The fourth-order total variation flow in $\mathbb{R}^n$

Dedicated to Professor Neil Trudinger on the occasion of his 80th birthday

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Abstract

We define rigorously a solution to the fourth-order total variation flow equation in $\mathbb{R}^n$. If $n \geq 3$, it can be understood as a gradient flow of the total variation energy in D^{-1}, the dual space of D_0^1, which is the completion of the space of compactly supported smooth functions in the Dirichlet norm. However, in the low dimensional case $n \leq 2$, the space D^{-1} does not contain characteristic functions of sets of positive measure, so we extend the notion of solution to a larger space. We characterize the solution in terms of what is called the Cahn-Hoffman vector field, based on a duality argument. This argument relies on an approximation lemma which itself is interesting.

We introduce a notion of calibrability of a set in our fourth-order setting. This notion is related to whether a characteristic function preserves its form throughout the evolution. It turns out that all balls are calibrable. However, unlike in the second-order total variation flow, the outside of a ball is calibrable if and only if $n \neq 2$. If $n \neq 2$, all annuli are calibrable, while in the case $n = 2$, if an annulus is too thick, it is not calibrable.

We compute explicitly the solution emanating from the characteristic function of a ball. We also provide a description of the solution emanating from any piecewise constant, radially symmetric datum in terms of a system of ODEs.

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1 Introduction

We consider the fourth-order total variation flow equation in $\mathbb{R}^n$ of the form

$$u_t = -\Delta \operatorname{div} \frac{\nabla u}{|\nabla u|}. \tag{1.1}$$

We aim to give explicit description of its solutions emanating from piecewise constant radial data. However, it turns out that the definition of a solution is itself non-trivial since $-\Delta$ does not have a bounded inverse on $L^2(\mathbb{R}^n)$. Our first goal is thus to provide a rigorous definition of a

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solution. Our second goal is to find explicit formula for the solution to (1.1) when the initial datum $u(0, x) = u_0(x)$ is the characteristic function of a ball or an annulus. In other words,

$$u_0 = a_0 \mathbf{1}_{B_{R_0}} \quad \text{or} \quad u_0 = a_0 \mathbf{1}_{A_{R_0^0}^{R_0^1}} \quad a_0 \in \mathbb{R},$$

where $\mathbf{1}_K$ is the characteristic function of a set $K \subset \mathbb{R}^n$, i.e.,

$$\mathbf{1}_K(x) = \begin{cases} 1, & x \in K \\ 0, & x \in \mathbb{R}^n \backslash K. \end{cases}$$

Here B_R denotes the open ball of radius R centered at $0 \in \mathbb{R}^n$ and $A_{R^0}^{R^1}$ denotes the annulus defined by $A_{R^0}^{R^1} = B_{R_1} \setminus \overline{B_{R_0}}$. Our major concern is whether or not the solution remains a characteristic function throughout the evolution. For example, in the case $u_0 = a_0 \mathbf{1}_{B_{R_0}}$, whether or not the solution u of (1.1) is of the form

$$u(t, x) = a(t) \mathbf{1}_{B_{R(t)}}$$

with a function $a = a(t)$. In other words, we are asking whether the speed u_t on the ball $B_{R(t)}$ and on its complement are constant in the spatial variable. As in the second-order problem [3] (see also [16]), this leads to the notion of calibrability of a set. In the case of the second-order problem $u_t = \operatorname{div}(\nabla u / |\nabla u|)$, a ball and its complement are always calibrable and $R(t) \equiv R_0$, i. e. the ball does not expand nor shrink [3]. In our problem, $R(t)$ may not be constant.

We first note that the definition of a solution itself is non-trivial. The fourth-order total variation flow has been mainly studied in the periodic setting [13], [11] or in a bounded domain with some boundary conditions [14]. Formally, it is a gradient flow of the total variation functional

$$TV(u) := \int_\Omega |\nabla u|$$

with respect to the inner product

$$(u, v)_{-1} = \int_\Omega u(-\Delta)^{-1} v$$

when Ω is a domain in $\mathbb{R}^n$ or a flat torus $\mathbb{T}^n$. In the periodic setting, i. e. $\Omega = \mathbb{T}^n$ as in [13], [11], it is interpreted as a gradient flow in H_{av}^{-1} which is the dual space of H_{av}^1, the space of average-free H^1 functions equipped with the inner product

$$(u, v)_1 = \int_\Omega \nabla u \cdot \nabla v.$$

For the homogeneous Dirichlet boundary condition with bounded Ω, H_{av}^{-1} is replaced by D^{-1}, the dual space of $D_0^1 = D_0^1(\Omega)$, which is the completion of $C_c^\infty(\Omega)$ in the norm associated with the inner product $(u, v)_1$; here $C_c^\infty(\Omega)$ denotes the space of all smooth functions compactly supported in Ω. By the Poincaré inequality, both H_{av}^1 and $D_0^1(\Omega)$ can be regarded as subspaces of $L^2(\Omega)$. However, if Ω equals $\mathbb{R}^n$, the situation is more involved. If $n \geq 3$, $D_0^1(\mathbb{R}^n)$ is continuously and densely embedded in $L^{2^*}(\mathbb{R}^n)$, where $p^* = np/(n-p)$ so that $2^* = 2n/(n-2)$, by the Sobolev inequality. In fact,

$$D_0^1(\mathbb{R}^n) = D^1(\mathbb{R}^n) \cap L^{2^*}(\mathbb{R}^n), \quad D^1(\mathbb{R}^n) = \left\{ u \in L_{loc}^1(\mathbb{R}^n) \;\middle|\; \nabla u \in L^2(\mathbb{R}^n) \right\}$$

see e. g. [10]. On the other hand, if $n \leq 2$, D_0^1 is isometrically identified with the quotient space $\dot{D}^1(\mathbb{R}^n) := D^1(\mathbb{R}^n)/\mathbb{R}$, when $D^1(\mathbb{R}^n)$ is equipped with inner product $(u, v)_1$ [10]. Thus, we need to be careful when $n \leq 2$ because an element of $D_0^1(\mathbb{R}^n)$ is determined only up to a constant. In any case, $D_0^1(\mathbb{R}^n)$ is a Hilbert space with the scalar product

$$(u, v)_{D_0^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \nabla u \cdot \nabla v.$$

Therefore, we can identify $D_0^1(\mathbb{R}^n)$ with its dual space by means of the isometry

$$-\Delta\colon u \mapsto (u,\cdot)_{D_0^1(\mathbb{R}^n)}.$$

On the other hand, let us define a subspace $\widetilde{D}^{-1}(\mathbb{R}^n) \subset D_0^1(\mathbb{R}^n)'$ by

$$\widetilde{D}^{-1}(\mathbb{R}^n) = \left\{ w \mapsto \int_{\mathbb{R}^n} uw \colon u \in C_c^\infty(\mathbb{R}^n) \right\} \quad \text{if } n \geq 3,$$

$$\widetilde{D}^{-1}(\mathbb{R}^n) = \left\{ w \mapsto \int_{\mathbb{R}^n} uw \colon u \in C_{c,av}^\infty(\mathbb{R}^n) \right\} \quad \text{if } n = 1 \text{ or } n = 2,$$

where

$$C_{c,av}^\infty(\mathbb{R}^n) = \left\{ u \in C_c^\infty(\mathbb{R}^n) \colon \int_{\mathbb{R}^n} u = 0 \right\}.$$

Then the closure $D^{-1}(\mathbb{R}^n)$ of $\widetilde{D}^{-1}(\mathbb{R}^n)$ coincides with $D_0^1(\mathbb{R}^n)'$ [10]. Note that the restriction to $C_{c,av}^\infty(\mathbb{R}^n)$ in the definition of $\widetilde{D}^{-1}(\mathbb{R}^n)$ in $n = 1,2$ is necessary for the functionals to be well-posed on $D^1(\mathbb{R}^n)/\mathbb{R}$. In any case, since (by definition) the space of test functions $\mathcal{D}(\mathbb{R}^n)$ is continuously and densely embedded in $D_0^1(\mathbb{R}^n)$, we also have a continuous embedding $D^{-1}(\mathbb{R}^n) = D_0^1(\mathbb{R}^n)' \subset \mathcal{D}'(\mathbb{R}^n)$. Throughout the paper, we will often drop $(\mathbb{R}^n)$ in the notation for spaces of functions on $\mathbb{R}^n$, e. g. $D^{-1} = D^{-1}(\mathbb{R}^n)$.

In the first step, we give a rigorous definition of the total variation functional TV on D^{-1}. Then we calculate the subdifferential of TV in D^{-1} space. Since it is a homogeneous functional, we are able to apply a duality method [3] to characterize the subdifferential, provided that TV is well approximated by nice functions in D^{-1}. We know that $C_{c,av}^\infty(\mathbb{R}^n)$ is dense in D^{-1} for $n \leq 2$; see e. g. [10]. However, it is not immediately clear whether TV is simultaneously approximable. Fortunately, it turns out that for any $w \in D^{-1}$, there is a sequence $w_k \in C_{c,av}^\infty(\mathbb{R}^n)$ which converges to w in D^{-1} and $TV(w_k) \to TV(w)$ as $k \to \infty$. This approximation part is relatively involved since we have to use an efficient cut-off function. Using the approximation, we are able to characterize the subdifferential $\partial_{D^{-1}} TV$ of TV in D^{-1} by adapting the argument in [3]. Namely, we have

$$\partial_{D^{-1}} TV(u) = \left\{ v = \Delta \operatorname{div} Z \;\middle|\; Z \in L^\infty(\mathbb{R}^n),\ |Z| \leq 1,\ -\langle u, \operatorname{div} Z \rangle = TV(u) \right\},$$

where $\langle\ ,\ \rangle$ denotes the canonical paring of D^{-1} and D_0^1. A vector field Z corresponding to an element of the subdifferential is often called a *Cahn-Hoffman vector field*. The equation (1.1) should be interpreted as the gradient flow of TV in D^{-1}, i. e.

$$u_t \in -\partial_{D^{-1}} TV(u), \tag{1.2}$$

and its unique solvability for any initial datum $u_0 \in D^{-1}$ is guaranteed by the classical theory of maximal monotone operators ([23], [6]). By our characterization of the subdifferential, we are able to give a more explicit definition of a solution which is consistent with that proposed in [14]. Namely, for $u_0 \in D^{-1}$ with $TV(u_0) < \infty$, a function $u \in C\left([0,T[, D^{-1}\right)$ is a solution to (1.2) with $u(0) = u_0$ if and only if there exists $Z \in L^\infty\left(]0,T[\times \mathbb{R}^n\right)$ satisfying $\operatorname{div} Z \in L^2\left(0,T; D_0^1(\mathbb{R}^n)\right)$ such that

$$u_t = -\Delta \operatorname{div} Z \quad \text{in} \quad D^{-1}(\mathbb{R}^n) \tag{1.3}$$

$$|Z(t,x)| \leq 1 \quad \text{for a. e. } x \in \mathbb{R}^n \tag{1.4}$$

$$\langle u, \operatorname{div} Z \rangle = -TV(u) \tag{1.5}$$

for a. e. $t \in]0,T[$. This is convenient for calculating explicit solutions.

Unfortunately, in $n \leq 2$, for a compactly supported square integrable function u_0, we know that $u_0 \in D^{-1}$ if and only if u_0 is average-free, i. e. $\int_{\mathbb{R}^n} u_0 = 0$ (see Lemma 16). Thus, the characteristic function of any bounded, measurable set of positive measure does not belong to D^{-1}. We have to extend a class of initial data u_0 such that $u_0 = \psi + w_0$ with $w_0 \in D^{-1}$ while ψ is a fixed compactly supported L^2 function. We consider a gradient flow $u_t \in -\partial_{D^{-1}} TV(u)$ in the affine space $\psi + D^{-1}$. Since $\partial_{D^{-1}}$ is a directional partial derivative in the direction of D^{-1}, it is more convenient to consider solutions to evolutionary variational inequality

$$\frac{1}{2}\frac{d}{dt}\|u(t)-g\|_{D^{-1}}^2 \le TV(g) - TV(u(t)) \quad \text{for a.e. } t>0 \tag{1.6}$$

for any $g \in \psi + D^{-1}$ [2]. In the case $\psi = 0$, it is easy to show that the evolutionary variational inequality is equivalent to (1.2). Indeed, by definition of the subdifferential, (1.2) is equivalent to

$$(-u_t, g-u(t))_{D^{-1}} \le TV(g) - TV(u(t))$$

for any $g \in D^{-1}$. The left-hand side equals $(d/dt)\left(\|u-g\|^2/2\right)$. Thus, the equivalence follows if $\psi = 0$.

It is easy to check that there is at most one solution to the evolutionary variational inequality (1.6). The solution u is constructed by solving

$$w_t \in -\partial_{D^{-1}} TV(w+\psi) \quad \text{with} \quad w(0) = w_0 = u_0 - \psi$$

and setting $u = w + \psi$. Characterization of the (directional) subdifferential is more involved since $w \mapsto TV(w+\psi)$ is no more positively one-homogeneous. We identify the one dimensional space $\{c\psi | c \in \mathbb{R}\}$ with $\mathbb{R}$ and consider the Hilbert space E^{-1} defined as the orthogonal sum $D^{-1} \oplus \mathbb{R}$. We calculate the subdifferential by the duality method since TV is now positively one-homogeneous on E^{-1}. We then project this subdifferential onto D^{-1} to get a characterization of a (directional) subdifferential $\partial_{D^{-1}} TV$. We end up with a characterization of solution to (1.2) similar to (1.3)–(1.5), with (1.5) adjusted in a suitable way. If we also denote $E_0^1 = D^1$ in $n \le 2$, $E_0^1 = D_0^1$, $E^{-1} = D^{-1}$ in $n \ge 3$ and

$$\langle u, v\rangle_E = \begin{cases} \langle u, v\rangle \text{ if } n \ge 3, \\ \langle w, [v]\rangle + c\int \psi v, \text{ where } u = w + c\psi,\ w \in D^{-1} \text{ if } n \le 2, \end{cases} \tag{1.7}$$

for $u \in E^{-1}$, $v \in E_0^1$, we end up with the following definition of solution

Definition 1. *Assume that $u_0 \in E^{-1}$. We say that $u \in C([0,\infty[, E^{-1})$ with $u_t \in L^2_{loc}(]0,\infty[, D^{-1})$ is a solution to (1.1) with initial datum u_0 if there exists $Z \in L^\infty(]0,\infty[\times\mathbb{R}^n)$ with $\operatorname{div} Z(t,\cdot) \in E_0^1$ for a. e. $t > 0$ such that*

$$u_t = -\Delta \operatorname{div} Z \quad in \quad D^{-1}(\mathbb{R}^n) \tag{1.8}$$

$$|Z(t,x)| \le 1 \quad for\ a.\ e.\ x \in \mathbb{R}^n \tag{1.9}$$

$$-\langle u, \operatorname{div} Z\rangle_E = TV(u) \tag{1.10}$$

for a. e. $t > 0$.

and associated well-posedness result

Theorem 2. *Let $u_0 \in E^{-1}$. There exists a unique solution to (1.1) with initial datum u_0.*

Our next problem is whether or not the speed of a characteristic function of a set is spatially constant inside and outside of the set. By the general theory ([6], [23]), the speed is determined by the minimal section (canonical restriction) $\partial^0_{D^{-1}} TV$ of $\partial_{D^{-1}} TV$. In other words, $\partial^0_{D^{-1}} TV(u) = v_0$ minimizes $\|v\|_{D^{-1}}$ for $v \in \partial_{D^{-1}} TV(u)$, i.e.,

$$\partial^0_{D^{-1}} TV(u) := \arg\min\left\{\|v\|_{D^{-1}} \;\middle|\; v \in \partial_{D^{-1}} TV(u)\right\}.$$

To motivate the notion of calibrability, we consider a smooth function u such that

$$\overline{U} = \left\{x \in \mathbb{R}^n \;\middle|\; u(x) = 0\right\},$$

where U is a smooth open set. Outside $\overline{U}$, we assume that $\nabla u \neq 0$. To fix the idea, we assume that ∂U has negative signature (orientation) in the sense that $u < 0$ outside $\overline{U}$. By our specification of u, we see that

$$\partial^0_{D^{-1}} TV(u) = \arg\min \left\{ \| \operatorname{div} Z \|_{D^1_0} \;\middle|\; |Z| \leq 1 \text{ in } U,\ Z = \nabla u / |\nabla u| \text{ in } \overline{U}^c,\ \operatorname{div} Z \in D^1_0 \right\}.$$

Since $\operatorname{div} Z$ is locally integrable, $Z \cdot \nu$ does not jump across ∂U, where ν is the exterior unit normal of ∂U. In this case,

$$Z \cdot \nu = Z \cdot \nabla u / |\nabla u| = -1 \quad \text{on} \quad \partial U. \tag{1.11}$$

Since $\nabla \operatorname{div} Z$ does not have a singular part, $\operatorname{div} Z$ does not jump across ∂U. In this case,

$$\operatorname{div} Z = -\operatorname{div} \nu(x) \quad \text{on} \quad \partial U. \tag{1.12}$$

However, $v = \Delta \operatorname{div} Z$ may have a non-zero singular part concentrated on ∂U even if $v = v_0$, i.e., v is the minimizer. This phenomenon is observed in [19], [20], [11] in a one-dimensional periodic setting. Different from the second-order problem, this causes expansion or shrinking of the ball when the solution u is of the form $u(t,x) = a(t)\mathbf{1}_{B_{R(t)}}$. If $u > 0$ outside U, the minus in (1.11) (1.12) should be replaced by the plus.

If $\partial^0_{D^{-1}} TV(u)$ is constant on $B_{R(t)}$ and $\left(\overline{B_{R(t)}}\right)^c$, this property is preserved under the evolution, which leads us to definition of calibrability. We say that U (with negative signature) is *calibrable* if $\Delta \operatorname{div} Z_0$ is constant on U, where Z_0 belongs to

$$\arg\min \left\{ \|\nabla \operatorname{div} Z\|_{L^2(U)} \;\middle|\; Z \text{ satisfies (1.11), (1.12) and } |Z| \leq 1,\ \text{a.e. } x \in U \right\}. \tag{1.13}$$

Note that the value of $\partial^0_{D^{-1}} TV(u)$ on U is determined by U and its signature does not depend on particular value of u. In the second-order problem, we say that U (with negative signature) is calibrable if $-\operatorname{div} \widetilde{Z}_0$ is constant on U for

$$\widetilde{Z}_0 = \arg\min \left\{ \| \operatorname{div} Z\|_{L^2(U)} \;\middle|\; z \text{ satisfies (1.11) and } |Z| \leq 1,\ \text{a.e. } x \in U \right\}$$

as in [16]. This is formally equivalent to *–calibrability* in [4], [3].

In our fourth-order problem, if one minimizes $\|\nabla \operatorname{div} Z\|_{L^2(U)}$ under (1.12) but without the constraint $|Z| \leq 1$, a minimizer Z_1 must satisfy

$$-\Delta w = \lambda \quad \text{in} \quad U \tag{1.14}$$

$$w = -\operatorname{div} \nu \quad \text{on} \quad \partial U \tag{1.15}$$

for $w = \operatorname{div} Z_1$ with some constant λ. If U is bounded, this constant λ is determined by (1.11) since

$$\int_U w \, \mathrm{d}\mathcal{L}^n = \int_U \operatorname{div} Z_1 \, \mathrm{d}\mathcal{L}^n = \int_{\partial U} Z_1 \cdot \nu \, \mathrm{d}\mathcal{H}^{n-1} = -\mathcal{H}^{n-1}(\partial U), \tag{1.16}$$

where $\mathcal{H}^{n-1}$ denotes the $n-1$ dimensional Hausdorff and $\mathcal{L}^n$ denotes the Lebesgue measure in $\mathbb{R}^n$. For bounded U, we prove that if $v = \Delta \operatorname{div} Z_2$ satisfies (1.14), (1.15), (1.16) for $w = \operatorname{div} Z_2$, and moreover it satisfies (1.11) and $|Z_2| \leq 1$ in U a. e., then v is the minimizer and U is calibrable (Theorem 26). We call such vector field Z_2 a calibration for U.

In the radially symmetric setting, it is not difficult to show that Z_0 in (1.13) can be chosen in the form $z(|x|)\frac{x}{|x|}$. Indeed, if Z_0 is belongs to the set of minimizers (1.13), then its rotational average $\overline{Z_0}$ belongs to (1.13) as well, because averaging preserves (1.11), (1.12) and the inequality $|Z| \leq 1$. Since the angular part of $\overline{Z_0}$ does not contribute to the divergence, it is possible to delete this part (Lemma 30). We thus conclude that there is an element of (1.13) of form $Z(x) = z(|x|)\frac{x}{|x|}$. Thus, the equation (1.14) can be written as the third-order ODE of the form

$$-r^{1-n}\left(r^{n-1}\left(r^{1-n}(r^{n-1}z)'\right)'\right)' = \lambda \tag{1.17}$$

since $\operatorname{div} Z = r^{1-n}(r^{n-1}z)'$. If U is B_R with negative signature, condition (1.11) implies

$$z(R) = -1. \tag{1.18}$$

Since $\operatorname{div} Z = z' + (n-1)z/r$, condition (1.12) implies that

$$z'(R) = 0. \tag{1.19}$$

Solving (1.17) under the assumption that z is smooth near zero under conditions (1.18), (1.19), we eventually get a unique solution (1.17)–(1.19) of the form

$$z(r) = \frac{1}{2}\left(\frac{r}{R}\right)^3 - \frac{3}{2}\frac{r}{R}, \quad \lambda = -\frac{n(n+2)}{R^3}$$

for all $n \geq 1$. It is easy to see that $Z(x) = z(|x|)\frac{x}{|x|}$ satisfies the constraint $|Z| \leq 1$ in B_R. We conclude that all balls are calibrable. More careful argument is necessary, but we are able to discuss calibrability of an annulus as well as a complement of a ball.

Theorem 3. *(i) All balls are calibrable for all $n \geq 1$.*

(ii) All complement of balls are calibrable except $n = 2$.

(iii) If $n = 2$, all complement of balls are not calibrable.

(iv) All annuli (with definite signature) are calibrable except in $n = 2$.

(v) For $n = 2$, there is $Q_* > 1$ such that an annulus (with definite signature) is calibrable if and only if the ratio of the exterior radius over the interior radius is smaller than or equal to Q_*. In other words, $A_{R_0}^{R_1}$ is calibrable if and only if $R_1/R_0 \leq Q_*$.

Theorem 3(v) is consistent with (iii) since $R_1 \to \infty$ implies $A_{R_0}^{R_1}$ converges to $\overline{B_{R_0}}^c$, a complement of the closure of the ball B_{R_0}. Note that in the case of an annulus, there is a possibility we take a signature which is different on the exterior boundary ∂B_{R_1} and the interior boundary ∂B_{R_0}. We also study such indefinite cases.

We now calculate an explicit solution of (1.1) starting from $u_0 = a_0 \mathbf{1}_{B_{R_0}}$. We first discuss the case $n \neq 2$. Since a ball and its complement is calibrable, the solution is of the form

$$u(t,x) = a(t)\mathbf{1}_{B_{R(t)}}. \tag{1.20}$$

We take the (radial) calibration Z_{in} in $B_{R(t)}$ and Z_{out} in $\mathbb{R}^n \backslash \overline{B_{R(t)}}$ and set

$$Z(x,t) = \begin{cases} Z_{in}(x), & x \in B_{R(t)} \\ Z_{out}(x), & x \in \mathbb{R}^n \backslash \overline{B_{R(t)}}. \end{cases}$$

Here $Z_{out}(x) = z_{out}(|x|)\frac{x}{|x|}$ can be calculated as

$$z_{out}(r) = -\frac{n-1}{2}\left(\frac{r}{R}\right)^3 + \frac{n-3}{2}\left(\frac{r}{R}\right)^{1-n}$$

while, as we already discussed, z_{in} for $Z_{in}(x) = z_{in}(|x|)\frac{x}{|x|}$ is of the form

$$z_{in}(r) = \frac{1}{2}\left(\frac{r}{R}\right)^3 - \frac{3}{2}\frac{r}{R}.$$

This Z satisfies (1.9) and (1.10), and moreover $\operatorname{div} Z \in D_0^1$ for any $t > 0$. Moreover, $\operatorname{div} Z$ is continuous across $\partial B_{R(t)}$. However, $\nabla \operatorname{div} Z$ may jump across $\partial B_{R(t)}$. Actually,

$$-\Delta \operatorname{div} Z = \lambda \mathbf{1}_{B_{R(t)}} + \nu \cdot (\nabla \operatorname{div} Z_{in} - \nabla \operatorname{div} Z_{out})\delta_{\partial B_{R(t)}},$$

where $\delta_\Gamma(\varphi) = \int_\Gamma \varphi \, d\mathcal{H}^{n-1}$ or $\delta_\Gamma = \mathcal{H}^{n-1} \llcorner \Gamma$ for a hypersurface Γ and ν is the exterior unit normal of $\partial B_{R(t)}$, i.e., $\nu = x/R(t)$. Here $\lambda = -\frac{n(n+2)}{R^3}$. By a direct calculation, the quantity $\nu \cdot (\nabla \operatorname{div} Z_{in} - \nabla \operatorname{div} Z_{out}) = -\frac{n(n-4)}{R^2}$. Since $u_t = -\Delta \operatorname{div} Z$, by

$$\partial_t \left(a \mathbf{1}_{B_R} \right) = \frac{da}{dt} \mathbf{1}_{B_R} + a \frac{dR}{dt} \delta_{\partial B_R},$$

we conclude that

$$\frac{da}{dt} = -\frac{n(n+2)}{R^3}, \quad \frac{dR}{dt} = -\frac{n(n-4)}{aR^2}.$$

Since

$$\frac{d}{dt}(aR^3) = -n(n+2) - 3n(n-4) = -n(4n-10),$$

an explicit form of a solution is given as

$$a(t) = a_0 \left(1 - \frac{n(4n-10)}{a_0 R_0^3} t \right)^{\frac{n+2}{4n-10}}, \quad R(t) = R_0 \left(1 - \frac{n(4n-10)}{a_0 R_0^3} t \right)^{\frac{n-4}{4n-10}}.$$

As noticed earlier, in the case $n = 2$, the complement of the disk is not calibrable. If u is a radially strictly decreasing function outside B_R, we expect $Z_{out}(x) = -x/|x|$ for $|x| > R(t)$. In [14], it is proposed that a solution u to (1.1) must satisfy

$$u_t = -\Delta \operatorname{div} Z_{out}.$$

Since $\operatorname{div} Z_{out} = -(n-1)/|x|^2$ and $\nabla \operatorname{div} Z_{out} = \frac{(n-1)x}{|x|^3}$, this implies

$$u_t(t,x) = -\frac{(n-1)(n-3)}{|x|^3}, \quad x \in \left(\overline{B_{R(t)}} \right)^c = \mathbb{R}^n \backslash \overline{B_{R(t)}}. \tag{1.21}$$

In the case $n = 2$, $\nabla \operatorname{div} Z_{out} \in L^2 \left(\left(\overline{B_{R(t)}} \right)^c \right)$ so Z_{out} is a Cahn-Hoffman vector field.

If we start with $u_0 = a_0 \mathbf{1}_{B_{R_0}}$ with $a_0 > 0$ for $n = 2$, the expected form of a solution is

$$u_t(t,x) = a(t) \mathbf{1}_{B_{R(t)}} + \frac{t}{|x|^3} \mathbf{1}_{\overline{B_{R(t)}}^c}, \tag{1.22}$$

where

$$\frac{da}{dt} = -\frac{2 \cdot 4}{R^3}, \quad \left(a(t) - \frac{t}{R(t)^3} \right) \frac{dR}{dt} = \frac{2 \cdot 2}{R^2}. \tag{1.23}$$

Analyzing this ODE system, we can deduce qualitative properties of the solution. Summing up our results yields

Theorem 4. *Let $u_0 = a_0 \mathbf{1}_{B_{R_0}}$ with $a_0 > 0$.*

If $n \neq 3$, then the solution u to (1.1) with initial datum u_0 is of the form

$$u(t,x) = a(t) \mathbf{1}_{B_R(t)} \quad \text{for} \quad t < t_* = a_0 R_0^3 \,/\, (n(4n-10))$$

and $u(t,x) \equiv 0$ for $t \geq t_*$. (The time t_* is called the extinction time.) Moreover, $a(t)$ is decreasing and $a(t) \to 0$ as $t \uparrow t_*$.

(i) $R(t)$ is increasing and $R(t) \to \infty$ as $t \uparrow t_*$ for $n = 3$.

(ii) $R(t) = R_0$ for $n = 4$.

(iii) $R(t)$ is decreasing and $R(t) \to 0$ as $t \uparrow t_*$ for $n \geq 5$.

If $n = 2$, then the solution is not a characteristic function for $t > 0$. It is of the form (1.21) and moves by (1.23). In particular, there is no extinction time, $R(t)$ is increasing and $a(t)$ is decreasing. Moreover, $R(t) \to \infty$ and $a(t) \to 0$ as $t \to \infty$. The gap $a(t) - \frac{t}{R(t)^3}$ is always positive.

If $n = 1$, then the solution is of the form $u(t,x) = a(t) \mathbf{1}_{B_R(t)}$ for $t > 0$. Moreover, $R(t)$ is increasing and $a(t)$ is decreasing with $R(t) \to \infty$ and $a(t) \to 0$ as $t \to \infty$.

We note that the infinite extinction time observed in $n \le 2$ is related to the fact that 0 is not an element of the affine space $u_0 + D^{-1}$ where the flow lives if $\int u_0 \neq 0$. In [13], finite time extinction for solution to (1.1) is proved in a periodic setting for average zero initial data when the space dimension $n \le 4$ Our result is unrelated to their result because we consider (1.1) in $\mathbb{R}^n$.

The formula (1.21) does not give a solution to (1.1) when $n \ge 4$ since $\nabla \operatorname{div} Z_{out}$ does not belong to $L^2\left(\left(\overline{B_{R(t)}}\right)^c\right)$. In the case $n = 3$, this formula is consistent with our definition. If we consider u_0 strictly radially decreasing for $|x| > R_0$ and $u_0(x) = u_*$ for $|x| \le R_0$, then u_0 does not belong to the domain of $\partial_{D^{-1}} TV$ for $n \ge 4$. In other words, there is no Cahn-Hoffman vector field.

These results contrast with the second-order total variation flow

$$u_t = \operatorname{div}\left(\nabla u / |\nabla u|\right).$$

In the second-order problem, a ball and an annulus are always calibrable with their complements, see e. g. [3] or [16, Section 5]. Furthermore, $u_t(t, \cdot)$ is a locally integrable function without singular part for $t > 0$. Thus, for example, the solution starting from $u_0 = a_0 \mathbf{1}_{B_{R_0}}$ ($a_0 > 0$) must be $u(t,x) = a(t)\mathbf{1}_{B_{R_0}}$ with $a(t) = -\lambda t + a_0$, where λ is the Cheeger ratio, i. e. $\lambda = \mathcal{H}^{n-1}(\partial B_{R_0})/\mathcal{L}^n(B_{R_0})$. In particular, the extinction time t_* equals $t_* = a_0/\lambda$.

We conclude this paper by deriving a system of ODEs prescribing the solution in the case when the initial datum is a piecewise constant, radially symmetric function, which we call a stack. To be precise, we say that $w \in E^{-1}$ is a stack if it is of the form

$$w = a^0 \mathbf{1}_{B_{R^0}} + a^1 \mathbf{1}_{A_{R^0}^{R^1}} + \ldots + a^{N-1} \mathbf{1}_{A_{R^{N-2}}^{R^{N-1}}} + a^N \mathbf{1}_{\mathbb{R}^n \setminus B_{R^{N-1}}},$$

$0 < R^0 < R^1 < \ldots < R^{N-1}$, $a^k \in \mathbb{R}$. In particular, we obtain

Theorem 5. *Let $n \neq 2$ and let u_0 be a stack. If u is the solution to* (1.1)*, then $u(t,\cdot)$ is a stack for $t > 0$.*

In the case $n = 2$, this result is no longer true, as evidenced by Theorem 4. However the solution can still be prescribed by a finite system of ODEs.

A total variation flow type equation

$$w_t = -\Delta \left(\operatorname{div} \frac{\nabla w}{|\nabla w|} + \beta \operatorname{div}\left(\nabla w |\nabla w|\right)\right) \tag{1.24}$$

was introduced by [30] to describe the height of crystal surface moved by relaxation dynamics below the roughening temperature, where $\beta > 0$. For this equation, characterization of the subdifferential of the corresponding energy was given by Y. Kashima in periodic setting [19], [20] and under Dirichlet condition on a bounded domain [20]. The speed of a facet (a flat part of the graph) is calculated for $n = 1$ in [19] and for a ball with the Dirichlet condition under radial symmetry [20]. Different from the second-order problem, the speed of a facet is determined not only by the shape of facet. Also it has been already observed in [19], that the minimal section may have a delta part although the behavior of the corresponding solution was not studied there. A numerical computation was given in [22]. The equation (1.24) was derived as a continuum limit of models describing motion of steps on crystal surface as discussed in [26], where numerical simulation was given; see also [21].

In [7], a crystalline diffusion flow was proposed and calculated numerically. In a special case, it is of the form $w_t = -\partial_x^2 \left(W'(w_x)\right)$, where W is a piecewise linear convex function, when the curve is given as the graph of a function. This equation was analyzed in [12] in a class of piecewise linear (in space) solutions.

Fourth-order equations of type (1.1) were proposed for image denoising as an improvement over the second-order total variation flow. For example, the equation

$$w_t = -\Delta \operatorname{div}\left(\nabla w / |\nabla w|\right) + \lambda (f - w),$$

where f is an original image which is given and $\lambda > 0$, corresponds to the Osher-Solé-Vese model [27]. The well-posedness of this equation was proved by using the Galerkin method by [8].

For (1.1), an extinction time estimate was given in [13] for $n = 1, 2, 3, 4$ in the periodic setting. It was extended to the Dirichlet problem in a bounded domain by [14]. In the review paper [11], it was proved that the solution u of (1.1) in $n = 1$ may become discontinuous instantaneously even if the initial datum is Lipschitz continuous, because the speed may have a delta part.

There are a few numerical studies for (1.1) in the periodic setting. A duality-based numerical scheme which applies the forward-backward splitting has been proposed in [15]. A split Bregman method was adjusted to (1.1) and also (1.24) in [17]. In these methods, the singularity of the equation at $\nabla u = 0$ is not regularized. However, all above studies deal with either periodic, Dirichlet or Neumann boundary condition for a bounded domain. It has never been rigorously studied in $\mathbb{R}^n$, although in [14] there are some preliminary calculations for radial solution in $\mathbb{R}^n$.

This paper is organized as follows. In Section 2, we discuss basic properties of the total variation on D^{-1}, notably we show strict density of $C^\infty_{c,av}$. In Section 3, we characterize the subdifferential of TV in D^{-1}, adjusting a duality argument in [3]. In Section 4, we give a rigorous definition of a solution to (1.1) in $\mathbb{R}^n$. In Section 5, we introduce the notion of calibrability. In Section 6, we discuss calibrability of rotationally symmetric sets in $\mathbb{R}^n$. In Section 7, we study solutions emanating from piecewise constant, radially symmetric data.

2 The total variation functional on D^{-1}

In this section, we give a rigorous definition of the total variation TV on D^{-1} and relate it to the usual total variation defined on L^1_{loc}. The main tool that we use here as well as in the following section is an approximation lemma, which for a given $w \in D^{-1}$ produces a sequence of nice functions $w_k \in D^{-1}$ that converges to w in D^{-1} and $TV(w_k) \to TV(w)$.

Let us denote

$$X_1 = \left\{\psi \in C^\infty_c(\mathbb{R}^n, \mathbb{R}^n),\ \|\psi\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)} \le 1\right\}.$$

We define $TV\colon D^{-1}(\mathbb{R}^n) \to [0, \infty]$ by

$$TV(u) = \sup_{\psi \in X_1} \langle u, \operatorname{div} \psi\rangle.$$

Let us compare this definition with the usual total variation, which we denote here by $\overline{TV}\colon L^1_{loc}(\mathbb{R}^n) \to [0, \infty]$, defined by

$$\overline{TV}(u) = \sup_{\psi \in X_1} \int_{\mathbb{R}^n} u \operatorname{div} \psi.$$

First of all, as in the case $\overline{TV}$, we easily check that TV is lower semicontinuous with respect to the weak-* (and, a fortiori, strong) convergence in $D^{-1}(\mathbb{R}^n)$. Indeed, if $v_k \overset{*}{\rightharpoonup} v$ in $D^{-1}(\mathbb{R}^n)$,

$$TV(v) = \sup_{\psi \in X_1} \{\langle v, \operatorname{div} \psi\rangle\} = \sup_{\psi \in X_1} \liminf_{k\to\infty} \{\langle v_k, \operatorname{div} \psi\rangle\} \le \liminf_{k\to\infty} \sup_{\psi \in X_1} \{\langle v_k, \operatorname{div} \psi\rangle\} = \liminf_{k\to\infty} TV(v_k).$$

In fact, we have

Lemma 6. *We have $D(TV) \subset L^1_{loc}$, and so $D(TV) \subset D(\overline{TV})$ with TV and $\overline{TV}$ coinciding on $D(TV)$. In particular, if $n \ge 2$, $D(TV) \subset L^{1^*}(\mathbb{R}^n)$. If $n = 1$,*

$$D(TV) \subset L^\infty_0(\mathbb{R}) = \{w \in L^\infty(\mathbb{R})\colon \lim_{x\to\pm\infty} w(x) = 0\}.$$

The proof of this fact is a consequence of the lemma below and we postpone it.

Lemma 7. *For any $w \in D^{-1}(\mathbb{R}^n)$ there exists a sequence $w_k \in C^\infty_{c,av}(\mathbb{R}^n)$ such that*

$$w_k \to w \text{ in } D^{-1}(\mathbb{R}^n)$$

and

$$TV(w_k) \to TV(w).$$

To prove it, we will use a special choice of cut-off function and associated variant of the Sobolev-Poincaré inequality. For $R>0$, let us denote by ϑ_R the element of minimal norm in $D^1_0(\mathbb{R}^n)$ among those $w\in D^1_0(\mathbb{R}^n)$ that satisfy $w(x)=1$ if $|x|\le \frac{R}{2}$, $w(x)=0$ if $|x|\ge R$. It is an easy exercise to show that for $\frac{R}{2}\le |x|\le R$

$$\vartheta_R(x)=\left(2^{n-2}-1\right)^{-1}\left(\left(\frac{|x|}{R}\right)^{2-n}-1\right) \text{ if } n\neq 2, \quad \vartheta_R(x)=\frac{\log\frac{R}{|x|}}{\log 2} \text{ if } n=2.$$

In either case,

$$\nabla\vartheta_R(x)=C_n\frac{|x|^{-n}x}{R^{2-n}} \text{ if } \frac{R}{2}\le |x|\le R. \tag{2.1}$$

Lemma 8. *If $p\in[1,n[$ and $q\in[1,p^*]$, then for all $w\in C^1(\mathbb{R}^n)$, $R>0$ there holds*

$$\left\| w-\frac{\int\vartheta_R w}{\int\vartheta_R}\right\|_{L^q(B_R)}\le CR^{1+\frac{n}{q}-\frac{n}{p}}\|\nabla w\|_{L^p(B_R)} \tag{2.2}$$

with $C=C(n,p)$ and

$$\|\nabla\vartheta_R\|_{L^p(\mathbb{R}^n)}=CR^{-\frac{1}{p}(n-1)(2-p)} \tag{2.3}$$

with a different $C=C(n,p)$.

Proof. Let $v\in C^1(\mathbb{R}^n)$. Following the proof of the standard Poincaré inequality by contradiction using Rellich-Kondrachov theorem, we obtain

$$\left\| v-\frac{\int\vartheta_1 v}{\int\vartheta_1}\right\|_{L^p(B_1)}\le C\|\nabla v\|_{L^p(B_1)}.$$

Applying the Sobolev inequality in B_1 to the function $v-\frac{\int\vartheta_1 v}{\int\vartheta_1}$, we upgrade this to

$$\left\| v-\frac{\int\vartheta_1 v}{\int\vartheta_1}\right\|_{L^q(B_1)}\le C\|\nabla v\|_{L^p(B_1)} \tag{2.4}$$

Next, let $v(x)=w(Rx)$ for a given $w\in C^1(\mathbb{R}^n)$. We observe that

$$\vartheta_1(x)=\vartheta_R(Rx)\quad\text{for } x\in\mathbb{R}^n$$

and so, by a change of variables $x=y/R$,

$$\int\vartheta_1=\frac{1}{R^n}\int\vartheta_R,\quad \int\vartheta_1 v=\frac{1}{R^n}\int\vartheta_R w.$$

Applying the same change of variables to both sides of (2.4) we conclude the proof of (2.2).

The proof of (2.3) is a matter of direct calculation. □

Let us now return to the proof of the approximation lemma.

Proof of Lemma 7. Given $w\in D^{-1}$, let

$$w_{\varepsilon,R}=\left(\varrho_\varepsilon * w-\frac{\int\vartheta_R\varrho_\varepsilon * w}{\int\vartheta_R}\right)\vartheta_R.$$

Equivalently, for $\varphi\in D^1_0(\mathbb{R}^n)$,

$$\langle w_{\varepsilon,R},\varphi\rangle=\left\langle w,\varrho_\varepsilon*\left(\left(\varphi-\frac{\int\vartheta_R\varphi}{\int\vartheta_R}\right)\vartheta_R\right)\right\rangle.$$

Denoting $\widetilde{w} = (-\Delta)^{-1}w$,

$$\langle w_{\varepsilon,R} - w, \varphi\rangle = \int \nabla\varrho_\varepsilon * \widetilde{w} \cdot \nabla\left(\left(\varphi - \frac{\int \vartheta_R\varphi}{\int \vartheta_R}\right)\vartheta_R - \varphi\right) + \int (\nabla\varrho_\varepsilon * \widetilde{w} - \nabla\widetilde{w})\cdot\nabla\varphi$$
$$= \int \nabla\varrho_\varepsilon * \widetilde{w}\cdot(\vartheta_R - 1)\nabla\varphi + \int \nabla\varrho_\varepsilon * \widetilde{w}\cdot\left(\varphi - \frac{\int\vartheta_R\varphi}{\int\vartheta_R}\right)\nabla\vartheta_R + \int(\nabla\varrho_\varepsilon * \widetilde{w} - \nabla\widetilde{w})\cdot\nabla\varphi.$$

We estimate the second term on the r. h. s. using the Poincaré inequality from Lemma 8, taking into account that the support of the integrand is contained in $\overline{A}_R$, where $A_R = B_R \setminus \overline{B}_{R/2}$,

$$\left|\int \nabla\varrho_\varepsilon * \widetilde{w}\cdot\left(\varphi - \frac{\int\vartheta_R\varphi}{\int\vartheta_R}\right)\nabla\vartheta_R\right| \le C\|\nabla\varrho_\varepsilon * \widetilde{w}\,\mathbf{1}_{A_R}\|_{L^2(\mathbb{R}^n)}\left\|\varphi - \frac{\int\vartheta_R\varphi}{\int\vartheta_R}\right\|_{L^2(\mathbb{R}^n)}\|\nabla\vartheta_R\|_{L^\infty(\mathbb{R}^n)}$$
$$\le C\|\nabla\varrho_\varepsilon * \widetilde{w}\,\mathbf{1}_{A_R}\|_{L^2(\mathbb{R}^n)}\|\nabla\varphi\|_{L^2(\mathbb{R}^n)}.$$

Thus,

$$\|w_{\varepsilon,R} - w\|_{D^{-1}(\mathbb{R}^n)} = \sup_{\|\varphi\|_{D_0^1(\mathbb{R}^n)}\le 1}\langle w_{\varepsilon,R} - w, \varphi\rangle$$
$$\le \|(1-\vartheta_R)\nabla\varrho_\varepsilon * \widetilde{w}\|_{L^2(\mathbb{R}^n)} + C\|\nabla\varrho_\varepsilon * \widetilde{w}\,\mathbf{1}_{A_R}\|_{L^2(\mathbb{R}^n)} + \|\nabla\varrho_\varepsilon * \widetilde{w} - \nabla\widetilde{w}\|_{L^2(\mathbb{R}^n)}$$

and so

$$\lim_{\varepsilon\to 0^+}\lim_{R\to\infty}\|w_{\varepsilon,R} - w\|_{D^{-1}(\mathbb{R}^n)} = 0. \tag{2.5}$$

Next we estimate $TV(w_{\varepsilon,R})$. Due to lower semicontinuity of TV, we can assume without loss of generality that $TV(w) < \infty$. First, we note that $\varrho_\varepsilon * w \in D^{-1}(\mathbb{R}^n)\cap C^\infty(\mathbb{R}^n)$ ([29], [28]) and

$$\|\varrho_\varepsilon * \psi\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)} \le \|\psi\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)}.$$

Thus, for any $\psi \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$,

$$TV(\varrho_\varepsilon * w) = \sup_{\psi\in X_1}\{\langle w, \operatorname{div}\ \varrho_\varepsilon * \psi\rangle\} \le TV(w).$$

In particular, this implies that $\nabla\varrho_\varepsilon * w \in L^1(\mathbb{R}^n)$ for $\varepsilon > 0$ and $\int|\nabla\varrho_\varepsilon * w| = TV(\varrho_\varepsilon * w) \le TV(w)$. By Lemma 8,

$$TV(w_{\varepsilon,R}) = \int|\nabla w_{\varepsilon,R}| \le \int\vartheta_R\,|\nabla\varrho_\varepsilon * w| + \int\left|\left(\varrho_\varepsilon * w - \frac{\int\vartheta_R\varrho_\varepsilon * w}{\int\vartheta_R}\right)\nabla\vartheta_R\right|$$
$$\le \int|\nabla\varrho_\varepsilon * w| + C\left\|\varrho_\varepsilon * w - \frac{\int\vartheta_R\varrho_\varepsilon * w}{\int\vartheta_R}\right\|_{L^{1^*}(\mathbb{R}^n)}\|\nabla\vartheta_R\|_{L^n(\mathbb{R}^n)}$$
$$\le \|\nabla\varrho_\varepsilon * w\|_{L^1(\mathbb{R}^n)} + C\|\nabla\varrho_\varepsilon * w\|_{L^1(\mathbb{R}^n)}R^{-\frac{(n-1)(n-2)}{n}} \le \left(1 + CR^{-\frac{(n-1)(n-2)}{n}}\right)TV(w). \tag{2.6}$$

If $n \ge 3$, together with lower semicontinuity of TV, this yields

$$\lim_{(\varepsilon,R)\to(0,\infty)} TV(w_{\varepsilon,R}).$$

Taking into account (2.5), by a diagonal procedure we can select sequences (ε_k), (R_k) such that $w_k := w_{\varepsilon_k,R_k}$ satisfies both requirements in the assertion. On the other hand, if $n = 1$ or $n = 2$, (2.6) only implies uniform boundedness of $TV(\nabla w_{\varepsilon,R})$.

Let now $n = 2$. Since $w_{\varepsilon,R}$ have compact support, uniform boundedness of $TV(\nabla w_{\varepsilon,R})$ implies uniform bound on $w_{\varepsilon,R}$ in $L^2(\mathbb{R}^n)$ by the Sobolev inequality. As $w_{\varepsilon,R}$ converges to $\varrho_\varepsilon * w$ in

$D^{-1}(\mathbb{R}^n)$, we have $\varrho_\varepsilon * w \in L^2(\mathbb{R}^n)$. This allows us to improve (2.6):

$$TV(w_{\varepsilon,R}) = \int |\nabla w_{\varepsilon,R}| \leq \int \vartheta_R \, |\nabla \varrho_\varepsilon * w| + \int |\varrho_\varepsilon * w \nabla \vartheta_R| + \frac{\int \vartheta_R \varrho_\varepsilon * w}{\int \vartheta_R} \int |\nabla \vartheta_R|$$
$$\leq \|\nabla \varrho_\varepsilon * w\|_{L^1(\mathbb{R}^2)} + C\|\varrho_\varepsilon * w \, \mathbf{1}_{A_R}\|_{L^2(\mathbb{R}^2)} \|\nabla \vartheta_R\|_{L^2(\mathbb{R}^2)} + C\frac{\|\vartheta_R\|_{D_0^1(\mathbb{R}^2)} \|\varrho_\varepsilon * w\|_{D^{-1}(\mathbb{R}^2)}}{\|\vartheta_R\|_{L^1(\mathbb{R}^2)}} \|\nabla \vartheta_R\|_{L^1(\mathbb{R}^2)}$$
$$\leq TV(w) + C\|\varrho_\varepsilon * w \, \mathbf{1}_{A_R}\|_{L^2(\mathbb{R}^2)} + \frac{C}{R}\|\varrho_\varepsilon * w\|_{D^{-1}(\mathbb{R}^2)}. \quad (2.7)$$

The r. h. s. of (2.7) converges to $TV(w)$ as $R \to \infty$ and we conclude as before.

Next, consider $n = 1$. In this case, finiteness of $TV(w)$ implies that $\varrho_\varepsilon * w \in L^\infty(\mathbb{R}^n)$ and there exist $g_\varepsilon^\pm \in \mathbb{R}$ such that

$$\lim_{x \to \pm\infty} \varrho_\varepsilon * w(x) = g_\varepsilon^\pm.$$

Now, let $\eta_R^\pm$ be the element of minimal norm in $D_0^1(\mathbb{R})$ under constraints

$$\eta_R^\pm(\pm x) = 1 \text{ if } x \in [2R, 3R], \quad \eta_R^\pm(\pm x) = 0 \text{ if } x \notin]R, 4R[.$$

(Clearly, $\eta_R^\pm$ is a continuous, piecewise affine function.) We have

$$|\langle \varrho_\varepsilon * w, \eta_R^\pm \rangle| \leq \|\varrho_\varepsilon * w\|_{D^{-1}(\mathbb{R})} \|\eta_R^\pm\|_{D_0^1(\mathbb{R})} \to 0 \text{ as } R \to \infty.$$

On the other hand, since $\eta_R^\pm$ are compactly supported and $\varrho_\varepsilon * w$ coincides as distribution with a locally integrable function, we can calculate

$$\langle \varrho_\varepsilon * w, \eta_R^\pm \rangle = \int \varrho_\varepsilon * w \, \eta_R^\pm \to \infty \cdot g_\varepsilon^\pm \text{ as } R \to \infty,$$

so $g_\varepsilon^\pm = 0$. Therefore, we can estimate

$$TV(w_{\varepsilon,R}) = \int |\nabla w_{\varepsilon,R}| \leq \int \vartheta_R \, |\nabla \varrho_\varepsilon * w| + \int |\varrho_\varepsilon * w \nabla \vartheta_R| + \frac{\int \vartheta_R \varrho_\varepsilon * w}{\int \vartheta_R} \int |\nabla \vartheta_R|$$
$$\leq TV(w) + \frac{2}{R} \int_{A_R} |\varrho_\varepsilon * w| + 2\frac{\int \vartheta_R \varrho_\varepsilon * w}{\int \vartheta_R}. \quad (2.8)$$

Since we have shown that $\varrho_\varepsilon * w(x) \to 0$ as $x \to \pm\infty$, the averages on the r. h. s. converge to 0 and we conclude as before.

□

As a first application of the approximation lemma, we demonstrate Lemma 6 announced before.

Proof of Lemma 6. Let $w \in D(TV)$ and let $(w_k) \subset C_{c,av}^\infty(\mathbb{R}^n)$ be the sequence provided by Lemma 7. Let first $n > 1$. Since ∇w_k is uniformly bounded in $L^1(\mathbb{R}^n, \mathbb{R}^n)$, by the Sobolev embedding w_k is uniformly bounded in $L^{1^*}(\mathbb{R}^n)$. Therefore, $w \in L^{1^*}(\mathbb{R}^n)$.

In case $n = 1$, since ∇w_k are compactly supported and uniformly bounded in $L^1(\mathbb{R})$, w_k is uniformly bounded in $L^\infty(\mathbb{R})$. From these two bounds it follows that $w \in L^\infty(\mathbb{R})$ $\nabla w \in M(\mathbb{R})$, $w_k \to w$ in $L_{loc}^1(\mathbb{R})$. Identifying w with its semicontinuous representative, there exist $g^\pm$ such that $w(x) \to g^\pm$ as $x \to \pm\infty$. Suppose that $g^+ \neq 0$ and let M be such that

$$\int_{]M,+\infty[} |\nabla w| < \frac{|g^+|}{2}.$$

In particular, $|w(x)| > \frac{|g^+|}{2}$ for $x > M$. Take any $x_0 > M$ such that $w_k(x_0) \to w(x_0)$ and $|\nabla w|(\{x_0\}) = 0$. Then

$$\begin{aligned}\liminf_{k\to\infty} \int_{\mathbb{R}} |\nabla w_k| &\geq \liminf_{k\to\infty} \int_{]-\infty,x_0[} |\nabla w_k| + \liminf_{k\to\infty} \int_{]x_0,\infty[} |\nabla w_k| \\ &\geq \int_{]-\infty,x_0[} |\nabla w| + \liminf_{k\to\infty} |w_k(x_0)| = \int_{\mathbb{R}} |\nabla w| - \int_{]x_0,\infty[} |\nabla w| + |w(x_0)| > \int_{\mathbb{R}} |\nabla w|\end{aligned}$$

which contradicts strict convergence of w_k, hence $g^+ = 0$. Similarly we prove that $g^- = 0$. □

3 Subdifferential of the total variation

We give a characterization of the total variation in the space $D^{-1} = D^{-1}(\mathbb{R}^n)$. The basic idea of the proof is a duality argument, which has been carried out in the case of L^2 subdifferentials. In the case of L^2 setting, the idea goes back to the unpublished note of F. Alter and a detailed proof is given in [3]. Let $\mathcal{E}$ be a functional on a real Hilbert space H equipped with an inner product $(\cdot,\cdot)_H$. The main idea is to characterize the subdifferential $\partial\mathcal{E}$ by the polar $\mathcal{E}^0$ of $\mathcal{E} : H \to [-\infty,\infty]$ is defined by

$$\mathcal{E}^0(v) := \sup\left\{(u,v)_H \;\middle|\; u \in H,\ \mathcal{E}(u) \leq 1\right\} = \sup\left\{(u,v)/\mathcal{E}(u) \;\middle|\; u \in D(\mathcal{E}),\ \mathcal{E}(u) \neq 0\right\},$$

where $D(\mathcal{E}) = \left\{u \in H \;\middle|\; |\mathcal{E}(u)| < \infty\right\}$. We first recall a lemma [3, Lemma 1.7].

Lemma 9. *Let $\mathcal{E}$ be convex. Assume that $\mathcal{E}$ is positively one-homogeneous, i.e.,*

$$\mathcal{E}(\lambda u) = \lambda \mathcal{E}(u)$$

for all $\lambda > 0$, $u \in H$. Then, $v \in \partial\mathcal{E}(u)$ if and only if $\mathcal{E}^0(v) \leq 1$ and $(u,v)_H = \mathcal{E}(u)$.

Proof. We give a proof for completeness. Since $\mathcal{E}$ is convex and positively one-homogeneous,

$$\mathcal{E}(u+w) \leq \mathcal{E}(u) + \mathcal{E}(w). \tag{3.1}$$

Indeed,

$$\frac{1}{2}\mathcal{E}(u+w) = \mathcal{E}\left((u+w)/2\right) \leq \left(\mathcal{E}(u) + \mathcal{E}(w)\right)/2.$$

"Only if" part. If $v \in \partial\mathcal{E}(u)$, then

$$\mathcal{E}(w) \geq (v,w)_H \quad \text{for all} \quad w \in H \tag{3.2}$$

which is equivalent to $\mathcal{E}^0(v) \leq 1$. Indeed, by (3.1) and the definition of v, we see

$$\mathcal{E}(w) \geq \mathcal{E}(u+w) - \mathcal{E}(u) \geq (v,w)_H \tag{3.3}$$

for all $w \in H$.

The second identity $(u,v)_H = \mathcal{E}(u)$ is nothing but the Euler equation for a homogeneous functional. We take $w = u$ in (3.3) to get

$$\mathcal{E}(u) \geq (v,u)_H.$$

If we take $w = -u/2$, then

$$\frac{1}{2}\mathcal{E}(u) - \mathcal{E}(u) \geq -(v,u)_H/2,$$

which implies $\mathcal{E}(u) \leq (v,u)_H$. Thus $\mathcal{E}(u) = (v,u)_H$.

"If" part. Since $\mathcal{E}^0(v) \le 1$, we see

$$\mathcal{E}(u+w) \ge (v, u+w)_H.$$

Thus,

$$\mathcal{E}(u+w) - \mathcal{E}(u) \ge (v, u+w)_H - \mathcal{E}(u) = (v, u+w)_H - (v, u)_H = (v, w)_H$$

by the Euler equation $\mathcal{E}(u) = (v, u)_H$, i.e. $v \in \partial\mathcal{E}(u)$. □

Remark 10. *By general theory of convex functionals, we know that*

$$(\mathcal{E}^0)^0 = \mathcal{E}$$

if $\mathcal{E}$ is a lower semicontinuous, convex, positively one-homogeneous functional [3, Proposition 1.6].

This property is essential for the proof of

Theorem 11. *Let $\Psi\colon D^{-1} \to [0, \infty]$ by defined by*

$$\Psi(v) = \inf\left\{ \|z\|_\infty \;\middle|\; v = \Delta \operatorname{div} Z,\ Z \in L^\infty(\mathbb{R}^n),\ \operatorname{div} Z \in D_0^1 \right\}.$$

Then $(TV)^0 = \Psi$.

Remark 12. *(i) By definition, Ψ is a convex, lower semi-continuous, positively one-homogeneous function, so $(\Psi^0)^0 = \Psi$.*

(ii) if $\Psi(v) < \infty$, the infimum is attained. Theorem 11 together with Lemma 9 implies the following characterization of the subdifferential of TV.

Theorem 13. *An element $v \in D^{-1}$ belongs to $\partial TV(u)$ if and only if there is $Z \in L^\infty(\mathbf{R}^n)$ with $\operatorname{div} Z \in D_0^1$ such that*

(i) $|Z| \le 1$

(ii) $v = \Delta \operatorname{div} Z$

(iii) $-\langle u, \operatorname{div} Z\rangle = TV(u)$.

Proof. By Lemma 9 and Theorem 11,

$$v \in \partial TV(u) \iff \Psi(v) \le 1 \text{ and } (v, u)_{D^{-1}} = TV(u).$$

The property $\Psi(v) \le 1$ together with Remark 12(ii) implies (i), (ii) and $\operatorname{div} Z \in D_0^1$.

$$(v, u)_{D^{-1}} = \left\langle u, (-\Delta)^{-1} v \right\rangle = -\langle u, \operatorname{div} Z\rangle.$$

It is not difficult to check the converse. □

Proof of Theorem 11. The inequality $TV^0 \le \Psi$:

We take $v \in D^{-1}$ with $\Psi(v) < \infty$. By Remark 12(ii), there is $Z \in L^\infty(\mathbb{R}^n)$ with $v = \Delta \operatorname{div} Z$ with $\operatorname{div} Z \in D_0^1$ such that $\Psi(v) = \|Z\|_\infty$. By Lemma 7, there is $u_k \in C_{c,av}^\infty$ such that $TV(u_k) \to TV(u)$, $u_k \to u$ in D^{-1}. We observe that

$$\begin{aligned}(u_k, v)_{D^{-1}} &= \left\langle u_k, (-\Delta)^{-1} v \right\rangle = -\langle u_k, \operatorname{div} Z\rangle \\ &= \int_{\mathbb{R}^n} Z \cdot \nabla u_k dx \le \|Z\|_\infty TV(u_k).\end{aligned}$$

Sending $k \to \infty$, we conclude that

$$(u, v)_{D^{-1}} \le \|Z\|_\infty \quad \text{for all} \quad u \in D^{-1} \quad \text{with} \quad TV(u) \le 1.$$

By definition of Ψ, this implies $TV^0 \le \Psi$.

The inequality $\Psi \leq TV^0$:

By definition,

$$\begin{aligned} TV(u) &= \sup \left\{ \langle u, -\operatorname{div} z \rangle \;\middle|\; z \in C_c^\infty(\mathbb{R}^n),\ |z| \leq 1 \right\} \\ &= \sup \left\{ \frac{\langle u, -\operatorname{div} z \rangle}{\|z\|_\infty} \;\middle|\; z \in C_c^\infty(\mathbb{R}^n),\ z \neq 0 \right\}. \end{aligned}$$

Since

$$\langle u, -\operatorname{div} z \rangle = \left\langle u, (-\Delta)^{-1} \Delta \operatorname{div} z \right\rangle = (u, \Delta \operatorname{div} z)_{D^{-1}},$$

we observe that

$$\begin{aligned} TV(u) &= \sup \left\{ \frac{(u, \Delta \operatorname{div} z)_{D^{-1}}}{\|z\|_\infty} \;\middle|\; z \in C_c^\infty(\mathbb{R}^n),\ z \neq 0 \right\} \\ &\leq \sup \left\{ \frac{(u, \Delta \operatorname{div} z)_{D^{-1}}}{\Psi(\Delta \operatorname{div} z)} \;\middle|\; z \in C_c^\infty(\mathbb{R}^n),\ \Psi(\Delta \operatorname{div} z) \neq 0 \right\} \\ &\leq \Psi^0(u). \end{aligned}$$

This implies that $TV^0 \geq (\Psi^0)^0 = \Psi$. □

4 Definition of solution

For a gradient flow of a convex functional, there is a general theory initiated by Y. Kōmura [23] and developed by H. Brézis [6] and others. It is summarized as follows.

Proposition 14 ([6]). *Let H be a real Hilbert space. Let $\mathcal{E}$ be a lower semicontinuous, convex functional on H with values in $]-\infty, \infty]$. Assume that $D(\mathcal{E})$ is dense in H. Then, for any $u_0 \in H$, there exists a unique solution $u \in C([0,\infty[\,, H)$ which is absolutely continuous in (δ, T) (for any $\delta < T < \infty$) satisfying*

$$\begin{cases} u_t \in -\partial\mathcal{E}(u) & \text{a.e.} \quad t > 0 \\ u(0) = u_0. \end{cases}$$

Moreover,

$$\int_s^t \|u_t\|_H^2 \, d\tau \leq \mathcal{E}(u(s)) - \mathcal{E}(u(t)) \quad \text{for all} \quad t \geq s > 0.$$

If $\mathcal{E}(u_0) < \infty$, then $s = 0$ is allowed. In particular, $u_t \in L^2(0, \infty; H)$.

As in [2], this solution satisfies the evolutionary variational inequality

$$\frac{1}{2} \frac{d}{dt} \|u - f\|_H^2 \leq \mathcal{E}(f) - \mathcal{E}(u) \quad \text{a.e.} \quad t > 0$$

for any $f \in H$. Indeed, by definition, $u_t \in -\partial\mathcal{E}(u)$ is equivalent to saying

$$(-u_t, f - u)_H \leq \mathcal{E}(f) - \mathcal{E}(u)$$

for any $f \in H$. Since the left-hand side equals to

$$\frac{1}{2} \frac{d}{dt} \|u - f\|_H^2,$$

we obtain the evolutionary variational inequality. The evolutionary variational inequality is not only an equivalent formulation of the gradient flow $u_t \in -\partial\mathcal{E}(u)$, but also apply to a gradient flow of a metric space by replacing $\|u - f\|_H$ by distance between u and f; see [2] for the theory.

Since the subdifferential of TV in D^{-1} is now calculated, we are able to justify an explicit definition of a solution proposed in [14].

Theorem 15. *Assume that* $u \in C\left([0,\infty[\,, D^{-1}\right)$. *Then* u *is a solution of* $u_t \in -\partial_{D^{-1}} TV(u)$ *with* $u_0 = u(0)$ *in the sense of Proposition 14 if and only if there exists* $Z \in L^\infty\left(]0,\infty[\times\mathbb{R}^n\right)$ *satisfying*

$$\operatorname{div} Z \in L^2\left(\delta,\infty; D_0^1(\mathbb{R}^n)\right) \quad \textit{for any} \quad \delta > 0$$

such that for a.e. $t \in]0,\infty[$ *there holds*

$$u_t = -\Delta \operatorname{div} Z \quad \textit{in} \quad D^{-1}(\mathbb{R}^n),$$

$$|Z| \le 1 \quad \mathcal{L}^n\text{-}a.e.$$

and

$$\langle u, \operatorname{div} Z\rangle = -TV(u).$$

(If $TV(u_0) < \infty$, $\delta = 0$ *is allowed.)*

Proof. The Theorem essentially follows from Theorem 13. We only need to justify that a Cahn-Hoffman vector field Z defined separately for every time instance by Theorem 13 can be chosen to be jointly measurable, i. e. $Z \in L^\infty\left(]0,\infty[\times\mathbb{R}^n\right)$. As in the second-order case [3], this can be done by recalling that u is a limit of time-discretizations. □

Unfortunately, in the case $n \le 2$, the characteristic function 1_A of a set A of positive measure is not in D^{-1} since $\int 1_A\, dx \ne 0$. We shall define a new space containing 1_A as follows. We take a function $\psi \in L^2(\mathbb{R}^n)$ with compact support such that $\int_{\mathbb{R}^n} \psi = 1$. We introduce a vector space

$$E_\psi^{-1} = \left\{ w + c\psi \;\middle|\; w \in D^{-1}(\mathbb{R}^n),\ c \in \mathbb{R} \right\}.$$

This space is independent of the choice of ψ. Indeed, let $\psi_i \in L^2(\mathbb{R}^n)$ be compactly supported and $\int \psi_i\, dx = 1$ $(i = 1,2)$. An element $w + c\psi_1 \in E_{\psi_1}^{-1}$ can be rewritten as

$$w + c\psi_1 = w + c(\psi_1 - \psi_2) + c\psi_2.$$

The next lemma implies $q = c(\psi_1 - \psi_2) \in D^{-1}(\mathbb{R}^n)$ since $\int q = 0$. We then conclude that $w + c\psi_1 \in E_{\psi_2}^{-1}$.

Lemma 16. *Assume that* $n \le 2$. *A compactly supported function* $q \in L^2(\mathbb{R}^n)$ *belongs to* $D^{-1}(\mathbb{R}^n)$ *if and only if* $\int_{\mathbb{R}^n} q = 0$.

Proof. If $q \in D^{-1} \cap L^1$, then

$$\int q = \langle q, [1]\rangle = 0,$$

where $[1]$ stands for the element of D_0^1 whose representatives are 1 as well as 0.

Now suppose that a compactly supported function $q \in L^2(\mathbb{R}^n)$ satisfies $\int_{\mathbb{R}^n} q = 0$. Given a $[\varphi] \in D_0^1$, by the Poincaré inequality, we have for any $R > 0$, independently of the representative $\varphi \in D^1$,

$$\left\| \varphi - \frac{1}{|B_R|} \int_{B_R} \varphi \right\|_{L^2(B_R)} \le C_R \|\nabla\varphi\|_{L^2(B_R)} \ \le C_R \|[\varphi]\|_{D_0^1}.$$

In particular, $\varphi \in L^2_{loc}$. Taking into account this and the assumption $\int q = 0$, we see that the linear functional

$$\langle q, [\varphi]\rangle = \int q\,\varphi \tag{4.1}$$

on D_0^1 is well defined. Moreover, if $R > 0$ is large enough that $\operatorname{supp} q \subset B_R$,

$$\left| \int q\,\varphi \right| = \left| \int_{B_R} q \left(\varphi - \frac{1}{|B_R|}\int_{B_R} \varphi \right) \right| \le \|q\|_{L^2(B_R)} \left\| \varphi - \frac{1}{|B_R|}\int_{B_R}\varphi \right\|_{L^2(B_R)} \le C_R \|q\|_{L^2} \|[\varphi]\|_{D_0^1}.$$

Thus, the functional defined by (4.1) is bounded, i. e. $q \in D^{-1}$.

□

Since E_ψ^{-1} is independent of the choice of ψ, we suppress ψ and denote this space by E^{-1}. In case $n \geq 3$, we will use notation $E^{-1} = D^{-1}$. We also denote $E_0^1 = D^1$ if $n \leq 2$, $E_0^1 = D_0^1$ if $n \geq 3$. For $u \in E^{-1}$, $v \in E_0^1$, we denote

$$\langle u, v\rangle_E = \begin{cases} \langle w, v\rangle \text{ if } n \geq 3, \\ \langle w, [v]\rangle + c \int \psi v \text{ if } n \leq 2, \end{cases} \tag{4.2}$$

where $u = w + c\psi$, $w \in D^{-1}$, $\psi \in L_c^2$, $\int \psi = 1$. As before, we check that the value of $\langle u, v\rangle_E$ does not depend on the choice of this decomposition.

We recall that if $n \geq 3$, $E_0^1 = D_0^1$ and $E^{-1} = D^{-1}$ come with a Hilbert space structure. We also define inner products on E_0^1, E^{-1} in case $n \leq 2$ by

$$(v_1, v_2)_{E_0^1} := ([v_1], [v_2])_{D_0^1} + \int \psi v_1 \int \psi v_2,$$

$$(u_1, u_2)_{E^{-1}} := (w_1, w_2)_{D^{-1}} + c_1 c_2$$

for $u_i = w_i + c_i\psi$, $w_i \in D^{-1}$, $c_i \in \mathbb{R}$ $(i = 1, 2)$. This gives an orthogonal decomposition

$$E^{-1} = D^{-1} \oplus \mathbb{R}.$$

We note that although the values of those products may depend on the choice of ψ, the topologies they induce on E_0^1, E^{-1} do not. Formula (4.2) associates to any $u \in E^{-1}$ a continuous linear functional on E_0^1. The resulting mapping is an isometric isomorphism between E^{-1} and the continuous dual to E_0^1.

We extend TV onto E^{-1} by defining

$$TV(u) := \sup_{\psi \in X_1} \langle u, -\operatorname{div} \psi\rangle_E.$$

As usual, we check that TV is a convex, weakly-* (and strongly) lower semicontinuous functional. In particular, for a fixed $g \in E^{-1}$, the functional $w \mapsto TV(w + g)$ is convex and lower semicontinuous on D^{-1}. We next give a definition of a solution of $u_t \in -\partial TV(u)$ in the space E^{-1}. It turns out the idea of evolutionary variational inequality is very convenient since it is a flow in an affine space $g + D^{-1}$ for some $g \in E^{-1}$.

Definition 17. *Assume that $u_0 \in E^{-1}$. We say that $u : [0, \infty[\to E^{-1}$ is a solution to*

$$u_t \in -\partial_{D^{-1}} TV(u) \tag{4.3}$$

in the sense of EVI (evolutionary variational inequality) with initial datum u_0 if

- *(i) $u - g$ is absolutely continuous on $[\delta, T]$ (for any $\delta < T < \infty$) with values in D^{-1} and continuous up to zero and*
- *(ii) $u - g$ satisfies the evolutionary variational inequality*

$$\frac{1}{2}\frac{d}{dt}\|u(t) - g\|_{D^{-1}}^2 \leq TV(g) - TV(u(t)), \quad u(0) = u_0$$

holds for a. e. $t > 0$,

provided that $u_0 - g \in D^{-1}$ and $g \in E^{-1}$.

Theorem 18. *For any $u_0 \in E^{-1}$, there exists a unique solution u of (4.3) in the sense of EVI.*

Proof. The uniqueness part is standard [2]. We give a full proof for the reader's convenience and for completeness. Let u^i $(i = 1, 2)$ be a solution to (4.3) in the sense of EVI with initial datum u_0^i such that $u_0^1 - u_0^2 \in D^{-1}$. Since u^2 is a solution, $u^2(s) - u_0^2 \in D^{-1}$ for all $s \geq 0$ by setting $g = u_0^2$.

This implies that $u_0^1 - u^2(s) \in D^{-1}$ since $u_0^1 - u_0^2 \in D^{-1}$. Since u^1 is a solution, we take $g = u^2(s)$ and conclude that

$$\frac{1}{2}\frac{d}{dt}\left\|u^1(t) - u^2(s)\right\|_{D^{-1}}^2 \le TV\left(u^2(s)\right) - TV\left(u^1(t)\right) \text{ for a.e. } t > 0, \quad \text{all } s > 0.$$

Interchanging u^1 and u^2, we obtain

$$\frac{1}{2}\frac{d}{dt}\left\|u^2(t) - u^1(s)\right\|_{D^{-1}}^2 \le TV\left(u^1(s)\right) - TV\left(u^2(t)\right) \text{ for a.e. } t > 0, \quad \text{all } s > 0.$$

Adding these two inequalities, we end up with

$$\frac{1}{2}\frac{\partial}{\partial t}\left\|u^1(t) - u^2(s)\right\|_{D^{-1}}^2 + \frac{1}{2}\frac{\partial}{\partial t}\left\|u^2(t) - u^1(s)\right\|_{D^{-1}}^2 \le 0.$$

Evaluating the left-hand side at $t = s$, we observe that it equals

$$\frac{1}{2}\frac{d}{dt}\left\|u^1(t) - u^2(t)\right\|_{D^{-1}}^2.$$

We now conclude that $\left\|u^1(t) - u^2(t)\right\|^2$ is non-increasing so that

$$\left\|u^1(t) - u^2(t)\right\|_{D^{-1}} \le \|u_0^1 - u_0^2\|_{D^{-1}} \quad \text{for all} \quad t \ge 0. \tag{4.4}$$

This yields uniqueness of solution.

The existence is more involved. For $u_0 = w_0 + g_0 \in E^{-1}$ with $w_0 \in D^{-1}$, we consider the gradient flow of the form

$$w_t \in -\partial_{D^{-1}} TV(w + g_0), \quad w(0) = w_0. \tag{4.5}$$

Applying Proposition 14, there is a unique solution w to (4.5) for $w_0 \in D^{-1}$. This solution satisfies the evolutionary variational inequality

$$\frac{1}{2}\frac{d}{dt}\|w - f\|_{D^{-1}}^2 \le TV(f + g_0) - TV(w + g_0) \quad \text{for a.e.} \quad t > 0$$

for any $f \in D^{-1}$. Setting $u = w + g_0$, $g = f + g_0$, we end up with

$$\frac{1}{2}\frac{d}{dt}\|w - g\|_{D^{-1}}^2 \le TV(g) - TV\left(u(t)\right).$$

Since g can be taken arbitrary such that $u_0 - g \in D^{-1}$, this shows that u is the solution of (4.3) in the sense of EVI; see condition (i) is easy by Proposition 14. □

From the proof of uniqueness we have a contraction property (4.4).

Corollary 19. *Let u^i be the solution to* (4.3) *in the sense of EVI with $u^i(0) = u_0^1$ for $i = 1, 2$ where $u_0^i \in E^{-1}$. Then*

$$\left\|u^1(t) - u^2(t)\right\|_{D^{-1}} \le \|u_0^1 - u_0^2\|_{D^{-1}} \quad \textit{for all} \quad t \ge 0$$

provided that $u_0^1 - u_0^2 \in D^{-1}$.

It is non-trivial to characterize the subdifferential $\partial_{D^{-1}} TV$. For this purpose, we introduce a mapping I which plays a role analogous to $-\Delta$ in $n \ge 3$.

Lemma 20. *Let $n \le 2$. The mapping $I\colon E_0^1 \to E^{-1}$ defined by*

$$I(f) = (-\Delta)[f] + \left(\int_{\mathbb{R}^n} f\psi\right)\psi$$

is an isometric isomorphism.

Proof. It is clear that $I(f) \in E^{-1}$ and I is linear. For given $u = w + c\psi \in E^{-1}$ with $w \in D^{-1}$, $c \in \mathbb{R}$, there is $\overline{f} \in D^1_0$ such that $(-\Delta)\overline{f} = w$. Since a representative f of $\overline{f}$ is determined up to an additive constant, there is a unique representative f such that

$$\int_{\mathbb{R}^n} f\psi = c.$$

Thus, the mapping I is surjective. If $I(f) = 0$, then $(-\Delta)[f] = 0$ so $[f] = 0$. Thus f is a constant. Since $\int f\,\psi = 0$, this constant must be zero, so $f = 0$. Thus, I is injective. Recalling our definitions of inner products on E^1_0, E^{-1}, it is easy to check that I is an isometry. □

We have a characterization of the polar of TV in E^{-1} as in Theorem 11.

Theorem 21. *Let $n \leq 2$. Let Ψ be given by*

$$\Psi(v) = \inf\left\{ \|Z\|_\infty \;\middle|\; v = I(-\operatorname{div} Z),\ Z \in L^\infty(\mathbb{R}^n),\ \operatorname{div} Z \in E^1_0 \right\}$$

for $v \in E^{-1}$. Then $(TV)^0 = \Psi$.

Admitting this fact, we are able to give a characterization of the subdifferential.

Theorem 22. *Let $n \leq 2$. An element $v \in E^{-1}$ belongs to $\partial_{E^{-1}} TV(u)$ if and only if there is $Z \in L^\infty(\mathbb{R}^n)$ with $\operatorname{div} Z \in E^1_0$ such that*

(i) $|Z| \leq 1$

(ii) $v = I(-\operatorname{div} Z)$

(iii) $-\langle u, \operatorname{div} Z\rangle_E = TV(u)$.

Proof of Theorem 22. The proof parallels that of Theorem 13. By Lemma 9 and Theorem 21

$$v \in \partial TV(u) \iff \Psi(v) \leq 1 \text{ and } (u, v)_{E^{-1}} = TV(u).$$

The properties (i), (ii) together with $\operatorname{div} Z \in E^1_0$ are equivalent to $\Psi(v) \leq 1$. Since

$$(u, v)_{E^{-1}} = (w, (-\Delta)[v])_{D^{-1}} + c\int_{\mathbb{R}^n} v\,\psi = \langle w, [v]\rangle + c\int_{\mathbb{R}^n} v\,\psi, \tag{4.6}$$

the Euler equation $(u, v)_{E^{-1}} = TV(u)$ is equivalent to (iii). □

Proof of Theorem 21. The proof parallels that of Theorem 11. We first prove that

$$(u, v)_{E^{-1}} \leq \|Z\|_\infty \quad \text{for all} \quad u \in E^{-1} \quad \text{with} \quad TV(u) \leq 1$$

for $v = I(-\operatorname{div} Z)$. This implies $TV^0 \leq \Psi$. The estimate $(u, v)_{E^{-1}} \leq \|Z\|_\infty$ formally follows from the identity (4.6). Indeed, by (4.6), we see

$$(u, v)_{E^{-1}} = -\langle w, [\operatorname{div} Z]\rangle - c\int_{\mathbb{R}^n} \psi \operatorname{div} Z.$$

If u is in $C^\infty_c(\mathbb{R}^n)$, then, by this formula, we obtain

$$(u, v)_{E^{-1}} = -\int_{\mathbb{R}^n} u \operatorname{div} Z\,dx = \int_{\mathbb{R}^n} \nabla u \cdot Z\,dx \leq \|Z\|_\infty TV(u).$$

By approximation, as in the proof of Theorem 11, we conclude the desired estimate.

The other inequality $\Psi \leq TV^0$ follows from $TV \leq \Psi^0$. The proof of $TV \leq \Psi^0$ is parallel to that of Theorem 11 by replacing $\Delta \operatorname{div} Z$ by $I(-\operatorname{div} Z)$ and the D^{-1} inner product by the E^{-1} inner product, respectively, if one notes the identity (4.6). Since I is an isometry, Ψ is lower semicontinuous, and we conclude that $\Psi = TV^0$ by Remark 10.

□

We have to be careful, since the E^{-1} gradient flow

$$u_t \in -\partial_{E^{-1}} TV(u)$$

does not correspond to the total variation flow $u_t = (-\Delta)\,\mathrm{div}\,(\nabla u/|\nabla u|)$. By Theorem 22(iii), $Z = \nabla u/|\nabla u|$ if $\nabla u \neq 0$. Thus the E^{-1} gradient flow is formally of the form

$$u_t = (-\Delta)\,\mathrm{div}\,(\nabla u/|\nabla u|) + \psi \int_{\mathbb{R}^n} \psi\,\mathrm{div}\,(\nabla u/|\nabla u|)\;dx.$$

To recover the original total variation flow, we consider "partial" subdifferential in the direction of D^{-1}. Let P be the orthogonal projection from E^{-1} to D^{-1}. Then, by definition,

$$\partial_{D^{-1}} TV(w + c\psi) = P\partial_{E^{-1}} TV(u).$$

The equation

$$w_t \in -\partial_{D^{-1}} TV(w + c\psi)$$

is now formally of the form

$$u_t = (-\Delta)\,\mathrm{div}\,(\nabla u/|\nabla u|)$$

since $c\psi$ is time-independent. Here is a precise statement.

Theorem 23. *Let $n \leq 2$ Consider the functional $\mathcal{F} : w \mapsto TV(w + c\psi)$ on D^{-1} for a fixed $c \in \mathbb{R}$ and ψ. Then, $\partial_{D^{-1}} TV(w+c\psi) = P\partial_{E^{-1}} TV(u)$ for $u = w+c\psi$. In particular, an element $v \in D^{-1}$ belongs to $\partial_{D^{-1}}\mathcal{F}(w + c\psi)$ if and only if there is $Z \in L^\infty(\mathbb{R}^n)$ with $\mathrm{div}\, Z \in E_0^1$ such that*

- *(i)* $|Z| \leq 1$,
- *(ii)* $v = \Delta\,\mathrm{div}\, Z$,
- *(iii)* $-\langle u, \mathrm{div}\, Z\rangle_E = TV(u)$.

(In case $n \leq 2$, by $\Delta\,\mathrm{div}\, Z$ we understand $\Delta[\mathrm{div}\, Z]$.) This characterization is important to calculate the solution of $u_t = (-\Delta)\,\mathrm{div}\,(\nabla u/|\nabla u|)$ for $n \leq 2$ explicitly. In fact, we recover the characterization of a solution in the sense of EVI as in Theorem 15. This amounts to Theorem 2.

5 The notion of calibrability

We are interested in sets where the speed of solution u_t is spatially constant. The speed is given as minus the minimal section of the subdifferential, i. e.

$$\partial^0_{D^{-1}} TV(u) := \arg\min\left\{ \|v\|_{D^{-1}} \;\middle|\; v \in \partial_{D^{-1}} TV(u)\right\}.$$

Since $\partial_{D^{-1}} TV(u)$ is closed and convex, $\partial^0_{D^{-1}} TV(u)$ is uniquely determined if $\partial_{D^{-1}} TV(u) \neq \emptyset$. Since we have characterized the subdifferential, we end up with

$$\partial^0_{D^{-1}} TV(u) = \arg\min\big\{ \|v\|_{D^{-1}} \;\big|\; v = \Delta\,\mathrm{div}\, Z,\ Z \in L^\infty(\mathbb{R}^n),\ |Z| \leq 1, \\ \mathrm{div}\, Z \in E_0^1(\mathbb{R}^n),\ -\langle u, \mathrm{div}\, Z\rangle_E = TV(u)\big\}.$$

Although the minimizer v is unique, the corresponding Z may not be unique. Let U be a smooth open set in $\mathbb{R}^n$. We consider a smooth function u such that

$$\overline{U} = \left\{ x \in \mathbb{R}^n \;\middle|\; u(x) = 0\right\}$$

and $\partial_{D^{-1}} TV(u) \neq \emptyset$. Such a closed set is often called a facet. Assume further that $\nabla u \neq 0$ outside $\overline{U}$. Let Z be a vector field satisfying $v = \Delta\,\mathrm{div}\, Z$ for $v \in \partial TV(u)$. It is easy to see that outside the facet $\overline{U}$,

$$Z(x) = \nabla u(x) \;/\; |\nabla u(x)|$$

by $-\langle u, \operatorname{div} Z\rangle_E = TV(u)$. Since $\|v\|_{D^{-1}} = \|\operatorname{div} Z\|_{D_0^1}$, we see that

$$\partial^0_{D^{-1}} TV(u) = \arg\min \left\{ \|\operatorname{div} Z\|_{D_0^1} \;\middle|\; |Z| \le 1 \text{ in } U,\ Z = \nabla u/|\nabla u| \text{ in } \overline{U}^c,\ \operatorname{div} Z \in E_0^1 \right\}.$$

Since $\operatorname{div} Z$ is locally integrable, the normal trace is well-defined from inside as an element of $L^\infty(\partial U)$ [3] and it must agree with that from outside, i. e.

$$\nu \cdot Z(x) = \nu(x) \cdot \nabla u \,/\, |\nabla u| = \nu(x) \cdot \chi\nu(x) = \chi(x),$$

where $\nu(x)$ is the exterior unit normal of ∂U and

$$\chi(x) = \begin{cases} 1 & \text{if } u > 0 \text{ outside } \overline{U} \text{ near } x \in \partial U, \\ -1 & \text{otherwise.} \end{cases}$$

Since $\operatorname{div} Z$ is in E_0^1, its trace from inside and outside must agree, i.e.,

$$\operatorname{div} Z(x) = \chi \operatorname{div} \nu(x).$$

Let Z_0 be a minimizer corresponding to $v = \partial^0_{D^{-1}} TV(u)$. Since the value Z_0 outside $\overline{U}$ is always the same, we consider its restriction on U and still denote by Z_0. Then,

$$Z_0 = \arg\min \left\{ \int_U |\nabla \operatorname{div} Z|^2 \;\middle|\; |Z| \le 1 \text{ in } U \quad \nu \cdot Z = \chi \text{ on } \partial U,\ \operatorname{div} Z = \chi \operatorname{div} \nu \text{ on } \partial U \right\}.$$

In the case $u > 0$ outside $\overline{U}$, $\operatorname{div} Z$ on ∂U must equal to the minus mean curvature (the sum of all principal curvatures) in the direction of ν. Although $\operatorname{div} Z_0 \in D_0^1(\mathbb{R}^n)$ so that $\nabla \operatorname{div} Z_0 \in L^2(\mathbb{R}^n)$, the quantity $\nabla \operatorname{div} Z_0$ may jump across ∂U. Thus $\Delta \operatorname{div} Z_0$ may contain singular part which is a driving force to move the facet boundary "horizontally" during its evolution under the fourth-order total variation equation as observed in the previous section and earlier in [11]. In the second-order problem, the speed does not contain any singular part so the jump discontinuity does not move.

We are interested in a situation where $\Delta \operatorname{div} Z_0$ is constant over U. In the spirit of [24], we call any continuous function $\chi\colon \partial U \to \{-1, 1\}$ a *signature* for U.

Definition 24. *Let U be a smooth open set in $\mathbb{R}^n$ with signature χ. We say that U is (D^{-1}-)*calibrable *(with signature χ) if there exists a minimizer Z_0 of*

$$\int_U |\nabla \operatorname{div} Z|^2 \tag{5.1}$$

under the constraint

$$|Z| \le 1 \text{ a. e. on } U \tag{5.2}$$

with boundary conditions

$$\nu \cdot Z = \chi, \quad \operatorname{div} Z = \chi \operatorname{div} \nu \quad \text{on} \quad \partial U, \tag{5.3}$$

with the property that

$$\Delta \operatorname{div} Z_0 \text{ is constant over } U. \tag{5.4}$$

*We call any such Z_0 a (D^{-1}-)*calibration *for U (with signature χ).*

We shall study this variational problem. We first set $w = \operatorname{div} Z$, $w_0 = \operatorname{div} Z_0$. Then, $\int_U |\nabla \operatorname{div} Z|^2$ is nothing but the Dirichlet energy for w, i.e.,

$$e(w) = \int_U |\nabla w|^2$$

and w_0 minimizes this $e(w)$ under the constraint. Since $\nu \cdot Z = \chi$,

$$\int_U w \, d\mathcal{L}^n = \int_{\partial U} \chi \, d\mathcal{H}^{n-1}.$$

The other boundary condition can be rewritten as

$$w = \chi \operatorname{div} \nu \quad \text{on} \quad \partial U.$$

Proposition 25. *Let U be a smooth bounded domain in $\mathbb{R}^n$. Assume that Z_0 is a calibration for U with signature χ. Then, $w_0 = \operatorname{div} Z_0$ must satisfy the Saint-Venant problem*

$$\begin{cases} -\Delta w = \lambda \quad \text{in} \quad U & (5.5) \\ w = \chi \operatorname{div} \nu \quad \text{on} \quad \partial U & (5.6) \end{cases}$$

with the constraint

$$\int_U w \, \mathrm{d}\mathcal{L}^n = \int_{\partial U} \chi \, \mathrm{d}\mathcal{H}^{n-1}, \tag{5.7}$$

where λ is some constant.

We are able to prove the converse.

Theorem 26. *Let U be a smooth bounded domain in $\mathbb{R}^n$. Assume that $Z_* \in L^\infty(U)$ satisfies $|Z_*| \le 1$ in U and that $w_* = \operatorname{div} Z_* \in D_0^1(U)$ is a solution to the Saint-Venant problem* (5.5), (5.6), *where λ is a constant. Assume that Z_* satisfies $\nu \cdot Z_* = \chi$. Then Z_* is a minimal Cahn-Hoffman vector field. In particular, U is calibrable with signature χ and Z_* is a calibration.*

Proof. We first note that $w_* = \operatorname{div} Z_*$ must satisfy (5.7). We consider the variational problem of $e(w)$ under the Dirichlet condition (5.6) and the constraint (5.7). Since the problem is strictly convex, there is a unique minimizer $\overline{w}$ in $D_0^1(U)$. By Lagrange's multiplier method, $\overline{w}$ must satisfy (5.5) because of the constraint (5.7). (Actually, a weak solution $\overline{w}$ of (5.5) is a smooth solution of (5.5), (5.6) by the standard regularity theory of linear elliptic partial differential equations [18, Chapter 6].) As we see below, the constant λ is uniquely determined by (5.5) and (5.7). For $\overline{w}$, there always exists $Z \in C^\infty(\overline{U})$ such that

$$\operatorname{div} Z = \overline{w} \quad \text{in} \quad U, \quad \nu \cdot Z = \chi \quad \text{on} \quad \partial U. \tag{5.8}$$

Indeed, let p be a solution of the Neumann problem

$$\Delta p = \overline{w} \quad \text{in} \quad U, \quad \nu \cdot \nabla p = \chi \quad \text{on} \quad \partial U.$$

Such a solution p always exists since $\overline{w}$ satisfies the compatibility condition (5.7) and it is smooth up to $\overline{U}$; see e. g. [10, 18]. If we set $Z = \nabla p$, then Z satisfies the desired property (5.8). Thus, the minimum $e(\overline{w})$ of the Dirichlet energy under the constraint (5.7) agrees with

$$\min \left\{ \int_U |\nabla \operatorname{div} Z|^2 \;\middle|\; \nu \cdot Z = \chi, \ \operatorname{div} Z = \chi \operatorname{div} \nu \text{ on } \partial U \right\}.$$

Since $w_* = \overline{w}$ and $|Z_*| \le 1$, this shows that Z_* is a minimal Cahn-Hoffman vector field. Thus, U is calibrable and Z_* is a calibration. □

Lemma 27. *Let U be a smooth bounded domain in $\mathbb{R}^n$. Let w solve*

$$\begin{aligned} -\Delta w &= \lambda \quad \text{in} \quad U \\ w &= f \quad \text{on} \quad \partial U \end{aligned}$$

for $\lambda \in \mathbb{R}$, $f \in C(\partial U)$. This solution can be written as

$$w = \lambda w_{\mathrm{sv}} + h_f$$

where w_{sv} solves the Saint-Venant problem

$$\begin{aligned} -\Delta w_{\mathrm{sv}} &= 1 \quad \text{in} \quad U \\ w_{\mathrm{sv}} &= 0 \quad \text{on} \quad \partial U \end{aligned}$$

and h_f is the harmonic extension of f to U. In particular, if $\int_U w = c$ is given then λ is uniquely determined by

$$\lambda \int_U w_{\mathrm{sv}} + \int_U h_f = c,$$

since $w_{\mathrm{sv}} > 0$ in U.

The decomposition $w = \lambda w_{\mathrm{sv}} + h_f$ is rather clear. The property $w_{\mathrm{sv}} > 0$ in U follows from the maximum principle [18].

We now compare the definition of calibrability for the second-order problem.

Definition 28. *Let U be a smooth open set in $\mathbb{R}^n$ with signature χ. We say that $\overline{U}$ is (L^2-)*calibrable *if there is a minimizer Z_0 of*

$$\int_U |\operatorname{div} Z|^2$$

under the constraint $|Z| \leq 1$ in U and the boundary condition $\nu \cdot Z = \chi$ with the property that $\operatorname{div} Z_0$ *is a constant over U.*

This definition is slightly weaker than the calibrability used in [1, 25], where $a(t)1_U$ is a solution of the total variation flow in $\mathbb{R}^n$ with some function $a(t)$ of t; see also [3]. This requires that $\partial^0_{L^2} TV(u)$ is constant not only on U but also U^c. Our definition follows from that of [5].

6 Calibrability of rotationally symmetric sets

Definition 29. *We say that a Lebesgue measurable subset U (defined up to a set of measure zero) of $\mathbb{R}^n$ is a* generalized annulus *if U is non-empty, open, connected and rotationally symmetric, i. e. invariant under the linear action of $SO(n)$ on $\mathbb{R}^n$.*

It is easy to see that any generalized annulus is a ball, an annulus, the complement of a ball or the whole space $\mathbb{R}^n$. In other words, any generalized annulus is of form

$$A^{R_1}_{R_0} = \{x \in \mathbb{R}^n \colon R_0 < x < R_1\}$$

with $0 \leq R_0 < R_1 \leq \infty$. We note that

$$A^R_0 = B_R$$

as measurable sets for $R > 0$. In this section we will settle the question which generalized annuli are calibrable.

Lemma 30. *Let U be a generalized annulus. Suppose that U is calibrable with signature χ. Then there exists a calibration $\overline{Z}$ for (U, χ) of form $\overline{Z}(x) = z(|x|)\frac{x}{|x|}$.*

Proof. Let Z be any calibration for (U, χ). Let μ_n be the Haar measure on $SO(n)$. We define $\overline{Z}$ as the average

$$\overline{Z}(x) = \int LZ(L^{-1}x)\, \mathrm{d}\mu_n(L).$$

It is an exercise in vector calculus to check that $\overline{Z}$ satisfies boundary conditions (5.3) and that $\Delta \operatorname{div} \overline{Z}$ is a constant (equal to $\Delta \operatorname{div} Z$) on U. By convexity, $|\overline{Z}| \leq 1$ and $\overline{Z}$ is also a minimizer of (5.1). Thus $\overline{Z}$ is a calibration for (U, χ). By definition, it is invariant under rotations, i. e.

$$L\overline{Z}(L^{-1}x) = \overline{Z}(x)$$

for $L \in SO(n)$, $x \in \mathbb{R}^n$. In the case $n = 1$ this already shows that $\overline{Z}$ is in the desired form. In higher dimensions, we consider the orthogonal decomposition

$$\overline{Z}(x) = \overline{Z}^{\perp}(x) + \overline{Z}^{T}(x) := \frac{x}{|x|} \otimes \frac{x}{|x|}\, \overline{Z}(x) + \left(I - \frac{x}{|x|} \otimes \frac{x}{|x|} \right) \overline{Z}(x).$$

Both $\overline{Z}^{\perp}$ and $\overline{Z}^{T}$ are invariant under rotations. In particular, for any given $R > 0$, the restriction of $\overline{Z}^{T}$ to $\mathbb{S}^{n-1}_R$ is an invariant tangent vector field on $\mathbb{S}^{n-1}_R$. Note that any such vector field is smooth. If $n = 3$, it follows by the hedgehog uncombability theorem [9, Proposition 7.15] that

$\overline{Z}^T \equiv 0$. If $n > 3$, any vector field invariant on $\mathbb{S}^{n-1}$ is in particular invariant on a sphere $\mathbb{S}^2$ containing any given point in $\mathbb{S}^{n-1}$, so the same conclusion follows. Thus, we have

$$\overline{Z}(x) = \overline{Z}^{\perp}(x) = \frac{x}{|x|} \otimes \frac{x}{|x|} \overline{Z}(x) = \frac{x}{|x|} \cdot \overline{Z}(x) \frac{x}{|x|} =: \overline{z}(x) \frac{x}{|x|}.$$

By rotational invariance, we have $\overline{z}(x) = z(|x|)$, which concludes the proof.

We are left with the case $n = 2$ in which there exists a one-dimensional space of invariant tangent fields on $\mathbb{S}^{n-1} = \mathbb{S}^1$ spanned by $e^T(x) := (x_2, -x_1)$. Thus, we have

$$\overline{Z}^T(x) = z^T(|x|) e^T(x).$$

We calculate

$$\operatorname{div} \overline{Z}^T(x) = z^T(|x|) \operatorname{div} e^T(x) + (z^T)'(|x|) \frac{x}{|x|} \cdot e^T(x) = 0.$$

Thus, we can disregard $\overline{Z}^T$ and choose $\overline{Z}^{\perp}$ as our calibration, since it satisfies conditions (5.2)-(5.4) (recall that $\overline{Z}^T$ and $\overline{Z}^{\perp}$ are orthogonal) and $\overline{Z}^T$ does not contribute to the value of (5.1). As before, we see that

$$\overline{Z}^{\perp}(x) = z(|x|) \frac{x}{|x|}.$$

□

Let U be a generalized annulus. By Lemma 30, if U is calibrable, then there exists a calibration Z for U of form $Z = z(|x|) \frac{x}{|x|}$. It follows from (5.5) that z needs to satisfy the ODE

$$-r^{1-n} \left(r^{n-1} \left(r^{1-n} \left(r^{n-1} z \right)' \right)' \right)' = \lambda. \tag{6.1}$$

The general solution to this ODE is

$$z(r) = c_0 r^3 + c_1 r^{3-n} + c_2 r + c_3 r^{1-n} \tag{6.2}$$

where $c_0 = -\frac{\lambda}{2n(n+2)}$ if $n \neq 2$ and

$$z(r) = c_0 r^3 + c_1 r \log r + c_2 r + c_3 r^{-1}. \tag{6.3}$$

where $c_0 = -\frac{\lambda}{16}$ if $n = 2$. We will now try to find a calibration for U by solving a suitable boundary value problem for (6.1).

6.1 Balls

Let $U = B_R(0)$. To focus attention, we choose $\chi = -1$ on ∂U. In this case, boundary conditions (5.3) lead to

$$z(R) = -1, \quad z'(R) = 0. \tag{6.4}$$

If $n \geq 2$, in order to satisfy the requirements $|Z| \leq 1$ and $\nabla \operatorname{div} Z$, we need to restrict to $c_1 = c_3 = 0$ in (6.2). We make the same choice also in case $n = 1$, as it leads to the right result. Then, applying (6.4) in (6.2) or (6.3), we obtain a system of two affine equations for two unknowns λ, c_2. We solve it obtaining

$$z(r) = \frac{1}{2} \left(\frac{r}{R} \right)^3 - \frac{3}{2} \frac{r}{R}, \tag{6.5}$$

$$\lambda = -\frac{n(n+2)}{R^3}. \tag{6.6}$$

We check that z satisfies $|z| \leq 1$ on $[0, R]$, so Z is a calibration for B_R. Thus, all balls are calibrable in any dimension.

6.2 Complements of balls

Let $U = \mathbb{R}^n \setminus B_R$. For consistency with the previous case, we choose $\chi = 1$ on ∂U. In this case, boundary conditions (5.3) also lead to

$$z(R) = -1, \quad z'(R) = 0. \tag{6.7}$$

Let us first assume that $n \geq 3$. In order to satisfy the requirement $|Z| \leq 1$, we need to restrict to $\lambda = c_2 = 0$ in (6.2). Again, applying (6.4) in (6.2) leads to a system of two affine equations for two unknowns c_1, c_3. We solve it obtaining

$$z(r) = -\frac{n-1}{2}\left(\frac{r}{R}\right)^{3-n} + \frac{n-3}{2}\left(\frac{r}{R}\right)^{1-n}. \tag{6.8}$$

Again, we easily check that z satisfies $|z| \leq 1$ on $[0, R]$, so Z is a calibration for B_R.

In the omitted cases $n = 1, 2$, requirement $|Z| \leq 1$ implies $\lambda = c_1 = c_2 = 0$ in (6.2). If $n = 1$, there exists z of such form satisfying (6.7): $z(r) \equiv -1$, consistently with (6.8). On the other hand, if $n = 2$, applying (6.7) to (6.2) with $\lambda = c_1 = c_2 = 0$ leads to a contradiction.

Summing up, all complements of balls are calibrable if $n \neq 2$. On the other hand, if $n = 2$ all complements of balls turn out not to be calibrable.

6.3 Annuli

Let now $U = A_{R_0}^{R_1} = B_{R_1} \setminus B_{R_0}$, $0 < R_0 < R_1$. In this case ∂U has two connected components, so there exist two distinct choices of signature: constant and non-constant. Let us first consider the former. To focus attention, we choose $\chi \equiv -1$. Then, boundary conditions (5.3) take form

$$z(R_0) = 1, \quad z(R_1) = -1, \quad z'(R_0) = z'(R_1) = 0. \tag{6.9}$$

Applying (6.9) to (6.2) or (6.3) leads to a system of four affine equations with four unknowns. In the case $n \neq 2$, the solution is

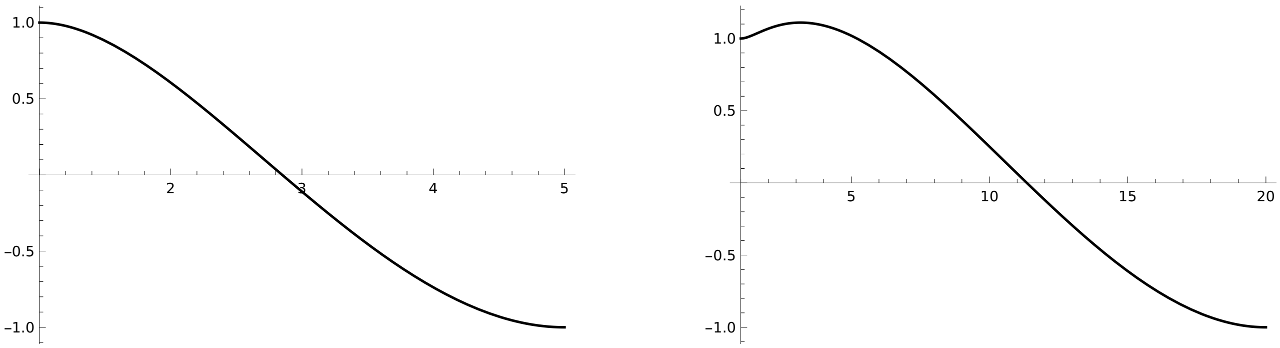
$$\begin{aligned}
c_0 &= \frac{-(R_1-R_0)R_1^n R_0^n\big((n-1)(n-2)(R_1+R_0)^2-2R_0R_1\big)+2R_1^3R_0^{2n}-2R_0^3R_1^{2n}}{R_1R_0\Big(R_1^nR_0^n\Big(n^2\big(R_1^2-R_0^2\big)^2+8R_1^2R_0^2\Big)-4R_1^2R_0^{2n+2}-4R_0^2R_1^{2n+2}\Big)},\\
c_1 &= \frac{(R_1+R_0)\Big(R_1^n\big(2(n-1)R_1^2+(n+2)R_1R_0-(n+2)R_0^2\big)+R_0^n\big((n+2)R_1^2-(n+2)R_1R_0-2(n-1)R_0^2\big)\Big)}{2(n^2-4)R_1^3R_0^3+4R_1^{3-n}R_0^{n+3}+4R_1^{n+3}R_0^{3-n}-n^2R_1^5R_0-n^2R_1R_0^5},\\
c_2 &= \frac{(R_1-R_0)R_1^nR_0^n\big(6R_1^2R_0^2+(n-1)nR_1^3R_0+(n-1)nR_1^4+(n-1)nR_1R_0^3+(n-1)nR_0^4\big)-6R_1^3R_0^{2n+2}+6R_0^3R_1^{2n+2}}{R_1R_0\Big(R_1^nR_0^n\Big(n^2\big(R_1^2-R_0^2\big)^2+8R_1^2R_0^2\Big)-4R_1^2R_0^{2n+2}-4R_0^2R_1^{2n+2}\Big)},\\
c_3 &= \frac{(R_1+R_0)\Big(R_0^2R_1^n\big(-2(n-3)R_1^2-nR_1R_0+nR_0^2\big)-R_1^2R_0^n\big(nR_1^2-R_0(nR_1+2(n-3)R_0)\big)\Big)}{2(n^2-4)R_1^3R_0^3+4R_1^{3-n}R_0^{n+3}+4R_1^{n+3}R_0^{3-n}-n^2R_1^5R_0-n^2R_1R_0^5}.
\end{aligned} \tag{6.10}$$

This can be rewritten in a form emphasizing homogeneity:

$$\begin{aligned}
c_0 &= \frac{-(Q-1)Q^n\big((n-1)(n-2)(Q+1)^2-2Q\big)+2Q^3-2Q^{2n}}{Q^{n-2}\big(n^2(Q^2-1)^2+8Q^2\big)-4-4Q^{2n}}R_1^{-3},\\
c_1 &= \frac{(Q+1)\Big(Q^n\big(2(n-1)Q^2+(n+2)Q-(n+2)\big)+\big((n+2)Q^2-(n+2)Q-2(n-1)\big)\Big)}{2(n^2-4)Q^n+4+4Q^{2n}-n^2Q^{n+2}-n^2Q^{n-2}}R_1^{n-3},\\
c_2 &= \frac{(Q-1)Q^n\big(6Q^2+(n-1)nQ^3+(n-1)nQ^4+(n-1)nQ+(n-1)n\big)-6Q^3+6Q^{2n+2}}{Q^n\big(n^2(Q^2-1)^2+8Q^2\big)-4Q^2-4Q^{2n+2}}R_1^{-1},\\
c_3 &= \frac{(Q+1)\Big(Q^n\big(-2(n-3)Q^2-nQ+n\big)-Q^2\big(nQ^2-(nQ+2(n-3))\big)\Big)}{2(n^2-4)Q^{n+2}+4Q^2+4Q^{2n+2}-n^2Q^{n+4}-n^2Q^n}R_1^{n-1},
\end{aligned} \tag{6.11}$$

where we denoted $Q = R_1/R_0$. We can further simplify it to

$$\begin{aligned}
c_0 &= \frac{2Q^3(Q^{2n-3}-1)+(Q-1)Q^n((n-1)(n-2)(Q+1)^2-2Q)}{4(Q^n-1)^2-n^2(Q^2-1)^2Q^{n-2}}R_1^{-3},\\
c_1 &= \frac{(Q+1)(2(n-1)(Q^{n+2}-1)+(n+2)Q(Q-1)(Q^{n-1}+1))}{4(Q^n-1)^2-n^2(Q^2-1)^2Q^{n-2}}R_1^{n-3},\\
c_2 &= -\frac{6Q(Q^{2n-1}-1)+(Q-1)Q^{n-2}(6Q^2+n(n-1)(1+Q)(1+Q^3))}{4(Q^n-1)^2-n^2(Q^2-1)^2Q^{n-2}}R_1^{-1},\\
c_3 &= -\frac{(Q+1)(2(n-3)(Q^n-1)+nQ(Q-1)(Q^{n-3}+1))}{4(Q^n-1)^2-n^2(Q^2-1)^2Q^{n-2}}R_1^{n-1}.
\end{aligned} \tag{6.12}$$



(a) $R_0 = 1,\ R_1 = 5$. (b) $R_0 = 1,\ R_1 = 20$.

Figure 1: Plots of z for an annulus with constant signature for two different values of Q in case $n = 2$.

We need to check whether condition $|Z| \le 1$ is satisfied. We calculate

$$\begin{aligned} z''(r) &= 6c_0 r + (n-3)(n-2)c_1 r^{1-n} + n(n-1)c_3 r^{-n-1} \\ &= r^{-n-1}(6c_0 r^{n+2} + (n-3)(n-2)c_1 r^2 + n(n-1)c_3) =: r^{-n-1}w(r). \end{aligned} \tag{6.13}$$

Using the form (6.12), we can check that $c_0 > 0$, $c_1 > 0$ for all $Q > 1$. Therefore, w has at most one zero on the half-line $r > 0$. Consequently, z'' has at most one zero, so z has at most one inflection point. Taking into account (6.9), z cannot have a local extremum on $]R_0, R_1[$. Thus, $|z| \le 1$ on $]R_0, R_1[$ and Z is a valid calibration.

In the case $n = 2$, the solution is

$$\begin{aligned} c_0 &= \tfrac{R_1^2 - R_0^2 + 2R_1R_0\log(R_1/R_0)}{4R_1(R_1-R_0)R_0(-R_1^2+R_0^2+(R_1^2+R_0^2)\log(R_1/R_0))} \\ c_1 &= \tfrac{-R_1^3 - 3R_1^2R_0 - 3R_1R_0^2 - R_0^3}{2R_1R_0(-R_1^2+R_0^2+(R_1^2+R_0^2)\log(R_1/R_0))} \\ c_2 &= \tfrac{-3R_1^4+3R_0^4+2(R_1^4-R_1^3R_0+R_1^2R_0^2-3R_1R_0^3)\log(R_1)+2(3R_1^3R_0-R_1^2R_0^2+R_1R_0^3-R_0^4)\log(R_0)}{4R_1(R_1-R_0)R_0(-R_1^2+R_0^2+(R_1^2+R_0^2)\log(R_1/R_0))} \\ c_3 &= \tfrac{R_1(-3R_1^2R_0+3R_0^3+2(R_1^2R_0-R_1R_0^2+R_0^3)\log(R_1/R_0))}{4(R_1-R_0)(-R_1^2+R_0^2+(R_1^2+R_0^2)\log(R_1/R_0))} \end{aligned}$$

which can be rewritten (again, denoting $Q = R_1/R_0$) as

$$\begin{aligned} c_0 &= \tfrac{Q^2\left(Q^2-1+2Q\log Q\right)}{4(Q-1)(-Q^2+1+(Q^2+1)\log Q)} R_1^{-3}, \\ c_1 &= \tfrac{-(Q+1)^3}{2(-Q^2+1+(Q^2+1)\log Q)} R_1^{-1}, \\ c_2 &= \tfrac{-3(Q^4-1)-2(3Q^3-Q^2+Q-1)\log Q}{4(Q-1)(-Q^2+1+(Q^2+1)\log Q)} R_1^{-1} + \tfrac{(Q+1)^3}{2(-Q^2+1+(Q^2+1)\log Q)} R_1^{-1}\log R_1, \\ c_3 &= \tfrac{-3Q^2+3+2(Q^2-Q+1)\log Q}{4(Q-1)(-Q^2+1+(Q^2+1)\log Q)} R_1. \end{aligned} \tag{6.14}$$

As before, we calculate the second derivative of z:

$$z''(r) = 6c_0 r + c_1 r^{-1} + 2c_3 r^{-3} = r^{-3}(6c_0 r^4 + c_1 r^2 + 2c_3) =: r^{-3}w(r).$$

The polynomial w has at most 2 positive roots, and so does z''. By (6.9), at least one of them belongs to $]R_0, R_1[$. Furthermore, since $c_0 > 0$ for $Q > 1$, $z''(r)$ is positive for large values of r. Taking into account these observations, we deduce that $|z| \le 1$ on $[R_0, R_1]$ if and only if $z''(R_0) \le 0$. This inequality is equivalent to

$$m(Q) := \log Q - \frac{(Q^2-1)(2Q-1)}{Q(Q^2-2Q+3)} \le 0.$$

We compute

$$m(1)=0, \qquad \lim_{Q\to+\infty} m(Q)=+\infty, \qquad m'(Q)=\frac{(Q-3)(Q-1)(Q+1)^3}{Q^2(Q^2-2Q+3)^2}. \tag{6.15}$$

We observe that m has exactly one zero Q_* on $]1,+\infty[$, and $m(Q)\le 0$ if and only if $Q\le Q_*$. Therefore, Z is a valid calibration for $A_{R_0}^{R_1}$ with signature -1 if and only if $R_1/R_0\le Q_*$. By (6.15) it is evident that $Q_*>3$. Numerical computation using Wolfram Mathematica shows that $Q_*\approx 9.7$. Thus, $A_{R_0}^{R_1}$ with constant signature is calibrable if and only if $R_1/R_0\le Q_*$. This concludes the proof of Theorem 3.

Now, let us consider non-constant signature. We assume that $\chi=1$ on ∂B_{R_0} and $\chi=-1$ on ∂B_{R_1}. This choice leads to

$$z(R_0)=-1, \quad z(R_1)=-1, \quad z'(R_0)=z'(R_1)=0. \tag{6.16}$$

If $n\neq 2$, the solution to the resulting affine system is

$$\begin{aligned}
c_0 &= \frac{-2R_1^3R_0^{2n}-2R_0^3R_1^{2n}+R_0^nR_1^n(R_0+R_1)\big((n-2)(n-1)R_0^2-2((n-3)n+1)R_0R_1+(n-2)(n-1)R_1^2\big)}{R_0R_1\Big(R_0^nR_1^n\Big(n^2\big(R_0^2-R_1^2\big)^2+8R_0^2R_1^2\Big)-4R_1^2R_0^{2n+2}-4R_0^2R_1^{2n+2}\Big)}\\
c_1 &= -\frac{-3nR_1R_0^{n+2}+2(n-1)R_0^{n+3}+(n+2)R_1^3R_0^n+(n+2)R_0^3R_1^n-3nR_0R_1^{n+2}+2(n-1)R_1^{n+3}}{4R_0^{3-n}R_1^{3-n}\big(R_0^n-R_1^n\big)^2-n^2R_0R_1\big(R_0^2-R_1^2\big)^2}\\
c_2 &= \frac{6R_1^3R_0^{2n+2}+6R_0^3R_1^{2n+2}-R_0^nR_1^n(R_0+R_1)\big((n-1)nR_0^4-(n-1)nR_0^3R_1-(n-1)nR_0R_1^3+(n-1)nR_1^4+6R_0^2R_1^2\big)}{R_0R_1\Big(R_0^nR_1^n\Big(n^2\big(R_0^2-R_1^2\big)^2+8R_0^2R_1^2\Big)-4R_1^2R_0^{2n+2}-4R_0^2R_1^{2n+2}\Big)}\\
c_3 &= \frac{(R_0-R_1)\Big(R_0^2R_1^n\big(n(R_0-R_1)(R_0+2R_1)+6R_1^2\big)-R_1^2R_0^n\big(-2(n-3)R_0^2+nR_0R_1+nR_1^2\big)\Big)}{4R_0^{3-n}R_1^{3-n}\big(R_0^n-R_1^n\big)^2-n^2R_0R_1\big(R_0^2-R_1^2\big)^2}
\end{aligned} \tag{6.17}$$

which we rewrite as

$$\begin{aligned}
c_0 &= \frac{-2Q^3-2Q^{2n}+Q^n(1+Q)\big((n-2)(n-1)-2((n-3)n+1)Q+(n-2)(n-1)Q^2\big)}{Q^{n-2}\big(n^2(1-Q^2)^2+8Q^2\big)-4-4Q^{2n}}R_1^{-3}\\
c_1 &= -\frac{-3nQ+2(n-1)+(n+2)Q^3+(n+2)Q^n-3nQ^{n+2}+2(n-1)Q^{n+3}}{4(1-Q^n)^2-n^2Q^{n-2}(1-Q^2)^2}R_1^{n-3}\\
c_2 &= \frac{6Q^3+6Q^{2n+2}-Q^n(1+Q)\big((n-1)n-(n-1)nQ-(n-1)nQ^3+(n-1)nQ^4+6Q^2\big)}{Q^n\big(n^2(1-Q^2)^2+8Q^2\big)-4Q^2-4Q^{2n+2}}R_1^{-1}\\
c_3 &= \frac{(1-Q)\Big(Q^n\big(n(1-Q)(1+2Q)+6Q^2\big)-Q^2\big(-2(n-3)+nQ+nQ^2\big)\Big)}{4Q^2(1-Q^n)^2-n^2Q^n(1-Q^2)^2}R_1^{n-1}
\end{aligned} \tag{6.18}$$

We note that in case $n=1$ the solution reduces to

$$c_0=0, \quad c_1=0, \quad c_2=0, \quad c_3=-1,$$

while in case $n=3$ it reduces to

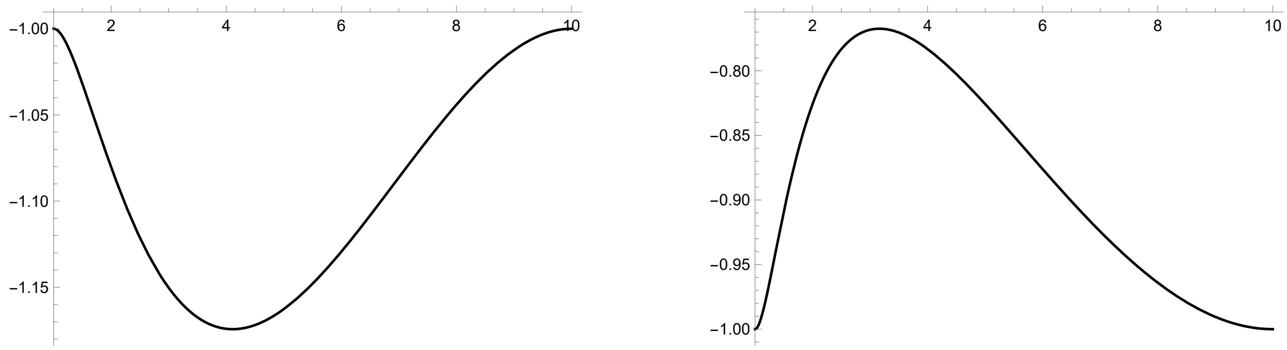
$$c_0=0, \quad c_1=-1, \quad c_2=0, \quad c_3=0.$$

In both of these cases z is constant and we have $\lambda=c_0=0$. On the other hand, if $n\ge 4$, we can check that $c_0>0$ for $Q>1$. Recalling (6.13), we observe that z'' has at most two zeros on the positive half-line and $z''(r)>0$ for large values of r. On the other hand, by (6.16), if z has N local extrema on $]R_0,R_1[$, it needs to have at least $N+1$ inflection points. We deduce from these conditions that z has exactly one local maximum and no local minima, and therefore $z\ge -1$ on $]R_0,R_1[$. It remains to check whether $z\le 1$ on $]R_0,R_1[$. Let now

$$f(r)=r^{1-n}(r^{n-1}z(r))'=f'(r)+(n-1)\frac{f(r)}{r}.$$

Then, by (6.1), (6.16), f is a solution to the second-order elliptic problem

$$\mathcal{A}f=\lambda, \quad f(R_0)=-\frac{n-1}{R_0}, \quad f(R_1)=-\frac{n-1}{R_1},$$



(a) $n = 2$.

(b) $n = 4$.

Figure 2: Plots of z for an annulus with non-constant signature for two values of n, with $R_0 = 1$, $R_1 = 10$.

where

$$\mathcal{A}f = -r^{1-n}(r^{n-1}f'(r))' = -f''(r) - (n-1)\frac{f'(r)}{r}.$$

Since $c_0 \geq 0$, we have $\lambda < 0$ for $Q > 1$. By the classical weak maximum principle [18, Theorem 3.1.],

$$\max_{[R_0,R_1]} f = \max_{\{R_0,R_1\}} f = -\frac{n-1}{R_0}.$$

Now, if z has a local maximum at r_0, then $z'(r_0) = 0$, so $f(r_0) = \frac{z(r_0)}{r_0}$. Consequently,

$$\frac{z(r_0)}{r_0} \leq -\frac{n-1}{R_0} < 0,$$

so $z < 0$ on $[R_0, R_1]$. Thus, if $n \neq 2$, all annuli with non-constant signature are calibrable.

We move to the case $n = 2$. Now, the solution to the affine system for coefficients of z is

$$c_0 = \frac{-R_0^2 - 2R_0R_1\log(R_1/R_0) + R_1^2}{4R_0R_1(R_0+R_1)\left(-R_0^2+R_1^2-\left(R_0^2+R_1^2\right)\log(R_1/R_0)\right)},$$
$$c_1 = \frac{(R_0-R_1)^3}{2R_0R_1\left(-R_0^2+R_1^2-\left(R_0^2+R_1^2\right)\log(R_1/R_0)\right)},$$
$$c_2 = \frac{(R_0-R_1)(R_0+R_1)\left(R_0^2+4R_0R_1+R_1^2\right)-2R_0\left(R_0^3+R_0^2R_1+R_0R_1^2+3R_1^3\right)\log(R_0)+2R_1\left(3R_0^3+R_0^2R_1+R_0R_1^2+R_1^3\right)\log(R_1)}{4R_0R_1(R_0+R_1)\left(-R_0^2+R_1^2-\left(R_0^2+R_1^2\right)\log(R_1/R_0)\right)},$$
$$c_3 = \frac{R_0R_1\left(2\left(R_0^2+R_0R_1+R_1^2\right)\log(R_1/R_0))+3(R_0-R_1)(R_0+R_1)\right)}{4(R_0+R_1)\left(-R_0^2+R_1^2-\left(R_0^2+R_1^2\right)\log(R_1/R_0)\right)}$$

or equivalently

$$c_0 = \frac{Q^2\left(-1-2Q\log Q+Q^2\right)}{4(1+Q)(-1+Q^2-(1+Q^2)\log Q)}R_1^{-3},$$
$$c_1 = \frac{(1-Q)^3}{2(-1+Q^2-(1+Q^2)\log Q)}R_1^{-1},$$
$$c_2 = \frac{(1-Q)(1+Q)\left(1+4Q+Q^2\right)+2\left(1+Q+Q^2+3Q^3\right)\log Q}{4(1+Q)(-1+Q^2-(1+Q^2)\log Q)}R_1^{-1} + \frac{(Q-1)^3}{2(-1+Q^2-(1+Q^2)\log Q)}R_1^{-1}\log R_1,$$
$$c_3 = \frac{2\left(1+Q+Q^2\right)\log Q+3(1-Q)(1+Q)}{4(1+Q)(-1+Q^2-(1+Q^2)\log Q)}R_1.$$

We can check that in this case $c_0 < 0$ for $Q > 1$. By the same argument as in the previous case, we show that $z < -1$ in $]R_0, R_1[$, so it does not define a valid calibration. Thus, in the case $n = 2$ all annuli with non-constant signature are not calibrable.

7 Explicit solutions

7.1 Balls

In this section, our goal is to provide explicit description of solutions to (1.1) emanating from the characteristic function of a ball

$$u_0 = a_0 \mathbf{1}_{B_{R_0}}. \tag{7.1}$$

In the case of second-order total variation flow, the solutions with initial datum (7.1) are known to be of form

$$u(t) = a(t)\mathbf{1}_{B_{R_0}}$$

with finite extinction time, i. e. there exists $t_* > 0$ such that $a(t) = 0$ for $t \geq t_*$. In the fourth order case, based on the treatment of case $n = 1$ in [11], we would expect the solutions to have the form

$$u(t) = a(t)\mathbf{1}_{B_{R(t)}}, \tag{7.2}$$

at least until an extinction time beyond which $u(t, \cdot) \equiv 0$. This intuition turns out to be correct in every dimension except $n = 2$.

Let first $n \geq 3$. As we have checked in section 6, in this case both balls and complements of balls are calibrable. Thus, as long as the solution is of form (7.2) in time instance $t \geq 0$, we expect a valid Cahn-Hoffman vector field Z to be given by

$$Z(x) = \begin{cases} Z_{in}(x) \text{ if } |x| \in [0, R[\\ Z_{out}(x) \text{ if } |x| > R, \end{cases} \tag{7.3}$$

where Z_{in} is the calibration w constructed for a ball B_R and Z_{out} is the calibration we constructed for the complement of that ball, recall:

$$\lambda = -\frac{n(n+2)}{R^3}, \tag{7.4}$$

$$Z_{in}(x) = \frac{1}{2}\left(\frac{|x|}{R}\right)^3 \frac{x}{|x|} - \frac{3}{2}\frac{x}{R}, \qquad Z_{out}(x) = -\frac{n-1}{2}\left(\frac{|x|}{R}\right)^{3-n} \frac{x}{|x|} + \frac{n-3}{2}\left(\frac{|x|}{R}\right)^{1-n} \frac{x}{|x|}. \tag{7.5}$$

We further calculate:

$$\operatorname{div} Z_{in}(x) = \frac{n+2}{2}\frac{|x|^2}{R^3} - \frac{3n}{2}\frac{1}{R}, \qquad \operatorname{div} Z_{out}(x) = -(n-1)\frac{|x|^{2-n}}{R^{3-n}},$$

$$\nabla \operatorname{div} Z_{in}(x) = (n+2)\frac{x}{R^3}, \qquad \nabla \operatorname{div} Z_{out}(x) = (n-1)(n-2)\frac{|x|^{-n}x}{R^{3-n}}.$$

It is straightforward to check that $\operatorname{div} Z \in D^1(\mathbb{R}^n) \cap L^{2^*}(\mathbb{R}^n) = D^1_0(\mathbb{R}^n)$. Next, we deduce

$$\begin{aligned} u_t &= -\Delta \operatorname{div} Z \\ &= -\Delta \operatorname{div} Z_{in}\, \mathcal{L}^n \llcorner_{B_R} - \Delta \operatorname{div} Z_{out}\, \mathcal{L}^n \llcorner_{\mathbb{R}^n \setminus B_R} + \frac{x}{|x|} \cdot (\nabla \operatorname{div} Z_{in} - \nabla \operatorname{div} Z_{out})\, \mathcal{H}^{n-1} \llcorner_{\partial B_R} \\ &= -\frac{n(n+2)}{R^3}\, \mathcal{L}^n \llcorner_{B_R} - \frac{n(n-4)}{R^2}\, \mathcal{H}^{n-1} \llcorner_{\partial B_R}. \end{aligned} \tag{7.6}$$

Then, using the identity $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} u = \int_{\mathbb{R}^n} u_t$, we obtain (recall notation (7.2))

$$a(t)\mathcal{H}^{n-1}\left(\partial B_{R(t)}\right)\frac{\mathrm{d}R}{\mathrm{d}t} = a(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}^n\left(B_{R(t)}\right) = -\frac{n(n-4)}{R^2}\mathcal{H}^{n-1}\left(\partial B_{R(t)}\right).$$

Summing up, evolution of initial datum (7.1) is given by (7.2) with a, R satisfying

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\frac{n(n+2)}{R^3}, \qquad \frac{\mathrm{d}R}{\mathrm{d}t} = -\frac{n(n-4)}{R^2 a}. \tag{7.7}$$

This system can be explicitly solved by noticing that

$$\frac{\mathrm{d}}{\mathrm{d}t}(aR^3) = -n(n+2) - 3n(n-4) = -n(4n-10)$$

and therefore

$$aR^3 = a_0R_0^3 - n(4n-10)t$$

along trajectories. The solution is

$$a(t) = a_0\left(1 - \frac{n(4n-10)}{a_0R_0^3}t\right)^{\frac{n+2}{4n-10}}, \qquad R(t) = R_0\left(1 - \frac{n(4n-10)}{a_0R_0^3}t\right)^{\frac{n-4}{4n-10}}. \tag{7.8}$$

We note that the solution satisfies

$$\left(\frac{a}{a_0}\right)^{n-4} = \left(\frac{R}{R_0}\right)^{n+2}$$

along trajectories. (This "first integral" could also have been used to solve the system (7.7).) Let us point out a few observations concerning the solutions:

- the extinction time is equal to $t_* = \frac{a_0R_0^3}{n(4n-10)}$,
- if $n=3$, $R(t)$ is increasing and $R(t)\to+\infty$ as $t\to t_*^-$,
- if $n=4$, $R(t)=R_0$ is constant,
- in higher dimensions, $R(t)$ is decreasing and $R(t)\to 0$ as $t\to t_*^-$.

In the case $n=2$ we were able to exhibit a calibration for the ball B_R, but not for its complement. Another possible ansatz on the Cahn-Hoffman vector field of form (7.3) is one where Z_{in} is the calibration we constructed for B_R and Z_{out} is the choice considered in [14]:

$$Z_{in}(x) = \frac{1}{2}\left(\frac{|x|}{R}\right)^3\frac{x}{|x|} - \frac{3}{2}\frac{x}{R}, \qquad Z_{out}(x) = -\frac{x}{|x|}. \tag{7.9}$$

We calculate

$$\operatorname{div} Z_{out} = -\frac{(n-1)}{|x|}, \qquad \nabla\operatorname{div} Z_{out} = \frac{(n-1)x}{|x|^3}, \tag{7.10}$$

hence

$$u_t(t,x) = -\frac{(n-1)(n-3)}{|x|^3} \quad \text{in } \mathcal{D}'\left(\mathbb{R}^n\setminus\overline{B}_{R(t)}\right). \tag{7.11}$$

If $n\geq 4$, this would lead to $u(t)$ being radially strictly increasing for positive t and large values of $|x|$, which would be at odds with our choice of Z_{out}. In fact, if $n\geq 4$, $\operatorname{div} Z\notin D_0^1(\mathbb{R}^n)$ for any Z of this form. However, in smaller dimensions this ansatz remains a viable option. If $n=3$, it leads to the same solution as before. On the other hand, if $n=2$, we obtain a solution which is not of form (7.2). Instead, we are led to assume

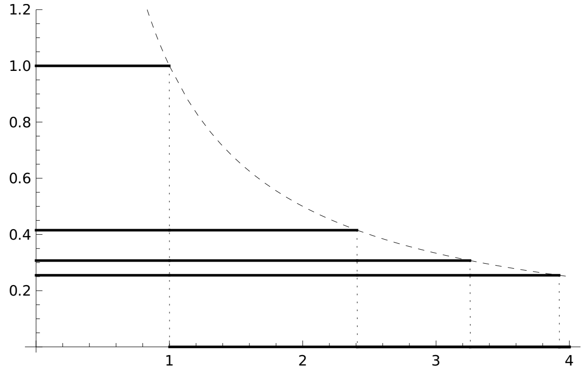
$$u(t,x) = a(t)\mathbf{1}_{B_{R(t)}} + \frac{t}{|x|^3}\mathbf{1}_{\mathbb{R}^2\setminus B_{R(t)}}. \tag{7.12}$$

We have:

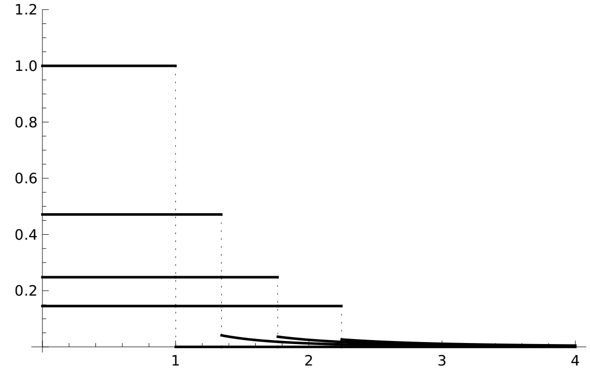
$$\operatorname{div} Z_{in}(x) = 2\frac{|x|^2}{R^3} - 3\frac{1}{R}, \qquad \operatorname{div} Z_{out}(x) = -\frac{1}{|x|},$$

$$\nabla\operatorname{div} Z_{in}(x) = 4\frac{x}{R^3}, \qquad \nabla\operatorname{div} Z_{out}(x) = \frac{x}{|x|^3},$$

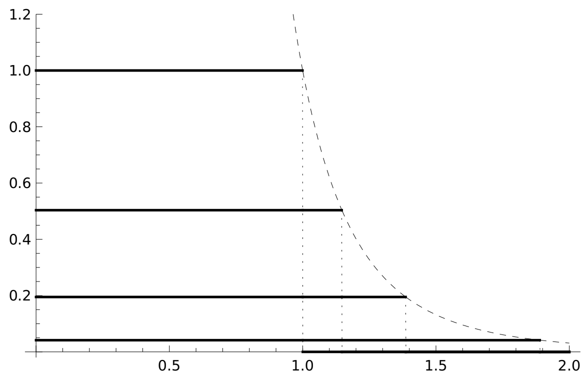
$$u_t = -\Delta\operatorname{div} Z = -\frac{8}{R^3}\,\mathcal{L}^2 \llcorner_{B_R} + \frac{1}{|x|^3}\,\mathcal{L}^2\llcorner_{\mathbb{R}^2\setminus B_R} + \frac{3}{R^2}\,\mathcal{H}^1\llcorner_{\partial B_R}$$



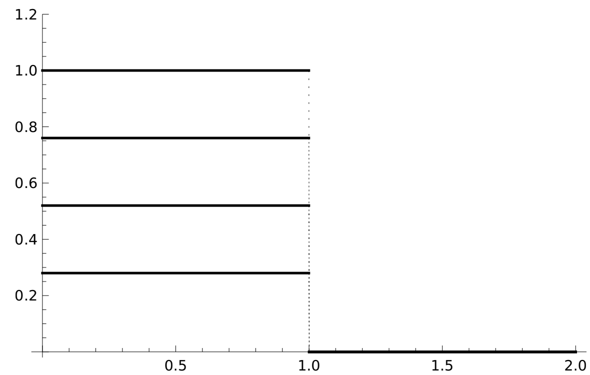
(a) Case $n = 1$. Solid lines: plots of $u(t, \cdot)$ for $t = 0$, $t = 0.8$, $t = 1.6$, $t = 2.4$.



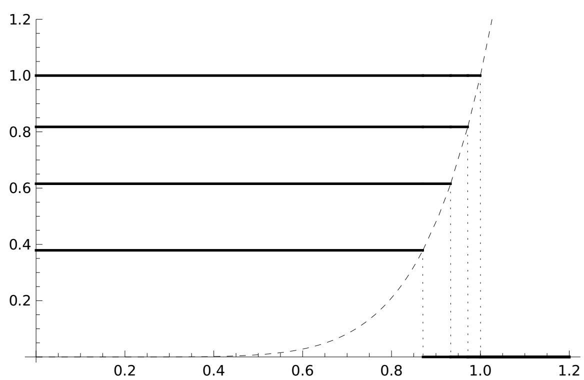
(b) Case $n = 2$. Solid lines: plots of $u(t, \cdot)$ for $t = 0$, $t = 0.1$, $t = 0.2$, $t = 0.3$.



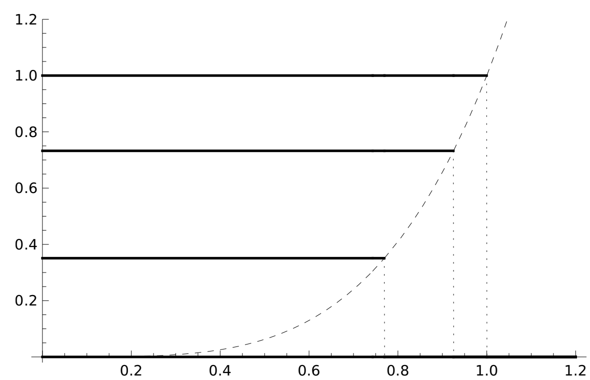
(c) Case $n = 3$. Solid lines: plots of $u(t, \cdot)$ for $t = 0$, $t = 0.04$, $t = 0.08$, $t = 0.12$.



(d) Case $n = 4$. Solid lines: plots of $u(t, \cdot)$ for $t = 0$, $t = 0.01$, $t = 0.02$, $t = 0.03$.



(e) Case $n = 5$. Solid lines: plots of $u(t, \cdot)$ for $t = 0$, $t = 0.005$, $t = 0.01$, $t = 0.015$.



(f) Case $n = 6$. Solid lines: plots of $u(t, \cdot)$ for $t = 0$, $t = 0.005$, $t = 0.01$, $t = 0.015$.

Figure 3: Plots of the solution $u(t, x)$ emanating from the characteristic function of the unit ball as a function of $|x|$ for chosen values of t.

and, recalling (7.12),

$$\left(a(t)-\frac{t}{R(t)^3}\right)\mathcal{H}^1(\partial B_{R(t)})\frac{\mathrm{d}R}{\mathrm{d}t}=\left(a(t)-\frac{t}{R(t)^3}\right)\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}^2(B_{R(t)})=\frac{3}{R(t)^2}\mathcal{H}^1(\partial B_{R(t)}).$$

Thus, we arrive at ODE system

$$\frac{\mathrm{d}a}{\mathrm{d}t}=-\frac{8}{R^3},\qquad \frac{\mathrm{d}R}{\mathrm{d}t}=\frac{3R}{aR^3-t}. \tag{7.13}$$

This system is not autonomous, but it can be integrated by noticing that along trajectories

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(aR^3-t\right)=\frac{9aR^3}{aR^3-t}-8-1=\frac{9t}{aR^3-t}$$

and so

$$aR^3=\sqrt{a_0^2R_0^6+9t^2}+t.$$

This implies, first of all, that

$$a(t)>\frac{t}{R(t)^3}$$

for all $t>0$ and the form of solution (7.12) is preserved as long as the solution does not vanish. Furthermore, we can rewrite the system (7.13) in decoupled form

$$\frac{\mathrm{d}}{\mathrm{d}t}\log a=\frac{-8}{\sqrt{a_0^2R_0^6+9t^2}+t},\qquad \frac{\mathrm{d}}{\mathrm{d}t}\log R^3=\frac{9}{\sqrt{a_0^2R_0^6+9t^2}}. \tag{7.14}$$

These equations can be explicitly integrated:

$$a(t)=a_0\frac{a_0^4R_0^{12}+8a_0^2R_0^{12}t^2}{\left(\sqrt{a_0^2R_0^6+9t^2}+3t\right)^2\left(a_0^2R_0^6+6t^2+2t\sqrt{a_0^2R_0^6+9t^2}\right)},$$

$$R(t)=R_0\sqrt{1+6t\frac{3t+\sqrt{a_0^2R_0^6+9t^2}}{a_0^2R_0^6}}.$$

We observe that the solutions exist globally and

$$\lim_{t\to\infty}a(t)=0,\qquad \lim_{t\to\infty}R(t)=\infty.$$

In particular u stays in the form (7.12) for all $t>0$.

Finally we consider $n=1$. In this case, both ansätze considered before lead to the same solution:

$$Z_{in}(x)=\frac{1}{2}\left(\frac{x}{R}\right)^3-\frac{3}{2}\frac{x}{R},\qquad Z_{out}(x)=-\operatorname{sgn}x \tag{7.15}$$

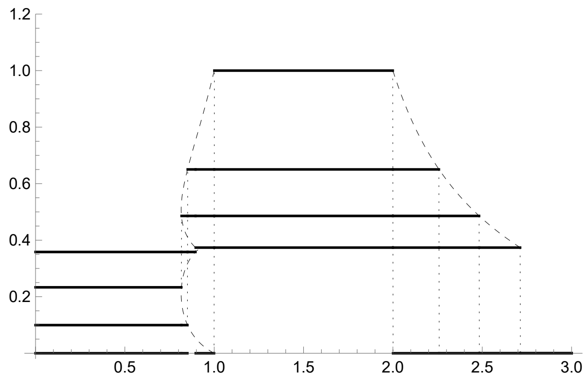
which coincides with (7.5). Repeating the calculations following (7.5), we obtain a solution of form (7.2) satisfying (7.8), i. e.

$$u(t)=a(t)\mathbf{1}_{B_{R(t)}},\qquad a(t)=a_0\left(1+\frac{6}{a_0R_0^3}t\right)^{-\frac{1}{2}},\qquad R(t)=R_0\left(1+\frac{6}{a_0R_0^3}t\right)^{\frac{1}{2}}. \tag{7.16}$$

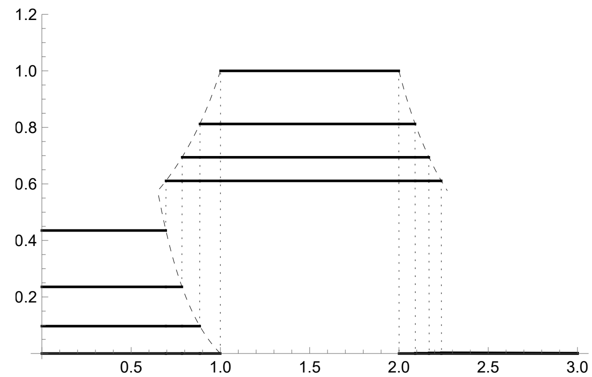
Note that now, as opposed to the case $n\geq 3$, the coefficient multiplying t is positive. Like in $n=2$, the extinction time is infinite and we have

$$\lim_{t\to\infty}a(t)=0,\qquad \lim_{t\to\infty}R(t)=\infty.$$

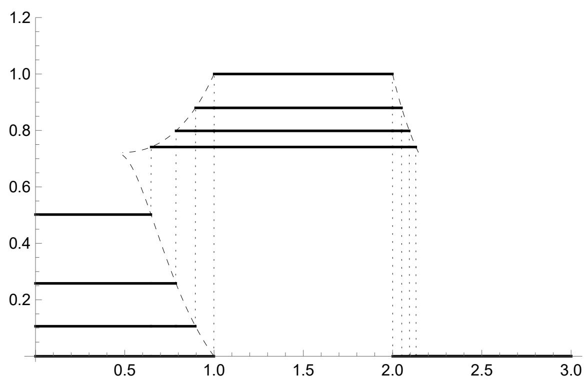
This concludes the proof of Theorem 4.



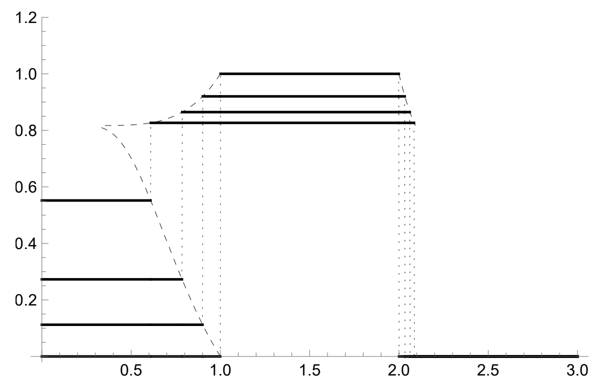
(a) Case $n = 1$. Solid lines: plots of $u(t, \cdot)$ for $t = 0$, $t = 0.025$, $t = 0.5$, $t = 0.075$.



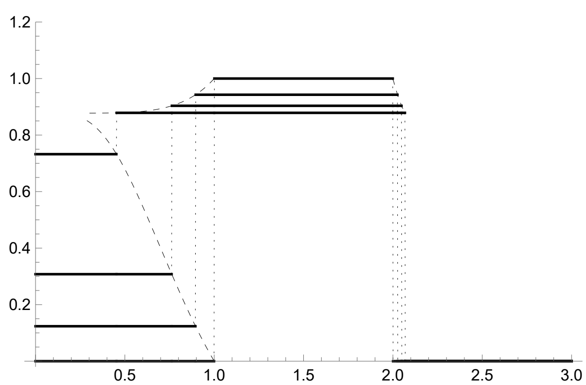
(b) Case $n = 2$. Solid lines: plots of $u(t, \cdot)$ for $t = 0$, $t = 0.01$, $t = 0.02$, $t = 0.03$.



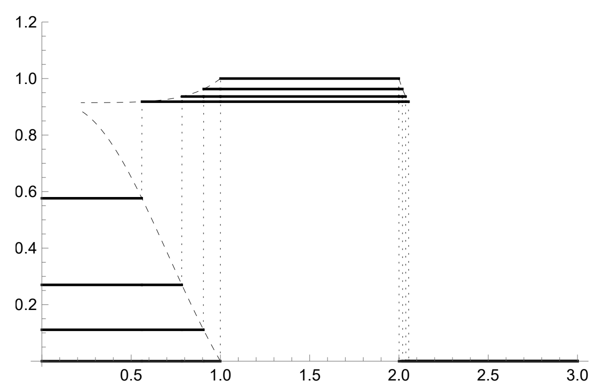
(c) Case $n = 3$. Solid lines: plots of $u(t, \cdot)$ for $t = 0$, $t = 0.006$, $t = 0.012$, $t = 0.018$.



(d) Case $n = 4$. Solid lines: plots of $u(t, \cdot)$ for $t = 0$, $t = 0.004$, $t = 0.008$, $t = 0.012$.



(e) Case $n = 5$. Solid lines: plots of $u(t, \cdot)$ for $t = 0$, $t = 0.003$, $t = 0.006$, $t = 0.009$.



(f) Case $n = 6$. Solid lines: plots of $u(t, \cdot)$ for $t = 0$, $t = 0.002$, $t = 0.004$, $t = 0.006$.

Figure 4: Plots of the solution $u(t, x)$ emanating from the characteristic function of annulus $A_{R^1}^{R^2}$ as a function of $|x|$ for chosen values of t with $R^1 = 1$, $R^2 = 2$.

7.2 Stacks

Using the calibrations we constructed for generalized annuli, we will now derive a system of ODEs locally prescribing the solution emanating from any piecewise constant, radially symmetric datum (a *stack*).

Definition 31. *Let $w \in D(TV)$. We say that w is a stack if there exists a number $N \in \mathbb{N}$ and sequences $0 < R^0 < R^1 < \ldots < R^{N-1}$, $a^0, a^1, \ldots, a^N$ with $a^k \in \mathbb{R}$ such that*

$$w = a^0 \mathbf{1}_{B_{R^0}} + a^1 \mathbf{1}_{A_{R^0}^{R^1}} + \ldots + a^{N-1} \mathbf{1}_{A_{R^{N-2}}^{R^{N-1}}} + a^N \mathbf{1}_{\mathbb{R}^n \setminus B_{R^{N-1}}}.$$

Suppose first that $n \neq 2$, in which case all connected components of level sets of any stack w are calibrable. Let u_0 be a stack

$$u_0 = a_0^0 \mathbf{1}_{B_{R_0^0}} + a_0^1 \mathbf{1}_{A_{R_0^0}^{R_0^1}} + \ldots + a_0^{N-1} \mathbf{1}_{A_{R_0^{N-2}}^{R_0^{N-1}}} + a_0^N \mathbf{1}_{\mathbb{R}^n \setminus B_{R_0^{N-1}}}, \tag{7.17}$$

where $a^{k-1} \neq a^k$ for $k = 1, \ldots, N$, $a_0^N = 0$. We expect that if u is the solution emanating from u_0, then $u(t, \cdot)$ is a stack of form

$$u(t, \cdot) = a^0(t) \mathbf{1}_{B_{R^0(t)}} + a^1(t) \mathbf{1}_{A_{R^0(t)}^{R^1(t)}} + \ldots + a^{N-1}(t) \mathbf{1}_{A_{R^{N-2}(t)}^{R^{N-1}(t)}} + a^N(t) \mathbf{1}_{\mathbb{R}^n \setminus B_{R^{N-1}(t)}}, \tag{7.18}$$

with $a^N(t) = 0$ for all $t > 0$, and that $a^{k-1} \neq a^k$, $k = 1, \ldots, N$ for small t. We construct a Cahn-Hoffman vector field $Z(t, \cdot)$ for $u(t, \cdot)$ by pasting together calibrations Z^k for $B_{R^0(t)}$, $A_{R^k(t)}^{R^{k+1}(t)}$, $\mathbb{R}^n \setminus B_{R^{N-1}(t)}$ with suitable choice of signatures. We have

$$\begin{aligned} u_t = -\Delta \operatorname{div} Z = -\Delta \operatorname{div} Z^0 \, \mathcal{L}^n \llcorner B_{R^0} - \sum_{k=1}^{n} \Delta \operatorname{div} Z^k \, \mathcal{L}^n \llcorner A_{R^{k-1}}^{R^k} - \Delta \operatorname{div} Z^N \, \mathcal{L}^n \llcorner \mathbb{R}^n \setminus B_{R^N} \\ + \sum_{k=0}^{n} \frac{x}{|x|} \cdot (\nabla \operatorname{div} Z^k - \nabla \operatorname{div} Z^{k+1}) \, \mathcal{H}^{n-1} \llcorner S_{R^k}. \end{aligned} \tag{7.19}$$

We denote

$$\begin{aligned} &\frac{x}{|x|} \cdot (\nabla \operatorname{div} Z^k - \nabla \operatorname{div} Z^{k+1}) \Big|_{S_{R^k}} \\ &\quad = z_{rr}^k(R^k) - z_{rr}^{k+1}(R^k) + \frac{n-1}{R^k}(z_r^k(R^k) - z_r^{k+1}(R^k)) - \frac{n-1}{(R^k)^2}(z^k(R^k) - z^{k+1}(R^k)) \\ &\quad = z_{rr}^k(R^k) - z_{rr}^{k+1}(R^k) =: d^k. \end{aligned}$$

The values of d^k are functions of $R^0, \ldots, R^{N-1}$. Assuming that R^k are regular enough and ε, $|t - s|$ are small enough, we have

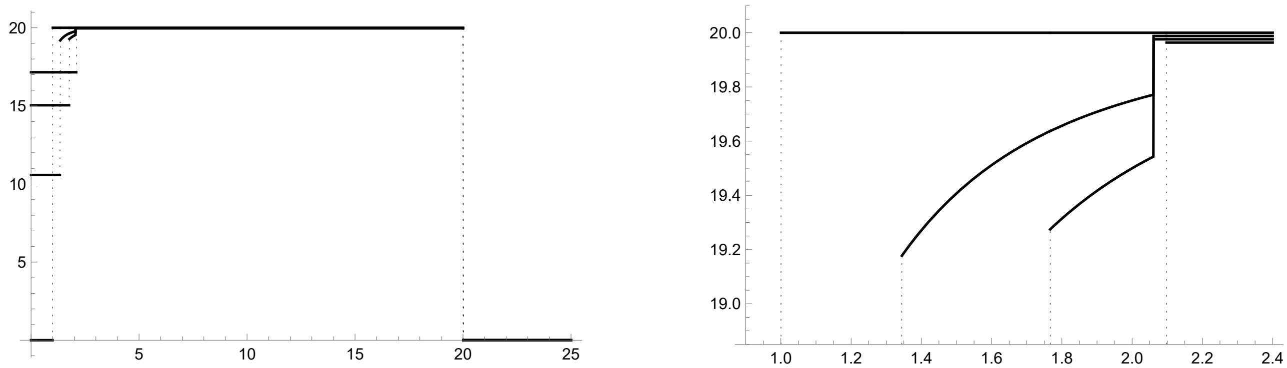
$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{A_{R^k(s)-\varepsilon}^{R^k(s)+\varepsilon}} u = \int_{A_{R^k(s)-\varepsilon}^{R^k(s)+\varepsilon}} u_t,$$

whence

$$(a^k(t) - a^{k-1}(t)) \mathcal{H}^{n-1}(S_{R^k(t)}) \frac{\mathrm{d}R^k}{\mathrm{d}t} = (a^k(t) - a^{k-1}(t)) \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{L}^n(A_{R^k(s)-\varepsilon}^{R^k(t)}) = d^k(t) \mathcal{H}^{n-1}(S_{R^k(t)}).$$

Further, for $k = 0, \ldots, N$, we denote by λ^k the value of $-\Delta \operatorname{div} Z^k(t, \cdot)$ which is constant since Z^k is a calibration. Then, we can write down the system of ODEs for a^k and R^k:

$$\frac{\mathrm{d}a^k}{\mathrm{d}t} = \lambda^k \text{ for } k = 0, \ldots N, \qquad \frac{\mathrm{d}R^k}{\mathrm{d}t} = \frac{d^k}{a^k - a^{k+1}} \text{ for } k = 0, \ldots N-1. \tag{7.20}$$



(a) Plot over $[0, 25]$.

(b) Zoomed in plot over $[0.9, 2.4]$.

Figure 5: Plots of the solution $u(t, x)$ emanating from the characteristic function of annulus $A_{R^1}^{R^2}$ with $R_0 = 1$, $R_1 = 20$ in $n = 2$ as a function of $|x|$ for $t = 0$, $t = 2$, $t = 4$, $t = 6$.

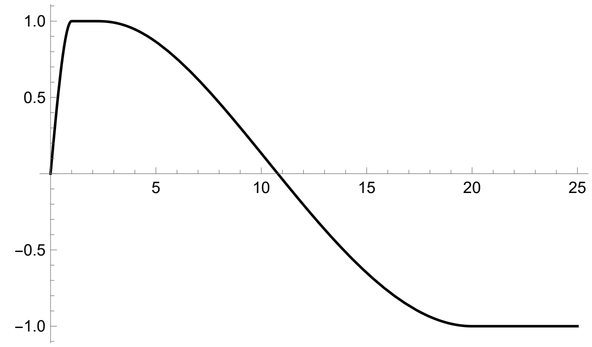


Figure 6: Plot of the Cahn-Hoffman vector field $Z(t, x)$ for the characteristic function of annulus $A_{R^1}^{R^2}$ with $R_0 = 1$, $R_1 = 20$ in $n = 2$ as a function of $|x|$.

Let c_0^k denote c_0 given by (6.17) if $\operatorname{sgn}(a^{k+1} - a^k) = \operatorname{sgn}(a^k - a^{k-1})$ or by (6.10) if $\operatorname{sgn}(a^{k+1} - a^k) \neq \operatorname{sgn}(a^k - a^{k-1})$, with R^{k+1} and R^k in place of R_1 and R_0. Then, we have

$$\lambda^0 = \operatorname{sgn}(a^1 - a^0)\frac{n(n+2)}{(R^0)^3}, \quad \lambda^k = 2n(n+2)\operatorname{sgn}(a^{k+1} - a^k)c_0^k \text{ for } k = 1, \dots, N-1, \quad \lambda^N = 0. \tag{7.21}$$

We observe that in a neighborhood of any initial datum $R_0^0, \dots, R_0^{N-1}$, $a_0^0, \dots, a_0^N$, $R_0^k < R_0^{k+1}$, $a_0^k \neq a_0^{k+1}$, the r. h. s. of (7.20) is regular in $R^0, \dots, R^{N-1}$, $a^0, \dots, a^N$, so locally the system has a unique solution. Unique solvability fails when a time instance $t > 0$ is reached such that $a^k(t) = a^{k+1}(t)$, $R^k(t) = R^{k+1}(t)$ or $R_0 = 0$. In such case $u(t, \cdot)$ is again a stack with a smaller N, and we can restart our procedure. This concludes the proof of Theorem 5.

Next we deal with the remaining case of dimension $n = 2$. In this case, our attempt to obtain a radial calibration failed for complements of balls and for some annuli. Again, let u_0 be a stack of form (7.17). For $k = 1, \dots, N$, let $\sigma^k = \operatorname{sgn}(a_0^k - a_0^{k-1})$. We assume the following ansatz on the solution u and the associated field Z for small $t > 0$:

$$u(t, \cdot) = a^0(t) \text{ on } B_{R^0(t)}, u(t, \cdot) = a^k(t) \text{ on } A_{\max(R^{k-1}(t), R^k(t)/Q_*)}^{R^k(t)} \text{ if } \sigma^{k+1} \neq \sigma^k,\ k = 1, \dots, N-1, \tag{7.22}$$

$$Z(t,x) = \sigma^k \tfrac{x}{|x|} \text{ on } A^{R^k(t)}_{R^{k-1}(t)} \text{ if } \sigma^{k+1} \neq \sigma^k \text{ or on } A^{R^k(t)/Q_*}_{R^{k-1}(t)} \text{ if } \sigma^{k+1} = \sigma^k,\ k = 1, \ldots, N-1,$$
$$Z(t,x) = \sigma^{k+1} \tfrac{x}{|x|} \text{ on } \mathbb{R}^n \setminus B_{R^{N-1}(t)}. \quad (7.23)$$

We complete the definition of a Cahn-Hoffman field Z consistent with (7.22), (7.23) by pasting the calibrations Z^k with suitable choice of signatures into the gaps left in (7.23). This leads to

$$u_t(t,\cdot) = \lambda^0(t) \text{ in } \mathcal{D}'(B_{R^0(t)}),$$

$$u_t(t,x) = \lambda^k(t) \text{ in } \mathcal{D}'\left(A^{R_k(t)}_{\max(R^{k-1}(t), R^k(t)/Q_*)}\right),$$
$$u_t(t,x) = \frac{\sigma^k}{|x|^3} \text{ in } \mathcal{D}'\left(A^{R^k(t)/Q_*}_{R^{k-1}(t)}\right) \text{ if } \sigma^k \neq \sigma^{k+1} \text{ or in } \mathcal{D}'\left(A^{R^k(t)}_{R^{k-1}(t)}\right) \text{ if } \sigma^k = \sigma^{k+1},\ k = 1, \ldots, N-1,$$
$$u_t(t,x) = \frac{\sigma^N}{|x|^3} \text{ in } \mathcal{D}'(\mathbb{R}^2 \setminus B_{R^N(t)}).$$

Moreover, $u_t(t,\cdot) \in M(\mathbb{R}^2)$ and

$$u_t \llcorner S_{R^k} = \frac{x}{|x|} \cdot ((\nabla \operatorname{div} Z)^- - (\nabla \operatorname{div} Z)^+)\mathcal{H}^1 \llcorner S_{R^k} =: d^k$$

for $k = 0, \ldots, N-1$, where $(\nabla \operatorname{div} Z)^\pm$ are the one sided limits as $|x| \to (R^k)^\pm$. The values of d^k are functions of $R^0, \ldots, R^{N-1}$. Reasoning as in the case $n \neq 2$, the evolution of R^k is governed by equations

$$\frac{\mathrm{d}R^k}{\mathrm{d}t} = \frac{d^k}{u(t,x)\big|_{|x|=(R^k)^-} - u(t,x)\big|_{|x|=(R^k)^+}}. \quad (7.24)$$

The values $u(t,x)\big|_{|x|=(R^k)^+}$ are either prescribed by ODEs

$$\frac{\mathrm{d}a^k}{\mathrm{d}t} = \lambda^k \quad (7.25)$$

with λ^k functions of $R^0, \ldots, R^{N-1}$ in calibrable regions where $u(t,x) = a^k(t)$, or explicitly determined by $u_t(t,x) = \sigma^k/|x|^3$ in bending regions. It is important to note that in the case $\sigma^k \neq \sigma^{k+1}$, $R^{k-1} \leq R^k/Q_*$ the functions d^k, λ^k do not depend on R^{k-1}. Thus, one can first solve a part of the system (7.24), (7.25) for the outer annuli, then calculate u in the bending region (without knowing a priori its inner boundary) and move on to solving innermore parts of (7.24), (7.25). This way, finding the solution is indeed again reduced to solving a system of ODEs.

References

[1] F. Alter, V. Caselles and A. Chambolle, A characterization of convex calibrable sets in $\mathbb{R}^N$, *Math. Ann.*, **332** (2005), 329–366.

[2] L. Ambrosio, N. Luigi and G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*, Second edition, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2008.

[3] F. Andreu-Vaillo, V. Caselles and J. M. Mazón, *Parabolic quasilinear equations minimizing linear growth functionals*, Progress in Mathematics, 223, Birkhäuser Verlag, Basel, 2004.

[4] G. Bellettini, V. Caselles and M. Novaga, The total variation flow in $\mathbb{R}^N$, *J. Differential Equations*, **184** (2002), 475–525.

[5] G. Bellettini, M. Novaga and M. Paolini, Characterization of facet breaking for nonsmooth mean curvature flow in the convex case, *Interfaces Free Bound.*, **3** (2001), 415–446.

[6] H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland Mathematics Studies, No. 5, Notas de Matemática, No. 50, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973.

[7] W. C. Carter, A. R. Roosen, J. W. Cahn and J. E. Taylor, Shape evolution by surface diffusion and surface attachment limited kinetics on completely faceted surfaces, *Acta Metall. Mater.*, **43** (1995), 4309–4323.

[8] C. M. Elliott and S. A. Smitheman, Analysis of the TV regularization and H^{-1} fidelity model for decomposing an image into cartoon plus texture, *Commun. Pure Appl. Anal.*, **6** (2007), 917–936.

[9] W. Fulton, *Algebraic topology. A first course*, Graduate Texts in Mathematics, 153, Springer-Verlag, New York, 1995.

[10] G. P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations. Steady-state problems*, Second edition, Springer Monographs in Mathematics, Springer, New York, 2011.

[11] M.-H. Giga and Y. Giga, Very singular diffusion equations: second and fourth order problems, *Japanese J. Ind. Appl. Math.*, **27** (2010), 323–345.

[12] M.-H. Giga and Y. Giga, Crystalline surface diffusion flow for graph-like curves, *Hokkaido University Preprint Series in Mathematics*, **1142** (2022).

[13] Y. Giga and R. V. Kohn, Scale-invariant extinction time estimates for some singular diffusion equations, *Discrete Contin. Dyn. Syst.*, **30** (2011), 509–535.

[14] Y. Giga, H. Kuroda and H. Matsuoka, Fourth-order total variation flow with Dirichlet condition: Characterization of evolution and extinction time estimates, *Adv. Math. Sci. Appl.*, **24** (2014), 499–534.

[15] Y. Giga, M. Muszkieta and P. Rybka, A duality based approach to the minimizing total variation flow in the space H^{-s}, *Jpn. J. Ind. Appl. Math.*, **36** (2019), 261–286.

[16] Y. Giga and N. Požár, Motion by crystalline-like mean curvature: a survey, *Hokkaido University Preprint Series in Mathematics*, **1140** (2021).

[17] Y. Giga and Y. Ueda, Numerical computations of split Bregman method for fourth order total variation flow, *J. Comput. Phys.*, **405** (2020), 109114.

[18] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[19] Y. Kashima, A subdifferential formulation of fourth order singular diffusion equations, *Adv. Math. Sci. Appl.*, **14** (2004), 49–74.

[20] Y. Kashima, Characterization of subdifferentials of a singular convex functional in Sobolev spaces of order minus one, *J. Funct. Anal.*, **262** (2012), 2833–2860.

[21] R. V. Kohn, Surface relaxation below the roughening temperature: some recent progress and open questions, In: *Nonlinear partial differential equations*, Abel Symp., 7, Springer, Heidelberg, 2012, 207–221.

[22] R. V. Kohn and H. M. Versieux, Numerical analysis of a steepest-descent PDE model for surface relaxation below the roughening temperature, *SIAM J. Numer. Anal.*, **48** (2010), 1781–1800.

[23] Y. Kōmura, Nonlinear semi-groups in Hilbert space, *J. Math. Soc. Japan*, **19** (1967), 493–507.

[24] M. Łasica, S. Moll and P. B. Mucha, Total variation denoising in l^1 anisotropy. *SIAM J. Imaging Sci.*, **10** (2017), 1691–1723.

[25] G. P. Leonardi, An overview on the Cheeger problem, In: *New trends in shape optimization*, Internat. Ser. Numer. Math., 166, Birkhäuser/Springer, Cham, 2015, 117–139.

[26] I. V. Odisharia, Simulation and analysis of the relaxation of a crystalline surface, Ph.D. thesis, New York University, New York 2006.

[27] S. Osher, A. Solé and L. Vese, Image decomposition and restoration using total variation minimization and the H^{-1} norm, *Multiscale Model. Simul.*, **1** (2003), 349–370.

[28] W. Rudin, *Functional analysis*, Second edition, International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, 1991.

[29] L. Schwartz, *Théorie des distributions*, Nouvelle édition, entiérement corrigée, refondue et augmentée, Publications de l'Institut de Mathématique de l'Université de Strasbourg, IX-X Hermann, Paris 1966.

[30] H. Spohn, Surface dynamics below the roughening transition, *J. Phys. I France*, **3** (1993), 69–81.

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