that is, $(E)DV_R$ can be reduced to $(E)IV_R$.

Proper Scoring Rules

- Although deducing itself the logarithmic scoring rules from properness and locality is clear, the statuses of properness and locality are not clear to us.
- So we would like to consider these statuses in terms of comparing the logarithmic scoring rules with other proper scoring rules.
- Besides the logarithmic scoring rules, there are at least two kinds of frequently-used proper scoring rules for probability functions:
  1. quadratic: $U(P(A)) := 2P(A) - \sum_{B \in R} P(B)^2$, and
  2. spherical: $U(P(A)) := \dfrac{P(A)}{\sqrt{\sum_{B \in R} P(B)^2}}$.

- Both the quadratic and spherical scoring rules are not local.
- Among these three types of functions, the logarithmic scoring rules only are both proper and local.
- Which of these three utility functions should be chosen?

Bickel (2007) criticizes the quadratic and spherical scoring rule, whereas Selten (1998) criticizes the logarithmic scoring rules.

J. E. Bickel.
Some comparisons among quadratic, spherical, and logarithmic scoring rules.
Decision Analysis, 4:49–65, 2007.

R. Selten.
Axiomatic characterization of the quadratic scoring rule.
Experimental Economics, 1:43–62, 1998.

Problem

- According to Carvalho (2016, p.4), "the choice of the most appropriate proper scoring rule is dependent on the desired properties, which in turn is dependent on the underlying context."

A. Carvalho.
An overview of applications of proper scoring rules.
Decision Analysis, Articles in Advance, 2016.

- Properness and locality can be considered to be examples of "desired properties".
- Because the statuses of properness and locality are not clear to us as we said before, we would like to change our viewpoint from properness and locality to the "underlying context" to determine when U is a logarithmic function.
- Then the following problem arises:

Problem (Reducibility and Underlying Context)

What is an underlying context to determine when $(E)DV_R$ can be reduced to $(E)IV_R$, that is, when U is a logarithmic scoring rule and so both $DV_R(A) = IV_R(A)$ and $EDV_R(Q) = EIV_R(Q)$ hold?

Contents of This Talk

Measurement Theory

- In this talk, we try to cope with this problem in terms of measurement theory.
- Measurement theory includes such important concepts as
  1. scale types,
  2. representation and uniqueness theorems, and
  3. measurement types.

Scale Types

- In this talk, we resort to scale types.
- Scale types have such categories as
  - ratio scale (unique up to $\varphi(x) = \alpha x (\alpha > 0)$), and
  - interval scale (unique up to $\varphi(x) = \alpha x + \beta (\alpha > 0)$).

Luce's Theorems (1)

- Indeed the concept of (underlying) context is ambiguous.
- But when $U := \psi(P)$, ψ can be considered to be an underlying context to connect P to U and to determine when U is a logarithmic scoring rule and so both $DV_R(A) = IV_R(A)$ and $EDV_R(Q) = EIV_R(Q)$ hold.
- Luce (1959) proves the theorems on the types of psychophysical laws that connect the physical scales to psychological scales in terms of measurement theory.
- First, Luce proves the following theorem that connects ratio scales as physical scales to ratio scales as psychological scales:

Fact (From Ratio Scale to Ratio Scale)

Suppose that $f : A \to \mathbb{R}^+$ and $g : A \to \mathbb{R}^+$ are both ratio scales and that $g(a) = \psi(f(a))$ for any $a \in A$ and that ψ is continuous. Then $\psi(x) = \alpha x^\beta$, where $\alpha > 0$.

Luce's Theorems (2)

Second, Luce proves the following theorem that connects ratio scales as physical scales to interval scales as psychological scales:

Fact (From Ratio Scale to Interval Scale)

Suppose that $f : A \to \mathbb{R}^+$ is a ratio scale and $g : A \to \mathbb{R}$ is an interval scale and that $g(a) = \psi(f(a))$ for any $a \in A$ and that ψ is continuous. Then $\psi(x) = \alpha x^\beta + \gamma$ or $\psi(x) = \alpha \log x + \gamma$.

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- Luce proves Fact (From Ratio Scale to Interval Scale) independently of Fact (From Ratio Scale to Ratio Scale).
- In addition, he proves Fact (From Ratio Scale to Interval Scale) as a corollary of Fact (From Ratio Scale to Ratio Scale) on the assumption that ψ is not only continuous but also differentiable in such a way that since the indefinite integral of a ratio scale is an interval scale, if f is considered to be a ratio scale and g is an interval scale, then either $\psi(x) = \frac{\alpha}{\beta+1}x^{\beta+1} + \gamma$ if $\beta \neq -1$ or $\psi(x) = \alpha \log x + \gamma$ if $\beta = -1$.
- Facts (From Ratio Scale to Ratio Scale) and (From Ratio Scale to Interval Scale) may be originally intended to determine the psychophysical laws that connect the physical scales to psychological scales.
- But we can regard Luce's theorems as the theorems which have wider applicability in the sense that these theorems can make clear connection between scales in general.

Now we would like to use these theorems in order to furnish a solution to Problem (Reducibility and Underlying Context):

Proposition (Reducibility and Scale Types)

Suppose

- *that a ratio scale P (in a wide sense) is given, and*
- *that an underlying context $\psi(x) := \alpha x^{\beta}; \alpha > 0$ is given connecting P to a ratio scale (stronger cardinal utility) U^*, and*
- *that R has more than two cells.*

Then

- *if $\beta = -1$ and the integral constant of $\int U^*(P)dP$ equals 0, then*
  - *such interval scale (weaker cardinal utility) U as $U(P) := \int U^*(P)dP$ is a logarithmic function and,*
  - *when DV_R is defined by U, both $DV_R(A) = IV_R(A)$ and $EDV_R(Q) = EIV_R(Q)$ hold, that is, $(E)DV_R$ can be reduced to $(E)IV_R$, and*
- *if $\beta \neq -1$, then U is no logarithmic function—it may be quadratic or spherical function—and either $DV_R(A) = IV_R(A)$ or $EDV_R(Q) = EIV_R(Q)$ does not always hold, that is, $(E)DV_R$ cannot be reduced to $(E)IV_R$.*

Remark (Solution to Problem (Reducibility and Underlying Context))

Proposition (Reducibility and Scale Types) states that such conditions as especially the value of β (i.e., $\beta = -1$ or not) concerning the underlying context $\psi(x) := \alpha x^{\beta}$ connecting a ratio scale P to a ratio scale U^* determines when the decision-theoretic value of questions and answers can be reduced to their information-theoretic values, which furnishes a solution to Problem (Reducibility and Underlying Context).

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Concluding Remarks

- Van Rooij observes on the reducibility of the decision-theoretic values of questions and answers to their information-theoretic values in terms of rather unclear concepts of properness and locality on the basis of Fischer's theorem.
- On the other hand, in this talk, we have remarked in terms of measurement-theoretic concepts, particularly, scale types on the reducibility of the decision-theoretic values of questions and answers to their information-theoretic values on the basis of Luce's theorems.

Thank You for Your Attention!