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Accurate Results for Free Vibration of Doubly Curved Shallow Shells of Rectangular Planform (Part. 2 Thickness effect)

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Abstract

This paper presents a follow-up study of a previous work that deals with the free vibration of moderately thin isotropic shallow shells under general edge conditions. The same semi-analytical method is used in this study for identical shape and degree of curvature in doubly curved geometry, and accurate natural frequencies are tabulated for a wide range of the shell edge conditions. Emphasis is made, however, to present the frequency parameters for the shallow shells with very thin thickness (representative length/shell thickness=100). In numerical experiments, convergence test is made against series terms in the case of very thin shallow shells. Twenty-one sets of frequency parameters are tabulated for three shell shapes (spherical, cylindrical and hyperbolic paraboloidal shells) and two curvature ratios. These two papers (Part. 1 and 2) will constitute the accurate standard in the area of shallow shell vibration of rectangular planform and serve for future comparison and practical design purpose.

Keywords: Accuracy; free vibration; natural frequency; shallow shell; small thickness

1. Introduction

There has been active and increasing usage of shallow shell structures in mechanical and structural engineering. A growth in the literature on shallow shell vibration has reflected this technical trend [1]. This author has, however, noticed in the published literature a significant lack of comprehensive sets of accurate natural frequencies to cover a wide range of shallow shell geometries and boundary conditions.

In a previous work [2], an attempt was made to present a semi-analytical method and to provide comprehensive lists of natural frequencies of open shallow shells of rectangular planform. As distinctive feature, not found in other types of closed shells, open shallow shell can take wide variations of geometric form, such as spherical shell, cylindrical shell and parabolic hyperbolic shell, each with different degree of curvature. In other words, there are a large number of combinations in the shape parameters.

For shallow shell vibration, relevant previous works are summarized in Ref. [2]. Practically the first landmark paper on this topic is one [3] published by Leissa and Kadi that formulates the exact solution of shallow shells of rectangular planform supported along four edges by shear diaphragm. This paper is followed by other works [4-7] in the 1980's, and by many works [8-18] up to the present. From the viewpoint of providing comprehensive lists of natural frequencies for general boundary conditions, it is noted that two papers [19, 20] present methods and numerical results under various combinations of in-plane and out-of-plane boundary conditions. In Ref. [19], Mochida and his co-workers use a superposition method and present natural frequencies of various shallow shells, but shallow shells with free edges are not included. Qatu and Asadi [20] present frequencies of the shells with twenty-one different sets of boundary conditions, but it seems to the present author that the numerical results are not well converged.

The objective of this work (Part. 2) is to present comprehensive lists of accurate frequency parameters of very thin shallow shells for twenty-one sets of the boundary conditions. With the two studies, reasonably sufficient free vibration information can be summarized to cover shallow shells with relatively thick case (representative edge length/shell thickness=20) in Part. 1 [2] and very thin case in Part. 2 (thickness ratio=100). The convergence of the solution and comparison with other methods are severely checked, and effect of thickness is discussed by comparing these sets of results.

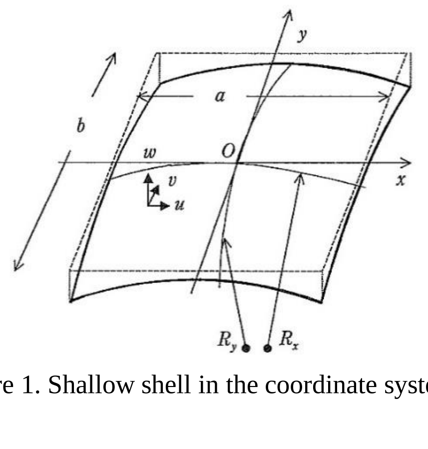

Figure 1. Shallow shell in the coordinate system

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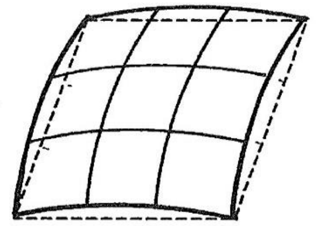
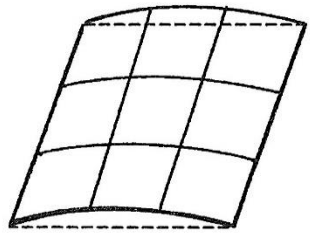
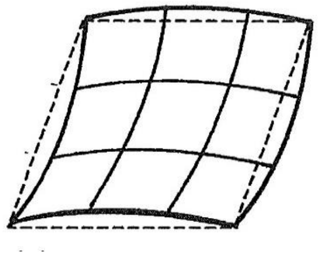


(a) Spherical shallow shell



(b) Cylindrical shallow shell



(c) Hyperbolic paraboloidal shallow shell

Figure 2. Shallow shells of rectangular planform

2. Outline of Analytical Method

The geometry of quadratic mid-surface can be expressed for a doubly curved shallow shell in a rectangular coordinate system (see Fig. 1) by

$$\phi\left(x,y\right)=-\frac{1}{2}\left(\frac{x^2}{R_x}+\frac{y^2}{R_y}\right) \tag{1}$$

where R_x and R_y are the radii of curvature in the x and y directions, respectively. The dimension of its planform is given by $a\times b$ and the thickness is h. The four sides are subjected to uniform in-plane (i.e., stretching) and out-of-plane (bending) boundary conditions.

This shell takes geometric form of a spherical shell for $1/R_x=1/R_y$=(finite) in Fig. 2(a), and takes form of a cylindrical shell for "$1/R_x$=(finite) and $1/R_y=0$ ($R_y=\infty$)" or "$1/R_y$=(finite) and $1/R_x=0$ ($R_x=\infty$)" in Fig. 2(b). When positive curvature exists in x direction and negative curvature in y direction, or vice versa, it takes form of a hyperbolic paraboloidal shell for $1/R_x= -1/R_y$=(finite) in Fig. 2(c).

In the previous study [2], details of extended Ritz method are presented based on Donnell-type shallow shell theory. The same method is used here. The stretching, stretching-bending coupling and bending stiffness matrices are given, respectively, by

$$\left[A\right]=\begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \tag{2a}$$

$$\left[B\right]=\begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \tag{2b}$$

$$\left[D\right]=\begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \tag{2c}$$

For isotropic material, they are simply reduced to

$$A_{ij}=hQ_{ij}\ \ ,\ B_{ij}=0\ ,\ D_{ij}=\left(\frac{h^3}{12}\right)Q_{ij} \tag{3}$$

(i,j=1,2,6), where the coefficients are elastic constants

$$Q_{11}=Q_{22}=\frac{E}{1-\nu^2},\ \ Q_{12}=\nu Q_{11},\ \ Q_{66}=G \tag{4}$$

Here, E is the modulus of elasticity, $G=E/2(1+\nu)$ is the shear modulus and ν is Poisson's ratio.

This semi-analytical method requires the evaluation of energy functional

$$L=T_{max}-\left(V_{s,max}+V_{bs,max}+V_{b,max}\right) \tag{5}$$

where V_s , V_{bs} and V_b are the parts of the total strain energy due to stretching, bending- stretching coupling and bending, respectively, and T is translational kinetic energy. The stationary value is determined in the functional by

$$\frac{\partial L}{\partial P_{ij}}=0\,,\ \frac{\partial L}{\partial Q_{kl}}=0\,,\ \frac{\partial L}{\partial R_{mn}}=0 \tag{6}$$

$(\,i,k,m=0,1,2,..,\left(M-1\right);\,j,l,n=0,1,2,...,\left(N-1\right)$

where P_{ij} , Q_{kl} and R_{mn} are unknown coefficients in the displacement functions. The displacement functions are formulated to satisfy at least the kinematical boundary conditions along the edges. Use of boundary index makes it possible to accommodate any combination of in-plane and out-of-plane boundary conditions [2].

After applying the process in Eq. (6), frequency equation is derived as

$$det\left(\left[K\right]-\Omega^2\left[M\right]\right)=0 \tag{7}$$

where [K] and [M] are global stiffness and mass (inertia) matrices, respectively. The Ω is a frequency parameter

$$\Omega=\omega a^2\sqrt{\frac{\rho h}{D}}\ \text{(dimensionless frequency)} \tag{8}$$

$$D=Eh^3\,/\,12\left(1-\nu^2\right)\ \text{(reference plate stiffness)} \tag{9}$$

The lowest six eigenvalues from Eq.(7) are frequency parameters to be listed in the following tables. It should be noted again that arbitrary sets of boundary conditions can be specified, and details are given in previous study [2]. For edge condition, "F" and ""C" indicate all four displacements are unconstrained and constrained, respectively, and "S" does simply support with in-plane displacement parallel to the edge is zero but one perpendicular to the edge is unconstrained [2].

3. Numerical Examples and Accuracy of Solution

3.1. Convergence and comparison of the solution

In the present numerical examples, very thin thickness (a/h=100) is assumed to study the thickness effect by comparing the results with those in previous study [2] for relatively thick case (a/h=20). Square planform (a/b=1), except in Table.2, and Poisson's ratio =0.3 are used.

Table 1 presents convergence study of frequency parameters of spherical (R_x/R_y=1), cylindrical (R_x/R_y=0, i.e., $R_y=\infty$) and hyperbolic paraboloidal (R_x/R_y= -1) shells of square planform. The shells (SSSS) in this table are supported by shear diaphragm along four edges, and the exact solution is available [3]. For each shell configuration, two degrees of curvature a/R_x=0.2 and 0.5 are used. The present results are calculated for the number of terms 8×8, 10×10 and 12×12 for each of u, v and w, and as in the relatively thick shell [2], the present parameters similarly converge well within five significant figures, and are in very good agreement with the exact values. A pair of half wave number of out-of-plane displacement w is given by (m,n) in the table.

Table 1 Convergence and comparison of frequency parameters Ω of simply supported shallow shells (SSSS), a/b =1, a/h =100, ν =0.3.

	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
Spherical shell (R_x/R_y=1, a/R_x=0.2)						
8×8	68.858	82.426	82.426	102.92	118.77	118.77
10×10	68.857	82.425	82.425	102.92	118.74	118.74
12×12	68.857	82.425	82.425	102.92	118.74	118.74
Exact	68.858	82.425	82.425	102.92	118.74	118.74
(m,n)	(1,1)	(1,2)	(2,1)	(2,2)	(1,3)	(3,1)
Spherical shell (R_x/R_y=1, a/R_x=0.5)						
8×8	164.63	171.70	171.70	182.64	192.09	192.09
10×10	164.63	171.70	171.70	182.63	192.05	192.05
12×12	164.63	171.70	171.70	182.63	192.05	192.05
Exact	164.63	171.70	171.70	182.63	192.05	192.05
(m,n)	(1,1)	(1,2)	(2,1)	(2,2)	(1,3)	(3,1)
Cylindrical shell (R_x/R_y=0, a/R_x=0.2)						
8×8	38.437	51.061	72.299	85.563	98.921	115.24
10×10	38.437	51.059	72.299	85.562	98.893	115.22
12×12	38.437	51.059	72.299	85.562	98.892	115.22
Exact	38.437	51.059	72.299	85.562	98.893	115.22
(m,n)	(1,1)	(1,2)	(2,1)	(2,2)	(3,1)	(1,3)
Cylindrical shell (R_x/R_y=0, a/R_x=0.5)						
8×8	59.191	84.179	99.987	114.02	137.86	140.79
10×10	59.184	84.179	99.915	114.02	137.81	140.79
12×12	59.184	84.179	99.914	114.02	137.81	140.79
Exact	59.184	84.179	99.914	114.02	137.81	140.79
(m,n)	(2,1)	(1,1)	(3,1)	(2,2)	(3,2)	(1,2)
Hyperbolic paraboloidal shell (R_x/R_y=-1, a/R_x=0.2)						
8×8	19.660	63.238	63.238	78.879	111.95	111.95
10×10	19.659	63.236	63.236	78.877	111.93	111.93
12×12	19.660	63.236	63.236	78.877	111.93	111.93
Exact	19.660	63.236	63.236	78.877	111.93	111.93
(m,n)	(1,1)	(1,2)	(2,1)	(2,2)	(1,3)	(3,1)
Hyperbolic paraboloidal shell (R_x/R_y=-1, a/R_x=0.5)						
8×8	19.257	78.472	109.98	109.98	142.75	142.75
10×10	19.257	78.461	109.98	109.98	142.70	142.70
12×12	19.258	78.461	109.98	109.98	142.70	142.70
Exact	19.257	78.461	109.98	109.98	142.70	142.70
(m,n)	(1,1)	(2,2)	(1,2)	(2,1)	(1,3)	(3,1)

Table 2 is a comparison study with values of Ref. [19] by Mochida and his co-workers. They use the method of superposition that is known as a method to provide numerical solutions in good accuracy. They provide frequency parameters only for combinations of two types of in-plane constraints, where displacement normal to the edge is zero and displacement parallel to the edge is zero, and two type of out-of-plane displacement, i.e., simple support and clamped edge. In their work, no results are presented for cases involving free edges.

In a previous study [2], comparison is made also with their values, but for avoiding duplication in this work, different sets of boundary conditions are used. Also result for CCCC is given for a rectangle (a/b=0.5). It is found in the table that the present values exactly agree with their values, when the present ones are rounded with four significant figures. Accuracy of the present solution is

Table 2 Comparison of frequency parameters Ω of shallow shells, a/Rx =0.5, a/h =100, ν =0.3.

	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
SCSS (a/b=1)						
Spherical shell (R_x/R_y=1)						
Present	168.71	173.78	181.43	188.92	193.37	204.54
Ref.[19]	168.7	173.8	181.4	188.9	193.4	204.6
Cylindrical shell (R_x/R_y=0)						
Present	64.987	87.899	102.51	120.76	144.25	144.86
Ref.[19]	64.98	87.90	102.5	120.8	144.2	144.9
Hyperbolic paraboloidal shell (R_x/R_y=-1)						
Present	82.623	90.895	130.08	131.01	150.55	169.19
Ref.[19]	82.62	90.90	130.1	131.0	150.5	169.2
SCSC (a/b=1)						
Spherical shell (R_x/R_y=1)						
Present	177.61	182.78	185.75	195.19	196.05	219.24
Ref.[19]	177.6	182.8	185.8	195.2	196.1	219.2
Cylindrical shell (R_x/R_y=0)						
Present	71.325	92.637	105.67	127.94	149.84	151.31
Ref.[19]	71.32	92.63	105.7	127.9	149.8	151.3
Hyperbolic paraboloidal shell (R_x/R_y=-1)						
Present	127.84	133.29	135.91	142.72	170.73	171.02
Ref.[19]	127.9	133.3	135.9	142.7	170.7	171.0
CCCC (a/b=0.5)						
Spherical shell (R_x/R_y=1)						
Present	179.41	179.90	180.47	187.88	189.62	192.52
Ref.[19]	179.4	179.9	180.5	187.9	189.6	192.5
Cylindrical shell (R_x/R_y=0)						
Present	72.277	95.799	106.45	116.43	120.96	132.05
Ref.[19]	72.27	95.78	106.4	116.4	120.9	132.0
Hyperbolic paraboloidal shell (R_x/R_y=-1)						
Present	131.04	131.20	139.19	139.90	154.35	155.52
Ref.[19]	131.0	131.2	139.2	139.9	154.4	155.5

thus established, and the following results are obtained in using the 12×12 solution that are presented in five significant figures.

3.2. Comprehensive results of shallow shells

Table 3(a) presents the lowest six frequency parameters Ω of shallow spherical shell (R_x/R_y=1) of square planform (*a/b*=1) with very small thickness (*a/h*=100) for twenty-one sets of boundary conditions. The degree of curvature is taken as a/R_x=a/R_y=0.2. Table 3(b) has the same format as Table 3(a), except that the curvature is larger in a/R_x/=a/R_y=0.5. Addition of curvature causes frequencies to be increased. In Table 3(a), the average increase from flat plates for the fundamental frequencies in Ω_1 is 105.5 percent, including the highest increase 20 percent of CSSS shell. These increases are much larger than the case of relatively thin shell (*a/h*=20) in [2], and roughly speaking, are almost ten times larger. One should note, however, that the frequency parameter defined in Eqs. (8) and (9) include the thickness *h* explicitly, and direct comparison between two sets of the parameters with different thickness ratios (*a/h*=20 and 100) may not be appropriate. This will be discussed later in this paper.

In Table 3(b), the deeper curvature *a/R*=0.5 causes the average increase of 252 percent in Ω_1 with the maximum 734 percent of SSSS shell, when they are compared to the flat plate frequencies.

Table 4(a) and (b) tabulate the lowest six frequency parameters of shallow cylindrical shells with R_x=(infinity) and a/R_y=0.2 and a/R_y=0.5, respectively. This shell takes straight edges of the shell along the *x* axis, and curvature is given only along *y* direction. When these tables are compared with those of spherical shell in Tab. 3(a) and (b), the effect of unidirectional curvature is a half of the spherical shells, and generally the effect of curvature increase in one direction is a half of curvatures in two directions of spherical shells.

Table 5(a) and (b) also tabulate the lowest six frequency parameters of shallow cylindrical shells, but with R_y=(infinity) and a/R_x=0.2 and a/R_x=0.5, respectively. Straight edges of the shell exist along the *y* axis, and curvature exists only along *x* direction. Likewise in results in [2], For cylindrical shells with FFFF, SSSS and CCCC, the frequency values are the identical as in Table 4(a) and (b) due to uniform boundary condition along four edges. Similarly, cylindrical shells with SSFF, CCFF and CCSS give the identical results as in Table 4(a) and (b) since the 90 degree rotation of the shell gives essentially the same boundary conditions. The same results in six cases are underlined.

Table 6(a) and (b) list up the lowest six parameters of shallow hyperbolic paraboloidal shells with a/R_y=0.2 and a/R_y=0.5, respectively. Just like in [2], the negative curvature ratio (R_x/R_y= -1) gives rise unusual response in frequency. Namely, for shell of hyperbolic paraboloidal shell, negative curvature causes decrease of frequencies, when shell has free edges. Four cases among twenty-one, the shell gives lower frequencies than frequencies of flat plate given in Table 7. As the constrained is increased along the edges, this tendency disappears.

Table 3(a) Frequency parameters Ω of shallow spherical shells, R_x/R_y=1, a/R_x=a/R_y=0.2, a/b=1, a/h=100, ν=0.3.

B.C.	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
FFFF	13.521	19.753	35.878	35.878	42.331	69.570
SFFF	6.6712	17.335	26.490	35.783	61.072	61.267
CFFF	6.5822	8.8500	24.876	32.141	38.908	68.677
SSFF	3.4094	18.027	28.038	57.296	66.252	74.143
CSFF	8.4529	21.169	33.288	60.520	74.132	78.481
CCFF	16.271	26.213	43.803	68.672	79.180	83.149
SFSF	12.424	16.938	47.450	49.994	75.167	93.448
CFSF	21.671	37.610	58.211	64.713	77.247	100.30
SSSF	14.395	48.653	71.242	86.457	90.390	98.601
CSSF	31.481	61.465	74.028	92.204	93.843	113.78
CCSF	31.773	61.515	79.182	97.294	101.32	113.82
CFCF	61.320	61.437	69.751	74.382	82.536	103.95
SCSF	15.689	48.666	76.431	90.101	98.635	99.237
CSCF	61.391	71.924	80.512	94.731	102.02	127.78
CCCF	61.426	71.952	87.036	104.06	105.42	132.40
SSSS	68.857	82.425	82.425	102.92	118.74	118.74
CSSS	72.522	84.443	90.352	109.41	120.20	131.99
CCSS	76.255	91.412	93.796	115.75	133.02	133.89
CSCS	80.177	87.332	98.107	117.07	122.04	147.14
CCCS	84.183	96.662	100.01	123.14	135.28	148.86
CCCC	96.717	102.74	102.74	130.31	148.49	154.07

Table3(b) Frequency parameters Ω of shallow spherical shells, R_x/R_y=1, a/R_x=a/R_y=0.5, a/b=1, a/h=100, ν=0.3.

B.C.	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
FFFF	13.574	19.983	36.858	36.858	49.526	70.625
SFFF	6.6235	17.914	27.470	38.759	64.412	71.594
CFFF	9.0041	9.7546	30.402	33.937	49.019	71.839
SSFF	3.3656	18.488	30.212	61.617	78.160	123.22
CSFF	12.169	23.168	42.208	67.551	88.251	128.57
CCFF	21.593	32.629	68.321	78.368	110.04	140.52
SFSF	13.116	17.328	53.474	54.929	110.64	110.95
CFSF	31.378	53.208	73.001	98.427	127.89	138.31
SSSF	14.987	54.190	110.80	166.05	173.40	175.80
CSSF	48.905	87.488	132.54	169.13	177.43	180.71
CCSF	49.647	87.592	132.58	174.10	182.71	188.86
CFCF	89.330	98.185	114.66	114.86	172.38	173.84
SCSF	20.617	54.193	110.80	172.06	176.98	182.58
CSCF	93.828	114.76	173.01	180.06	184.06	186.22
CCCF	93.835	114.77	173.26	184.38	189.33	195.28
SSSS	164.63	171.70	171.70	182.63	192.05	192.05
CSSS	168.71	173.78	181.43	188.92	193.37	204.54
CCSS	171.60	180.27	186.99	195.32	204.04	210.09
CSCS	177.61	182.78	185.75	195.19	196.05	219.24
CCCS	179.90	187.88	192.50	202.47	208.02	228.73
CCCC	191.99	191.99	196.93	209.96	216.19	242.22

Table 4(a) Frequency parameters Ω of shallow cylindrical shells, R_x =(infinity), a/R_y =0.2, a/b =1, a/h =100, ν =0.3.

B.C.	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
FFFF	13.483	21.903	34.850	37.642	38.468	61.117
SFFF	6.6792	25.137	27.266	29.361	53.625	62.120
CFFF	8.3592	8.9008	26.822	33.231	35.102	58.709
SSFF	3.3876	18.187	28.597	49.805	52.146	63.917
CSFF	9.5159	21.294	34.312	52.845	54.308	73.867
CCFF	11.007	29.916	35.356	62.297	65.099	74.651
SFSF	17.370	19.560	39.632	50.600	52.218	75.590
CFSF	22.359	25.244	43.750	60.101	61.232	77.857
SSSF	19.168	36.550	51.489	62.571	73.691	97.950
CSSF	24.151	40.977	60.740	65.130	80.745	104.66
CCSF	24.529	50.278	60.808	77.810	85.915	113.99
CFCF	28.583	31.660	48.648	70.829	71.612	80.604
SCSF	19.697	46.614	51.545	75.749	79.201	99.759
CSCF	30.225	45.975	68.284	71.291	89.195	112.37
CCCF	30.533	54.804	71.342	80.478	94.018	121.44
SSSS	38.437	51.059	72.299	85.562	98.892	115.22
CSSS	41.773	54.061	79.051	92.603	100.63	127.85
CCSS	48.570	67.079	81.803	100.93	116.03	129.48
CSCS	45.944	57.772	87.426	100.72	102.78	142.05
CCCS	51.987	69.788	90.013	108.44	117.90	143.61
CCCC	67.681	78.294	94.610	116.46	135.00	145.76

Table 4(b) Frequency parameters Ω of shallow cylindrical shells, R_x =(infinity), a/R_y =0.5, a/b =1, a/h =100, ν =0.3.

B.C.	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
FFFF	13.507	22.073	34.868	48.702	54.308	61.193
SFFF	6.7626	25.729	34.960	41.383	64.227	79.883
CFFF	10.588	16.980	30.638	42.203	47.659	65.439
SSFF	3.3702	18.226	37.799	52.273	79.338	83.783
CSFF	14.358	32.691	44.477	54.022	86.787	91.932
CCFF	14.983	42.872	49.155	73.302	89.760	96.679
SFSF	22.529	29.269	61.011	64.354	65.594	77.232
CFSF	29.383	34.058	71.553	73.837	74.882	80.774
SSSF	25.569	65.016	66.151	71.716	116.30	116.48
CSSF	31.551	70.922	74.274	79.607	118.44	123.64
CCSF	31.933	74.264	76.427	108.31	131.26	132.00
CFCF	36.942	39.735	82.233	84.268	84.971	85.003
SCSF	26.099	65.100	70.344	104.38	118.41	124.46
CSCF	38.287	76.311	84.646	87.429	121.03	131.22
CCCF	38.528	82.521	84.699	113.45	138.63	143.98
SSSS	59.184	84.179	99.914	114.02	137.81	140.79
CSSS	64.986	87.899	102.51	120.76	144.25	144.86
CCSS	72.375	105.48	127.24	133.02	147.49	168.62
CSCS	71.325	92.637	105.67	127.94	149.84	151.31
CCCS	78.068	108.33	134.01	134.72	153.32	173.98
CCCC	99.263	119.00	151.13	156.35	172.52	192.43

Table 5(a) Frequency parameters Ω of shallow cylindrical shells, R_y=(infinity), a/R_x=0.2, a/b =1, a/h =100, ν =0.3.

B.C.	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
FFFF	13.483	21.903	34.850	37.642	38.468	61.117
SFFF	6.6303	15.153	25.367	38.286	49.549	59.200
CFFF	3.4740	8.4730	21.673	30.718	38.552	61.123
SSFF	3.3876	18.187	28.597	49.805	52.146	63.917
CSFF	5.3711	23.986	29.189	58.193	62.305	65.517
CCFF	11.007	29.916	35.356	62.297	65.099	74.651
SFSF	9.7012	16.074	38.965	46.722	62.804	75.604
CFSF	15.183	36.307	49.399	61.353	66.142	86.169
SSSF	11.672	41.185	51.040	62.204	83.750	90.277
CSSF	26.655	53.502	56.075	74.830	86.146	105.96
CCSF	30.034	54.215	59.721	78.205	94.146	106.29
CFCF	58.315	60.258	61.144	70.589	73.150	96.003
SCSF	19.980	42.029	54.733	66.491	90.640	91.840
CSCF	58.926	63.964	67.510	84.225	89.974	118.23
CCCF	59.122	64.590	70.454	87.422	97.705	125.27
SSSS	38.437	51.059	72.299	85.562	98.892	115.22
CSSS	45.859	64.956	75.151	94.571	114.52	116.94
CCSS	48.570	67.079	81.803	100.93	116.03	129.48
CSCS	63.541	73.706	80.634	103.59	119.33	131.90
CCCS	65.171	75.764	86.896	109.46	131.64	133.36
CCCC	67.681	78.294	94.610	116.46	135.00	145.76

Table 5(b) Frequency parameters Ω of shallow cylindrical shells, R_y=(infinity), a/R_x=0.5, a/b =1, a/h =100, ν =0.3.

B.C.	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
FFFF	13.507	22.073	34.868	48.702	54.308	61.193
SFFF	6.5659	15.172	25.255	48.942	51.902	59.303
CFFF	3.4483	8.2887	21.400	29.527	51.692	60.834
SSFF	3.3702	18.226	37.799	52.273	79.338	83.783
CSFF	5.3196	23.755	38.350	62.825	80.388	89.447
CCFF	14.983	42.872	49.155	73.302	89.760	96.679
SFSF	9.6518	15.726	38.968	46.600	87.930	95.924
CFSF	14.858	46.682	49.066	90.447	102.91	106.34
SSSF	11.528	41.099	76.463	90.188	109.96	111.74
CSSF	39.866	69.777	85.768	109.29	117.32	140.36
CCSF	46.746	74.034	90.546	109.96	120.78	147.64
CFCF	60.841	85.618	101.77	106.69	129.75	137.60
SCSF	39.743	43.622	82.174	90.776	112.50	115.78
CSCF	69.222	103.23	112.13	124.49	156.98	158.29
CCCF	71.051	104.02	115.45	127.91	163.30	163.49
SSSS	59.184	84.179	99.914	114.02	137.81	140.79
CSSS	67.289	103.11	120.99	131.59	142.40	164.20
CCSS	72.375	105.48	127.24	133.02	147.49	168.62
CSCS	92.641	114.28	140.12	145.75	168.92	182.49
CCCS	95.798	116.43	145.24	150.67	170.30	187.31
CCCC	99.263	119.00	151.13	156.35	172.52	192.43

Table 6(a) Frequency parameters Ω of shallow hyperbolic paraboloidal shells, R_x/R_y= -1, a/R_y=0.2, a/b =1, a/h =100, ν =0.3.

B.C.	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
FFFF	13.462	24.738	36.955	36.955	52.563	63.898
SFFF	6.6380	21.641	26.605	43.730	55.348	63.658
CFFF	6.4996	8.7953	29.914	32.664	46.110	64.991
SSFF	3.3656	20.411	38.508	39.592	63.965	66.883
CSFF	7.6241	28.830	42.288	52.961	65.428	75.539
CCFF	8.6992	33.048	50.320	63.530	74.322	78.713
SFSF	16.528	16.753	50.102	54.907	56.478	70.884
CFSF	26.302	37.366	59.804	64.066	64.665	84.047
SSSF	17.199	38.042	54.112	63.519	77.934	95.426
CSSF	33.152	48.479	64.771	76.666	81.357	105.33
CCSF	41.686	56.645	65.357	83.766	92.208	115.14
CFCF	61.217	63.091	70.497	73.361	74.448	94.428
SCSF	34.797	46.701	54.358	72.630	89.198	100.62
CSCF	62.678	66.374	74.210	86.152	87.457	114.99
CCCF	63.201	71.908	74.659	92.908	97.131	124.50
SSSS	19.660	63.236	63.236	78.877	111.93	111.93
CSSS	41.441	68.114	74.443	91.002	114.11	125.71
CCSS	52.386	77.989	79.478	100.61	127.67	127.81
CSCS	65.750	77.903	83.122	100.84	117.48	141.59
CCCS	69.936	86.409	87.335	109.50	130.71	143.33
CCCC	79.599	94.110	94.110	117.77	145.72	145.92

Table 6(b) Frequency parameters Ω of shallow hyperbolic paraboloidal shells, R_x/R_y= -1, a/R_y=0.5, a/b =1, a/h =100, ν =0.3.

B.C.	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
FFFF	13.424	25.664	38.909	38.909	64.227	79.301
SFFF	6.5797	23.152	28.121	48.293	72.323	88.044
CFFF	8.2199	9.4072	36.242	36.427	72.068	81.721
SSFF	3.3264	21.403	38.639	72.319	82.019	94.838
CSFF	8.9011	34.299	75.355	80.171	88.219	112.27
CCFF	9.8110	38.144	85.107	95.412	110.78	127.38
SFSF	17.348	18.533	60.794	71.367	107.19	111.31
CFSF	43.390	58.519	89.161	102.31	113.00	123.96
SSSF	18.753	64.362	73.281	96.120	111.74	122.48
CSSF	66.052	71.660	107.99	112.92	125.04	141.99
CCSF	77.735	94.160	109.74	126.07	145.85	151.05
CFCF	98.601	116.10	121.74	127.34	128.31	140.44
SCSF	73.294	74.852	88.379	98.575	123.21	146.89
CSCF	105.30	124.20	130.30	140.39	141.58	146.16
CCCF	106.46	126.11	137.44	143.08	165.67	166.80
SSSS	19.258	78.461	109.98	109.98	142.70	142.70
CSSS	82.623	90.895	130.08	131.01	150.55	169.19
CCSS	94.181	122.38	136.98	149.24	173.01	173.62
CSCS	127.84	133.29	135.91	142.72	170.73	171.02
CCCS	131.20	139.90	154.35	158.42	186.02	189.83
CCCC	157.35	157.35	157.41	166.52	204.03	208.69

Table 7 Frequency parameters Ω of flat square plates, a/b =1, ν =0.3.

B.C.	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
FFFF	13.468	19.596	24.270	34.801	34.801	61.093
SFFF	6.6433	14.902	25.376	26.001	48.449	50.579
CFFF	3.4711	8.5065	21.286	27.199	30.958	54.189
SSFF	3.3674	17.316	19.293	38.211	51.035	53.487
CSFF	5.3512	19.076	24.671	43.089	52.707	63.762
CCFF	6.9200	23.907	26.586	47.655	62.709	65.537
SFSF	9.6313	16.135	36.726	38.945	46.738	70.740
CFSF	15.192	20.584	39.736	49.449	56.280	77.324
SSSF	11.685	27.756	41.197	59.065	61.861	90.294
CSSF	16.792	31.114	51.397	64.021	67.540	101.12
CCSF	17.537	36.023	51.812	71.077	74.326	105.79
CFCF	22.168	26.407	43.597	61.176	67.176	79.817
SCSF	12.687	33.065	41.702	63.015	72.398	90.611
CSCF	23.371	35.571	62.875	66.762	77.374	108.87
CCCF	23.921	39.998	63.221	76.710	80.572	116.66
SSSS	19.739	49.348	49.348	78.957	98.696	98.696
CSSS	23.646	51.674	58.646	86.134	100.27	113.23
CCSS	27.054	60.538	60.786	92.836	114.56	114.70
CSCS	28.951	54.743	69.327	94.585	102.22	129.10
CCCS	31.826	63.331	71.076	100.79	116.36	130.35
CCCC	35.985	73.394	73.394	108.22	131.58	132.20

3.3. Discussion on thickness effect

Although the frequency parameter Ω defined in Eqs. (8) and (9) has been used for shallow shell vibration in the past literature, it turns out in this study that direct comparison is not appropriate because thickness h is included in the parameter Ω. A new frequency parameter is therefore proposed here as

$$\Omega^* = \Omega\left(\frac{h}{a}\right) = \omega a\sqrt{\frac{12\rho\left(1-\nu^2\right)}{E}} \tag{10}$$

that is still nondimensional and proportional to ω, but excludes thickness ratio (a/h) in the parameter. In other words, comparison of Ω^* is more reasonable between two cases with different thickness.

Figure 3 illustrates variations of new frequency parameter Ω* for three lowest modes of spherical shell versus four different thickness ratios of a/h=10 (thick shell), 20, 100 and 1000 (extremely thin shell). Theoretically, a/h=10 might be almost limit of applicable range of the thin shell theory. Figure 3(a) represents small curvature of a/R_x=a/R_y=0.2, and lower figure (b) does large curvature a/R_x=a/R_y=0.5. Values of Ω^* are inserted for a/h=10 and 20 in each figure, but omitted for a/h=100 and 1000 due to lack of space. It is clearly seen in both figures that all frequency parameters Ω* monotonically decrease with decreasing bending stiffness, but interestingly the difference due to curvature increase from a/R_x=a/R_y=0.2 to a/R_x=a/R_y=0.5 is not significant.

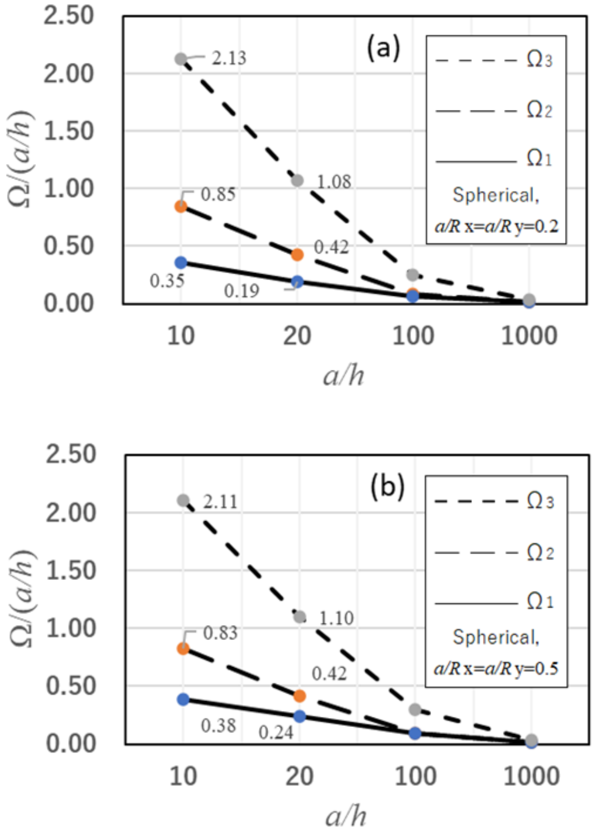


Figure 3. Thickness effect (Spherical shell, CFFF)

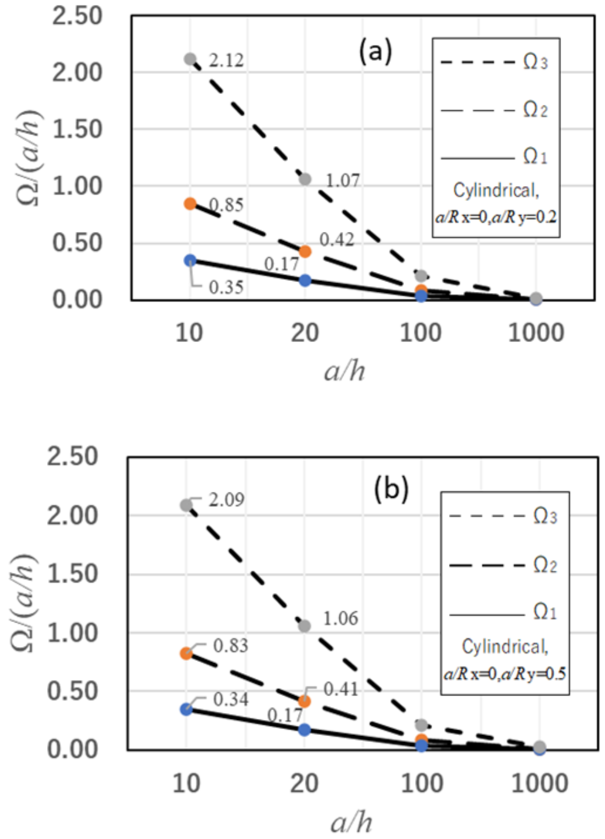


Figure 4. Thickness effect (Cylindrical shell, CFFF)

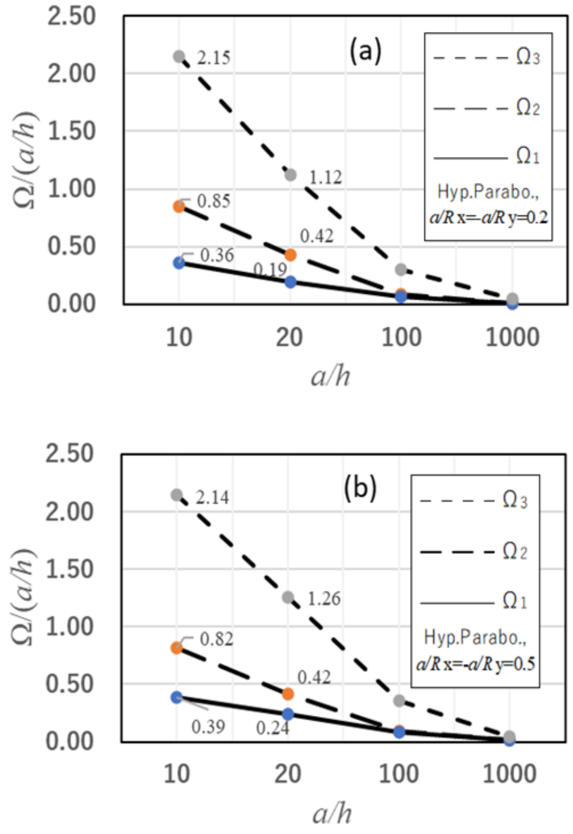


Figure 5. Thickness effect (Hyp.Para. shell, CFFF)

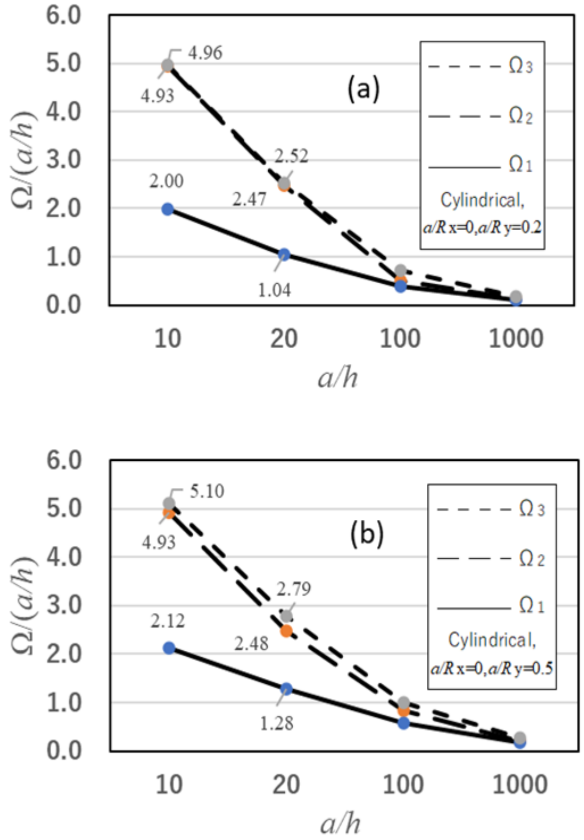


Figure 6. Thickness effect (Cylindrical shell, SSSS)

Figures 4 and 5 present in the same format the results of frequency $\Omega^*=\Omega(h/a)$ for cylindrical shell and hyperbolic paraboloidal shell, respectively. When these six sets of variations in Ω^* in Fig.3-5 are compared, the difference in frequencies stays within a small range of parameter, for example, the first frequency parameter changes between $\Omega_1^*=0.34$ and 0.39 for $a/h=10$, and does between $\Omega_1^*=0.17$ and 0.24 for $a/h=20$. Generally speaking, these six figures present quite similar forms of variation.

Figure 6 illustrates the variation of Ω_1^* for cylindrical shell with totally simply supported edges (SSSS). Figures 6(a) and (b) show that frequency behavior and values of the parameters differ from shell with cantilever type boundary condition (CFFF) in in Fig.3-5, but the frequency decrease takes similar tendency.

4. Conclusions

As Part. 2 with a previous study (Part. 1) [2], this paper tabulated accurate natural frequencies for free vibration of doubly curved, isotropic shallow shells of rectangular (square) planform, when the shell thickness is small (representative length/shell thickness=100). Twenty-one different sets of boundary conditions are included. The same mathematical procedure was used and was briefly outlined.

In the process of this study, it was found that the traditional representation in frequency parameter may not be appropriate to evaluate the effect of changing shell thickness on free vibration of the shells. Based on this finding, a new frequency parameter was proposed by excluding thickness in the parameter. With this new parameter, rather unified behavior was identified, and more effective use of this parameter will be studied.

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