Representation of Geometric Objects by Path Integrals

(経路積分による幾何的対象の表現)

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Abstract

In this thesis, we study two problems by using physical models with supersymmetry and their path integral. Since this thesis deals with these topics from a common perspective, but in different ways, we have divided this thesis into two parts. One is deriving fixed-point theorem using path integral [25]. The other is the Euler number and Mathai-Quillen formalism in the Grassmann manifold [20]. This thesis is based on [20], [25] and [28].

In the part I, we derive the Bott residue formula by using the topological sigma model (A-model) that describes dynamics of maps from $\mathbb{C}P^1$ to a Kähler manifold M, with potential terms induced from a holomorphic vector field K on M [25]. The Bott residue formula represents the intersection number of Chern classes of holomorphic vector bundles on M as the sum of contributions from fixed point sets of K on M. Our strategy is to represent the integral of differential form on M by a correlation function and show that the correlation function is obtained by collecting contributions from the zero set of K. It is realized by showing the invariance of correlation function for the parameter of potential terms. As an effect of adding a potential term to the topological sigma model, we are forced to modify the BRST symmetry of the original topological sigma model.

In the part II, we provide a recipe for computing Euler number of Grassmann manifold $G(k, N)$ by using Mathai-Quillen formalism [33] and Atiyah-Jeffrey construction [3]. Especially, we construct the path integral representation of Euler number of $G(k, N)$ [20]. As a by-product, we construct free fermion realization of cohomology ring of $G(k, N)$. It means that the cohomology ring of $G(k, N)$ can be represented by fermionic fields that appear in our model. As an application, we calculate some integrals of cohomology classes by using fermion integrals [28].

Contents

About this thesis

In physics, the systematic transition procedure from classical understanding of phenomena to quantum mechanics is called quantization. It can be done in two major ways. One is the method using operators, and the other is using path integrals by Feynman. We focus on path integrals in this thesis. A path integral is given by the sum of contributions from all possible configurations of the fields can be taken in the physical model we are considering. In many situations, it can be computed by a Gaussian integral. However, the measure of the path integral needs to be discussed for each model. In this thesis, we focus the physical models and supersymmetry. Physical models are given by the Lagrangian. In physical models, there are two types of fields (variables) called bosonic fields and fermionic fields. Bosonic fields are like usual functions or variables. Fermionic fields have anticommutative property. Supersymmetric transformation is a transformation that exchanges bosonic fields and fermionic fields in the Lagrangian. If the Lagrangian is invariant under supersymmetric transformations, the model is called having supersymmetry. In a physical model having supersymmetry, the principle of localization is used. It claims that path integrals can be evaluated by the sum of contributions from field configurations satisfying the conditions that supersymmetric transformations of fermionic fields are zero [26]. By this principle, the measure of path integrals may become a few simple. In this thesis, we study supersymmetric transformations and physical models.

We explain why we consider the connection between geometric quantities and physical models having supersymmetry. A sigma model is one of the well-known physical models that describes dynamics of maps from a space to a target space. The sigma model having supersymmetry is called a supersymmetric sigma model. Various topological indices of a Riemann manifold M, such as Euler number, Hirzebruch signature, Atiyah-Singer index etc., are computed by using a path integral of the supersymmetric sigma model with target space M [36, 2, 13]. The characteristic of a sigma model is that observables that is constructed by fields in the model correspond to differential forms on M. And the supersymmetry (or more precisely, its charge) corresponds to the exterior derivative on M[18]. And the path integral of the observable multiplied $\exp(-\text{Lagrangian})$ is called the correlation function. Then, we expect that when we consider observables that vanish under supersymmetry, it becomes an element of De Rham cohomology of M, and its correlation function represents the intersection number. It is implemented by the topological sigma model (A-model). The A-model describes dynamics of maps from Riemann surface Σ to a Kähler manifold M. Please refer [23]. It is given by the $N = (2, 2)$ supersymmetric sigma model after operating A-twist on the fields of the model. Supersymmetric transformation of the $N = (2, 2)$ supersymmetric sigma model has four fermionic infinitesimal deformation parameters $\alpha_+, \alpha_-, \tilde{\alpha}_+, \tilde{\alpha}_-$. The supersymmetry of the A-model is obtained from setting $\alpha_- = \tilde{\alpha}_+ = \alpha,\ \alpha_+ = \tilde{\alpha}_- = 0$ after operating A-twist on the fields. And the supersymmetry of the A-model is called BRST-symmetry. The BRST-symmetry is nilpotent. If the observable transformed by the BRST-symmetry is zero, it called the BRST-closed observable. Since the BRST-symmetry and BRST-closed observables correspond to the exterior derivative and elements of the De Rham cohomology (differential form) on M, we consider BRST-closed observables. In general, correlation functions for the A-model are decomposed into the degree of map. Then, the path integral becomes the integral on the moduli space of the holomorphic map of the degree d from Σ to M. Especially, when we consider the A-model for a map from $\mathbb{C}P^1$ to M, the correlation function of degree 0 represents the integral of the cohomology class on M.

We explain the first half of this thesis. In [38], Witten considered the supersymmetric sigma model with various potential terms. Especially, he suggested that the fixed-point formula for signature of even-dimensional M can be obtained by using this model with a potential term induced from a Killing vector field on M. On the other hand, various fixed-point formulas, such as Duistermaat-Heckman formula etc., have been derived by using this kind of potential terms [1, 2, 35]. In the first half of this thesis, we focus the Bott residue formula. It is one of the well-known fixed-point theorems. It represents the intersection number of Chern classes of holomorphic vector bundles on M as the sum of contributions from fixed point sets of a holomorphic vector field K on a Kähler manifold M. The topic of the first-half of this thesis is deriving the Bott residue formula by using the topological sigma model (A-model) with potential terms induced from K and the path integral [25]. Here, we use a map from $\mathbb{C}P^1$ to M. In order to extend the BRST-symmetry to the A-model with the potential terms, we have to modify the BRST-transformation of the usual topological sigma model. Our BRST symmetry is obtained from setting $\tilde{\alpha}_+ = \bar{\alpha},\ \alpha_- = \alpha_+ = \tilde{\alpha}_- = 0$ after operating A-twist on the fields. Hence our BRST symmetry uses half of the supersymmetry used in constructing the usual BRST symmetry of the A-model. But our BRST symmetry still has nilpotency, which will be shown later by explicit computations. The new BRST symmetry is still conserved in the original Lagrangian of the A-model, and it can be extended to the A-model with the potential term. Since we modify the BRST-symmetry of the A-model to extend the A-model with potential terms, BRST-closed observables correspond to Dolbeault cohomology of M instead of De Rham cohomology of M. But Chern classes of holomorphic vector bundles on M are given by (i, i) forms of M, and they are automatically Dolbeault cohomology classes of M. Therefore, change of BRST-symmetry causes no problem for our purpose. When potential terms in our model are parameterized, correlation functions are independent of the parameter. From this property, we have derived the Bott residue formula by computing the correlation function of degree 0 in our model. More precisely, when the contribution from potential terms is very small, since the Lagrangian becomes original Lagrangian of the A-model, as mentioned above, the correlation function of degree 0 represent the integral on the M. On the contrary, when the contribution from potential terms is a large, the path integral is localized at the zero set of K. Potential terms similar to our model have already proposed by Labastida and Llatas in [31]. They construct a supersymmetric sigma model and derived Poincar*é*-Hopf theorem. However, we note that their supersymmetry is not BRST-symmetry and their result is different from our aim.

Recently, Beasley and Witten considered supersymmetry closely related to our new BRST-transformation for $(0,2)$ liner sigma models with a potential term induced from holomorphic section of a holomorphic vector bundle [4]. Our new BRST-symmetry seems to be closely related to their idea applied to the case when the holomorphic vector bundle is given by the holomorphic tangent vector bundle $T'M$. Of course, they derive a kind of fixed-point formula, but their result is different from our goal of first half in this thesis: "deriving the Bott-residue formula by using the topological sigma model (A-model) with the potential term induced from a holomorphic tangent vector field". Our model can be used not only for the map of degree 0 but also for higher degrees. For example, let Σ and M be a $\mathbb{C}P^1$ and $\mathbb{C}P^4$ (4-dimensional complex projective space) [22]. Let $z_l(l=1,\cdots,n)$ be different points each other. Then, the degree d correlation function of its A-model is interpreted by

$$\int_{\overline{\mathcal{M}}_{\mathbb{C}P^1}(\mathbb{C}P^4,d)} \bigwedge_{l=1}^{N} \mathrm{ev}_l^*(h^{b_l})$$

$\overline{\mathcal{M}}_{\mathbb{C}P^1}(\mathbb{C}P^4,d)$ is a compactified moduli space holomorphic map of degree d from $\mathbb{C}P^1$ to $\mathbb{C}P^4$. h is the first Chern class of hyperplane bundle of $\mathbb{C}P^4$. h^{b_l} is an element of $H^{b_l,b_l}(\mathbb{C}P^4,\mathbb{C})$. ev_l^* is a pull back of an evaluation map $\mathrm{ev}_l:\overline{\mathcal{M}}_{\mathbb{C}P^1}(\mathbb{C}P^4,d)\to\mathbb{C}P^4(l=1,\cdots,n)$. It is defined by $\mathrm{ev}_l(\phi)=\phi(z_l)$ for $\phi\in\overline{\mathcal{M}}_{\mathbb{C}P^1}(\mathbb{C}P^4,d)$ and $z_l\in\mathbb{C}P^1$. If we take $\mathbb{C}P^{5d+4}$ as $\overline{\mathcal{M}}_{\mathbb{C}P^1}(\mathbb{C}P^4,d)$ (see [22] for details), $\mathrm{ev}_l^*(h^{b_l})\in H^{b_l,b_l}(\mathbb{C}P^{5d+4},\mathbb{C})$. Then,

$$\int_{\overline{\mathcal{M}}_{\mathbb{C}P^1}(\mathbb{C}P^4,d)} \bigwedge_{l=1}^{N} \mathrm{ev}_l^*(h^{b_l}) = \int_{\mathbb{C}P^{5d+4}} \bigwedge_{l=1}^{N} \tilde{h}^{b_l},$$

where $\tilde{h}$ is the first Chern class of hyperplane bundle of $\mathbb{C}P^{5d+4}$. When $\sum_{l=1}^{n} b_l = 5d+4$, its result is 1. It is computed by the Bott residue formula on $\mathbb{C}P^{5d+4}$. On the other hand, the holomorphic vector field in our model is an element of the holomorphic tangent vector bundle on M. So, the question arises as how to compute the integral over $\overline{\mathcal{M}}_{\mathbb{C}P^1}(M,d)$. In other words, how should we treat the contribution from each field on a zero set of K? Since we can use our model for a map from Σ to M, we may be able to calculate correlation functions for all A-model by solving these problems.

The second half of this thesis is the topic of the representation of the Euler number for complex Grassmann manifold via Mathai-Quillen formalism (MQ-formalism). This topic is based on [20, 28]. We construct the physical toy-model which path integral representing the Euler number for Grassmann manifold. Toy-models mean that its path integral for each variable is the usual integral. So, integrals for bosonic fields and fermionic fields become Gaussian integrals and integrals for fermionic fields. Therefore, we do not discuss integral measures in this part. The Euler class of E is given by pull-back of the Thom class by the section $s:M\to E$. It does not depend on choice of the section of s as a cohomology class because of homotopy invariance of De Rham cohomology. Therefore, we can compute the Euler number of E by choosing a convenient section. Mathai-Quillen formalism [33] is a method for constructing

Thom class of finite dimensional vector bundle E on a manifold M, that decreases like Gaussian along fiber direction. This Thom class plays the same role as the original Thom class which has compact support along fiber direction. Suppose M is given as an orbit space X/G where a Lie group G acts freely on a manifold X. Atiyah and Jefferey extended the MQ-formalism for an orbit space X/G [3]. We call this method "Atiyah-Jeffrey construction". They extended it to the case when X/G is given by infinite-dimensional space of gauge equivalence classes of connections of $SU(2)$ bundle on a 4-dimensional manifold, in order to study mechanism behind Witten's construction of topological Yang-Mills theory [3, 39]. The motivation in the second half is "On finite dimension manifold, can its Euler number really be given by Atiyah-Jeffrey construction?". Imanishi and Jinzenji chose zero-section as s and construct the toy-model by applying Atiyah Jeffrey construction to the case when X/G is complex Grassmann manifold $G(2,N)$ and E is holomorphic tangent bundle of $G(2,N)$. Then, they compute the Euler number of $G(2,N)$ by using their model [19]. We have extended their model to the general $G(k,N)$ and obtained the representation of Euler number in [20]. As a by-product, we construct free fermion realization of cohomology ring of $G(k,N)$. It means that Chern classes of the dual bundle of the tautological bundle on $G(k,N)$ are represented by fermionic variables. In general, these Chern classes correspond to the Poincaré dual of Schubert cycles [17]. Therefore, an integral of Chern classes over $G(k,N)$ is known as an intersection number of Schubert cycles. From free fermion realization of cohomology ring of $G(k,N)$, we found that an integral of Chern classes over $G(k,N)$ can be computed using a fermion integral. In the second half, we calculate some intersection numbers using this result [28].

To summarize, this thesis is based on [25], [20] and [28]. In particular, a common topic in paper [25] and [20] is the use of physical models and path integrals to reproduce geometric objects. However, these researches are not directly related, this thesis is divided into two parts. In the first half (the topic of [25]), we derive the Bott residue formula by using the A-model with potential terms induced from a holomorphic vector field. In the second half (topics of [20] and [28]), we provide a recipe for computing Euler number of Grassmann manifold $G(k,N)$ by using Mathai-Quillen formalism [33] and Atiyah-Jeffrey construction [3] and calculate some intersection numbers of Schubert cycles. Our research is characterized by the use a supersymmetry and physical model to prove geometric objects and quantities. One of the aims of the research are to review these quantities from a physical aspect, to obtain new computational methods and to enable intuitive understanding.

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[Journal URL: https://www.worldscientific.com/worldscinet/ijmpa].

Part I

Holomorphic Vector Field and Topological Sigma Model on $\mathbb{C}\mathrm{P}^1$ World Sheet

This part is an edited and reprinted version of the contents of [25], with the exception of Subsections 3.1 and 3.2. Subsection 3.1 and Subsection 3.2 are reviews of the A-model [22, 23].

1 Main result and organization of Part I

1.1 Main result of Part I

The purpose of this part is to provide a derivation of the Bott residue formula by using the topological sigma model (A-model) with potential terms induced from a holomorphic tangent vector field. First, we introduce assertion of the Bott residue formula [8].

The Bott Residue Formula

$$\int_M \underline{\varphi}(E) = \sum_{\alpha} \int_{N_\alpha} \frac{\underline{\varphi}(\Lambda_\alpha)}{\det(\theta^\nu_\alpha + tR^\nu_\alpha)}\bigg|_{t=\frac{i}{2\pi}}.$$

Here, M is a compact Kähler manifold with $\dim_{\mathbb{C}}(M) = m$. Let E be a holomorphic vector bundle over M with $\mathrm{rank}_{\mathbb{C}} E = q$ and $\underline{\varphi}(E)$ is a wedge product of Chern classes of E. It is represented by symmetric homogeneous polynomial $\varphi(x_1, \cdots, x_q)$ of degree m. Let K be a holomorphic tangent vector field on M and $\{N_\alpha\}$ be a set of connected components of the zero set of K. $\underline{\varphi}(\Lambda_\alpha)$ is a cohomology class of N_α. θ^ν_α is the automorphism induced by action of K on the normal bundle of N_α. R^ν_α is the curvature $(1,1)$-form of the normal bundle of N_α. The details are descrived in Subsection 2.1.

Next, we introduce the topological sigma model (A-model). We use the topological sigma model (A-model) from $\mathbb{C}P^1$ to the Kähler manifold M with potential terms induced from the holomorphic tangent vector field $K = K^i \frac{\partial}{\partial z^i}$. Fields that appear in the model is given as follows.

- $\phi^i\ \phi^{\bar{i}}$: bosonic fields given as C^∞-map from $\mathbb{C}P^1$ to M
- χ^i: fermionic fields that takes values in C^∞ section of $\phi^{-1}T'M$
- $\chi^{\bar{i}}$: fermionic fields that takes values in C^∞ section of $\phi^{-1}\overline{T'}M$

- $\psi_z^{\bar{i}}$: fermionic fields that takes values in C^∞ section of $T'^*\mathbb{C}P^1 \otimes \phi^{-1}\overline{T'}M$
- $\psi_{\bar{z}}^i$: fermionic fields that takes values in C^∞ section of $\overline{T'^*}\mathbb{C}P^1 \otimes \phi^{-1}T'M$

Let $g_{i\bar{j}}$ be Kähler metric of M and $R_{i\bar{j}k\bar{l}}$ be its curvature tensor. Then, we introduce the Lagrangian of our model. Let us define L and V as follows.

$$L := \int_{\mathbb{CP}^1} dzd\bar{z}\big[\frac{t}{2}g_{i\bar{j}}(\partial_z\phi^i\partial_{\bar{z}}\phi^{\bar{j}} + \partial_{\bar{z}}\phi^i\partial_z\phi^{\bar{j}}) + \sqrt{t}ig_{i\bar{j}}\psi_z^{\bar{j}}D_{\bar{z}}\chi^i + \sqrt{t}ig_{i\bar{j}}\psi_{\bar{z}}^iD_z\chi^{\bar{j}} - R_{i\bar{j}k\bar{l}}\psi_{\bar{z}}^i\psi_z^{\bar{j}}\chi^k\chi^{\bar{l}}\big], \tag{1.1}$$

$$V := \int_{\mathbb{CP}^1} dzd\bar{z}\big[ts^2\beta g_{i\bar{j}}K^i\bar{K}^{\bar{j}} + tsg_{i\bar{j}}\nabla_{\bar{\mu}}\bar{K}^{\bar{j}}\chi^{\bar{\mu}}\chi^i + s\beta g_{i\bar{j}}\nabla_\mu K^i\psi_{\bar{z}}^\mu\psi_z^{\bar{j}}\big]. \tag{1.2}$$

L is the original Lagrangian of the A-model [23] and V is the potential terms induced from K. t is a parameter called the coupling constant ($t \in \mathbb{R}_{\geq 0}$). $s \in \mathbb{R}, \beta \in \mathbb{C}$ is parameters for V. The Lagrangian of our model is

$$L + V = \int_{\mathbb{CP}^1} dzd\bar{z}\big[\frac{t}{2}g_{i\bar{j}}(\partial_z\phi^i\partial_{\bar{z}}\phi^{\bar{j}} + \partial_{\bar{z}}\phi^i\partial_z\phi^{\bar{j}}) + \sqrt{t}ig_{i\bar{j}}\psi_z^{\bar{j}}D_{\bar{z}}\chi^i + \sqrt{t}ig_{i\bar{j}}\psi_{\bar{z}}^iD_z\chi^{\bar{j}} -R_{i\bar{j}k\bar{l}}\psi_{\bar{z}}^i\psi_z^{\bar{j}}\chi^k\chi^{\bar{l}} + ts^2\beta g_{i\bar{j}}K^i\bar{K}^{\bar{j}} + tsg_{i\bar{j}}\nabla_{\bar{\mu}}\bar{K}^{\bar{j}}\chi^{\bar{\mu}}\chi^i + s\beta g_{i\bar{j}}\nabla_\mu K^i\psi_{\bar{z}}^\mu\psi_z^{\bar{j}}\big]. \tag{1.3}$$

A detailed explanation is given in Section 3 and 4. For simplicity, also only L is referred to as the Lagrangian. From now on, we set $\beta := 2\pi i$. Covariant derivatives are given by

$$D_{\bar{z}}\chi^i = \partial_{\bar{z}}\chi^i + \Gamma^i_{\mu\nu}\partial_{\bar{z}}\phi^\mu\chi^\nu, \tag{1.4}$$

$$D_z\chi^{\bar{i}} = \partial_z\chi^{\bar{i}} + \Gamma^{\bar{i}}_{\bar{\mu}\bar{\nu}}\partial_z\phi^{\bar{\mu}}\chi^{\bar{\nu}}. \tag{1.5}$$

Our BRST-transformation for this model is given as follows. ($\bar{\alpha}$ is a fermionic variable.)

$$\delta\phi^{\bar{i}} = i\bar{\alpha}\chi^{\bar{i}}, \qquad \delta\psi_z^{\bar{i}} = -i\bar{\alpha}\Gamma^{\bar{i}}_{\bar{\mu}\bar{\nu}}\chi^{\bar{\mu}}\psi_z^{\bar{\nu}},$$
$$\delta\psi_{\bar{z}}^i = -\sqrt{t}\bar{\alpha}\partial_{\bar{z}}\phi^i, \qquad \delta\chi^i = is\bar{\alpha}\beta K^i, \qquad \delta\phi^i = \delta\chi^{\bar{i}} = 0. \tag{1.6}$$

The above potential terms and BRST-symmetry are closely related to the supersymmetry used in [4]. In this part, we prove the following proposition that play important roles in our derivation.

Proposition 1. *Correlation functions of BRST-closed observables are invariant under variation of s.*

Please see Subection 4.1 for details. Explanations for correlation functions and BRST-closed observables are given in Section 3. Then we can derive the Bott residue formula by evaluating correlation function of degree zero map both in the limit $s \to 0$ and $s \to \infty$.

1.2 Organization of Part I

This part is organized as follows.

In Section 2, we introduce the Bott residue formula and notations used in this part. And to demonstrate usage of the Bott resideuformula, we compute the integral of Chern classes of a hyperplane bundle on $\mathbb{C}P^n$.

In Section 3, we introduce our topological sigma model. In Subsection 3.1 and 3.2, we review outline of the topological sigma model (A-model) and observables for A-model. Discussions in these subsection is besed on [22], [23] and [27]. Then, we introduce our BRST-symmetry that uses half of the supersymmetry in Subsection 3.3. We show that our new BRST-symmetry conserves the Lagrangian of the topological sigma model without potential terms. Under the new BRST-symmetry, BRST-closed observables become elements of Dolbeault cohomology of the target Kähler manifold. Next, we include potential terms induced from a holomorphic tangent vector field and extend the new BRST-symmetry to this case. Then BRST-closed observables become elements of equivariant Dolbeault cohomology under the action of the holomorphic vector field. Mathematical relationship between the Bott residue formula [7] and this cohomology is discussed in [10, 32].

Section 4 is the main section of this part. We evaluate the degree 0 (i.e., homotopic to constant maps) correlation function of our model. It corresponds to the correlation function represented by the Bott residue formula. Proposition 1 claims that the correlation function is invariant under change of the parameter s. Hence, we can compare the results evaluated under the $s \to 0$ limit and the $s \to \infty$ limit.

In the $s \to 0$ limit, the degree 0 correlation function turns into the classical integration on M of differential forms that represent Chen classes by the standard argument of weak coupling limit of the topological sigma model. In the $s \to \infty$ limit, evaluation of the degree 0 correlation function reduces to the evaluation of contributions of from connected components of the zero set of the holomorphic tangent vector field K. This follows from the localization principle. The localization principle insists that path integrals of a model having supersymmetry can be evaluated by the sum of contributions from field configurations satisfying the conditions that supersymmetric transformations of fermionic fields are zero. We perform standard localization computation. The result of evaluation from each connected component turns out to be the contribution in the Bott residue formula from the same connected component. By equating these two results, we obtain the desired Bott residue formula.

In appendix, we prove Proposition 1.

2 Introducton to the Bott residue formula

2.1 Notation and The Bott Residue Formula

We introduce here our basic notations. Let us denote a compact Kähler manifold by M ($\dim_{\mathbb{C}}(M) = m$) and a holomorphic tangent vector field on M by K. Let E be a holomorphic vector bundle on M with $\mathrm{rank}_{\mathbb{C}}(E) = q$ and $\Gamma(E)$ be set of C^∞ sections of E.

First, we note the basic facts of E and introduce an action of K on E used in [8]. The holomorphic vector bundle E has a canonical connection compatible with Hermitian metric. (For more details of holomorphic vector bundles, please see [29].) Let $\tilde{\nabla}$ be the canonical connection on E and $\Omega^{p,r}(E)$ be complex vector space of E-valued (p,r)-forms. We also introduce the exterior holomorphic covariant derivative $D' : \Omega^{p,r}(E) \to \Omega^{p+1,r}(E)$. Then the canonical connection is decomposed into $\tilde{\nabla} = D' + \bar{\partial}$ where $\bar{\partial}$ is the anti-holomorphic part of the exterior derivative operator d of M. Let $\{e_a \mid a = 1, \cdots, q\}$ be local holomorphic frame of E. Then the following relation holds.

$$\begin{aligned} &\bar{\partial} e_a = 0, \\ &\tilde{\nabla} e_a = D' e_a = \Theta^b_{ak} dz^k e_b, \end{aligned} \tag{2.7}$$

where $(z^1, \cdots, z^m)$ is a local coordinate of M and $\Theta^b_{ak} dz^k$ is the connection $(1,0)$-form of E. Curvature $(1,1)$-form $F = (F^b_a), F^b_a = F^b_{ak\bar{l}} dz^k \wedge \overline{dz^l}$, is given by $F^b_a = \bar{\partial}(\Theta^b_{ak} dz^k) = -\bar{\partial}_l(\Theta^b_{ak}) dz^k \wedge \overline{dz^l}$. Here, we define $\Lambda : \Gamma(E) \to \Gamma(E)$ as a differential operator which acts on fs ($f :$ C^∞ a function on M, $s \in \Gamma(E)$) in the following way,

$$\Lambda(fs) = (Kf) \cdot s + f\Lambda(s), \qquad \bar{\partial}\Lambda = \Lambda\bar{\partial}. \tag{2.8}$$

This Λ defines the action of K on E. In the case when E is the holomorphic tangent bundle $T'M$, Λ is explicitly given by holomorphic Lie derivative of the holomorphic tangent vector field Y by K:

$$\theta(K) : Y \to [K, Y]. \tag{2.9}$$

Then, we can check that $\theta(K)$ satisfies the condition (2.8) by using local coordinate on M.

$$\begin{aligned} &\theta(K)(fY) = [K, fY] = K^i \frac{\partial}{\partial z^i}(fY^j \frac{\partial}{\partial z^j}) - fY^j \frac{\partial}{\partial z^j}(K^i \frac{\partial}{\partial z^i}) \\ &= K^i \frac{\partial f}{\partial z^i}(Y^j \frac{\partial}{\partial z^j}) + fK^i \frac{\partial}{\partial z^i}(Y^j \frac{\partial}{\partial z^j}) - fY^j \frac{\partial}{\partial z^j}(K^i \frac{\partial}{\partial z^i}) \\ &= K^i \frac{\partial f}{\partial z^i}(Y^j \frac{\partial}{\partial z^j}) + f[K, Y] \\ &= (Kf)Y + f[K, Y]. \end{aligned} \tag{2.10}$$

Second, we introduce notations for characteristic classes of E. Let $\varphi(x_1, \cdots, x_q)$ be a symmetric homogeneous polynomial in $x_1, \cdots, x_q$ with complex coefficients of homogeneous degree $m = \dim_{\mathbb{C}}(M)$. We define $\varphi(A)$ where A is an endomorphism $A : V \to V$. (V: a complex q-dimensional vector space). Let λ_i ($I = 1, \cdots q$) be eigenvalues of A. Then it is defined by

$$\varphi(A) := \varphi(\lambda_1, \cdots, \lambda_q). \tag{2.11}$$

We then regard $x_1, \cdots, x_q$ as Chern roots of E defined by

$$\prod_{i=1}^{q}(1+tx_i) := 1 + tc_1(E) + t^2c_2(E) + \cdots + t^q c_q(E). \tag{2.12}$$

With this set-up, a characteristic class $\underline{\varphi}(E)$ is defined as follows.

$$\underline{\varphi}(E) := \varphi(x_1, \cdots, x_q). \tag{2.13}$$

Let $\{N_\alpha\}$ be the set of connected components of the zero set of K. We assume that each N_α is a compact Kähler submanifold of M. In the following, we define $\underline{\varphi}(\Lambda_\alpha)$, which is given as a cohomology class of N_α. Let Λ_α be $\Lambda|_{N_\alpha}$, i.e., restriction of Λ to N_α. By the first condition in (2.8), Λ_α becomes an endmorphism of $E_\alpha := E|_{N_\alpha}$. We say that Λ is constant type if eigenvalues of Λ_α are constant on each connected component N_α. In this part, we assume that Λ is constant type. Then we introduce the following notations. Let $\{\lambda_i^\alpha \mid i = 1, \cdots, r\}$ be distinct eigenvalues of Λ_α ($r \leq q$) and m_i^α be multiplicity of λ_i^α ($\sum_{i=1}^r m_i^\alpha = q$). We denote the largest sub-bundle of E_α on which $(\Lambda_\alpha - \lambda_i^\alpha)$ is nilpotent by $E_\alpha(\lambda_i^\alpha)$ ($\mathrm{rank}_{\mathbb{C}}(E_\alpha(\lambda_i^\alpha)) = m_i^\alpha$). Then, E_α canonically decomposes into a direct sum, $E_\alpha = \bigoplus_{i=1}^{r} E_\alpha(\lambda_i^\alpha)$. Let $c_i(E_\alpha(\lambda_i^\alpha))$ be the i-th Chern class of $E_\alpha(\lambda_i^\alpha)$ and $x_j(\lambda_i^\alpha)$ $(j = 1, \cdots, m_i^\alpha)$ be Chern roots of $E_\alpha(\lambda_i^\alpha)$ defined by

$$\prod_{j=1}^{m_i^\alpha}(1+tx_j(\lambda_i^\alpha)) := 1 + tc_1(E_\alpha(\lambda_i^\alpha)) + t^2c_2(E_\alpha(\lambda_i^\alpha)) + \cdots + t^q c_q(E_\alpha(\lambda_i^\alpha)). \tag{2.14}$$

With these set-up's, the cohomology class $\underline{\varphi}(\Lambda_\alpha)$ is defined by

$$\begin{aligned}\underline{\varphi}(\Lambda_\alpha) \;:=\; & \varphi(\lambda_1^\alpha + x_1(\lambda_1^\alpha), \cdots, \lambda_1^\alpha + x_{m_1^\alpha}(\lambda_1^\alpha), \lambda_2^\alpha + x_1(\lambda_2^\alpha), \cdots, \lambda_2^\alpha + x_{m_2^\alpha}(\lambda_2^\alpha), \\ & \cdots, \lambda_r^\alpha + x_1(\lambda_r^\alpha), \cdots, \lambda_r^\alpha + x_{m_r^\alpha}(\lambda_r^\alpha)).\end{aligned} \tag{2.15}$$

This is the original definition of $\underline{\varphi}(\Lambda_\alpha)$ used in [8]. Let F_α be curvature $(1,1)$-form (valued in $\mathrm{End}(E|_{N_\alpha})$) of $E|_{N_\alpha}$. If we regard $\Lambda_\alpha + \frac{i}{2\pi}F_\alpha$ as $\mathrm{End}(E|_{N_\alpha})$ valued form on N_α, $\underline{\varphi}(\Lambda_\alpha)$ is rewritten by using (2.11) as follows.

$$\underline{\varphi}(\Lambda_\alpha) = \varphi(\Lambda_\alpha + \frac{i}{2\pi}F_\alpha). \tag{2.16}$$

With these set-up's, we introduce the Bott residue formula. We assume that the endomorphism $\theta|_{N_\alpha}$, induced by the action of K on the holomorphic tangent bundle $T'M|_{N_\alpha}$ has precisely $T'N_\alpha$ for its kernel; i.e., the sequence

$$0 \to T'N_\alpha \to T'M|_{N_\alpha} \xrightarrow{\theta|_{N_\alpha}} T'M|_{N_\alpha} \tag{2.17}$$

is exact. From the above exact sequence, $\mathrm{Im}(\theta|_{N_\alpha}) \cong T'M|_{N_\alpha}/T'N_\alpha$ follows. Hence we obtain an automorphism $\theta_\alpha^\nu := \theta^\nu|_{N_\alpha} : T'M|_{N_\alpha}/T'N_\alpha \to T'M|_{N_\alpha}/T'N_\alpha$ (where the subscript "ν" means that we consider holomorphic normal bundle $T'M|_{N_\alpha}/T'N_\alpha$ instead of $T'M|_{N_\alpha}$). Let R^ν_α be the curvature (1,1)-form of the holomorphic normal bundle $T'M|_{N_\alpha}/T'N_\alpha$ on N_α. Then the Bott residue formula is given as follows.

$$\begin{aligned}\int_M \underline{\varphi}(E) &= \sum_\alpha \int_{N_\alpha} \frac{\underline{\varphi}(\Lambda_\alpha)}{\det\left(\theta^\nu_\alpha + \frac{i}{2\pi}R^\nu_\alpha\right)} \\ &= \sum_\alpha \int_{N_\alpha} \frac{\varphi(\Lambda_\alpha + tF_\alpha)}{\det(\theta^\nu_\alpha + tR^\nu_\alpha)}\Bigg|_{t=\frac{i}{2\pi}}. \end{aligned} \tag{2.18}$$

In Section 3 and Section 4, we will derive the Bott Residue formula in the form of the second line of the above equality.

2.2 Usage of the Bott residue formula: An Example

(The contents of this subsection are based on discussions with Professor Masao Jinzenji.) In order to demonstrate usage of the Bott residue formula, we compute the following integral of Chern classes by using the formula (2.18):

$$\int_{\mathbb{C}P^n} c_1(H)^n. \tag{2.19}$$

Here, H is a hyperplane bundle on $\mathbb{C}P^n$.

First, we explain the case of $\mathbb{C}P^1$. Let $(X_1 : X_2)$ be homogeneous coordinates of $\mathbb{C}P^1$. Then, $\mathbb{C}P^1$ is covered by two open sets $U_1 := \{(X_1 : X_2) \in \mathbb{C}P^1 | X_1 \neq 0\}$ and $U_2 := \{(X_1 : X_2) \in \mathbb{C}P^1 | X_2 \neq 0\}$. We use $z := \frac{X_2}{X_1}$ and $w := \frac{X_1}{X_2}$ as local holomorphic coordinates on U_1 and U_2 respectively. Then we introduce the following holomorphic vector field K on $\mathbb{C}P^1$.

$$K = \alpha_1 z \frac{d}{dz} \qquad \text{(on } U_1\text{)}, \tag{2.20}$$

$$= -\alpha_1 w \frac{d}{dw} \qquad \text{(on } U_2\text{)}, \tag{2.21}$$

where we assume $\alpha_1 \neq 0$. Then actions of $\theta(K)$ on $\dfrac{d}{dz}$ and $\dfrac{d}{dw}$ are given by

$$\theta(K)(\frac{d}{dz}) := [K, \frac{d}{dz}] = [\alpha_1 z\frac{d}{dz}, \frac{d}{dz}] = -\alpha_1 \frac{d}{dz}, \qquad \theta(K)(\frac{d}{dw}) = \alpha_1 \frac{d}{dw}. \tag{2.22}$$

Zero set of K is given by $\{(1:0),(0:1)\}$ and we denote $(1:0)$ ($\leftrightarrow\ z = 0$) and $(0:1)$ ($\leftrightarrow\ w = 0$) by p_1 and p_2 respectively. Then (2.22) tells us that $\theta^\nu|_{p_1} = -\alpha_1$ and $\theta^\nu|_{p_2} = \alpha_1$ because $T'\mathbb{C}P^1|_{p_i}/T'p_i = T'\mathbb{C}P^1|_{p_i}$.

Next, we construct local frame of $H = S^*$ in order to determine $\Lambda_1 = \Lambda|_{p_1}$ and $\Lambda_2 = \Lambda|_{p_2}$ (S is the tautological line bundle on $\mathbb{C}P^1$). The fiber of S on $(X_1 : X_2) \in \mathbb{C}P^1$

is given by a complex 1-dimensional linear subspace of $\mathbb{C}^2$ spanned by (X_1, X_2). Let e_1 and e_2 be local frames on U_1 and U_2 given by

$$e_1 = (1, z), \qquad\qquad e_2 = (w, 1). \tag{2.23}$$

Hence transition functions of S that satisfy $e_\alpha = g^S_{\alpha\beta} e_\beta$ on $U_1 \cap U_2$ are obtained as $g^S_{12} = \frac{1}{w}$ and $g^S_{21} = \frac{1}{z}$. Let f_1 and f_2 be local frames of H on U_1 and U_2 respectively. Since hyperplane bundle H is dual line bundle of S, transition functions of H are given as $g^H_{12} = (g^S_{12})^{-1} = w$ and $g^H_{21} = z$. Then a global holomorphic section s is given by f_1 on U_1 and wf_2 on U_2 because $f_1 = wf_2$ holds on $U_1 \cap U_2$. With this set-up, note that Λ satisfies (2.8) and that $\Lambda(s^1 f_1) = \Lambda(s^2 f_2)$ on $U_1 \cap U_2$. Since $p_1 = (1 : 0) \in U_1 \setminus U_2$ and $p_2 = (0 : 1) \in U_2 \setminus U_1$, we set $\Lambda(f_1) =: \Lambda_1 f_1$ and $\Lambda(f_2) =: \Lambda_2 f_2$. Then we apply (2.8) to the above global holomorphic section s.

$$\Lambda(f_1) = \Lambda(f_1) = \Lambda_1 f_1, \tag{2.24}$$

$$\Lambda(wf_2) = K(w)f_2 + w\Lambda(f_2) = -\alpha_1 w \frac{dw}{dw} f_2 + w\Lambda_2 f_2 = (-\alpha_1 + \Lambda_2) w f_2. \tag{2.25}$$

Since $\Lambda(f_1) = \Lambda(wf_2)$ holds on $U_1 \cap U_2$, we obtain

$$\Lambda_1 = C, \quad \Lambda_2 = C + \alpha_1, (C \text{ is an arbitrary constant}). \tag{2.26}$$

With these set-up's, we compute $\int_{\mathbb{C}P^1} c_1(H)$. Since zero set of K is given by $\{p_1,\ p_2\}$ and $c_1(H)$ is given by trace of curvature form of H, the assertion of Bott Residue Formula in this case becomes

$$\int_{\mathbb{C}P^1} c_1(H) = \sum_{i=1}^{2} \frac{\mathrm{tr}(\Lambda|_{p_i})}{\det(\theta^\nu|_{p_i})}. \tag{2.27}$$

On the other hand, we have the following table from the discussions so far:

Table 1: **Summary of the case of $\mathbb{C}P^1$**

	p_1	p_2
$\mathrm{tr}(\Lambda\|_{p_i})$	C	$C + \alpha_1$
$\det(\theta^\nu\|_{p_i})$	$-\alpha_1$	α_1

Hence we obtain

$$\int_{\mathbb{C}P^1} c_1(H) = -\frac{C}{\alpha_1} + \frac{C + \alpha_1}{\alpha_1} = 1. \tag{2.28}$$

Then we turn into the $\mathbb{C}P^n$ case. Let $(X_1 : \cdots : X_{n+1})$ be homogeneous coordinates of $\mathbb{C}P^n$. $\mathbb{C}P^n$ is covered by $(n+1)$ open sets $U_i := \{(X_1 : \cdots : X_{n+1}) \in \mathbb{C}P^n \mid X_i \neq$

$0\}$ $(i = 1, 2, \cdots, n+1)$. Local coordinate systems on U_i: $\phi_i : U_i \to \mathbb{C}^n$ are defined in the following way.

$$\phi_i(X_1 : \cdots : X_{n+1}) = \Big(\frac{X_1}{X_i}, \cdots, \frac{X_{i-1}}{X_i}, \frac{X_{i+1}}{X_i}, \cdots, \frac{X_{n+1}}{X_i}\Big) = (z^1_{(i)}, \cdots, z^n_{(i)}). \tag{2.29}$$

Coordinate transformations between U_1 and $U_j (j = 2, 3, \cdots, n+1)$ are explicitly given by

$$z^1_{(j)} = \frac{1}{z^{j-1}_{(1)}}, \quad z^i_{(j)} = \frac{z^{i-1}_{(1)}}{z^{j-1}_{(1)}} \quad (1 < i \leq j-1), \quad z^i_{(j)} = \frac{z^i_{(1)}}{z^{j-1}_{(1)}} \quad (j \leq i). \tag{2.30}$$

From these results, we obtain the following transformation rules on $U_1 \cap U_i$:

$$\frac{\partial}{\partial z^{j-1}_{(1)}} = \frac{-1}{\partial z^{j-1}_{(1)}} \sum_{l=1}^{n} z^l_{(j)} \frac{\partial}{\partial z^l_{(j)}}, \tag{2.31}$$

$$\frac{\partial}{\partial z^i_{(1)}} = \frac{z^{i+1}_{(j)}}{\partial z^i_{(1)}} \frac{\partial}{\partial z^{i+1}_{(j)}} \quad (1 \leq i \leq j-2), \tag{2.32}$$

$$\frac{\partial}{\partial z^i_{(1)}} = \frac{z^i_{(j)}}{\partial z^i_{(1)}} \frac{\partial}{\partial z^i_{(j)}} \quad (j \leq i). \tag{2.33}$$

Then, we introduce the following holomorphic tangent vector field K on $\mathbb{C}P^n$:

$$K = \sum_{i=1}^{n} \alpha_i z^i_{(1)} \frac{\partial}{\partial z^i_{(1)}} \quad \big(\text{on } U_1\big) \tag{2.34}$$

$$\begin{aligned} &= -\alpha_{j-1} z^1_{(j)} \frac{\partial}{\partial z^1_{(j)}} + \sum_{i=2}^{j-1} (\alpha_{i-1} - \alpha^{j-1}) z^i_{(j)} \frac{\partial}{\partial z^i_{(j)}} \\ &+ \sum_{i=j}^{n} (\alpha_i - \alpha_{j-1}) z^i_{(j)} \frac{\partial}{\partial z^i_{(j)}} \quad \big(\text{on } U_i \ (i = 2, 3, \cdots, n+1)\big), \end{aligned} \tag{2.35}$$

where we assume $\alpha_i \neq 0$ $(i = 1, 2, \cdots, n)$ and $\alpha^i \neq \alpha^j$ $(i \neq j)$. Zero set of K is given by

$$\begin{aligned} &\{p_1, p_2, \cdots, p_{n+1}\}, \\ &\big(p_i := \overbrace{0 : \cdots : 0}^{i-1} : 1 : \overbrace{0 : \cdots : 0}^{n-i+1}\big) \ \ (i = 1, 2, \cdots, n+1)\big). \end{aligned} \tag{2.36}$$

Then we determine action of $\theta(K)$ on each U_i. On U_1, it is given by

$$\theta(K)(\frac{\partial}{\partial z^i_{(1)}}) = -\alpha^i \frac{\partial}{\partial z^i_{(1)}}. \tag{2.37}$$

On U_j $(j = 2, 3, \cdots, n+1)$, it is given as follows.

$$\theta(K)(\frac{\partial}{\partial z^1_{(j)}}) = \alpha^{j-1}\frac{\partial}{\partial z^1_{(j)}},$$
$$\theta(K)(\frac{\partial}{\partial z^i_{(j)}}) = (\alpha^{j-1} - \alpha^{i-1})\frac{\partial}{\partial z^i_{(j)}} \quad (i = 2, 3, \cdots, j-1),$$
$$\theta(K)(\frac{\partial}{\partial z^i_{(j)}}) = (\alpha^{j-1} - \alpha^{i})\frac{\partial}{\partial z^i_{(j)}} \quad (i = j, j+1, \cdots, n+1). \tag{2.38}$$

Since p_i $(i = 1, 2, \cdots, n+1)$ is a point in $\mathbb{C}P^n$ and $p_i \in U_i \setminus (\cup_{j\neq i} U_j)$, $\theta^\nu_i := \theta^\nu|_{p_i}$ is given by the above representations of $\theta(K)$ on U_i.

Next, we construct local frames of hyperplane bundle $H = S^*$ on U_i in order to determine $\Lambda_i := \Lambda|_{p_i}$. In the same way as the $\mathbb{C}P^1$ case, the local frame of S on U_i is given by

$$e_i = (z^1_{(i)}, \cdots, z^{i-1}_{(i)}, 1, z^i_{(i)}, \cdots, z^n_{(i)}). \tag{2.39}$$

Then we can determine transition function g^S_{1j} that satisfies $e_1 = g^S_{1j} e_j$ on $U_1 \cap U_j$.

$$\begin{aligned} e_1 = (1, z^1_{(1)}, \cdots, z^n_{(1)}) &= z^{j-1}_{(1)}\Big(\frac{1}{z^{j-1}_{(1)}}, \frac{z^1_{(1)}}{z^{j-1}_{(1)}}, \cdots, \frac{z^{j-2}_{(1)}}{z^{j-1}_{(1)}}, 1, \frac{z^j_{(1)}}{z^{j-1}_{(1)}}, \cdots, \frac{z^n_{(1)}}{z^{j-1}_{(1)}}\Big) \\ &= z^{j-1}_{(1)} e_j := g^S_{1j} e_j \quad (j = 2, 3, \cdots, n+1). \end{aligned} \tag{2.40}$$

Let $f_i (i = 1, \cdots, n+1)$ be the local frame of $H = S^*$ on U_i. Then transition functions g^H_{1j} is given by $(g^S_{1j})^{-1} = \frac{1}{z^{j-1}_{(1)}} = z^1_{(j)}$ and we have $f_1 = g_{1j} f_j = z^1_{(j)} f_j$. Then we can construct a global holomorphic section s of H which is reperesented as $s^i f_i$ on U_i, by setting $s^1 = 1$, $s^j = z^1_{(j)}$ $(j = 2, 3, \cdots, n+1)$ Since $p_i \in U_i \setminus (\cup_{j\neq i} U_j)$. we can set $\Lambda(f_i) = \Lambda_i f_i$. Then by applying (2.8) to s, the following equality:

$$\begin{aligned} &\Lambda(s^1 f_1) = \Lambda_1 f_1 \\ &= \Lambda(s^i f_i) = K(z^1_{(i)}) f_i + z^1_{(i)} \Lambda(f_i) = -\alpha^{i-1} z^1_{(i)} f_i + z^1_{(i)} \Lambda_i f_i = (-\alpha^{i-1} + \Lambda_i) f_1, \end{aligned}$$

holds on $U_1 \cap U_i$. Hence Λ_i's are given as $\Lambda^1 = C$, $\Lambda^i = C + \alpha^{i-1}$ $(i = 2, 3, \cdots, n+1)$ where C is an arbitrary constant. We summarize the above results in the following table.

Table 2: **Summary of the case of $\mathbb{C}P^n$**

	p_1	p_i $(i = 2, \cdots, n+1)$
$\mathrm{tr}(\Lambda\|_{p_i})$	C	$C + \alpha^{i-1}$
$\det(\theta^\nu\|_{p_i})$	$\prod_{i=1}^n (-\alpha^i)$	$\alpha^{i-1} \prod_{j=2}^{i-1} (\alpha^{i-1} - \alpha^{j-1}) \prod_{j=i}^n (\alpha^{i-1} - \alpha^j)$

Then the assertion of Bott residue formula is given by

$$\begin{aligned}\int_{\mathbb{C}P^n}(c_1(H))^n &= \sum_{i=1}^{n+1}\frac{(\mathrm{tr}(\Lambda|_{p_i})^n}{\det(\theta^\nu|_{p_i})}\\ &= \frac{C^n}{\prod_{i=1}^n(-\alpha^i)}+\sum_{i=2}^{n+1}\frac{(C+\alpha^{i-1})^n}{\alpha^{i-1}\prod_{j=1}^{i-1}(\alpha^{i-1}-\alpha^{j-1})\prod_{j=i}^n(\alpha^{i-1}-\alpha^j)}\\ &= \frac{C^n}{\prod_{i=1}^n(-\alpha^i)}+\sum_{i=1}^{n}\frac{(C+\alpha^{i})^n}{\alpha^{i}\prod_{j=1}^{i}(\alpha^{i}-\alpha^{j-1})\prod_{j=i+1}^n(\alpha^{i}-\alpha^j)}. \end{aligned} \quad (2.41)$$

Let us consider a melomorphic function $f(z)=\frac{(C+z)^n}{z\prod_{j=1}^n(z-\alpha^j)}$ on $\mathbb{C}\cup\{\infty\}$. $f(z)$ has simple poles at $z=0,\infty,\alpha^i\ \ (i=1,\cdots,n)$ and sum of residues at these points equals zero. Therefore, we have

$$\frac{C^n}{\prod_{i=1}^n(-\alpha^i)}+\sum_{i=1}^{n}\frac{(C+\alpha^{i})^n}{\alpha^{i}\prod_{j=1}^{i}(\alpha^{i}-\alpha^{j-1})\prod_{j=i+1}^n(\alpha^{i}-\alpha^j)}-1=0. \quad (2.42)$$

By combining (2.41) with (2.42), we obtain

$$\int_{\mathbb{C}P^n}(c_1(H))^n=1. \quad (2.43)$$

3 Models in Part I

For the properties of the connection, $\triangledown$ the covariant derivative and the curvature tensor of a Kähler manifold, please refer to [34].

3.1 Original Topological Sigma Model (A-Model)

We introduce the topological sigma model (A-model) and its correlation function. This subsection and Subsection 3.2 are reviews of the A-model [22, 23]. The discussion and description in this subsection and Subsection 3.2 are based on [22], [23] and [27]. (Please refer to [22] and [23] for details.) First, we consider the Lagrangian. The Lagrangian of the topological sigma model (A-model) is given by

$$\begin{aligned}L=\int_{\mathbb{C}\mathrm{P}^1}dzd\bar{z}\big[&\frac{t}{2}g_{i\bar{j}}(\partial_z\phi^i\partial_{\bar{z}}\phi^{\bar{j}}+\partial_{\bar{z}}\phi^i\partial_z\phi^{\bar{j}})+\sqrt{t}ig_{i\bar{j}}\psi_z^{\bar{j}}D_{\bar{z}}\chi^i+\sqrt{t}ig_{i\bar{j}}\psi_{\bar{z}}^{i}D_z\chi^{\bar{j}}\\ &-R_{i\bar{j}k\bar{l}}\psi_{\bar{z}}^i\psi_z^{\bar{j}}\chi^k\chi^{\bar{l}}\big].\end{aligned} \quad (3.44)$$

t is the coupling constant. Here, we rescaled $t\to\frac{t}{2}$ and $\psi\to\frac{\psi}{\sqrt{t}}$ in [23]. Fields in the Lagrangian and covariant derivatives are the same as the ones introduced in Section

1. Original supersymmetry (BRST-symmetry) of this model is given in the following form.

$$\begin{aligned} &\delta\phi^{i} = i\alpha\chi^{i}, && \delta\phi^{\bar{i}} = i\alpha\chi^{\bar{i}}, \\ &\delta\psi_{z}^{\bar{i}} = -\sqrt{t}\alpha\partial_{z}\phi^{\bar{i}} - i\alpha\Gamma^{\bar{i}}_{\bar{\mu}\bar{\nu}}\chi^{\bar{\mu}}\psi_{z}^{\bar{\nu}}, && \\ &\delta\psi_{\bar{z}}^{i} = -\sqrt{t}\alpha\partial_{\bar{z}}\phi^{i} - i\alpha\Gamma^{i}_{\mu\nu}\chi^{\mu}\psi_{\bar{z}}^{\nu}, && \delta\chi^{i} = \delta\chi^{\bar{i}} = 0, \end{aligned} \tag{3.45}$$

where α is a fermionic variable. We define the operator $\{Q_0, *\}$ via $\delta X =: i\alpha\{Q_0, X\}$($X$ is a field that appears in the A-model). And, we call Q_0 the generator of this transformation. Then $\{Q_0, \{Q_0, X\}\} = 0$ holds. And $\{Q_0, L\} = 0$ from $\delta L = 0$. When we rewrite $\{Q_0, X\}$ to $Q_0 X$, we identify the operator Q_0 and the exterior derivative d on M from $\{Q_0, \{Q_0, X\}\} = 0$. $Q_0^2 = 0$ is a characteristic of the BRST-transformation. Note that Q_0 and $\{Q_0, *\}$ may be equated.

Next, we discuss general characteristics of a correlation function in the A-model. The correlation function of an observable W is defined by

$$< W >:= \int \mathfrak{D}X e^{-L} W. \tag{3.46}$$

An observable is constructed by fields that appear in this model. Its detail is given by Subsection 3.2. If W is given by $W = \{Q_0, U\}$ (U is other observable) , its correlation function vanishes [22, 23].

$$< \{Q_0, U\} >= 0. \tag{3.47}$$

It is shown by rotating the field through supersymmetric transformation. The rotating is induced by acting the operator $\exp(\epsilon Q_0)$ on each field. ϵ is a fermionic parameter. The path integral is invariant under the rotating.

$$< U >= \int \mathfrak{D}X e^{-L} U = \int \mathfrak{D}(\exp(\epsilon Q_0)X)(\exp(\epsilon Q_0)e^{-L}U)$$

Since the transformation is given by rotating the field, the Jacobian for its transformation is 1.

$$< U > = \int \mathfrak{D}X(\exp(\epsilon Q_0)e^{-L}U) = \int \mathfrak{D}X(1 + \epsilon Q_0)(e^{-L}U).$$

From $Q_0 L = 0$,

$$\begin{aligned} < U > &= \int \mathfrak{D}X e^{-L}(1 + \epsilon Q_0)U = \int \mathfrak{D}X e^{-L}(U + \epsilon Q_0 U) \\ &=< U > +\epsilon \int \mathfrak{D}X e^{-L}\{Q_0 U\} =< U > +\epsilon < \{Q_0, U\} > . \end{aligned}$$

Therefore, we obtain $< \{Q_0, U\} >= 0$. (This proof is referenced to [23].) In the A-model, we consider the observable $\mathcal{O}$ that satisfies the relation$\{Q_0, \mathcal{O}\} = 0$. One of the reasons of this is to use the weak coupling limit in the calculation for the correlation function. We explain it later. Here, we introduce important formulas for observables. "If observables $\mathcal{O}_i (i = 1, 2, \cdots, n)$ satisfy $\{Q, \mathcal{O}_i\} = 0$, the equalities

$$\{Q, \mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n\} = 0, \quad \{Q, A\}\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n = \{Q, A\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n\}$$

hold for arbitrary observable A."[23] (Q in [23] corresponds to Q_0.) From now on, we assume that the observable satisfies $\{Q_0, \mathcal{O}\} = 0$ in this subsection. It is known that the phase space of a correlation function is decomposed in every degree of ϕ. We rewrite the Lagrangian as follows.

$$L = \frac{t}{2}\int_{\mathbb{C}P^1} \phi^*(\omega) + L_d. \tag{3.48}$$

Here,

$$L_d := \int_{\mathbb{C}\mathrm{P}^1} dzd\bar{z}\big[tg_{i\bar{j}}\partial_{\bar{z}}\phi^i\partial_z\phi^{\bar{j}} + \sqrt{t}ig_{i\bar{j}}\psi_z^{\bar{j}}D_{\bar{z}}\chi^i + \sqrt{t}ig_{i\bar{j}}\psi_{\bar{z}}^{i}D_z\chi^{\bar{j}} - R_{i\bar{j}k\bar{l}}\psi_{\bar{z}}^i\psi_z^{\bar{j}}\chi^k\chi^{\bar{l}}\big] \tag{3.49}$$

and

$$\int_{\mathbb{C}P^1} \phi^*(\omega) := \int_{\mathbb{C}P^1} d^2z[g_{i\bar{j}}(\partial_z\phi^i\partial_{\bar{z}}\phi^{\bar{j}} - \partial_{\bar{z}}\phi^i\partial_z\phi^{\bar{j}})]. \tag{3.50}$$

$\omega := g_{i\bar{j}}dz^i \wedge d\overline{z^j}$ is the Kähler form of M. And we note that ω is a representative of $H^2(M)$. $\phi^*(\omega)$ means the pull-back of ω by the map $\phi : \mathbb{C}P^1 \to M$. If $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$, we obtain

$$\frac{t}{2}\int_{\mathbb{C}P^1} \phi^*(\omega) = 2\pi idt \quad (d \in \mathbb{Z} \text{ and } d \geq 0) \tag{3.51}$$

by rescaling the metric $g_{i\bar{j}}$. d means the degree of ϕ. We introduce the equation of motion for ψ. We vary ψ in the Lagrangian.

$$\delta_{\psi_{\bar{z}}^i} L_d = \int_{\mathbb{C}\mathrm{P}^1} dzd\bar{z}\big[\sqrt{t}ig_{i\bar{j}}\delta\psi_{\bar{z}}^i D_z\chi^{\bar{j}} - R_{i\bar{j}k\bar{l}}\delta\psi_{\bar{z}}^i\psi_z^{\bar{j}}\chi^k\chi^{\bar{l}}\big] \tag{3.52}$$

$$= \int_{\mathbb{C}\mathrm{P}^1} dzd\bar{z}\big[-\sqrt{t}ig_{i\bar{j}}D_z\chi^{\bar{j}} + R_{i\bar{j}k\bar{l}}\psi_z^{\bar{j}}\chi^k\chi^{\bar{l}}\big]\delta\psi_{\bar{z}}^i \tag{3.53}$$

The equation of motion for $\psi_{\bar{z}}^i$ is given by $\delta_{\psi_{\bar{z}}^i} L_d = 0$. Therefore,

$$-\sqrt{t}ig_{i\bar{j}}D_z\chi^{\bar{j}} + R_{i\bar{j}k\bar{l}}\psi_z^{\bar{j}}\chi^k\chi^{\bar{l}} = 0. \tag{3.54}$$

The equation of $\psi_z^{\bar{j}}$ is also given in the same way.

$$-i\sqrt{t}g_{i\bar{j}}D_{\bar{z}}\chi^i - R_{i\bar{j}k\bar{l}}\psi_{\bar{z}}^i\chi^k\chi^{\bar{l}} = 0. \tag{3.55}$$

By using the equation of motion for ψ,

$$i\sqrt{t}g_{i\bar{j}}\psi_z^{\bar{j}}D_{\bar{z}}\chi^i = R_{i\bar{j}k\bar{l}}\psi_{\bar{z}}^i\psi_z^{\bar{j}}\chi^k\chi^{\bar{l}}, \quad i\sqrt{t}g_{i\bar{j}}\psi_{\bar{z}}^i D_z\chi^{\bar{j}} = R_{i\bar{j}k\bar{l}}\psi_{\bar{z}}^i\psi_z^{\bar{j}}\chi^k\chi^{\bar{l}}, \tag{3.56}$$

L_d is represented by

$$L_d = -i\sqrt{t}\int_{\mathbb{C}P^1} d^2z\{Q_0, P\}, P := \frac{1}{2}g_{i\bar{j}}(\psi_z^{\bar{j}}\partial_{\bar{z}}\phi^i + \psi_{\bar{z}}^i\partial_z\phi^{\bar{j}}). \tag{3.57}$$

Then, the phase space of the correlation function $< \mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n >$ can be decomposed into connected components P_d (d is a degree of ϕ. $d = 0, 1, 2, 3, \cdots$) as follows.

$$\begin{aligned} < \mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n > &= \int \mathfrak{D}Xe^{-L}\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n = \int \mathfrak{D}Xe^{-t\int_{\mathbb{C}P^1}\phi^{-1}(\omega)-L_d}\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n \\ &= \sum_{d=0}^{\infty} e^{-2\pi idt}\int_{P_d}\mathfrak{D}Xe^{-L_d}\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n. \end{aligned}$$

Let us define the degree d correlation function by

$$< \mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n >_d := \int_{P_d}\mathfrak{D}Xe^{-L_d}\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n. \tag{3.58}$$

$< \mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n >_d$ does not depend on the coupling constant t. We change t into $t + \delta t$ in (3.58). And $L_d = -i\sqrt{t}\int_{\mathbb{C}P^1} d^2z\{Q_0, P\}$ from equations of motion.

$$\begin{aligned} &\int_{P_d}\mathfrak{D}Xe^{i\sqrt{t+\delta t}\int_{\mathbb{C}P^1}d^2z\{Q_0,P\}}\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n \\ &= \int_{P_d}\mathfrak{D}Xe^{i\sqrt{t}\int_{\mathbb{C}P^1}d^2z\{Q_0,P\}+i\frac{\delta t}{2\sqrt{t}}\int_{\mathbb{C}P^1}d^2z\{Q_0,P\}}\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n \\ &= \int_{P_d}\mathfrak{D}Xe^{i\sqrt{t}\int_{\mathbb{C}P^1}d^2z\{Q_0,P\}}(1 + i\frac{\delta t}{2\sqrt{t}}\int_{\mathbb{C}P^1}d^2z\{Q_0, P\})\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n \\ &= \int_{P_d}\mathfrak{D}Xe^{i\sqrt{t}\int_{\mathbb{C}P^1}d^2z\{Q_0,P\}}\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n \\ &+ i\frac{\delta t}{2\sqrt{t}}\int_{p_d}\mathfrak{D}Xe^{i\sqrt{t}\int_\Sigma d^2z\{Q_0,P\}}\int_{\mathbb{C}P^1}d^2z\{Q_0, P\}\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n \\ &= \int_{P_d}\mathfrak{D}Xe^{i\sqrt{t}\int_{\mathbb{C}P^1}d^2z\{Q_0,P\}}\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n \\ &+ i\frac{\delta t}{2\sqrt{t}}\int_{P_d}\mathfrak{D}Xe^{it\int_{\mathbb{C}P^1}d^2z\{Q_0,P\}}\{Q_0, (\int_{\mathbb{C}P^1}d^2zP)\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n\} \\ &= \int_{P_d}\mathfrak{D}Xe^{i\sqrt{t}\int_{\mathbb{C}P^1}d^2z\{Q_0,P\}}\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n. \end{aligned}$$

We ignored terms above the second or higher order of δt and used $< \{Q_0, U\} >_d = 0$. Therefore, in $< \mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n >_d$, we can take the limit of $t \to \infty$. "In field theory, this operation is called “taking weak coupling limit”.”[23] Under the limit $t \to \infty$, the exact value of the path integral is given by expanding each variable in a neighborhood of the solution of equation of motion. So, an observable $\mathcal{O}$ has to satisfy $\{Q_0, \mathcal{O}\} = 0$ to take the weak coupling limit.

Lastly, we consider the phase space of path integral into connected component P_d and explain the weak coupling limit. The lagrangian of the degree d correlation function L_d is given as follows.

$$L_d = \int_{\mathbb{C}P^1} dz d\bar{z} \big[t g_{i\bar{j}} \partial_{\bar{z}} \phi^i \partial_z \phi^{\bar{j}} + i\sqrt{t} g_{i\bar{j}} \psi_z^{\bar{j}} D_{\bar{z}} \chi^i + i\sqrt{t} g_{i\bar{j}} \psi_{\bar{z}}^i D_z \chi^{\bar{j}} - R_{i\bar{j}k\bar{l}} \psi_{\bar{z}}^i \psi_z^{\bar{j}} \chi^k \chi^{\bar{l}} \big] \quad (3.59)$$

For the time being, we neglect $-R_{i\bar{j}k\bar{l}} \psi_{\bar{z}}^i \psi_z^{\bar{j}} \chi^k \chi^{\bar{l}}$. (According to [22, 23], this operation does not change the result.) Here, we introduce the principle of localization [26].

> The principle of localization
>
> Path integrals of a model having supersymmetry can be evaluated by the sum of contributions from field configurations satisfying the conditions that supersymmetric transformations of fermionic fields are zero.

Since $\delta \psi_z^{\bar{i}} = -\sqrt{t}\alpha \partial_z \phi^{\bar{i}}, \delta \psi_{\bar{z}}^i = -\sqrt{t}\alpha \partial_{\bar{z}} \phi^i$ (we neglect the term of Γ) in $t \to \infty$, ϕ must satisfy $\partial_z \phi^{\bar{i}} = 0$ and $\partial_{\bar{z}} \phi^i = 0$. Equations of motion for each ferminic field are

$$D_z \chi^{\bar{i}} = 0, D_{\bar{z}} \chi^i = 0, D_z \psi_{\bar{z}}^i = 0, D_{\bar{z}} \psi_z^{\bar{i}} = 0. \quad (3.60)$$

Here, $D_z \psi_{\bar{z}}^i = 0$ and $D_{\bar{z}} \psi_z^{\bar{i}} = 0$ are given by the equation of motion for χ as follows. (We ignore $R_{i\bar{j}k\bar{l}} \psi_{\bar{z}}^i \psi_z^{\bar{j}} \chi^k \chi^{\bar{l}}$.)

$$\begin{aligned}
\delta_{\chi^i} L_d &= \delta_{\chi^i} \int_{\mathbb{C}P^1} dz d\bar{z} \big[i\sqrt{t} g_{i\bar{j}} \psi_z^{\bar{j}} D_{\bar{z}} \chi^i \big] && (3.61)\\
&= \int_{\mathbb{C}P^1} dz d\bar{z} \big[i\sqrt{t} g_{i\bar{j}} \psi_z^{\bar{j}} \partial_{\bar{z}} (\delta \chi^i) + i\sqrt{t} g_{i\bar{j}} \psi_z^{\bar{j}} \Gamma^i_{\mu\nu} \delta\chi^\mu \partial_{\bar{z}} \phi^\nu \big] && (3.62)\\
&= \int_{\mathbb{C}P^1} dz d\bar{z} \big[i\sqrt{t} \{ (-\partial_l g_{i\bar{j}} \partial_{\bar{z}} \phi^l - \partial_{\bar{l}} g_{i\bar{j}} \partial_{\bar{z}} \phi^{\bar{l}}) \psi_z^{\bar{j}} - g_{i\bar{j}} \partial_{\bar{z}} \psi_z^{\bar{j}} \} \delta\chi^i \\
&\quad + i\sqrt{t} g_{i\bar{j}} \psi_z^{\bar{j}} \Gamma^i_{\mu\nu} \delta\chi^\mu \partial_{\bar{z}} \phi^\nu \big] && (3.63)\\
&= \int_{\mathbb{C}P^1} dz d\bar{z} \big[i\sqrt{t} \{ (-g_{\lambda\bar{j}} \Gamma^\lambda_{il} \partial_{\bar{z}} \phi^l - g_{i\bar{\lambda}} \Gamma^{\bar{\lambda}}_{\bar{l}\bar{j}} \partial_{\bar{z}} \phi^{\bar{l}}) \psi_z^{\bar{j}} - g_{i\bar{j}} \partial_{\bar{z}} \psi_z^{\bar{j}} \} \delta\chi^i \\
&\quad + i\sqrt{t} g_{i\bar{j}} \psi_z^{\bar{j}} \Gamma^i_{\mu\nu} \delta\chi^\mu \partial_{\bar{z}} \phi^\nu \big] && (3.64)\\
&= \int_{\mathbb{C}P^1} dz d\bar{z} \big[i\sqrt{t} (-g_{i\bar{\lambda}} \Gamma^{\bar{\lambda}}_{\bar{l}\bar{j}} \partial_{\bar{z}} \phi^{\bar{l}} \psi_z^{\bar{j}} - g_{i\bar{j}} \partial_{\bar{z}} \psi_z^{\bar{j}}) \delta\chi^i \big] && (3.65)\\
&= \int_{\mathbb{C}P^1} dz d\bar{z} \big[-i\sqrt{t} g_{i\bar{j}} (\Gamma^{\bar{j}}_{\bar{l}\bar{\nu}} \partial_{\bar{z}} \phi^{\bar{l}} \psi_z^{\bar{\nu}} + \partial_{\bar{z}} \psi_z^{\bar{j}}) \delta\chi^i \big] && (3.66)\\
&= \int_{\mathbb{C}P^1} dz d\bar{z} \big[-i\sqrt{t} g_{i\bar{j}} D_{\bar{z}} \psi_z^{\bar{j}} \delta\chi^i \big] = 0 \Rightarrow D_{\bar{z}} \psi_z^{\bar{j}} = 0. && (3.67)
\end{aligned}$$

In the same way

$$\delta_{\chi^{\bar{j}}} L_d = \delta_{\chi^{\bar{j}}} \int_{\mathbb{C}P^1} dz d\bar{z} \big[i\sqrt{t} g_{i\bar{j}} \psi^i_{\bar{z}} D_z \chi^{\bar{j}} \big] \tag{3.68}$$

$$= \int_{\mathbb{C}P^1} dz d\bar{z} \big[i\sqrt{t} g_{i\bar{j}} \psi^i_{\bar{z}} \partial_z (\delta\chi^{\bar{j}}) + i\sqrt{t} g_{i\bar{j}} \psi^i_{\bar{z}} \Gamma^{\bar{j}}_{\bar{\mu}\bar{\nu}} \delta\chi^{\bar{\mu}} \partial_z \phi^{\bar{\nu}} \big] \tag{3.69}$$

$$= \int_{\mathbb{C}P^1} dz d\bar{z} \big[-i\sqrt{t} \{ \partial_l g_{i\bar{j}} \partial_z \phi^l \psi^i_{\bar{z}} + \partial_{\bar{l}} g_{i\bar{j}} \partial_z \phi^{\bar{l}} \psi^i_{\bar{z}} + g_{i\bar{j}} \partial_z \psi^i_{\bar{z}} \} \delta\chi^{\bar{j}}$$
$$+ i\sqrt{t} g_{i\bar{j}} \psi^i_{\bar{z}} \Gamma^{\bar{j}}_{\bar{\mu}\bar{\nu}} \delta\chi^{\bar{\mu}} \partial_z \phi^{\bar{\nu}} \big] \tag{3.70}$$

$$= \int_{\mathbb{C}P^1} dz d\bar{z} \big[-i\sqrt{t} \{ g_{\lambda\bar{j}} \Gamma^{\lambda}_{il} \partial_z \phi^l \psi^i_{\bar{z}} + g_{i\bar{\lambda}} \Gamma^{\bar{\lambda}}_{\bar{l}\bar{j}} \partial_z \phi^{\bar{l}} \psi^i_{\bar{z}} + g_{i\bar{j}} \partial_z \psi^i_{\bar{z}} \} \delta\chi^{\bar{j}}$$
$$+ i\sqrt{t} g_{i\bar{j}} \psi^i_{\bar{z}} \Gamma^{\bar{j}}_{\bar{\mu}\bar{\nu}} \delta\chi^{\bar{\mu}} \partial_z \phi^{\bar{\nu}} \big] \tag{3.71}$$

$$= \int_{\mathbb{C}P^1} dz d\bar{z} \big[-i\sqrt{t} (g_{\lambda\bar{j}} \Gamma^{\lambda}_{il} \partial_z \phi^l \psi^i_{\bar{z}} + g_{i\bar{j}} \partial_z \psi^i_{\bar{z}}) \delta\chi^{\bar{j}} \big] \tag{3.72}$$

$$= \int_{\mathbb{C}P^1} dz d\bar{z} \big[-i\sqrt{t} g_{i\bar{j}} D_z \psi^i_{\bar{z}} \delta\chi^{\bar{j}} \big] = 0 \Rightarrow D_z \psi^i_{\bar{z}} = 0. \tag{3.73}$$

We used $\partial_l g_{i\bar{j}} = g_{\lambda\bar{j}} \Gamma^{\lambda}_{il}$ and $\partial_{\bar{l}} g_{i\bar{j}} = g_{i\bar{\lambda}} \Gamma^{\bar{\lambda}}_{\bar{l}\bar{j}}$. (In a Kähler manifold, $\Gamma^{\lambda}_{il} = \Gamma^{\lambda}_{li}$ and $\Gamma^{\bar{\lambda}}_{\bar{l}\bar{j}} = \Gamma^{\bar{\lambda}}_{\bar{j}\bar{l}}$.) We consider $\partial_{\bar{z}} \phi^i = 0, D_{\bar{z}} \chi^i = 0$ and $D_{\bar{z}} \psi^{\bar{i}}_z = 0$. Other equations are complex conjugate of these equations. (We derive the condition $\partial_{\bar{z}} \phi^i = 0$ from the principle of localization in this thesis [27]. However, in [22] and [23], $\partial_{\bar{z}} \phi^i = 0$ is given by the equation of motion for ϕ. So, we call $\partial_{\bar{z}} \phi^i = 0$ the equation of motion for ϕ, here.) $\partial_{\bar{z}} \phi^i = 0$ means that ϕ is a holomorphic map. Then, $D_{\bar{z}} \chi^i = 0$ means that χ is a holomorphic section of the holomorphic vector bundle $\phi^{-1}(T'M)$ on $\mathbb{C}P^1$. In other words, χ is an element of $H^0(\mathbb{C}P^1, \phi^{-1}(T'M))$. In the same way, ψ_z is an element of $H^0(\mathbb{C}P^1, (T'\mathbb{C}P^1)^* \otimes \phi^{-1}(\overline{T'}M))$ from $D_{\bar{z}} \psi^{\bar{i}}_z = 0$. When M is a Kähler manifold, the isomorphism $\overline{T'}M \simeq (T'M)^*$ is obtaind from Kähler metric. From the Serre duality,

$$H^0(\mathbb{C}P^1, (T'\mathbb{C}P^1)^* \otimes \phi^{-1}(\overline{T'}M)) \simeq H^0(\mathbb{C}P^1, (T'\mathbb{C}P^1)^* \otimes \phi^{-1}((T'M)^*) \tag{3.74}$$

$$\simeq (H^1(\mathbb{C}P^1, T'\mathbb{C}P^1 \otimes \phi^{-1}(T'M) \otimes (T'\mathbb{C}P^1)^*))^* \simeq (H^1(\mathbb{C}P^1, \phi^{-1}(T'M)))^*. \tag{3.75}$$

Therefore, ψ_z is an element of $(H^1(\mathbb{C}P^1, \phi^{-1}(T'M)))^*$. "If M is a compact Kähler manifold, the set of holomorphic maps of degree d from Riemann surface Σ to M becomes a complex space of finite dimension."[23] "This space is called the "moduli space of holomorphic maps of degree d from Σ to M". Let us denote it by $\mathcal{M}_{\Sigma}(M, d)$ temporarily. The complex dimension of $\mathcal{M}_{\Sigma}(M, d)$ in a small open neighboorhood of a point $\phi \in \mathcal{M}_{\Sigma}(M, d))$ is given by $\dim(H^0(\Sigma, \phi^{-1}(T'M)))$."[23] The relationship between the number of solutions to the equations of motion for χ and ψ_z is given by the Riemann-Roch theorem

$$\dim(H^0(\mathbb{C}P^1, \phi^{-1}(T'M))) - \dim(H^1(\mathbb{C}P^1, \phi^{-1}(T'M)))$$
$$= \dim_{\mathbb{C}}(M) + \int_{\mathbb{C}P^1} \phi^*(c_1(M)). \tag{3.76}$$

If degree $d = 0$, the holomorphic map ϕ is a constant map. Since $\phi^{-1}(T'M)$ is a trivial bundle $\mathbb{C}P^1 \times \mathbb{C}^{\dim_{\mathbb{C}}(M)}$, $\dim(H^0(\mathbb{C}P^1, \phi^{-1}(T'M))) = \dim_{\mathbb{C}}(M)$. From the Riemann-Roch theorem, $\dim(H^1(\mathbb{C}P^1, \phi^{-1}(T'M))) = 0$. In other words, the solution of the equation of motion for ψ_z is trivial. We consider the weak coupling limit. Let ϕ_0, χ_0 and ψ_0 be solutions of equations of motion ($\partial_{\bar{z}}\phi^i = 0, D_{\bar{z}}\chi^i = 0$ and $D_{\bar{z}}\psi_z^{\bar{i}} = 0$). We call them zero modes. And we expand each variable around a zero mode as follows.

$$\phi^i = \phi_0^i + \phi'^i \qquad \chi^i = \chi_0^i + \chi'^i \qquad \psi_{\bar{z}}^i = \psi_{0\bar{z}}^i + \psi'^i_{\bar{z}} \tag{3.77}$$

$$\phi^{\bar{i}} = \phi_0^{\bar{i}} + \phi'^{\bar{i}} \qquad \chi^{\bar{i}} = \chi_0^{\bar{i}} + \chi'^{\bar{i}} \qquad \psi_z^{\bar{i}} = \psi_{0z}^{\bar{i}} + \psi'^{\bar{i}}_z \tag{3.78}$$

We call ϕ', χ' and ψ' oscillation modes. We expand Lagrangian by using this result. (We neglect third or higher terms involving the oscillation mode.)

$$L_d = L_0 + L' \tag{3.79}$$

$$L_0 = -\int_{\mathbb{C}P^1} dz d\bar{z} R_{i\bar{j}k\bar{l}} \psi_{0\bar{z}}^i \psi_{0z}^{\bar{j}} \chi_0^k \chi_0^{\bar{l}} \tag{3.80}$$

$$L' = \int_{\mathbb{C}P^1} dz d\bar{z} \big[t g_{i\bar{j}} \partial_{\bar{z}} \phi'^i \partial_z \phi'^{\bar{j}} + i\sqrt{t} g_{i\bar{j}} \psi'^{\bar{j}}_z \partial_{\bar{z}} \chi'^i + i\sqrt{t} g_{i\bar{j}} \psi'^i_{\bar{z}} \partial_z \chi'^{\bar{j}} \big] \tag{3.81}$$

From $\partial_{\bar{z}}\phi_0^i = \partial_z \phi_0^{\bar{i}} = 0$, D_z and $D_{\bar{z}}$ become ∂_z and $\partial_{\bar{z}}$. "More ever, in the $t \to \infty$, the radius of M becomes infinity and M can be regarded as a locally flat space. Therefore, we can replace the Kähler metric $g_{i\bar{j}}$ by $\delta_{i\bar{j}}$."[23] Therefore, the oscillation part of Lagrangian is given as follows.

$$L' = \sum_{i=1}^{\dim_{\mathbb{C}} M} \int_{\mathbb{C}P^1} dz d\bar{z} \big[t \partial_{\bar{z}} \phi'^i \partial_z \phi'^{\bar{i}} + i\sqrt{t} \psi'^{\bar{i}}_z \partial_{\bar{z}} \chi'^i + i\sqrt{t} \psi'^i_{\bar{z}} \partial_z \chi'^{\bar{i}} \big] \tag{3.82}$$

The integral measure $\mathfrak{D}X = \mathfrak{D}\phi\mathfrak{D}\chi\mathfrak{D}\psi$ is also divided into the zero mode part $\mathfrak{D}\phi_0\mathfrak{D}\chi_0\mathfrak{D}\psi_0$ and the oscillation mode part $\mathfrak{D}\phi'\mathfrak{D}\chi'\mathfrak{D}\psi'$. Therefore,

$$\left\langle \prod_{I=1}^n \mathcal{O}_I \right\rangle_d = \int_{\mathcal{M}_{\mathbb{C}P^1}(M,d)} \mathfrak{D}\phi_0\mathfrak{D}\chi_0\mathfrak{D}\psi_0 e^{-L_0} \prod_{I=1}^n \mathcal{O}_I \Big(\int \mathfrak{D}\phi'\mathfrak{D}\chi'\mathfrak{D}\psi' e^{-L'} \Big). \tag{3.83}$$

Since L' is the same form (3.29) in [23], we use the result of Gaussian integration of (3.29) in [23]. " The result of Gaussian integration of ϕ', χ' and ψ' is given by

$$\frac{(\det{}'(\partial_z))^{\dim_{\mathbb{C}}(M)} (\det{}'(\partial_{\bar{z}}))^{\dim_{\mathbb{C}}(M)}}{(\det{}'(\partial_z \partial_{\bar{z}}))^{\dim_{\mathbb{C}}(M)}} = \frac{\det{}'(\partial_z \partial_{\bar{z}}))^{\dim_{\mathbb{C}}(M)}}{\det{}'(\partial_z \partial_{\bar{z}}))^{\dim_{\mathbb{C}}(M)}} = 1,$$

where $\det'(A)$ is the product of non-zero eigenvalues of A. In sum, contributions of oscillation modes of bosons and fermions cancel each other."[23] As a result,

$$\left\langle \prod_{I=1}^n \mathcal{O}_I \right\rangle_d = \int_{\mathcal{M}_{\mathbb{C}P^1}(M,d)} \mathfrak{D}\phi_0\mathfrak{D}\chi_0\mathfrak{D}\psi_0 e^{-L_0} \prod_{I=1}^n \mathcal{O}_I. \tag{3.84}$$

In this thesis, we consider holomorphic map of degree 0 from $\mathbb{C}P^1$ to M (ϕ zero mode is constant map) and its correlation function. Since ψ zero mode is an element of $(H^1(\mathbb{C}P^1, \phi^{-1}(T'M)))^*$ and $\dim(H^1(\mathbb{C}P^1, \phi^{-1}(T'M))) = 0$ in degree 0, ψ has no zero mode. Then, $L_0 = 0$ and integral measures of ϕ_0 and χ_0 are

$$\mathfrak{D}\phi_0 = d\phi_0^1 d\phi_0^{\bar{1}} \cdots d\phi_0^{\dim_{\mathbb{C}}(M)} d\phi_0^{\overline{\dim_{\mathbb{C}}(M)}}, \quad \mathfrak{D}\chi_0 = d\chi_0^1 d\chi_0^{\bar{1}} \cdots d\chi_0^{\dim_{\mathbb{C}}(M)} d\chi_0^{\overline{\dim_{\mathbb{C}}(M)}}. \tag{3.85}$$

3.2 Interpretation of Observable $\mathcal{O}$

In this subsection, we explain observables for A-model. From above discussion, they have to satisfy $\{Q_0, \mathcal{O}\} = 0$. From the analogy of Q_0 and exterior derivative d, an observable corresponds to an element of cohomology ring of M [22, 23]. We introduce a differential (p, q)-form W on M as follows.

$$W = \frac{1}{p!q!} W_{i_1 i_2 \cdots i_p \bar{j}_1 \bar{j}_2 \cdots \bar{j}_q}(z^1, \cdots z^m) dz^{i_1} \wedge dz^{i_2} \wedge \cdots dz^{i_p} \wedge d\bar{z}^{\bar{j}_1} \wedge d\bar{z}^{\bar{j}_2} \cdots \wedge d\bar{z}^{\bar{j}_q}. \tag{3.86}$$

Here, z^i is a local coordinate of M and $z^{\bar{i}}$ is the complex conjugate of z^i. Then, we correspond χ^i with dz^i and $\chi^{\bar{i}}$ with $dz^{\bar{i}}$, and represent $\mathcal{O}$ associated with W.

$$\mathcal{O}_W = \frac{1}{p!q!} W_{i_1 i_2 \cdots i_p \bar{j}_1 \bar{j}_2 \cdots \bar{j}_q}(\phi) \chi^{i_1} \chi^{i_2} \cdots \chi^{i_p} \chi^{\bar{j}_1} \chi^{\bar{j}_2} \cdots \chi^{\bar{j}_q}. \tag{3.87}$$

Since

$$\delta \mathcal{O}_W = \frac{i\alpha}{p!q!} (\partial_l W_{i_1 i_2 \cdots i_p \bar{j}_1 \bar{j}_2 \cdots \bar{j}_q}(\phi) \chi^l + \partial_{\bar{k}} W_{i_1 i_2 \cdots i_p \bar{j}_1 \bar{j}_2 \cdots \bar{j}_q}(\phi) \chi^{\bar{k}}) \chi^{i_1} \chi^{i_2} \cdots \chi^{i_p} \chi^{\bar{j}_1} \chi^{\bar{j}_2} \cdots \chi^{\bar{j}_q} \tag{3.88}$$

$$= i\alpha \mathcal{O}_{dW} \tag{3.89}$$

from the supersymmetry,

$$\{Q_0, \mathcal{O}_W\} = \mathcal{O}_{dW}. \tag{3.90}$$

If W is a closed form, $\{Q_0, \mathcal{O}_W\} = 0$. On the other hand, if W is an exact form or there is a differential form A that satisfies $W = dA$, $\mathcal{O}_W = \{Q_0, \mathcal{O}_A\}$. An observable $\mathcal{O}$ has to satisfy $\{Q_0, \mathcal{O}\} = 0$. When $\{Q_0, \mathcal{O}_i\} = 0 (i = 1, 2, \cdots, n)$,

$$\{Q_0, A\} \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_n = \{Q_0, A \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_n\}.$$

If correlation function involves an observable associated with exact form, its correlation function vanishes. So, the observable of the A-model regarded as an element of the De Rham cohomology of M. In other words, observables for the degree d correlation function must be associated with $W_I \in H^{p_I, q_I}(M, \mathbb{C})$. From the discussion in Subsection 3.1,

$$\left\langle \prod_{I=1}^{n} \mathcal{O}_{W_I}(z_I) \right\rangle_d = \int_{\mathcal{M}_{\mathbb{C}P^1}(M,d)} \mathfrak{D}\phi_0 \mathfrak{D}\chi_0 \mathfrak{D}\psi_0 e^{-L_0} \prod_{I=1}^{n} \mathcal{O}_{W_I}(z_I). \tag{3.91}$$

In the above expression, $\mathcal{O}_{W_I}(z_I)(I=1,2,\cdots,n)$ consists only zero modes.

$$\mathcal{O}_{W_I}(z_I) = \frac{1}{p_I!q_I!}(W_I)_{i_1 i_2 \cdots i_{p_I}\bar{j}_1\bar{j}_2\cdots\bar{j}_{q_I}}(\phi_0(z_I))\chi_0^{i_1}(z_I)\chi_0^{i_2}(z_I)\cdots\chi_0^{i_{p_I}}(z_I) \times \chi_0^{\bar{j}_1}(z_I)\chi_0^{\bar{j}_2}(z_I)\cdots\chi_0^{\bar{j}_{q_I}}(z_I). \tag{3.92}$$

ϕ_0 and χ_0 are solutions of equation of motion. z_I means a point in $\mathbb{C}P^1$. In the case of degree $d=0$, since ϕ is a constant map, $\mathcal{M}_{\mathbb{C}P^1}(M,0)=M$. And ψ has no zero mode.

$$\left\langle \prod_{I=1}^{n} \mathcal{O}_{W_I}(z_I) \right\rangle_0 = \int_M \mathfrak{D}\phi_0 \mathfrak{D}\chi_0 \prod_{I=1}^{n} \mathcal{O}_{W_I}(z_I). \tag{3.93}$$

From (3.85), the integral measure of χ_0 , p_i and $q_i(i=1,2,\cdots,n)$ satisfy

$$\dim_{\mathbb{C}}(M) = \sum_{I=1}^{n} p_I = \sum_{I=1}^{n} q_I. \tag{3.94}$$

It means that the degree 0 correlation function represents the integration of some differential form on M.

$$\left\langle \prod_{I=1}^{n} \mathcal{O}_{W_I}(z_I) \right\rangle_0 = \int_M W_1 \wedge W_2 \wedge \cdots \wedge W_n. \tag{3.95}$$

3.3 Base Model (Topological Sigma Model (A-Model) with Half BRST-Symmetry)

We introduce the model for deriving the Bott residue formula. The base model is a topological sigma model (A-model). Let us represent the Lagrangian again.

$$L = \int_{\mathbb{C}\mathrm{P}^1} dz d\bar{z} \big[\frac{t}{2} g_{i\bar{j}}(\partial_z \phi^i \partial_{\bar{z}} \phi^{\bar{j}} + \partial_{\bar{z}} \phi^i \partial_z \phi^{\bar{j}}) + \sqrt{t} i g_{i\bar{j}} \psi_z^{\bar{j}} D_{\bar{z}} \chi^i + \sqrt{t} i g_{i\bar{j}} \psi_{\bar{z}}^i D_z \chi^{\bar{j}} - R_{i\bar{j}k\bar{l}} \psi_{\bar{z}}^i \psi_z^{\bar{j}} \chi^k \chi^{\bar{l}}\big]. \tag{3.96}$$

This model has a supersymmetry. However, in order to include potential terms induced from a holomorphic tangent vector field K, we have to change the BRST-symmetry in the following way (we observed that this change is inevitable to extend BRST-symmetry to the Lagrangian with potential terms).

$$\begin{aligned} &\delta\phi^{\bar{i}} = i\bar{\alpha}\chi^{\bar{i}}, && \delta\psi_z^{\bar{i}} = -i\bar{\alpha}\Gamma^{\bar{i}}_{\bar{\mu}\bar{\nu}}\chi^{\bar{\mu}}\psi_z^{\bar{\nu}}, \\ &\delta\psi_{\bar{z}}^i = -\sqrt{t}\bar{\alpha}\partial_{\bar{z}}\phi^i, && \delta\phi^i = \delta\chi^i = \delta\chi^{\bar{i}} = 0, \end{aligned} \tag{3.97}$$

where $\bar{\alpha}$ is also a fermionic parameter. In the next subsection, we prove that the Lagrangian (3.96) remains invariant under this new BRST-symmetry. We can confirm that this transformation is the BRST-transformation in the following way. Let Q be the generator of this transformation defined via the relation $\delta X =: i\bar{\alpha}\{Q, X\}$ (X is a field that appears in the theory). If Q has nilpotency, i.e. $\{Q, \{Q, X\}\} = 0$, this transformation is called the BRST-transformation. The most non-trivial part comes from deriving $\{Q, \{Q, \psi_z^{\bar{i}}\}\} = 0$. In this case, we have

$$\delta\psi_z^{\bar{i}} = i\bar{\alpha}\{Q, \psi_z^{i}\} \Rightarrow \{Q, \psi_z^{\bar{i}}\} = -\Gamma^{\bar{i}}_{\bar{\mu}\bar{\nu}}\chi^{\bar{\mu}}\psi_{\bar{z}}^{\bar{\nu}}. \tag{3.98}$$

By using the following relation that holds for the curvature tensor of Kăhler manifold,

$$\begin{aligned} R^{\bar{i}}_{\bar{\nu}\bar{l}\bar{\mu}} = \partial_{\bar{l}}\Gamma^{\bar{i}}_{\bar{\mu}\bar{\nu}} - \Gamma^{\bar{i}}_{\bar{\mu}\bar{\alpha}}\Gamma^{\bar{\alpha}}_{\bar{l}\bar{\nu}} - \partial_{\bar{\mu}}\Gamma^{\bar{i}}_{\bar{l}\bar{\nu}} + \Gamma^{\bar{i}}_{\bar{l}\bar{\alpha}}\Gamma^{\bar{\alpha}}_{\bar{\mu}\bar{\nu}} = 0 \\ \Rightarrow \partial_{\bar{l}}\Gamma^{\bar{i}}_{\bar{\mu}\bar{\nu}} - \Gamma^{\bar{i}}_{\bar{\mu}\bar{\alpha}}\Gamma^{\bar{\alpha}}_{\bar{l}\bar{\nu}} = \partial_{\bar{\mu}}\Gamma^{\bar{i}}_{\bar{l}\bar{\nu}} - \Gamma^{\bar{i}}_{\bar{l}\bar{\alpha}}\Gamma^{\bar{\alpha}}_{\bar{\mu}\bar{\nu}}, \end{aligned}$$

we can show

$$\begin{aligned} \delta\big(-\Gamma^{\bar{i}}_{\bar{\mu}\bar{\nu}}\chi^{\bar{\mu}}\psi_{\bar{z}}^{\bar{\nu}}\big) &= -i\bar{\alpha}\partial_{\bar{l}}\Gamma^{\bar{i}}_{\bar{\mu}\bar{\nu}}\chi^{\bar{l}}\chi^{\bar{\mu}}\psi_{\bar{z}}^{\bar{\nu}} - i\bar{\alpha}\Gamma^{\bar{i}}_{\bar{\mu}\bar{\alpha}}\Gamma^{\bar{\alpha}}_{\bar{l}\bar{\nu}}\chi^{\bar{\mu}}\chi^{\bar{l}}\psi_{\bar{z}}^{\bar{\nu}} = -i\bar{\alpha}\big(\partial_{\bar{l}}\Gamma^{\bar{i}}_{\bar{\mu}\bar{\nu}} - \Gamma^{\bar{i}}_{\bar{\mu}\bar{\alpha}}\Gamma^{\bar{\alpha}}_{\bar{l}\bar{\nu}}\big)\chi^{\bar{l}}\chi^{\bar{\mu}}\psi_{\bar{z}}^{\bar{\nu}} \\ &= -i\bar{\alpha}\big[\frac{1}{2}\big(\partial_{\bar{l}}\Gamma^{\bar{i}}_{\bar{\mu}\bar{\nu}} - \Gamma^{\bar{i}}_{\bar{\mu}\bar{\alpha}}\Gamma^{\bar{\alpha}}_{\bar{l}\bar{\nu}}\big)\chi^{\bar{l}}\chi^{\bar{\mu}}\psi_{\bar{z}}^{\bar{\nu}} + \frac{1}{2}\big(\partial_{\bar{l}}\Gamma^{\bar{i}}_{\bar{\mu}\bar{\nu}} - \Gamma^{\bar{i}}_{\bar{\mu}\bar{\alpha}}\Gamma^{\bar{\alpha}}_{\bar{l}\bar{\nu}}\big)\chi^{\bar{l}}\chi^{\bar{\mu}}\psi_{\bar{z}}^{\bar{\nu}}\big] \\ &= -i\bar{\alpha}\big[\frac{1}{2}\big(\partial_{\bar{l}}\Gamma^{\bar{i}}_{\bar{\mu}\bar{\nu}} - \Gamma^{\bar{i}}_{\bar{\mu}\bar{\alpha}}\Gamma^{\bar{\alpha}}_{\bar{l}\bar{\nu}}\big)\chi^{\bar{l}}\chi^{\bar{\mu}}\psi_{\bar{z}}^{\bar{\nu}} + \frac{1}{2}\big(\partial_{\bar{\mu}}\Gamma^{\bar{i}}_{\bar{l}\bar{\nu}} - \Gamma^{\bar{i}}_{\bar{l}\bar{\alpha}}\Gamma^{\bar{\alpha}}_{\bar{\mu}\bar{\nu}}\big)\chi^{\bar{l}}\chi^{\bar{\mu}}\psi_{\bar{z}}^{\bar{\nu}}\big] \\ &= -i\bar{\alpha}\big[\frac{1}{2}\big(\partial_{\bar{l}}\Gamma^{\bar{i}}_{\bar{\mu}\bar{\nu}} - \Gamma^{\bar{i}}_{\bar{\mu}\bar{\alpha}}\Gamma^{\bar{\alpha}}_{\bar{l}\bar{\nu}}\big)\chi^{\bar{l}}\chi^{\bar{\mu}}\psi_{\bar{z}}^{\bar{\nu}} + \frac{1}{2}\big(\partial_{\bar{l}}\Gamma^{\bar{i}}_{\bar{\mu}\bar{\nu}} - \Gamma^{\bar{i}}_{\bar{\mu}\bar{\alpha}}\Gamma^{\bar{\alpha}}_{\bar{l}\bar{\nu}}\big)\chi^{\bar{\mu}}\chi^{\bar{l}}\psi_{\bar{z}}^{\bar{\nu}}\big] \\ &= -i\bar{\alpha}\big[\frac{1}{2}\big(\partial_{\bar{l}}\Gamma^{\bar{i}}_{\bar{\mu}\bar{\nu}} - \Gamma^{\bar{i}}_{\bar{\mu}\bar{\alpha}}\Gamma^{\bar{\alpha}}_{\bar{l}\bar{\nu}}\big)\chi^{\bar{l}}\chi^{\bar{\mu}}\psi_{\bar{z}}^{\bar{\nu}} - \frac{1}{2}\big(\partial_{\bar{l}}\Gamma^{\bar{i}}_{\bar{\mu}\bar{\nu}} - \Gamma^{\bar{i}}_{\bar{\mu}\bar{\alpha}}\Gamma^{\bar{\alpha}}_{\bar{l}\bar{\nu}}\big)\chi^{\bar{l}}\chi^{\bar{\mu}}\psi_{\bar{z}}^{\bar{\nu}}\big] = 0 \\ &\Rightarrow \{Q, \{Q, \psi_z^{\bar{i}}\}\} = 0. \end{aligned}$$

Check for other fields is straightforward.

3.4 Proof of $\delta L = 0$

In this subsection, we check invariance of the Lagrangian in (3.96) under the BRST-transformation given in (3.97) , i.e., the equality $\delta L = 0$. We first evaluate variation of $g_{i\bar{j}}\partial_z\phi^i\partial_{\bar{z}}\phi^{\bar{j}}$.

$$\begin{aligned} \delta\big(tg_{i\bar{j}}\partial_z\phi^i\partial_{\bar{z}}\phi^{\bar{j}}\big) &= t\partial_{\bar{l}}(g_{i\bar{j}})\delta\phi^{\bar{l}}\partial_z\phi^i\partial_{\bar{z}}\phi^{\bar{j}} + tg_{i\bar{j}}\partial_z\phi^i\partial_{\bar{z}}(\delta\phi^{\bar{j}}) \\ &= it\bar{\alpha}\big(g_{i\bar{\lambda}}\Gamma^{\bar{\lambda}}_{\bar{j}\bar{l}}\partial_z\phi^i\partial_{\bar{z}}\phi^{\bar{j}}\chi^{\bar{l}} + g_{i\bar{j}}\partial_z\phi^i\partial_{\bar{z}}\chi^{\bar{j}}\big) \\ &= it\bar{\alpha}g_{i\bar{j}}\partial_z\phi^i\big(\partial_{\bar{z}}\chi^{\bar{j}} + \Gamma^{\bar{j}}_{\bar{\mu}\bar{\nu}}\partial_{\bar{z}}\phi^{\bar{\mu}}\chi^{\bar{\nu}}\big) \\ &= it\bar{\alpha}g_{i\bar{j}}\partial_z\phi^i D_{\bar{z}}\chi^{\bar{j}}. \end{aligned} \tag{3.99}$$

By integration by parts, we obtain the following (we neglect total differential).

$$\int_\Sigma dzd\bar{z}\delta\big(\frac{1}{2}tg_{i\bar{j}}\partial_z\phi^i\partial_{\bar{z}}\phi^{\bar{j}}\big) \tag{3.100}$$
$$= it\bar{\alpha}\int_\Sigma dzd\bar{z}\frac{1}{2}\big(g_{i\bar{j}}\partial_z\phi^i\partial_{\bar{z}}\chi^{\bar{j}} + g_{i\bar{j}}\partial_z\phi^i\Gamma^{\bar{j}}_{\bar{\mu}\bar{l}}\partial_{\bar{z}}\phi^{\bar{\mu}}\chi^{\bar{l}}\big)$$
$$= it\bar{\alpha}\int_\Sigma dzd\bar{z}\frac{1}{2}\big(-\partial_l g_{i\bar{j}}\partial_{\bar{z}}\phi^l\partial_z\phi^i\chi^{\bar{j}} - \partial_{\bar{l}}g_{i\bar{j}}\partial_{\bar{z}}\phi^{\bar{l}}\partial_z\phi^i\chi^{\bar{j}}$$
$$- g_{i\bar{j}}\partial_{\bar{z}}\partial_z\phi^i\chi^{\bar{j}} + \partial_{\bar{\mu}}g_{i\bar{l}}\partial_{\bar{z}}\phi^{\bar{\mu}}\partial_z\phi^i\chi^{\bar{l}}\big)$$
$$= it\bar{\alpha}\int_\Sigma dzd\bar{z}\frac{1}{2}\big[g_{i\bar{j}}\partial_{\bar{z}}\phi^i\big(\partial_z\chi^{\bar{j}} + \Gamma^{\bar{j}}_{\bar{\mu}\bar{\nu}}\partial_z\phi^{\bar{\mu}}\chi^{\bar{\nu}}\big)\big]$$
$$= it\bar{\alpha}\int_\Sigma dzd\bar{z}\frac{1}{2}\big(g_{i\bar{j}}\partial_{\bar{z}}\phi^i D_z\chi^{\bar{j}}\big). \tag{3.101}$$

Variation of other terms are evaluated as follows.

$$\delta\big(\frac{1}{2}tg_{i\bar{j}}\partial_z\phi^{\bar{j}}\partial_{\bar{z}}\phi^i\big) = \frac{1}{2}tg_{i\bar{\lambda}}\Gamma^{\bar{\lambda}}_{\bar{l}\bar{j}}\delta\phi^{\bar{l}}\partial_z\phi^{\bar{j}}\partial_{\bar{z}}\phi^i + \frac{1}{2}tg_{i\bar{j}}\partial_z(\delta\phi^{\bar{j}})\partial_{\bar{z}}\phi^i$$
$$= \frac{1}{2}it\bar{\alpha}\big(g_{i\bar{j}}\Gamma^{\bar{j}}_{\bar{l}\bar{\mu}}\chi^{\bar{l}}\partial_z\phi^{\bar{\mu}}\partial_{\bar{z}}\phi^i + g_{i\bar{j}}\partial_{\bar{z}}\phi^i\partial_z\chi^{\bar{j}}\big)$$
$$= \frac{1}{2}it\bar{\alpha}g_{i\bar{j}}\partial_{\bar{z}}\phi^i D_z\chi^{\bar{j}}. \tag{3.102}$$
$$\delta\big(i\sqrt{t}g_{i\bar{j}}\psi^{\bar{j}}_z D_{\bar{z}}\chi^i\big) = i\sqrt{t}g_{i\bar{\lambda}}\Gamma^{\bar{\lambda}}_{\bar{l}\bar{j}}\delta\phi^{\bar{l}}\psi^{\bar{j}}_z D_{\bar{z}}\chi^i + i\sqrt{t}g_{i\bar{j}}\delta\psi^{\bar{j}}_z D_{\bar{z}}\chi^i + i\sqrt{t}g_{i\bar{j}}\psi^{\bar{j}}_z\partial_{\bar{l}}\Gamma^i_{\mu\nu}\delta\phi^{\bar{l}}\partial_{\bar{z}}\phi^\mu\chi^\nu$$
$$= -\bar{\alpha}\sqrt{t}g_{i\bar{j}}\Gamma^{\bar{j}}_{\bar{\mu}\bar{\nu}}\chi^{\bar{\mu}}\psi^{\bar{\nu}}_z D_{\bar{z}}\chi^i + \bar{\alpha}\sqrt{t}g_{i\bar{j}}\Gamma^{\bar{j}}_{\bar{\mu}\bar{\nu}}\chi^{\bar{\mu}}\psi^{\bar{\nu}}_z D_{\bar{z}}\chi^i$$
$$+ \bar{\alpha}\sqrt{t}g_{i\bar{j}}\psi^{\bar{j}}_z R^i_{\mu\bar{l}\nu}\chi^{\bar{l}}\partial_{\bar{z}}\phi^\mu\chi^\nu$$
$$= \bar{\alpha}\sqrt{t}R_{\bar{j}\mu\bar{l}\nu}\psi^{\bar{j}}_z\partial_{\bar{z}}\phi^\mu\chi^{\bar{l}}\chi^\nu = -\bar{\alpha}\sqrt{t}R_{\mu\bar{j}\nu\bar{l}}\psi^{\bar{j}}_z\partial_{\bar{z}}\phi^\mu\chi^\nu\chi^{\bar{l}}$$
$$= -\bar{\alpha}\sqrt{t}R_{i\bar{j}k\bar{l}}\partial_{\bar{z}}\phi^i\psi^{\bar{j}}_z\chi^k\chi^{\bar{l}}. \tag{3.103}$$
$$\delta\big(i\sqrt{t}g_{i\bar{j}}\psi^i_{\bar{z}}D_z\chi^{\bar{j}}\big) = i\sqrt{t}g_{i\bar{\lambda}}\Gamma^{\bar{\lambda}}_{\bar{l}\bar{j}}\delta\phi^{\bar{l}}\psi^i_{\bar{z}}D_z\chi^{\bar{j}} + i\sqrt{t}g_{i\bar{j}}\delta\psi^i_{\bar{z}}D_z\chi^{\bar{j}} + i\sqrt{t}g_{i\bar{j}}\psi^i_{\bar{z}}\partial_{\bar{l}}\Gamma^{\bar{j}}_{\bar{\mu}\bar{\nu}}\delta\phi^{\bar{l}}\partial_z\phi^{\bar{\mu}}\chi^{\bar{\nu}}$$
$$+ i\sqrt{t}g_{i\bar{j}}\psi^i_{\bar{z}}\Gamma^{\bar{j}}_{\bar{\mu}\bar{\nu}}\partial_z(\delta\phi^{\bar{\mu}})\chi^{\bar{\nu}}$$
$$= -\bar{\alpha}\sqrt{t}g_{i\bar{j}}\Gamma^{\bar{j}}_{\bar{\mu}\bar{\nu}}\chi^{\bar{\mu}}\psi^i_{\bar{z}}D_z\chi^{\bar{\nu}} - i\bar{\alpha}tg_{i\bar{j}}\partial_{\bar{z}}\phi^i D_z\chi^{\bar{j}}$$
$$+ \bar{\alpha}\sqrt{t}g_{i\bar{j}}\psi^i_{\bar{z}}\partial_{\bar{l}}\Gamma^{\bar{j}}_{\bar{\mu}\bar{\nu}}\chi^{\bar{l}}\partial_z\phi^{\bar{\mu}}\chi^{\bar{\nu}} + \bar{\alpha}\sqrt{t}g_{i\bar{j}}\psi^i_{\bar{z}}\Gamma^{\bar{j}}_{\bar{\mu}\bar{\nu}}\partial_z\chi^{\bar{\mu}}\chi^{\bar{\nu}}$$
$$= -\bar{\alpha}\sqrt{t}g_{i\bar{j}}\Gamma^{\bar{j}}_{\bar{\mu}\bar{\nu}}\chi^{\bar{\nu}}\psi^i_{\bar{z}}\partial_z\chi^{\bar{\mu}} - \bar{\alpha}\sqrt{t}g_{i\bar{j}}\Gamma^{\bar{j}}_{\bar{\beta}\bar{\nu}}\chi^{\bar{\nu}}\psi^i_{\bar{z}}\Gamma^{\bar{\beta}}_{\bar{\mu}\bar{l}}\partial_z\phi^{\bar{\mu}}\chi^{\bar{l}} - i\bar{\alpha}tg_{i\bar{j}}\partial_{\bar{z}}\phi^i D_z\chi^{\bar{j}}$$
$$+ \bar{\alpha}\sqrt{t}g_{i\bar{j}}\psi^i_{\bar{z}}\partial_{\bar{l}}\Gamma^{\bar{j}}_{\bar{\mu}\bar{\nu}}\chi^{\bar{l}}\partial_z\phi^{\bar{\mu}}\chi^{\bar{\nu}} + \bar{\alpha}\sqrt{t}g_{i\bar{j}}\psi^i_{\bar{z}}\Gamma^{\bar{j}}_{\bar{\mu}\bar{\nu}}\partial_z\chi^{\bar{\mu}}\chi^{\bar{\nu}}$$
$$= \bar{\alpha}\sqrt{t}g_{i\bar{j}}(\partial_{\bar{l}}\Gamma^{\bar{j}}_{\bar{\mu}\bar{\nu}} + \Gamma^{\bar{j}}_{\bar{\beta}\bar{l}}\Gamma^{\bar{\beta}}_{\bar{\mu}\bar{\nu}})\psi^i_{\bar{z}}\chi^{\bar{l}}\partial_z\phi^{\bar{\mu}}\chi^{\bar{\nu}} - i\bar{\alpha}tg_{i\bar{j}}\partial_{\bar{z}}\phi^i D_z\chi^{\bar{j}}. \tag{3.104}$$

By using the following relation that holds for the curvature tensor of Kähler manifold,

$$R^{\bar{j}}_{\bar{\nu}\bar{l}\bar{\mu}} = \partial_{\bar{l}}\Gamma^{\bar{j}}_{\bar{\nu}\bar{\mu}} - \partial_{\bar{\mu}}\Gamma^{\bar{j}}_{\bar{l}\bar{\nu}} + \Gamma^{\bar{j}}_{\bar{\beta}\bar{l}}\Gamma^{\bar{\beta}}_{\bar{\mu}\bar{\nu}} - \Gamma^{\bar{j}}_{\bar{\beta}\bar{\mu}}\Gamma^{\bar{\beta}}_{\bar{\nu}\bar{l}} = 0, \tag{3.105}$$

$\partial_{\bar{l}}\Gamma^{\bar{j}}_{\bar{\nu}\bar{\mu}} = \partial_{\bar{l}}\Gamma^{\bar{j}}_{\bar{\mu}\bar{\nu}}$ and $\Gamma^{\bar{\beta}}_{\bar{\mu}\bar{\nu}} = \Gamma^{\bar{\beta}}_{\bar{\nu}\bar{\mu}}$, the first term is

$$\begin{aligned}
&g_{i\bar{j}}\psi^i_{\bar{z}}\partial_z\phi^{\bar{\nu}}(\partial_{\bar{l}}\Gamma^{\bar{j}}_{\bar{\nu}\bar{\mu}} + \Gamma^{\bar{j}}_{\bar{\beta}\bar{l}}\Gamma^{\bar{\beta}}_{\bar{\nu}\bar{\mu}})\chi^{\bar{l}}\chi^{\bar{\mu}} \\
&= \frac{1}{2}g_{i\bar{j}}\psi^i_{\bar{z}}\partial_z\phi^{\bar{\nu}}(\partial_{\bar{l}}\Gamma^{\bar{j}}_{\bar{\nu}\bar{\mu}} + \Gamma^{\bar{j}}_{\bar{\beta}\bar{l}}\Gamma^{\bar{\beta}}_{\bar{\nu}\bar{\mu}})\chi^{\bar{l}}\chi^{\bar{\mu}} + \frac{1}{2}g_{i\bar{j}}\psi^i_{\bar{z}}\partial_z\phi^{\bar{\nu}}(\partial_{\bar{\mu}}\Gamma^{\bar{j}}_{\bar{l}\bar{\nu}} + \Gamma^{\bar{j}}_{\bar{\beta}\bar{\mu}}\Gamma^{\bar{\beta}}_{\bar{\nu}\bar{l}})\chi^{\bar{l}}\chi^{\bar{\mu}} \\
&= \frac{1}{2}g_{i\bar{j}}\psi^i_{\bar{z}}\partial_z\phi^{\bar{\nu}}(\partial_{\bar{l}}\Gamma^{\bar{j}}_{\bar{\nu}\bar{\mu}} + \Gamma^{\bar{j}}_{\bar{\beta}\bar{l}}\Gamma^{\bar{\beta}}_{\bar{\nu}\bar{\mu}})\chi^{\bar{l}}\chi^{\bar{\mu}} + \frac{1}{2}g_{i\bar{j}}\psi^i_{\bar{z}}\partial_z\phi^{\bar{\nu}}(\partial_{\bar{l}}\Gamma^{\bar{j}}_{\bar{\mu}\bar{\nu}} + \Gamma^{\bar{j}}_{\bar{\beta}\bar{l}}\Gamma^{\bar{\beta}}_{\bar{\nu}\bar{\mu}})\chi^{\bar{\mu}}\chi^{\bar{l}} \\
&= \frac{1}{2}g_{i\bar{j}}\psi^i_{\bar{z}}\partial_z\phi^{\bar{\nu}}(\partial_{\bar{l}}\Gamma^{\bar{j}}_{\bar{\nu}\bar{\mu}} + \Gamma^{\bar{j}}_{\bar{\beta}\bar{l}}\Gamma^{\bar{\beta}}_{\bar{\nu}\bar{\mu}})\chi^{\bar{l}}\chi^{\bar{\mu}} + \frac{1}{2}g_{i\bar{j}}\psi^i_{\bar{z}}\partial_z\phi^{\bar{\nu}}(\partial_{\bar{l}}\Gamma^{\bar{j}}_{\bar{\nu}\bar{\mu}} + \Gamma^{\bar{j}}_{\bar{\beta}\bar{l}}\Gamma^{\bar{\beta}}_{\bar{\mu}\bar{\nu}})\chi^{\bar{\mu}}\chi^{\bar{l}} = 0. \quad (3.106)
\end{aligned}$$

Then,

$$\delta\big(i\sqrt{t}g_{i\bar{j}}\psi^i_{\bar{z}}D_z\chi^{\bar{j}}\big) = -i\bar{\alpha}tg_{i\bar{j}}\partial_{\bar{z}}\phi^i D_z\chi^{\bar{j}}. \quad (3.107)$$

$$\begin{aligned}
\delta\big(-R_{i\bar{j}k\bar{l}}\psi^i_{\bar{z}}\psi^{\bar{j}}_z\chi^k\chi^{\bar{l}}\big) &= -\partial_{\bar{\lambda}}R_{i\bar{j}k\bar{l}}\delta\phi^{\bar{\lambda}}\psi^i_{\bar{z}}\psi^{\bar{j}}_z\chi^k\chi^{\bar{l}} - R_{i\bar{j}k\bar{l}}\delta\psi^i_{\bar{z}}\psi^{\bar{j}}_z\chi^k\chi^{\bar{l}} - R_{i\bar{j}k\bar{l}}\psi^i_{\bar{z}}\delta\psi^{\bar{j}}_z\chi^k\chi^{\bar{l}} \\
&= -i\bar{\alpha}\partial_{\bar{\lambda}}R_{i\bar{j}k\bar{l}}\psi^i_{\bar{z}}\psi^{\bar{j}}_z\chi^k\chi^{\bar{l}}\chi^{\bar{\lambda}} + \bar{\alpha}\sqrt{t}R_{i\bar{j}k\bar{l}}\partial_{\bar{z}}\phi^i\psi^{\bar{j}}_z\chi^k\chi^{\bar{l}} \quad (3.108) \\
&\quad - i\bar{\alpha}R_{i\bar{j}k\bar{l}}\psi^i_{\bar{z}}\Gamma^{\bar{j}}_{\bar{\mu}\bar{\nu}}\chi^{\bar{\mu}}\psi^{\bar{\nu}}_z\chi^k\chi^{\bar{l}} \\
&= -i\bar{\alpha}(\partial_{\bar{\lambda}}R_{i\bar{j}k\bar{l}} - R_{i\bar{\beta}k\bar{l}}\Gamma^{\bar{\beta}}_{\bar{j}\bar{\lambda}})\psi^i_{\bar{z}}\psi^{\bar{j}}_z\chi^k\chi^{\bar{l}}\chi^{\bar{\lambda}} + \bar{\alpha}\sqrt{t}R_{i\bar{j}k\bar{l}}\partial_{\bar{z}}\phi^i\psi^{\bar{j}}_z\chi^k\chi^{\bar{l}}. \quad (3.109)
\end{aligned}$$

Next, we use the following formula of covariant derivative of the curvature,

$$\nabla_{\bar{\lambda}}R_{i\bar{j}k\bar{l}} = \partial_{\bar{\lambda}}R_{i\bar{j}k\bar{l}} - R_{i\bar{\beta}k\bar{l}}\Gamma^{\bar{\beta}}_{\bar{\lambda}\bar{j}} - R_{i\bar{j}k\bar{\beta}}\Gamma^{\bar{\beta}}_{\bar{\lambda}\bar{l}}, \quad (3.110)$$

and Bianchi's identity,

$$\nabla_{\bar{\lambda}}R_{i\bar{j}k\bar{l}} = \nabla_{\bar{l}}R_{i\bar{j}k\bar{\lambda}}. \quad (3.111)$$

$$\begin{aligned}
\delta\big(-R_{i\bar{j}k\bar{l}}\psi^i_{\bar{z}}\psi^{\bar{j}}_z\chi^k\chi^{\bar{l}}\big) &= -i\bar{\alpha}\nabla_{\bar{\lambda}}R_{i\bar{j}k\bar{l}}\psi^i_{\bar{z}}\psi^{\bar{j}}_z\chi^k\chi^{\bar{l}}\chi^{\bar{\lambda}} + \bar{\alpha}\sqrt{t}R_{i\bar{j}k\bar{l}}\partial_{\bar{z}}\phi^i\psi^{\bar{j}}_z\chi^k\chi^{\bar{l}} \\
&= \bar{\alpha}\sqrt{t}R_{i\bar{j}k\bar{l}}\partial_{\bar{z}}\phi^i\psi^{\bar{j}}_z\chi^k\chi^{\bar{l}}.
\end{aligned} \quad (3.112)$$

As a result, all the variations cancel each other. Therefore, we have shown the equality: $\delta L = 0 \Leftrightarrow \{Q, L\} = 0$.

3.5 BRST-Closed Observables of the Base Model with Half Symmetry

In this subsection, we consider BRST-closed observable of this model, i.e., observable $\mathcal{O}$ that satisfies $\{Q, \mathcal{O}\} = 0$. Here we restrict observables that are obtained from differential forms on M. Let W be a (p,q)-form on M:

$$W = \frac{1}{p!q!}W_{i_1 i_2 \cdots i_p \bar{j}_1 \bar{j}_2 \cdots \bar{j}_q}(z^1, \cdots z^m)dz^{i_1}dz^{i_2}\cdots dz^{i_p}d\bar{z}^{\bar{j}_1}d\bar{z}^{\bar{j}_2}\cdots d\bar{z}^{\bar{j}_q}, \quad (3.113)$$

then we consider the following observable $\mathcal{O}_W$:

$$\mathcal{O}_W = \frac{1}{p!q!} W_{i_1 i_2 \cdots i_p \bar{j}_1 \bar{j}_2 \cdots \bar{j}_q}(\phi) \chi^{i_1} \chi^{i_2} \cdots \chi^{i_p} \chi^{\bar{j}_1} \chi^{\bar{j}_2} \cdots \chi^{\bar{j}_q}. \quad (3.114)$$

Variation of $\mathcal{O}_W$ under the BRST-transformation:

$$\{Q, \phi^i\} = 0, \{Q, \phi^{\bar{i}}\} = \chi^{\bar{i}}, \{Q, \chi^i\} = 0, \{Q, \chi^{\bar{i}}\} = 0,$$

is given by

$$\begin{aligned} \delta \mathcal{O}_W &= \frac{1}{p!q!} \partial_{\bar{l}} W_{i_1 i_2 \cdots i_p \bar{j}_1 \bar{j}_2 \cdots \bar{j}_q} \delta \phi^{\bar{l}} \chi^{i_1} \chi^{i_2} \cdots \chi^{i_p} \chi^{\bar{j}_1} \chi^{\bar{j}_2} \cdots \chi^{\bar{j}_q} \\ &= i\bar{\alpha} \frac{1}{p!q!} \partial_{\bar{l}} W_{i_1 i_2 \cdots i_p \bar{j}_1 \bar{j}_2 \cdots \bar{j}_q} \chi^{\bar{l}} \chi^{i_1} \chi^{i_2} \cdots \chi^{i_p} \chi^{\bar{j}_1} \chi^{\bar{j}_2} \cdots \chi^{\bar{j}_q}. \end{aligned} \quad (3.115)$$

The above result is summarized as follows.

$$\delta \mathcal{O}_W = i\bar{\alpha} \bar{\partial} \mathcal{O}_W = i\bar{\alpha} \{Q, \mathcal{O}_W\}. \quad (3.116)$$

$\{Q, \ \}$ is represented as follows.

$$\{Q, \mathcal{O}_W\} = \mathcal{O}_{\bar{\partial} W}.$$

Therefore, a BRST-closed observable $\mathcal{O}_W$ is obtained from a differential form W that satisfies $\bar{\partial} W = 0$. By standard discussion of topological field theory, correlation function of BRST-closed observables with insertion of a observable of type $\{Q, \mathcal{O}_W\} = \mathcal{O}_{\bar{\partial} W}$ automatically vanishes. Hence, physical observables of the base model correspond to elements of Dolbeault cohomology. Let us recall Dolbeault's theorem and Hodge's decomposition theorem.

Theorem 1. *(Dolbeault's theorem)*

$$H^q(M, \wedge^p T'^* M) \simeq H^{p,q}_{\bar{\partial}}(M). \quad (3.117)$$

Theorem 2. *(Hodge's decomposition thorem)[17]*

$$H^r(M, \mathbb{C}) \simeq \bigoplus_{p+q=r} H^{p,q}(M). \quad (3.118)$$

$$H^{p,q}(M) \simeq H^{p,q}_{\bar{\partial}}(M) \simeq H^q(M, \wedge^p T'^* M). \quad (3.119)$$

If W is a Chern class of a holomorphic vector bundle of M, it is given as a closed (i,i)-form on M. Therefore $\partial W = \bar{\partial} W = 0$ follows from $dW = (\partial + \bar{\partial}) W = 0$ and $\mathcal{O}_W$ is a BRST-closed observable of the base model.

3.6 Potential Terms Induced from Holomorphic Tangent Vector Field

In this subsection, we include potential terms induced from the holomorphic tangent vector field K. The potential terms are given as follows (we use a parameter β that equals $2\pi i$ for brevity).

$$V = \int_\Sigma dzd\bar{z}\big[ts^2\beta g_{i\bar{j}}K^i\bar{K}^{\bar{j}} + tsg_{i\bar{j}}\nabla_{\bar{\mu}}\bar{K}^{\bar{j}}\chi^{\bar{\mu}}\chi^i + s\beta g_{i\bar{j}}\nabla_\mu K^i\psi^\mu_{\bar{z}}\psi^{\bar{j}}_z\big]. \tag{3.120}$$

V contains a parameter $s \in \mathbb{R}$ that controls scale of the vector field K. We can extend the BRST-transformation to the new Lagrangian $L + V$ as follows.

$$\delta\phi^{\bar{i}} = i\bar{\alpha}\chi^{\bar{i}}, \qquad \delta\psi^{\bar{i}}_z = -i\bar{\alpha}\Gamma^{\bar{i}}_{\bar{\mu}\bar{\nu}}\chi^{\bar{\mu}}\psi^{\bar{\nu}}_z, \qquad \delta\psi^i_{\bar{z}} = -\sqrt{t}\bar{\alpha}\partial_{\bar{z}}\phi^i, \tag{3.121}$$

$$\delta\chi^i = is\bar{\alpha}\beta K^i, \qquad \delta\phi^i = \delta\chi^{\bar{i}} = 0. \tag{3.122}$$

Let Q be the generator of this transformation whose action is defined via the relation $\delta X =: i\bar{\alpha}\{Q, X\}$ (X is a field that appears in the theory). We check nilpotency of this generator $\{Q, \{Q, X\}\} = 0$. Non-trivial parts caused by appearance of K in (3.122) is given as follows.

$$\begin{aligned}
\delta\phi^{\bar{i}} &= i\bar{\alpha}\{Q, \phi^{\bar{i}}\} \Rightarrow \{Q, \phi^{\bar{i}}\} = \chi^{\bar{i}},\\
\delta\chi^{\bar{i}} &= i\bar{\alpha}\{Q, \{Q, \phi^{\bar{i}}\}\} \Rightarrow \{Q, \{Q, \phi^{\bar{i}}\}\} = 0,\\
\delta\chi^i &= i\bar{\alpha}\{Q, \chi^i\} \Rightarrow \{Q, \chi^i\} = \beta s K^i,\\
\delta K^i &= i\bar{\alpha}\{Q, \{Q, \chi^i\}\} \Rightarrow \{Q, \{Q, \chi^i\}\} = 0.
\end{aligned}$$

Hence the relation $\{Q, \{Q, X\}\} = 0$ also holds in this case.

3.7 Proof of $\delta(L + V) = 0$

In this subsection, we check invariance of the Lagrangian $L+V$ under (3.122), i.e., the relation $\delta(L + V) = 0$. Let us recall some properties of covariant derivatives of K.

$$\nabla_{\bar{\mu}}K^j = 0, \qquad \nabla_\mu\bar{K}^{\bar{j}} = 0, \tag{3.123}$$

$$\partial_{\bar{l}}\big(g_{i\bar{j}}\nabla_{\bar{\mu}}\bar{K}^{\bar{j}}\big) = g_{i\bar{j}}\nabla_{\bar{l}}\nabla_{\bar{\mu}}\bar{K}^{\bar{j}} + g_{i\bar{j}}\Gamma^{\bar{\alpha}}_{\bar{l}\bar{\mu}}\nabla_{\bar{\alpha}}\bar{K}^{\bar{j}}, \tag{3.124}$$

$$\partial_{\bar{l}}\big(g_{i\bar{j}}\nabla_\mu K^i\big) = g_{i\bar{\lambda}}\Gamma^{\bar{\lambda}}_{\bar{j}\bar{l}}\nabla_\mu K^i + R_{i\bar{j}\mu\bar{l}}K^i. \tag{3.125}$$

We use the above formulas and standard property of Kähler metric. Then additional terms that appear in checking $\delta(L+V)$ are given as follows.

$$\begin{aligned}\delta\big(i\sqrt{t}g_{i\bar{j}}\psi_z^{\bar{j}}D_{\bar{z}}\chi^i\big) &= i\sqrt{t}g_{i\bar{j}}\psi_z^{\bar{j}}\partial_{\bar{z}}(\delta\chi^i) + i\sqrt{t}g_{i\bar{j}}\psi_z^{\bar{j}}\Gamma^i_{\mu\nu}\partial_{\bar{z}}\phi^\mu\delta\chi^\nu \\ &= \bar{\alpha}s\sqrt{t}\beta g_{i\bar{j}}\psi_z^{\bar{j}}\partial_l K^i\partial_{\bar{z}}\phi^l + \bar{\alpha}s\sqrt{t}s\beta g_{i\bar{j}}\psi_z^{\bar{j}}\Gamma^i_{\mu\nu}\partial_{\bar{z}}\phi^\mu K^\nu \\ &= \bar{\alpha}s\beta\sqrt{t}g_{i\bar{j}}\psi_z^{\bar{j}}\nabla_l K^i\partial_{\bar{z}}\phi^l. \end{aligned} \tag{3.126}$$

$$\delta\big(-R_{i\bar{j}k\bar{l}}\psi_{\bar{z}}^i\psi_z^{\bar{j}}\chi^k\chi^{\bar{l}}\big) = -i\bar{\alpha}s\beta R_{i\bar{j}k\bar{l}}\psi_{\bar{z}}^i\psi_z^{\bar{j}}K^k\chi^{\bar{l}}. \tag{3.127}$$

$$\begin{aligned}\delta\big(ts^2\beta g_{i\bar{j}}K^i\bar{K}^{\bar{j}}\big) &= ts^2\beta g_{i\bar{\lambda}}\Gamma^{\bar{\lambda}}_{\bar{j}\bar{l}}\delta\phi^{\bar{l}}K^i\bar{K}^{\bar{j}} + ts^2\beta g_{i\bar{j}}K^i\partial_{\bar{l}}\bar{K}^{\bar{j}}\delta\phi^{\bar{l}} \\ &= ts^2\beta g_{i\bar{j}}K^i\nabla_{\bar{l}}\bar{K}^{\bar{j}}\delta\phi^{\bar{l}} = i\bar{\alpha}ts^2\beta g_{i\bar{j}}K^i\nabla_{\bar{l}}\bar{K}^{\bar{j}}\chi^{\bar{l}}. \end{aligned} \tag{3.128}$$

$$\begin{aligned}\delta\big(tsg_{i\bar{j}}\nabla_{\bar{\mu}}\bar{K}^{\bar{j}}\chi^{\bar{\mu}}\chi^i\big) &= ts\partial_{\bar{l}}\big(g_{i\bar{j}}\nabla_{\bar{\mu}}\bar{K}^{\bar{j}}\big)\delta\phi^{\bar{l}}\chi^{\bar{\mu}}\chi^i + tsg_{i\bar{j}}\nabla_{\bar{\mu}}\bar{K}^{\bar{j}}\chi^{\bar{\mu}}\delta\chi^i \\ &= i\bar{\alpha}ts\big(g_{i\bar{j}}\nabla_{\bar{l}}\nabla_{\bar{\mu}}\bar{K}^{\bar{j}} + g_{i\bar{j}}\Gamma^{\bar{\alpha}}_{\bar{l}\bar{\mu}}\nabla_{\bar{\alpha}}\bar{K}^{\bar{j}}\big)\chi^{\bar{l}}\chi^{\bar{\mu}}\chi^i - i\bar{\alpha}ts^2\beta g_{i\bar{j}}\nabla_{\bar{\mu}}\bar{K}^{\bar{j}}K^i\chi^{\bar{\mu}} \\ &= -i\bar{\alpha}ts^2\beta g_{i\bar{j}}\nabla_{\bar{\mu}}\bar{K}^{\bar{j}}K^i\chi^{\bar{\mu}}. \end{aligned} \tag{3.129}$$

Here, we used $\nabla_{\bar{l}}\nabla_{\bar{\mu}}\bar{K}^{\bar{j}} = \nabla_{\bar{\mu}}\nabla_{\bar{l}}\bar{K}^{\bar{j}}$.

$$\begin{aligned}\delta\big(s\beta g_{i\bar{j}}\nabla_\mu K^i\psi_{\bar{z}}^\mu\psi_z^{\bar{j}}\big) &= s\beta\partial_{\bar{l}}\big(g_{i\bar{j}}\nabla_\mu K^i\big)\delta\phi^{\bar{l}}\psi_{\bar{z}}^\mu\psi_z^{\bar{j}} + s\beta g_{i\bar{j}}\nabla_\mu K^i\delta\psi_{\bar{z}}^\mu\psi_z^{\bar{j}} + s\beta g_{i\bar{j}}\nabla_\mu K^i\psi_{\bar{z}}^\mu\delta\psi_z^{\bar{j}} \\ &= is\bar{\alpha}\beta\big(g_{i\bar{\lambda}}\Gamma^{\bar{\lambda}}_{\bar{j}\bar{l}}\nabla_\mu K^i + R_{i\bar{j}\mu\bar{l}}K^i\big)\chi^{\bar{l}}\psi_{\bar{z}}^\mu\psi_z^{\bar{j}} - s\bar{\alpha}\beta\sqrt{t}g_{i\bar{j}}\nabla_\mu K^i\partial_{\bar{z}}\phi^\mu\psi_z^{\bar{j}} \\ &\quad + i\bar{\alpha}s\beta g_{i\bar{j}}\Gamma^{\bar{j}}_{\bar{\rho}\bar{\nu}}\nabla_\mu K^i\psi_{\bar{z}}^\mu\chi^{\bar{\rho}}\psi_z^{\bar{\nu}} \\ &= is\bar{\alpha}\beta R_{i\bar{j}k\bar{l}}\psi_{\bar{z}}^i\psi_z^{\bar{j}}K^k\chi^{\bar{l}} - s\bar{\alpha}\beta\sqrt{t}g_{i\bar{j}}\nabla_\mu K^i\partial_{\bar{z}}\phi^\mu\psi_z^{\bar{j}}. \end{aligned} \tag{3.130}$$

From the above results, we can conclude that $\delta(L+V)=0$ holds. From now on, we only consider the model given by $L+V$.

3.8 BRST-Closed Observable of the Model

In this subsection, we construct BRST-closed observable of the model with potential terms. In the same way as the previous discussions, we restrict our selves to observables obtained from a (p,q)-form W on M. It is represented in the following form.

$$\mathcal{O}_W = \frac{1}{p!q!}W_{i_1i_2\cdots i_p\bar{j}_1\bar{j}_2\cdots\bar{j}_q}\chi^{i_1}\chi^{i_2}\cdots\chi^{i_p}\chi^{\bar{j}_1}\chi^{\bar{j}_2}\cdots\chi^{\bar{j}_q}. \tag{3.131}$$

Variation $\delta\mathcal{O}_W$.under the BRST-transformation is given by

$$\begin{aligned}\delta\mathcal{O}_W &= \frac{1}{p!q!}\partial_{\bar{l}}W_{i_1i_2\cdots i_p\bar{j}_1\bar{j}_2\cdots\bar{j}_q}\delta\phi^{\bar{l}}\chi^{i_1}\chi^{i_2}\cdots\chi^{i_p}\chi^{\bar{j}_1}\chi^{\bar{j}_2}\cdots\chi^{\bar{j}_q} \\ &\quad + \frac{1}{(p-1)!q!}W_{li_1i_2\cdots i_{p-1}\bar{j}_1\bar{j}_2\cdots\bar{j}_q}\delta\chi^l\chi^{i_1}\chi^{i_2}\cdots\chi^{i_{(p-1)}}\chi^{\bar{j}_1}\chi^{\bar{j}_2}\cdots\chi^{\bar{j}_q} \\ &= i\bar{\alpha}\Big[\frac{1}{p!q!}\partial_{\bar{l}}W_{i_1i_2\cdots i_p\bar{j}_1\bar{j}_2\cdots\bar{j}_q}\chi^{\bar{l}}\chi^{i_1}\chi^{i_2}\cdots\chi^{i_p}\chi^{\bar{j}_1}\chi^{\bar{j}_2}\cdots\chi^{\bar{j}_q} \\ &\quad + \frac{s\beta}{(p-1)!q!}W_{li_1i_2\cdots i_{p-1}\bar{j}_1\bar{j}_2\cdots\bar{j}_q}K^l\chi^{i_1}\chi^{i_2}\cdots\chi^{i_{p-1}}\chi^{\bar{j}_1}\chi^{\bar{j}_2}\cdots\chi^{\bar{j}_q}\Big]. \end{aligned} \tag{3.132}$$

Let $i(K)$ be the inner-product operator by K. Then, the above result is rewritten as follows.

$$\delta \mathcal{O}_W = i\bar{\alpha}\mathcal{O}_{(\bar{\partial}+\beta i(sK))W} =: i\bar{\alpha}\{Q, \mathcal{O}_W\}. \tag{3.133}$$

Hence BRST-closed observable is obtained from a differential form W that satisfies $(\bar{\partial} + \beta i(sK))W = 0$. Note that this condition reduces to $\bar{\partial}W = 0$ if $s = 0$.

We comment on mathematical background to this condition. In general, differential form ω on M is graded by the following operators:

$$F_A\omega = (p+q)\omega, \qquad F_V\omega = (q-p)\omega.$$

This says that ω is a (p,q)-form on M. For the operator $\bar{\partial} + \beta i(sK)$, we adopt F_V as the grading operator and consider the following vector space:

$$A^{(k)} = \bigoplus_{q-p=k} \Omega^{p,q}(M)$$

where k ranges from $-m$ to m. From the condition (3.133), we can see that observables correspond to elements of cohomology of the complex $(A^{(k)}, \bar{\partial} + \beta i(sK))$. This complex is called Liu's complex. In [32], it is shown that cohomology of this complex is independent of s $(s \neq 0)$. This property is closely related to Proposition 1.

4 Derivation of the Bott Residue Formula

4.1 Overview

In this subsection, we explain our strategy of deriving the Bott residue formula by using the topological sigma model with potential terms.

Since the Bott residue formula is a fixed point formula for integration of Chern classes of the holomorphic vector bundle E, we consider observable that corresponds to wedge product of Chern classes in the $s \to 0$ limit. We first construct observable $\mathcal{O}_W$ that satisfies $(\bar{\partial} + \beta i(sK))W = 0$ and $\lim_{s\to 0} W = \underline{\varphi}(E)$. Let us recall Proposition 1 introduced in Subsection 1.1.

Proposition 1

Correlation functions of BRST-closed observables are invariant under variation of s.

Assuming its proposition, we evaluate the degree 0 correlation function $\langle \mathcal{O}_W \rangle_0$ both in the $s \to 0$ limit and $s \to \infty$ limit.

As for the $s \to 0$ limit, the observable becomes $\mathcal{O}_{\underline{\varphi}(E)}$ and we can use standard weak coupling limit. Moreover, the potential terms vanish in this limit. Then we expand each field around the solution of the classical equations of motion ($\phi = \phi_0$ (constant map),

$\chi = \chi_0$ (constant solution), $\psi = 0$) and perform Gaussian integration of oscillation modes. We show that contributions from Gaussian integral is trivial. Hence $\langle \mathcal{O}_W \rangle_0$ turns out to be classical integration of (m, m)-differential form $\underline{\varphi}(E)$ on M, i.e., the l.h.s of (2.18).

In the $s \to \infty$ limit, the path integral is localized around neighborhood of zero set $\{M_\alpha\}$ of the holomorphic vector field K where the condition $\delta\chi^i = is\alpha\beta K^i = 0$ holds. We also expand each field around the solution of the classical equations of motion ($\phi = \phi_0 \in M_\alpha$ (constant map), $\chi = \chi_0$ (constant solution), $\psi = 0$). In this case, we carefully discuss integration measure of oscillation modes by using eigenvalue decomposition by Laplacian for differential forms on $\mathbb{C}P^1$. Since contributions from oscillation modes do not affect correlation functions, contribution from a connected component M_α turns out to be integration of differential form on M_α given in the r.h.s of (2.18). Summing up all the connected components, $\langle \mathcal{O}_W \rangle_0$ becomes the r.h.s of (2.18).

We can equate these two results by using Proposition 1 and obtain the Bott residue formula.

4.2 The BRST-Closed Observable Used for Derivation

In this subsection, we construct the observable $\mathcal{O}_W$ that satisfies $\lim_{s\to 0} W = \underline{\varphi}(E)$ and $(\bar{\partial} + \beta i(K))W = 0$. We use the notation in Subsection 2.1. If we rescale K into sK, $\theta(K)$ is also rescaled to $s\theta(K)$. In general, Λ is also rescaled $s\Lambda$. Let $\{e_a\}$ be the local holomorphic frame of E. Then, we define $L = (L^b_a) : \Gamma(E) \to \Gamma(E)$ by

$$L(s) := s\Lambda(s) - i(sK)\tilde{\nabla}(s) \tag{4.134}$$

($s \in \Gamma(E)$), where we rescale K into sK. For the local holomorphic frame,

$$L^b_a e_b := s\Lambda^b_a e_b - s\Theta^b_{ak} K^k e_b. \tag{4.135}$$

Let us note that the following relations hold.

$$\begin{aligned}
&\bar{\partial} i(sK)\tilde{\nabla} e_a = s\partial_{\bar{l}}(\Theta^b_{ak})K^k \overline{dz^l} e_b = -i(sK)(F^b_{ak\bar{l}} dz^k \wedge \overline{dz^l} e_b),\\
&\bar{\partial}(L^b_a e_b) = \bar{\partial}(s\Lambda^b_a e_b - s\Theta^b_{ak}K^k e_b) = -s\partial_{\bar{l}}(\Theta^b_{ak})K^k \overline{dz^l} e_b = i(sK)(F^b_{ak\bar{l}} dz^k \wedge \overline{dz^l} e_b),\\
&\bar{\partial}(F^a_{bk\bar{l}} dz^k \wedge \overline{dz^l} e^b) = 0,\\
&i(sK)(L^a_b e^b) = 0.
\end{aligned} \tag{4.136}$$

By using the above relations, we obtain ($\beta = 2\pi i$),

$$(\bar{\partial} + \beta i(sK))(L^b_a e_b + \frac{i}{2\pi} F^b_a e_b) = \bar{\partial}(L^b_a e_b) - i(sK)(F^b_{ak\bar{l}} dz^k \wedge \overline{dz^l} e_b) = 0. \tag{4.137}$$

If we define matrix valued form:

$$A = (A^a_b),\ A = L + \frac{i}{2\pi} F, \tag{4.138}$$

(4.137) says $(\bar{\partial}+\beta i(K))A=0$. Therefore $(\bar{\partial}+\beta i(K))\mathrm{tr}(A^m)=0$ holds for arbitrary positive integer m. Let U be a linear automorphism of a complex vector space V with $\dim_{\mathbb{C}}=\mathrm{rank}(E)=q$. It is well-known that $\varphi(U)$ defined in Subsection 2.1 can be represented in the following way.

$$\varphi(U)=\sum_{m_i\geq 0,\ \sum_{i=1}^{l}m_i=m}\alpha_{m_1m_2\cdots m_l}\mathrm{tr}(U^{m_1})\mathrm{tr}(U^{m_2})\cdots\mathrm{tr}(U^{m_l}), \tag{4.139}$$

where m is the complex dimension of M. Therefore, $\varphi(A)=\varphi(L+\frac{i}{2\pi}F)$ is annihilated by the operator $\bar{\partial}+\beta i(sK)$. Obviously, $\varphi(L+\frac{i}{2\pi}F)$ reduces to $\varphi(\frac{i}{2\pi}F)=\underline{\varphi}(E)$ under the $s\to 0$ limit. In this way, we have constructed the operator $\mathcal{O}_{\varphi(L+\frac{i}{2\pi}F)}$ that is used in derivation of the Bott residue formula. From now on, we simply denote it by φ for brevity.

4.3 Degree 0 Correlation Function and Integral Measure in the $s\to 0$ Limit

From now on, we consider the degree 0 correlation function $<\varphi>_0$. First, we rewrite the Lagrangian in the following form.

$$L+V=\frac{t}{2}\int_{\Sigma}\phi^*(\omega)+L'+V', \tag{4.140}$$

$$\begin{aligned}L'+V' :=& \int_{\mathbb{C}\mathrm{P}^1}dzd\bar{z}\big[tg_{i\bar{j}}\partial_{\bar{z}}\phi^i\partial_z\phi^{\bar{j}}+\sqrt{t}ig_{i\bar{j}}\psi_z^{\bar{j}}D_{\bar{z}}\chi^i+\sqrt{t}ig_{i\bar{j}}\psi_{\bar{z}}^{i}D_z\chi^{\bar{j}}\\ &-R_{i\bar{j}k\bar{l}}\psi_{\bar{z}}^i\psi_z^{\bar{j}}\chi^k\chi^{\bar{l}}+ts^2\beta g_{i\bar{j}}K^i\bar{K}^{\bar{j}}+tsg_{i\bar{j}}\nabla_{\bar{\mu}}\bar{K}^{\bar{j}}\chi^{\bar{\mu}}\chi^i+s\beta g_{i\bar{j}}\nabla_{\mu}K^i\psi_{\bar{z}}^{\mu}\psi_z^{\bar{j}}\big]\end{aligned} \tag{4.141}$$

$\int_{\Sigma}\phi^*(\omega)$ is a topological term that gives mapping degree of $\phi:\mathbb{C}P^1\to M$. Since we focus on the degree 0 correlation function, we use the Lagrangian $L'+V'$ instead of $L+V$. By using Proposition 1, we obtain the following equality.

$$\lim_{s\to 0}<\varphi>_0=\lim_{s\to\infty}<\varphi>_0\ . \tag{4.142}$$

In this subsection, we focus on the left hand side. In the $s\to 0$ limit, the Lagrangian $L+V$ becomes Lagrangian of the usual topological sigma model (A-model). Moreover, φ turns into $\mathcal{O}_{\underline{\varphi}(E)}$, which is also a standard BRST-closed observable of the A-model. Then we can apply standard result of the weak coupling limit $t\to\infty$ [23]. It says that the path integral reduces to Gaussian integration around the constant map $\phi(z,\bar{z})=\phi_0$ $(\in M)$. From the discussion in Subsection 3.1 and 3.2, the correlation function becomes,

$$\lim_{s\to 0}<\varphi>_0=\int_M d\phi_0d\chi_0\mathcal{O}_{\underline{\varphi}(E)}=\int_M\underline{\varphi}(E)=:\underline{\varphi}(E)[M],$$

where $d\phi_0d\chi_0$ is the measure for integration of position of $\phi_0\in M$ and the corresponding zero-mode of χ, that can be interpreted as integration of (m,m)-form on M.

4.4 Integration Measure for the Degree 0 Correlation Function in the $s \to \infty$ Limit

(This section is based on a discussion with Professor Masao Jinzenji.) We discuss integration measure in evaluating the correlation function in the $s \to \infty$ limit with fixed t. Since we are considering the degree 0 correlation function, the map ϕ is homotopic to a constant map $\phi(z,\bar{z}) = \phi_0$ $(\in M)$. Therefore, we expand the fields ϕ, χ and ψ around the constant map ϕ_0. Then χ (resp. ψ) becomes section of $\phi_0^{-1}(T'M)$ (resp. $\phi_0^{-1}(T'M)\otimes\overline{T'^{*}}\mathbb{C}\mathrm{P}^1$) and its complex conjugate. But $\phi_0^{-1}(T'M)$ is isomorphic to trivial bundle $\mathbb{C}\mathrm{P}^1\times\mathbb{C}^m$. Hence we can simply regard $\chi^i, \chi^{\bar{i}}$ (resp. $\psi^i_{\bar{z}}, \psi^{\bar{i}}_z$) as $(0,0)$-form (resp. $(0,1)$-form, $(1,0)$-form) on $\mathbb{C}\mathrm{P}^1$ By using standard Kähler metric of $\mathbb{C}\mathrm{P}^1$, we can apply eigenvalue decomposition by Laplacian for differential forms on $\mathbb{C}\mathrm{P}^1$ to the expansion. The Laplacian is represented as follows († means adjoint defined by Hodge operator of $\mathbb{C}\mathrm{P}^1$).

$$\triangle := dd^\dagger + d^\dagger d, \qquad \triangle_\partial := \partial\partial^\dagger + \partial^\dagger\partial, \qquad \triangle_{\bar{\partial}} := \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial},$$

$$\triangle = 2\triangle_\partial = 2\triangle_{\bar{\partial}}.$$

Let $\triangle^{(p,q)}$ be restriction to $\Omega^{(p,q)}(\mathbb{C}\mathrm{P}^1)$. Vector space of (p,q)-forms with zero eigenvalue is known as $H^{p,q}(\mathbb{C}\mathrm{P}^1)$: the vector space of (p,q) harmonic forms. The following result is well-known.

$$\dim_{\mathbb{C}}(H^{0,0}(\mathbb{C}\mathrm{P}^1)) = 1, \ \ \dim_{\mathbb{C}}(H^{1,0}(\mathbb{C}\mathrm{P}^1)) = \dim_{\mathbb{C}}(H^{0,1}(\mathbb{C}\mathrm{P}^1)) = 0. \qquad (4.143)$$

Let $\{E_n \mid n > 0\}$ be set of positive eigenvalues of $\frac{1}{2}\triangle^{(0,0)}$ ordered as follows.

$$0 < E_1 \le E_2 \le E_3 \le \cdots . \qquad (4.144)$$

Then we denote by $f_n(z,\bar{z})$ the $(0,0)$-form that satisfy

$$\frac{1}{2}\triangle^{(0,0)} f_n(z,\bar{z}) = E_n f_n(z,\bar{z}). \qquad (4.145)$$

Lemma 1. *Sets of positive eigenvalues of* $\frac{1}{2}\triangle^{(1,0)}$ *and* $\frac{1}{2}\triangle^{(0,1)}$ *are both given by* $\{E_n \mid n > 0\}$ *and* $(1,0)$ *and* $(0,1)$ *forms with eigenvalue* E_n *are given by* $\partial f_n(z,\bar{z})$ *and* $\bar{\partial} f_n(z,\bar{z})$ *respectively.*

Proof)
Since $\frac{1}{2}\triangle$ equals $(\partial\partial^\dagger + \partial^\dagger\partial)$, we obtain

$$\begin{aligned}
\frac{1}{2}\triangle\partial f_n(z,\bar{z}) &= (\partial\partial^\dagger + \partial^\dagger\partial)\partial f_n(z,\bar{z}) \\
&= \partial\partial^\dagger\partial f_n(z,\bar{z}) \\
&= \partial(\partial\partial^\dagger + \partial^\dagger\partial) f_n(z,\bar{z}) \\
&= \frac{1}{2}\partial\triangle f_n(z,\bar{z}) \\
&= E_n\partial f_n(z,\bar{z}). \qquad (4.146)
\end{aligned}$$

Hence $\partial f_n(z,\bar{z})$ is $(1,0)$-form with eigenvalue E_n. On the contrary, let ω be $(1,0)$ form with eigenvalue E. Then $(0,0)$ form $\partial^\dagger\omega$ satisfy

$$\begin{aligned} \frac{1}{2}\triangle\partial^\dagger\omega &= (\partial\partial^\dagger+\partial^\dagger\partial)\partial^\dagger\omega \\ &= \partial^\dagger\partial\partial^\dagger\omega \\ &= \partial^\dagger(\partial\partial^\dagger+\partial^\dagger\partial)\omega \\ &= \frac{1}{2}\partial^\dagger\triangle\omega \\ &= E\partial^\dagger\omega. \end{aligned} \tag{4.147}$$

Therefore, $\partial^\dagger\omega$ must coincide some $f_n(z,\bar{z})$. This completes proof for $\triangle^{(1,0)}$. Proof for $\frac{1}{2}\triangle^{(0,1)}$ goes in the same way by using the equality $\frac{1}{2}\triangle=(\bar{\partial}\bar{\partial}^\dagger+\bar{\partial}^\dagger\bar{\partial})$. □

Variation $\delta\phi$ from the constant map ϕ_0 can also be regard as $(0,0)$ form on $\mathbb{C}\mathrm{P}^1$. Combining

$$\phi^i=\phi^i_0+\sum_{n>0}\phi'^i_n f_n(z,\bar{z}),\quad \phi^{\bar{i}}=\phi^{\bar{i}}_0+\sum_{n>0}\phi'^{\bar{i}}_n \bar{f}_n(z,\bar{z}), \tag{4.148}$$

$$\chi^i=\chi^i_0+\sum_{n>0}\chi'^i_n f_n(z,\bar{z}),\quad \chi^{\bar{i}}=\chi^{\bar{i}}_0+\sum_{n>0}\chi'^{\bar{i}}_n \bar{f}_n(z,\bar{z}), \tag{4.149}$$

$$\psi^i_{\bar{z}}=\sum_{n>0}\psi'^i_n\frac{1}{\sqrt{E_n}}\partial_{\bar{z}}f_n(z,\bar{z}),\quad \psi^{\bar{i}}_z=\sum_{n>0}\psi'^{\bar{i}}_n\frac{1}{\sqrt{E_n}}\partial_z\bar{f}_n(z,\bar{z}). \tag{4.150}$$

We set the volume of $\mathbb{C}\mathrm{P}^1$ to 1. Since $\triangle$ is Hermitian, $\{f_0(z,\bar{z})=1, f_1(z,\bar{z}), f_2(z,\bar{z}),\cdots\}$ can be considered as orthonormal basis of $\Omega^{(0,0)}$.

$$(f_n,f_m):=\int_{\mathbb{C}\mathrm{P}^1}\bar{f}_n(z,\bar{z})f_m(z,\bar{z})dz\wedge d\bar{z}=\delta_{n,m}\quad(n,m\geq 0). \tag{4.151}$$

Since $(f_n,(\bar{\partial}^\dagger\bar{\partial}+\bar{\partial}\bar{\partial}^\dagger)f_m)=E_m\delta_{n,m}=(f_n,\bar{\partial}^\dagger\bar{\partial}f_m)=(\bar{\partial}f_n,\bar{\partial}f_m)$, we obtain

$$\int_{\mathbb{C}\mathrm{P}^1}\partial_z\bar{f}_n(z,\bar{z})\partial_{\bar{z}}f_m(z,\bar{z})dz\wedge d\bar{z}=E_m\delta_{n,m}\quad(n,m>0). \tag{4.152}$$

This forces us to adopt $\{\frac{1}{\sqrt{E_n}}\partial_z f_n(z,\bar{z})\mid n>0\}$ and $\frac{1}{\sqrt{E_n}}\partial_{\bar{z}}f_n(z,\bar{z})\mid n>0\}$ as expansion basis of ψ. Therefore, integration measure for path-integral is given by

$$\begin{aligned} \mathfrak{D}\phi\mathfrak{D}\chi\mathfrak{D}\psi &= \left(\prod_{i=1}^m d\phi^i_0 d\phi^{\bar{i}}_0 d\chi^i_0 d\chi^{\bar{i}}_0\right)\left(\prod_{i=1}^m\prod_{n=1}^\infty\frac{d\phi'^i_n d\phi'^{\bar{i}}_n}{2\pi i}d\chi'^i_n d\chi'^{\bar{i}}_n d\psi'^i_n d\psi'^{\bar{i}}_n\right), \\ &= d\phi_0 d\chi_0\mathfrak{D}\phi'\mathfrak{D}\chi'\mathfrak{D}\psi'. \end{aligned} \tag{4.153}$$

4.5 The case when $E = T'M$

In this subsection, we discuss the case when the vector bundle E equals $T'M$ and the zero set of K is given by a finite set of discrete points $\{p_1, \cdots, p_N\}$. In this case, the action Λ on $T'M$ is given by $\theta(sK) : Y \to [sK, Y]$. On the zero set of K, it is explicitly given as follows.

$$\theta(sK)(\frac{\partial}{\partial z^j}) = [sK^i\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}] = -s\partial_j K^i \frac{\partial}{\partial z^i}. \tag{4.154}$$

Hence we set $\Lambda^i_j = -\partial_j K^i$ in this subsection.

4.5.1 Explicit Construction of the BRST-closed Observable

First, we construct the observable φ. For this purpose, we have only to determine the explicit form of A^a_b. Since the local holomorphic frame $\{e_a\}$ is given by $\{\frac{\partial}{\partial z^a}\}$, we do not distinguish subscripts of local frame from ones of local coordinate. Then canonical connection becomes $\tilde{\nabla}\frac{\partial}{\partial z^i} = \Gamma^j_{ik} dz^k \frac{\partial}{\partial z^j}$. Then, we obtain

$$\begin{aligned}
&i(sK)\tilde{\nabla}\frac{\partial}{\partial z^i} = sK^l i(dz^l)\Gamma^j_{ik}\chi^k\frac{\partial}{\partial z^j} = sK^l\Gamma^j_{il}\frac{\partial}{\partial z^j}, \\
&F^i_j\frac{\partial}{\partial z^i} = \bar{\partial}(\Gamma^i_{jk}dz^k\frac{\partial}{\partial z^i}) = \partial_{\bar{l}}\Gamma^i_{jk}d\overline{z^l}\wedge dz^k\frac{\partial}{\partial z^i} = R^i_{j\bar{l}k}d\overline{z^l}\wedge dz^k\frac{\partial}{\partial z^i} = R^i_{jk\bar{l}}dz^k\wedge d\overline{z^l}\frac{\partial}{\partial z^i}
\end{aligned}$$

and

$$\begin{aligned}
A^i_j\frac{\partial}{\partial z^i} &:= L^i_j\frac{\partial}{\partial z^i} + \frac{i}{2\pi}F^i_j\frac{\partial}{\partial z^i} = s\Lambda^i_j\frac{\partial}{\partial z^i} - i(sK)\tilde{\nabla}\frac{\partial}{\partial z^i} - F^i_j\frac{\partial}{\partial z^i} \\
&= (-s\partial_j K^i - s\Gamma^i_{j\mu}K^\mu + \frac{i}{2\pi}R^i_{jk\bar{l}}dz^k\wedge d\overline{z^l})\frac{\partial}{\partial z^i}.
\end{aligned}$$

This is the formula we use in this subsection.

4.5.2 Expansion of the Lagrangian up to the Second Order

In the $s \to \infty$ limit, path-integral is localized on neighborhood of p_α. Therefore, we use expansion of the fields in (4.148), (4.149) and (4.150) with $\phi_0 = p_\alpha$. Then we expand the Lagrangian up to the second order of the expansion variables. In the neighborhood of p_α, K is expanded in the form: $-K_{\alpha}{}^i_j z^j + \cdots$. We can also assume that $g_{i\bar{j}} = \delta_{i\bar{j}}$ and $\Gamma^k_{ij} = \Gamma^{\bar{k}}_{\bar{i}\bar{j}} = 0$. Therefore, in expanding the Lagrangian, we can use the following simplification.

$$D_z \to \partial_z, \qquad D_{\bar{z}} \to \partial_{\bar{z}}, \qquad \nabla_j \to \partial_j, \qquad \nabla_{\bar{j}} \to \partial_{\bar{j}}. \tag{4.155}$$

We also have $\nabla_{\bar{\mu}}\bar{K}^{\bar{j}} = -\bar{K}^{\bar{j}}_{\alpha\bar{\mu}} + \cdots$. Then expansion of the Lagrangian up to the second order is given as follows.

$$
\begin{aligned}
(L+V)_{\text{2nd.}} &:= L_0^{\alpha} + L^{\alpha\prime}, \\
L_0^{\alpha} &:= t\big[\beta s^2 \delta_{i\bar{j}} K_{\alpha\mu}^{\ \ i} \bar{K}^{\bar{j}}_{\alpha\bar{\nu}} \phi_0^{\mu}\phi_0^{\bar{\nu}} - s\delta_{i\bar{j}} \bar{K}^{\bar{j}}_{\alpha\bar{\mu}} \chi_0^{\bar{\mu}} \chi_0^{i}\big], \\
L^{\alpha\prime} &:= \sum_{n>0}\big[t\delta_{i\bar{j}}\phi_n^{\prime i}\phi_n^{\prime\bar{j}}E_n + \sqrt{t}i\delta_{i\bar{j}}(\psi_n^{\prime\bar{j}}\chi_n^{\prime i} + \psi_n^{\prime i}\chi_n^{\prime\bar{j}})\sqrt{E_n}\big] \\
&\quad + \sum_{n>0}\Big\{t\beta s^2\delta_{i\bar{j}}K_{\alpha\mu}^{\ \ i}\bar{K}^{\bar{j}}_{\alpha\bar{\nu}}\phi_n^{\prime\mu}\phi_n^{\prime\bar{\nu}} - ts\delta_{i\bar{j}}\bar{K}^{\bar{j}}_{\alpha\bar{\mu}}\chi_n^{\prime\bar{\mu}}\chi_n^{\prime i} - s\beta\delta_{i\bar{j}}K_{\alpha\mu}^{\ \ i}\psi_n^{\prime\mu}\psi_n^{\prime\bar{j}}\Big\},
\end{aligned}
$$

where L_0 is zero mode part and L' is oscillation mode part.

4.5.3 Evaluation of $\lim_{s\to\infty} < \varphi >_0$

We represent the correlation function in the following form:

$$
\lim_{s\to\infty} < \varphi >_0 = \sum_{\alpha} \lim_{s\to\infty} \int \mathfrak{D}\phi\mathfrak{D}\chi\mathfrak{D}\psi\, \varphi|_{p_\alpha} e^{-L_0^{\alpha} - L^{\alpha\prime}}, \tag{4.156}
$$

On p_α, $A^i_j = -s\partial_j K^i - s\Gamma^i_{j\mu}K^{\mu} + \frac{i}{2\pi}R^i_{jk\bar{l}}dz^k \wedge d\overline{z^l}$ becomes $sK^i_{\alpha j} - \frac{i}{2\pi}R^i_{jk\bar{l}}dz^k \wedge d\overline{z^l}$. Let K_α be $m\times m$ matrix defined by $K^i_{\alpha j}$. Then we have $\varphi|_{p_\alpha} = s^m\varphi(K_\alpha - \frac{1}{s}\frac{i}{2\pi}R|_{p_\alpha}) =: s^m\varphi(p_\alpha, s)$. With this set-up, we evaluate the contribution from p_α. First, we integrate oscillation modes. $\varphi|_{p_\alpha}$ does not contain ψ oscillation modes and we neglect the third and higher order terms that contain oscillation modes. The part of oscillation modes integration is given as follows.

$$
\lim_{s\to\infty}\int \mathfrak{D}\phi\mathfrak{D}\chi\mathfrak{D}\psi\,\varphi|_{p_\alpha}e^{-L^{\alpha\prime}} = \lim_{s\to\infty}\varphi|_{p_\alpha}\int\prod_{i=1}^{m}\prod_{n=1}^{\infty}\frac{d\phi_n^{\prime i}d\phi_n^{\prime\bar{i}}}{2\pi i}d\chi_n^{\prime i}d\chi_n^{\prime\bar{i}}d\psi_n^{\prime i}d\psi_n^{\prime\bar{i}}e^{-L^{\alpha\prime}}.
$$

At this stage, we transform integration variables in the following way ($n > 0$).

$$
\phi_n^{\prime\mu} = \frac{1}{s}\phi_n^{\mu} \qquad \phi_n^{\prime\bar{\mu}} = \frac{1}{s}\phi_n^{\bar{\mu}} \qquad \chi_n^{\prime\mu} = \frac{1}{\sqrt{s}}\chi_n^{\mu} \qquad \chi_n^{\prime\bar{\mu}} = \frac{1}{\sqrt{s}}\chi_n^{\bar{\mu}}, \tag{4.157}
$$

$$
\psi_n^{\prime\mu} = \frac{1}{\sqrt{s}}\psi_n^{\mu} \qquad \psi_n^{\prime\bar{\mu}} = \frac{1}{\sqrt{s}}\psi_n^{\bar{\mu}}. \tag{4.158}
$$

Then integral measures of oscillation modes are invariant under the transformation and $L^{\alpha\prime}$ is transformed in the following form.

$$
\prod_{i=1}^{m}\prod_{n=1}^{\infty}\frac{d\phi_n^{\prime i}d\phi_n^{\prime\bar{i}}}{2\pi i}d\chi_n^{\prime i}d\chi_n^{\prime\bar{i}}d\psi_n^{\prime i}d\psi_n^{\prime\bar{i}} = \prod_{i=1}^{m}\prod_{n=1}^{\infty}\frac{d\phi_n^{i}d\phi_n^{\bar{i}}}{2\pi i}d\chi_n^{i}d\chi_n^{\bar{i}}d\psi_n^{i}d\psi_n^{\bar{i}}, \tag{4.159}
$$

$$
L^{\alpha\prime} := \sum_{n>0}\big[\frac{t}{s^2}\delta_{i\bar{j}}\phi_n^{i}\phi_n^{\bar{j}}E_n + \frac{\sqrt{t}i}{s}\delta_{i\bar{j}}(\psi_n^{\bar{j}}\chi_n^{i} + \psi_n^{i}\chi_n^{\bar{j}})\sqrt{E_n}\big] \tag{4.160}
$$

$$
+\sum_{n>0}\Big\{t\beta\delta_{i\bar{j}}K_{\alpha\mu}^{\ \ i}\bar{K}^{\bar{j}}_{\alpha\bar{\nu}}\phi_n^{\mu}\phi_n^{\bar{\nu}} - t\delta_{i\bar{j}}\bar{K}^{\bar{j}}_{\alpha\bar{\mu}}\chi_n^{\bar{\mu}}\chi_n^{i} - \beta\delta_{i\bar{j}}K_{\alpha\mu}^{\ \ i}\psi_n^{\mu}\psi_n^{\bar{j}}\Big\}. \tag{4.161}
$$

We neglect $O(s^{-1})$ part since we take the $s \to \infty$ limit. As a result, integration of oscillation modes is given by

$$\lim_{s\to\infty} \varphi|_{p_\alpha} \int \prod_{i=1}^{m}\prod_{n=1}^{\infty} \frac{d\phi_n^i d\phi_n^{\bar{i}}}{2\pi i} d\chi_n^i d\chi_n^{\bar{i}} d\psi_n^i d\psi_n^{\bar{i}}$$
$$\times \exp\Big[\sum_{n>0}\Big\{-t\beta\delta_{i\bar{j}}K_{\alpha_\mu}{}^{i}\bar{K}^{\bar{j}}_{\alpha\bar{\nu}}\phi_n^\mu\phi_n^{\bar{\nu}} + t\delta_{i\bar{j}}\bar{K}^{\bar{j}}_{\alpha\bar{\mu}}\chi_n^{\bar{\mu}}\chi_n^i + \beta\delta_{i\bar{j}}K_{\alpha_\mu}{}^{i}\psi_n^\mu\psi_n^{\bar{j}}\Big\}\Big] \quad (4.162)$$
$$= \lim_{s\to\infty}\varphi|_{p_\alpha}\prod_{n=1}^{\infty}\frac{1}{(2\pi i)^m}\int\prod_{i=1}^{m} d\phi_n^i d\phi_n^{\bar{i}} d\chi_n^i d\chi_n^{\bar{i}} d\psi_n^i d\psi_n^{\bar{i}}$$
$$\times \exp\Big[-t\beta\delta_{i\bar{j}}K_{\alpha_\mu}{}^{i}\bar{K}^{\bar{j}}_{\alpha\bar{\nu}}\phi_n^\mu\phi_n^{\bar{\nu}} + t\delta_{i\bar{j}}\bar{K}^{\bar{j}}_{\alpha\bar{\mu}}\chi_n^{\bar{\mu}}\chi_n^i + \beta\delta_{i\bar{j}}K_{\alpha_\mu}{}^{i}\psi_n^\mu\psi_n^{\bar{j}}\Big]. \quad (4.163)$$

By using the following integral formulas,

$$\int d\phi_0 \exp\big[-uM_{ij}\phi_0^i\phi_0^{\bar{j}}\big] = (\frac{-2\pi i}{u})^m(\det M)^{-1}, \quad (4.164)$$
$$\int d\chi_0 \exp\big[M_{i\bar{j}}\chi_0^i\chi_0^{\bar{j}}\big] = \det M, \quad (4.165)$$

we proceed as follows.

$$\lim_{s\to\infty}\varphi|_{p_\alpha}\prod_{n=1}^{\infty}\frac{1}{(2\pi i)^m}\int\prod_{i=1}^{m} d\phi_n^i d\phi_n^{\bar{i}} d\chi_n^i d\chi_n^{\bar{i}} d\psi_n^i d\psi_n^{\bar{i}}$$
$$\times \exp\Big[-t\beta\delta_{i\bar{j}}K_{\alpha_\mu}{}^{i}\bar{K}^{\bar{j}}_{\alpha\bar{\nu}}\phi_n^\mu\phi_n^{\bar{\nu}} + t\delta_{i\bar{j}}\bar{K}^{\bar{j}}_{\alpha\bar{\mu}}\chi_n^{\bar{\mu}}\chi_n^i + \beta\delta_{i\bar{j}}K_{\alpha_\mu}{}^{i}\psi_n^\mu\psi_n^{\bar{j}}\Big] \quad (4.166)$$
$$= \lim_{s\to\infty}\varphi|_{p_\alpha}\prod_{n=1}^{\infty}\Big(\frac{-1}{t\beta}\Big)^m\frac{(-t\beta)^m\det(\delta_{i\bar{j}}\bar{K}^{\bar{j}}_{\alpha\bar{\mu}})\det(\delta_{i\bar{j}}K_{\alpha_\mu}{}^{i})}{\det(\delta_{i\bar{j}}K_{\alpha_\mu}{}^{i}\bar{K}^{\bar{j}}_{\alpha\bar{\nu}})} \quad (4.167)$$
$$= \lim_{s\to\infty}\varphi|_{p_\alpha}. \quad (4.168)$$

Contribution from oscillation modes turn out to be 1. Next, we calculate integral of zero mode part.

$$\lim_{s\to\infty}\int d\phi_0 d\chi_0\, s^m\varphi(p_\alpha,s)e^{-L_0^\alpha}$$
$$= \lim_{s\to\infty} s^m\varphi(p_\alpha,s)\int d\phi_0 d\chi_0 \exp\big[-t\big(s^2\beta\delta_{i\bar{j}}K^i_{\alpha\mu}\bar{K}^{\bar{j}}_{\alpha\bar{l}}\phi_0^\mu\phi_0^{\bar{l}} - s\delta_{i\bar{j}}\bar{K}^{\bar{j}}_{\alpha\bar{\mu}}\chi_0^{\bar{\mu}}\chi_0^i\big)\big]$$
$$= \lim_{s\to\infty} s^m\varphi(p_\alpha,s)\int d\phi_0 \exp\big[-ts^2\beta\delta_{i\bar{j}}K^i_{\alpha\mu}\bar{K}^{\bar{j}}_{\alpha\bar{l}}\phi_0^\mu\phi_0^{\bar{l}}\big] \quad (4.169)$$
$$\times \int d\chi_0 \exp\big[ts\delta_{i\bar{j}}\bar{K}^{\bar{j}}_{\alpha\bar{\mu}}\chi_0^{\bar{\mu}}\chi_0^i\big]$$
$$= \lim_{s\to\infty}\frac{s^m\varphi(p_\alpha,s)\cdot(2\pi its)^m\det\Big(\delta_{i\bar{j}}\bar{K}^{\bar{j}}_{\alpha\bar{\mu}}\Big)}{(ts^2\beta)^m\det(\delta_{i\bar{j}}K^i_{\alpha\mu}\bar{K}^{\bar{j}}_{\alpha\bar{l}})} = \lim_{s\to\infty}\frac{\varphi(p_\alpha,s)\det(\delta_{i\bar{j}}\bar{K}^{\bar{j}}_{\alpha\bar{\mu}})}{\det(\delta_{i\bar{j}}K^i_{\alpha\mu}\bar{K}^{\bar{j}}_{\alpha\bar{l}})}, \quad (4.170)$$

where we used $\beta = 2\pi i$. Since $\lim_{s\to\infty}\varphi(p_\alpha, s) = \lim_{s\to\infty}\varphi(K_\alpha - \frac{1}{s}\frac{i}{2\pi}R|_{p_\alpha}) = \varphi(K_\alpha)$, we obtain

$$\lim_{s\to\infty}\int \mathfrak{D}\phi_0\mathfrak{D}\chi_0\,\varphi(p)e^{-L_0} = \frac{\varphi(K_\alpha)\det(\delta_{i\bar{j}}\bar{K}^{\bar{j}}_{\alpha\bar{\mu}})}{\det(\delta_{i\bar{j}}K^i_{\alpha\mu}\bar{K}^{\bar{j}}_{\alpha\bar{l}})} = \frac{\varphi(K_\alpha)}{\det(K^i_{\alpha j})}. \tag{4.171}$$

Since we already know $\theta|_{p_\alpha} = K^i_{\alpha j}$, the above result is rewritten by

$$\lim_{s\to\infty} <\varphi>=\sum_{\alpha=1}^{N}\frac{\varphi(K_\alpha)}{\det(\theta|_{p_\alpha})}. \tag{4.172}$$

By combining Proposition 1 and the result in the $s\to 0$ limit, we obtain the Bott residue formula in the case of this subsection.

$$\underline{\varphi}(T'M)[M] = \sum_{\alpha=1}^{N}\frac{\varphi(K_\alpha)}{\det(\theta|_{p_\alpha})}. \tag{4.173}$$

Let us assume that E is a general holomorphic vector bundle on M and that zero set of K is given by a discrete point set $\{p_1, \cdots, p_N\}$. In the same way as the discussion of this subsection, we can derive the Bott residue formula:

$$\underline{\varphi}(E)[M] = \sum_{\alpha=1}^{N}\frac{\varphi(\Lambda|_{p_\alpha})}{\det(\theta|_{p_\alpha})}. \tag{4.174}$$

This result corresponds to the example given in [8].

4.6 Derivation in General Case

In this subsection, we derive general case of the Bott residue formula, i.e., zero set of K is given by $\{N_\alpha\}$ where N_α is a connected compact Kähler submanifold of M. For simplicity, we focus on one connected component $N := N_\alpha$ in the following discussion. We set $\mathrm{codim}_{\mathbb{C}}(N) = \nu$. In the $s\to\infty$ limit, the path integral is localized to neighborhood of N, we apply expansion given in Subsection 4.4 around the constant map $\phi_0 \in N$. But one subtlety occurs in this case. Since N is a Kähler submanifold of M, local coordinates around $\phi_0 \in N$ can be taken in the following form:

$$(z^1_\perp, \cdots, z^\nu_\perp, z^{\nu+1}_\parallel, \cdots, z^m_\parallel),$$

where points in N is described by the condition $z^1_\perp = \cdots = z^\nu_\perp = 0$. Then fields ϕ, χ and ψ are also decomposed into $\phi_\perp + \phi_\parallel$, $\chi_\perp + \chi_\parallel$ and $\psi_\perp + \psi_\parallel$ respectively. From now on, we use alphabets for $\perp$ directions and Greek characters for $\|$ directions. Then expansion in Subsection 4.4 is changed as follows.

$$\begin{aligned}
&\phi^i_\perp = \phi^i_{\perp 0} + \sum_{n>0} \phi'^i_{\perp n} f_n(z,\bar z), \quad \phi^{\bar i}_\perp = \phi^{\bar i}_{\perp 0} + \sum_{n>0} \phi'^{\bar i}_{\perp n} \bar f_n(z,\bar z), \\
&\chi^i_\perp = \chi^i_{\perp 0} + \sum_{n>0} \chi'^i_{\perp n} f_n(z,\bar z), \quad \chi^{\bar i}_\perp = \chi^{\bar i}_{\perp 0} + \sum_{n>0} \chi'^{\bar i}_{\perp n} \bar f_n(z,\bar z), \\
&\psi^i_{\bar z \perp} = \sum_{n>0} \frac{1}{\sqrt{E_n}} \partial_{\bar z} \psi'^i_{\perp n} f_n(z,\bar z), \quad \psi^{\bar i}_{z\perp} = \sum_{n>0} \psi'^{\bar i}_{\perp n} \frac{1}{\sqrt{E_n}} \partial_z \bar f_n(z,\bar z),
\end{aligned} \tag{4.175}$$

$$\begin{aligned}
&\phi^\nu_\| = \phi^\nu_{\|0} + \sum_{n>0} \phi'^\nu_{\|n} f_n(z,\bar z), \quad \phi^{\bar\nu}_\| = \phi^{\bar\nu}_{\|0} + \sum_{n>0} \phi'^{\bar\nu}_{\|n} \bar f_n(z,\bar z), \\
&\chi^\nu_\| = \chi^\nu_{\|0} + \sum_{n>0} \chi'^\nu_{\|n} f_n(z,\bar z), \quad \chi^{\bar\nu}_\| = \chi^{\bar\nu}_{\|0} + \sum_{n>0} \chi'^{\bar\nu}_{\|n} \bar f_n(z,\bar z), \\
&\psi^\nu_{\bar z \|} = \sum_{n>0} \psi'^\nu_{\|n} \frac{1}{\sqrt{E_n}} \partial_{\bar z} f_n(z,\bar z), \quad \psi^{\bar\nu}_{z\|} = \sum_{n>0} \psi'^{\bar\nu}_{\|n} \frac{1}{\sqrt{E_n}} \partial_z \bar f_n(z,\bar z).
\end{aligned} \tag{4.176}$$

In other words, we decompose the integration measure as follows.

$$\int_N \mathfrak{D}\phi_{\|0} \mathfrak{D}\chi_{\|0} \int \mathfrak{D}\phi'_\| \mathfrak{D}\chi'_\| \mathfrak{D}\psi'_\| \int_{N_\perp} \mathfrak{D}\phi_{\perp 0} \mathfrak{D}\chi_{\perp 0} \int \mathfrak{D}\phi'_\perp \mathfrak{D}\chi'_\perp \mathfrak{D}\psi'_\perp .$$

4.6.1 Expansion of L up to the Second Order

By using the orthonormal relation (4.151) and (4.152), expansion L up to the second order is guven by

$$\begin{aligned}
L = &\sum_{n>0} \Big[tE_n \delta_{i\bar j} \phi'^i_{\perp n} \phi'^{\bar j}_{\perp n} + \sqrt{t} i \sqrt{E_n} \delta_{i\bar j} \psi'^{\bar j}_{\perp n} \chi'^i_{\perp n} + \sqrt{t} i \sqrt{E_n} \delta_{i\bar j} \psi'^i_{\perp n} \chi'^{\bar j}_{\perp n} \\
&+ tE_n \delta_{\iota\bar\tau} \phi'^\iota_{\|n} \phi'^{\bar\tau}_{\|n} + \sqrt{t} i \sqrt{E_n} \delta_{\iota\bar\tau} \psi'^{\bar\tau}_{\|n} \chi'^\iota_{\|n} + \sqrt{t} i \sqrt{E_n} \delta_{\iota\bar\tau} \psi'^\iota_{\|n} \chi'^{\bar\tau}_{\|n} \Big],
\end{aligned} \tag{4.177}$$

where we used local coordinates that make $g_{i\bar j}$ and $\Gamma^k_{ij}(\Gamma^{\bar k}_{\bar i \bar j})$ into $\delta_{i\bar j}$ and 0 respectively. $(g_{i\bar j} \to \delta_{i\bar j}, g_{\mu\bar\nu} \to \delta_{\mu\bar\nu}, g_{i\bar\nu} = 0)$

4.6.2 Expansion of the Potential V

In this subsection, we expand the potential term V around neighborhood of $p \in N$.

$$V = \int_{\mathbb{CP}^1} dz d\bar z \big[ts^2 \beta g_{I\bar J} K^I \bar K^{\bar J} + ts g_{I\bar J} \nabla_{\bar M} \bar K^{\bar J} \chi^{\bar M} \chi^I + s g_{I\bar J} \nabla_\mu K^I \psi^M_{\bar z} \psi^{\bar J}_z \big]. \tag{4.178}$$

where subscripts $I, J, M, \cdots$ run through all the directions $1, 2, \cdots, m$.

First, we consider expansion of the second term of V. Let us consider the following Taylor expansion around p . We use the coordinate system that satisfies

$$g_{I\bar{J}}(p) = \delta_{I\bar{J}} + \cdots \ (g_{i\bar{j}}(p) = \delta_{i\bar{j}}, g_{\mu\bar{\nu}}(p) = \delta_{\mu\bar{\nu}}, g_{i\bar{\nu}}(p) = 0), \tag{4.179}$$

$$\Gamma^{\bar{J}}_{\bar{I}\bar{M}} = R^{\bar{J}}_{\bar{I}L\bar{M}}\phi^L + \partial_{\bar{N}}\Gamma^{\bar{J}}_{\bar{I}\bar{M}}(p)\phi^{\bar{N}} + \cdots . \tag{4.180}$$

Moreover, p is in the zero set of K, we can use $K^i = -K^i_l z^l_\perp + \cdots$. We neglect second order or higher term each.

$$\begin{aligned} sg_{I\bar{J}}\nabla_{\bar{M}}\bar{K}^{\bar{J}} &= sg_{I\bar{J}}(\partial_{\bar{M}}\bar{K}^{\bar{J}} + \Gamma^{\bar{J}}_{\bar{M}\bar{l}}\bar{K}^{\bar{l}}) \\ &= -s\delta_{i\bar{j}}\bar{K}^{\bar{j}}_{\bar{m}} - sR_{I\bar{l}N\bar{M}}\bar{K}^{\bar{l}}_{\bar{k}}\phi^{\bar{k}}_\perp\phi^N - s\delta_{I\bar{J}}\partial_{\bar{N}}\Gamma^{\bar{J}}_{\bar{I}\bar{M}}\bar{K}^{\bar{l}}_{\bar{k}}\phi^{\bar{k}}_\perp\phi^{\bar{N}}\cdots \\ &= -s\delta_{i\bar{j}}\bar{K}^{\bar{j}}_{\bar{m}} - sR_{I\bar{l}\mu\bar{M}}\bar{K}^{\bar{l}}_{\bar{k}}\phi^{\bar{k}}_\perp\phi^\mu_\parallel - sR_{I\bar{j}l\bar{M}}\bar{K}^{\bar{j}}_{\bar{k}}\phi^{\bar{k}}_\perp\phi^l_\perp \\ &\quad - s\delta_{I\bar{J}}\partial_{\bar{N}}\Gamma^{\bar{J}}_{\bar{I}\bar{M}}\bar{K}^{\bar{l}}_{\bar{k}}\phi^{\bar{k}}_\perp\phi^{\bar{N}} + \cdots . \end{aligned}$$

Since the last term does not contain holomorphic part of ϕ and contain $\bar{\phi}_\perp$, it does not contribute to Gaussian integration of $\phi_{\perp 0}$ that will be done later. Hence we neglect this term. Then we obtain

$$sg_{I\bar{J}}\nabla_{\bar{M}}\bar{K}^{\bar{J}} = -s\delta_{i\bar{j}}\bar{K}^{\bar{j}}_{\bar{m}} - sR_{I\bar{j}\mu\bar{M}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_\perp\phi^\mu_\parallel - sR_{I\bar{j}l\bar{M}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_\perp\phi^l_\perp, \tag{4.181}$$

and

$$\begin{aligned} sg_{I\bar{J}}\nabla_{\bar{M}}\bar{K}^{\bar{J}}\chi^{\bar{M}}\chi^I \quad = \quad & -s\delta_{i\bar{j}}\bar{K}^{\bar{j}}_{\bar{m}}\chi^{\bar{m}}_\perp\chi^i_\perp - sR_{I\bar{j}\mu\bar{M}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_\perp\phi^\mu_\parallel\chi^{\bar{M}}\chi^I \\ & -sR_{I\bar{j}l\bar{M}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_\perp\phi^l_\perp\chi^{\bar{M}}\chi^I. \end{aligned} \tag{4.182}$$

Since $\chi = \chi_\parallel + \chi_\perp$, we decompose

$$\chi^{\bar{M}}\chi^I = \chi^{\bar{\mu}}_\parallel\chi^\iota_\parallel + \chi^{\bar{m}}_\perp\chi^\iota_\parallel + \chi^{\bar{\mu}}_\parallel\chi^i_\perp + \chi^{\bar{m}}_\perp\chi^i_\perp.$$

Then the part that corresponds to (4.182) is rewritten as follows.

$$\begin{aligned} &\int_{\mathbb{CP}^1} dzd\bar{z}\Big\{sg_{I\bar{J}}\nabla_{\bar{M}}\bar{K}^{\bar{J}}\chi^{\bar{M}}\chi^I\Big\} \\ &= \int_{\mathbb{CP}^1} dzd\bar{z}\Big\{-s\delta_{i\bar{j}}\bar{K}^{\bar{j}}_{\bar{m}}\chi^{\bar{m}}_\perp\chi^i_\perp - sR_{I\bar{j}\mu\bar{M}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_\perp\phi^\mu_\parallel\chi^{\bar{M}}\chi^I \\ &- sR_{I\bar{j}l\bar{M}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_\perp\phi^l_\perp\chi^{\bar{M}}\chi^I\Big\}. \end{aligned}$$

Then we use the expansion (4.175) and (4.176).

$$
\begin{aligned}
&\int_{\mathbb{C}\mathrm{P}^1} dzd\bar{z}\Big\{-s\delta_{i\bar{j}}\bar{K}^{\bar{j}}_{\bar{m}}\chi^{\bar{m}}_{\perp}\chi^{i}_{\perp}\Big\}\\
&= -s\delta_{i\bar{j}}\bar{K}^{\bar{j}}_{\bar{m}}\chi^{\bar{m}}_{\perp 0}\chi^{i}_{\perp 0} - s\delta_{i\bar{j}}\bar{K}^{\bar{j}}_{\bar{m}}\sum_{n\in\mathbb{Z}(n\neq 0)}\chi'^{\bar{m}}_{\perp n}\chi'^{i}_{\perp n},\\
&\int_{\mathbb{C}\mathrm{P}^1} dzd\bar{z}\Big\{-sR_{I\bar{j}\mu\bar{M}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp}\phi^{\mu}_{\parallel}\chi^{\bar{M}}\chi^{I}\Big\}\\
&= \int_{\mathbb{C}\mathrm{P}^1} dzd\bar{z}\Big\{-sR_{I\bar{j}\mu\bar{M}}\bar{K}^{\bar{j}}_{\bar{m}}\{\phi^{\bar{m}}_{\perp 0} + \sum_{n\in\mathbb{Z}(n\neq 0)}\phi'^{\bar{m}}_{\perp n}\bar{f}_n(z,\bar{z})\}\phi^{\mu}_{\parallel 0}\chi^{\bar{M}}\chi^{I}\Big\}\\
&= \int_{\mathbb{C}\mathrm{P}^1} dzd\bar{z}\Big\{-sR_{\iota\bar{j}\mu\bar{\nu}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp 0}\phi^{\mu}_{\parallel 0}\chi^{\bar{\nu}}_{\parallel}\chi^{\iota}_{\parallel} - sR_{\iota\bar{j}\mu\bar{l}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp 0}\phi^{\mu}_{\parallel 0}\chi^{\bar{l}}_{\perp}\chi^{\iota}_{\parallel}\\
&- sR_{i\bar{j}\mu\bar{\nu}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp 0}\phi^{\mu}_{\parallel 0}\chi^{\bar{\nu}}_{\parallel}\chi^{i}_{\perp} - sR_{i\bar{j}\mu\bar{l}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp 0}\phi^{\mu}_{\parallel 0}\chi^{\bar{l}}_{\perp}\chi^{i}_{\perp}\Big\}\\
&= -sR_{\iota\bar{j}\mu\bar{\nu}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp 0}\phi^{\mu}_{\parallel 0}\chi^{\bar{\nu}}_{\parallel 0}\chi^{\iota}_{\parallel 0} - sR_{\iota\bar{j}\mu\bar{l}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp 0}\phi^{\mu}_{\parallel 0}\chi^{\bar{l}}_{\perp 0}\chi^{\iota}_{\parallel 0}\\
&- sR_{i\bar{j}\mu\bar{\nu}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp 0}\phi^{\mu}_{\parallel 0}\chi^{\bar{\nu}}_{\parallel 0}\chi^{i}_{\perp 0} - sR_{i\bar{j}\mu\bar{l}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp 0}\phi^{\mu}_{\parallel 0}\chi^{\bar{l}}_{\perp 0}\chi^{i}_{\perp 0},\\
&\int_{\mathbb{C}\mathrm{P}^1} dzd\bar{z}\Big\{-sR_{I\bar{j}l\bar{M}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp}\phi^{l}_{\perp}\chi^{\bar{M}}\chi^{I}\Big\}\\
&= \int_{\mathbb{C}\mathrm{P}^1} dzd\bar{z}\Big\{-sR_{I\bar{j}l\bar{M}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp 0}\phi^{l}_{\perp 0}\chi^{\bar{M}}\chi^{I}\Big\}\\
&= -sR_{\iota\bar{j}l\bar{\mu}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp 0}\phi^{l}_{\perp 0}\chi^{\bar{\mu}}_{\parallel 0}\chi^{\iota}_{\parallel 0} - sR_{\iota\bar{j}l\bar{n}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp 0}\phi^{l}_{\perp 0}\chi^{\bar{n}}_{\perp 0}\chi^{\iota}_{\parallel 0}\\
&- sR_{i\bar{j}l\bar{\mu}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp 0}\phi^{l}_{\perp 0}\chi^{\bar{\mu}}_{\parallel 0}\chi^{i}_{\perp 0} - sR_{i\bar{j}l\bar{n}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp 0}\phi^{l}_{\perp 0}\chi^{\bar{n}}_{\perp 0}\chi^{i}_{\perp 0}.
\end{aligned}
$$

In this expansion, we neglect third and higher terms that contain oscillation modes. Next, we expand the first term by using the same rule.

$$
\begin{aligned}
&\int_{\mathbb{C}\mathrm{P}^1} dzd\bar{z}\Big\{s^2\beta g_{i\bar{j}}K^{i}\bar{K}^{\bar{j}}\Big\} = \int_{\mathbb{C}\mathrm{P}^1} dzd\bar{z}\Big\{s^2\beta\delta_{i\bar{j}}K^{i}_{m}\bar{K}^{\bar{j}}_{\bar{l}}\phi^{m}_{\perp}\phi^{\bar{l}}_{\perp}\Big\}\\
&= \int_{\mathbb{C}\mathrm{P}^1} dzd\bar{z}\Big\{s^2\beta\delta_{i\bar{j}}K^{i}_{m}\bar{K}^{\bar{j}}_{\bar{l}}\{\phi^{m}_{\perp 0} + \sum_{n\in\mathbb{Z}(n\neq 0)}\phi'^{m}_{\perp n}f_n(z,\bar{z})\}\\
&\times\{\phi^{\bar{l}}_{\perp 0} + \sum_{n\in\mathbb{Z}(n\neq 0)}\phi'^{\bar{l}}_{\perp n}\bar{f}_n(z,\bar{z})\}\Big\}\\
&= s^2\beta\delta_{i\bar{j}}K^{i}_{m}\bar{K}^{\bar{j}}_{\bar{l}}\phi^{m}_{\perp 0}\phi^{\bar{l}}_{\perp 0} + \sum_{n\in\mathbb{Z}(n\neq 0)} s^2\beta\delta_{i\bar{j}}K^{i}_{m}\bar{K}^{\bar{j}}_{\bar{l}}\phi'^{m}_{\perp n}\phi'^{\bar{l}}_{\perp n}.
\end{aligned}
$$

We can neglect the third term by applying tha same rule for expansion. As a result, we obtain the following form.

$$
\int_{\mathbb{C}\mathrm{P}^1} dzd\bar{z}\Big\{s^2\beta g_{i\bar{j}}K^{i}\bar{K}^{\bar{j}} + sg_{I\bar{J}}\nabla_{\bar{M}}\bar{K}^{\bar{J}}\chi^{\bar{M}}\chi^{I}\Big\}
$$

$$
\begin{aligned}
&= s^2\beta\delta_{i\bar{j}}K^i_m\bar{K}^{\bar{j}}_{\bar{l}}\phi^m_{\perp 0}\phi^{\bar{l}}_{\perp 0} + \sum_{n>0} s^2\beta\delta_{i\bar{j}}K^i_m\bar{K}^{\bar{j}}_{\bar{l}}\phi'^m_{\perp n}\phi'^{\bar{l}}_{\perp n} - s\delta_{i\bar{j}}\bar{K}^{\bar{j}}_{\bar{m}}\chi^{\bar{m}}_{\perp 0}\chi^i_{\perp 0} \\
&- s\delta_{i\bar{j}}\bar{K}^{\bar{j}}_{\bar{m}}\sum_{n>0}\chi'^{\bar{m}}_{\perp n}\chi'^i_{\perp n} - sR_{\iota\bar{j}\mu\bar{\nu}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp 0}\phi^{\mu}_{\|0}\chi^{\bar{\nu}}_{\|0}\chi^{\iota}_{\|0} - sR_{\iota\bar{j}\mu\bar{l}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp 0}\phi^{\mu}_{\|0}\chi^{\bar{l}}_{\perp 0}\chi^{\iota}_{\|0} \\
&- sR_{i\bar{j}\mu\bar{\nu}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp 0}\phi^{\mu}_{\|0}\chi^{\bar{\nu}}_{\|0}\chi^{i}_{\perp 0} - sR_{i\bar{j}\mu\bar{l}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp 0}\phi^{\mu}_{\|0}\chi^{\bar{l}}_{\perp 0}\chi^{i}_{\perp 0} \\
&- sR_{\iota\bar{j}l\bar{\mu}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp 0}\phi^{l}_{\perp 0}\chi^{\bar{\mu}}_{\|0}\chi^{\iota}_{\|0} - sR_{\iota\bar{j}l\bar{n}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp 0}\phi^{l}_{\perp 0}\chi^{\bar{n}}_{\perp 0}\chi^{\iota}_{\|0} \\
&- sR_{i\bar{j}l\bar{\mu}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp 0}\phi^{l}_{\perp 0}\chi^{\bar{\mu}}_{\|0}\chi^{i}_{\perp 0} - sR_{i\bar{j}l\bar{n}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp 0}\phi^{l}_{\perp 0}\chi^{\bar{n}}_{\perp 0}\chi^{i}_{\perp 0}.
\end{aligned}
$$

From the above result, oscillation mode part is the same as the one in the previous subsection. So, new things that we have to consider is integration of zero mode part. We summarize the result of expansion of $L+V$ in the following form.

$$
L + V = L_0 + L'_{\|} + L'_{\perp}, \tag{4.183}
$$

$$
\begin{aligned}
L_0 := t\Big[&s^2\beta\delta_{i\bar{j}}K^i_m\bar{K}^{\bar{j}}_{\bar{l}}\phi^m_{\perp 0}\phi^{\bar{l}}_{\perp 0} - s\delta_{i\bar{j}}\bar{K}^{\bar{j}}_{\bar{m}}\chi^{\bar{m}}_{\perp 0}\chi^i_{\perp 0} \\
&- sR_{\iota\bar{j}\mu\bar{\nu}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp 0}\phi^{\mu}_{\|0}\chi^{\bar{\nu}}_{\|0}\chi^{\iota}_{\|0} - sR_{\iota\bar{j}\mu\bar{l}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp 0}\phi^{\mu}_{\|0}\chi^{\bar{l}}_{\perp 0}\chi^{\iota}_{\|0} \\
&- sR_{i\bar{j}\mu\bar{\nu}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp 0}\phi^{\mu}_{\|0}\chi^{\bar{\nu}}_{\|0}\chi^{i}_{\perp 0} - sR_{i\bar{j}\mu\bar{l}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp 0}\phi^{\mu}_{\|0}\chi^{\bar{l}}_{\perp 0}\chi^{i}_{\perp 0} \\
&- sR_{\iota\bar{j}l\bar{\mu}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp 0}\phi^{l}_{\perp 0}\chi^{\bar{\mu}}_{\|0}\chi^{\iota}_{\|0} - sR_{\iota\bar{j}l\bar{n}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp 0}\phi^{l}_{\perp 0}\chi^{\bar{n}}_{\perp 0}\chi^{\iota}_{\|0} \\
&- sR_{i\bar{j}l\bar{\mu}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp 0}\phi^{l}_{\perp 0}\chi^{\bar{\mu}}_{\|0}\chi^{i}_{\perp 0} - sR_{i\bar{j}l\bar{n}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\bar{m}}_{\perp 0}\phi^{l}_{\perp 0}\chi^{\bar{n}}_{\perp 0}\chi^{i}_{\perp 0}\Big],
\end{aligned} \tag{4.184}
$$

$$
L'_{\|} := \sum_{n>0}\Big[tE_n\delta_{\iota\bar{\tau}}\phi'^{\iota}_{\|n}\phi'^{\bar{\tau}}_{\|n} + \sqrt{tE_n}i\delta_{\iota\bar{\tau}}\psi'^{\bar{\tau}}_{\|n}\chi'^{\iota}_{\|n} + \sqrt{tE_n}i\delta_{\iota\bar{\tau}}\psi'^{\iota}_{\|n}\chi'^{\bar{\tau}}_{\|n}\Big], \tag{4.185}
$$

$$
\begin{aligned}
L'_{\perp} := \sum_{n>0}\Big\{&tE_n\delta_{i\bar{j}}\phi'^{i}_{\perp n}\phi'^{\bar{j}}_{\perp n} + \sqrt{tE_n}i\delta_{i\bar{j}}\psi'^{\bar{j}}_{\perp n}\chi'^{i}_{\perp n} + \sqrt{tE_n}i\delta_{i\bar{j}}\psi'^{i}_{\perp n}\chi'^{\bar{j}}_{\perp n} \\
&+ t\beta s^2\delta_{i\bar{j}}K^i_{\mu}\bar{K}^{\bar{j}}_{\bar{\nu}}\phi'^{\mu}_{\perp n}\phi'^{\bar{\nu}}_{\perp n} - s\delta_{i\bar{j}}\bar{K}^{\bar{j}}_{\bar{\mu}}\chi'^{\bar{\mu}}_{\perp n}\chi'^{i}_{\perp n} - s\beta\delta_{i\bar{j}}K^i_{\mu}\psi'^{\mu}_{\perp n}\psi'^{\bar{j}}_{\perp n}\Big\}.
\end{aligned} \tag{4.186}
$$

4.6.3 Evaluation of $\lim_{s\to\infty} < \varphi >_0$

First, we decompose L_0 into $L_{\perp 0} + L_{\|0}$. For this purpose, we transform variables in the following way $(n > 0)$.

$$
\begin{aligned}
&\phi^{\iota}_{\|0} = \phi''^{\iota}_{\|0}, \qquad \phi^{\bar{\iota}}_{\|0} = \phi''^{\bar{\iota}}_{\|0}, \qquad \phi^{i}_{\perp 0} = \frac{1}{s}\phi''^{i}_{\perp 0}, \qquad \phi^{\bar{i}}_{\perp 0} = \frac{1}{s}\phi''^{\bar{i}}_{\perp 0}, \\
&\chi^{\iota}_{\|0} = \sqrt{s}\chi''^{\iota}_{0}, \qquad \chi^{\bar{\iota}}_{\|0} = \sqrt{s}\chi''^{\bar{\iota}}_{0}, \qquad \chi^{i}_{\perp 0} = \frac{1}{\sqrt{s}}\chi''^{i}_{\perp 0}, \qquad \chi^{\bar{i}}_{\perp 0} = \frac{1}{\sqrt{s}}\chi''^{\bar{i}}_{\perp 0}, \\
&\phi'^{\iota}_{\|n} = \phi'''^{\iota}_{\|n}, \qquad \phi'^{\bar{\iota}}_{\|n} = \phi'''^{\bar{\iota}}_{\|n}, \qquad \phi'^{i}_{\perp n} = \frac{1}{s}\phi'''^{i}_{\perp n}, \qquad \phi'^{\bar{i}}_{\perp n} = \frac{1}{s}\phi''^{\bar{i}}_{\perp 0},
\end{aligned} \tag{4.187}
$$

$$
\begin{aligned}
&\chi'^{\iota}_{\|n} = \chi'''^{\iota}_{\|n}, \qquad &&\chi'^{\bar{\iota}}_{\|n} = \chi'''^{\bar{\iota}}_{\|n}, \qquad &&\chi'^{i}_{\perp n} = \frac{1}{\sqrt{s}}\chi''^{i}_{\perp 0}, \qquad &&\chi'^{\bar{i}}_{\perp n} = \frac{1}{\sqrt{s}}\chi''^{\bar{i}}_{\perp n}, \\
&\psi'^{\iota}_{\|n} = \psi'''^{\iota}_{\|n}, \qquad &&\psi'^{\bar{\iota}}_{\|n} = \psi'''^{\bar{\iota}}_{\|n}, \qquad &&\psi'^{i}_{\perp n} = \frac{1}{\sqrt{s}}\psi'''^{i}_{\perp n}, \qquad &&\psi'^{\bar{i}}_{\perp n} = \frac{1}{\sqrt{s}}\psi''^{\bar{i}}_{\perp 0}.
\end{aligned} \tag{4.188}
$$

Let us focus on the measure:

$$
d\phi_{\|0} d\chi_{\|0} = d\phi^{\nu+1}_{\|0} d\phi^{\overline{\nu+1}}_{\|0} \cdots d\phi^{m}_{\|0} d\phi^{\overline{m}}_{\|0} d\chi^{\nu+1}_{\|0} d\chi^{\overline{\nu+1}}_{\|0} \cdots d\chi^{m}_{\|0} d\chi^{\overline{m}}_{\|0}.
$$

Transformation rule of the measure is given by Berezinian as follows.

$$
d\phi_{\|0} d\chi_{\|0} = \frac{1}{s^{m-\nu}} d\phi''_{\|0} d\chi''_{\|0}.
$$

As for the measure:

$$
d\phi_{\perp 0} d\chi_{\perp 0} = d\phi^{1}_{\perp 0} d\phi^{\bar{1}}_{\perp 0} \cdots d\phi^{\nu}_{\perp 0} d\phi^{\bar{\nu}}_{\perp 0} d\chi^{1}_{\perp 0} d\chi^{\bar{1}}_{\perp 0} \cdots d\chi^{\nu}_{\perp 0} d\chi^{\bar{\nu}}_{\perp 0},
$$

transformation rule is given by

$$
d\phi_{\perp 0} d\chi_{\perp 0} = \frac{1}{s^{\nu}} d\phi''_{\perp 0} d\chi''_{\perp 0}.
$$

Integral measures of oscillation modes are defined as follows.

$$
\begin{aligned}
\mathfrak{D}\phi'_{\|}\mathfrak{D}\chi'_{\|}\mathfrak{D}\psi'_{\|} = \prod_{n>0} \frac{1}{(2\pi i)^{m-\nu}} & d\phi'^{\nu+1}_{\|n} d\phi'^{\overline{\nu+1}}_{\|n} \cdots d\phi'^{m}_{\|n} d\phi'^{\overline{m}}_{\|n} d\chi'^{\nu+1}_{\|n} d\chi'^{\overline{\nu+1}}_{\|n} \cdots d\chi'^{m}_{\|n} d\chi'^{\overline{m}}_{\|n} \\
& \times d\psi'^{\nu+1}_{\|n} d\psi'^{\overline{\nu+1}}_{\|n} \cdots d\psi'^{m}_{\|n} d\psi'^{\overline{m}}_{\|n},
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{D}\phi'_{\perp}\mathfrak{D}\chi'_{\perp}\mathfrak{D}\psi'_{\perp} = \prod_{n>0} \frac{1}{(2\pi i)^{\nu}} & d\phi'^{1}_{\perp n} d\phi'^{\bar{1}}_{\perp n} \cdots d\phi'^{\nu}_{\perp n} d\phi'^{\bar{\nu}}_{\perp n} d\chi'^{1}_{\perp n} d\chi'^{\bar{1}}_{\perp n} \cdots d\chi'^{\nu}_{\perp n} d\chi'^{\bar{\nu}}_{\perp n} \\
& \times d\psi'^{1}_{\perp n} d\psi'^{\bar{1}}_{\perp n} \cdots d\psi'^{\nu}_{\perp n} d\psi'^{\bar{\nu}}_{\perp n}.
\end{aligned}
$$

These are invariant under the transformation.

$$
\mathfrak{D}\phi'_{\|}\mathfrak{D}\chi'_{\|}\mathfrak{D}\psi'_{\|} = \mathfrak{D}\phi'''_{\|}\mathfrak{D}\chi'''_{\|}\mathfrak{D}\psi'''_{\|},
$$

$$
\mathfrak{D}\phi'_{\perp}\mathfrak{D}\chi'_{\perp}\mathfrak{D}\psi'_{\perp} = \mathfrak{D}\phi'''_{\perp}\mathfrak{D}\chi'''_{\perp}\mathfrak{D}\psi'''_{\perp}.
$$

In sum, transformation of the whole integral measure is given by

$$
\begin{aligned}
&\int_N \mathfrak{D}\phi_{\|0}\mathfrak{D}\chi_{\|0} \int \mathfrak{D}\phi'_{\|}\mathfrak{D}\chi'_{\|}\mathfrak{D}\psi'_{\|} \int_{N_\perp} \mathfrak{D}\phi_{\perp 0}\mathfrak{D}\chi_{\perp 0} \int \mathfrak{D}\phi'_{\perp}\mathfrak{D}\chi'_{\perp}\mathfrak{D}\psi'_{\perp} \\
&= \frac{1}{s^m} \int_N d\phi''_{\|0} d\chi''_{\|0} \int_{N_\perp} d\phi''_{\perp 0} d\chi''_{\perp 0} \int \mathfrak{D}\phi'''_{\|}\mathfrak{D}\chi'''_{\|}\mathfrak{D}\psi'''_{\|} \int \mathfrak{D}\phi'''_{\perp}\mathfrak{D}\chi'''_{\perp}\mathfrak{D}\psi'''_{\perp}.
\end{aligned}
$$

Let us consider φ at $p \in N$. By using the above variables, A_a^b is expanded in the following form:

$$A_a^b = s\Lambda_a^b + \frac{i}{2\pi}F|_{p}{}^{b}_{aI\bar{J}}\chi^I\chi^J = s(\Lambda_a^b + \frac{i}{2\pi}F^b_{a\nu\bar{\mu}}\chi_0^{\prime\prime\nu}\chi_0^{\prime\prime\bar{\mu}}) + \sqrt{s}\{\cdots\} + \cdots + s^{-1}\{\cdots\}$$

Since we neglect the third and higher terms that contain oscillation modes, $F|_p$ does not depend on $\phi_\perp$. Then we expand $\varphi(p)$ in the form: $\varphi(p) = \sum_{k=-2m}^{2m} s^{\frac{k}{2}}\varphi_k(p)$. Note that $\varphi_{2m}(p)$ is written as $\varphi(\Lambda_a^b + \frac{i}{2\pi}F^b_{a\nu\bar{\mu}}\chi_0^{\prime\prime\nu}\chi_0^{\prime\prime\bar{\mu}})$. We then look back at the expansion of the potential term. Since we neglect terms that have negative powers in s, it is represented as follows.

$$\begin{aligned} L_0 = t\Big\{&\beta\delta_{i\bar{j}}K^i_m\bar{K}^{\bar{j}}_{\bar{l}}\phi^{\prime\prime m}_{\perp 0}\phi^{\prime\prime\bar{l}}_{\perp 0} - \delta_{i\bar{j}}\bar{K}^{\bar{j}}_{\bar{m}}\chi^{\prime\prime\bar{m}}_{\perp 0}\chi^{\prime\prime i}_{\perp 0} \\ &- sR_{\iota\bar{j}\mu\bar{\nu}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\prime\prime\bar{m}}_{\perp 0}\phi^{\prime\prime\mu}_{\|0}\chi^{\prime\prime\bar{\nu}}_{\|0}\chi^{\prime\prime\iota}_{\|0} - R_{\iota\bar{j}\mu\bar{l}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\prime\prime\bar{m}}_{\perp 0}\phi^{\prime\prime\mu}_{\|0}\chi^{\prime\prime\bar{l}}_{\perp 0}\chi^{\prime\prime\iota}_{\|0} \\ &- R_{i\bar{j}\mu\bar{\nu}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\prime\prime\bar{m}}_{\perp 0}\phi^{\prime\prime\mu}_{\|0}\chi^{\prime\prime\bar{\nu}}_{\|0}\chi^{i}_{\perp 0} - R_{\iota\bar{j}l\bar{\mu}}\bar{K}^{\bar{j}}_{\bar{m}}\phi^{\prime\prime\bar{m}}_{\perp 0}\phi^{\prime\prime l}_{\perp 0}\chi^{\prime\prime\bar{\mu}}_{\|0}\chi^{\prime\prime\iota}_{\|0}\Big\}, \end{aligned}$$

$$L'_{\|} = \sum_{n>0}\Big[tE_n\delta_{\iota\bar{\tau}}\phi^{\prime\prime\prime\iota}_{\|n}\phi^{\prime\prime\prime\bar{\tau}}_{\|n} + \sqrt{tE_n}i\delta_{\iota\bar{\tau}}\psi^{\prime\prime\prime\bar{\tau}}_{\|n}\chi^{\prime\prime\prime\iota}_{\|n} + \sqrt{tE_n}i\delta_{\iota\bar{\tau}}\psi^{\prime\prime\prime\iota}_{\|n}\chi^{\prime\prime\prime\bar{\tau}}_{\|n}\Big], \tag{4.189}$$

$$L'_{\perp} = \sum_{n>0}\Big[t\beta\delta_{i\bar{j}}K^i_m\bar{K}^{\bar{j}}_{\bar{l}}\phi^{\prime\prime\prime m}_{\perp n}\phi^{\prime\prime\prime\bar{l}}_{\perp n} - t\delta_{i\bar{j}}\bar{K}^{\bar{j}}_{\bar{m}}\chi^{\prime\prime\prime\bar{m}}_{\perp n}\chi^{\prime i}_{\perp n} - \beta\delta_{i\bar{j}}K^i_m\psi^{\prime\prime\prime m}_{\perp n}\psi^{\prime\prime\prime\bar{j}}_{\perp n}\Big]. \tag{4.190}$$

Let us evaluate the contribution from the component N to $\lim_{s\to\infty} < \varphi >_0$. First, we integrate oscillation modes in $\|$-part. Since each $\varphi_k(p)$ does not contain $\psi'''_{\|}$ and $\phi'''_{\|}$, we have only to perform simple Gaussian integral.

$$\begin{aligned} &\int \mathfrak{D}\phi'''_{\|}\mathfrak{D}\chi'''_{\|}\mathfrak{D}\psi'''_{\|}\varphi(p)e^{-L'_{\|}} \\ &= \varphi'(p)\int \mathfrak{D}\phi'''_{\|}\mathfrak{D}\chi'''_{\|}\mathfrak{D}\psi'''_{\|}\exp\Big\{-\sum_{n>0}\Big[tE_n\delta_{\iota\bar{\tau}}\phi^{\prime\prime\prime\iota}_{\|n}\phi^{\prime\prime\prime\bar{\tau}}_{\|n} + \sqrt{tE_n}i\delta_{\iota\bar{\tau}}\psi^{\prime\prime\prime\bar{\tau}}_{\|n}\chi^{\prime\prime\prime\iota}_{\|n} + \sqrt{tE_n}i\delta_{\iota\bar{\tau}}\psi^{\prime\prime\prime\iota}_{\|n}\chi^{\prime\prime\prime\bar{\tau}}_{\|n}\Big]\Big\} \\ &= \varphi'(p)\prod_{n>0}\frac{1}{(2\pi i)^{m-\nu}}\int d\phi'''_{\|n}d\chi'''_{\|n}d\psi'''_{\|n} \\ &\times \exp\Big\{-\Big[tE_n\delta_{\iota\bar{\tau}}\phi^{\prime\prime\prime\iota}_{\|n}\phi^{\prime\prime\prime\bar{\tau}}_{\|n} + \sqrt{tE_n}i\delta_{\iota\bar{\tau}}\psi^{\prime\prime\prime\bar{\tau}}_{\|n}\chi^{\prime\prime\prime\iota}_{\|n} + \sqrt{tE_n}i\delta_{\iota\bar{\tau}}\psi^{\prime\prime\prime\iota}_{\|n}\chi^{\prime\prime\prime\bar{\tau}}_{\|n}\Big]\Big\} \\ &= \varphi'(p)\prod_{n>0}\Big(\frac{tE_n}{tE_n}\Big)^{m-\nu} = \varphi'(p). \end{aligned}$$

We mean by $\varphi'(p)$ the operator obtained from removing $\chi'''_{\|}$ from $\varphi(p)$. Next, we integrate oscillation modes in $\perp$-part. We expand $\varphi'(p) = \sum_{k=-2m}^{2m} s^{\frac{k}{2}}\varphi'_k(p)$, $(\varphi_m(p) =$

$\varphi'_{2m}(p) = \varphi(\Lambda^b_a + \frac{i}{2\pi} F^b_{a\nu\bar{\mu}} \chi''^{\nu}_0 \chi''^{\bar{\mu}}_0))$. Then we obtain

$$
\begin{aligned}
&\int \mathfrak{D}\phi'''_\perp \mathfrak{D}\chi'''_\perp \mathfrak{D}\psi'''_\perp \varphi'(p) \exp(-L'_\perp) \\
&= \int \mathfrak{D}\phi'''_\perp \mathfrak{D}\chi'''_\perp \mathfrak{D}\psi'''_\perp \varphi'(p) \exp\Big\{-\sum_{n>0}\Big\{t\beta\delta_{i\bar{j}} K^i_m \bar{K}^{\bar{j}}_{\bar{l}} \phi'''^m_{\perp n} \phi'''^{\bar{l}}_{\perp n} - t\delta_{i\bar{j}} \bar{K}^{\bar{j}}_{\bar{m}} \chi'''^{\bar{m}}_{\perp n} \chi'^i_{\perp n} \\
&- \beta\delta_{i\bar{j}} K^i_m \psi'''^m_{\perp n} \psi'''^{\bar{j}}_{\perp n}\Big\}\Big\} \\
&= s^m \varphi'_{2m}(p) \prod_{n>0} \frac{1}{(2\pi i)^\nu} \int d\phi'''_{\perp n} \exp\Big\{-t\beta\delta_{i\bar{j}} K^i_m \bar{K}^{\bar{j}}_{\bar{l}} \phi'''^m_{\perp n} \phi'''^{\bar{l}}_{\perp n}\Big\} \\
&\times \int d\chi'''_{\perp n} d\psi'''_{\perp n} \exp\Big\{t\delta_{i\bar{j}} \bar{K}^{\bar{j}}_{\bar{m}} \chi'''^{\bar{m}}_{\perp n} \chi'^i_{\perp n} + \beta\delta_{i\bar{j}} K^i_m \psi'''^m_{\perp n} \psi'''^{\bar{j}}_{\perp n}\Big\} \\
&= s^m \varphi'_{2m}(p) \prod_{n>0} \Big(\frac{\det\big(\delta_{i\bar{j}} \bar{K}^{\bar{j}}_{\bar{m}}\big) \det\big(\delta_{i\bar{j}} K^i_l\big)}{\det\big(\delta_{i\bar{j}} K^i_m \bar{K}^{\bar{j}}_{\bar{l}}\big)} \Big) + (\text{terms of lower power in } s) \\
&= s^m \varphi'_{2m}(p) + (\text{terms of lower power in } s).
\end{aligned}
$$

At this stage, we can represent the contribution from N in the following form.

$$
\int_{N_\alpha} d\phi''_{\|0} d\chi''_{\|0} \varphi'_{2m}(p) \int_{N_{\perp\alpha}} d\phi''_{\perp 0} d\chi''_{\perp 0} \; \exp\{-L_0\} + (\text{terms of lower power in } s).
$$

Finally, we integrate out zero modes in $\perp$-part.

$$
\begin{aligned}
&\int_{N_\perp} d\phi''_{\perp 0} d\chi''_{\perp 0} \; \exp\{-L_0\} \\
&= \int_{N_\perp} d\phi''_{\perp 0} \exp\Big[-t\phi''^m_{\perp 0}\Big\{\beta\delta_{i\bar{j}} K^i_m \bar{K}^{\bar{j}}_{\bar{l}} - R_{\iota\bar{j}m\bar{\mu}} \bar{K}^{\bar{j}}_{\bar{l}} \chi''^{\bar{\mu}}_{\|0} \chi''^{\iota}_{\|0}\Big\} \phi''^{\bar{l}}_{\perp 0} + ts R_{\iota\bar{j}\mu\bar{\nu}} \bar{K}^{\bar{j}}_{\bar{m}} \phi''^{\bar{m}}_{\perp 0} \phi''^{\mu}_{\|0} \chi''^{\bar{\nu}}_{\|0} \chi''^{\iota}_{\|0}\Big] \\
&\times \int d\chi''_{\perp 0} \exp\Big[t\Big\{\delta_{i\bar{j}} \bar{K}^{\bar{j}}_{\bar{m}} \chi''^{\bar{m}}_{\perp 0} \chi''^i_{\perp 0} + R_{\iota\bar{j}\mu\bar{l}} \bar{K}^{\bar{j}}_{\bar{m}} \phi''^{\bar{m}}_{\perp 0} \phi''^{\mu}_{\|0} \chi''^{\bar{l}}_{\perp 0} \chi''^{\iota}_{\|0} + R_{i\bar{j}\mu\bar{\nu}} \bar{K}^{\bar{j}}_{\bar{m}} \phi''^{\bar{m}}_{\perp 0} \phi''^{\mu}_{\|0} \chi''^{\bar{\nu}}_{\|0} \chi^i_{\perp 0}\Big\}\Big] \\
&= \int_{N_\perp} d\phi''_{\perp 0} \exp\Big[-t\phi''^m_{\perp 0}\Big\{\beta\delta_{i\bar{j}} K^i_m \bar{K}^{\bar{j}}_{\bar{l}} - R_{\iota\bar{j}m\bar{\mu}} \bar{K}^{\bar{j}}_{\bar{l}} \chi''^{\bar{\mu}}_{\|0} \chi''^{\iota}_{\|0}\Big\} \phi''^{\bar{l}}_{\perp 0} + ts R_{\iota\bar{j}\mu\bar{\nu}} \bar{K}^{\bar{j}}_{\bar{m}} \phi''^{\bar{m}}_{\perp 0} \phi''^{\mu}_{\|0} \chi''^{\bar{\nu}}_{\|0} \chi''^{\iota}_{\|0}\Big] \\
&\times \Big[(-t)^\nu \det\big(\delta_{i\bar{j}} \bar{K}^{\bar{j}}_{\bar{m}}\big) + \cdots \\
&+ \int d\chi''_{\perp 0} \exp\big(t R_{\iota\bar{j}\mu\bar{l}} \bar{K}^{\bar{j}}_{\bar{m}} \phi''^{\bar{m}}_{\perp 0} \phi''^{\mu}_{\|0} \chi''^{\bar{l}}_{\perp 0} \chi''^{\iota}_{\|0} + t R_{i\bar{j}\mu\bar{\nu}} \bar{K}^{\bar{j}}_{\bar{m}} \phi''^{\bar{m}}_{\perp 0} \phi''^{\mu}_{\|0} \chi''^{\bar{\nu}}_{\|0} \chi^i_{\perp 0}\big)\Big].
\end{aligned}
$$

Then we preform Gaussian integral of $\exp\Big(-t\phi''^m_{\perp 0}\Big\{\beta\delta_{i\bar{j}} K^i_m \bar{K}^{\bar{j}}_{\bar{l}} - R_{\iota\bar{j}m\bar{\mu}} \bar{K}^{\bar{j}}_{\bar{l}} \chi''^{\bar{\mu}}_{\|0} \chi''^{\iota}_{\|0}\Big\} \phi''^{\bar{l}}_{\perp 0}\Big)$. Note that terms except for $(t)^\nu \det\big(\delta_{i\bar{j}} \bar{K}^{\bar{j}}_{\bar{m}}\big)$ include fermionic variables. Hence by expanding exponential, we only have to consider polynomial correlation function of $\phi^*_{\perp 0}$ for these terms. But the matrix $\Big\{\beta\delta_{i\bar{j}} K^i_m \bar{K}^{\bar{j}}_{\bar{l}} - R_{\iota\bar{j}m\bar{\mu}} \bar{K}^{\bar{j}}_{\bar{l}} \chi''^{\bar{\mu}}_{\|0} \chi''^{\iota}_{\|0}\Big\}$ takes the form $A_{i\bar{j}}$, and these correlation function all vanishes because they only have anti-holomorphic variable $\phi^{\bar{m}}_{\perp 0}$. As a result, we obtain

$$\int_{N_\perp} d\phi''_{\perp 0} d\chi''_{\perp 0}\ \exp\{-L_0\} = \Big(\frac{-2\pi i}{t}\Big)^\nu \frac{(-t)^\nu \det\{\delta_{i\bar{j}}\bar{K}^{\bar{j}}_{\bar{m}}\}}{\det\{\beta\delta_{i\bar{j}}K^i_m\bar{K}^{\bar{j}}_{\bar{l}} - R_{\iota\bar{j}m\bar{\mu}}\bar{K}^{\bar{j}}_{\bar{l}}\chi''^{\bar{\mu}}_{\|0}\chi''^{\iota}_{\|0}\}}$$

$$= \frac{(2\pi i)^\nu \det\{\delta_{i\bar{j}}\bar{K}^{\bar{j}}_{\bar{m}}\}}{(\beta)^\nu \det\{\delta_{i\bar{j}}K^i_m\bar{K}^{\bar{j}}_{\bar{l}} + \frac{i}{2\pi}R_{\iota\bar{j}m\bar{\mu}}\bar{K}^{\bar{j}}_{\bar{l}}\chi''^{\bar{\mu}}_{\|0}\chi''^{\iota}_{\|0}\}}.$$

From $R_{\iota\bar{j}m\bar{\mu}} = R_{m\bar{j}\iota\bar{\mu}} = -\delta_{i\bar{j}}R^i_{m\iota\bar{\mu}}$,

$$\int_{N_\perp} d\phi''_{\perp 0} d\chi''_{\perp 0}\ \exp\{-L_0\} = \frac{1}{\det\{K^i_m + \frac{i}{2\pi}R^i_{m\iota\bar{\mu}}\chi''^{\iota}_{\|0}\chi''^{\bar{\mu}}_{\|0}\}}.$$

We remark $\varphi'_{2m}(p) = \varphi(\Lambda^b_a + \frac{i}{2\pi}F^b_{a\nu\bar{\mu}}\chi''^{\nu}_0\chi''^{\bar{\mu}}_0) = \underline{\varphi}(\Lambda_\alpha)$. By adding up contributions from all the connected components, we obtain the correlation function in the following form.

$$\begin{aligned}\lim_{s\to\infty} <\varphi>_0 &= \lim_{s\to\infty}\sum_\alpha\Big[\int_{N_\alpha} d\phi''_{\|0}d\chi''_{\|0}\frac{\underline{\varphi}(\Lambda_\alpha)}{\det\{K^i_m + \frac{i}{2\pi}R^i_{m\iota\bar{\mu}}\chi''^{\iota}_{\|0}\chi''^{\bar{\mu}}_{\|0}\}} \\ &\qquad + (\text{terms of negative power in } s)\Big] \\ &= \sum_\alpha\int_{N_\alpha} d\phi''_{\|0}d\chi''_{\|0}\frac{\underline{\varphi}(\Lambda_\alpha)}{\det\{K^i_m + \frac{i}{2\pi}R^i_{m\iota\bar{\mu}}\chi''^{\iota}_{\|0}\chi''^{\bar{\mu}}_{\|0}\}}.\end{aligned}$$

Lastly, we rewrite the determinant in denominator. K^i_j corresponds to the map $\theta^\nu|_N : T'M|_N/T'N \to T'M|_N/T'N$. and $R^i_{l\iota\bar{\mu}}\chi'^{\iota}_0\chi'^{\bar{\mu}}_0$ is nothing but the curvature $(1,1)$-form of $T'M|_N/T'N$ ($R^i_l = R^\nu_\alpha$). As a result, we can rewrite the above result into the form:

$$\lim_{s\to\infty} <\varphi>_0 = \sum_\alpha\int_{N_\alpha}\frac{\underline{\varphi}(\Lambda_\alpha)}{\det\{\theta^\nu_\alpha + \frac{i}{2\pi}R^\nu_\alpha\}}.$$

By combining the above result with Proposition 1, we finally obtain the Bott residue formula:

$$\underline{\varphi}(E)[M] = \sum_\alpha\int_{N_\alpha}\frac{\underline{\varphi}(\Lambda_\alpha)}{\det\{\theta^\nu_\alpha + \frac{i}{2\pi}R^\nu_\alpha\}}.$$

Appendix of Part I (Proof of Proposition 1)

We prove the correlation function is independent of parameter s. The basic idea comes from [38] and [18]. Sigma model has two charge for fermion F_A and F_V. These charge acts on the operator $\mathcal{O}_\omega$ obtained from (p,q)-form ω as follows.

$$F_A\mathcal{O}_\omega = (p+q)\mathcal{O}_\omega, \qquad F_V\mathcal{O}_\omega = (-p+q)\mathcal{O}_\omega. \tag{4.191}$$

The symmetry F_A is broken by the potential term. So, observables are graded by F_V. However, since F_A is counting total degree of differential forms, we can use it for taking conjugation of operators (this idea was used in the discussion on Landau-Ginzburg model in [18]). Let us consider $e^{\lambda F_A}$ ($\lambda \in \mathbb{R}$). Then $e^{-\lambda F_A} Q_s e^{\lambda F_A}$ is evaluated as follows.

$$e^{-\lambda F_A} Q_s e^{\lambda F_A} \mathcal{O}_\omega = e^{-\lambda} Q_{se^{2\lambda}} \mathcal{O}_\omega. \tag{4.192}$$

Next, we focus on the observable φ_s. We decompose observable φ_s into $\varphi_s = \sum_{k=0}^{m} s^k \varphi_{m-k}$. Since φ_{m-k} corresponds to $(m-k, m-k)$-form, we can compute $e^{-\lambda F_A} \varphi_s e^{\lambda F_A} \mathcal{O}_\omega$.

$$e^{-\lambda F_A} s^k \varphi_{m-k} e^{\lambda F_A} \mathcal{O}_\omega = e^{-2m\lambda} (se^{2\lambda})^k \varphi_{m-k} \mathcal{O}_\omega. \tag{4.193}$$

Hence we obtain

$$e^{-\lambda F_A} \varphi_s e^{\lambda F_A} = e^{-2m\lambda} \varphi_{se^{2\lambda}}. \tag{4.194}$$

Let us introduce vacuum vector $|0>$ and its dual $<0|$. Then we can represent the correlation function $<\varphi_s>$ as $<0|\varphi_s|0>$. By using the relation (4.194), we obtain

$$\begin{aligned} <\varphi_s> &= <0|\varphi_s|0>=<0|e^{\lambda F_A} e^{-\lambda F_A} \varphi_s e^{\lambda F_A} e^{-\lambda F_A}|0> \\ &= <0|e^{\lambda F_A} \varphi_{se^{2\lambda}} e^{-\lambda F_A}|0> e^{-2m\lambda}. \end{aligned}$$

Since our theory has $2m$ fermion zero modes χ_0^i and $\chi_0^{\bar{i}}$ ($i = 1, \cdots, m$), it is anomalous. Therefore, if we assign $|0>$ charge $(0,0)$, we have to assign $<0|$ charge (m,m). Therefore, we have $F_A|0>=0$ and $<0|F_A = 2m<0|$. Hence we obtain

$$\begin{aligned} <\varphi_s> &= <0|e^{\lambda F_A} \varphi_{se^{2\lambda}} e^{-\lambda F_A}|0> e^{-2m\lambda} \\ &= e^{2m\lambda} <0|\varphi_{se^{2\lambda}}|0> e^{-2m\lambda} \\ &= <0|\varphi_{se^{2\lambda}}|0> \\ &= <\varphi_{se^{2\lambda}}> . \end{aligned}$$

This completes proof of the proposition.

Part II

Evaluation of Euler Number of Complex Grassmann Manifold $G(k, N)$ via Mathai-Quillen Formalism and Schubert Calculus by Fermionic variables

This part is an edited and reprinted version of the contents of [20] and [28].

1 Main result of Part II

1.1 Cohomology Ring of Complex Grassmann manifold

In this section, we introduce some fundamental results of the cohomology ring. Complex Grassmann manifold $G(k, N)$ is a space which parametrizes k-dimensional linear subspaces of N-dimensional complex vector space. First, let us introduce well-known facts on Chern classes of holomorphic tangent bundle $T'G(k, N)$ of $G(k, N)$ and the cohomology ring $G(k, N)$. Let S be the tautological bundle of $G(k, N)$ whose fiber of $\Lambda \in G(k, N)$ is given by complex k-dimensional subspace $\Lambda \subset \mathbb{C}^N$ itself ($\mathrm{rk}(S) = k$). Then universal quotient bundle Q ($\mathrm{rk}Q = N - k$) is defined by the following exact sequence

$$0 \to S \to \mathbb{C}^N \to Q \to 0. \tag{1.1}$$

where $\mathbb{C}^N$ means trivial bundle $G(k, N) \times \mathbb{C}^N$. Since $T'G(k, N)$ can be identified with $Q \otimes S^*$, we obtain the following exact sequence:

$$0 \to S \otimes S^* \to \mathbb{C}^N \otimes S^* \to T'G(k, N) \to 0. \tag{1.2}$$

Let $c(E)$ be a total Chern class of vector bundle E. Hence the total Chern class of $T'G(k, N)$ is given by $\dfrac{(c(S^*))^N}{c(S \otimes S^*)}$ (Euler number of $G(k, N)$ is obtained from integration of top Chern class of $T'G(k, N)$). If we decompose S^* formally by the line bundle L_i^* ($i = 1, 2, \cdots, k$):

$$S^* = \overset{k}{\underset{i=1}{\oplus}} L_i^*, \tag{1.3}$$

$$c(S^*) = 1 + \sum_{n=1}^{k} t^n c_n(S^*) = \prod_{i=1}^{k}(1 + x_i), \quad (x_i := c_1(L_i^*)). \tag{1.4}$$

Here, $c_i(S^*)(i=1,\cdots,k)$ is the i-th Chern class of S^*. Since the relation of the i-th Chern class of a vector bundle E and the one of its dual bundle E^* is $c_i(E^*) = (-1)^i c_i(E)$,

$$\begin{aligned} c(S\otimes S^*) &= c\left(\left(\overset{k}{\underset{i=1}{\oplus}} L_i\right)\otimes\left(\overset{k}{\underset{j=1}{\oplus}} L_j^*\right)\right) = c\left(\overset{k}{\underset{i,j=1}{\oplus}}(L_i\otimes L_j^*)\right) = \prod_{i,j=1}^{k} c(L_i\otimes L_j^*) \\ &= \prod_{i,j=1}^{k}(1+c_1(L_i\otimes L_j^*)) = \prod_{i\neq j}(1+(x_j-x_i)) \\ &= \prod_{i<j}(1+(x_j-x_i))(1+(x_i-x_j)) = \prod_{i>j}(1-(x_i-x_j)^2). \end{aligned} \tag{1.5}$$

Then, we obtain

$$c(T'G(k,N)) = \frac{\prod_{i=1}^{k}(1+x_i)^N}{\prod_{l>j}(1-(x_l-x_j)^2)}. \tag{1.6}$$

Next, we recall the cohomology ring of $G(k,N)$ [9]. The cohomology ring of $G(k,N)$ is given by

$$H^*(G(k,N)) = \frac{\mathbb{R}[c_1(S),\cdots,c_k(S),c_1(Q),\cdots,c_{N-k}(Q)]}{(c(S)c(Q)=1)}. \tag{1.7}$$

To match our theorem, we denote this by $c_i(S^*)(i=1,\cdots,k)$. Since $c_i(E^*) = (-1)^i c_i(E)$, we can take $c_i(S^*)$'s and $c_i(Q^*)$'s as generators of $H^*(G(k,N))$.

$$H^*(G(k,N)) = \frac{\mathbb{R}[c_1(S^*),\cdots,c_k(S^*),c_1(Q^*),\cdots,c_{N-k}(Q^*)]}{(c(S^*)c(Q^*)=1)}. \tag{1.8}$$

$c(S^*)c(Q^*)=1$ is obtained by considering the dual of (1.1). On the other hand, the relation $c(S^*)c(Q^*)=1$ is rewritten by

$$c(Q^*) = \frac{1}{c(S^*)}. \tag{1.9}$$

Then, if we expand $\dfrac{1}{c(S^*)} = 1/(1+c_1(S^*)t+c_2(S^*)t^2+\cdots+c_k(S^*)t^k)$ in powers of t,

$$\frac{1}{1+c_1(S^*)t+c_2(S^*)t^2+\cdots+c_k(S^*)t^k} = \sum_{i=0}^{\infty} a_i t^i, \tag{1.10}$$

from rank$Q = N-k$, we can rewrite (1.9) as follows:

$$c_i(Q^*) = a_i \ \ (i=1,2,\cdots,N-k), \ \ a_i = 0 \ \ (i>N-k). \tag{1.11}$$

Note that a_i is degree i homogeneous polynomial of $c_j(S^*)$'s ($j = 1, 2, \cdots, k$). Hence we can eliminate generators $c_j(Q^*)$'s from (1.8) and obtain another representation of $H^*(G(k, N))$.

$$H^*(G(k, N)) = \frac{\mathbb{R}[c_1(S^*), \cdots, c_k(S^*)]}{(a_i = 0 \ \ (i > N - k))}. \tag{1.12}$$

Lastly, we include the Schubert cycles of $G(k, N)$ [17]. It is the well known fact that the Chern class of $G(k, N)$ corresponds to the Poincaré dual of Schubert cycle. Here, we introduce Schubert cycles. For a more detailed discussion, please refer to [17, 21]. For any flag $V : 0 \subset V_1 \subset V_2 \subset \cdots \subset V_N = \mathbb{C}^N$, the Schubert manifold $\tau_a(V)$ is defined by

$$\tau_a(V) := \{\Lambda \in G(k, N) | \dim(\Lambda \cap V_{N-k+i-a_i}) \geq i (1 \leq i \leq k)\}, \tag{1.13}$$

where $a = (a_1, \cdots, a_k)$ is a sequence of natural number that satisfies $0 \leq a_k \leq a_{k-1} \leq \cdots \leq a_1 \leq N - k$. $\tau_a(V)$ is an analytic subvariety of $G(k, N)$ of codimension $\sum_{l=1}^{k} a_l$. Then, the homology class of $\tau_a(V)$ is independent of the flag chosen. Therefore, let $\tau_a(V)$ as the homology class be denoted by τ_a. Let τ_a^* be the Poincaré dual of the cycle τ_a. And we abbreviate 0 in a. For example, $\tau_{(a_1, a_2, \cdots, a_n, 0, \cdots, 0)}$ is denoted by $\tau_{a_1, a_2, \cdots, a_n}$. From the Gauss-Bonnet theorem,

$$c_i(S^*) = (-1)^i c_i(S) = \tau^*_{\underbrace{1, \cdots, 1}_{i}} =: \tau^*_{1^{(i)}}. \tag{1.14}$$

Therefore, the integral of Chern classes over Grassmann manifold represents the intersection number of Schubert cycles. One of the results of the part II is that we can perform this calculation using only fermionic variables. More details in Subsection 1.3.

1.2 Representation of Euler Number by Our Model (Main Theorem in Part II)

In this section, we introduce Main theorem in this part. It is given in [20]. We denote by $U(k)$ unitary group that acts on complex k-dimensional vector space and by $V_k(\mathbb{C}^N)$ Stiefel manifold of orthonormal k-frames in $\mathbb{C}^N$. Then $G(k, N) \simeq V_k(\mathbb{C}^N)/U(k)$. We introduce detailed informations of $G(k, N)$ in Section 3. By applying Atiyah-Jeffrey construction to the case when a manifold X and a Lie group G are given by $V_k(\mathbb{C}^N)$ and $U(k)$, Euler number of $G(k, N)$ is represented by the following finite-dimensional path integral (unless otherwise noted, we use the Einstein convention).

Theorem 1. (Main Theorem of Part II) *Euler number of $G(k, N)$ is evaluated by finite dimensional path integral:*

$$\begin{aligned}\chi(G(k, N)) &= \binom{N}{k} = \chi(V_k(\mathbb{C}^N)/U(k)) \\ &= \beta \int_{V_k(\mathbb{C}^N)} Dz \int D\psi D\phi D\bar{\phi} DA D\eta D\psi_A D\chi DH \ \omega \exp(-\mathcal{L}_{MQ}),\end{aligned} \tag{1.15}$$

where the Lagrangian $\mathcal{L}_{MQ}$, the projection operator ω and the normalization factor β are given by

$$\mathcal{L}_{MQ} = \delta < \chi, H > + \delta < \psi_A, A > + \frac{i}{2}\delta\Big[< \psi, iz^t\bar{\phi} > + \overline{< \psi, iz^t\bar{\phi} >}\Big] \tag{1.16}$$

$$\begin{aligned} &= \sum_{s=1}^{N}\Big[\sum_{i=1}^{k} H_{\bar{s}}^{\bar{i}} H_s^i - \big(\chi_{\bar{s}}^{\bar{i}}(\delta_{\bar{i}j} + i\phi_{\bar{i}j})\chi_s^j\big) + \psi_{\bar{s}}^{\bar{i}}\bar{\phi}_{\bar{i}l}\psi_s^l\Big] + i\mathrm{tr}(\phi\bar{\phi}) \\ &+ \frac{1}{2}\sum_{s=1}^{N}\Big\{\psi_{\bar{s}}^{\bar{i}} z_s^j - \psi_s^j z_{\bar{s}}^{\bar{i}}\Big\}\eta_{\bar{i}j} + \mathrm{tr}(A^\dagger(A + [i\phi, A]) - \psi_A^\dagger\psi_A)., \end{aligned} \tag{1.17}$$

$$\omega = \prod_{i,j=1}^{k}\Big[\sum_{s=1}^{N}(\psi_s^{\bar{i}} z_s^j + z_s^{\bar{i}}\psi_s^j)\Big], \quad \beta = \frac{\prod_{j=0}^{k-1} j!}{2^{2k}(-\pi)^{k^2+kN}\pi^{kN+\frac{k(k+1)}{2}}(-1)^{\frac{k}{2}(k-1)}}. \tag{1.18}$$

In the above Lagrangian, δ represents supersymmetric transformation whose detailed construction will be explained in Section 2. In this section, we briefly introduce our notations and supersymmetric transformation. Let

$$z := (\boldsymbol{z}^1 \cdots \boldsymbol{z}^k), \tag{1.19}$$

$$\boldsymbol{z}^j := \ {}^t(z_1^j, z_2^j, \cdots, z_N^j) \ \ (j = 1, 2, \cdots, k), \tag{1.20}$$

be local complex coordinate system of $\mathbb{C}^{kN}$. Here, we introduce some notation for our model (this notation is used in this section, Section 3, Section 4, and Section 5).

Definition 1. *Let X and Y be matrix variables used in our Lagrangian.*
(1) We represent complex conjugate of X by $\overline{X}$ or ${}^*X$. We also use the notation $(\overline{X})_i^j =: X_{\bar{i}}^{\bar{j}}$ or $(\overline{X})_{\bar{i}j} =: X_{i\bar{j}}$. In our Lagrangian, ϕ and $\bar{\phi}$ are diferent independent fields, and we use ${}^*\phi$ and ${}^*\bar{\phi}$ to represent the complex conjugate of ϕ and $\bar{\phi}$.
(2) We represent transpose of X (resp. adjoint of X) by ${}^tX$ (resp. $X^\dagger$). ($X^\dagger := {}^{t*}X$).
(3) We define inner product of matrix variables X and Y of the same type by $< X, Y >:= \mathrm{tr}((X^\dagger)Y)$.

Then, $V_k(\mathbb{C}^N)$ is given by a set of points in $\mathbb{C}^{kN}$ that satisfy

$$\sum_{s=1}^{N} z_{\bar{s}}^{\bar{i}} z_s^j - \delta_{ij} = 0 \quad (i, j = 1, 2, \cdots, k), \tag{1.21}$$

where $z_{\bar{s}}^{\bar{i}}$ represents $\overline{z_s^i}$, and δ_{ij} is the Kronecker's delta. ψ's are complex fermionic variables that correspond to super-partner of z:

$$\psi := (\boldsymbol{\psi}^1, \boldsymbol{\psi}^2, \cdots, \boldsymbol{\psi}^k), \qquad \boldsymbol{\psi}^j := \ {}^t(\psi_1^j, \psi_2^j, \cdots, \psi_N^j). \tag{1.22}$$

ϕ and $\bar{\phi}$ are $k \times k$ Hermite matrices ($\phi = \phi^\dagger$, $\bar{\phi} = \bar{\phi}^\dagger$). $i\phi$ and $i\bar{\phi}$ plays the role of generator of Lie algebra of the gauge group $U(k)$. Note that $\bar{\phi}$ is not complex conjugate of ϕ. η is super-partner of ϕ. Hence η is a grassmann and Hermite matrix ($\eta = \eta^\dagger$).

$$\phi := \begin{pmatrix} \phi_{\bar{1}1} & \cdots & \phi_{\bar{1}k} \\ \vdots & \ddots & \vdots \\ \phi_{\bar{k}1} & \cdots & \phi_{\bar{k}k} \end{pmatrix}. \qquad \bar{\phi} := \begin{pmatrix} \bar{\phi}_{\bar{1}1} & \cdots & \bar{\phi}_{\bar{1}k} \\ \vdots & \ddots & \vdots \\ \bar{\phi}_{\bar{k}1} & \cdots & \bar{\phi}_{\bar{k}k} \end{pmatrix}. \tag{1.23}$$

$$\eta := \begin{pmatrix} \eta_{\bar{1}1} & \cdots & \eta_{\bar{1}k} \\ \vdots & \ddots & \vdots \\ \eta_{\bar{k}1} & \cdots & \eta_{\bar{k}k} \end{pmatrix}. \tag{1.24}$$

H is $k \times N$ complex matrix which plays the role of auxiliary variable in MQ-formalism. χ is super-partner of H. A is $k \times k$ complex matrix and ψ_A is super-partner of A.

$$H := \begin{pmatrix} \boldsymbol{H}^1 & \cdots & \boldsymbol{H}^k \end{pmatrix}, \qquad \boldsymbol{H}^j := {}^t(H_1^j, \cdots, H_N^j). \tag{1.25}$$

$$\chi := \begin{pmatrix} \boldsymbol{\chi}^1 & \cdots & \boldsymbol{\chi}^k \end{pmatrix}, \qquad \boldsymbol{\chi}^j := {}^t(\chi_1^j, \cdots, \chi_N^j). \tag{1.26}$$

$$A := \begin{pmatrix} A_{\bar{1}1} & \cdots & A_{\bar{1}k} \\ \vdots & \ddots & \vdots \\ A_{\bar{k}1} & \cdots & A_{\bar{k}k} \end{pmatrix}, \qquad \psi_A := \begin{pmatrix} \psi_{A}{}_1^1 & \cdots & \psi_{A}{}_1^k \\ \vdots & \ddots & \vdots \\ \psi_{A}{}_k^1 & \cdots & \psi_{A}{}_k^k \end{pmatrix}. \tag{1.27}$$

The supersymmetric transformation δ of our model is given as follows.

$$\begin{aligned} &\delta z = \psi,\ \delta\psi = \delta_\phi z = iz^t\phi,\ \delta\chi = H, \delta H = \chi + \delta_\phi\chi = \chi + i\chi^t\phi, \\ &\delta\psi_A = A + \delta_\phi A = A + [A, i\phi],\ \delta A = \psi_A, \delta\phi = 0,\ \delta\bar{\phi} = \eta,\ \delta\eta = \delta_\phi\bar{\phi} = i[\phi, \bar{\phi}]. \end{aligned} \tag{1.28}$$

δ_ϕ and $\delta_{\bar{\phi}}$ are infinitesimal gauge transformation generated by $i\phi$ and $i\bar{\phi}$. In (1.16), the terms Supersymmetric transformation for complex conjugate of X is defined as complex conjugate of δX ($\delta\overline{X} := \overline{(\delta X)}$). δ behaves like a fermionic variable. By applying (1.28) to (1.16), we can obtain explicit form of the Lagrangian (1.17). Note that the z variables satisfy (1.21), the defining equations of the Stiefel manifold $V_k(\mathbb{C}^N)$. On the other hand, ψ, the super-partner of z, plays the role of the 1-form dz in the supersymmetric path integral. Since $(\boldsymbol{z}^i)^\dagger \boldsymbol{z}^j - \delta^{\bar{i}j} = 0$ ($i, j = 1, 2, \cdots, k$), we also have the constraint for dz:

$$\sum_{s=1}^{N} (dz_{\bar{s}}^{\bar{i}} z_s^j + z_{\bar{s}}^{\bar{i}} dz_s^j) = 0 \quad (i, j = 1, 2, \cdots, k). \tag{1.29}$$

By identifying dz_s^i and $dz_{\bar{s}}^{\bar{i}}$ with ψ_s^i and $\psi_{\bar{s}}^{\bar{i}}$. respectively, the above constraint is realized by insertion of the following projection operator ω:

$$\omega = \prod_{i,j=1}^{k} \Big[\sum_{s=1}^{N} (\psi_{\bar{s}}^{\bar{i}} z_s^j + z_{\bar{s}}^{\bar{i}} \psi_s^j) \Big]. \tag{1.30}$$

β is the normalization factor that normalizes volume of $G(k,N)$ into 1.

$$\beta = \frac{\prod_{j=0}^{k-1} j!}{2^{2k}(-\pi)^{k^2+kN}\pi^{kN+\frac{k(k+1)}{2}}(-1)^{\frac{k}{2}(k-1)}}. \tag{1.31}$$

Note that the volume $U(k)$ is given by

$$\mathrm{vol}(U(k)) = \prod_{j=1}^{k} \mathrm{vol}(S^{2j-1}) = \frac{2^k \pi^{\frac{k(k+1)}{2}}}{\prod_{j=1}^{k-1} j!}. \tag{1.32}$$

Then, we can rewrite β into the following form:

$$\beta = \frac{1}{2^k(-\pi)^{k^2+kN}\pi^{kN}(-1)^{\frac{k}{2}(k-1)}\mathrm{vol}(U(k))}. \tag{1.33}$$

Lastly, we remark that the supersymmetry and ghost numbers for each variable. (This topic is based on discussion with Professor Masao Jinzenji.) Atiyah-Jeffrey construction is used to study mechanism behind Witten's construction of the topological Yang-Mills theory [3, 39]. Then, each field in this theory is given a ghost number in [39]. In this part, it is given by $(z, \psi, \chi, H, \phi, \bar{\phi}, \eta) = (0, 1, -1, 0, 2, -2, -1)$ [37]. Supersymmetric transformation also has a ghost number 1. However, since $\delta H = \chi + \delta_\phi \chi$, the ghost number of δH is not fixed. We introduce a bosonic variable c for a ghost number 2. And we modify δH to $\delta H = c\chi + \delta_\phi \chi$. In the same way, $\delta\psi_A$ is modified by $\delta\psi_A = cA + \delta_\phi A$. Then, the ghost number of δ is fixed 1. Related to this, our Lagrangian is not strictly invariant under the supersymmetric transformation, i.e., $\delta\mathcal{L}_{MQ} \neq 0$, because our supersymmetric transformation is not nilpotent. For example, we have the following relation.

$$\begin{aligned}
\delta^2 < \psi_A, A > &= \delta \mathrm{tr}(cA^\dagger(A + [i\phi, A]) - \psi_A^\dagger \psi_A) && (1.34)\\
&= \mathrm{tr}(c\psi_A^\dagger(A + [i\phi, A]) + cA^\dagger(\psi_A + [i\phi, \psi_A]) \\
&- (cA + [A, i\phi])^\dagger \psi_A + \psi_A^\dagger(cA + [A, i\phi])) && (1.35)\\
&= \mathrm{tr}(2c\psi_A^\dagger A + cA^\dagger(\psi_A + [i\phi, \psi_A]) - (cA^\dagger + [A^\dagger, i\phi])\psi_A + (c-1)\psi_A^\dagger[i\phi, A]) \\
&= \mathrm{tr}(2c\psi_A^\dagger A + cA^\dagger[i\phi, \psi_A] - [A^\dagger, i\phi]\psi_A + (c-1)\psi_A^\dagger[i\phi, A]) \\
&= \mathrm{tr}(2c\psi_A^\dagger A + A^\dagger[i\phi, \psi_A] - iA^\dagger\phi\psi_A + i\phi A^\dagger\psi_A + (c-1)\psi_A^\dagger[i\phi, A]). && (1.36)
\end{aligned}$$

Since $\mathrm{tr}\big(i\phi A^\dagger \psi_A\big) = \mathrm{tr}\big(A^\dagger \psi_A i\phi\big)$,

$$\begin{aligned}
\delta^2 < \psi_A, A > &= \mathrm{tr}(2c\psi_A^\dagger A + cA^\dagger[i\phi, \psi_A] - A^\dagger[i\phi, \psi_A] + (c-1)\psi_A^\dagger[i\phi, A]) \\
&= \mathrm{tr}(2c\psi_A^\dagger A + (c-1)\big(A^\dagger[i\phi, \psi_A] + \psi_A^\dagger[i\phi, A]\big)). && (1.37)
\end{aligned}$$

However, since our aim is that constructing a recipe for the Euler number of $G(k,N)$, we put $c=1$. Then,

$$\delta^2 < \psi_A, A >= 2 < \psi_A, A > . \tag{1.38}$$

The reason for introducing of A and ψ_A and the modifying of supersymmetric transformations for$A$ and H comes from our requirement that integration of A and H variables produces top Chern class (Euler class) of tangent bundle of $G(k,N)$. More explanation will be given in Subsection 1.4. In some sense, we consider central extension of supersymmetric transformation in order to obtain top Chern class of tangent bundle of $G(k,N)$ from the toy model version of topological Yang-Mills theory.

1.3 Schubert Calculus by Fermionic integral

As mentioned in Subsection 1.1, integrals of Chern classes can be computed using integrals of fermion. In this section, we give an overview of the method and the results. In our model, $\psi_s^i(1 \leq i \leq k1 \leq s \leq N-k)$ are important. We define the $k \times k$ matrix whose elements are given by fermionic fields.

$$\Phi' := \sum_{s=1}^{N-k} \begin{pmatrix} \omega_s^{1\bar{1}} & \dots & \omega_s^{1\bar{k}} \\ \vdots & \ddots & \vdots \\ \omega_s^{k\bar{1}} & \dots & \omega_s^{k\bar{k}} \end{pmatrix} \qquad (\omega_s^{i\bar{j}} := \psi_s^i \psi_{\bar{s}}^{\bar{j}}). \tag{1.39}$$

Let λ_i $(i=1,\cdots,k)$ be eigenvalues of Φ'. Then, we define $\sigma_i(i=1,\cdots,k)$ by

$$1 + t\sigma_1 + t^2\sigma_2 + \cdots t^k\sigma_k := \det\left(I_k + t\Phi'\right) = \prod_{i=1}^{k}(1+\lambda_i). \tag{1.40}$$

Then, we obtain following theorems. (They come from [20].)

Theorem 2. *Let us define b_i $(i=0,1,2,\cdots)$ by*

$$\frac{1}{\det(I_k + t\Phi')} = \sum_{m=0}^{\infty} b_m t^m. \tag{1.41}$$

Then, $b_m = 0$ if $m > N-k$.

Theorem 3.

$$\frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!} \int D\psi' \left(\det(\Phi')\right)^{N-k} = 1. \tag{1.42}$$

Theorem 4.

$$H^*(G(k,N)) = \frac{\mathbb{R}[c_1(S^*),\cdots,c_k(S^*)]}{(a_i = 0 \ \ (i > N-k))} \simeq \mathbb{R}[\sigma_1,\cdots,\sigma_k]. \tag{1.43}$$

These theorems are proven in Section 4. Theorem 3 corresponds to normalization condition of integration on $G(k,N)$[17, 40],

$$\int_{G(k,N)} (c_k(S^*))^{N-k} = 1. \tag{1.44}$$

And Theorem 4 claims that x_i and λ_i are identified. Since an element of is represented by a symmetric polynomial $g(x_1,\cdots,x_k)$,

$$\int_{G(k,N)} g(x_1,\cdots,x_k) = \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!} \int D\psi' g(\lambda_1,\cdots,\lambda_k). \tag{1.45}$$

Here, $D\psi' = \prod_{s=1}^{N-k} d\psi_s^1 d\psi_{\bar{s}}^{\bar{1}} \cdots d\psi_s^k d\psi_{\bar{s}}^{\bar{k}}$. In Section 5, we prove next result by using this result. (This result comes from [28].)

Theorem 5.

$$\int_{G(k,N)} (\tau^*_{1(1)})^{Nk-k^2} = (Nk-k^2)! \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!}. \tag{1.46}$$

$$\int_{G(k,N)} (\tau^*_{1(1)})^{kN-k^2-2}(\tau^*_{1(2)}) = \frac{(kN-k^2-2)!(N-k)(N-k+1)k(k-1)}{2} \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!}. \tag{1.47}$$

$$\begin{aligned}\int_{G(k,N)} (\tau^*_{1(1)})^{kN-k^2-4}(\tau^*_{1(2)})^2 = {} & \frac{(kN-k^2-4)!}{4} k(k-1)(N-k)(N-k+1) \\ & \times \Big[k(k-1)(N-k)(N-k-1) + 2(k-2)(k-3)(N-k) \\ & + 4(k-2)(N-k-1)\Big] \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!}.\end{aligned} \tag{1.48}$$

Here, we assume that N and k in (1.47) and (1.48) satisfy $kN-k^2-2 \geq 0$ and $kN-k^2-4 \geq 0$, respectively.

We note that these are intersection numbers of $\tau_{1(1)}$ and $\tau_{1(2)}$. Results of (1.46) are already known [16, 11], but we prove these results by using fermionic variables.

1.4 The New Feature of Our Model

Explicit evaluation of the path integral in the Main Theorem of Part II will be given in Section 3 and Section 4, but in this subsection, we briefly explain feature of our model.

The new feature of our model is that integration of A, H and their super partner ψ_A, χ produces total Chern class of $T'G(k,N)$ instead of top Chern class. This is because it was difficult for us to produce top Chern class of $T'G(k,N)$ by using only H and χ, which are standard in usual MQ formalism, with Atiyah-Jeffrey construction.

Instead, we consider two vector bundles, $\mathbb{C}^N \otimes S^*$ and $S \otimes S^*$ that correspond to H, χ and A, ψ_A respectively. Moreover, we introduce the extension of the supersymmetric transformation in order to produce total Chern classes of these vector bundles instead of top Chern classes. Precisely speaking, integration of H and χ results in $\prod_{i=1}^{k}(1+x_i)^N$ and integration of A and ψ_A produces $\frac{1}{\prod_{l>j}(1-(x_l-x_j)^2)}$. Then, $x_i (i = 1, \cdots, k)$ are represented by elements of $i\phi$ But after the integration of $\bar{\phi}$, the matrix $i\phi$ whose (i,j)-element is replaced by $< \boldsymbol{\psi}^i, \boldsymbol{\psi}^j >$. More details are provided in Section 3. The integration of the fermionic variable ψ corresponds to integration of differential form on $G(k,N)$, only the contribution from the top Chern class survives and we obtain the Euler number of $G(k,N)$.

1.5 Organization of Part II

This part is organized as follows. In Section from 2 to 4, we prove the main theorem. These discussions are presented in [20]. In Section 2, we give an overview of MQ formalism and Atiyah-Jeffrey construction. In Section 3, we construct Lagrangian that counts Euler number of $G(k,N)$ by applying these techniques. Then we integrate out fields except for ψ and show that the Euler number is represented by fermion integral of the Chern class represented by the matrix $i\phi$ whose (i,j)-element is given by $< \boldsymbol{\psi}^i, \boldsymbol{\psi}^j >$. This calculation is a bit complicated, so we will introduce integration results of $\bar{\phi}$ and A as formulas and calculate the omitted parts in Appendix. In Section 4, we prove our main theorem by showing that the representation of the Chern class by the fermionic variables give the desired Chern class as an elements of cohomology ring of $G(k,N)$. In the process we prove Theorem 2, 3 and 4. Especially, validity of normalization factor β will be verified. We think that combinatorial aspects in the discussions in Section 4 is quite interesting for mathematicians. Since Chern classes are represented by fermion fields, the integration of Chern classes is given by the fermion integral. In Section 5, we compute some integral of Chern claases using the fermion integral. This process is shown in[28].

2 Mathai-Quillen formalism and Atiyah Jeffrey construction

In this section, we explain outline of Mathai-Quillen formalism and Atiyah Jeffrey construction. For more details, see the literatures [5, 6, 30, 37, 41]. (The discussion in this section is based on Imanishi [19] and discussion with Professor Masao Jinzenji.)

2.1 Overview of Mathai-Quillen formalism

Mathai-Quillen formalism provides us with a recipe to construct Thom from of a vector bundle by Gaussian integrals and fermion integrals. Here, we briefly explain outline of Mathai-Quillen formalism. Let $\pi : E \to M$ be a vector bundle of rank n on n-dimensional compact manifold M. We assume that each fiber $\pi^{-1}(x)$ has metric (or inner product) that varies smoothly as $x \in M$ varies. We denote by $\{f_1, \cdots, f_n\}$ a local orthonormal frame of $\pi^{-1}(U)$ (U is some open subset of M) with respect to this metric. Let $\Omega^q(M, E)$ be vector space of E-valued differntial q-form on M ($\Omega^0(M, E) \simeq \Gamma(E)$ is vector space of smooth section of E) and $\nabla^E : \Omega^q(M, E) \to \Omega^{q+1}(M, E)$ be a connection compatible with inner product on E. ∇^E satisfies Leibniz rule for $g \in \Omega^q(M), s \in \Gamma(E)$

$$\nabla^E(gs) = (d_M g)s + (-1)^q g \wedge (\nabla^E s), \tag{2.49}$$

where d_M is exterier derivative on M and $\Omega^q(M)$ is vector space of smooth q-form on M. Let us define a connection form ω_i^j by

$$\nabla^E f_i = \omega_i^j f_j. \tag{2.50}$$

where $\omega_i^j = -\omega_j^i$. Curvature of ∇^E is given by $(\nabla^E)^2 := R^E$. R^E for the local orthonormal frame is represented in the following form:

$$\begin{aligned}(R^E)_i^j f_j &:= (\nabla^E)^2 f_i = \nabla^E(\omega_i^j f_j) = d_M \omega_i^j f_j - \omega_i^j \wedge \omega_j^k f_k \\ &= (d_M \omega_i^j - \omega_i^k \wedge \omega_k^j) f_j. \end{aligned} \tag{2.51}$$

Next, Let $\pi^* E$ be a pullback bundle of E by π. Its tautological section $\boldsymbol{x} \in \Gamma(E, \pi^* E)$ is defined by the smooth section whose image of $u \in E$ is given by $\boldsymbol{x}(u) = (u, u) \in \pi^* E$. The connection form $\omega_i'^j$of $\nabla : \Omega^q(E, \pi^* E) \to \Omega^{q+1}(E, \pi^* E)$ is then given by pull-backing ω_i^j by π. Let $\{e_1, \cdots, e_n\}$ be local orthonormal frame for $\Gamma(\pi^{-1}(U), \pi^*(\pi^{-1}(U)))$ that corresponds to $\{f_1, \cdots, f_n\}$ of $\Gamma(U, \pi^{-1}(U))$. Then, $\boldsymbol{x}(u) \in \Gamma(\pi^{-1}(U), \pi^*(\pi^{-1}(U)))$ is represented by $\boldsymbol{x}(u) = u^i e_i$ where $(x, u) = (x, u^i f_i) \in \pi^{-1}(U)$ $(x \in U)$. Accordingly, we introduce connection of $\pi^* E$,

$$\nabla e_i = \omega_i'^j e_j, \tag{2.52}$$

and the curvature form R' of ∇. R' under the local orthonormal frame is given by

$$\begin{aligned} R'^j_i e_j &:= (\nabla)^2 e_i = \nabla(\omega_i'^j e_j) = (d_E \omega_i'^j) e_j - \omega_i'^k \wedge \omega_k'^j e_j \\ &= \{d_E \omega_i'^j - \omega_i'^k \wedge \omega_k'^j\} e_j \end{aligned} \tag{2.53}$$

Since $\omega_i'^j = \pi^* \omega_i^j$ and $d_E \pi^* = \pi^* d_M$, R'^j_i is given by $\pi^*(R^E)_i^j$.

From now on, we assume $n = 2m$. Let $\mathbf{u} = {}^t(u^1, \cdots, u^{2m})$ be coordinates of fiber $\pi^{-1}(x)$ of E and χ be a fermionic variable: $\chi = {}^t(\chi^1, \cdots, \chi^{2m})$, which correspond to super-partner of **u**. And let R_{ij} be $(R')_i^j$ (R_{ij} is skew-symmetric). With this set-up, Thom form $\Phi_\nabla(E)$ constructed in MQ-formalism is given as follows.

$$\Phi_\nabla(E) = \frac{1}{(2\pi)^m} e^{-|\boldsymbol{u}|^2/2} \int D\chi \exp\left(\frac{1}{2}\ {}^t\chi R\chi + i\ {}^t(\nabla u)\chi\right). \tag{2.54}$$

$$|\boldsymbol{u}|^2 := \sum_{i=1}^{2m} (u^i)^2,\ {}^t(\nabla u)\chi := \sum_{i=1}^{2m} (\nabla u)^i \chi^i := \sum_{i=1}^{2m} (d_E u^i + \omega'^i_j u^j)\chi^i. \tag{2.55}$$

Let $\mathcal{L}_0$ be a Lagrangian defined by

$$\mathcal{L}_0 := |\boldsymbol{u}|^2/2 - \frac{1}{2}\ {}^t\chi R\chi - i\ {}^t(\nabla u)\chi. \tag{2.56}$$

Then we define supersymmetric transformation as follows:

$$\delta\chi^i := iu^i,\ \delta u^i := \nabla u^i. \tag{2.57}$$

Here, we assume the following.

Assumptions for δ

1. δ behaves fermionic. Hence δ is anti-commutative with d_E.
2. δ acts only fiber variables and $\delta\omega = \delta R = 0$.

Then we can show that $\mathcal{L}_0$ invariant under δ transformation.

$$\delta(|\boldsymbol{u}|^2/2) = \sum_{i=1}^{2m} u^i (\nabla u)^i. \tag{2.58}$$

$$\delta(-\frac{1}{2}\ {}^t\chi R\chi) = \frac{1}{2}(-\delta(\chi^i) R_{ij}\chi^j + \chi^i R_{ij}\delta(\chi^j)) = -iu^i R_{ij}\chi^j. \tag{2.59}$$

$$\delta(-i\ {}^t(\nabla u)\chi) = -i\sum_{i=1}^{2m} (\delta((\nabla u)^i)\chi^i - (\nabla u)^i \delta(\chi^i)) \tag{2.60}$$

$$= -i\sum_{i=1}^{2m} ((-d_E\delta(u^i) - \omega'^i_j \delta(u^j))\chi^i - i(\nabla u)^i u^i) \tag{2.61}$$

$$= -i\sum_{i=1}^{2m} ((-d_E(\omega'^i_j u^j) - \omega'^i_j (d_E u^j + \omega'^j_k u^k))\chi^i - i(\nabla u)^i u^i) \tag{2.62}$$

$$= -i\sum_{i=1}^{2m} (-(d_E(\omega'^i_j) - \omega'^k_j \omega'^i_k u^j))\chi^i - i(\nabla u)^i u^i) \tag{2.63}$$

$$= \sum_{i=1}^{2m} (iR_{ji} u^j \chi^i - (\nabla u)^i u^i) = iR_{ji} u^j \chi^i - \sum_{i=1}^{2m} (\nabla u)^i u^i. \tag{2.64}$$

Hence $\delta\mathcal{L}_0 = 0$. Let us integrate out $\Phi_\nabla(E)$ on a fiber $\pi^{-1}(x)$. Since $x \in M$ is fixed, $R = \omega = 0$. Then we can derive

$$\int_{\pi^{-1}(x)} \Phi_\nabla(E) = 1. \tag{2.65}$$

Explicit derivation is given as follows.

$$\begin{aligned}\int_{\pi^{-1}(x)} \Phi_\nabla(E) &= (2\pi)^{-m}(-1)^{m+\frac{(2m-1)(2m)}{2}} \int_{\pi^{-1}(x)} e^{-|\boldsymbol{u}|^2/2} \int \mathcal{D}\chi \prod_{a=1}^{2m}(1 + idu^a\chi^a) \\ &= \frac{(-1)^{\frac{(2m-1)(2m)}{2}}}{(2\pi)^m} \int_{\pi^{-1}(x)} e^{-|\boldsymbol{u}|^2/2} \int \mathcal{D}\chi (du^1\chi^1)\cdots(du^{2m}\chi^{2m}) \\ &= \frac{1}{(2\pi)^m} \int_{\pi^{-1}(x)} e^{-|\boldsymbol{u}|^2/2} du^1 \cdots du^{2m} = 1. \end{aligned} \tag{2.66}$$

(2.65) is one of the two features that characterizes Thom form of E. The other one is given by

$$s_0^*(\Phi_\nabla(E)) = e_{0,\nabla}(E), \tag{2.67}$$

where $s_0 : M \to E$ is the zero section of E and $e_{0,\nabla}(E)$ is Euler class of E. This can be easily seen as follows:

$$\begin{aligned} s_0^*(\Phi_\nabla(E)) &= \frac{1}{(2\pi)^m} \int D\chi \exp\left(\frac{1}{2} {}^t\chi R^E \chi\right) = \frac{1}{(2\pi)^m}\mathrm{Pfaff}(R^E) \\ &= e_{0,\nabla}(E), \end{aligned} \tag{2.68}$$

where we used that $u^i(s_0(x)) = 0, s^*(R')_i^j = (s^*(\pi^* R^E))_i^j = R_i^{Ej}$ and that R^E is skew symmetric. Integration of Euler class $e_{0,\nabla}(E)$ on M gives Euler number of E, which is denoted by $\chi(E)$. Therefore, we have

$$\chi(E) = \int_M s_0^*(\Phi_\nabla(E)). \tag{2.69}$$

At this stage, we include auxiliary bosonic variable H^i and modify the supersymmetric transformation as follows.

$$\delta\chi^i := H^i, \ \ \delta H^i := R_{ij}\chi^j. \tag{2.70}$$

Now we intoroduce $\Psi := \left\langle \chi, \frac{H}{2} - iu \right\rangle$ where $< A, B >:= {}^tAB$ is inner product of A and B. We also use the notation $|A|^2 :=< A, A >$. Then $\delta\Psi$ is given as follows.

$$\begin{aligned}\delta\Psi = \delta\left\langle \chi, \frac{H}{2} - iu \right\rangle &= {}^t(\delta\chi)\left(\frac{H}{2} - iu\right) - {}^t\chi\left(\frac{\delta H}{2} - i(\delta u)\right) \\ &= \frac{1}{2}\sum_{i=1}^{2m}(H^i - iu^i)^2 + \frac{|\boldsymbol{u}|^2}{2} - \frac{1}{2}\,{}^t\chi R\chi - i\,{}^t(\nabla u)\chi = \frac{1}{2}|H - iu|^2 + \mathcal{L}_0. \end{aligned} \tag{2.71}$$

Here, we assume χ and ∇u are anticommutative. $\delta\Psi$ can be identified with $\mathcal{L}_0$ modulo the relation $H^a = iu^a$ (equation of motion of H). We can easily see that $\Phi_\nabla(E)$ is obtained by integrating $\exp(-\delta\Psi)$ by H and χ.

$$\begin{aligned}\Phi_\nabla(E) &:= \frac{1}{(2\pi)^{2m}}\int \mathcal{D}\chi \int \mathcal{D}H \exp\left(-\frac{1}{2}|H - iu|^2 - \mathcal{L}_0\right)\\ &= \frac{1}{(2\pi)^{2m}}\int \mathcal{D}\chi \int \mathcal{D}H \exp\left(-\delta\left\langle \chi, \frac{H}{2} - iu\right\rangle\right). \end{aligned} \tag{2.72}$$

Let $s : M \to E$ is any smooth section of E, then we can easily derive

$$e_{s,\nabla}(E) := s^*\Phi_\nabla(E) = \frac{1}{(2\pi)^{2m}}\int \mathcal{D}\chi\mathcal{D}H \exp\left(-\delta\left\langle \chi, \frac{H}{2} - is\right\rangle\right). \tag{2.73}$$

Since s is homotopic to the zero section s_0, $e_{s,\nabla}$ belongs to the same cohomology class as $e_{0,\nabla}(E)$. Hence we obtain the following equality.

$$\begin{aligned}\chi(E) &= \int_M e_{0,\nabla}(E) = \int_M e_{s,\nabla}(E)\\ &= \frac{1}{(2\pi)^{2m}}\int_M \mathcal{D}x\mathcal{D}\psi\mathcal{D}\chi\mathcal{D}H \exp\left(-\delta\left\langle \chi, \frac{H}{2} - is\right\rangle\right). \end{aligned} \tag{2.74}$$

where x is local coordinate of M and ψ is a fermionic variable that plays the role of differential form dx on M.

2.2 The Case when M is an Orbit Space

In this subsection, we search for the Lagrangian that produces Euler class of M when M is given as an orbit space, by using Atiyah-Jeffrey construction. Atiyah-Jeffrey construction is an extension of MQ formalism for vector bundle whose base space M is given as an orbit space X/G (G: Lie group). Let us consider first behavior of δ^2 of the transformation (2.70).

$$\delta^2\chi^i := R_{ij}\chi^j, \quad \delta^2 H^i := R_{ij}H^j. \tag{2.75}$$

Therefore, δ^2 corresponds to "infinitessimal rotation of fiber coordinates generated by R_{ij}". Then Atiyah and Jeffrey modified the above relation into the following:

$$\delta^2 = \delta_\phi, \tag{2.76}$$

where δ_ϕ is the infinitesimal gauge transformation generated by ϕ. It corresponds to infinitesimal rotation of infinite dimensional Lie group $\mathcal{G}$. Note that δ is nilpotent when we consider orbit space X/G. Then δ can be regarded as infinite dimensional version of equivalent derivative $d - \iota_\omega$, $\omega \in \mathrm{Lie}(\mathcal{G})$ on X/G.

$$\delta \Leftrightarrow d - \iota_\omega, \tag{2.77}$$

$$\delta^2 = \delta_\phi \Leftrightarrow (d - \iota_\omega)^2 = -d\iota_\omega - \iota_\omega d = -\mathcal{L}_\omega, \tag{2.78}$$

where $\mathcal{L}_\omega$ is the Lie derivative. With these considerations, δ transformation is modified as follows.

$$\delta x = \psi, \qquad \delta\psi = \delta_\phi x, \qquad \delta\chi = H, \qquad \delta H = \delta_\phi \chi, \qquad \delta\phi = 0, \tag{2.79}$$

where x is coordinate of $\mathcal{A}$, ψ is the fermion coordinate that plays the role of dx, χ is fermion coordinate of $\mathcal{E}$ and H is auxiliary field and super-partner of χ.

In the previous subsection, $\mathcal{L}_{MQ}$, the Lagrangian obtained from the MQ-formalism was represented as $\delta\Psi := \delta < \chi, \frac{H}{2} - is >$. In Atiyah-Jeffrey construction, Lagrangian $\mathcal{L}_{MQ}$, which is expected to produce Euler number $\chi(\mathcal{E})$ of the vector bundle $\mathcal{E}$ on $\mathcal{A}/\mathcal{G}$,

$$\chi(\mathcal{E}) \overset{?}{=} \int \mathcal{D}x\mathcal{D}\psi\mathcal{D}H\mathcal{D}\chi\mathcal{D}\phi \exp(-\mathcal{L}_{MQ}), \tag{2.80}$$

is given by

$$\mathcal{L}_{MQ} = \delta(\Psi + \Psi_{\text{proj}}). \tag{2.81}$$

The term $\delta\Psi_{\text{proj}}$ plays the role of projecting out gauge horizontal direction (direction parallel to orbit of $\mathcal{G}$) in integrating ψ over $T^*\mathcal{A}$. In other words, $\exp(-\delta\Psi_{\text{proj}})$ can be regarded as projection operator from $T^*\mathcal{A}$ to $T^*(\mathcal{A}/\mathcal{G})$. In (2.80), we add ? above = because $\mathcal{E}$, the vector bundle of infinite rank, is not clearly stated in [3] and $\chi(\mathcal{E})$ is not well-defined. Indeed, the Lagrangian $\mathcal{L}_{MQ}$ is used to produce Donaldson invariants of M_4 in context of topological Yang-Mills theory [39].

Let us explain outline of construction of Ψ_{proj}. On $\mathcal{G}$-principle bundle $\mathcal{A} \to \mathcal{A}/\mathcal{G}$, group action for $x \in \mathcal{A}$ is given by $\mathcal{G}$. Let $C : \mathfrak{g}$ (Lie algebra of $\mathcal{G}$) $\to T_x\mathcal{A}$ be differentiation of group action on $x \in \mathcal{A}$. Let θ be an element of $\mathfrak{g}$. Then, $C\theta$ is given by

$$C\theta = \delta_\theta x. \tag{2.82}$$

$C^\dagger$ is defined as adjoint operator of C

$$\left\langle C^\dagger\psi, \theta \right\rangle = \langle \psi, C\theta \rangle, \tag{2.83}$$

where $\langle *, * \rangle$ in the l.h.s. is inner product of $\mathfrak{g}$ and $\langle *, * \rangle$ in the r.h.s is inner product of $T^*\mathcal{A}$. If $\psi \in \text{Ker}C^\dagger$, we obtain

$$C^\dagger\psi = 0 \Rightarrow 0 = \left\langle C^\dagger\psi, \theta \right\rangle = \langle \psi, C\theta \rangle = \langle \psi, \delta_\theta x \rangle. \tag{2.84}$$

So, $\text{Ker}C^\dagger \subset T^*\mathcal{A}$ corresponds to vertical direction of gauge transformation. Then we have to restrict integration of ψ into $\text{Ker}C^\dagger$. At this stage, we introduce additional boson field $\bar{\phi}$ and fermion field η. Supersymmetric transformation for these fields are defined by

$$\delta\bar{\phi} = \eta, \qquad \delta\eta = \delta_\phi\bar{\phi}, \tag{2.85}$$

where ϕ is gauge transformation parameter (element of $\mathfrak{g}$). Then Ψ_{proj} is defined in the following form.

$$\Psi_{\mathrm{proj}} := \left\langle \psi, C\bar{\phi} \right\rangle . \tag{2.86}$$

We obtain

$$\delta\Psi_{\mathrm{proj}} = \delta \left\langle C^\dagger\psi, \bar{\phi} \right\rangle = \left\langle \delta(C^\dagger\psi), \bar{\phi} \right\rangle - \left\langle C^\dagger\psi, \delta(\bar{\phi}) \right\rangle = \left\langle \delta(C^\dagger\psi), \bar{\phi} \right\rangle - \left\langle C^\dagger\psi, \eta \right\rangle . \tag{2.87}$$

From equation of motion of η, we obtain

$$\frac{\delta}{\delta\eta}\delta\Psi_{\mathrm{proj}} = 0 \Leftrightarrow C^\dagger\psi = 0. \tag{2.88}$$

Hence multiplying $\exp\left(-\delta\Psi_{\mathrm{proj}}\right)$ can restrict ψ integration to $\mathrm{Ker}C^\dagger$. When we consider the case of zero section ($s \equiv 0$), the Lagrangian becomes

$$\mathcal{L}_{MQ} = \delta\left\langle \chi, \frac{H}{2} \right\rangle + i\delta \left\langle \psi, C\bar{\phi} \right\rangle , \tag{2.89}$$

where supersymmetric transformation is given by

$$\begin{aligned} &\delta x = \psi, && \delta\psi = \delta_\phi x, && \delta\chi = H, && \delta H = \delta_\phi \chi, \\ &\delta\phi = 0, && \delta\bar{\phi} = \eta, && \delta\eta = \delta_\phi\bar{\phi}. && \end{aligned} \tag{2.90}$$

This is our starting point of construction of the Lagrangian (1.16) and of the supersymmetric transformation (1.28). But what we aim to compute is $\chi(G(k,N)) = \chi(T'G(k,N))$, we have to modify the above settings further. Some points of modification are already mentioned in Subsection 1.1 and Subsection 1.2. We will discuss again points of modification in Subsection 3.2.

3 Construction of Lagrangian and First Half of Evaluation of Path Integral

In order to apply MQ formalism on the Grassmann manifold, it is necessary to consider the Grassmann manifold $G(k,N)$ as an orbit space X/G. $G(k,N)$ is a space which parametrizes all k-dimensional linear subspaces of the N-dimensional complex vector space $\mathbb{C}^N$.

$$G(k,N) := \{W \subset \mathbb{C}^N \,|\, \dim_{\mathbb{C}} W = k\}. \tag{3.91}$$

We consider the Stiefel manifold $V_k(\mathbb{C}^N)$. Its point is given by a set of k unit vectors which are orthogonal system on $\mathbb{C}^N$.

$$V_k(\mathbb{C}^N) := \{(\boldsymbol{z}^1 \cdots \boldsymbol{z}^k) \in \mathbb{C}^{kN} | \boldsymbol{z}^i \in \mathbb{C}^N,\, <\boldsymbol{z}^i, \boldsymbol{z}^j> = \delta_{ij}\ (i,j = 1, \cdots, k)\}. \tag{3.92}$$

Let $< \boldsymbol{z}^1, \cdots, \boldsymbol{z}^k >_{\mathbb{C}}$ be a vector space spanned by $(\boldsymbol{z}^1 \cdots \boldsymbol{z}^k)$. $(\boldsymbol{z}^1 \cdots \boldsymbol{z}^k), (\boldsymbol{z}'^1 \cdots \boldsymbol{z}'^k) \in V_k(\mathbb{C}^N)$ satisfy the relation $< \boldsymbol{z}^1, \cdots, \boldsymbol{z}^k >_{\mathbb{C}} = < \boldsymbol{z}'^1, \cdots, \boldsymbol{z}'^k >_{\mathbb{C}}$ if

$$\text{There exists } (U_{ij})_{1\leq i,j\leq k} \in U(k) \text{ such that } \boldsymbol{z}^i = \sum_{j=1}^{k} U_{ij}\boldsymbol{z}'^j \ (i = 1, \cdots, k). \tag{3.93}$$

Hence, we define the equivalence relation of $U(k)$ by above form. $G(k, N) \simeq V_k(\mathbb{C}^N)/U(k)$.

Since $V_k(\mathbb{C}^N)$ is regraded as quotient space $U(N)/U(N-k)$ and volume of $U(N)$ is given by $\prod_{j=1}^{N} \mathrm{vol}(S^{2j-1})$ (S^{2j-1} is the $(2j-1)$-dimensional unit sphere) [12, 14], we obtain

$$\mathrm{vol}(V_k(\mathbb{C}^N)) = \prod_{j=N-k+1}^{N} \mathrm{vol}(S^{2j-1}) = \frac{2^k(\pi)^{kN-\frac{k(k-1)}{2}}}{\prod_{j=N-k+1}^{N}(j-1)!}. \tag{3.94}$$

3.1 Lagrangian that counts Euler Number $\chi(G(k,N))$

In this section, we construct the Lagrangian for $\mathcal{L}_{MQ}$ by applying MQ-formalism and Atiyah-Jeffrey construction outlined in the previous section to the orbit space $G(k, N) = V_k(\mathbb{C}^N)/U(k)$ From now on, we set section s used in Subsection 2.2 to zero section.

Let us mention again the fields used in our Lagrangian. The variable z that describes a point of $V_k(\mathbb{C}^N)$ is given as follows.

$$z := (\boldsymbol{z}^1 \cdots \boldsymbol{z}^k) \in \mathbb{C}^{kN}, \tag{3.95}$$

$$\boldsymbol{z}^j := {}^t(z^j_1, z^j_2, \cdots, z^j_N), \ (j = 1, 2, \cdots, k), \ (\boldsymbol{z}^i)^\dagger \boldsymbol{z}^j - \delta^{\bar{i}j} = 0 \, (i, j = 1, 2, \cdots, k). \tag{3.96}$$

This corresponds to the variable x in Subsection 2.2. Other fields are represented in the following form.

$$\psi := (\boldsymbol{\psi}^1 \cdots \boldsymbol{\psi}^k), \qquad \boldsymbol{\psi}^j := {}^t(\psi^j_1, \psi^j_2, \cdots, \psi^j_N). \tag{3.97}$$

$$\phi := \begin{pmatrix} \phi_{\bar{1}1} & \cdots & \phi_{\bar{1}k} \\ \vdots & \ddots & \vdots \\ \phi_{\bar{k}1} & \cdots & \phi_{\bar{k}k} \end{pmatrix}. \qquad \bar{\phi} := \begin{pmatrix} \bar{\phi}_{\bar{1}1} & \cdots & \bar{\phi}_{\bar{1}k} \\ \vdots & \ddots & \vdots \\ \bar{\phi}_{\bar{k}1} & \cdots & \bar{\phi}_{\bar{k}k} \end{pmatrix}, \tag{3.98}$$

$$\eta := \begin{pmatrix} \eta_{\bar{1}1} & \cdots & \eta_{\bar{1}k} \\ \vdots & \ddots & \vdots \\ \eta_{\bar{k}1} & \cdots & \eta_{\bar{k}k} \end{pmatrix}. \tag{3.99}$$

$$\chi := \begin{pmatrix} \boldsymbol{\chi}^1 & \cdots & \boldsymbol{\chi}^k \end{pmatrix}, \qquad \boldsymbol{\chi}^j := {}^t(\chi^j_1, \cdots, \chi^j_N). \tag{3.100}$$

$$H := \begin{pmatrix} \boldsymbol{H}^1 & \cdots & \boldsymbol{H}^k \end{pmatrix}, \qquad \boldsymbol{H}^j := {}^t(H^j_1, \cdots, H^j_N). \tag{3.101}$$

ϕ and $\bar{\phi}$ are Hermite matrix ($\phi_{\bar{i}j} = \phi_{j\bar{i}}, \bar{\phi}_{\bar{i}j} = \bar{\phi}_{j\bar{i}}$). They are the generators of the elements of $U(k)$. $\bar{\phi}$ is not the complex conjugate of ϕ. These fields play the same roles as the corresponding fields in Subsection 2.2. Next, we introduce the field A and its superpartner ψ_A in order to produce the part $\dfrac{1}{c(S\otimes S^*)}$ in the total Chern class $\dfrac{c(S^*)^N}{c(S\otimes S^*)}$.

$$A := \begin{pmatrix} A_{\bar{1}1} & \cdots & A_{\bar{1}k} \\ \vdots & \ddots & \vdots \\ A_{\bar{k}1} & \cdots & A_{\bar{k}k} \end{pmatrix}, \qquad \psi_A := \begin{pmatrix} \psi^1_{A1} & \cdots & \psi^k_{A1} \\ \vdots & \ddots & \vdots \\ \psi^1_{Ak} & \cdots & \psi^k_{Ak} \end{pmatrix}. \tag{3.102}$$

At this stage, we define supersymmetric transformation for each variables. We define supersymmetry for each variables. Supersymmetric transformations are given by

$$\begin{aligned} &\delta z = \psi, \quad \delta\psi = \delta_\phi z, \quad \delta\chi = H, \quad \delta H = \chi + \delta_\phi \chi, \quad \delta A = \psi_A, \\ &\delta\phi = 0, \quad \delta\bar{\phi} = \eta, \quad \delta\eta = \delta_\phi \bar{\phi}, \quad \delta\psi_A = A + \delta_\phi A = A + [A, i\phi]. \end{aligned} \tag{3.103}$$

Let the gauge transformation of $U(k)$ for z is defined by

$$\begin{pmatrix} z^1_s \\ \vdots \\ z^k_s \end{pmatrix} \to \begin{pmatrix} z'^1_s \\ \vdots \\ z'^k_s \end{pmatrix} := e^{i\phi} \begin{pmatrix} z^1_s \\ \vdots \\ z^k_s \end{pmatrix} = (I_k + i\phi) \begin{pmatrix} z^1_s \\ \vdots \\ z^k_s \end{pmatrix} + \mathcal{O}(\phi^2),\ (s = 1, 2, \cdots, N). \tag{3.104}$$

We neglect second or higher order of ϕ. Then, $\delta_\phi z := iz\ {}^t\phi$. We define $\delta_\phi A$ as

$$e^{-i\phi} A e^{i\phi} = A + [A, i\phi] + \mathcal{O}(\phi^2) =: A + \delta_\phi A + \mathcal{O}(\phi^2). \tag{3.105}$$

Therefore,

$$\begin{aligned} &\delta z^i_s = \psi^i_s, \quad \delta\psi^i_s = i\phi_{\bar{i}m} z^m_s, \quad \delta\chi^i_s = H^i_s, \quad \delta H^i_s = (\delta_{\bar{i}j} + i\phi_{\bar{i}j})\chi^j_s, \\ &\delta A^i_j = \psi^i_{Aj}, \quad \delta\phi_{\bar{i}j} = 0, \quad \delta\bar{\phi}_{\bar{i}j} = \eta_{\bar{i}j}, \quad \delta\psi_A = A + [A, i\phi]. \end{aligned} \tag{3.106}$$

These are fundamentally obtained from applying the construction in Subsection 2.2 to the orbit space $G(k,N) = V_k(\mathbb{C}^N)/U(k)$ but we modified the transformation of H and ψ_A from the standard version $\delta H^i_s = i\phi_{\bar{i}j}\chi^j_s$ and $\delta\psi_A = [A, i\phi]$. This modification is done in order to produce total Chern class instead of top Chern class.

With these set-up's Lagrangian is defined as follows:

$$\mathcal{L}_{MQ} = \delta < \chi, H > + \delta < \psi_A, A > + \frac{i}{2}\delta\{< \psi, C\bar{\phi} > + {}^* < \psi, C\bar{\phi} >\}. \tag{3.107}$$

where $C\bar{\phi} = \delta_{\bar{\phi}} z = iz^t\bar{\phi}$. Except for the term $\delta < \psi_A, A >$, this Lagrangian is obtained from applying discussion of Subsection 2.2 with $s = 0$ to the orbit space $G(k, N) =$

$V_k(\mathbb{C}^N)/U(k)$. Then we can derive (1.17) from (1.16) by straightforward computation. Each term is calculated as follows.

$$\Big\langle \chi, H \Big\rangle = \sum_{s=1}^{N} \sum_{i=1}^{k} \chi^{\bar{i}}_{\bar{s}} H^{i}_{s}. \tag{3.108}$$

$$\begin{aligned} \delta\Big\langle \chi, H \Big\rangle &= \sum_{s=1}^{N} \sum_{i=1}^{k} \{\delta\chi^{\bar{i}}_{\bar{s}} H^{i}_{s} - \chi^{\bar{i}}_{\bar{s}} \delta H^{i}_{s}\} \\ &= \sum_{s=1}^{N} \{\sum_{i=1}^{k} H^{\bar{i}}_{\bar{s}} H^{i}_{s} - \chi^{\bar{i}}_{\bar{s}} (\delta_{\bar{i}j} + i\phi_{\bar{i}j}) \chi^{j}_{s}\} &(3.109) \\ &= \sum_{i=1}^{k} \Big\langle \boldsymbol{H}^{i}, \boldsymbol{H}^{i} \Big\rangle - \Big\langle \chi, \chi(I_k + i\ {}^t\phi) \Big\rangle &(3.110) \end{aligned}$$

$$\Big\langle \psi, C\bar{\phi} \Big\rangle = \Big\langle \psi, \delta_{\bar{\phi}} z \Big\rangle = \mathrm{tr}(\psi^{\dagger} i z\ {}^t\bar{\phi}) = i \sum_{s=1}^{N} \psi^{\bar{i}}_{\bar{s}} \bar{\phi}_{\bar{i}l} z^{l}_{s}, \tag{3.111}$$

$$\begin{aligned} i\delta\Big\langle \psi, C\bar{\phi} \Big\rangle &= -\sum_{s=1}^{N} \{-i\phi_{i\bar{m}} z^{\bar{m}}_{\bar{s}} \bar{\phi}_{\bar{i}l} z^{l}_{s} - \psi^{\bar{i}}_{\bar{s}} \eta_{\bar{i}l} z^{l}_{s} - \psi^{\bar{i}}_{\bar{s}} \bar{\phi}_{\bar{i}l} \psi^{l}_{s}\} \\ &= \sum_{s=1}^{N} \{i\phi_{i\bar{m}} z^{\bar{m}}_{\bar{s}} \bar{\phi}_{\bar{i}l} z^{l}_{s} + \psi^{\bar{i}}_{\bar{s}} \eta_{\bar{i}l} z^{l}_{s} + \psi^{\bar{i}}_{\bar{s}} \bar{\phi}_{\bar{i}l} \psi^{l}_{s}\} \end{aligned} \tag{3.112}$$

From $\phi_{i\bar{m}} = \phi_{\bar{m}i}$ and $\sum_{s=1}^{N} z^{\bar{i}} z^{j} - \delta^{\bar{i}j} = 0$,

$$i\delta\Big\langle \psi, C\bar{\phi} \Big\rangle = \sum_{i,j=1}^{k} i\phi_{\bar{i}j} \bar{\phi}_{\bar{j}i} + \sum_{s=1}^{N} \{\psi^{\bar{i}}_{\bar{s}} \eta_{\bar{i}l} z^{l}_{s} + \psi^{\bar{i}}_{\bar{s}} \bar{\phi}_{\bar{i}l} \psi^{l}_{s}\} = i\mathrm{tr}(\phi\bar{\phi}) + \sum_{s=1}^{N} \{\psi^{\bar{i}}_{\bar{s}} \eta_{\bar{i}l} z^{l}_{s} + \psi^{\bar{i}}_{\bar{s}} \bar{\phi}_{\bar{i}l} \psi^{l}_{s}\} \tag{3.113}$$

$$= i\Big\langle \phi, \bar{\phi} \Big\rangle + \Big\langle \boldsymbol{\psi}^{i}, \boldsymbol{z}^{j} \Big\rangle \eta_{\bar{i}j} + \Big\langle \boldsymbol{\psi}^{i}, \boldsymbol{\psi}^{j} \Big\rangle \bar{\phi}_{\bar{i}j} = i\Big\langle \phi, \bar{\phi} \Big\rangle + \Big\langle \psi, z\ {}^t\eta \Big\rangle + \Big\langle \psi, \psi\ {}^t\bar{\phi} \Big\rangle. \tag{3.114}$$

In the same way,

$$\begin{aligned} i\delta\ {}^*\Big\langle \psi, C\bar{\phi} \Big\rangle &= \delta\left(\sum_{s=1}^{N} \psi^{i}_{s} \bar{\phi}_{i\bar{l}} z^{\bar{l}}_{\bar{s}}\right) = \sum_{s=1}^{N} \Big\{ i\phi_{\bar{i}m} z^{m}_{s} \bar{\phi}_{i\bar{l}} z^{\bar{l}}_{\bar{s}} - \psi^{i}_{s} \eta_{i\bar{l}} z^{\bar{l}}_{\bar{s}} - \psi^{i}_{s} \bar{\phi}_{i\bar{l}} \psi^{\bar{l}}_{\bar{s}} \Big\} \\ &= i\phi_{\bar{i}j} \bar{\phi}_{\bar{j}i} - \sum_{s=1}^{N} \Big\{ \psi^{i}_{s} \eta_{i\bar{l}} z^{\bar{l}}_{\bar{s}} + \psi^{i}_{s} \bar{\phi}_{i\bar{l}} \psi^{\bar{l}}_{\bar{s}} \Big\} = i\mathrm{tr}(\phi\bar{\phi}) - \sum_{s=1}^{N} \Big\{ \psi^{i}_{s} \eta_{i\bar{j}} z^{\bar{j}}_{\bar{s}} - \psi^{\bar{j}}_{\bar{s}} \bar{\phi}_{\bar{j}i} \psi^{i}_{s} \Big\} &(3.115) \\ &= i\Big\langle \phi, \bar{\phi} \Big\rangle - \overline{\Big\langle \boldsymbol{\psi}^{i}, \boldsymbol{z}^{j} \Big\rangle \eta_{\bar{i}j}} - \overline{\Big\langle \boldsymbol{\psi}^{i}, \boldsymbol{\psi}^{j} \Big\rangle \bar{\phi}_{\bar{i}j}} = i\Big\langle \phi, \bar{\phi} \Big\rangle - \overline{\Big\langle \psi, z\ {}^t\eta \Big\rangle} - \overline{\Big\langle \psi, \psi\ {}^t\bar{\phi} \Big\rangle}. &(3.116) \end{aligned}$$

Therefore,

$$\frac{i}{2}\delta\{<\psi, C\bar{\phi}> + ^* <\psi, C\bar{\phi}>\} = i\mathrm{tr}(\phi\bar{\phi}) + \sum_{s=1}^{N}\psi_{\bar{s}}^{\bar{i}}\bar{\phi}_{\bar{i}j}\psi_s^j + \frac{1}{2}\sum_{s=1}^{N}\left\{\psi_{\bar{s}}^{\bar{i}}z_s^j - \psi_s^j z_{\bar{s}}^{\bar{i}}\right\}\eta_{\bar{i}j}. \tag{3.117}$$

$$\delta <\psi_A, A> = \delta(\mathrm{tr}(\psi_A^\dagger A)) = \mathrm{tr}((A^\dagger + [A^\dagger, i\phi])A - \psi_A^\dagger\psi_A) \tag{3.118}$$

Since $\mathrm{tr}\big(i\phi A^\dagger A\big) = \mathrm{tr}\big(A^\dagger A i\phi\big)$,

$$\begin{aligned}\delta <\psi_A, A> &= \mathrm{tr}(A^\dagger A + A^\dagger i\phi A - A^\dagger A i\phi - \psi_A^\dagger\psi_A)\\ &= \mathrm{tr}(A^\dagger(A + [i\phi, A]) - \psi_A^\dagger\psi_A).\end{aligned} \tag{3.119}$$

Lastly, we write again the path integral that will be evaluated in the remaining part of this part.

$$Z_{MQ} := \beta\int_{V_k(\mathbb{C}^N)} Dz\int D\psi D\phi D\bar{\phi}DAD\eta D\chi DH\ \omega\exp(-\mathcal{L}_{MQ}) \tag{3.120}$$

$$\begin{aligned}\mathcal{L}_{MQ} &= \delta <\chi, H> + \delta <\psi_A, A> + \frac{i}{2}\delta\Big[<\psi, iz\bar{\phi}> + \overline{<\psi, iz\bar{\phi}>}\Big]\\ &= \Big\langle H, H\Big\rangle - \Big\langle\chi, \chi(I_k + i\ {}^t\phi)\Big\rangle + i\Big\langle\phi, \bar{\phi}\Big\rangle + \Big\langle\psi, \psi\ {}^t\bar{\phi}\Big\rangle\\ &+ \frac{1}{2}\Big\{\Big\langle\psi, z\ {}^t\eta\Big\rangle - \overline{\Big\langle\psi, z\ {}^t\eta\Big\rangle}\Big\} + \mathrm{tr}(A^\dagger(A + [i\phi, A]) - \psi_A^\dagger\psi_A)\\ &= \sum_{s=1}^{N}\Big[\sum_{i=1}^{k} H_{\bar{s}}^{\bar{i}}H_s^i - \big(\chi_{\bar{s}}^{\bar{i}}(\delta_{\bar{i}j} + i\phi_{\bar{i}j})\chi_s^j\big) + \psi_{\bar{s}}^{\bar{i}}\bar{\phi}_{\bar{i}l}\psi_s^l\Big] + i\mathrm{tr}(\phi\bar{\phi})\\ &+ \frac{1}{2}\sum_{s=1}^{N}\Big\{\psi_{\bar{s}}^{\bar{i}}z_s^j - \psi_s^j z_{\bar{s}}^{\bar{i}}\Big\}\eta_{\bar{i}j} + \mathrm{tr}(A^\dagger(A + [i\phi, A]) - \psi_A^\dagger\psi_A).\end{aligned} \tag{3.121}$$

As we have already mentioned, β is normalization factor for Chern class. and ω is projective operator along tangent space of $V_k(\mathbb{C}^N)$. They are given as follows.

$$\omega = \prod_{i,j=1}^{k}\Big[\sum_{s=1}^{N}(\psi_s^{\bar{i}}z_s^j + z_s^{\bar{i}}\psi_s^j)\Big], \quad \beta = \frac{\prod_{j=0}^{k-1} j!}{2^{2k}(-\pi)^{k^2+kN}\pi^{kN+\frac{k(k+1)}{2}}(-1)^{\frac{k}{2}(k-1)}}. \tag{3.122}$$

3.2 First Half of Evaluation of the Path Integral

3.2.1 $U(N)\times U(k)$ Symmetry of the Lagrangian

Lemma 1.

$$Z_{MQ} = \beta\int_{V_k(\mathbb{C}^N)} Dz\int D\psi D\phi D\bar{\phi}DAD\eta D\psi_A D\chi DH\ \omega\exp(-\mathcal{L}_{MQ})$$

$$= \beta \mathrm{vol}(V_k(\mathbb{C}^N)) \int D\psi D\phi D\bar{\phi} DAD\eta D\psi_A D\chi DH \; \omega' \exp\bigl(-\mathcal{L}'_{MQ}\bigr).$$

$$\mathcal{L}'_{MQ} = \sum_{s=1}^{N} \Bigl[\sum_{i=1}^{k} H^{\bar{i}}_{\bar{s}} H^{i}_{s} - \bigl(\chi^{\bar{i}}_{\bar{s}}(\delta_{\bar{i}j} + i\phi_{\bar{i}j})\chi^{j}_{s}\bigr) + \psi^{\bar{i}}_{\bar{s}} \bar{\phi}_{\bar{i}l} \psi^{l}_{s}\Bigr] + i\mathrm{tr}(\phi\bar{\phi})$$

$$+ \frac{1}{2} \sum_{i,j=1}^{k} \left\{\psi^{\bar{i}}_{\bar{j}} - \psi^{j}_{i}\right\} \eta_{\bar{i}j} + \mathrm{tr}(A^{\dagger}(A + [i\phi, A]) - \psi^{\dagger}_{A}\psi_{A}).$$

$$\omega' = \prod_{i=1}^{k} \prod_{j=1}^{k} \Bigl[(\psi^{\bar{i}}_{\bar{j}} + \psi^{j}_{i})\Bigr]. \tag{3.123}$$

Proof. (The proof is based on discussion with Professor Masao Jinzenji.) First, we note that projection operator ω is rewritten as follows:

$$\omega = \prod_{i=1}^{k} \prod_{j=1}^{k} (\sum_{s=1}^{N} (\psi^{\bar{i}}_{\bar{s}} z^{j}_{s} + \psi^{j}_{s} z^{\bar{i}}_{\bar{s}}))$$

$$= (-1)^{\frac{k^2}{2}(k^2+1)} \int D\theta \exp\left(< \psi, z\, {}^{t}\theta > + \overline{< \psi, z\, {}^{t}\theta >}\right), \tag{3.124}$$

where θ is fermionic Hermite matrix.

Hence we can rewrite Z_{MQ} by

$$Z_{MQ} = (-1)^{\frac{k^2}{2}(k^2+1)} \beta \int_{V_k(\mathbb{C}^N)} Dz$$

$$\times \int D\psi D\phi D\bar{\phi} DAD\eta D\psi_A D\chi DHD\theta \quad \exp\Bigl(-\widetilde{\mathcal{L}_{MQ}}(z, \psi, \phi, \bar{\phi}, A, \psi_A, \eta, \chi, H, \theta)\Bigr)$$

$$\widetilde{\mathcal{L}_{MQ}}(z, \psi, \phi, \bar{\phi}, A, \psi_A \eta, \chi, H, \theta)$$

$$:= \mathcal{L}_{MQ}(z, \psi, \phi, \bar{\phi}, A, \psi_A, \eta, \chi, H) + < \psi, z\, {}^{t}\theta > + \overline{< \psi, z\, {}^{t}\theta >}. \tag{3.125}$$

If we transform $\chi, \psi, \phi, \bar{\phi}, \eta, A, z, H, \theta$ in the following way,

$$\begin{aligned} &\chi = U^N \chi' U^k, && \psi = U^N \psi' U^k, && \phi = {}^{t}U^k \phi' \, {}^{*}U^k, && \bar{\phi} = {}^{t}U^k \bar{\phi}' \, {}^{*}U^k, \\ &\eta = {}^{t}U^k \eta' \, {}^{*}U^k, && A = {}^{t}U^k A' \, {}^{*}U^k, && z = U^N z' U^k, && H = U^N H' U^k, \\ &\theta = {}^{t}U^k \theta' \, {}^{*}U^k, && \psi_A = \psi'_A, \end{aligned} \tag{3.126}$$

we can easily see that $\widetilde{\mathcal{L}_{MQ}}$ has $U(N) \times U(k)$ symmetry. From $U^k U^{k\dagger} = U^{k\dagger} U^k = I_k$ and $U^N U^{N\dagger} = U^{N\dagger} U^N = I_N$, we obtain $\phi' = {}^{*}U^k \phi\, {}^{t}U^k$, $\bar{\phi}' = {}^{*}U^k \bar{\phi}\, {}^{t}U^k$, $\theta' = {}^{*}U^k \theta\, {}^{t}U^k$ and $\eta' = {}^{*}U^k \eta\, {}^{t}U^k$. Since $\eta,\ \theta,\ \phi,\ \bar{\phi}$ are Hermitian matrices, $\eta',\ \phi',\ \bar{\phi}'$ are Hermitian matrices. The Lagrangian can be expressed in the following form.

$$\sum_{s=1}^{N}\sum_{i=1}^{k} H_{\bar{s}}^{\bar{i}} H_s^i = \mathrm{tr}\big(H^\dagger, H\big) = \mathrm{tr}\big((U^N H' U^k)^\dagger U^N H' U^k\big) = \mathrm{tr}\big(H'^\dagger H'\big) \tag{3.127}$$

$$\begin{aligned}
\Big\langle \chi, \chi(I_k + i\ {}^t\phi)\Big\rangle &= \mathrm{tr}\Big(\chi^\dagger \chi(I_k + i\ {}^t\phi)\Big) \qquad (3.128)\\
&= \mathrm{tr}\Big((U^N \chi' U^k)^\dagger U^N \chi' U^k (I_k + i\ {}^t(\ {}^tU^k \phi'\ {}^*U^k))\Big)\\
&= \mathrm{tr}\Big(U^{k\dagger}\chi'^\dagger \chi' U^k (I_k + iU^{k\dagger}\ {}^t\phi' U^k)\Big)\\
&= \mathrm{tr}\Big(U^{k\dagger}\chi'^\dagger \chi' U^k (U^{k\dagger}U^k + iU^{k\dagger}\ {}^t\phi' U^k)\Big)\\
&= \mathrm{tr}\Big(\chi'^\dagger \chi'(I_k + i\ {}^t\phi')\Big) = \Big\langle \chi', \chi'(I_k + i\ {}^t\phi')\Big\rangle. \qquad (3.129)
\end{aligned}$$

$$\begin{aligned}
\Big\langle \phi, \bar{\phi}\Big\rangle &= \mathrm{tr}(\phi^\dagger \bar{\phi}) = \mathrm{tr}((\ {}^tU^k \phi'\ {}^*U^k)^\dagger\ {}^tU^k \bar{\phi}'\ {}^*U^k)\\
&= \mathrm{tr}(\ {}^tU^k \phi'^\dagger\ {}^*U^k\ {}^tU^k \bar{\phi}'\ {}^*U^k) = \mathrm{tr}(\phi'^\dagger \bar{\phi}') = \Big\langle \phi', \bar{\phi}'\Big\rangle. \qquad (3.130)
\end{aligned}$$

$$\begin{aligned}
\Big\langle \psi, \psi\ {}^t\bar{\phi}\Big\rangle &= \mathrm{tr}(\psi^\dagger \psi\ {}^t\bar{\phi}) = \mathrm{tr}((U^N \psi' U^k)^\dagger U^N \psi' U^k\ {}^t(\ {}^tU^k \bar{\phi}'\ {}^*U^k))\\
&= \mathrm{tr}(U^{k\dagger}\psi'^\dagger \psi'\ {}^t\bar{\phi}' U^k) = \mathrm{tr}(\psi'^\dagger \psi'\ {}^t\bar{\phi}') = \Big\langle \psi', \psi'\ {}^t\bar{\phi}'\Big\rangle. \qquad (3.131)
\end{aligned}$$

$$\begin{aligned}
\Big\langle \psi, z\ {}^t\eta\Big\rangle &= \mathrm{tr}(\psi^\dagger z\ {}^t\eta) = \mathrm{tr}((U^N \psi' U^k)^\dagger U^N z' U^k\ {}^t(\ {}^tU^k \eta'\ {}^*U^k))\\
&= \mathrm{tr}(U^{k\dagger}\psi'^\dagger z'\ {}^t\eta' U^k) = \mathrm{tr}(\psi'^\dagger z'\ {}^t\eta') = \Big\langle \psi', z'\ {}^t\eta'\Big\rangle. \qquad (3.132)
\end{aligned}$$

$$\begin{aligned}
< \psi, z\ {}^t\theta > &= \mathrm{tr}\big(\psi^\dagger z\ {}^t\theta\big) = \mathrm{tr}\big((U^N \psi' U^k)^\dagger U^N z' U^k\ {}^t(\ {}^tU^k \theta'\ {}^*U^k)\big)\\
&= \mathrm{tr}\big(U^{k\dagger}\psi'^\dagger z'\ {}^t\theta' U^k\big) = \mathrm{tr}\big(\psi'^\dagger z'\ {}^t\theta'\big) = \Big\langle \psi', z'\ {}^t\theta'\Big\rangle. \qquad (3.133)
\end{aligned}$$

$$\begin{aligned}
\mathrm{tr}(A^\dagger(A + [i\phi, A]) &= \mathrm{tr}((\ {}^tU^k A'\ {}^*U^k)^\dagger(\ {}^tU^k A'\ {}^*U^k + [i(\ {}^tU^k \phi'\ {}^*U^k),\ {}^tU^k A'\ {}^*U^k])\\
&= \mathrm{tr}(A'^\dagger A') + i\mathrm{tr}(\ {}^tU^k A'^\dagger\ {}^*U^k(\ {}^tU^k \phi'\ {}^*U^k\ {}^tU^k A'\ {}^*U^k\\
&-\ {}^tU^k A'\ {}^*U^k\ {}^tU^k \phi'\ {}^*U^k))\\
&= \mathrm{tr}(A'^\dagger A') + i\mathrm{tr}(\ {}^tU^k A'^\dagger\ {}^*U^k(\ {}^tU^k \phi' A'\ {}^*U^k -\ {}^tU^k A' \phi'\ {}^*U^k))\\
&= \mathrm{tr}(A'^\dagger A') + i\mathrm{tr}(A'^\dagger(\phi' A' - A'\phi')) = \mathrm{tr}(A'^\dagger(A'^\dagger + i[\phi' A'])).
\end{aligned} \tag{3.134}$$

Therefore,

$$\begin{aligned}
&\widetilde{\mathcal{L}_{MQ}}(z', \psi', \phi', \bar{\phi}', A', \psi'_A, \eta', \chi', H', \theta')\\
&= \widetilde{\mathcal{L}_{MQ}}(z, \psi, \phi, \bar{\phi}, A, \psi_A, \eta, \chi, H, \theta).
\end{aligned} \tag{3.135}$$

On the other hand, we can confirm that the integral measures of each variable is also $U(N) \times U(k)$ invariant: $DX = DX'$ (X is each variable). We confirm it for z and ϕ.

To do so, consider the following equation.

$$\left(\prod_{s=1}^{N} dz_s^1 \cdots dz_s^k\right) = \det\left(\frac{\partial z}{\partial z'}\right)\prod_{s=1}^{N} dz_s'^1 \cdots dz_s'^k. \tag{3.136}$$

$$\left(\prod_{s=1}^{N} d\bar{z}_{\bar{s}}^{\bar{1}} \cdots d\bar{z}_{\bar{s}}^{\bar{k}}\right) = \overline{\det\left(\frac{\partial z}{\partial z'}\right)}\prod_{s=1}^{N} d\bar{z}_{\bar{s}}'^{\bar{1}} \cdots d\bar{z}_{\bar{s}}'^{\bar{k}}. \tag{3.137}$$

$$\prod_{i=1}^{k} d\phi_{\bar{i}1} \cdots d\phi_{\bar{i}k} = \det\left(\frac{\partial \phi}{\partial \phi'}\right)\prod_{i=1}^{k} d\phi'_{\bar{i}1} \cdots d\phi'_{\bar{i}k}. \tag{3.138}$$

We compute $\det\left(\frac{\partial z}{\partial z'}\right)$ and $\det\left(\frac{\partial \phi}{\partial \phi'}\right)$. From $z = U^N z' U^k$ and $\phi = {}^tU^k \phi' \ {}^*U^k$, $z_s^i = U_{\bar{s}a}^N z_a'^j U_{\bar{j}i}^k (1 \leq s, a \leq N,\ 1 \leq i, j \leq k)$ and $\phi_{\bar{i}j} = ({}^tU^k)_{\bar{i}l}\phi'_{\bar{l}m}({}^*U^k)_{\bar{m}j} = U_{\bar{l}i}^k \phi'_{\bar{l}m} U_{m\bar{j}}^k$. Then $\frac{\partial z_s^i}{\partial z_a'^j} = U_{\bar{s}a}^N U_{\bar{j}i}^k$, $\frac{\partial \phi_{\bar{i}j}}{\partial \phi'_{\bar{l}m}} = U_{\bar{l}i}^k U_{m\bar{j}}^k$.

$$\left(\frac{\partial z}{\partial z'}\right) = \begin{pmatrix} \frac{\partial z_1^1}{\partial z_1'^1} & \cdots & \frac{\partial z_1^1}{\partial z_1'^k} & \cdots & \frac{\partial z_1^1}{\partial z_N'^1} & \cdots & \frac{\partial z_1^1}{\partial z_N'^k} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial z_1^k}{\partial z_1'^1} & \cdots & \frac{\partial z_1^k}{\partial z_1'^k} & \cdots & \frac{\partial z_1^k}{\partial z_N'^1} & \cdots & \frac{\partial z_1^k}{\partial z_N'^k} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial z_N^1}{\partial z_1'^1} & \cdots & \frac{\partial z_N^1}{\partial z_1'^k} & \cdots & \frac{\partial z_N^1}{\partial z_N'^1} & \cdots & \frac{\partial z_N^1}{\partial z_N'^k} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial z_N^k}{\partial z_1'^1} & \cdots & \frac{\partial z_N^k}{\partial z_1'^k} & \cdots & \frac{\partial z_N^k}{\partial z_N'^1} & \cdots & \frac{\partial z_N^k}{\partial z_N'^k} \end{pmatrix} \tag{3.139}$$

$$= \begin{pmatrix} U_{\bar{1}1}^N U_{\bar{1}1}^k & \cdots & U_{\bar{1}1}^N U_{\bar{k}1}^k & \cdots & U_{\bar{1}N}^N U_{\bar{1}1}^k & \cdots & U_{\bar{1}N}^N U_{\bar{k}1}^k \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ U_{\bar{1}1}^N U_{\bar{1}k}^k & \cdots & U_{\bar{1}1}^N U_{\bar{k}k}^k & \cdots & U_{\bar{1}N}^N U_{\bar{1}k}^k & \cdots & U_{\bar{1}N}^N U_{\bar{k}k}^k \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ U_{\bar{N}1}^N U_{\bar{1}1}^k & \cdots & U_{\bar{N}1}^N U_{\bar{k}1}^k & \cdots & U_{\bar{N}N}^N U_{\bar{1}1}^k & \cdots & U_{\bar{N}N}^N U_{\bar{k}1}^k \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ U_{\bar{N}1}^N U_{\bar{1}k}^k & \cdots & U_{\bar{N}1}^N U_{\bar{k}k}^k & \cdots & U_{\bar{N}N}^N U_{\bar{1}k}^k & \cdots & U_{\bar{N}N}^N U_{\bar{k}k}^k \end{pmatrix} \tag{3.140}$$

$$= \begin{pmatrix} U_{\bar{1}1}^N\ {}^tU^k & \cdots & U_{\bar{1}N}^N\ {}^tU^k \\ \vdots & \ddots & \vdots \\ U_{\bar{N}1}^N\ {}^tU^k & \cdots & U_{\bar{N}N}^N\ {}^tU^k \end{pmatrix} = U^N \otimes\ {}^tU^k \tag{3.141}$$

$\otimes$ means Kronecker product. $\det\left(\frac{\partial z}{\partial z'}\right) = (\det U^N)^k (\det U^k)^N$. U^N and U^k are Unitary matrixes:

$$\det\left(\frac{\partial z}{\partial z'}\right)\overline{\det\left(\frac{\partial z}{\partial z'}\right)} = 1. \tag{3.142}$$

Therefore, $Dz = Dz'$. In the same way,

$$\left(\frac{\partial \phi}{\partial \phi'}\right) = \begin{pmatrix} \frac{\partial \phi_{\bar{1}1}}{\partial \phi'_{\bar{1}1}} & \cdots & \frac{\partial \phi_{\bar{1}1}}{\partial \phi'_{\bar{1}k}} & \cdots & \frac{\partial \phi_{\bar{1}1}}{\partial \phi'_{\bar{k}1}} & \cdots & \frac{\partial \phi_{\bar{1}1}}{\partial \phi'_{\bar{k}k}} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial \phi_{\bar{1}k}}{\partial \phi'_{\bar{1}1}} & \cdots & \frac{\partial \phi_{\bar{1}k}}{\partial \phi'_{\bar{1}k}} & \cdots & \frac{\partial \phi_{\bar{1}k}}{\partial \phi'_{\bar{k}1}} & \cdots & \frac{\partial \phi_{\bar{1}k}}{\partial \phi'_{\bar{k}k}} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial \phi_{\bar{k}1}}{\partial \phi'_{\bar{1}1}} & \cdots & \frac{\partial \phi_{\bar{k}1}}{\partial \phi'_{\bar{1}k}} & \cdots & \frac{\partial \phi_{\bar{k}1}}{\partial \phi'_{\bar{k}1}} & \cdots & \frac{\partial \phi_{\bar{k}1}}{\partial \phi'_{\bar{k}k}} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial \phi_{\bar{k}k}}{\partial \phi'_{\bar{1}1}} & \cdots & \frac{\partial \phi_{\bar{k}k}}{\partial \phi'_{\bar{1}k}} & \cdots & \frac{\partial \phi_{\bar{k}k}}{\partial \phi'_{\bar{k}1}} & \cdots & \frac{\partial \phi_{\bar{k}k}}{\partial \phi'_{\bar{k}k}} \end{pmatrix} \tag{3.143}$$

$$= \begin{pmatrix} U^k_{\bar{1}1}U^k_{1\bar{1}} & \cdots & U^k_{\bar{1}1}U^k_{k\bar{1}} & \cdots & U^k_{\bar{k}1}U^k_{1\bar{1}} & \cdots & U^k_{\bar{k}1}U^k_{k\bar{1}} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ U^k_{\bar{1}1}U^k_{1\bar{k}} & \cdots & U^k_{\bar{1}1}U^k_{k\bar{k}} & \cdots & U^k_{\bar{k}1}U^k_{1\bar{k}} & \cdots & U^k_{\bar{k}1}U^k_{k\bar{k}} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ U^k_{\bar{1}k}U^k_{1\bar{1}} & \cdots & U^k_{\bar{1}k}U^k_{k\bar{1}} & \cdots & U^k_{\bar{k}k}U^k_{1\bar{1}} & \cdots & U^k_{\bar{k}k}U^k_{k\bar{1}} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ U^k_{\bar{1}k}U^k_{1\bar{k}} & \cdots & U^k_{\bar{1}k}U^k_{k\bar{k}} & \cdots & U^k_{\bar{k}k}U^k_{1\bar{k}} & \cdots & U^k_{\bar{k}k}U^k_{k\bar{k}} \end{pmatrix} \tag{3.144}$$

$$= \begin{pmatrix} U^k_{\bar{1}1}U^{k\dagger} & \cdots & U^k_{\bar{k}1}U^{k\dagger} \\ \vdots & \ddots & \vdots \\ U^k_{\bar{1}k}U^{k\dagger} & \cdots & U^k_{\bar{k}k}U^{k\dagger} \end{pmatrix} = U^k \otimes U^{k\dagger}. \tag{3.145}$$

Since $\det\left(\frac{\partial \phi}{\partial \phi'}\right) = (\det U^k)^k (\det U^{k\dagger})^k = 1$, $D\phi = D\phi'$. Integral measures for other variables are confirmed in the same way (note that the Jacobian for fermionic variables acts in reciprocal). Let us define $F_{MQ}(z)$ by

$$F_{MQ}(z) := (-1)^{\frac{k^2}{2}(k^2+1)}\beta \int D\psi D\phi D\bar{\phi} DAD\eta D\psi_A D\chi DHD\theta \\ \times \exp\Big(-\widetilde{\mathcal{L}_{MQ}}(z,\psi,\phi,\bar{\phi},A,\psi_A,\eta,\chi,H,\theta)\Big). \tag{3.146}$$

Then we obviously have

$$Z_{MQ} = \int_{V_k(\mathbb{C}^N)} Dz F_{MQ}(z). \tag{3.147}$$

Then by using (3.135) and invariance of integral measure, we obtain,

$$F_{MQ}(z) \quad = (-1)^{\frac{k^2}{2}(k^2+1)}\beta \int D\psi D\phi D\bar{\phi} DAD\eta D\psi_A D\chi DHD\theta \\ \exp\Big(-\widetilde{\mathcal{L}_{MQ}}(z,\psi,\phi,\bar{\phi},A,\psi_A,\eta,\chi,H,\theta)\Big)$$

$$
\begin{aligned}
&= (-1)^{\frac{k^2}{2}(k^2+1)}\beta \int D\psi D\phi D\bar{\phi} DAD\eta D\psi_A D\chi DHD\theta \\
&\exp\Big(-\widetilde{\mathcal{L}_{MQ}}(z', \psi', \phi', \bar{\phi}', A', \psi'_A, \eta', \chi', H', \theta')\Big) \\
&= (-1)^{\frac{k^2}{2}(k^2+1)}\beta \int D\psi' D\phi' D\bar{\phi}' DA' D\eta' D\psi'_A D\chi' DH' D\theta' \\
&\times \exp\Big(-\widetilde{\mathcal{L}_{MQ}}(z', \psi', \phi', \bar{\phi}', A', \psi'_A, \eta', \chi', H', \theta')\Big) \\
&= F_{MQ}(z') \qquad (3.148)
\end{aligned}
$$

For each $z \in V_k(\mathbb{C}^N)$, we can choose $U^N \in U(N)$ and $U^k \in U(k)$ that satisfy

$$
U^N z U^k = \begin{pmatrix} I_k \\ 0_{N-k,k} \end{pmatrix} =: z_0, \qquad (3.149)
$$

where I_k is $k \times k$-type unit matrix and $0_{N-k,k}$ is $(N-k) \times k$-type zero matrix. Hence we obtain

$$
Z_{M,Q} = \int_{V_k(\mathbb{C}^N)} Dz F_{MQ}(z) = \int_{V_k(\mathbb{C}^N)} Dz F_{MQ}(z_0) = \mathrm{vol}(V_k(\mathbb{C}^N)) F_{MQ}(z_0). \qquad (3.150)
$$

This is nothing but the assertion of the Lemma. □

3.2.2 Integration of Fields except for ψ

Note that we have integrated out z in the previous subsubsection. Next, we integrate out ψ_A, H and χ.

$$
\begin{aligned}
&\int D\psi_A \exp\Big(\mathrm{tr}(\psi_A^\dagger \psi_A)\Big) \\
&= \int d\psi_{A}{}_1^1 d\psi_{A}{}_{\bar{1}}^{\bar{1}} \cdots d\psi_{A}{}_1^k d\psi_{A}{}_{\bar{1}}^{\bar{k}} \cdots d\psi_{A}{}_k^1 d\psi_{A}{}_{\bar{k}}^{\bar{1}} \cdots d\psi_{A}{}_k^k d\psi_{A}{}_{\bar{k}}^{\bar{k}} \\
&\times \prod_{i=1}^{k} \{(1 + \psi_{A}{}_{\bar{1}}^{\bar{i}} \psi_{A}{}_1^i) \cdots (1 + \psi_{A}{}_{\bar{k}}^{\bar{i}} \psi_{A}{}_k^i)\} = (-1)^{k^2}. \qquad (3.151) \\
&\int DH \exp\Bigg(-\sum_{s=1}^{N}\sum_{i=1}^{k} H_{\bar{s}}^{\bar{i}} H_s^i\Bigg) \\
&:= \int \prod_{s=1}^{N} (\frac{i}{2})^k dH_s^1 dH_{\bar{s}}^{\bar{1}} \cdots dH_s^k dH_{\bar{s}}^{\bar{k}} \exp\Bigg(-\sum_{s=1}^{N}\sum_{i=1}^{k} H_{\bar{s}}^{\bar{i}} H_s^i\Bigg) = \pi^{kN}. \qquad (3.152) \\
&\int D\chi \exp\Bigg(\sum_{s=1}^{N} \chi_{\bar{s}}^{\bar{i}} (\delta_{\bar{i}j} + i\phi_{\bar{i}j}) \chi_s^j\Bigg) \\
&:= \int \prod_{s=1}^{N} d\chi_s^1 d\chi_{\bar{s}}^{\bar{1}} \cdots d\chi_s^k d\chi_{\bar{s}}^{\bar{k}} \exp\Bigg(\sum_{s=1}^{N} \chi_{\bar{s}}^{\bar{i}} (\delta_{\bar{i}j} + i\phi_{\bar{i}j}) \chi_s^j\Bigg)
\end{aligned}
$$

$$= (-1)^{kN} \int \prod_{s=1}^{N} d\chi_{\bar{s}}^{\bar{1}} d\chi_s^1 \cdots d\chi_{\bar{s}}^{\bar{k}} d\chi_s^k \exp\left(\sum_{s=1}^{N} \chi_{\bar{s}}^{\bar{i}} (\delta_{\bar{i}j} + i\phi_{\bar{i}j}) \chi_s^j \right)$$
$$= (-1)^{kN} \left(\det(I_k + i\phi)\right)^N . \tag{3.153}$$

Then let us integrate out η. We set

$$I_\eta := \int D\eta \exp\left(\frac{-1}{2} \sum_{i,j=1}^{k} \left\{ \psi_{\bar{j}}^{\bar{i}} - \psi_i^j \right\} \eta_{\bar{i}j} \right) \tag{3.154}$$

We abbreviate $\left\{ \psi_{\bar{j}}^{\bar{i}} - \psi_i^j \right\}$ as $\alpha^{\bar{i}j}$ and obtain

$$I_\eta = \int D\eta \left(\frac{-1}{2}\right)^{k^2} \prod_{i=1}^{k} \prod_{j=1}^{k} \alpha^{\bar{i}j} \eta_{\bar{i}j}$$
$$= \left(\frac{-1}{2}\right)^{k^2} (-1)^{\frac{k^2}{2}(k^2+1)} \int D\eta \left[\prod_{i=1}^{k} \prod_{j=1}^{k} \eta_{\bar{i}j} \right] \left[\prod_{i=1}^{k} \prod_{j=1}^{k} \alpha^{\bar{i}j} \right] \tag{3.155}$$
$$= \left(\frac{-1}{2}\right)^{k^2} (-1)^{\frac{k^2}{2}(k^2+1)} \left[\prod_{i=1}^{k} \prod_{j=1}^{k} \alpha^{\bar{i}j} \right]. \tag{3.156}$$

We also abbreviate $\left\{ \psi_{\bar{j}}^{\bar{i}} + \psi_i^j \right\}$ as $\beta^{\bar{i}j}$ and evaluate $I_\eta \omega$. Since

$$\left[\prod_{i=1}^{k} \prod_{j=1}^{k} \alpha^{\bar{i}j} \right] \left[\prod_{i=1}^{k} \prod_{j=1}^{k} \beta^{\bar{i}j} \right] = (-1)^{\frac{k^2}{2}(k^2+1)} \left[\prod_{i=1}^{k} \prod_{j=1}^{k} \alpha^{\bar{i}j} \beta^{\bar{i}j} \right], \tag{3.157}$$

and $\alpha^{\bar{i}j}\beta^{\bar{i}j} = (\psi_{\bar{j}}^{\bar{i}} - \psi_i^j)(\psi_{\bar{j}}^{\bar{i}} + \psi_i^j) = -2\psi_i^j \psi_{\bar{j}}^{\bar{i}}$, we obtain

$$I_\eta \omega = \left(\frac{-1}{2}\right)^{k^2} \left[\prod_{i=1}^{k} \prod_{j=1}^{k} \alpha^{\bar{i}j} \beta^{\bar{i}j} \right] = \prod_{i=1}^{k} \prod_{j=1}^{k} \psi_i^j \psi_{\bar{j}}^{\bar{i}}$$
$$= \left(\prod_{i=1}^{k} \psi_i^i \psi_{\bar{i}}^{\bar{i}} \right) \left(\prod_{i<j} \psi_i^j \psi_{\bar{j}}^{\bar{i}} \right) \left(\prod_{i>j} \psi_i^j \psi_{\bar{j}}^{\bar{i}} \right)$$
$$= \left(\prod_{i=1}^{k} \psi_i^i \psi_{\bar{i}}^{\bar{i}} \right) \left(\prod_{i<j} \psi_i^j \psi_{\bar{j}}^{\bar{i}} \psi_j^i \psi_{\bar{i}}^{\bar{j}} \right) = \left(\prod_{i=1}^{k} \psi_i^i \psi_{\bar{i}}^{\bar{i}} \right) \left(\prod_{i<j} (-\psi_i^j \psi_{\bar{i}}^{\bar{j}} \psi_j^i \psi_{\bar{j}}^{\bar{i}}) \right)$$
$$= (-1)^{\frac{k(k-1)}{2}} \prod_{i=1}^{k} \prod_{j=1}^{k} \psi_i^j \psi_{\bar{i}}^{\bar{j}}. \tag{3.158}$$

Then the result of these integrations is given as follows.

$$Z_{MQ} = \beta \int_{V_k(\mathbb{C}^N)} DzD\psi D\phi D\bar{\phi}DA \ (-1)^{k^2+\frac{k(k-1)}{2}} (-\pi)^{kN} \left(\det(I_k + i\phi)\right)^N$$
$$\exp\left(-\Big\{ \sum_{s=1}^{N} \psi^{\bar{i}}_{\bar{s}} \bar{\phi}_{\bar{i}l} \psi^l_s + i\mathrm{tr}(\phi\bar{\phi}) + \mathrm{tr}(A^\dagger(A + [i\phi, A])) \Big\} \right) \prod_{i=1}^{k}\prod_{j=1}^{k} \psi^j_i \psi^{\bar{j}}_{\bar{i}}. \tag{3.159}$$

In order to integrate out ϕ and $\bar{\phi}$, we we decompose complex fields ϕ, $\bar{\phi}$ and ψ into real parts and imaginary parts:

$$\phi_{\bar{i}j} = \phi^R_{ij} + i\phi^I_{ij}, \qquad \phi_{i\bar{j}} = \phi^R_{ij} - i\phi^I_{ij}. \tag{3.160}$$
$$\bar{\phi}_{\bar{i}j} = \bar{\phi}^R_{ij} + i\bar{\phi}^I_{ij}, \qquad \bar{\phi}_{i\bar{j}} = \bar{\phi}^R_{ij} - i\bar{\phi}^I_{ij}. \tag{3.161}$$
$$\psi^i_s = \psi^i_{Rs} + i\psi^i_{Is}, \qquad \psi^{\bar{i}}_{\bar{s}} = \psi^i_{Rs} - i\psi^i_{Is}, \tag{3.162}$$

and define integration measures for $\bar{\phi}$ and ϕ as

$$D\bar{\phi} := \prod_{i=1}^{k} d\bar{\phi}^R_{\bar{i}i} \prod_{j=i+1}^{k} d\bar{\phi}^R_{ij} d\bar{\phi}^I_{ij}, \tag{3.163}$$
$$D\phi := \prod_{i=1}^{k} d\phi^R_{\bar{i}i} \prod_{j=i+1}^{k} d\phi^R_{ij} d\phi^I_{ij}. \tag{3.164}$$

Note that $\bar{\phi}^I_{ii} = \phi^I_{ii} = 0$ and $\phi^R_{ij} = \phi^R_{ji}$, $\bar{\phi}^R_{ij} = \bar{\phi}^R_{ji}$, $\phi^I_{ij} = -\phi^I_{ji}$, $\phi^I_{ij} = -\phi^I_{ji}(i \neq j)$ since both $\bar{\phi}$ and ϕ are both Hermitian matrices. Normally, we should consider the integral measure of ψ because ψ^i_s and $\psi^{\bar{i}}_{\bar{s}}$ are Grassmann variables. However, we skip this process. Since the reverse transformation is done, the integral measure of ψ is not changed. Then integration of ϕ and $\bar{\phi}$ results in the following lemma:

Lemma 2. *Let $\lambda_i (i = 1, 2, \cdots, k)$ be eigenvalues of ϕ. Then we have,*

$$\int D\bar{\phi} \exp\left(-i\mathrm{tr}(\phi\bar{\phi}) - \sum_{s=1}^{N} \psi^{\bar{i}}_{\bar{s}} \bar{\phi}_{\bar{i}l} \psi^l_s \right) = \frac{(2\pi)^{k^2}}{2^{k(k-1)}} \left\{ \prod_{i=1}^{k} \delta\left(\phi_{\bar{i}i} + 2\sum_{s=1}^{N} \psi^i_{Rs}\psi^i_{Is} \right) \right\}$$
$$\times \left\{ \prod_{i=1}^{k-1}\prod_{j=i+1}^{k} \delta\left(\phi^R_{ji} + \sum_{s=1}^{N} (\psi^i_{Rs}\psi^j_{Is} - \psi^i_{Is}\psi^j_{Rs}) \right) \times \delta\left(\phi^I_{ji} - \sum_{s=1}^{N} (\psi^i_{Rs}\psi^j_{Rs} + \psi^i_{Is}\psi^j_{Is}) \right) \right\}. \tag{3.165}$$

$$\int DA \exp\left(-\mathrm{tr}(A^\dagger A + A^\dagger[i\phi, A]) \right) = \frac{\pi^{k^2}}{\prod_{l<j}(1 - (i\lambda_l - i\lambda_j)^2)}. \tag{3.166}$$

We prove it in Appendix. By using the above lemma, we obtain

$$Z_{MQ} = \beta \int D\psi' DzD\phi \frac{(-1)^{\frac{k}{2}(k-1)}(2\pi)^{k^2}(-\pi)^{k^2+kN}\left(\det(I_k+i\phi)\right)^N}{2^{k(k-1)}\prod_{l>j}\left(1-(i\lambda_l-i\lambda_j)^2\right)}$$
$$\left\{\prod_{i=1}^{k}\delta\left(\phi_{\bar{i}i}+2\sum_{s=1}^{N}\psi^i_{Rs}\psi^i_{Is}\right)\right\}\left\{\prod_{i=1}^{k-1}\prod_{j=i+1}^{k}\delta\left(\phi^R_{ji}+\sum_{s=1}^{N}(\psi^i_{Rs}\psi^j_{Is}-\psi^i_{Is}\psi^j_{Rs})\right)\right.$$
$$\left.\times\,\delta\left(\phi^I_{ji}-\sum_{s=1}^{N}(\psi^i_{Rs}\psi^j_{Rs}+\psi^i_{Is}\psi^j_{Is})\right)\right\}\prod_{i=1}^{k}\prod_{j=1}^{k}\psi^j_i\psi^{\bar{j}}_{\bar{i}}. \tag{3.167}$$

Then, integration of ϕ results in replacement of ϕ fields by the following composite of ψ fields.

$$\phi_{\bar{i}i} \to -2\sum_{s=1}^{N}\psi^i_{Rs}\psi^i_{Is} = -\sum_{s=1}^{N}(\psi^i_{Rs}\psi^i_{Is}-\psi^i_{Is}\psi^i_{Rs})$$
$$= -i\sum_{s=1}^{N}(\psi^i_{Rs}+i\psi^i_{Is})(\psi^i_{Rs}-i\psi^i_{Is}) = -i\sum_{s=1}^{N}\psi^i_s\psi^{\bar{i}}_{\bar{s}}, \tag{3.168}$$

$$\phi_{\bar{j}i} = \phi^R_{ji}+i\phi^I_{ji} \to \sum_{s=1}^{N}\{-(\psi^i_{Rs}\psi^j_{Is}-\psi^i_{Is}\psi^j_{Rs})+i(\psi^i_{Rs}\psi^j_{Rs}+\psi^i_{Is}\psi^j_{Is})\}$$
$$= i\sum_{s=1}^{N}\{\psi^i_{Rs}\psi^j_{Rs}+\psi^i_{Is}\psi^j_{Is}+i(\psi^i_{Rs}\psi^j_{Is}-\psi^i_{Is}\psi^j_{Rs}) \tag{3.169}$$
$$= i\sum_{s=1}^{N}\psi^{\bar{i}}_{\bar{s}}\psi^j_s = -i\sum_{s=1}^{N}\psi^j_s\psi^{\bar{i}}_{\bar{s}}. \tag{3.170}$$

At this stage, let us introduce the following anti-Hermitian matrix:

$$\Phi_s := \begin{pmatrix} \psi^1_s\psi^{\bar{1}}_{\bar{s}} & \cdots & \psi^1_s\psi^{\bar{k}}_{\bar{s}} \\ \vdots & \ddots & \vdots \\ \psi^k_s\psi^{\bar{1}}_{\bar{s}} & \cdots & \psi^k_s\psi^{\bar{k}}_{\bar{s}} \end{pmatrix}, \quad \Phi := \sum_{s=1}^{N}\Phi_s. \tag{3.171}$$

Then the result of integration of ϕ is summarized by replacement of ϕ by $-i\Phi$. Let λ'_i $(i=1,\cdots,k)$ be eigenvalues of the Φ. Then integration of $\bar{\phi}$ and ϕ results in,

$$Z_{MQ} = \beta\mathrm{vol}(V_k(\mathbb{C}^N))\int D\psi\frac{(-1)^{\frac{k}{2}(k-1)}2^k\pi^{k^2}(-\pi)^{k^2+kN}\left(\det(I_k+\Phi)\right)^N}{\prod_{l>j}\left(1-(\lambda'_l-\lambda'_j)^2\right)}\prod_{i=1}^{k}\prod_{j=1}^{k}\psi^j_i\psi^{\bar{j}}_{\bar{i}}. \tag{3.172}$$

On the other hand, we obtain from (3.94),

$$\beta \mathrm{vol}(V_k(\mathbb{C}^N))(-1)^{\frac{k}{2}(k-1)}2^k\pi^{k^2}(-\pi)^{k^2+kN} = \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!}, \tag{3.173}$$

and reach the final expression of Z_{MQ} in this section:

$$Z_{MQ} = \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!}\int D\psi \frac{(\det(I_k+\Phi))^N}{\prod_{l>j}\left(1-(\lambda'_l-\lambda'_j)^2\right)}\prod_{i=1}^{k}\prod_{j=1}^{k}\psi_i^j\psi_{\bar{i}}^{\bar{j}}. \tag{3.174}$$

4 Second Half: Proof of the Main Theorem

4.1 Free Fermion Realization of Cohomology Ring of $G(k,N)$

In the previous section, we reached the expression (3.174). Then what remains to prove is the following equality:

$$\begin{aligned} Z_{MQ} &= \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!}\int D\psi \frac{(\det(I_k+\Phi))^N}{\prod_{l>j}\left(1-(\lambda'_l-\lambda'_j)^2\right)}\prod_{i=1}^{k}\prod_{j=1}^{k}\psi_i^j\psi_{\bar{i}}^{\bar{j}} && (4.175)\\ &= \int_{G(k,N)} c(T'G(k,N)) && (4.176)\\ &= \int_{G(k,N)} \frac{\prod_{i=1}^{k}(1+x_i)}{\prod_{l>j}(1-(x_l-x_j)^2)} && (4.177)\\ &= \binom{N}{k}, && (4.178)\end{aligned}$$

where $c(S^*) = \prod_{i=1}^{k}(1+x_i)$. First, we note that the factor $\prod_{i=1}^{k}\psi_i^j\psi_{\bar{i}}^{\bar{j}}$ allows us to neglect ψ_i^j, $\psi_{\bar{i}}^{\bar{j}}$ $(i,j=1,2,\cdots,k)$ in the integral measure and the remaining part of the integrand. Hence we only have to consider the fields ψ_i^j, $\psi_{\bar{i}}^{\bar{j}}$ $(i=k+1,\cdots,N, j=1,\cdots,k)$. At this stage, we redefine ψ_{k+i}^j, $\psi_{\overline{k+i}}^{\bar{j}}$ by ψ_i^j, $\psi_{\bar{i}}^{\bar{j}}$ $(i=1,\cdots,N-k,\ j=1,\cdots k)$ and introduce

$$\Phi' := \sum_{s=1}^{N-k}\begin{pmatrix} \omega_s^{1\bar{1}} & \ldots & \omega_s^{1\bar{k}} \\ \vdots & \ddots & \vdots \\ \omega_s^{k\bar{1}} & \ldots & \omega_s^{k\bar{k}} \end{pmatrix} \qquad (\omega_s^{i\bar{j}} := \psi_s^i\psi_{\bar{s}}^{\bar{j}}), \tag{4.179}$$

$$D\psi' = \prod_{s=1}^{N-k} d\psi_s^1 d\psi_{\bar{s}}^{\bar{1}}\cdots d\psi_s^k d\psi_{\bar{s}}^{\bar{k}}. \tag{4.180}$$

Let λ_i $(i=1,\cdots,k)$ be eigenvalues of the matrix Φ'. Then (3.174) is rewritten as follows.

$$Z_{MQ} = \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!}\int D\psi' \frac{(\det(I_k+\Phi'))^N}{\prod_{l>j}(1-(\lambda_l-\lambda_j)^2)}. \tag{4.181}$$

Here, we prove Theorem 2. We represent Theorem 2 again.

Theorem. *Let us define* b_i $(i = 0, 1, 2, \cdots)$ *by*

$$\frac{1}{\det(I_k + t\Phi')} = \sum_{m=0}^{\infty} b_m t^m. \tag{4.182}$$

Then, $b_m = 0$ *if* $m > N - k$.

Proof. (Proof of Theorem 2. (The proof is based on discussion with Professor Masao Jinzenji.))

By using Gaussian integral of complex variables $X_1, \cdots, X_k$, we obtain the equality:

$$\frac{1}{\det(I_k + t\Phi')} = \int_{\mathbb{C}^n} \mathcal{D}X \exp\big\{(-\,{}^t\bar{X}(I_k + t\Phi')X)\big\}. \tag{4.183}$$

where $X = {}^t(X_1, \cdots, X_k)$ and integral measure is given by $\mathcal{D}X := \frac{1}{(-2\pi i)^k} dX_1 dX_{\bar{1}} \cdots dX_k dX_{\bar{k}}$.

$$\begin{aligned}
(R.H.S) &= \int_{\mathbb{C}^n} \mathcal{D}X \exp\left\{(-X_{\bar{i}}(\delta_{\bar{i}j} + t\sum_{\mu=1}^{N-k}\omega_{\mu}^{i\bar{j}})X_j)\right\} \\
&= \int_{\mathbb{C}^n} \mathcal{D}X \{e^{-|X|^2} e^{-t\sum_{\mu=1}^{N-k}\omega_{\mu}^{i\bar{j}} X_{\bar{i}} X_j}\} \\
&= \int_{\mathbb{C}^n} \mathcal{D}X e^{-|X|^2} \sum_{m=0}^{\infty} \frac{1}{m!}\Big(-t\sum_{\mu=1}^{N-k}\omega_{\mu}^{i\bar{j}} X_{\bar{i}} X_j\Big)^m \\
&= \sum_{m=0}^{\infty} \frac{(-t)^m}{m!} \sum_{(\mu_1,\cdots,\mu_m)} \sum_{\substack{(i_1,\cdots,i_m)\\(j_1,\cdots,j_m)}} \omega_{\mu_1}^{i_1\bar{j}_1}\cdots\omega_{\mu_m}^{i_m\bar{j}_m} \int_{\mathbb{C}^n} \mathcal{D}X e^{-|X|^2} X_{\bar{i}_1} X_{j_1} \cdots X_{\bar{i}_m} X_{j_m}.
\end{aligned} \tag{4.184}$$

Here, $|X|^2$ represents $\sum_{i=1}^{k} X_{\bar{i}} X_i$. We remark that $1 \leq \mu_l \leq N-k$, $1 \leq i_l, j_l \leq k$, $(l = 1, \cdots, m)$. By Wick's theorem of Gaussian integral, we can easily see that the integral in the last line of (4.184) does not vanish if and only if $\{i_1, \cdots, i_m\} = \{j_1, \cdots, j_m\}$. Hence we obtain

$$\begin{aligned}
&\int_{\mathbb{C}^n} \mathcal{D}X \exp\big\{(-\,{}^t\bar{X}(I_k + t\Phi')X)\big\} \\
&= \sum_{m=0}^{\infty} \frac{(-t)^m}{m!} \sum_{(\mu_1,\cdots,\mu_m)} \sum_{(i_1,\cdots,i_m)} \sum_{\sigma\in S_m} \mathrm{Sym}(i_1,\cdots,i_m)\omega_{\mu_1}^{i_1\bar{i}_{\sigma(1)}}\cdots\omega_{\mu_m}^{i_m\bar{i}_{\sigma(m)}} \\
&\times \int_{\mathbb{C}^n} \mathcal{D}X e^{-|X|^2} \prod_{j=1}^{m} |X_{i_j}|^2.
\end{aligned} \tag{4.185}$$

where $\mathrm{Sym}(i_1, \cdots, i_m)$ is symmetric factor of the m-tuple $(i_1, \cdots, i_m)$ given by

$$\begin{aligned}
\mathrm{Sym}(i_1,\cdots,i_m) &= \prod_{j=1}^{k} \frac{1}{\mathrm{mul}((i_1,\cdots,i_m);j)!}, \\
\mathrm{mul}((i_1,\cdots,i_m);j) &:= (\text{ number of } l\text{'s that satisfy } i_l = j).
\end{aligned} \tag{4.186}$$

Then let us fix $(\mu_1,\cdots,\mu_m)$ and $(i_1,\cdots,i_m)$, and assume that there exists a pair (i,j) $(1\le i<j\le m)$ that satisfy $\mu_i=\mu_j$. Without loss of generality, we can further assume that $\mu_1=\mu_2=\mu$. Obviously, for any $\sigma\in S_m$ we can uniquely take $\sigma'\in S_m$ that satisfy $\sigma'(1)=\sigma(2),\ \sigma'(2)=\sigma(1),\ \sigma'(i)=\sigma(i)(i=3,4,\cdots m)$. Then, we can easily see

$$\begin{aligned}
&\omega_{\mu}^{i_1\bar{i}_{\sigma(1)}}\omega_{\mu}^{i_2\bar{i}_{\sigma(2)}}\omega_{\mu_3}^{i_3\bar{i}_{\sigma(3)}}\cdots\omega_{\mu_m}^{i_m\bar{i}_{\sigma(m)}}+\omega_{\mu}^{i_1\bar{i}_{\sigma'(1)}}\omega_{\mu}^{i_2\bar{i}_{\sigma'(2)}}\omega_{\mu_3}^{i_3\bar{i}_{\sigma'(3)}}\cdots\omega_{\mu_m}^{i_m\bar{i}_{\sigma'(m)}}\\
=\;&\omega_{\mu}^{i_1\bar{i}_{\sigma(1)}}\omega_{\mu}^{i_2\bar{i}_{\sigma(2)}}\omega_{\mu_3}^{i_3\bar{i}_{\sigma(3)}}\cdots\omega_{\mu_m}^{i_m\bar{i}_{\sigma(m)}}+\omega_{\mu}^{i_1\bar{i}_{\sigma(2)}}\omega_{\mu}^{i_2\bar{i}_{\sigma(1)}}\omega_{\mu_3}^{i_3\bar{i}_{\sigma(3)}}\cdots\omega_{\mu_m}^{i_m\bar{i}_{\sigma(m)}}\\
=\;&0 \qquad (4.187)
\end{aligned}$$

because $\omega_{\mu}^{i_1\bar{i}_{\sigma(2)}}\omega_{\mu}^{i_2\bar{i}_{\sigma(1)}}=\psi_{\mu}^{i_1}\psi_{\bar{\mu}}^{\bar{i}_{\sigma(2)}}\psi_{\mu}^{i_2}\psi_{\bar{\mu}}^{\bar{i}_{\sigma(1)}}=-\psi_{\mu}^{i_1}\psi_{\bar{\mu}}^{\bar{i}_{\sigma(1)}}\psi_{\mu}^{i_2}\psi_{\bar{\mu}}^{\bar{i}_{\sigma(2)}}=-\omega_{\mu}^{i_1\bar{i}_{\sigma(1)}}\omega_{\mu}^{i_2\bar{i}_{\sigma(2)}}$. Hence

$$\sum_{\sigma\in S_m}\mathrm{Sym}(i_1,\cdots,i_m)\omega_{\mu_1}^{i_1\bar{i}_{\sigma(1)}}\cdots\omega_{\mu_m}^{i_m\bar{i}_{\sigma(m)}}$$

in the last line of (4.185) vanishes if some μ_j's in the m-tuple $(\mu_1,\cdots,\mu_m)$ coincide. Since $1\le\mu_j\le N-k$, it follows that the summand in the last line of (4.185) vanishes if $m>N-k$. □

Here, we note the explicit form of b_i in Theorem 2. Since

$$\frac{i}{2\pi}\int_{\mathbb{C}}dzd\bar{z}|z|^{2m}e^{-|z|^2}=m!\ \ (m=0,1,2,\cdots), \qquad (4.188)$$

in (4.185),

$$\int_{\mathbb{C}^n}\mathcal{D}Xe^{-|X|^2}\prod_{j=1}^{m}|X_{i_j}|^2=\prod_{j=1}^{k}\mathrm{mul}((i_1,\cdots,i_m);j)!. \qquad (4.189)$$

Therefore, Theorem 2 is rewritten as follows.

$$\frac{1}{\det(I_k+t\Phi')}=\sum_{m=0}^{N-k}t^m\frac{(-1)^m}{m!}\sum_{(\mu_1,\cdots,\mu_m)}\sum_{(i_1,\cdots,i_m)}\sum_{\sigma\in S_m}\omega_{\mu_1}^{i_1\bar{i}_{\sigma(1)}}\cdots\omega_{\mu_m}^{i_m\bar{i}_{\sigma(m)}}. \qquad (4.190)$$

Next, we prove Theorem 4. We represent it again.

Theorem.

$$H^*(G(k,N))=\frac{\mathbb{R}[c_1(S^*),\cdots,c_k(S^*)]}{(a_i=0\ \ (i>N-k))}\simeq\mathbb{R}[\sigma_1,\cdots,\sigma_k] \qquad (4.191)$$

Proof. (Proof of Theorem 4. (The proof is given by Professor Masao Jinzenji.))
σ_j $(j=1,2\cdots,k)$ is defined by

$$1+\sigma_1t+\cdots+\sigma_kt^k:=\det(I_k+t\Phi')=\prod_{j=1}^{k}(1+\lambda_jt). \qquad (4.192)$$

Then, let us define a ring homomorphism $f : \mathbb{R}[c_1(S^*), \cdots, c_k(S^*)] \to \mathbb{R}[\sigma_1, \cdots, \sigma_k]$ by

$$f(c_j(S^*)) = \sigma_j \quad (j = 1, 2, \cdots, k). \tag{4.193}$$

It is a surjection by definition. Then,

$$\mathbb{R}[c_1(S^*), \cdots, c_k(S^*)]/\ker(f) \simeq \mathbb{R}[\sigma_1, \cdots, \sigma_k]. \tag{4.194}$$

On the other hand, the cohomology ring of $G(k, N)$ is given by

$$H^*(G(k, N)) = \frac{\mathbb{R}[c_1(S^*), \cdots, c_k(S^*)]}{(a_i = 0 \ \ (i > N - k))}. \tag{4.195}$$

from Subsection 1.1. Then let us consider the ring $\mathbb{R}[\sigma_1, \cdots, \sigma_k]$. Since in

$$\sum_{i=0}^{N-k} b_i t^i = \frac{1}{\det(I_k + t\Phi')} = \frac{1}{1 + \sigma_1 t + \cdots + \sigma_k t^k}, \tag{4.196}$$

$b_i = 0 \ \ (i > N - k)$ from Theorem 2. Since $f(a_l) = b_l = 0 (i \geq N - k + 1)$ from (1.9) (the expansion of $\frac{1}{c(S^*)}$ by $c_i(S^*)$) and Theorem 2, $\{a_i = 0 \ \ (i > N - k)\} \subset \ker(f)$. Therefore, the surjective map $\tilde{f} : \frac{\mathbb{R}[c_1(S^*), \cdots, c_k(S^*)]}{(a_i = 0 \ \ (i > N-k))} \to \mathbb{R}[c_1(S^*), \cdots, c_k(S^*)]/\ker(f) \simeq \mathbb{R}[\sigma_1, \cdots, \sigma_k]$ is induced by f.

We prove that $\tilde{f}$ is injective. We assume that $\ker(\tilde{f}) \neq \{0\}$. Then $\exists a \in \frac{\mathbb{R}[c_1(S^*), \cdots, c_1(S^*)]}{(a_i = 0 \ \ (i > N-k))}$ such that $a \neq 0$ and $\tilde{f}(a) = 0$. Here, we note about degree of $\mathbb{R}[c_1(S^*), \cdots, c_k(S^*)]$ and $\mathbb{R}[\sigma_1, \cdots, \sigma_k]$. Let us define the degree of $\sigma_j (j = 1, \cdots, k)$ and $c_j(S^*)$ by j. The degree of σ_j is the number of $\omega^{i\bar{j}}$ contained in each term of σ_j. Then, $\mathbb{R}[c_1(S^*), \cdots, c_k(S^*)]$ and $\mathbb{R}[\sigma_1, \cdots, \sigma_k]$ are graded ring and f preserves degree. Therefore, $\frac{\mathbb{R}[c_1(S^*), \cdots, c_k(S^*)]}{(a_i = 0 \ \ (i > N-k))}$ is a graded ring and $\tilde{f}$ preserve degree. We fix the degree of a as $i (1 \leq i \leq kN - k^2)$. Since $G(k, N)$ is a compact oriented manifold, from the Poincaré duality theorem, $\exists b \in \frac{\mathbb{R}[c_1(S^*), \cdots, c_1(S^*)]}{(a_i = 0 \ \ (i > N-k))}$ such that the degree of b is $Nk - k^2 - i$ and b satisfies

$$a \cdot b = (c_k(S^*))^{N-k}. \tag{4.197}$$

$\cdot$ means the exterior product of differential form. Therefore, $\tilde{f}(a \cdot b) = (\sigma_k)^{N-k}$. Here, we remark Theorem 3.

Theorem.

$$\frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!} \int D\psi' \left(\det(\Phi')\right)^{N-k} = 1. \tag{4.198}$$

It tells us that $(\sigma_k)^{N-k}$ is a non-vanishing element of $\mathbb{R}[\sigma_1, \cdots, \sigma_k]$. Therefore, $\tilde{f}(a) \neq 0$. □

In Subsection 1.1, we have introduced formal line bundle decomposition $S^* = \oplus_{i=1}^{k} L_i$ and the relation:

$$c(S^*) = \prod_{j=1}^{k}(1 + x_j t) \ \ (x_j = c_1(L_j)). \tag{4.199}$$

Theorem 4 shows that x_j and λ_j are identical. Normalization condition of integration on $G(k, N)$ is given by

$$\int_{G(k,N)} (c_k(S^*))^{N-k} = 1.$$

Therefore, Theorem 3 also leads us to the following equality:(1.45)

$$\int_{G(k,N)} g(x_1, \cdots, x_k) = \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!} \int D\psi' g(\lambda_1, \cdots, \lambda_k),$$

where $g(x_1, \cdots, x_k)$ is a symmetric polynomial of $x_1, \cdots, x_k$ that represents an element of $H^*(G(k, N))$. By combining (4.181) with (1.45), we obtain

$$\begin{aligned}
Z_{MQ} &= \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!} \int D\psi' \frac{(\det(I_k + \Phi'))^N}{\prod_{l>j}(1 - (\lambda_l - \lambda_j)^2)} \\
&= \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!} \int D\psi' \frac{\prod_{i=1}^{k}(1 + \lambda_i)^N}{\prod_{l>j}(1 - (\lambda_l - \lambda_j)^2)} = \int_{G(k,N)} \frac{\prod_{i=1}^{k}(1 + x_i)^N}{\prod_{l>j}(1 - (x_l - x_j)^2)} \\
&= \int_{G(k,N)} c(T'G(k, N)) = \int_{G(k,N)} c_{top}(T'G(k, N)) \\
&= \chi(G(k, N)) = \binom{N}{k}.
\end{aligned} \tag{4.200}$$

In the last line, $\chi(G(k, N)) = \binom{N}{k}$ is given by the cell decomposition [24]. ($\chi(G(k, N))$ is given by solving the excersice 2.13 (ii) in [24] but it is given by only counting the number of bases of each chain. So, we omit its detail.) This completes proof of the main theorem. □

4.2 Proof of Theorem 3

(The discussion of this section is given by Professor Masao Jinzenji.) We consider how to represent coefficients of the power of the determinant of $k \times k$ matrix X.

Definition 2. *Let $\mathcal{M}_l$ be set of $k \times k$ matrix M_l:*

$$M_l := \begin{pmatrix} m_{1,1}^l & \dots & m_{1,k}^l \\ \vdots & \ddots & \vdots \\ m_{k,1}^l & \dots & m_{k,k}^l \end{pmatrix}, \tag{4.201}$$

whose (i,j)-element $m^l_{i,j}$ is given by non-negative integer that satisfies the following conditions:

$$\sum_{i=1}^{k} m^l_{i,j} = l, \quad (j = 1, \cdots, k), \quad \sum_{j=1}^{k} m^l_{i,j} = l \quad (i = 1, \cdots, k). \tag{4.202}$$

Definition 3. *Let S_k be symmetric group of size k. For $\sigma \in S_k$, we define $k \times k$ matrix $R(\sigma)$:*

$$R(\sigma) := \begin{pmatrix} \delta_{\sigma(1),1} & \dots & \delta_{\sigma(1),k} \\ \vdots & \ddots & \vdots \\ \delta_{\sigma(k),1} & \dots & \delta_{\sigma(k),k} \end{pmatrix}, \tag{4.203}$$

where $\delta_{i,j}$ is Kronecker's delta symbol.

Then, we obtain the poroposition as follows.

Proposition 1. *We denote by $(n_\sigma)_{\sigma \in S_k}$ a sequence of $k!$ non-negative integers labeled by $\sigma \in S_k$. Let $\mathcal{N}_l$ be set of $(n_\sigma)_{\sigma \in S_k}$'s that satisfy $\sum_{\sigma \in S_k} n_\sigma = l$. Then $\varphi : \mathcal{N}_l \to \mathcal{M}_l$ defined by*

$$\varphi((n_\sigma)_{\sigma \in S_k}) = \sum_{\sigma \in S_k} n_\sigma R(\sigma) \in \mathcal{M}_l, \tag{4.204}$$

is a surjection.

Proof. In the $l = 1$ case, assertion of the proposition is obvious because $\varphi : \mathcal{N}_1 \to \mathcal{M}_1$ is a bijection. Then we can prove the proposition by induction of l. We assume that $\varphi : \mathcal{N}_l \to \mathcal{M}_l$ is a surjection. By definition, any element of $\mathcal{M}_{l+1}$ is represented by $M_l + R(\tau), M_l \in \mathcal{M}_l, \tau \in S_k$. By the assumption, there is $(n_\sigma)_{\sigma \in S_k} \in \mathcal{N}_l$ such that $\varphi((n_\sigma)_{\sigma \in S_k}) = \sum_{\sigma \in S_k} n_\sigma R(\sigma) = M_l$. Then, let us define $(m_\sigma)_{\sigma \in S_k}$ by $m_\tau = n_\tau + 1$ and $m_\sigma = n_\sigma (\sigma \neq \tau)$. Since $\sum_{\sigma \in S_k} m_\sigma = l + 1$, $(m_\sigma)_{\sigma \in S_k} \in \mathcal{N}_{l+1}$. □

Remark 1. *For general k and l, φ is not injective. For example, in the $k = l = 3$ case, we have the following equalities:*

$$\begin{aligned} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned} \tag{4.205}$$

Definition 4. *Let X be $k \times k$ matrix whose (i,j)-element is given by $x_{i,j}$. For $M_l \in \mathcal{M}_l$, we define integer $\widetilde{\mathrm{mul}}(M_l)$ by the following expansion:*

$$|X|^l = \sum_{M_l \in \mathcal{M}_l} \widetilde{\mathrm{mul}}(M_l) \prod_{a,b=1}^{k} x_{a,b}^{m^l_{a,b}}. \tag{4.206}$$

Proposition 2. $\widetilde{\text{mul}}(M_l)$ *is explicitly evaluated as follows.*

$$\widetilde{\text{mul}}(M_l) = \sum_{\varphi((n_\sigma)_{\sigma\in S_k})=M_l} \frac{l!\prod_{\sigma\in S_k}\Big(\text{sgn}(\sigma)\Big)^{n_\sigma}}{\prod_{\sigma\in S_k} n_\sigma!} \tag{4.207}$$

Proof. First, we explicitly expand $|X|^l$ by using definition of $|X|$:

$$\begin{aligned}
|X|^l &= \Big(\sum_{\sigma\in S_k}\text{sgn}(\sigma)\prod_{a=1}^{k} x_{a,\sigma(a)}\Big)^l \\
&= \sum_{(n_\sigma)_{\sigma\in S_k}\in\mathcal{N}_l} \frac{l!}{\prod_{\sigma\in S_k} n_\sigma!}\prod_{\sigma\in S_k}\Big((\text{sgn}(\sigma))^{n_\sigma}\prod_{a=1}^{k} x_{a,\sigma(a)}^{n_\sigma}\Big) \\
&= \sum_{(n_\sigma)_{\sigma\in S_k}\in\mathcal{N}_l} \frac{l!}{\prod_{\sigma\in S_k} n_\sigma!}\prod_{\sigma\in S_k}\Big((\text{sgn}(\sigma))^{n_\sigma}\prod_{a,b=1}^{k} x_{a,b}^{n_\sigma\delta_{\sigma(a),b}}\Big) \\
&= \sum_{(n_\sigma)_{\sigma\in S_k}\in\mathcal{N}_l} \frac{l!}{\prod_{\sigma\in S_k} n_\sigma!}\prod_{\sigma\in S_k}\Big((\text{sgn}(\sigma))^{n_\sigma}\Big)\prod_{a,b=1}^{k} x_{a,b}^{\sum_{\sigma\in S_k} n_\sigma\delta_{\sigma(a),b}}.
\end{aligned} \tag{4.208}$$

$\sum_{\sigma\in S_k} n_\sigma\delta_{\sigma(a),b}$ is nothing but the (a,b)-element of $\sum_{\sigma\in S_l} n_\sigma R(\sigma)$ and $\sum_{\sigma\in S_l} n_\sigma R(\sigma) = \varphi((n_\sigma)_{\sigma\in S_k}) \in \mathcal{M}_l$. Then assertion of proposition immediately follows from (4.208). □

Here, we introduce important property of $\widetilde{\text{mul}}(M_l)$.

Lemma 3. *The following equality holds,*

$$\widetilde{\text{mul}}(M_{l+1}) = \sum_{\sigma\in S_k}\text{sgn}(\sigma)\widetilde{\text{mul}}(M_{l+1}-R(\sigma)), \tag{4.209}$$

where we set $\widetilde{\text{mul}}(M_{l+1}-R(\sigma)) = 0$ *if* $M_{l+1}-R(\sigma)\notin\mathcal{M}_l$.

Proof.

$$\begin{aligned}
|X|^{l+1} &= \sum_{M_{l+1}\in\mathcal{M}_{l+1}} \widetilde{\text{mul}}(M_{l+1})\prod_{a,b=1}^{k} x_{a,b}^{m_{a,b}^{l+1}} \\
&= |X|^l\cdot|X| \\
&= \Bigg[\sum_{M_l\in\mathcal{M}_l}\widetilde{\text{mul}}(M_l)\prod_{a,b=1}^{k} x_{a,b}^{m_{a,b}^{l}}\Bigg]\times\Bigg[\sum_{\sigma\in S_k}\text{sgn}(\sigma)\prod_{a=1}^{k} x_{a\sigma(a)}\Bigg] \\
&= \sum_{M_l\in\mathcal{M}_l}\sum_{\sigma\in S_k}\text{sgn}(\sigma)\widetilde{\text{mul}}(M_l)\prod_{a,b=1}^{k} x_{a,b}^{m_{a,b}^{l}+\delta_{\sigma(a)b}} \\
&= \sum_{M_{l+1}\in\mathcal{M}_{l+1}}\sum_{\sigma\in S_k}\text{sgn}(\sigma)\widetilde{\text{mul}}(M_{l+1}-R(\sigma))\prod_{a,b=1}^{k} x_{a,b}^{m_{a,b}^{l+1}}.
\end{aligned}$$

□

With above preparation, Theorem 3 is shown.

Definition 5. *Let* ψ_s^i $(s = 1, \cdots, l,\ i = 1, \cdots, k)$ *be complex fermionic variable and* $\psi_{\bar{s}}^{\bar{i}}$ *be its complex conjugate. We denote by* Φ' *a* $k \times k$ *matrix whose* (i, j)*-element is given by* $\sum_{s=1}^{l} \psi_s^i \psi_{\bar{s}}^{\bar{j}}$. *Then we define* $C(l, k)$ *as follows.*

$$C(l,k) := \int \Big(\prod_{s=1}^{l} \prod_{i=1}^{k} d\psi_s^i d\psi_{\bar{s}}^{\bar{i}} \Big) \det(\Phi')^l. \tag{4.210}$$

We prove Theorem 3 by finding the recurrence relation of $C(l, k)$ for l. For brevity, we introduce the following notations.

$$\begin{aligned} D\psi &:= \prod_{s=1}^{l} \prod_{i=1}^{k} d\psi_s^i d\psi_{\bar{s}}^{\bar{i}}, \\ \omega_s^{i\bar{j}} &:= \psi_s^i \psi_{\bar{s}}^{\bar{j}}. \end{aligned} \tag{4.211}$$

We note here the equalities:

$$(\omega_s^{i\bar{j}})^2 = \omega_s^{i\bar{j}} \omega_s^{i\bar{k}} = \omega_s^{i\bar{j}} \omega_s^{k\bar{j}} = 0, \tag{4.212}$$

which plays an important role in proof of the next lemma.

Lemma 4.

$$C(l,k) = \sum_{M_l \in \mathcal{M}_l} \left[\prod_{a,b=1}^{k} (m_{a,b}^l)! \right] \widetilde{\mathrm{mul}}(M_l)^2. \tag{4.213}$$

Proof. By substituting Φ' for X in (4.206), we rewrite (4.210) as follows.

$$\begin{aligned} C(l,k) &= \int D\psi (\det \Phi')^l = \sum_{M_l \in \mathcal{M}_l} \widetilde{\mathrm{mul}}(M_l) \int D\psi \prod_{a,b=1}^{k} \Big(\sum_{s=1}^{l} \omega_s^{a\bar{b}} \Big)^{m_{a,b}^l} \\ &= \sum_{M_l \in \mathcal{M}_l} \widetilde{\mathrm{mul}}(M_l) \int D\psi \prod_{a,b=1}^{k} \Bigg((m_{a,b}^l)! \Big(\sum_{1 \le s_{a,b}^1 < \cdots < s_{a,b}^{m_{a,b}^l} \le l} \omega_{s_{a,b}^1}^{a\bar{b}} \cdots \omega_{s_{a,b}^{m_{a,b}^l}}^{a\bar{b}} \Big) \Bigg) \\ &= \sum_{M_l \in \mathcal{M}_l} \widetilde{\mathrm{mul}}(M_l) \left[\prod_{a,b=1}^{k} (m_{a,b}^l)! \right] \int D\psi \Big(\sum_{1 \le s_{a,b}^1 < \cdots < s_{a,b}^{m_{a,b}^l} \le l} \prod_{a,b=1}^{k} \omega_{s_{a,b}^1}^{a\bar{b}} \cdots \omega_{s_{a,b}^{m_{a,b}^l}}^{a\bar{b}} \Big). \end{aligned} \tag{4.214}$$

In going from the first line to the second line, we used an equality:

$$\Big(\sum_{s=1}^{l}\omega_s^{a\bar{b}}\Big)^{m^l_{a,b}} = (m^l_{a,b})!\Big(\sum_{1\leq s^1_{a,b}<\cdots<s^{m^l_{a,b}}_{a,b}\leq l}\omega^{a\bar{b}}_{s^1_{a,b}}\cdots\omega^{a\bar{b}}_{s^{m^l_{a,b}}_{a,b}}\Big), \tag{4.215}$$

which follows from $(\omega_s^{a\bar{b}})^2 = 0$. Let us define $S_{a,b} := \{s^1_{a,b},\cdots,s^{m^l_{a,b}}_{a,b}\}$ $(|S_{a,b}| = m^l_{a,b})$ associated with the sequence $1\leq s^1_{a,b}<\cdots<s^{m^l_{a,b}}_{a,b}\leq l$ in the last line of (4.214). Since $\omega_s^{i\bar{j}}\omega_s^{i\bar{k}} = \omega_s^{i\bar{j}}\omega_s^{k\bar{j}} = 0$, S_{ab}'s associated with non-vanishing $\prod_{a,b=1}^{k}\omega^{a\bar{b}}_{s^1_{a,b}}\cdots\omega^{a\bar{b}}_{s^{m^l_{a,b}}_{a,b}}$ satisfy the following condition:

$$\begin{aligned}\coprod_{a=1}^{k} S_{a,b} &= \{1,\cdots,l\} \quad (b=1,\cdots,k),\\ \coprod_{b=1}^{k} S_{a,b} &= \{1,\cdots,l\} \quad (a=1,\cdots,k).\end{aligned} \tag{4.216}$$

We denote by $\{S_{a,b}\}$ set of $S_{a,b}$'s $(a,b=1,\cdots,k)$ that satisfy $|S_{a,b}| = m^l_{a,b}$ and the condition (4.216). We also denote by $S(M_l)$ set of $\{S_{a,b}\}$'s associated with the matrix $\big(m^l_{a,b}\big) = M_l \in \mathcal{M}_l$. Then we can further rewrite $C(l,k)$ into the following form:

$$C(l,k) = \sum_{M_l\in\mathcal{M}_l}\widetilde{\mathrm{mul}}(M_l)\left[\prod_{a,b=1}^{k}(m^l_{a,b})!\right]\int D\psi\Big(\sum_{\{S_{a,b}\}\in S(M_l)}\prod_{a,b=1}^{k}\omega^{a\bar{b}}_{s^1_{a,b}}\cdots\omega^{a\bar{b}}_{s^{m^l_{a,b}}_{a,b}}\Big). \tag{4.217}$$

We take a closer look at the sum $\sum_{\{S_{a,b}\}\in S(M_l)}\prod_{a,b=1}^{k}\omega^{a\bar{b}}_{s^1_{a,b}}\cdots\omega^{a\bar{b}}_{s^{m^l_{a,b}}_{a,b}}$. For a fixed element $\{S_{a,b}\}\in S(M_l)$, we can construct a sequence $(\sigma_1,\sigma_2,\cdots,\sigma_l)$ of permutations $\sigma_s \in S_k$ $(s=1,\cdots,l)$. This is because for each $s\in\{1,2,\cdots,l\}$, we can fix unique permutation $\sigma_s\in S_k$ that satisfy $s\in S_{a,\sigma_s(a)}$ $(a=1,\cdots,k)$ by using the condition (4.216). Obviously, the sequence $(\sigma_1,\sigma_2,\cdots,\sigma_l)$ satisfy the following condition:

$$\sum_{s=1}^{l} R(\sigma_s) = M_l. \tag{4.218}$$

Conversely, for a sequence $(\sigma_1,\cdots,\sigma_l)$ that satisfy (4.218), we can construct unique $\{S_{a,b}\}\in S(M_l)$ that satisfies $s\in S_{a,\sigma_s(a)}$ $(a=1,\cdots,k)$. Hence we have one to one correspondence between $\{S_{a,b}\}\in S(M_l)$ and a sequence $(\sigma_1,\cdots,\sigma_l)$ that satisfy (4.218). Since we can construct $\frac{l!}{\prod_{\sigma\in S_k}n_\sigma!}$ different elements of $S(M_l)$ from a fixed

$(n_\sigma)_{\sigma\in S_k}$ that satisfies $\varphi((n_\sigma)_{\sigma\in S_k}) = M_l$, the following equality holds.

$$\int D\psi\Big(\sum_{\{S_{a,b}\}\in S(M_l)}\prod_{a,b=1}^{k}\omega^{a\bar{b}}_{s^1_{a,b}}\cdots\omega^{a\bar{b}}_{s^{m^l_{a,b}}_{a,b}}\Big)$$

$$= \sum_{\varphi((n_\sigma)_{\sigma\in S_k})=M_l}\frac{l!}{\prod_{\sigma\in S_k} n_\sigma!}\int D\psi\prod_{s=1}^{l}\omega_s^{1\overline{\sigma_s(1)}}\cdots\omega_s^{k\overline{\sigma_s(k)}}$$

Then we obtain

$$\begin{aligned}
&C(l,k)\\
&= \sum_{M_l\in\mathcal{M}_l}\widetilde{\mathrm{mul}}(M_l)\left[\prod_{a,b=1}^{k}(m^l_{a,b})!\right]\left[\sum_{\varphi((n_\sigma)_{\sigma\in S_k})=M_l}\frac{l!}{\prod_{\sigma\in S_k} n_\sigma!}\int D\psi\prod_{s=1}^{l}\Big(\omega_s^{1\overline{\sigma_s(1)}}\cdots\omega_s^{k\overline{\sigma_s(k)}}\Big)\right]\\
&= \sum_{M_l\in\mathcal{M}_l}\widetilde{\mathrm{mul}}(M_l)\left[\prod_{a,b=1}^{k}(m^l_{a,b})!\right]\left[\sum_{\varphi((n_\sigma)_{\sigma\in S_k})=M_l}\frac{l!}{\prod_{\sigma\in S_k} n_\sigma}\prod_{\sigma}\mathrm{sgn}(\sigma)^{n_\sigma}\right]\\
&= \sum_{M_l\in\mathcal{M}_l}\widetilde{\mathrm{mul}}(M_l)\left[\prod_{a,b=1}^{k}(m^l_{a,b})!\right]\widetilde{\mathrm{mul}}(M_l). \qquad (4.219)
\end{aligned}$$

In going from the third line to the last line, we used (4.207). □

Here we introduce the following Lemma [15]. In [20], an other proof is given using a fermion integral. But we do not present here. For details, please refer to [20].

Lemma 5. *(cf.* **[15]***)*

$$\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)\Big(\prod_{a=1}^{k}\frac{\partial}{\partial x_{a,\sigma(a)}}\Big)|X|^{l+1} = \frac{(k+l)!}{l!}|X|^l. \qquad (4.220)$$

The recurrence relation of $C(l,k)$ for l is given as follows.

Lemma 6.

$$C(l+1,k) = \frac{(l+k)!}{l!}C(l,k). \qquad (4.221)$$

Proof. We start from Lemma 4 applied to $C(l+1,k)$.

$$C(l+1,k) = \sum_{M_{l+1}\in\mathcal{M}_{l+1}}\left[\prod_{a,b=1}^{k}(m^{l+1}_{a,b})!\right]\widetilde{\mathrm{mul}}(M_{l+1})^2 \qquad (4.222)$$

By combining the above equality with (4.209), we obtain

$$\begin{aligned} C(l+1,k) &= \sum_{M_{l+1}\in\mathcal{M}_{l+1}} \sum_{\sigma,\tau\in S_k} \left[\prod_{a,b=1}^{k} (m_{a,b}^{l+1})!\right] \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) \\ &\quad \times \widetilde{\mathrm{mul}}(M_{l+1}-R(\sigma))\widetilde{\mathrm{mul}}(M_{l+1}-R(\tau)). \end{aligned} \tag{4.223}$$

We set $M_l = M_{l+1} - R(\sigma)$ and rewrite further the above equality.

$$\begin{aligned} C(l+1,k) &= \sum_{M_l\in\mathcal{M}_l} \sum_{\sigma,\tau\in S_k} \left[\prod_{a,b=1}^{k} (m_{a,b}^{l}+\delta_{\sigma(a),b})!\right] \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau)\widetilde{\mathrm{mul}}(M_l) \\ &\quad \times \widetilde{\mathrm{mul}}(M_l+R(\sigma)-R(\tau)) \\ &= \sum_{M_l\in\mathcal{M}_l} \sum_{\sigma,\tau\in S_k} \left[\prod_{a,b=1}^{k} (m_{a,b}^{l}+\delta_{\sigma(a),b})!\right] \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau)\widetilde{\mathrm{mul}}(M_l) \\ &\quad \times \widetilde{\mathrm{mul}}(M_l+R(\sigma)-R(\tau)) \\ &= \sum_{M_l\in\mathcal{M}_l} \left[\prod_{a,b=1}^{k} (m_{a,b}^{l})!\right] \widetilde{\mathrm{mul}}(M_l) \sum_{\sigma,\tau\in S_k} \left[\prod_{a=1}^{k} (m_{a,\sigma(a)}^{l}+1)\right] \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) \\ &\quad \times \widetilde{\mathrm{mul}}(M_l+R(\sigma)-R(\tau)). \end{aligned} \tag{4.224}$$

Hence in order to prove the lemma, we only have to confirm the following equality:

$$\sum_{\sigma,\tau\in S_k} \left[\prod_{a=1}^{k} (m_{a,\sigma(a)}^{l}+1)\right] \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau)\widetilde{\mathrm{mul}}(M_l+R(\sigma)-R(\tau)) = \frac{(k+l)!}{l!}\widetilde{\mathrm{mul}}(M_l). \tag{4.225}$$

If $\sigma=\tau$, $\mathrm{sgn}(\sigma)\mathrm{sgn}(\tau)\widetilde{\mathrm{mul}}(M_l+R(\sigma)-R(\tau))$ obviously equals $\widetilde{\mathrm{mul}}(M_l)$. Let us use the representation $M_l = \sum_{\sigma'\in S_k} n_{\sigma'} R(\sigma')$. If $\sigma\neq\tau$, we have the following representation:

$$M_l + R(\sigma) - R(\tau) = \sum_{\sigma'\neq\sigma,\tau} n_{\sigma'}R(\sigma') + (n(\sigma)+1)R(\sigma) + (n(\tau)-1)R(\tau). \tag{4.226}$$

Applying (4.207) carefully to these two cases, we obtain

$$\begin{aligned} &\mathrm{sgn}(\sigma)\mathrm{sgn}(\tau)\widetilde{\mathrm{mul}}(M_l+R(\sigma)-R(\tau)) \\ &= \begin{cases} \displaystyle\sum_{\varphi((n_{\sigma'})_{\sigma'\in S_k})=M_l} \frac{l!}{\prod_{\sigma'\in S_k} n_{\sigma'}!} \prod_{\sigma'\in S_k} \Big(\mathrm{sgn}(\sigma')\Big)^{n_{\sigma'}} & (\sigma=\tau), \\ \displaystyle\sum_{\varphi((n_{\sigma'})_{\sigma'\in S_k})=M_l} \frac{l!}{\prod_{\sigma'\in S_k} n_{\sigma'}!} \times \frac{n_\tau}{n_\sigma+1} \prod_{\sigma'\in S_k} \Big(\mathrm{sgn}(\sigma')\Big)^{n_{\sigma'}} & (\sigma\neq\tau). \end{cases} \end{aligned} \tag{4.227}$$

Hence we can rewrite the l.h.s. of (4.225) as follows.

$$\begin{aligned}
&\sum_{\sigma,\tau\in S_k}\Big(\prod_{a=1}^{k}(m^{l}_{a,\sigma(a)}+1)\Big)\mathrm{sgn}(\sigma)\mathrm{sgn}(\tau)\widetilde{\mathrm{mul}}(M_l+R(\sigma)-R(\tau))\\
&=\sum_{\sigma\in S_k}\Big(\prod_{a=1}^{k}(m^{l}_{a,\sigma(a)}+1)\Big)\Big\{\sum_{\varphi((n_{\sigma'})_{\sigma'\in S_k})=M_l}\frac{l!}{\prod_{\sigma'\in S_k}n_{\sigma'}!}\prod_{\sigma'\in S_k}\Big(\mathrm{sgn}(\sigma')\Big)^{n_{\sigma'}}\\
&+\sum_{\tau\neq\sigma}\sum_{\varphi((n_{\sigma'})_{\sigma'\in S_k})=M_l}\frac{l!}{\prod_{\sigma'\in S_k}n_{\sigma'}!}\times\frac{n_\tau}{n_\sigma+1}\prod_{\sigma'\in S_k}\Big(\mathrm{sgn}(\sigma')\Big)^{n_{\sigma'}}\Big\}\\
&=\sum_{\sigma\in S_k}\Big(\prod_{a=1}^{k}(m^{l}_{a,\sigma(a)}+1)\Big)\Big\{\sum_{\varphi((n_{\sigma'})_{\sigma'\in S_k})=M_l}\frac{l!}{\prod_{\sigma'\in S_k}n_{\sigma'}!}\prod_{\sigma'\in S_k}\Big(\mathrm{sgn}(\sigma')\Big)^{n_{\sigma'}}\\
&+\sum_{\varphi((n_{\sigma'})_{\sigma'\in S_k})=M_l}\frac{l!}{\prod_{\sigma'\in S_k}n_{\sigma'}!}\times\frac{l-n_\sigma}{n_\sigma+1}\prod_{\sigma'\in S_k}\Big(\mathrm{sgn}(\sigma')\Big)^{n_{\sigma'}}\Big\}\\
&=\sum_{\sigma\in S_k}\Big(\prod_{a=1}^{k}(m^{l}_{a,\sigma(a)}+1)\Big)\sum_{\varphi((n_{\sigma'})_{\sigma'\in S_k})=M_l}\frac{(l+1)!}{(n_\sigma+1)\prod_{\sigma'\in S_k}n_{\sigma'}!}\prod_{\sigma'\in S_k}\Big(\mathrm{sgn}(\sigma')\Big)^{n_{\sigma'}}.
\end{aligned}\tag{4.228}$$

In going from the second expression to the third expression, we used the equality:

$$\sum_{\tau\neq\sigma}n_\tau=\sum_{\tau\in S_k}n_\tau-n_\sigma=l-n_\sigma.\tag{4.229}$$

At this stage, we explicitly compute the l.h.s. of Lemma 5 $\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)\Big(\prod_{a=1}^{k}\frac{\partial}{\partial x_{a,\sigma(a)}}\Big)|X|^{l+1}$ by using the expansion $|X|^{l+1}=\sum_{M_{l+1}\in\mathcal{M}_{l+1}}\widetilde{\mathrm{mul}}(M_{l+1})\prod_{a,b=1}^{k}x_{a,b}^{m^{l+1}_{a,b}}$.

$$\begin{aligned}
&\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)\Big(\prod_{a=1}^{k}\frac{\partial}{\partial x_{a\sigma(a)}}\Big)|X|^{l+1}\\
&=\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)\Big(\prod_{a=1}^{k}\frac{\partial}{\partial x_{a\sigma(a)}}\Big)\sum_{M_{l+1}\in\mathcal{M}_{l+1}}\sum_{\varphi((n_\tau)_{\tau\in S_k})=M_{l+1}}\frac{(l+1)!}{\prod_{\tau\in S_k}n_\tau!}\\
&\times\Big[\prod_{\tau\in S_k}\Big(\mathrm{sgn}(\tau)\Big)^{n_\tau}\Big]\prod_{a,b=1}^{k}x_{a,b}^{m^{l+1}_{a,b}}\\
&=\sum_{\sigma\in S_k}\sum_{M_{l+1}\in\mathcal{M}_{l+1}}\sum_{\varphi((n_\tau)_{\tau\in S_k})=M_{l+1}}\frac{(l+1)!\mathrm{sgn}(\sigma)}{\prod_{\tau\in S_k}n_\tau!}\Big[\prod_{\tau\in S_k}\Big(\mathrm{sgn}(\tau)\Big)^{n_\tau}\Big]\Big[\prod_{a=1}^{k}m^{l+1}_{a,\sigma(a)}x_{a,\sigma(a)}^{m^{l+1}_{a,\sigma(a)}-1}\Big]\\
&\times\prod_{a=1}^{k}\prod_{\substack{b=1\\(b\neq\sigma(a))}}^{k}x_{a,b}^{m^{l+1}_{a,b}}.
\end{aligned}\tag{4.230}$$

In the last expression, $\varphi((n_\tau)_{\tau\in S_k}) = M_{l+1}$ that corresponds to non vanishing summand satisfies the condition $n_\sigma \geq 1$. Hence we can set $M_{l+1} = M_l + R(\sigma)$ with $\varphi((m_\tau)_{\tau\in S_k}) = M_l$. Then we can further rewrite the above expression

$$
\begin{aligned}
&= \sum_{\sigma\in S_k} \sum_{M_l\in\mathcal{M}_l} \sum_{\varphi((m_\tau)_{\tau\in S_k})=M_l} \frac{(l+1)!}{(m_\sigma+1)\prod_\tau m_\tau!} (\mathrm{sgn}(\sigma))^2 \left[\prod_{\tau\in S_k} \Big(\mathrm{sgn}(\tau)\Big)^{m_\tau}\right] \\
&\times \left[\prod_{a=1}^{k} (m^l_{a,\sigma(a)}+1) x_{a,\sigma(a)}^{m^l_{a,\sigma(a)}}\right] \prod_{a=1}^{k} \prod_{\substack{b=1 \\ (b\neq\sigma(a))}}^{k} x_{a,b}^{m^l_{a,b}} \\
&= \sum_{\sigma\in S_k} \sum_{M_l\in\mathcal{M}_l} \sum_{\varphi((m_\tau)_{\tau\in S_k})=M_l} \frac{(l+1)!}{(m_\sigma+1)\prod_\tau m_\tau!} \left[\prod_{\tau\in S_k} \Big(\mathrm{sgn}(\tau)\Big)^{m_\tau}\right] \\
&\times \left[\prod_{a=1}^{k} (m^l_{a,\sigma(a)}+1)\right] \prod_{a,b=1}^{k} x_{a,b}^{m^l_{a,b}}.
\end{aligned} \tag{4.231}
$$

By combining, the above derivation with assertion of Lemma 5,

$$
\begin{aligned}
&\sum_{\sigma\in S_k} \mathrm{sgn}(\sigma) \Big(\prod_{a=1}^{k} \frac{\partial}{\partial x_{a\sigma(a)}}\Big) |X|^{l+1} \\
&= \sum_{M_l\in\mathcal{M}_l} \sum_{\varphi((m_\tau)_{\tau\in S_k})=M_l} \sum_{\sigma\in S_k} \frac{(l+1)!}{(m_\sigma+1)\prod_\tau m_\tau!} \left[\prod_{\tau\in S_k} \Big(\mathrm{sgn}(\tau)\Big)^{m_\tau}\right] \\
&\times \left[\prod_{a=1}^{k} (m^l_{a,\sigma(a)}+1)\right] \prod_{a,b=1}^{k} x_{a,b}^{m^l_{a,b}} = \frac{(k+l)!}{l!} \sum_{M_l\in\mathcal{M}_l} \widetilde{\mathrm{mul}}(M_l) \prod_{a,b=1}^{k} x_{a,b}^{m^l_{a,b}},
\end{aligned} \tag{4.232}
$$

we obtain

$$
\begin{aligned}
&\sum_{\substack{M_l= \\ \sum_\tau m_\tau R(\tau)}} \sum_{\sigma\in S_k} \frac{(l+1)!}{(m_\sigma+1)\prod_\tau m_\tau!} \left[\prod_{\tau\in S_k} \Big(\mathrm{sgn}(\tau)\Big)^{m_\tau}\right] \left[\prod_{a=1}^{k} (m^l_{a,\sigma(a)}+1)\right] \\
&= \frac{(k+l)!}{l!} \widetilde{\mathrm{mul}}(M_l).
\end{aligned} \tag{4.233}
$$

By comparing (4.228) with (4.233), we reach the equality:

$$
\sum_{\sigma,\tau\in S_k} \left[\prod_{a=1}^{k} (m^l_{a,\sigma(a)}+1)\right] \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) \widetilde{\mathrm{mul}}(M_l + R(\sigma) - R(\tau)) = \frac{(k+l)!}{l!} \widetilde{\mathrm{mul}}(M_l), \tag{4.234}
$$

which completes the proof of the lemma. □

Proof of Theorem 3

Proof. $C(1,k)$ is calculated as follows.

$$C(1,k) = \sum_{\sigma\in S_k} \mathrm{sgn}(\sigma) \int D\psi \psi_1^1 \psi_{\bar{1}}^{\overline{\sigma(1)}} \cdots \psi_1^k \psi_{\bar{1}}^{\overline{\sigma(k)}}$$
$$= \sum_{\sigma\in S_k} (\mathrm{sgn}(\sigma))^2 \int D\psi \psi_1^1 \psi_{\bar{1}}^{\bar{1}} \cdots \psi_1^k \psi_{\bar{1}}^{\bar{k}} = \sum_{\sigma\in S_k} 1 = k!. \tag{4.235}$$

Then successive use of Lemma 6 leads us to

$$C(l,k) = \frac{(l+k-1)!}{(l-1)!} \cdots \frac{(k+1)!}{1!} C(1,k) = \frac{\prod_{j=0}^{k+l-1} j!}{\prod_{j=0}^{k-1} j! \prod_{j=0}^{l-1} j!}. \tag{4.236}$$

Hence in the case of Theorem 3, the l.h.s. $\frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!} \int D\psi' \left(\det(\Phi')\right)^{N-k}$ equals $\frac{\prod_{j=0}^{k-1} j! \prod_{j=0}^{N-k-1} j!}{\prod_{j=0}^{N-1} j!} C(N-k,k) = \frac{\prod_{j=0}^{k-1} j! \prod_{j=0}^{N-k-1} j!}{\prod_{j=0}^{N-1} j!} \cdot \frac{\prod_{j=0}^{N-1} j!}{\prod_{j=0}^{k-1} j! \prod_{j=0}^{N-k-1} j!} = 1.$

□

5 Application of the Fermion Representation

In this section, we prove Theorem 5.

Proof. First, we prove (1.46). From (4.179),

$$\det(I_k + t\Phi') = \exp\{\mathrm{tr}(\log(I_k + t\Phi'))\} = \exp\left\{ \mathrm{tr}\left(\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} t^j \Phi'^j \right) \right\}$$
$$= \exp\left\{ \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} t^j \,\mathrm{tr}\big(\Phi'^j\big) \right\} = \sum_{m=0}^{\infty} \frac{1}{m!} \left\{ \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} t^j \,\mathrm{tr}\big(\Phi'^j\big) \right\}^m$$
$$= \sum_{m=0}^{\infty} \frac{1}{m!} \left\{ t\,\mathrm{tr}(\Phi') - \frac{1}{2} t^2 \,\mathrm{tr}\big(\Phi'^2\big) + \cdots \right\}^m$$
$$= 1 + t\,\mathrm{tr}(\Phi') - \frac{1}{2} t^2 \,\mathrm{tr}\big(\Phi'^2\big) + \frac{1}{2} \left\{ t\,\mathrm{tr}(\Phi') - \frac{1}{2} t^2 \,\mathrm{tr}\big(\Phi'^2\big) \right\}^2 + \cdots$$
$$= 1 + t\,\mathrm{tr}(\Phi') + \frac{1}{2} t^2 \left((\mathrm{tr}(\Phi'))^2 - \mathrm{tr}\big(\Phi'^2\big) \right) + \cdots \tag{5.237}$$

Therefore, $\sigma_1 = \mathrm{tr}(\Phi')$ and $\sigma_2 = \frac{1}{2}\left((\mathrm{tr}(\Phi'))^2 - \mathrm{tr}(\Phi'^2)\right)$. From (4.180), (4.192) and (1.45),

$$\int_{G(k,N)} \left(\tau^*_{1(1)}\right)^{kN-k^2} = \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!} \int D\psi \left(\mathrm{tr}\left(\Phi'\right)\right)^{kN-k^2}$$
$$= \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!} \int D\psi \left(\sum_{s=1}^{N-k}\sum_{j=1}^{k} \psi^j_s \psi^{\bar{j}}_{\bar{s}}\right)^{kN-k^2}. \tag{5.238}$$

From the multinomial theorem and conditions of fermionic variables $\psi^j_s\psi^j_s = \psi^{\bar{j}}_{\bar{s}}\psi^{\bar{j}}_{\bar{s}} = 0$, we get the following result.

$$\int_{G(k,N)} \left(\tau^*_{1(1)}\right)^{kN-k^2} = (Nk-k^2)!\frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!} \int D\psi \prod_{s=1}^{N-k}\prod_{j=1}^{k} \psi^j_s \psi^{\bar{j}}_{\bar{s}}$$
$$= (kN-k^2)!\frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!}. \tag{5.239}$$

Second, we show (1.47) and (1.48). In the same way as for (1.46),

$$\int_{G(k,N)} (\tau^*_{1(1)})^{kN-k^2-2l}(\tau^*_{1(2)})^l = \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!} \int D\psi (\sigma_1)^{kN-k^2-2l}(\sigma_2)^l \ (l=1,2). \tag{5.240}$$
$$\tag{5.241}$$

Since $\sigma_2 = \frac{1}{2}\{(\mathrm{tr}\,(\Phi'))^2 - \mathrm{tr}\,(\Phi'^2)\}$,

$$\int D\psi(\sigma_1)^{kN-k^2-2l}(\sigma_2)^l = \frac{1}{2^l}\int D\psi \left\{\mathrm{tr}\,(\Phi')\right\}^{kN-k^2-2l}\left\{(\mathrm{tr}\,(\Phi'))^2 - \mathrm{tr}\left(\Phi'^2\right)\right\}^l \tag{5.242}$$
$$= \frac{1}{2^l}\sum_{m=0}^{l}\binom{l}{m}(-1)^m \int D\psi \left(\mathrm{tr}\,(\Phi')\right)^{kN-k^2-2m}\left(\mathrm{tr}\left(\Phi'^2\right)\right)^m. \tag{5.243}$$

Let us define

$$P_m := \int D\psi \left(\mathrm{tr}\,(\Phi')\right)^{kN-k^2-2m}\left(\mathrm{tr}\left(\Phi'^2\right)\right)^m \ (m=0,1,2). \tag{5.244}$$

$P_0 = (kN-k^2)!$ from the calculation of (1.46). We obtain the following result for P_1 and P_2.

Proposition 3.

$$P_1 = (kN-k^2-2)!k(N-k)(N-2k). \tag{5.245}$$
$$P_2 = (kN-k^2-4)!k(N-k)\Big[k(N-k)^3 - 2(N-k)^2(k^2+2)$$
$$+ (N-k)(k^3+10k) - 4k^2 - 2\Big]. \tag{5.246}$$

We prove these results later. Since

$$\int D\psi(\tau_1)^{kN-k^2-2}(\tau_2) = \frac{1}{2}(P_0 - P_1)$$
$$= \frac{1}{2}(kN-k^2-2)!k(N-k)\{(kN-k^2-1)-(N-2k)\}$$
$$= \frac{1}{2}(kN-k^2-2)!(N-k)(N-k+1)k(k-1), \qquad (5.247)$$

We obtain (1.46).

$$\int D\psi(\tau_1)^{kN-k^2-4}(\tau_2)^2 = \frac{1}{4}(P_0 - 2P_1 + P_2) \qquad (5.248)$$
$$= \frac{1}{4}\Big((kN-k^2)! - 2\Big[(kN-k^2-2)!k(N-k)(N-2k)\Big]$$
$$+ (kN-k^2-4)!k(N-k)\Big[k(N-k)^3 - 2(N-k)^2(k^2+2)$$
$$+ (N-k)(k^3+10k) - 4k^2 - 2\Big]\Big) \qquad (5.249)$$

We put $A := N - k$. Then,

$$\frac{1}{4}(P_0 - 2P_1 + P_2) = \frac{1}{4}\Big((kA)! - 2\Big[(kA-2)!kA(A-k)\Big]$$
$$+ (kA-4)!kA\Big[kA^3 - 2A^2(k^2+2) + A(k^3+10k) - 4k^2 - 2\Big]\Big) \qquad (5.250)$$
$$= \frac{(kA-4)!}{4}kA\Big((kA-1)(kA-2)(kA-3) - 2\Big[(kA-2)(kA-3)(A-k)\Big]$$
$$+ kA^3 - 2A^2(k^2+2) + A(k^3+10k) - 4k^2 - 2\Big) \qquad (5.251)$$
$$= \frac{(kA-4)!}{4}kA\Big((kA-1)(k^2A^2 - 5kA + 6) - 2\Big[(kA-2)(kA-3)(A-k)\Big]$$
$$+ kA^3 - 2A^2(k^2+2) + A(k^3+10k) - 4k^2 - 2\Big) \qquad (5.252)$$
$$= \frac{(kA-4)!}{4}kA\Big((k^3A^3 - 5k^2A^2 + 6kA - k^2A^2 + 5kA - 6)$$
$$- 2\Big[(k^2A^2 - 5kA + 6)(A-k)\Big] + kA^3 - 2A^2(k^2+2) + A(k^3+10k) - 4k^2 - 2\Big)$$
$$(5.253)$$
$$= \frac{(kA-4)!}{4}kA\Big((k^3A^3 - 6k^2A^2 + 11kA - 6)$$
$$- 2\Big[k^2A^3 - 5kA^2 + 6A - k^3A^2 + 5k^2A - 6k\Big]$$
$$+ kA^3 - 2A^2(k^2+2) + A(k^3+10k) - 4k^2 - 2\Big) \qquad (5.254)$$
$$= \frac{(kA-4)!}{4}kA\Big((k^3A^3 - 6k^2A^2 + 11kA - 6)$$

$$-2\Big[k^2A^3 + A^2(-k^3-5k) + A(5k^2+6) - 6k\Big]$$
$$+ kA^3 - 2A^2(k^2+2) + A(k^3+10k) - 4k^2 - 2\Big) \tag{5.255}$$
$$= \frac{(kA-4)!}{4}kA\Big((k^3-2k^2+k)A^3 + A^2(-6k^2+2k^3+10k-2k^2-4)$$
$$+ A(11k-10k^2-12+k^3+10k) - 4k^2 + 12k - 2 - 6\Big) \tag{5.256}$$
$$= \frac{(kA-4)!}{4}kA\Big(k(k-1)^2A^3 + A^2(2k^3-8k^2+10k-4)$$
$$+ A(k^3-10k^2+21k-12) - 4k^2 + 12k - 8\Big) \tag{5.257}$$
$$\tag{5.258}$$

From the identity $A^3-A = A(A+1)(A-1)$ and $(k-1)(k-2)(k-3) = k^3-6k^2+11k-6$,

$$= \frac{(kA-4)!}{4}kA\Big(k(k-1)^2A(A+1)(A-1) + A^2(2(k-1)(k-2)(k-3)$$
$$+ 2(6k^2-11k+6) - 8k^2 + 10k - 4)$$
$$+ A(k^3-10k^2+21k-12+k(k-1)^2) - 4(k^2-3k+2)\Big) \tag{5.259}$$
$$= \frac{(kA-4)!}{4}kA\Big(k(k-1)^2A(A+1)(A-1) + A^2(2(k-1)(k-2)(k-3)$$
$$+ 4k^2 - 12k + 8) + A(k^3-10k^2+21k-12+k^3-2k^2+k) - 4(k-2)(k-1)\Big) \tag{5.260}$$
$$= \frac{(kA-4)!}{4}kA\Big(k(k-1)^2A(A+1)(A-1) + A^2(2(k-1)(k-2)(k-3)$$
$$+ 4(k-2)(k-1)) + A(2k^3-12k^2+22k-12) - 4(k-2)(k-1)\Big) \tag{5.261}$$
$$= \frac{(kA-4)!}{4}kA\Big(k(k-1)^2A(A+1)(A-1) + 2(k-1)(k-2)(k-3)A^2$$
$$+ 4A^2(k-2)(k-1) + 2(k-1)(k-2)(k-3)A - 4(k-2)(k-1)\Big) \tag{5.262}$$
$$= \frac{(kA-4)!}{4}kA\Big(k(k-1)^2A(A+1)(A-1) + 2(k-1)(k-2)(k-3)A(A+1)$$
$$+ 4(k-2)(k-1)(A^2-1)\Big) \tag{5.263}$$
$$= \frac{(kA-4)!}{4}k(k-1)A(A+1)\Big(k(k-1)A(A-1) + 2(k-2)(k-3)A + 4(k-2)(A-1)\Big) \tag{5.264}$$
$$= \frac{(kN-k^2-4)!}{4}k(k-1)(N-k)(N-k+1)\Big(k(k-1)(N-k)(N-k-1)$$
$$+ 2(k-2)(k-3)(N-k) + 4(k-2)(N-k-1)\Big). \tag{5.265}$$

□

5.1 Proof of Proposition 3

Proof. (Proposition 3)

Let ω^{ij} be $\sum_{s=1}^{N-k}\psi_s^i\psi_{\bar{s}}^{\bar{j}}$. By definition,

$$\mathrm{tr}\left(\Phi'^2\right) = \sum_{i,j=1}^{k}\omega^{ij}\omega^{ji} = \sum_{i=1}^{k}(\omega^{ii})^2 + \sum_{i\neq j}\omega^{ij}\omega^{ji}, \tag{5.266}$$

$$P_1 = \int D\psi\,(\mathrm{tr}\,(\Phi'))^{kN-k^2-2}\left(\mathrm{tr}\left(\Phi'^2\right)\right), \tag{5.267}$$

$$= \sum_{i=1}^{k}\int D\psi\left(\sum_{n=1}^{k}\omega^{nn}\right)^{kN-k^2-2}(\omega^{ii})^2 + \sum_{i\neq j}\int D\psi\left(\sum_{n=1}^{k}\omega^{nn}\right)^{kN-k^2-2}\omega^{ij}\omega^{ji} \tag{5.268}$$

$$= \sum_{i=1}^{k}\sum_{p_n}\frac{(kN-k^2-2)!}{\prod_{n=1}^{k}p_n!}\int D\psi\left(\prod_{n=1}^{k}(\omega^{nn})^{p_n}\right)(\omega^{ii})^2$$
$$+\sum_{i\neq j}\sum_{p_n}\frac{(kN-k^2-2)!}{\prod_{n=1}^{k}p_n!}\int D\psi\left(\prod_{n=1}^{k}(\omega^{nn})^{p_n}\right)\omega^{ij}\omega^{ji}. \tag{5.269}$$

$\sum_{p_n}$ means summing so that $kN-k^2-2\geq p_1,\cdots,p_k\geq 0$ satisfy condition $\sum_{n=1}^{k}p_n = kN-k^2-2$. For the fermion integral to be non-zero, in the first term $p_n = N-k(n\neq i)$ and $p_i = N-k-2$, since each ω^{ii} $(i=1,\cdots,k)$ needs to be $N-k$. And $p_n = N-k(n\neq i,j)$ and $p_i = p_j = N-k-1$ in the second term.

$$P_1 = \sum_{i=1}^{k}\frac{(kN-k^2-2)!}{((N-k)!)^{k-1}(N-k-2)!}\int D\psi\left(\prod_{n=1}^{k}(\omega^{nn})^{N-k}\right)$$
$$+\sum_{i\neq j}\frac{(kN-k^2-2)!}{((N-k)!)^{k-2}((N-k-1)!)^2}\int D\psi\left(\prod_{n\neq i,j}(\omega^{nn})^{N-k}\right)\left(\omega^{ii}\omega^{jj}\right)^{N-k-1}\omega^{ij}\omega^{ji}. \tag{5.270}$$

From $\omega^{ii} = \sum_{s=1}^{N-k}\psi_s^i\psi_{\bar{s}}^{\bar{i}}$, multinomial theorem and conditions of fermionic variables $\psi_s^j\psi_s^j = \psi_{\bar{s}}^{\bar{j}}\psi_{\bar{s}}^{\bar{j}} = 0$,

$$P_1 = \sum_{i=1}^{k}\frac{(kN-k^2-2)!}{(N-k-2)!}(N-k)!$$
$$+\sum_{i\neq j}\frac{(kN-k^2-2)!}{((N-k-1)!)^2}\int D\psi\left(\prod_{n\neq i,j}\prod_{l=1}^{N-k}\psi_l^n\psi_{\bar{l}}^{\bar{n}}\right)\left(\omega^{ii}\omega^{jj}\right)^{N-k-1}\left(\sum_{s,t=1}^{N-k}\psi_s^i\psi_{\bar{s}}^{\bar{j}}\psi_t^j\psi_{\bar{t}}^{\bar{i}}\right). \tag{5.271}$$

In the second term, $(\omega^{ii}\omega^{jj})^{N-k-1}$ contains $N-k-1$ $\psi_s^i\psi_{\bar{s}}^{\bar{i}}$'s and $\psi_t^j\psi_{\bar{t}}^{\bar{i}}$'s, it must be

$s=t$ from conditions of fermionic variables.

$$P_1 = (kN-k^2-2)!k(N-k)(N-k-1)$$
$$-\sum_{i\neq j}\sum_{s=1}^{N-k} k\frac{(kN-k^2-2)!}{((N-k-1)!)^2}\int D\psi \left(\prod_{n\neq i,j}\prod_{l=1}^{N-k}\psi_l^n\psi_{\bar{l}}^{\bar{n}}\right)\left(\omega^{ii}\omega^{jj}\right)^{N-k-1}\left(\psi_s^i\psi_{\bar{s}}^{\bar{i}}\psi_s^j\psi_{\bar{s}}^{\bar{j}}\right) \tag{5.272}$$
$$= (kN-k^2-2)!k(N-k)(N-k-1)$$
$$-\sum_{i\neq j}\sum_{s=1}^{N-k}(kN-k^2-2)!\int D\psi\left(\prod_{n\neq i,j}\prod_{l=1}^{N-k}\psi_l^n\psi_{\bar{l}}^{\bar{n}}\right)\left(\prod_{\substack{q=1\\ q\neq s}}^{N-k}\psi_q^i\psi_{\bar{q}}^{\bar{i}}\psi_q^j\psi_{\bar{q}}^{\bar{j}}\right)\left(\psi_s^i\psi_{\bar{s}}^{\bar{i}}\psi_s^j\psi_{\bar{s}}^{\bar{j}}\right) \tag{5.273}$$
$$= (kN-k^2-2)!k(N-k)(N-k-1) - \sum_{i\neq j}\sum_{s=1}^{N-k}(kN-k^2-2)! \tag{5.274}$$
$$= (kN-k^2-2)!\{k(N-k)(N-k-1)-(N-k)k(k-1)\} \tag{5.275}$$
$$= (kN-k^2-2)!k(N-k)(N-2k). \tag{5.276}$$

We compute P_2.

$$P_2 = \int D\psi\left(\sum_{n=1}^k\omega^{nn}\right)^{kN-k^2-4}\left(\sum_{i=1}^k(\omega^{ii})^2+\sum_{i\neq j}\omega^{ij}\omega^{ji}\right)^2 \tag{5.277}$$
$$= \int D\psi\left(\sum_{n=1}^k\omega^{nn}\right)^{kN-k^2-4}\Big[\sum_{i,j}(\omega^{ii}\omega^{jj})^2 + 2\sum_{m=1}^k\sum_{i\neq j}(\omega^{mm})^2\omega^{ij}\omega^{ji}$$
$$+\sum_{a\neq b}\sum_{i\neq j}\omega^{ab}\omega^{ba}\omega^{ij}\omega^{ji}\Big]. \tag{5.278}$$

Let us define

$$Q_1 := \sum_{i,j}\int D\psi\left(\sum_{n=1}^k\omega^{nn}\right)^{kN-k^2-4}(\omega^{ii}\omega^{jj})^2,$$
$$Q_2 := 2\sum_{m=1}^k\sum_{i\neq j}\int D\psi\left(\sum_{n=1}^k\omega^{nn}\right)^{kN-k^2-4}(\omega^{mm})^2\omega^{ij}\omega^{ji}, \tag{5.279}$$
$$Q_3 := \sum_{a\neq b}\sum_{i\neq j}\int D\psi\left(\sum_{n=1}^k\omega^{nn}\right)^{kN-k^2-4}\omega^{ab}\omega^{ba}\omega^{ij}\omega^{ji}. \tag{5.280}$$

First, we consider Q_1.

$$
\begin{aligned}
Q_1 &= \sum_{i=1}^{k} \int D\psi \left(\sum_{n=1}^{k} \omega^{nn} \right)^{kN-k^2-4} (\omega^{ii})^4 + \sum_{i\neq j} \int D\psi \left(\sum_{n=1}^{k} \omega^{nn} \right)^{kN-k^2-4} (\omega^{ii}\omega^{jj})^2 \\
&= \sum_{i=1}^{k} \frac{(kN-k^2-4)!}{\{(N-k)!\}^{k-1}(N-k-4)!} \int D\psi \prod_{n=1}^{k} (\omega^{nn})^{N-k} \\
&+ \sum_{i\neq j} \frac{(kN-k^2-4)!}{\{(N-k)!\}^{k-2}\{(N-k-2)!\}^2} \int D\psi \prod_{n=1}^{k} (\omega^{nn})^{kN-k^2-4} \\
&= \sum_{i=1}^{k} \frac{(kN-k^2-4)!}{(N-k-4)!}(N-k)! + \sum_{i\neq j} \frac{(kN-k^2-4)!}{\{(N-k-2)!\}^2}((N-k)!)^2 \\
&= k\frac{(kN-k^2-4)!}{(N-k-4)!}(N-k)! + k(k-1)\frac{(kN-k^2-4)!}{\{(N-k-2)!\}^2}((N-k)!)^2 \\
&= (kN-k^2-4)!k(N-k)\{(N-k-1)(N-k-2)(N-k-3) \\
&+ (k-1)(N-k)(N-k-1)^2\}. \qquad (5.281)
\end{aligned}
$$

Next, we calculate Q_2.

$$
\begin{aligned}
Q_2 = 2\sum_{i\neq j} \int D\psi \left(\sum_{n=1}^{k} \omega^{nn} \right)^{kN-k^2-4} (\omega^{ii})^2\omega^{ij}\omega^{ji} \\
+ 2\sum_{i\neq j} \int D\psi \left(\sum_{n=1}^{k} \omega^{nn} \right)^{kN-k^2-4} (\omega^{jj})^2\omega^{ij}\omega^{ji} \\
+ 2\sum_{i\neq j}\sum_{m\neq i,j} \int D\psi \left(\sum_{n=1}^{k} \omega^{nn} \right)^{kN-k^2-4} (\omega^{mm})^2\omega^{ij}\omega^{ji}. \qquad (5.282)
\end{aligned}
$$

From $\omega^{ij}\omega^{ji} = \omega^{ji}\omega^{ij}$, if we replace i with j and j with i in the second term, it is the same as the first term.

$$
\begin{aligned}
Q_2 &= 4\sum_{i\neq j} \int D\psi \left(\sum_{n=1}^{k} \omega^{nn} \right)^{kN-k^2-4} (\omega^{ii})^2\omega^{ij}\omega^{ji} \\
&+ 2\sum_{i\neq j}\sum_{m\neq i,j} \int D\psi \left(\sum_{n=1}^{k} \omega^{nn} \right)^{kN-k^2-4} (\omega^{mm})^2\omega^{ij}\omega^{ji} \qquad (5.283) \\
&= 4\sum_{i\neq j}\sum_{p_n} \frac{(kN-k^2-4)!}{\prod_{q=1}^{k} p_q!} \int D\psi \left(\prod_{n=1}^{k} (\omega^{nn})^{p_n} \right) (\omega^{ii})^2\omega^{ij}\omega^{ji} \\
&+ 2\sum_{i\neq j}\sum_{m\neq i,j}\sum_{p_n} \frac{(kN-k^2-4)!}{\prod_{q=1}^{k} p_q!} \int D\psi \left(\prod_{n=1}^{k} (\omega^{nn})^{p_n} \right) (\omega^{mm})^2\omega^{ij}\omega^{ji}. \qquad (5.284)
\end{aligned}
$$

$\sum_{p_n}$ means summing so that $kN-k^2-4 \geq p_1, \cdots, p_k \geq 0$ satisfy condition $\sum_{n=1}^{k} p_n = kN-k^2-4$. From the condition of integration and the condition of fermionic variables $\psi_s^i \psi_s^i = 0$, in the first term, $p_n = N-k(n \neq i,j)$ and $p_i = N-k-3$, $p_j = N-k-1$. In the second term, $p_n = N-k(n \neq i,j,m)$ and $p_i = p_j = N-k-1$, $p_m = N-k-2$.

$$
\begin{aligned}
Q_2 &= 4\sum_{i\neq j} \frac{(kN-k^2-4)!}{(N-k-3)!(N-k-1)!} \int D\psi \left(\prod_{n\neq i,j} \prod_{l=1}^{N-k} \psi_l^n \psi_{\bar{l}}^{\bar{n}} \right) (\omega^{ii}\omega^{jj})^{N-k-1}\omega^{ij}\omega^{ji} \\
&+ 2\sum_{i\neq j}\sum_{m\neq i,j} \frac{(kN-k^2-4)!(N-k)!}{(N-k-2)!((N-k-1)!)^2} \\
&\times \int D\psi \left(\prod_{n\neq i,j} \prod_{l=1}^{N-k} \psi_l^n \psi_{\bar{l}}^{\bar{n}} \right) (\omega^{ii}\omega^{jj})^{N-k-1}\omega^{ij}\omega^{ji}. \qquad (5.285)
\end{aligned}
$$

Here, the integral value is calculated in the same way with P_1. In $i \neq j$,

$$
\int D\psi \left(\prod_{n\neq i,j} \prod_{l=1}^{N-k} \psi_l^n \psi_{\bar{l}}^{\bar{n}} \right) (\omega^{ii}\omega^{jj})^{N-k-1}\omega^{ij}\omega^{ji} = -(N-k)((N-k-1)!)^2. \qquad (5.286)
$$

$$
\begin{aligned}
Q_2 &= -4\sum_{i\neq j} \frac{(kN-k^2-4)!(N-k)!}{(N-k-3)!} \\
&- 2\sum_{i\neq j}\sum_{m\neq i,j} \frac{(kN-k^2-4)!(N-k)!(N-k)}{(N-k-2)!} \qquad (5.287) \\
&= -4\frac{(kN-k^2-4)!(N-k)!}{(N-k-3)!}k(k-1) \\
&- 2\frac{(kN-k^2-4)!(N-k)!(N-k)}{(N-k-2)!}k(k-1)(k-2) \\
&= (kN-k^2-4)!k(N-k)(k-1)[-4(N-k-1)(N-k-2) \\
&- 2(N-k)(N-k-1)(k-2)]. \qquad (5.288)
\end{aligned}
$$

Finally, we compute Q_3.

$$
Q_3 = \sum_{a\neq b}\sum_{i\neq j} \int D\psi \left(\sum_{n=1}^{k} \omega^{nn} \right)^{kN-k^2-4} \omega^{ab}\omega^{ba}\omega^{ij}\omega^{ji}. \qquad (5.289)
$$

The sum $\sum_{a\neq b}\sum_{i\neq j}$ can be divided into the following seven cases.

Sum patterns of (i,j) and (a,b)

(1) $i=a$, $j=b$. (2) $i=b$, $j=a$. (3) $i=a$, $j\neq b$. (4) $i=b$, $j\neq a$. (5) $i\neq a$, $j=b$. (6) $i\neq b$, $j=a$. (7) $i\neq a,b$ $j\neq a,b$.

From the symmetry of a, b and i, j, (1) and (2) have the same form. Similarly, (3), (4), (5) and (6) have the same form. Therefore,

$$
\begin{aligned}
Q_3 &= 2\sum_{i\neq j}\int D\psi\left(\sum_{n=1}^{k}\omega^{nn}\right)^{kN-k^2-4}(\omega^{ij}\omega^{ji})^2 \\
&+4\sum_{i\neq j}\sum_{b\neq i,j}\int D\psi\left(\sum_{n=1}^{k}\omega^{nn}\right)^{kN-k^2-4}\omega^{ib}\omega^{bi}\omega^{ij}\omega^{ji} \\
&+\sum_{(i,j,a,b)}{}'\int D\psi\left(\sum_{n=1}^{k}\omega^{nn}\right)^{kN-k^2-4}\omega^{ab}\omega^{ba}\omega^{ij}\omega^{ji}.
\end{aligned}
\tag{5.290}
$$

Here, $\sum'_{(i,j,a,b)}$ implies that i, j, a, and b sum so that they are different each other.

$$
\begin{aligned}
Q_3 &= 2\sum_{i\neq j}\frac{(kN-k^2-4)!}{((N-k)!)^{k-2}((N-k-2)!)^2} \\
&\times\int D\psi\left(\prod_{n\neq i,j}(\omega^{nn})^{N-k}\right)\left(\omega^{ii}\omega^{jj}\right)^{N-k-2}(\omega^{ij}\omega^{ji})^2 \\
&+4\sum_{i\neq j}\sum_{b\neq i,j}\frac{(kN-k^2-4)!}{((N-k)!)^{k-3}(N-k-2)!((N-k-1)!)^2} \\
&\times\int D\psi\left(\prod_{n\neq i,j,b}(\omega^{nn})^{N-k}\right)\left(\omega^{ii}\right)^{N-k-2}\left(\omega^{bb}\omega^{jj}\right)^{N-k-1}\omega^{ib}\omega^{bi}\omega^{ij}\omega^{ji} \\
&+\sum_{(i,j,a,b)}{}'\frac{(kN-k^2-4)!}{((N-k)!)^{k-4}((N-k-1)!)^4}\int D\psi\left(\prod_{n\neq a,b,i,j}(\omega^{nn})^{N-k}\right)\left(\omega^{aa}\omega^{bb}\omega^{ii}\omega^{jj}\right)^{N-k-1} \\
&\times\omega^{ab}\omega^{ba}\omega^{ij}\omega^{ji}
\end{aligned}
\tag{5.291}
$$

$$
\begin{aligned}
&= 2\sum_{i\neq j}\frac{(kN-k^2-4)!}{((N-k-2)!)^2}\int D\psi\left(\prod_{n\neq i,j}\prod_{l=1}^{N-k}\psi_l^n\psi_{\bar{l}}^{\bar{n}}\right)\left(\omega^{ii}\omega^{jj}\right)^{N-k-2} \\
&\times\left(\sum_{s_1,s_2,t_1,t_2}\psi_{s_1}^{i}\psi_{\bar{s}_1}^{\bar{j}}\psi_{t_1}^{j}\psi_{\bar{t}_1}^{\bar{i}}\psi_{s_2}^{i}\psi_{\bar{s}_2}^{\bar{j}}\psi_{t_2}^{j}\psi_{\bar{t}_2}^{\bar{i}}\right) \\
&+4\sum_{i\neq j}\sum_{b\neq i,j}\frac{(kN-k^2-4)!}{(N-k-2)!((N-k-1)!)^2} \\
&\times\int D\psi\left(\prod_{n\neq i,j}\prod_{l=1}^{N-k}\psi_l^n\psi_{\bar{l}}^{\bar{n}}\right)\left(\omega^{ii}\right)^{N-k-2}\left(\omega^{bb}\omega^{jj}\right)^{N-k-1} \\
&\times\left(\sum_{s_1,s_2,t_1,t_2}\psi_{s_1}^{i}\psi_{\bar{s}_1}^{\bar{b}}\psi_{t_1}^{b}\psi_{\bar{t}_1}^{\bar{i}}\psi_{s_2}^{i}\psi_{\bar{s}_2}^{\bar{j}}\psi_{t_2}^{j}\psi_{\bar{t}_2}^{\bar{i}}\right)
\end{aligned}
$$

$$+\sum_{(i,j,a,b)}^{\prime}\frac{(kN-k^2-4)!}{((N-k-1)!)^4}\int D\psi\left(\prod_{n\neq a,b,i,j}\prod_{l=1}^{N-k}\psi_l^n\psi_{\bar{l}}^{\bar{n}}\right)\left(\omega^{aa}\omega^{bb}\omega^{ii}\omega^{jj}\right)^{N-k-1}$$
$$\times\left(\sum_{s_1,s_2,t_1,t_2}\psi_{s_1}^a\psi_{\bar{s}_1}^{\bar{b}}\psi_{t_1}^b\psi_{\bar{t}_1}^{\bar{a}}\psi_{s_2}^i\psi_{\bar{s}_2}^{\bar{j}}\psi_{t_2}^j\psi_{\bar{t}_2}^{\bar{i}}\right). \tag{5.292}$$

We consider sum of s_1, s_2, t_1, t_2. In the first term, the summation can be divided into two ways, ($s_1 = t_1$, $s_2 = t_2$, $s_1 \neq s_2$) and ($s_1 = t_2$, $s_2 = t_1$,$s_1 \neq s_2$). In the second term, it must be ($s_1 = t_1$, $s_2 = t_2$, $s_1 \neq s_2$). In the third term, it must be ($s_1 = t_1$, $s_2 = t_2$). Since the first term is symmetric for s_1 and s_2, and t_1 and t_2,

$$Q_3 = 4\sum_{i\neq j}\sum_{s_1\neq s_2}\frac{(kN-k^2-4)!}{((N-k-2)!)^2}\int D\psi\left(\prod_{n\neq i,j}\prod_{l=1}^{N-k}\psi_l^n\psi_{\bar{l}}^{\bar{n}}\right)\left(\omega^{ii}\omega^{jj}\right)^{N-k-2}$$
$$\times\left(\psi_{s_1}^i\psi_{\bar{s}_1}^{\bar{i}}\psi_{s_1}^j\psi_{\bar{s}_1}^{\bar{j}}\psi_{s_2}^i\psi_{\bar{s}_2}^{\bar{i}}\psi_{s_2}^j\psi_{\bar{s}_2}^{\bar{j}}\right)$$
$$+4\sum_{i\neq j}\sum_{b\neq i,j}\sum_{s_1\neq s_2}\frac{(kN-k^2-4)!}{(N-k-2)!((N-k-1)!)^2}$$
$$\times\int D\psi\left(\prod_{n\neq i,j}\prod_{l=1}^{N-k}\psi_l^n\psi_{\bar{l}}^{\bar{n}}\right)\left(\omega^{ii}\right)^{N-k-2}\left(\omega^{bb}\omega^{jj}\right)^{N-k-1}\left(\psi_{s_1}^i\psi_{\bar{s}_1}^{\bar{i}}\psi_{s_1}^b\psi_{\bar{s}_1}^{\bar{b}}\psi_{s_2}^i\psi_{\bar{s}_2}^{\bar{i}}\psi_{s_2}^j\psi_{\bar{s}_2}^{\bar{j}}\right)$$
$$+\sum_{(i,j,a,b)}^{\prime}\sum_{s_1,s_2}\frac{(kN-k^2-4)!}{((N-k-1)!)^4}\int D\psi\left(\prod_{n\neq a,b,i,j}\prod_{l=1}^{N-k}\psi_l^n\psi_{\bar{l}}^{\bar{n}}\right)\left(\omega^{aa}\omega^{bb}\omega^{ii}\omega^{jj}\right)^{N-k-1}$$
$$(\psi_{s_1}^a\psi_{\bar{s}_1}^{\bar{a}}\psi_{s_1}^b\psi_{\bar{s}_1}^{\bar{b}}\psi_{s_2}^i\psi_{\bar{s}_2}^{\bar{i}}\psi_{s_2}^j\psi_{\bar{s}_2}^{\bar{j}}) \tag{5.293}$$
$$=4\sum_{i\neq j}\sum_{s_1\neq s_2}(kN-k^2-4)!+4\sum_{i\neq j}\sum_{b\neq i,j}\sum_{s_1\neq s_2}(kN-k^2-4)!$$
$$+\sum_{(i,j,a,b)}^{\prime}\sum_{s_1,s_2}(kN-k^2-4)! \tag{5.294}$$
$$=(kN-k^2-4)![4k(k-1)(N-k)(N-k-1)$$
$$+4k(k-1)(k-2)(N-k)(N-k-1)+k(k-1)(k-2)(k-3)(N-k)^2] \tag{5.295}$$
$$=(kN-k^2-4)!k(N-k)[4(k-1)^2(N-k-1)+(k-1)(k-2)(k-3)(N-k)]. \tag{5.296}$$

From above results,

$$P_3 = Q_1+Q_2+Q_3$$
$$=(kN-k^2-4)!k(N-k)\{(N-k-1)(N-k-2)(N-k-3)$$
$$+(k-1)(N-k)(N-k-1)^2\}$$
$$+(kN-k^2-4)!k(N-k)(k-1)[-4(N-k-1)(N-k-2)$$

$$
\begin{aligned}
&- 2(N-k)(N-k-1)(k-2)] \\
&+ (kN-k^2-4)!k(N-k)[4(k-1)^2(N-k-1) + (k-1)(k-2)(k-3)(N-k)] \\
&= (kN-k^2-4)!k(N-k)\Big[(N-k-1)(N-k-2)(N-k-3) \\
&+ (k-1)(N-k)(N-k-1)^2 + (k-1)\{-4(N-k-1)(N-k-2) \\
&- 2(N-k)(N-k-1)(k-2)\} \\
&+ 4(k-1)^2(N-k-1) + (k-1)(k-2)(k-3)(N-k)\Big]. \qquad (5.297)
\end{aligned}
$$

Here, we put $A := N - k$.

$$
\begin{aligned}
P_3 &= (kA-4)!kA\Big[(A-1)(A-2)(A-3) + (k-1)A(A-1)^2 \\
&+ (k-1)\{-4(A-1)(A-2) - 2A(A-1)(k-2)\} \\
&+ 4(k-1)^2(A-1) + (k-1)(k-2)(k-3)A\Big] \\
&= (kA-4)!kA\Big[(A-1)(A^2-5A+6) + (k-1)A(A^2-2A+1) \\
&- 4(k-1)(A^2-3A+2) - 2(A^2-A)(k-1)(k-2) \\
&+ 4(k-1)^2(A-1) + (k-1)(k-2)(k-3)A\Big] \\
&= (kA-4)!kA\Big[A^3-5A^2+6A-A^2+5A-6 + (k-1)(A^3-2A^2+A) \\
&- 4(k-1)(A^2-3A+2) - 2A^2(k-1)(k-2) + 2A(k-1)(k-2) \\
&+ 4(k-1)^2A - 4(k-1)^2 + (k-1)(k-2)(k-3)A\Big] \qquad (5.298) \\
&= (kA-4)!kA\Big[kA^3-6A^2+11A-6 + (k-1)(-2A^2+A) \\
&- 4(k-1)(A^2-3A+2) - 2A^2(k-1)(k-2) + 2A(k-1)(k-2) \\
&+ 4(k-1)^2A - 4(k-1)^2 + (k-1)(k-2)(k-3)A\Big] \qquad (5.299) \\
&= (kA-4)!kA\Big[kA^3-6A^2+11A-6-2(k-1)A^2 + (k-1)A \\
&- 4(k-1)A^2 + 12(k-1)A - 8(k-1) - 2A^2(k-1)(k-2) + 2A(k-1)(k-2) \\
&+ 4(k-1)^2A - 4(k-1)^2 + (k-1)(k-2)(k-3)A\Big] \\
&= (kA-4)!kA\Big[kA^3 + A^2(-6-2(k-1)-4(k-1)-2(k-1)(k-2)) \\
&+ A(11+(k-1)+12(k-1)+2(k-1)(k-2)+4(k-1)^2+(k-1)(k-2)(k-3)) \\
&- 6 - 8(k-1) - 4(k-1)^2\Big] \qquad (5.300) \\
&= (kA-4)!kA\Big[kA^3 + A^2(-6-6(k-1)-2(k-1)(k-2)) \\
&+ A(11+13(k-1)+(k-1)^2(k-2)+4(k-1)^2) - 8k + 2 - 4(k^2-2k+1)\Big] \qquad (5.301)
\end{aligned}
$$

$$= (kA-4)!kA\Big[kA^3 + A^2(-6k - 2(k^2-3k+2))$$

$$+ A(11 + 13(k-1) + (k-1)^2(k+2)) - 8k + 2 - 4k^2 + 8k - 4\Big] \tag{5.302}$$

$$= (kA-4)!kA\Big[kA^3 + A^2(-2k^2-4)) + A(13k - 2 + (k^2-2k+1)(k+2)) - 4k^2 - 2\Big] \tag{5.303}$$

$$= (kA-4)!kA\Big[kA^3 + A^2(-2k^2-4)) + A(13k - 2 + k^3 - 2k^2 + k + 2k^2 - 4k + 2)$$

$$- 4k^2 - 2\Big] \tag{5.304}$$

$$= (kA-4)!kA\Big[kA^3 - 2A^2(k^2+2)) + A(k^3+10k) - 4k^2 - 2\Big] \tag{5.305}$$

$$= (kN-k^2-4)!k(N-k)\Big[k(N-k)^3 - 2(N-k)^2(k^2+2))$$

$$+ (N-k)(k^3+10k) - 4k^2 - 2\Big]. \tag{5.306}$$

□

Appendix of Part II (Integral formula for ϕ and A)

We prove Lemma 2.

Proof. Here, we calculate the integral of $\bar{\phi}$ and A in section 3. First, we consider

$$I_{\bar{\phi}} := \int D\bar{\phi} \exp\left(-\sum_{s=1}^{N} \psi^{\bar{i}}_{\bar{s}} \psi^{j}_{s} \bar{\phi}_{\bar{i}j} - i\mathrm{tr}(\phi\bar{\phi})\right). \tag{5.307}$$

Since $\bar{\phi}$ is a Hermite matrix, we define

$$D\bar{\phi} := \prod_{i=1}^{k-1}\left(d\bar{\phi}_{\bar{i}i} \prod_{j=i+1}^{k} \frac{i}{2} d\bar{\phi}_{\bar{i}j} d\bar{\phi}_{\bar{j}i}\right) d\bar{\phi}_{\bar{k}k}. \tag{5.308}$$

The exponent term is summarized as follows.

$$\begin{aligned}
&- i\mathrm{tr}(\phi\bar{\phi}) - \sum_{s=1}^{N} \psi^{\bar{i}}_{\bar{s}} \psi^{j}_{s} \bar{\phi}_{\bar{i}j} \\
=& \sum_{i=1}^{k}\left[-i\phi_{\bar{i}i}\bar{\phi}_{\bar{i}i} - \sum_{s=1}^{N} \psi^{\bar{i}}_{\bar{s}} \psi^{i}_{s} \bar{\phi}_{\bar{i}i}\right] + \sum_{i>j}^{k}\left[-i\phi_{\bar{j}i}\bar{\phi}_{\bar{i}j} - \sum_{s=1}^{N} \psi^{\bar{i}}_{\bar{s}} \psi^{j}_{s} \bar{\phi}_{\bar{i}j}\right] \\
&+ \sum_{i<j}^{k}\left[-i\phi_{\bar{j}i}\bar{\phi}_{\bar{i}j} - \sum_{s=1}^{N} \psi^{\bar{i}}_{\bar{s}} \psi^{j}_{s} \bar{\phi}_{\bar{i}j}\right]
\end{aligned} \tag{5.309}$$

$$
\begin{aligned}
&= \sum_{i=1}^{k}\left[-i\phi_{\bar{i}i}\bar{\phi}_{\bar{i}i} - \sum_{s=1}^{N}\psi_{\bar{s}}^{\bar{i}}\psi_{s}^{i}\bar{\phi}_{\bar{i}i}\right] \\
&\quad + \sum_{i<j}^{k}\left[-i(\phi_{\bar{j}i}\bar{\phi}_{\bar{i}j} + \phi_{\bar{i}j}\bar{\phi}_{\bar{j}i}) - \sum_{s=1}^{N}\psi_{\bar{s}}^{\bar{j}}\psi_{s}^{i}\bar{\phi}_{\bar{j}i} - \sum_{s=1}^{N}\psi_{\bar{s}}^{\bar{i}}\psi_{s}^{j}\bar{\phi}_{\bar{i}j}\right].
\end{aligned} \tag{5.310}
$$

From $\phi_{\bar{i}j} = \phi_{j\bar{i}}$ and $\bar{\phi}_{\bar{i}j} = \bar{\phi}_{j\bar{i}}$,

$$
\begin{aligned}
&\sum_{i=1}^{k}\left[-i\phi_{\bar{i}i}\bar{\phi}_{\bar{i}i} - \sum_{s=1}^{N}\psi_{\bar{s}}^{\bar{i}}\psi_{s}^{i}\bar{\phi}_{\bar{i}i}\right] \\
&\quad + \sum_{i<j}^{k}\left[-i(\phi_{\bar{j}i}\bar{\phi}_{\bar{i}j} + \phi_{j\bar{i}}\bar{\phi}_{i\bar{j}}) - \sum_{s=1}^{N}\left\{\psi_{\bar{s}}^{\bar{j}}\psi_{s}^{i}\bar{\phi}_{i\bar{j}} + \psi_{\bar{s}}^{\bar{i}}\psi_{s}^{j}\bar{\phi}_{\bar{i}j}\right\}\right]
\end{aligned} \tag{5.311}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{k}\left[-i\phi_{\bar{i}i}\bar{\phi}_{\bar{i}i} - \sum_{s=1}^{N}\psi_{\bar{s}}^{\bar{i}}\psi_{s}^{i}\bar{\phi}_{\bar{i}i}\right] \\
&\quad + \sum_{i<j}^{k}\left[-i\{\phi_{\bar{j}i}\bar{\phi}_{\bar{i}j} + {}^{*}(\phi_{\bar{j}i}\bar{\phi}_{\bar{i}j})\} - \sum_{s=1}^{N}\left\{\psi_{\bar{s}}^{\bar{i}}\psi_{s}^{j}\bar{\phi}_{\bar{i}j} - {}^{*}(\psi_{\bar{s}}^{\bar{i}}\psi_{s}^{j}\bar{\phi}_{\bar{i}j})\right\}\right]
\end{aligned} \tag{5.312}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{k}\left[-i\left\{\phi_{\bar{i}i} - i\sum_{s=1}^{N}\psi_{\bar{s}}^{\bar{i}}\psi_{s}^{i}\right\}\bar{\phi}_{\bar{i}i}\right] \\
&\quad + \sum_{i<j}^{k}\left[-2i\mathrm{Re}(\phi_{\bar{j}i}\bar{\phi}_{\bar{i}j}) - 2i\sum_{s=1}^{N}\mathrm{Im}(\psi_{\bar{s}}^{\bar{i}}\psi_{s}^{j}\bar{\phi}_{\bar{i}j})\right\}\Bigg].
\end{aligned} \tag{5.313}
$$

To evaluate the integral of $\bar{\phi}$, we represent $\bar{\phi}_{\bar{i}j}, \phi_{\bar{i}j}(i \neq j)$ and ψ_{s}^{i} with $\phi_{ij}^{R}, \phi_{ij}^{I}$ (they are real variables) and $\psi_{Rs}^{i}, \psi_{Is}^{i}$ as follows. ($\phi_{\bar{i}i}$ is a real variable from $\phi_{\bar{i}j} = \phi_{j\bar{i}}$.)

$$
\phi_{\bar{i}j} = \phi_{ij}^{R} + i\phi_{ij}^{I}, \qquad \phi_{i\bar{j}} = \phi_{ij}^{R} - i\phi_{ij}^{I}. \tag{5.314}
$$

$$
\bar{\phi}_{\bar{i}j} = \bar{\phi}_{ij}^{R} + i\bar{\phi}_{ij}^{I}, \qquad \bar{\phi}_{i\bar{j}} = \bar{\phi}_{ij}^{R} - i\bar{\phi}_{ij}^{I}. \tag{5.315}
$$

$$
\psi_{s}^{i} = \psi_{Rs}^{i} + i\psi_{Is}^{i}, \qquad \psi_{\bar{s}}^{\bar{i}} = \psi_{Rs}^{i} - i\psi_{Is}^{i}. \tag{5.316}
$$

Then, $\phi_{ij}^{R} = \phi_{ji}^{R}, \phi_{ij}^{I} = -\phi_{ji}^{I}$ and $\bar{\phi}_{ij}^{R} = \bar{\phi}_{ji}^{R}, \quad \bar{\phi}_{ij}^{I} = -\bar{\phi}_{ji}^{I}$. Since $d\phi_{\bar{i}j} \wedge d\phi_{\bar{j}i} = -2id\phi_{ij}^{R} \wedge d\phi_{ij}^{I}$,

$$
D\phi = \prod_{i=1}^{k-1}\left(d\phi_{\bar{i}i}\prod_{j=i+1}^{k}d\phi_{ij}^{R}d\phi_{ij}^{I}\right)d\phi_{\bar{k}k}. \tag{5.317}
$$

$$\begin{aligned}\psi_{\bar{s}}^{\bar{i}}\psi_s^j &= (\psi_{Rs}^i - i\psi_{Is}^i)(\psi_{Rs}^j + i\psi_{Is}^j) \\ &= \psi_{Rs}^i\psi_{Rs}^j + \psi_{Is}^i\psi_{Is}^j + i(\psi_{Rs}^i\psi_{Is}^j - \psi_{Is}^i\psi_{Rs}^j). \end{aligned} \tag{5.318}$$

$$\mathrm{Im}(\psi_{\bar{s}}^{\bar{i}}\psi_s^j\bar{\phi}_{\bar{i}j}) = \bar{\phi}_{ij}^I(\psi_{Rs}^i\psi_{Rs}^j + \psi_{Is}^i\psi_{Is}^j) + \bar{\phi}_{ij}^R(\psi_{Rs}^i\psi_{Is}^j - \psi_{Is}^i\psi_{Rs}^j). \tag{5.319}$$

$$\begin{aligned}\phi_{\bar{j}i}\bar{\phi}_{\bar{i}j} &= (\phi_{ji}^R + i\phi_{ji}^I)(\bar{\phi}_{ij}^R + i\bar{\phi}_{ij}^I) \\ &= \phi_{ji}^R\bar{\phi}_{ij}^R - \phi_{ji}^I\bar{\phi}_{ij}^I + i(\cdots) \end{aligned} \tag{5.320}$$

$$\mathrm{Re}(\phi_{\bar{j}i}\bar{\phi}_{\bar{i}j}) = \phi_{ji}^R\bar{\phi}_{ij}^R - \phi_{ji}^I\bar{\phi}_{ij}^I. \tag{5.321}$$

We obtain the following result by changing the order of integral.

$$\begin{aligned} I_{\bar{\phi}} = &\left(\prod_{i=1}^k \int d\bar{\phi}_{\bar{i}i} \exp\left(-i\Big\{\phi_{\bar{i}i} + 2\sum_{s=1}^N \psi_{Rs}^i\psi_{Is}^i\Big\}\bar{\phi}_{\bar{i}i}\right)\right) \\ &\times \prod_{i=1}^{k-1}\prod_{j=i+1}^k \left(\int d\phi_{ij}^R d\phi_{ij}^I \exp\bigg(-2i\Big\{\Big(\phi_{ji}^R + \sum_{s=1}^N(\psi_{Rs}^i\psi_{Is}^j - \psi_{Is}^i\psi_{Rs}^j)\Big)\bar{\phi}_{ij}^R \right. \\ &\left. - \Big(\phi_{ji}^I - \sum_{s=1}^N(\psi_{Rs}^i\psi_{Rs}^j + \psi_{Is}^i\psi_{Is}^j)\Big)\bar{\phi}_{ij}^I)\Big\}\bigg)\right). \end{aligned} \tag{5.322}$$

By the integral formula

$$\int_{-\infty}^{\infty} dx \exp(ifx) = 2\pi\delta(f) \quad (f : \text{field}), \tag{5.323}$$

we obtain

$$\int d\bar{\phi}_{\bar{i}i} \exp\left(-i\Big\{\phi_{\bar{i}i} + 2\sum_{s=1}^N \psi_{Rs}^i\psi_{Is}^i\Big\}\bar{\phi}_{\bar{i}i}\right) = 2\pi\delta\left(\phi_{\bar{i}i} + 2\sum_{s=1}^N \psi_{Rs}^i\psi_{Is}^i\right) \tag{5.324}$$

and

$$\begin{aligned} &\int d\phi_{ij}^R d\phi_{ij}^I \exp\bigg(-2i\Big\{\Big(\phi_{ji}^R + \sum_{s=1}^N(\psi_{Rs}^i\psi_{Is}^j - \psi_{Is}^i\psi_{Rs}^j)\Big)\bar{\phi}_{ij}^R \\ &- \Big(\phi_{ji}^I - \sum_{s=1}^N(\psi_{Rs}^i\psi_{Rs}^j + \psi_{Is}^i\psi_{Is}^j)\Big)\bar{\phi}_{ij}^I)\Big\}\bigg) \\ &\frac{(2\pi)^2}{2^2}\delta\left(\phi_{ji}^R + \sum_{s=1}^N(\psi_{Rs}^i\psi_{Is}^j - \psi_{Is}^i\psi_{Rs}^j)\right)\delta\left(\phi_{ji}^I - \sum_{s=1}^N(\psi_{Rs}^i\psi_{Rs}^j + \psi_{Is}^i\psi_{Is}^j)\right). \end{aligned} \tag{5.325}$$

Therefore,

$$\begin{aligned} I_{\bar{\phi}} = &\frac{(2\pi)^{k^2}}{2^{k(k-1)}}\left\{\prod_{i=1}^k \delta\left(\phi_{\bar{i}i} + 2\sum_{s=1}^N \psi_{Rs}^i\psi_{Is}^i\right)\right\} \\ &\times \left\{\prod_{i=1}^{k-1}\prod_{j=i+1}^k \delta\left(\phi_{ji}^R + \sum_{s=1}^N(\psi_{Rs}^i\psi_{Is}^j - \psi_{Is}^i\psi_{Rs}^j)\right)\delta\left(\phi_{ji}^I - \sum_{s=1}^N(\psi_{Rs}^i\psi_{Rs}^j + \psi_{Is}^i\psi_{Is}^j)\right)\right\}. \end{aligned} \tag{5.326}$$

Next, we calculate

$$I_A := \int DA \exp\left(-\mathrm{tr}(A^\dagger A + A^\dagger [i\phi, A])\right). \quad DA := \left(\frac{i}{2}\right)^{k^2} \prod_{i=1}^{k} \left(\prod_{j=1}^{k} dA_{\bar{i}j} dA_{i\bar{j}}\right). \tag{5.327}$$

$$A := \begin{pmatrix} A_{\bar{1}1} & \cdots & A_{\bar{1}k} \\ \vdots & \ddots & \vdots \\ A_{\bar{k}1} & \cdots & A_{\bar{k}k} \end{pmatrix}. \tag{5.328}$$

ϕ is diagonalized to $\phi = P \Lambda P^\dagger$. P is unitary matrix.

$$P := \begin{pmatrix} P_{\bar{1}1} & \cdots & P_{\bar{1}k} \\ \vdots & \ddots & \vdots \\ P_{\bar{k}1} & \cdots & P_{\bar{k}k} \end{pmatrix}, \ \Lambda = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_k \end{pmatrix}. \tag{5.329}$$

($\lambda_i \, (i = 1, 2, \cdots k)$ are eigenvalue of ϕ. We transform $A := PA'P^\dagger$ $(A' = P^\dagger A P)$. Then,

$$\mathrm{tr}\left(A^\dagger A\right) = \mathrm{tr}\left(PA'^\dagger P^\dagger P A' P^\dagger\right) = \mathrm{tr}\left(A'^\dagger A'\right) = \sum_{i,j=1}^{k} |A'_{\bar{i}j}|^2, \tag{5.330}$$

$$\begin{aligned} \mathrm{tr}\left(A^\dagger \phi A\right) &= \mathrm{tr}\left(PA'^\dagger P^\dagger \phi P A' P^\dagger\right) = \mathrm{tr}\left(A'^\dagger \Lambda A'\right) = \sum_{i,l,p=1}^{k} A'^\dagger_{\bar{i}l} \Lambda_{\bar{l}p} A'_{\bar{p}i} \\ &= \sum_{i,l,p=1}^{k} A'_{l\bar{i}} \lambda_l \delta_{l,p} A'_{\bar{p}i} = \sum_{i,l=1}^{k} A'_{l\bar{i}} \lambda_l A'_{\bar{l}i} = \sum_{i,j=1}^{k} \lambda_i |A'_{\bar{i}j}|^2. \end{aligned} \tag{5.331}$$

$$\begin{aligned} \mathrm{tr}\left(A^\dagger A \phi\right) &= \mathrm{tr}\left(PA'^\dagger P^\dagger P A' P^\dagger \phi\right) = \mathrm{tr}\left(A'^\dagger A' \Lambda\right) = \sum_{i,l,p=1}^{k} A'^\dagger_{\bar{i}l} A'_{\bar{l}p} \Lambda_{\bar{p}i} \\ &= \sum_{i,l,p=1}^{k} A'_{l\bar{i}} A'_{\bar{l}p} \lambda_p \delta_{pi} = \sum_{i,l=1}^{k} A'_{l\bar{i}} A'_{\bar{l}i} \lambda_i = \sum_{i,j=1}^{k} |A'_{\bar{i}j}|^2 \lambda_j. \end{aligned} \tag{5.332}$$

$$\begin{aligned} \mathrm{tr}(A^\dagger A + A^\dagger [i\phi, A]) &= \sum_{i,j=1}^{k} |A'_{\bar{i}j}|^2 (1 + \sqrt{-1}(\lambda_i - \lambda_j)) \\ &= \sum_{i=1}^{k} |A'_{\bar{i}i}|^2 + \sum_{i<j} |A'_{\bar{i}j}|^2 (1 + \sqrt{-1}(\lambda_i - \lambda_j)) \\ &+ \sum_{j<i} |A'_{\bar{i}j}|^2 (1 + \sqrt{-1}(\lambda_i - \lambda_j)). \end{aligned} \tag{5.333}$$

We define

$$DA' := \left(\frac{i}{2}\right)^{k^2} \prod_{i=1}^{k} \left(\prod_{j=1}^{k} dA'_{\bar{i}j} dA'_{i\bar{j}}\right). \tag{5.334}$$

From $A_{\bar{i}j} = \sum_{l,p=1}^{k} P_{\bar{i}l} A'_{\bar{l}p} P_{j\bar{p}}$, the integral measure is invariant. $DA = DA'$. Therefore,

$$\begin{aligned}
I_A = &\int DA' \exp\Big(-\sum_{i=1}^{k} |A'_{\bar{i}i}|^2 - \sum_{i<j} |A'_{\bar{i}j}|^2 (1+\sqrt{-1}(\lambda_i - \lambda_j)) \\
&- \sum_{j<i} |A'_{\bar{i}j}|^2 (1+\sqrt{-1}(\lambda_i - \lambda_j))\Big) \\
&= \left[\prod_{i=1}^{k} \left\{\int dA'_{\bar{i}i} dA'_{i\bar{i}} \exp\big(-|A'_{\bar{i}i}|^2\big)\right\}\right] \\
&\times \left[\prod_{i<j} \left\{\int dA'_{\bar{i}j} dA'_{i\bar{j}} \exp\big(-|A'_{\bar{i}j}|^2 (1+\sqrt{-1}(\lambda_i - \lambda_j))\big)\right\}\right] \\
&\times \left[\prod_{i>j} \left\{\int dA'_{\bar{i}j} dA'_{i\bar{j}} \exp\big(-|A'_{\bar{i}j}|^2 (1+\sqrt{-1}(\lambda_i - \lambda_j))\big)\right\}\right]
\end{aligned} \tag{5.335}$$

$$= \pi^k \Big(\prod_{l<j} \frac{\pi}{(1+i(\lambda_l - \lambda_j))}\Big)\Big(\prod_{l>j} \frac{\pi}{(1+i(\lambda_l - \lambda_j))}\Big) = \frac{\pi^{k^2}}{\prod_{l<j}(1-(i\lambda_l - i\lambda_j)^2)}. \tag{5.336}$$

□

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