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博士学位論文

Explicit logarithmic formulas of hypergeometric functions 3F2
(超幾何関数 3F2 の明示対数公式)

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令和４年３月

EXPLICIT LOGARITHMIC FORMULAS FOR HYPERGEOMETRIC FUNCTIONS ${}_3F_2$

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1. Introduction

The series

$$\sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_{p-1})_n} \frac{t^n}{n!} \tag{1.1}$$

which is denoted by

$${}_pF_{p-1}(a_1, \ldots, a_p; b_1, \ldots, b_{p-1}; t) \text{ or } {}_pF_{p-1}\left(\begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_{p-1} \end{matrix}; t\right)$$

is called the (generalized) hypergeometric function where $a_i, b_j \in \mathbb{C}$ with $b_j \notin \mathbb{Z}_{\leq 0}$ are parameters and

$$(\alpha)_0 := 1, \quad (\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1)$$

denotes the Pochhammer symbol. When $p = 2$, this is often refered to as the Gaussian hypergeometric function. The radius of convergence is 1 unless it terminates, and it has the analytic continuation to $\mathbb{C} \setminus \{0, 1\}$ (see [8] for the general theory).

Hypergeometric function is one of the most traditional functions, and it has been studied by many people for more than a century. There are a number of interesting formulas on hypergeometric function (e.g. [16, Chap.15,16]). Very recently, Asakura, Otsubo and Terasoma discovered a new formula on ${}_3F_2$, which they call the *log formula* for ${}_3F_2$.

Theorem 1.1 (Asakura, Otsubo, Terasoma, [5, Theorem 2.1])**.** *Let $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$ be the beta function. For $x \in \mathbb{R}$, let $\{x\} := x - \lfloor x \rfloor$ denote the decimal part. Let $a, b, q \in \mathbb{Q}$ be non-integers such that none of $q - a, q - b, q - a - b$ is an integer. Assume that*

$$\{sq\} + \{s(-q + a)\} + \{s(-q + b)\} + \{s(q - a - b)\} = 2 \tag{1.2}$$

holds for all $s \in \mathbb{Z}$ prime to the denominators of a, b, q. Then

$$B(a, b) {}_3F_2\left(\begin{matrix} a, b, q \\ a + b, q + 1 \end{matrix}; 1\right) \in \overline{\mathbb{Q}} + \overline{\mathbb{Q}} \log \overline{\mathbb{Q}}^{\times}, \tag{1.3}$$

where the right hand side denotes the $\overline{\mathbb{Q}}$-linear subspace of $\mathbb{C}$ generated by 1, $2\pi i$ and $\log \alpha$'s, $\alpha \in \overline{\mathbb{Q}}^{\times}$.

The series (1.1) converges absolutely at $t = 1$ if $\mathrm{Re}(\sum b_j - \sum a_i) > 0$, and then the convergence value is denoted by ${}_pF_{p-1}(a_1, \ldots, a_p; b_1, \ldots, b_{p-1}; 1)$.

Moreover, Asakura and Otsubo obtained its functional version.

Theorem 1.2 (Asakura, Otsubo, [6, Theorem 2.1])**.** *Let $q, a, b \in \mathbb{Q}$ satisfy that none of $q, a, b, q-a, q-b, q-a-b$ is an integer. If*

$$\min(\{sa\}, \{sb\}) < \{sq\} < \max(\{sa\}, \{sb\}) \tag{1.4}$$

for any integers s prime to the denominators of a, b, q, then

$${}_3F_2\left(\begin{matrix}1,1,q\\a,b\end{matrix};t\right) \in \overline{\mathbb{Q}(t)} + \overline{\mathbb{Q}(t)}\log\overline{\mathbb{Q}(t)}^{\times}$$

where $\overline{\mathbb{Q}(t)}$ is the algebraic closure of the field of rational functions $\mathbb{Q}(t)$.

The condition (1.4) implies (1.2), but the converse is not true. Therefore, Theorem 1.1 is partially covered by Theorem 1.2 thanks to Thomae's formula

$$B(a,b){}_3F_2\left(\begin{matrix}a,b,q\\a+b,q+1\end{matrix};1\right) = \frac{q}{ab}{}_3F_2\left(\begin{matrix}1,1,a+b-q\\a+1,b+1\end{matrix};1\right) \quad (q>0),$$

while not every case is covered.

The proof in [5] or [6] is based on the observation on Beilinson's regulator on K_1, entirely algebro-geometric. It is worth noticing that they are "existence theorems". Namely they only assert that there exist algebraic numbers (or functions) describing ${}_3F_2$. On the other hand, there remains the problem exploring the explicit descriptions. If one can provide an explicit formula, then we call it an *explicit (functional) log formula* for ${}_3F_2$, which is the topic of the thesis.

Concerning the special value

$$B(a,b){}_3F_2\left(\begin{matrix}a,b,q\\a+b,q+1\end{matrix};1\right),$$

there is a classical formula of Watson [13] (see also [8] Ex.9, p.98) which says

$$2B(a,b){}_3F_2\left(\begin{matrix}a,b,\frac{a+b-1}{2}\\a+b,\frac{a+b+1}{2}\end{matrix};1\right) = \psi\left(\frac{a+1}{2}\right)+\psi\left(\frac{b+1}{2}\right)-\psi\left(\frac{a}{2}\right)-\psi\left(\frac{b}{2}\right)$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$. Gauss gives an explicit formula of the special values $\psi(\alpha)$ for $\alpha \in \mathbb{Q}\setminus\mathbb{Z}$ (see [16, 5.4.19]), and hence one sees that the case $q = (a+b-1)/2$ in Theorem 1.1 is covered by Watson's formula. The first example that is not covered by Watson was provided by the author's master thesis [14]. In the thesis, we give more thorough discussion, and give explicit log formulas in the following cases,

$$(a,b,q) = \left(\frac{1}{6},\frac{5}{6},\frac{k}{l}\right), \quad l = 2,3,4,5 \text{ and } 0<k<l$$

which includes the result in [14].

Theorem 1.3. (1) $B\left(\frac{1}{6},\frac{5}{6}\right){}_3F_2\left(\begin{matrix}\frac{1}{6},\frac{5}{6},\frac{1}{2}\\1,\frac{3}{2}\end{matrix};1\right) = 3\sqrt{3}\log(2+\sqrt{3}).$

(2) *Put*

$$A_1 = \log\left(\frac{(1-2^{-\frac{2}{3}})^2+(1+2^{-\frac{2}{3}}\sqrt{3})^2}{(1-2^{-\frac{2}{3}})^2+(1-2^{-\frac{2}{3}}\sqrt{3})^2}\right), \quad A_2 = \mathrm{Tan}^{-1}\left(\frac{3}{3+\sqrt[3]{2}+3\sqrt[3]{4}}\right).$$

Then, for $k=1,2$

$$B\left(\frac{1}{6},\frac{5}{6}\right){}_3F_2\left(\begin{matrix}\frac{1}{6},\frac{5}{6},\frac{k}{3}\\1,\frac{4}{3}\end{matrix};1\right) = \left(\frac{2}{3}\right)^{k-1}\sqrt{3}\sqrt[3]{2^k}A_1 - 2\left(-\frac{2}{3}\right)^{k-1}\sqrt[3]{2^k}A_2.$$

(3)

$$B\left(\frac{1}{6},\frac{5}{6}\right){}_3F_2\left(\begin{matrix}\frac{1}{6},\frac{5}{6},\frac{1}{4}\\ 1,\frac{5}{4}\end{matrix};1\right)=12^{3/4}\left(\frac{1}{2}\log\left(\frac{3^{5/4}-3^{3/4}+\sqrt{2}}{3^{5/4}-3^{3/4}-\sqrt{2}}\right)-\mathrm{Cos}^{-1}\left(\frac{3^{5/4}+3^{3/4}}{2\sqrt{5+3\sqrt{3}}}\right)\right),$$

$$B\left(\frac{1}{6},\frac{5}{6}\right){}_3F_2\left(\begin{matrix}\frac{1}{6},\frac{5}{6},\frac{3}{4}\\ 1,\frac{7}{4}\end{matrix};1\right)=\frac{9\cdot 12^{3/4}}{7\sqrt{3}}\left(\frac{1}{2}\log\left(\frac{3^{5/4}-3^{3/4}+\sqrt{2}}{3^{5/4}-3^{3/4}-\sqrt{2}}\right)+\mathrm{Cos}^{-1}\left(\frac{3^{5/4}+3^{3/4}}{2\sqrt{5+3\sqrt{3}}}\right)\right).$$

(4) *Let* $\zeta=e^{2\pi i/5}$, $\zeta_{20}=e^{2\pi i/20}$, $\alpha=1/\sqrt[10]{24}>0$, *and put*

$$e_j=\frac{\sqrt{2}\alpha^3\zeta_{20}^3\zeta^j+\frac{\sqrt{2}}{4}\alpha^{-3}\zeta_{20}^{-3}\zeta^j-\sqrt{3}(\alpha^2\zeta_{20}^2\zeta^j-1)}{\sqrt{2}\alpha^3\zeta_{20}^3\zeta^j+\frac{\sqrt{2}}{4}\alpha^{-3}\zeta_{20}^{-3}\zeta^j+\sqrt{3}(\alpha^2\zeta_{20}^2\zeta^j-1)},\quad j=0,1,2,3,$$

$$B_k=\frac{\Gamma(k/5+1/6)\Gamma(k/5+5/6)}{\Gamma(k/5)^2}.$$

Then, for $k=1,2,3,4$,

$$B\left(\frac{1}{6},\frac{5}{6}\right){}_3F_2\left(\begin{matrix}\frac{1}{6},\frac{5}{6},\frac{k}{5}\\ 1,\frac{k}{5}+1\end{matrix};1\right)=\frac{k(\zeta^{2k}-1)}{5B_k}\Big((\zeta^{2k}-1)\log e_0$$

$$+(\zeta^{2k}-\zeta^{3k})\log e_1+(\zeta^{2k}-\zeta^{k})\log e_2+(\zeta^{2k}-\zeta^{4k})\log e_3+4\pi i\zeta^{2k}\Big).$$

The proof of Theorem 1.3 mostly follows the argument in [5] or [6], while we need a new ingredient "ω_{Del}" (see §2.5 for the definition). Moreover, we need to find a suitable hypergeometric fibration, motivic elements in K_1 etc. The task is highly non-trivial, and the various computation (periods, regulator, etc.) is heavy.

Our second main result is to give the explicit *functional* log formulas that are the first non-trivial examples of Theorem 1.2 [1].

Theorem 1.4. *Put*

$$a_1(t)=\frac{t}{(1+\sqrt{1-t^6})^{\frac{1}{3}}}+\frac{(1+\sqrt{1-t^6})^{\frac{1}{3}}}{t},\quad a_2(t)=\frac{t}{(1+\sqrt{1-t^6})^{\frac{1}{3}}}-\frac{(1+\sqrt{1-t^6})^{\frac{1}{3}}}{t}.$$

Then

$${}_3F_2\left(\begin{matrix}1,1,\frac{1}{2}\\ \frac{4}{3},\frac{5}{3}\end{matrix};t^6\right)=\frac{-4i}{3\sqrt{3}t^3}\left(a_2(\mathrm{e}^{\frac{\pi i}{3}}t)\log\left(1+\frac{3}{2}t^3a_1(t)\right)+a_2(t)\log\left(1-\frac{3}{2}t^3a_1(\mathrm{e}^{\frac{\pi i}{3}}t)\right)\right).$$

Theorem 1.5.

$${}_3F_2\left(\begin{matrix}1,1,\frac{1}{2}\\ \frac{5}{4},\frac{7}{4}\end{matrix};t^4\right)=-\frac{3i\sqrt{1-t^2}}{2t^3}\log(\sqrt{1-t^2}-it)-\frac{3\sqrt{1+t^2}}{2t^3}\log(\sqrt{1+t^2}-t).$$

Theorem 1.6. *Put*

$$a_1(t)=\frac{t^2}{\left(1+\sqrt{1-t^6}\right)^{\frac{2}{3}}}+\frac{\left(1+\sqrt{1-t^6}\right)^{\frac{2}{3}}}{t^2},\quad a_2(t)=\frac{t^2}{(1+\sqrt{1-t^6})^{\frac{2}{3}}}-\frac{(1+\sqrt{1-t^6})^{\frac{2}{3}}}{t^2}.$$

Then

$${}_3F_2\left(\begin{matrix}1,1,\frac{1}{2}\\ \frac{7}{6},\frac{11}{6}\end{matrix};t^6\right)=\frac{5i}{12t^3}\left(a_2(\mathrm{e}^{\frac{\pi i}{3}}t)\log\frac{2-\sqrt{3}t^3(a_1(t)-1)}{2+\sqrt{3}t^3(a_1(t)-1)}+a_2(t)\log\frac{2+\sqrt{3}t^3(a_1(\mathrm{e}^{\frac{\pi i}{3}}t)-1)}{2-\sqrt{3}t^3(a_1(\mathrm{e}^{\frac{\pi i}{3}}t)-1)}\right).$$

[1] One finds in [6] the explicit log formula of ${}_3F_2(1,1,\frac{1}{2};\frac{7}{6},\frac{11}{6};t)$ without proof.

Using the contiguous operators, one can also obtain the explicit log formulas of ${}_3F_2\left({n_1,n_2,q+n_3 \atop a+n_4,b+n_5};t\right)$ with $n_i \in \mathbb{Z}$ (see §4.4 for detail).

Each proof of Theorems 1.4, ..., 1.6 goes in a similar way to that of Theorems 1.3. The key is to find a suitable elliptic fibration, a non-trivial divisor and a motivic element in K_1 in each (a,b,q), as is so in the proof of Theorem 1.3.

The paper is organized as follows. In §2, we give the recipe for obtaining the explicit log formulas, in particular, a rational differential form "$(\omega)_{\mathrm{Del}}$" plays an important role in the later section. In §3, we prove Theorem 1.3. The sections 2 and 3 are reproduction from a part of [7]. In §4, we prove Theorems 1.4, 1.5 and 1.6.

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2. Explicit Log formula

Throughout this paper, we fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

2.1. **Hypergeometric Fibrations.** We recall the hypergeometric fibrations introduced in [4] §3.1. Let R be a finite-dimensional semisimple $\mathbb{Q}$-algebra. Let $e : R \to E$ be a projection onto a number field E. Let X be a smooth projective variety over k_{dR}, and $f : X \to \mathbb{P}^1$ a surjective map endowed with a multiplication on $R^1 f_*\mathbb{Q}|_U$ by R where $U \subset \mathbb{P}^1$ is the maximal Zariski open set such that f is smooth over U. We say f is a *hypergeometric fibration with multiplication by* (R,e) (abbreviated HG fibration) if the following conditions hold. We fix an inhomogeneous coordinate $t \in \mathbb{P}^1$.

- **(a)** f is smooth over $\mathbb{P}^1 \setminus \{t = 0, 1, \infty\}$,
- **(b)** $\dim_E (R^1 f_*\mathbb{Q})(e) = 2$ where we write $V(e) := E \otimes_{e,R} V$ the e-part,
- **(c)** Let $\mathrm{Pic}^0_f \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ be the Picard fibration whose general fiber is the Picard variety $\mathrm{Pic}^0(f^{-1}(t))$, and let $\mathrm{Pic}^0_f(e)$ be the component associated to the e-part $(R^1 f_*\mathbb{Q})(e)$ (this is well-defined up to isogeny). Then $\mathrm{Pic}^0_f(e) \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ has totally degenerate semistable reduction at $t = 1$.

The last condition **(c)** is equivalent to saying that the local monodromy T on $(R^1 f_*\mathbb{Q})(e)$ at $t = 1$ is unipotent and the rank of log monodromy $N := \log(T)$ is maximal, namely $\mathrm{rank}(N) = \frac{1}{2}\dim_{\mathbb{Q}}(R^1 f_*\mathbb{Q})(e)$ ($= [E : \mathbb{Q}]$ by the condition **(b)**).

Example 2.1. Let $f : X \to \mathbb{P}^1$ be an elliptic fibration. Then f is a HG fibration with multiplication by $(\mathbb{Q}, \mathrm{id})$ if and only if f is smooth over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and the reduction at $t = 1$ is multiplicative (i.e. of type I_n, $n > 0$).

Example 2.2 ([4] §3.2)**.** Let N, A, B be integers such that $0 < A, B < N$ and $\gcd(A, N) = \gcd(B, N) = 1$. Let $f : X \to \mathbb{P}^1$ be a fibration whose general fiber $X_t = f^{-1}(t)$ is the projective nonsingular model of an affine curve

$$y^N = x^A(1-x)^B(1-tx)^{N-B}.$$

Then f is smooth over $\mathbb{P}^1 \setminus \{t = 0, 1, \infty\}$. Let μ_N be the group of N-th roots of unity. For $\zeta_N \in \mu_N$, the automorphism given by $(x, y, t) \mapsto (x, \zeta_N y, t)$ gives rise to the multiplication by the group ring $R=\mathbb{Q}[\mu_N]$. Let $e:R\to E$ be a projection onto a number field E. If $E\neq\mathbb{Q}$, then (R,e) satisfies the conditions **(b), (c)**. We call f the *HG fibration of Gauss type*.

2.2. **Motivic cohomology and Deligne-Beilinson cohomology.** The theory of the motivic cohomology groups

$$H^i_{\mathscr{M}}(X,\mathbb{Z}(j))$$

of a variety X over a field is developed by Suslin, Voevodsky et al. We here review $H^3_{\mathscr{M}}(X,\mathbb{Z}(2))$, which has an elementary description in the following way. Let X be a smooth quasi-projective variety over a field k. The Milnor K-theory is denoted by K_2^M. Then the *motivic cohomology group* $H^3_{\mathscr{M}}(X,\mathbb{Z}(2))$ can be identified with the cohomology at the middle term of of the following complex

$$K_2^M(\overline{\mathbb{Q}}(X))\xrightarrow{\delta_2}\bigoplus_D k(D)^\times\xrightarrow{\delta_1}\bigoplus_E\mathbb{Z}\tag{2.1}$$

at the middle term, where D and E run over all integral closed subschemes on X of codimension 1 and 2 respectively, and δ_i are given as follows

$$\delta_2\{f,g\}=\sum_D(-1)^{v_D(f)v_D(g)}\frac{f^{v_D(g)}}{g^{v_D(f)}}|_D,\quad \delta_1\left(\sum_D(f,D)\right)=\sum_D\mathrm{div}_D(f).$$

Here (f,D) denotes an element $f\in k(D)^\times\subset\oplus_D k(D)^\times$ placed in the D-component. Thus any element of $H^3_{\mathscr{M}}(X,\mathbb{Z}(2))$ is represented by an element $\sum_D(f,D)$ satisfying $\sum_D\mathrm{div}_D(f)=0$. Note that the Chow group $\mathrm{CH}^2(X)$ is defined to be the cokernel of δ_1. For a closed subscheme $Z\subset X$ of codimension 1, the motivic cohomology $H^3_{\mathscr{M},Z}(X,\mathbb{Z}(2))$ supported on Z is canonically isomorphic to the kernel of

$$\bigoplus_{D\subset Z}k(D)^\times\xrightarrow{\delta_1}\bigoplus_{E\subset Z}\mathbb{Z}.\tag{2.2}$$

Hence there is an exact sequence

$$H^3_{\mathscr{M},Z}(X,\mathbb{Z}(2))\to H^3_{\mathscr{M}}(X,\mathbb{Z}(2))\to H^3_{\mathscr{M}}(X\setminus Z,\mathbb{Z}(2)).$$

Let X be a projective smooth variety over $\mathbb{C}$, and $Z\subset X$ a closed subscheme. The *Deligne-Beilinson cohomology* group $H^\bullet_{\mathscr{D},Z}(X,\mathbb{Z}(r))$ is defined to be the cohomology $\mathbb{H}^\bullet_Z(X^{an},\mathbb{Z}(r)_{\mathscr{D}})$ of the complex

$$\mathbb{Z}(r)_{\mathscr{D}}:\mathbb{Z}(r)\to\mathscr{O}_X\to\Omega^1_X\to\cdots\to\Omega^{r-1}_X$$

of sheaves on the analytic site X^{an} (e.g. [11]). Write $H^\bullet_{\mathscr{D}}(X,\mathbb{Z}(r)):=H^\bullet_{\mathscr{D},X}(X,\mathbb{Z}(r))$. If the base field is $\overline{\mathbb{Q}}$, we simply write $H^\bullet_{\mathscr{D},Z}(X,\mathbb{Z}(r))=H^\bullet_{\mathscr{D},Z\times_{\overline{\mathbb{Q}}}\mathbb{C}}(X\times_{\overline{\mathbb{Q}}}\mathbb{C},\mathbb{Z}(r))$ (note that we fix an embedding $\overline{\mathbb{Q}}\hookrightarrow\mathbb{C}$ throughout the paper). There is the *Beilinson regulator map* (or higher Chern class map)

$$\mathrm{reg}:H^i_{\mathscr{M},Z}(X,\mathbb{Z}(r))\longrightarrow H^i_{\mathscr{D},Z}(X,\mathbb{Z}(r)).\tag{2.3}$$

We refer to [12] for the definition of regulator maps. We shall discuss the case $(i,r)=(3,2)$ in detail in §2.4. There is the exact sequence

$$0\to H^2_B(X,\mathbb{C})/F^2H^2_B(X,\mathbb{C})+H^2_B(X,\mathbb{Z}(2))\to H^3_{\mathscr{D}}(X,\mathbb{Z}(2))\xrightarrow{i}H^3_B(X,\mathbb{Z}(2))_{\mathrm{tor}}\to 0$$

where $F^\bullet$ denotes the Hodge filtration. Write $H^3_{\mathscr{D}}(X,\mathbb{Z}(2))' := \mathrm{Ker}(i)$. One has

$$H^3_{\mathscr{D}}(X,\mathbb{Z}(2))' \cong H^2_B(X,\mathbb{C})/F^2H^2_B(X,\mathbb{C}) + H^2_B(X,\mathbb{Z}(2)) \tag{2.4}$$

$$\cong \mathrm{Hom}_{\mathbb{C}}(F^{d-1}H^{2d-2}_B(X),\mathbb{C})/\mathrm{Im}H^B_{2d-2}(X,\mathbb{Z}(2-d)) \tag{2.5}$$

where $d = \dim X$.

2.3. **Relative de Rham cohomology.** For a smooth manifold M, $\mathscr{A}^q(M)$ denotes the complex of spaces of smooth differential q-forms on M with coefficients in $\mathbb{C}$.

Let X be a quasi-projective smooth variety over $\mathbb{C}$. The de Rham cohomology $H^q_{\mathrm{dR}}(X)$ is defined to be the cohomology of the complex $\mathscr{A}^\bullet(X)$

$$H^q_{\mathrm{dR}}(X) = H^q(\mathscr{A}^\bullet(X)).$$

By Grothendieck's comparison theorem, one may replace $\mathscr{A}^\bullet(X)$ with the algebraic de Rham complex,

$$H^q(\mathscr{A}^\bullet(X)) \cong H^q_{\mathrm{zar}}(X,\Omega^\bullet_X).$$

The right hand side is often referred as algebraic de Rham cohomology groups (and the left hand side as analytic de Rham cohomology). In this paper we identify the both sides, and simply call the de Rham cohomology.

In more general, the relative de Rham cohomology groups $H^q_{\mathrm{dR}}(X_\bullet, Y_\bullet)$ for an embedding $Y_\bullet \hookrightarrow X_\bullet$ of simplicial schemes are defined (e.g. [9] 8.3.8). We here review the definition of $H^2_{\mathrm{dR}}(V,D)$ in case that V is a quasi-projective smooth surface over $\mathbb{C}$ and $D \subset V$ a reduced curve (i.e. a reduced closed subscheme of codimension one). Let $\rho : \widetilde{D} \to D$ be the normalization and $\Sigma \subset D$ the set of singular points. Let $s : \widetilde{\Sigma} := \rho^{-1}(\Sigma) \hookrightarrow \widetilde{D}$ be the inclusion. There is an exact sequence

$$0 \longrightarrow \mathscr{O}_D \xrightarrow{\rho^*} \mathscr{O}_{\widetilde{D}} \xrightarrow{s^*} \mathbb{C}_{\widetilde{\Sigma}}/\mathbb{C}_\Sigma \longrightarrow 0$$

where $\mathbb{C}_{\widetilde{\Sigma}} = \mathrm{Maps}(\widetilde{\Sigma},\mathbb{C}) = \mathrm{Hom}(\mathbb{Z}\widetilde{\Sigma},\mathbb{C})$, ρ^* and s^* are the pull-back. We define $\mathscr{A}^\bullet(D)$ to be the mapping fiber of $s^* : \mathscr{A}^\bullet(\widetilde{D}) \to \mathbb{C}_{\widetilde{\Sigma}}/\mathbb{C}_\Sigma$:

$$\mathscr{A}^0(\widetilde{D}) \xrightarrow{s^*\oplus d} \mathbb{C}_{\widetilde{\Sigma}}/\mathbb{C}_\Sigma \oplus \mathscr{A}^1(\widetilde{D}) \xrightarrow{0\oplus d} \mathscr{A}^2(\widetilde{D})$$

where the first term is placed in degree 0. Then

$$H^q_{\mathrm{dR}}(D) = H^q(\mathscr{A}^\bullet(D))$$

is the de Rham cohomology of D, which fits into the exact sequence

$$\cdots \longrightarrow H^0_{\mathrm{dR}}(\widetilde{D}) \longrightarrow \mathbb{C}_{\widetilde{\Sigma}}/\mathbb{C}_\Sigma \longrightarrow H^1_{\mathrm{dR}}(D) \longrightarrow H^1_{\mathrm{dR}}(\widetilde{D}) \longrightarrow \cdots.$$

There is a natural pairing

$$H_1(D,\mathbb{Z}) \otimes H^1_{\mathrm{dR}}(D) \longrightarrow \mathbb{C}, \quad \gamma \otimes z \mapsto \int_\gamma z := \int_\gamma \eta - c(\partial(\rho^{-1}\gamma)) \tag{2.6}$$

where $z = (c,\eta) \in \mathbb{C}_{\widetilde{\Sigma}}/\mathbb{C}_\Sigma \oplus \mathscr{A}^1(\widetilde{D})$ with $d\eta = 0$ and $\partial : H_1(\widetilde{D},\widetilde{\Sigma}) \to H_0(\widetilde{\Sigma}) = \mathbb{Z}\widetilde{\Sigma}$ denotes the boundary map (note that $c(\partial(\rho^{-1}\gamma) = 0$ if $c \in \mathbb{C}_\Sigma$).

We define $\mathscr{A}^\bullet(V,D)$ to be the mapping fiber of $j^* : \mathscr{A}^\bullet(V) \to \mathscr{A}^\bullet(D)$ the pull-back by $j : D \hookrightarrow V$:

$$\mathscr{A}^0(V) \xrightarrow{\mathscr{D}_0} \mathscr{A}^0(\widetilde{D}) \oplus \mathscr{A}^1(V) \xrightarrow{\mathscr{D}_1} \mathbb{C}_{\widetilde{\Sigma}}/\mathbb{C}_\Sigma \oplus \mathscr{A}^1(\widetilde{D}) \oplus \mathscr{A}^2(V) \xrightarrow{\mathscr{D}_2} \cdots$$

where

$$\mathscr{D}_0 = (j\rho)^* \oplus d, \quad \mathscr{D}_1 = \begin{pmatrix} -(s^* \oplus d) & 0 \oplus (j\rho)^* \\ & d \end{pmatrix}, \quad \mathscr{D}_2 = \begin{pmatrix} -(0 \oplus d) & (j\rho)^* \\ & d \end{pmatrix}, \ldots$$

Then

$$H^q_{\mathrm{dR}}(V, D) = H^q(\mathscr{A}^\bullet(V, D)) \tag{2.7}$$

is the de Rham cohomology which fits into the exact sequence

$$\cdots \longrightarrow H^{q-1}_{\mathrm{dR}}(D) \longrightarrow H^q_{\mathrm{dR}}(V, D) \longrightarrow H^q_{\mathrm{dR}}(V) \longrightarrow H^q_{\mathrm{dR}}(D) \longrightarrow \cdots. \tag{2.8}$$

An arbitrary element of $H^2_{\mathrm{dR}}(V, D)$ is represented by

$$(c, \eta, \omega) \in \mathbb{C}_{\widetilde{\Sigma}}/\mathbb{C}_\Sigma \oplus \mathscr{A}^1(\widetilde{D}) \oplus \mathscr{A}^2(V) \tag{2.9}$$

which satisfies $j^*\omega = d\eta$ and $d\omega = 0$. They are subject to relations $(s^*f, df, 0) \sim 0$ and $(0, j^*\theta, d\theta) \sim 0$ for $f \in \mathscr{A}^0(\widetilde{D}_0)$ and $\theta \in \mathscr{A}^1(V)$. The natural pairing

$$H_2(V, D; \mathbb{Z}) \otimes H^2_{\mathrm{dR}}(V, D) \longrightarrow \mathbb{C}, \quad \Gamma \otimes z \longmapsto \int_\Gamma z \tag{2.10}$$

is given by

$$\int_\Gamma z := \int_\Gamma \omega - \int_{\partial\Gamma} (c, \eta) = \int_\Gamma \omega - \int_{\partial\Gamma} \eta + c(\rho^{-1}(\partial\Gamma)). \tag{2.11}$$

2.4. **The Beilinson regulator map by 1-extensions of mixed Hodge structures.** Let X be a smooth quasi-projective variety over $\mathbb{C}$. Let

$$\mathrm{reg} : H^3_{\mathscr{M}}(X, \mathbb{Z}(2)) \longrightarrow H^3_{\mathscr{D}}(X, \mathbb{Z}(2))$$

be the Beilinson regulator map to the Deligne-Beilinson cohomology group ([12]). We here describe it in terms of 1-extensions of mixed Hodge structures (abbreviated to MHS's). For simplicity we assume that X is a projective smooth surface. Let $Z \subset X$ be a curve. There is also the regulator map reg_Z on $H^3_{\mathscr{M},Z}(X, \mathbb{Z}(2))$ which fits into a commutative diagram

$$\begin{array}{ccc} H^3_{\mathscr{M},Z}(X, \mathbb{Z}(2)) & \xrightarrow{\mathrm{reg}_Z} & H^3_{\mathscr{D},Z}(X, \mathbb{Z}(2)) \\ \downarrow & & \downarrow \\ H^3_{\mathscr{M}}(X, \mathbb{Z}(2)) & \xrightarrow{\mathrm{reg}} & H^3_{\mathscr{D}}(X, \mathbb{Z}(2)). \end{array}$$

Let $\mathrm{Ext}^1(\mathbb{Z}, -)$ denote the group of 1-extensions of MHS's. There is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}^1(\mathbb{Z}, H_2(Z, \mathbb{Z})) & \longrightarrow & H^3_{\mathscr{D},Z}(X, \mathbb{Z}(2)) & \xrightarrow{\mathrm{can}} & H_1(Z, \mathbb{Z}) \cap H^{0,0} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow i & & \\ 0 & \longrightarrow & \mathrm{Ext}^1(\mathbb{Z}, H_2(X, \mathbb{Z})) & \longrightarrow & H^3_{\mathscr{D}}(X, \mathbb{Z}(2)) & \longrightarrow & H_1(X, \mathbb{Z})_{\mathrm{tor}} & \longrightarrow & 0 \end{array}$$

with exact rows where $H^{p,q} \subset H(X, \mathbb{C})$ denotes the Hodge (p,q)-component. We call the composition $c_Z := \mathrm{can} \circ \mathrm{reg}_Z$ the *cycle map*. The above diagram gives rise to a map

$$\Phi : \mathrm{Ker}(i) \longrightarrow \mathrm{Ext}^1(\mathbb{Z}, H_2(X, \mathbb{Z})/H_2(Z)).$$

This is explicitly described in the following way. Let

$$0 \longrightarrow H_2(X, \mathbb{Z})/H_2(Z) \longrightarrow H_2(X, Z; \mathbb{Z}) \xrightarrow{\partial} H_1(Z, \mathbb{Z})$$

be the exact sequence of homology. Then, for $\gamma \in H_1(Z,\mathbb{Z}) \cap H^{0,0}$ such that $\gamma \in \mathrm{Ker}(i)$ ($\Leftrightarrow \gamma \in \mathrm{Im}\partial$), $\Phi(\gamma)$ is the 1-extension corresponding to

$$0 \longrightarrow H_2(X,\mathbb{Z})/H_2(Z) \longrightarrow \partial^{-1}(\mathbb{Z}\gamma) \longrightarrow \mathbb{Z} \longrightarrow 0. \tag{2.12}$$

Summing up the above we have the following proposition.

Proposition 2.3. *Write the composition*

$$\mathrm{Ker}[H^3_{\mathscr{M}}(X,\mathbb{Z}(2)) \to H_1(X,\mathbb{Z})_{\mathrm{tor}}] \overset{\mathrm{reg}}{\to} \mathrm{Ext}^1(\mathbb{Z},H_2(X,\mathbb{Z})) \to \mathrm{Ext}^1(\mathbb{Z},H_2(X,\mathbb{Z})/H_2(Z))$$

by $\overline{\mathrm{reg}}$. *Let* $\xi \in H^3_{\mathscr{M},Z}(X,\mathbb{Z}(2))$ *and suppose that the homology cycle* $\gamma_\xi := c_Z(\xi) \in H_1(Z,\mathbb{Z})$ *lies in the image of* ∂. *Then* $\overline{\mathrm{reg}}(\xi)$ *is the 1-extension* (2.12) *for* $\gamma = \gamma_\xi$.

Writing down the 1-extension (2.12) in a down-to-earth way, we also have the following proposition.

Proposition 2.4. *Write* $H^2(X)_Z := \mathrm{Ker}[H^2(X) \longrightarrow H^2(Z)]$, *and consider the surjective map* $F^1H^2_{\mathrm{dR}}(X,Z) \to F^1H^2_{\mathrm{dR}}(X)_Z$. *We fix* $(c,\eta,\widetilde{\omega}) \in F^1H^2_{\mathrm{dR}}(X,Z)$ *a lifting for each* $\omega \in F^1H^2_{\mathrm{dR}}(X)_Z$. *Fix* $\Gamma_\xi \in H_2(X,Z;\mathbb{Z})$ *a lifting of* γ_ξ. *Then under the natural identification*

$$\mathrm{Ext}^1(\mathbb{Z},H_2(X,\mathbb{Z})/H_2(Z)) \cong \mathrm{Hom}(F^1H^2_{\mathrm{dR}}(X)_Z,\mathbb{C})/\mathrm{Im}H_2(X,\mathbb{Z}),$$

the Beilinson regulator is given as follows

$$\overline{\mathrm{reg}}(\xi) = [\omega \to \langle \Gamma_\xi, (c,\eta,\widetilde{\omega})\rangle]$$

where $\langle\, ,\, \rangle$ *denotes the natural pairing* $H_2(X,Z;\mathbb{Z}) \otimes_{\mathbb{Z}} H^2_{\mathrm{dR}}(X,Z) \to \mathbb{C}$.

Note

$$\langle \Gamma_\xi, (\widetilde{\omega},\eta)\rangle = \int_{\Gamma_\xi} \widetilde{\omega} - \int_{\gamma_\xi} (c,\eta)$$

and this does not depend on the choice of $(c,\eta,\widetilde{\omega})$ because $\gamma_\xi \in H^{0,0}$ and hence $\int_{\gamma_\xi}$ annihilates elements of $F^1H^1_{\mathrm{dR}}(Z)$. We should keep notice that, it is *not* true in general that $\int_{\Gamma_\xi} \widetilde{\omega}$ depends only on the cohomology class $\omega \in H^2_{\mathrm{dR}}(X)$.

2.5. **Special liftings of rational forms.** The crucial idea of the functional log formula in [6] is to relate the hypergeometric functions with the Beilinson regulators of K_1 of the hypergeometric fibrations. We mostly follows the discussion, while we need a new ingredient that does no appear in [5] or [6], namely a certain rational differential form

$$(\omega)_{\mathrm{Del}}.$$

This plays an essential role in our discussion. In this section, focusing on elliptic fibrations, we introduce $(\omega)_{\mathrm{Del}}$ and provide the recipe for the functional log formula.

We mean by an elliptic fibration a morphism

$$f : X \longrightarrow \mathbb{P}^1$$

from a smooth projective surface X onto the projective line such that there is a section $e : \mathbb{P}^1 \to X$. Let $T \subset \mathbb{P}^1$ be the set of points such that f is smooth over $S := \mathbb{P}^1 \setminus T$. Put $U := f^{-1}(S)$. Suppose that f has a multiplicative singular fiber (i.e. Kodaira symbol I_m, $m > 0$).

Let $\mathscr{H} := H^1_{\mathrm{dR}}(U/S)$ be a vector bundle over S endowed with the Gauss-Manin connection ∇. There is the comparison isomorphism,

$$H^1_{\mathrm{dR}}(S,\mathscr{H}) := H^1_{\mathrm{zar}}(S,\mathscr{H} \to \Omega^1_S \otimes \mathscr{H}) \cong H^1_B(S,R^1f_*\mathbb{Q}) \otimes \mathbb{C}$$

where $H_B^\bullet$ denotes the Betti cohomology. There is Deligne's canonical extension $\mathscr{H}_e$ on $\mathbb{P}^1$ (cf. [15, (17)]), so that the connection extends to

$$\nabla : \mathscr{H}_e \longrightarrow \Omega^1_{\mathbb{P}^1}(\log T) \otimes \mathscr{H}_e.$$

Then there is the natural isomorphism

$$H^1_{\mathrm{dR}}(\mathbb{P}^1, \mathscr{H}_e) := H^1_{\mathrm{zar}}(\mathbb{P}^1, \mathscr{H}_e \to \Omega^1_{\mathbb{P}^1}(\log T) \otimes \mathscr{H}_e) \cong H^1_{\mathrm{dR}}(S, \mathscr{H}). \tag{2.13}$$

Let $F^\bullet\mathscr{H}$ be the Hodge filtration, and put $F^1\mathscr{H}_e := \mathscr{H}_e \cup j_\star F^1\mathscr{H}$ where $j : S \to \mathbb{P}^1$ is the embedding. Consider a commutative diagram

$$\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
 & & \Omega^1_{\mathbb{P}^1}(\log T) \otimes F^1\mathscr{H}_e \\
 & & \downarrow \\
F^1\mathscr{H}_e & \xrightarrow{\nabla} & \Omega^1_{\mathbb{P}^1}(\log T) \otimes \mathscr{H}_e \\
\| & & \downarrow \\
F^1\mathscr{H}_e & \xrightarrow{\overline{\nabla}} & \Omega^1_{\mathbb{P}^1}(\log T) \otimes \mathscr{H}_e/F^1 \\
 & & \downarrow \\
 & & 0.
\end{array}$$

Let $(\mathbb{P}^1)^\circ \subset \mathbb{P}^1$ be the Zariski open set such that $\overline{\nabla}|_{(\mathbb{P}^1)^\circ}$ is bijective. Put $X^\circ = f^{-1}((\mathbb{P}^1)^\circ)$. The open set $(\mathbb{P}^1)^\circ$ contains every closed points at which f has a multiplicative fiber, and in particular, $(\mathbb{P})^\circ \neq \emptyset$ by the assumption. We note that neither $(\mathbb{P}^1)^\circ \subset S$ or $S \subset (\mathbb{P}^1)^\circ$ is true in general. From the above diagram, we get an isomorphism

$$H^1((\mathbb{P}^1)^\circ, F^1\mathscr{H}_e \to \Omega^1_{\mathbb{P}^1}(\log T) \otimes \mathscr{H}_e) \cong \Gamma((\mathbb{P}^1)^\circ, \Omega^1_{\mathbb{P}^1}(\log T) \otimes F^1\mathscr{H}_e). \tag{2.14}$$

The cohomology group $H^1_{\mathrm{dR}}(S, \mathscr{H})$ is endowed with the Hodge and weight filtrations written by $F^\bullet$ and $W_\bullet$ respectively, which are compatible with the mixed Hodge structure on $H^2_B(U, \mathbb{Q})$ under the (injective) map $H^1_B(S, R^1 f_* \mathbb{Q}) \to H^2_B(U, \mathbb{Q})$. Let Θ_{Del} be the composition of the arrows

$$\begin{aligned}
F^1 H^1_{\mathrm{dR}}(S, \mathscr{H}) &\cong H^1(\mathbb{P}^1, F^1\mathscr{H}_e \to \Omega^1_{\mathbb{P}^1}(\log T) \otimes \mathscr{H}_e) && \text{(by (2.13))} \\
&\hookrightarrow H^1((\mathbb{P}^1)^\circ, F^1\mathscr{H}_e \to \Omega^1_{\mathbb{P}^1}(\log T) \otimes \mathscr{H}_e) \\
&\cong \Gamma((\mathbb{P}^1)^\circ, \Omega^1_{\mathbb{P}^1}(\log T) \otimes F^1\mathscr{H}_e) && \text{(by (2.14))} \\
&\subset \Gamma(U \cap X^\circ, \Omega^2_X)
\end{aligned}$$

where the injectivity of the second arrow follows from the fact that the composition of the maps

$$F^1 H^1_{\mathrm{dR}}(S, \mathscr{H}) \to F^1 H^1_{\mathrm{dR}}(S \cap (\mathbb{P}^1)^\circ, \mathscr{H}) \to H^1(S \cap (\mathbb{P}^1)^\circ, F^1\mathscr{H} \to \Omega^1_S \otimes \mathscr{H})$$

is injective (cf. [2, Lemma 4.2]). Then Θ_{Del} induces an injective map

$$F^1 W_2 H^1_{\mathrm{dR}}(S, \mathscr{H}) \longrightarrow \Gamma(X^\circ, \Omega^2_X) \tag{2.15}$$

which we also write by Θ_{Del}. For

$$\omega \in F^1W_2H^1_{\mathrm{dR}}(S, \mathscr{H}) = \mathrm{Ker}[F^1W_2H^2_{\mathrm{dR}}(U/\mathbb{C}) \to H^2(X_t/\mathbb{C})],$$

we define

$$(\omega)_{\mathrm{Del}} := \Theta_{\mathrm{Del}}(\omega).$$

The following theorem plays the central role in the discussion towards the explicit functional log formula.

Theorem 2.5. *Let $C = \Sigma n_i C_i$ be a divisor in X with $\mathbb{Z}$-coefficients which is perpendicular to any components of singular fibers. Let $\omega_C \in H^2_{\mathrm{dR}}(X)$ be the cycle class of C. Let $j : \bigsqcup \tilde{C}_i \to \cup C_i \to X$ be the composition of normalization and the embedding. Let*

$$\mathrm{tr}_{\tilde{C}_i} : H^3_{\mathcal{M}}(\tilde{C}_i, \mathbb{Z}(2)) \longrightarrow H^1_{\mathcal{M}}(\mathrm{Spec}\,\mathbb{C}, \mathbb{Z}(1)) = \mathbb{C}^\times$$

be the transfer map, and put $\mathrm{tr}_C = \sum n_i \mathrm{tr}_{\tilde{C}_i}$. Let Z be a fibral divisor (e.g. a multiplicative fiber). Let $\xi \in H^3_{\mathcal{M},Z}(X, \mathbb{Z}(2))$ be arbitrary, and put $\gamma_\xi = c_Z(\xi) \in H^3_Z(X, \mathbb{Z}(2)) = H_1(Z, \mathbb{Z})$ where c_Z is the cycle map . Let $\Gamma_\xi \in H_2(X^\circ, Z; \mathbb{Z})$ be a lifting of γ_ξ. Then

$$\log \mathrm{tr}_C(j^*\xi) = \int_{\Gamma_\xi} (\omega_C|_U)_{\mathrm{Del}} \in \mathbb{C}/\mathbb{Z}(1). \tag{2.16}$$

3. Proof of Theorem 1.3

In this section, we prove Theorem 1.3. To do this, we employ Theorem 2.5 where we take an elliptic fibration $f_l : X_l \to \mathbb{P}^1$ to be defined by a Weierstrass equation

$$y^2 = 2x^3 - 3x^2 + t^l.$$

Since each case is similarly proven, we only demonstrate the case $l = 2$, namely

$${}_3F_2\left(\begin{matrix}\frac{1}{6}, \frac{5}{6}, \frac{1}{2} \\ 1, \frac{3}{2}\end{matrix}; 1\right) = \frac{3\sqrt{3}}{2\pi} \log(2 + \sqrt{3}). \tag{3.1}$$

The fibration f_l is endowed with an action of μ_l the group of l-th roots of 1. Namely, to $\zeta \in \mu_l$ we associate $\sigma_\zeta \in \mathrm{Aut}(X_l)$ an automorphism defined by $\sigma(x, y, t) = (x, y, \zeta t)$. We thus have $\mu_l \hookrightarrow \mathrm{Aut}(X_l)$ and $\mathbb{Q}[\mu_l] \hookrightarrow \mathrm{End}(R^1 f_{l*}\mathbb{Q})$. Let

$$M_l := H^2(X_l, \mathbb{Q})/\langle \text{fibral divisors}, \infty \rangle \cong W_2H^1(\mathbb{P}^1 \setminus \{0, 1, \ldots, \zeta_l^{l-1}, \infty\}, R^1 f_{l*}\mathbb{Q})$$

where $\infty \subset X_l$ denotes the section $y = \infty$. For a projector $e : \mathbb{Q}[\mu_l] \to F$ onto a number field F, $M_l(e) := F \otimes_{e, \mathbb{Q}[\mu_l]} M_l$ denotes the e-part. One easily shows,

$$\dim_F M_l(e) = \begin{cases} 1 & l/d \neq 1, 6 \\ 0 & l/d = 1, 6 \end{cases} \quad d := \sharp\mathrm{Ker}[e : \mu_l \to F^\times]. \tag{3.2}$$

This implies $\dim_F(M_l(e) \cap H^{0,0}) \leq 1$, and then

$$M_l(e) \cap H^{0,0} \neq 0 \Leftrightarrow F^2 M_l(e) = F^2 H^2_{\mathrm{dR}}(X_l)(e) = 0 \Leftrightarrow 2 \leq l/d \leq 5. \tag{3.3}$$

Let Z be the union of totally degenerate semistable fibers over $t^l = 1$, and consider elements

$$\xi_j := \left(\frac{y - \sqrt{3}(x-1)}{y + \sqrt{3}(x-1)}, f_l^{-1}(\zeta_l^j)\right) \in H^3_{\mathscr{M},Z}(X_l, \mathbb{Z}(2)), \quad j \in \{0, 1, \ldots, l-1\}.$$

It is straightforward to see that $c_Z(\xi_j) \in H_1(f_l^{-1}(\zeta_l^j), \mathbb{Z}) \cong \mathbb{Z}$ is a basis where $c_Z : H^3_{\mathscr{M},Z}(X_l, \mathbb{Z}(2)) \to H^3_Z(X_l, \mathbb{Z}(2)) = H_1(Z, \mathbb{Z})$ is the cycle map.

To prove (3.1) we apply Theorem 2.5 (2.16) to the elliptic fibration f_l in case that $l = 2$ and $e : \mathbb{Q}[\mu_2] \to \mathbb{Q}$ is the projector such that $e(\sigma_{-1}) = -1$ ($\Leftrightarrow d = 1$). Put $\xi := \xi_0$. By (3.2) and (3.3),

$$M_2(e) = M_2 = W_2 H^1(\mathbb{P}^1 \setminus \{0, \pm 1, \infty\}, R^1 f_{2*}\mathbb{Q}) \cong \mathbb{Q}, \tag{3.4}$$

and this is a Tate-Hodge structure of type $(1,1)$ (and hence generated by a cycle class).

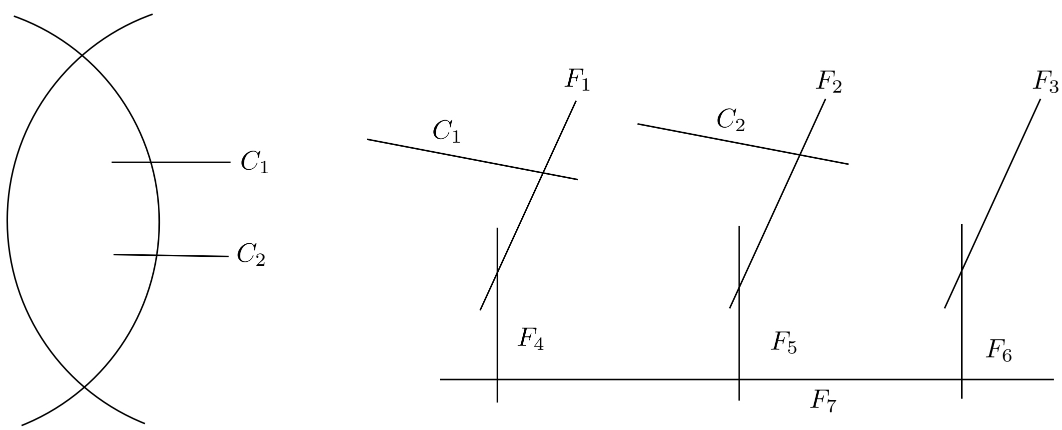
Step 1. The 1st step is to find a (nontrivial) divisor C which is perpendicular to all fibral divisors and generates the e-part $M_2(e)$. Let

$$C_1 : x = 0,\ y = t, \quad C_2 : x = 0,\ y = -t$$

be sections in X_2. Then $\sigma_{-1}(C_1) = C_2$, and hence the cycle class $[C_1] - [C_2] \in H^2(X_2)$ belongs to the e-part. Let $f_2^{-1}(\infty) = F_1 + F_2 + F_3 + 2(F_4 + F_5 + F_6) + 3F_7$ be the singular fiber at $t = \infty$ (see the figure in below). Put

$$C := 3(C_1 - C_2) + 2(F_1 - F_2) + F_4 - F_5.$$

Then this is perpendicular to all fibral divisors (see the following figure), and $M_2(e) = \mathbb{Q}[C]$.



$f_2^{-1}(0)$ $\qquad$ $f_2^{-1}(\infty) = F_1 + F_2 + F_3 + 2(F_4 + F_5 + F_6) + 3F_7$

Step 2 (Computing LHS of (2.16)).

$$\begin{aligned}\text{LHS of (2.16)} &= 3\log\left(\frac{y-\sqrt{3}(x-1)}{y+\sqrt{3}(x-1)}\Big|_{f_2^{-1}(1)\cap C_1}\right)\left(\frac{y-\sqrt{3}(x-1)}{y+\sqrt{3}(x-1)}\Big|_{f_2^{-1}(1)\cap C_2}\right)^{-1}\\ &= 3\log\left(\frac{1+\sqrt{3}}{1-\sqrt{3}}\right)\left(\frac{-1+\sqrt{3}}{-1-\sqrt{3}}\right)^{-1}\\ &= 6\log(2+\sqrt{3}).\end{aligned}$$

Step 3 (Computing $(\omega_C|_U)_{\mathrm{Del}}$). Let $S := \mathbb{P}^1 \setminus \{0, \pm 1, \infty\}$ and put $U := f_2^{-1}(S)$. Let $X_2^\circ = f_2^{-1}(\mathbb{P}^1 \setminus \{\infty\})$ be as in §2.5. Let $\omega_C \in H^2_{\mathrm{dR}}(X_2)_{\mathrm{fib}}$ be the cycle class. Then we claim

$$(\omega_C|_U)_{\mathrm{Del}} = \alpha dt \frac{dx}{y} \in \Gamma(X_2^\circ, \Omega^2_{X_2}), \quad \exists \alpha \in \mathbb{C}^\times. \tag{3.5}$$

This is proven in the following way. Let $\mathscr{H} := H^1_{\mathrm{dR}}(U/S)$ be the vector bundle on S equipped with the Gauss-Manin connection ∇. By (3.4), $W_2 H^1_{\mathrm{dR}}(S, \mathscr{H}) = F^1 W_2 H^1_{\mathrm{dR}}(S, \mathscr{H})$ is one-dimensional and moreover it is spanned by the cycle class $\omega_C|_U$ under the inclusion $H^2_{\mathrm{dR}}(X_2)_{\mathrm{fib}} \hookrightarrow W_2 H^1_{\mathrm{dR}}(S, \mathscr{H})$. Note that $(\omega_C|_U)_{\mathrm{Del}} \neq 0$ as Θ_{Del} is injective (see §2.5). Hence

$$\mathrm{Im}[\Theta_{\mathrm{Del}} : F^1 W_2 H^1_{\mathrm{dR}}(S, \mathscr{H}) \to \Gamma(X_2^\circ, \Omega^2_{X_2^\circ})] = \mathbb{C}(\omega_C|_U)_{\mathrm{Del}}. \tag{3.6}$$

On the other hand, we claim

$$\mathrm{Im}[\Theta_{\mathrm{Del}} : F^1 W_2 H^1_{\mathrm{dR}}(S, \mathscr{H}) \to \Gamma(X_2^\circ, \Omega^2_{X_2^\circ})] = \mathbb{C} dt \frac{dx}{y}. \tag{3.7}$$

The explicit description of ∇ is given as follows (e.g. [3] Theorem 6.4)

$$\begin{pmatrix} \nabla\left(\frac{dx}{y}\right) & \nabla\left(\frac{xdx}{y}\right) \end{pmatrix} = \begin{pmatrix} \frac{dx}{y} & \frac{xdx}{y} \end{pmatrix} A, \quad A := \frac{dt_0}{6(t_0 - t_0^2)} \begin{pmatrix} t_0 & t_0 \\ -1 & -t_0 \end{pmatrix} \tag{3.8}$$

where $t_0 = t^2$. Deligne's extension $\mathscr{H}_e$ of $\mathscr{H}$ is given by a local frame $\{dx/y, xdx/y\}$ on $\mathbb{P}^1 \setminus \{\infty\}$ and $\{dx/y, t^{-1}xdx/y\}$ on a neighborhood of $t = \infty$. Indeed one easily check that

$$\nabla(\mathscr{H}_e) \subset \Omega^1_{\mathbb{P}^1}(\log T) \otimes \mathscr{H}_e, \quad T := \{0, \pm 1, \infty\}$$

and any eigenvalue of $\mathrm{Res}(\nabla)$ at a point of T is $0, 1/6$ or $5/6$. Since $F^1 \mathscr{H}_e \cong \mathscr{O}_{\mathbb{P}^1}$ and $\mathscr{H}_e / F^1 \mathscr{H}_e \cong \mathscr{O}_{\mathbb{P}^1}(-1)$, one has an exact sequence

$$0 \to H^0(F^1 \mathscr{H}_e) \to H^0(\Omega^1_{\mathbb{P}^1}(\log T) \otimes \mathscr{H}_e) \to F^2 H^1_{\mathrm{dR}}(S, \mathscr{H}) \to 0$$

and $F^2 W_2 H^1_{\mathrm{dR}}(S, \mathscr{H})$ is generated by

$$\eta := \frac{dt}{t(t^2 - 1)} \left(\frac{t^2 dx}{y} - \frac{xdx}{y} \right).$$

Noticing

$$\nabla\left(t \frac{dx}{y}\right) = dt \frac{dx}{y} - \frac{dt}{6t(t^2 - 1)} \left(\frac{t^2 dx}{y} - \frac{xdx}{y} \right)$$

by (3.8), we have

$$\Theta_{\mathrm{Del}}(\eta) = 6 dt \frac{dx}{y}$$

by definition of Θ_{Del}. This shows (3.7). Now (3.5) is immediate from (3.6) and (3.7).

The coefficient "α" shall be determined in Step 5. Before this, we show a certain property of α.

Let $\delta_t \in H_1(f_2^{-1}(t), \mathbb{Z})$ be the vanishing cycle at $t = 1$, namely δ_t is a homology 1-cycle which is a generator of $\mathrm{Ker}[H_1(f_2^{-1}(t), \mathbb{Z}) \to H_1(f_2^{-1}(1), \mathbb{Z})] \cong \mathbb{Z}$. Then it defines a Lefschetz thimble Δ over $[0, 1] \subset \mathbb{P}^1(\mathbb{C})$, and hence a homology 2-cycle

$(1-\sigma_{-1})\Delta \in H_2(X_2^\circ, \mathbb{Z})$. Since $C|_{X_2^\circ}$ is a divisor with integral coefficients, one has $\omega_C|_{X_2^\circ} \in H^2(X_2^\circ, \mathbb{Z}(1))$ and hence

$$\int_{(1-\sigma_{-1})\Delta} (\omega_C|_U)_{\mathrm{Del}} = \int_{(1-\sigma_{-1})\Delta} \omega_C|_{X_2^\circ} \in \mathbb{Z}(1). \tag{3.9}$$

Lemma 3.1.

$$\int_{\delta_t} \frac{dx}{y} = \frac{2\pi i}{\sqrt{3}}\, {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1; 1-t^2\right)$$

Proof. Let $D_{t_0} = \nabla_{\frac{d}{dt_0}}$ be the composition $\mathscr{H} \to \Omega_S^1 \otimes \mathscr{H} \to \mathscr{H}$ where the second arrow given by $dt_0 \otimes v \mapsto v$. One can derive from (3.8) that

$$\left((t_0 - t_0^2)D_{t_0}^2 + (1-2t_0)D_{t_0} - \frac{5}{36}\right)\left(\frac{dx}{y}\right) = 0.$$

This implies that $\int_{\delta_t} \frac{dx}{y}$ is a solution of the differential equation

$$(t_0 - t_0^2)\frac{d^2u}{dt_0^2} + (1-2t_0)\frac{du}{dt_0} - \frac{5}{36}u = 0.$$

Therefore $\int_{\delta_t} \frac{dx}{y}$ is a $\mathbb{C}$-linear combination of

$${}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1; 1-t_0\right), \quad {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1; t_0\right).$$

Since δ_t is invariant by the local monodromy at $t_0 = 1$, there is a constant $K \in \mathbb{C}$ such that

$$\int_{\delta_t} \frac{dx}{y} = K \cdot {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1; 1-t_0\right).$$

One can compute the constant K in the following way. Let $2x^3 - 3x^2 + t^2 = 2(x-\alpha_t)(x-\beta_t)(x-\gamma_t)$ where $\alpha_t \to -\frac{1}{2}$ and $\beta_t, \gamma_t \to 1$ as $t \to 1$. Then

$$\begin{aligned} K &= \lim_{t\to 1} \int_{\delta_t} \frac{dx}{y} \\ &= \lim_{t\to 1} 2\int_{\beta_t}^{\gamma_t} \frac{dx}{\sqrt{2(x-\alpha_t)(x-\beta_t)(x-\gamma_t)}} \\ &= \lim_{t\to 1} \sqrt{2}i \int_0^{\gamma_t-\beta_t} \frac{dx}{\sqrt{(x+\beta_t-\alpha_t)x(\gamma_t-\beta_t-x)}} \\ &= \lim_{t\to 1} \sqrt{2}i \int_0^1 \frac{dx}{\sqrt{((\gamma_t-\beta_t)x+\beta_t-\alpha_t)x(1-x)}} \\ &= \sqrt{2}i \int_0^1 \frac{dx}{\sqrt{\frac{3}{2}x(1-x)}} \\ &= \frac{2\pi i}{\sqrt{3}}. \end{aligned}$$

□

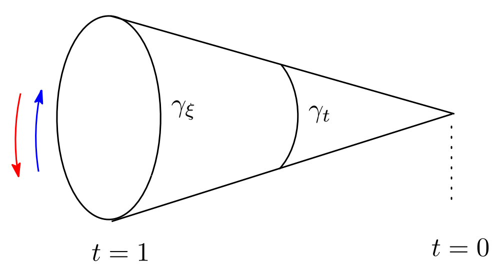
Now one computes

$$\begin{aligned}
\text{RHS of (3.9)} &= 2\alpha\int_0^1 dt\int_{\delta_t}\frac{dx}{y} \\
&= \frac{4\pi i\alpha}{\sqrt{3}}\int_0^1 {}_2F_1\left(\frac{1}{6},\frac{5}{6},1;1-t^2\right)dt && \text{(by Lemma 3.1)} \\
&= \frac{2\pi i\alpha}{\sqrt{3}}\int_0^1 t^{-\frac{1}{2}}\,{}_2F_1\left(\frac{1}{6},\frac{5}{6},1;1-t\right)dt \\
&= \frac{4\pi i\alpha}{\sqrt{3}}\cdot {}_3F_2\left(\begin{matrix}1,\frac{1}{6},\frac{5}{6}\\ \frac{3}{2},1\end{matrix};1\right) && \text{(by [16] 16.5.2)} \\
&= \frac{4\pi i\alpha}{\sqrt{3}}\cdot {}_2F_1\left(\begin{matrix}\frac{1}{6},\frac{5}{6}\\ \frac{3}{2}\end{matrix};1\right) \\
&= \frac{4\pi i\alpha}{\sqrt{3}}\frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2}-\frac{1}{6})\Gamma(\frac{3}{2}-\frac{5}{6})} && \text{(by [16] 15.4.20)} \\
&= 3\pi i\alpha && \text{(by [16] 5.5.6).}
\end{aligned}$$

Hence

$$\alpha\in\frac{2}{3}\mathbb{Z}. \tag{3.10}$$

Step 4 (Computing RHS of (2.16)). Let $\gamma_\xi = c(\xi)\in H_1(f_2^{-1}(1),\mathbb{Z})$ where $c: H^3_{\mathscr{M},f_2^{-1}(1)}(X_2,\mathbb{Z}(2))\to H^3_{f_2^{-1}(1)}(X_2,\mathbb{Z}(2))\cong H_1(f_2^{-1}(1),\mathbb{Z})$ is the cycle map. For $0\le t\le 1$, let $\gamma_t\in H_1(f_2^{-1}(t),\mathbb{Z})$ be the homology cycle such that $\gamma_t|_{t=1}=\gamma_\xi$ and $\gamma_t|_{t=0}=0$ the vanishing cycle at $t=0$. The family of $\{\gamma_t\}_t$ defines a Lefschetz thimble Γ_ξ over the line segment $[0,1]\subset\mathbb{P}^1(\mathbb{C})$. It defines a homology cycle $\Gamma_\xi\in H_2(X_2^\circ,Z;\mathbb{Z})$ with boundary $\partial\Gamma_\xi=\gamma_\xi=c(\xi)$. Note that the homology cycle $\gamma_\xi\in H_1(f_2^{-1}(1),\mathbb{Z})\cong\mathbb{Z}$ is a generator. The figure of the cycle Γ_ξ is as follows, where the orientation of γ_t is given by either the red arrow or the blue one (we omit to determine the orientation since it is not necessary in the discussion below).



Lemma 3.2.

$$\int_{\gamma_t}\frac{dx}{y}=\pm\frac{2\pi}{\sqrt{3}}\,{}_2F_1\left(\frac{1}{6},\frac{5}{6},1;t^2\right)$$

Proof. Similar to the proof of Lemma 3.1 (details are left to the reader). □

We now have

$$
\begin{aligned}
\text{RHS of (2.16)} &= \alpha \int_{\Gamma_\xi} dt \frac{dx}{y} && \text{(by (3.5))}\\
&= \alpha \int_0^1 dt \int_{\gamma_t} \frac{dx}{y}\\
&= \pm\frac{2\pi\alpha}{\sqrt{3}} \int_0^1 {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1; t^2\right) dt && \text{(by Lemma 3.2)}\\
&= \pm\frac{\pi\alpha}{\sqrt{3}} \int_0^1 t^{-\frac{1}{2}} {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1; t\right) dt\\
&= \pm\frac{2\pi\alpha}{\sqrt{3}} {}_3F_2\left(\begin{matrix}\frac{1}{6}, \frac{5}{6}, \frac{1}{2}\\ 1, \frac{3}{2}\end{matrix}; 1\right) && \text{(by [16] 16.5.2).}
\end{aligned}
$$

Step 5(final step) We apply Theorem 2.5 to the results in Step 2 and Step 4, and hence we have

$$\alpha \cdot {}_3F_2\left(\begin{matrix}\frac{1}{6}, \frac{5}{6}, \frac{1}{2}\\ 1, \frac{3}{2}\end{matrix}; 1\right) = \pm\frac{3\sqrt{3}}{\pi}\log(2+\sqrt{3}) \in \mathbb{C}/\mathbb{Z}(1).$$

Taking the absolute value of the real part we have

$$|\mathrm{Re}(\alpha)| \cdot {}_3F_2\left(\begin{matrix}\frac{1}{6}, \frac{5}{6}, \frac{1}{2}\\ 1, \frac{3}{2}\end{matrix}; 1\right) = \frac{3\sqrt{3}}{\pi}\log(2+\sqrt{3}) \in \mathbb{R},$$

$$(\Longrightarrow \quad \mathrm{Re}(\alpha) = \pm 2.0000000 \quad \text{by the aid of computer.})$$

Since $\alpha \in \frac{2}{3}\mathbb{Z}$ by (3.10) this yields $|\mathrm{Re}(\alpha)| = |\alpha| = 2$. This completes the proof of (3.1).

4. EXPLICIT FUNCTIONAL LOG FORMULA

In this section, we prove Theorems 1.4, 1.5 and 1.6, namely the explicit log formulas of

$${}_3F_2\left(\begin{matrix}1, 1, \frac{1}{2}\\ \frac{5}{4}, \frac{7}{4}\end{matrix}; s\right), \quad {}_3F_2\left(\begin{matrix}1, 1, \frac{1}{2}\\ \frac{7}{6}, \frac{11}{6}\end{matrix}; s\right), \quad {}_3F_2\left(\begin{matrix}1, 1, \frac{1}{2}\\ \frac{4}{3}, \frac{5}{3}\end{matrix}; s\right).$$

To do this, we compute the both side of (2.16) in Theorem 2.5 for a suitably chosen triplet "(X, C, ξ)".

4.1. **Explicit log formulas of ${}_3F_2\left(\begin{smallmatrix}1,1,\frac{1}{2}\\ \frac{5}{4},\frac{7}{4}\end{smallmatrix}; t^4\right)$.** We begin with an elliptic fibration $f_0 : X_0 \to \mathbb{P}^1(t_0)$ over the t_0-plane whose generic fiber $f_0^{-1}(t_0)$ is defined by a Weierstrass equation

$$y^2 = x((x-1)^2 - t_0).$$

The morphism f_0 is smooth outside $\{t_0 = 0, 1, \infty\}$ and the "period" of $f_0^{-1}(t_0)$ is given by the hypergeometric function

$${}_2F_1\left(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; t_0\right),$$

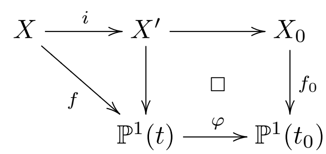
which can be seen from the fact that the connection matrix is given by

$$\begin{pmatrix}\nabla(\frac{dx}{y}) & \nabla(\frac{2xdx}{y})\end{pmatrix} = \begin{pmatrix}\frac{dx}{y} & \frac{2xdx}{y}\end{pmatrix} \frac{dt_0}{12t_0(1-t_0)} \begin{pmatrix}-1+3t_0 & -1-3t_0\\ 1 & 1-3t_0\end{pmatrix} \tag{4.1}$$

and hence the Picard-Fuchs equation is

$$\left((t_0 - t_0^2)\partial_{t_0}^2 + (1-2t_0)\partial_{t_0} - \frac{3}{16}\right)\left(\frac{dx}{y}\right) = 0, \quad \partial_{t_0} := \nabla_{\frac{d}{dt_0}}.$$

Setting. Let $s \in \mathbb{C} \setminus \{0,1\}$ be a constant. Let t be another parameter, and $\varphi : \mathbb{P}^1(t) \to \mathbb{P}^1(t_0)$ the morphism given by $\varphi^*(t_0) = s - t^2$. We consider a cartesian diagram



with i a desingularization. In more down-to-earth manner, the elliptic fibration $f : X \to \mathbb{P}^1$ is defined by an affine equation

$$y^2 = x((x-1)^2 - (s - t^2)) \quad (s \neq 0, 1).$$

The morphism f is smooth outside $T = \{t = \pm\sqrt{s-1}, \pm\sqrt{s}, \infty\} \subset \mathbb{P}^1(t)$. Put $S = \mathbb{P}^1 \setminus T$ and $U = f^{-1}(S)$.

Let $Z = f^{-1}(\sqrt{s})$ be the multiplicative fiber, which is of Kodaira type I_1. Let

$$\xi = \left(\frac{y-(x-1)}{y+(x-1)}, f^{-1}(\sqrt{s})\right) \in H^3_{\mathcal{M},Z}(X, \mathbb{Z}(2)) \tag{4.2}$$

be an element of motivic cohomology. We construct a divisor C in X which is perpendicular to any components of all singular fibers as follows. Let

$$C_1 : x = 1 + \sqrt{s},\ y = \sqrt{\sqrt{s}+1}t$$

be a section in X. Let (∞) be the infinity section. By Zariski's lemma ([10, III,(8.2)]) there exists a fibral divisor F supported in the singular fibers other than Z such that

$$C = C_1 - (\infty) + F$$

is perpendicular to any components of singular fibers. We are going to apply Theorem 2.5 to the triplet

$$(X, C, \xi).$$

Step 1. Let F_i be irreducible components of F and $j : C_1 \sqcup (\infty) \sqcup \bigsqcup F_i \to X$ the natural morphism. The fiber $Z = f^{-1}(\sqrt{s})$ intersects only with C_1 and (∞), and the intersection is transversal. Therefore $\mathrm{tr}_{F_i}(j^*\xi) = 0$, and

$$\mathrm{tr}_{C_1}(j^*\xi) = \left.\frac{y-(x-1)}{y+(x-1)}\right|_{C_1 \cap Z} = \frac{\sqrt{s(\sqrt{s}+1)} - \sqrt{s}}{\sqrt{s(\sqrt{s}+1)} + \sqrt{s}} = \frac{\sqrt{\sqrt{s}+1} - 1}{\sqrt{\sqrt{s}+1}+1},$$

$$\mathrm{tr}_{(\infty)}(j^*\xi) = \left.\frac{y-(x-1)}{y+(x-1)}\right|_{(\infty)\cap Z} = 1.$$

Therefore

$$(\text{LHS of } (2.16)) = \log\left(\frac{\sqrt{\sqrt{s}+1}-1}{\sqrt{\sqrt{s}+1}+1}\right).$$

Step 2 Let ω_C be the cycle class of C in the de Rham cohomology of X. Then we describe $(\omega_C|_U)_{\mathrm{Del}}$ explicitly. Let $\mathscr{H} := H^1_{\mathrm{dR}}(U/S)$ be the de Rham cohomology endowed with the Gauss-Manin connection ∇. Since C is perpendicular to components of all singular fibers, the cohomology class $\omega_C|_U \in H^2_{\mathrm{dR}}(U/S)$ lies in

$$\mathrm{Ker}[F^1W_2H^2_{\mathrm{dR}}(U/\mathbb{C}) \to H^2(X_t/\mathbb{C})] = F^1W_2H^1_{\mathrm{dR}}(S, \mathscr{H}).$$

Let $\mathscr{H}_e$ be Deligne's canonical extension on $\mathbb{P}^1$ so that the connection ∇ extends to

$$\nabla : \mathscr{H}_e \to \Omega^1_{\mathbb{P}^1}(\log T) \otimes \mathscr{H}_e.$$

A local frame of $\mathscr{H}_e$ is $\{\frac{dx}{y}, \frac{xdx}{y}\}$ on $\mathbb{P}^1(t) \setminus \{\infty\}$ and $\{\frac{dx}{y}, \frac{2t^{-1}xdx}{y}\}$ on $\mathbb{P}^1(t) \setminus \{0\}$, so we have isomorphisms

$$F^1\mathscr{H}_e \cong \mathscr{O}_{\mathbb{P}^1}, \quad \mathscr{H}_e/F^1 \cong \mathscr{O}_{\mathbb{P}^1}(-1)$$

of vector bundles. We then have

$$\Gamma(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(\log T) \otimes \mathscr{H}_e) = \left\langle \frac{t^i dt}{(s-1-t^2)(s-t^2)} \frac{dx}{y}, \frac{t^j dt}{(s-1-t^2)(s-t^2)} \frac{2xdx}{y} \,\middle|\, \begin{matrix} 0 \le i \le 3 \\ 0 \le j \le 2 \end{matrix} \right\rangle_{\mathbb{C}},$$

$$\Gamma(\mathbb{P}^1, \mathscr{H}_e) = \Gamma(\mathbb{P}^1, F^1\mathscr{H}_e) = \left\langle \frac{dx}{y} \right\rangle.$$

We have

$$\nabla\left(\frac{dx}{y}\right) = \frac{tdt}{6(s-1-t^2)(s-t^2)}\left((-1+3s-3t^2)\frac{dx}{y} + \frac{2xdx}{y}\right)$$

from (4.1), and hence $H^1(\mathbb{P}^1, \mathscr{H}_e \to \Omega^1_{\mathbb{P}^1}(\log T) \otimes \mathscr{H}_e) \cong \Gamma(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(\log T) \otimes \mathscr{H}_e)/\mathrm{Im}(\nabla)$ is generated by

$$\frac{t^i dt}{(s-1-t^2)(s-t^2)} \otimes \frac{dx}{y} \ (0 \le i \le 3), \frac{t^j dt}{(s-1-t^2)(s-t^2)} \otimes \frac{2xdx}{y} \ (j = 0, 2).$$

Recall from §2.2 the inclusion

$$\Theta_{\mathrm{Del}} : H^1(\mathbb{P}^1, F^1\mathscr{H}_e \to \Omega^1_{\mathbb{P}^1}(\log T) \otimes \mathscr{H}_e) \to \quad \Gamma(\mathbb{P}^1 \setminus \{0, \infty\}, \Omega^1_{\mathbb{P}^1}(\log T) \otimes F^1\mathscr{H}_e).$$

One can compute

$$\Theta_{\mathrm{Del}}\left(\frac{t^{j+1}dt}{(s-1-t^2)(s-t^2)} \frac{2xdx}{y}\right) = \left(-6jt^{j-1} + \frac{(-1+3s-3t^2)t^{j+1}}{(s-1-t^2)(s-t^2)}\right) dt\frac{dx}{y}$$

by definition of Θ_{Del} and (4.1), and hence

$$\mathrm{Im}(\Theta_{\mathrm{Del}}) \subset \left\langle \frac{dt}{t^2}\frac{dx}{y}, \frac{t^i}{(s-1-t^2)(s-t^2)} dt \frac{dx}{y} |\ 0 \le i \le 4 \right\rangle.$$

Letting $\mathrm{Res}_D : H^2_{\mathrm{dR}}(U) \to H^{\mathrm{dR}}_1(D)$ be the residue map along D, one has

$$\Theta_{\mathrm{Del}}(F^1W_2H^2_{\mathrm{dR}}(S, \mathscr{H})) = \mathrm{Im}(\Theta_{\mathrm{Del}}) \cap \mathrm{Ker}(\mathrm{Res}_D) = \left\langle \frac{dt}{t^2}\frac{dx}{y}, dt\frac{dx}{y} \right\rangle,$$

so that one can write

$$(\omega_C|_U)_{\mathrm{Del}} = c_1 \frac{dt}{t^2}\frac{dx}{y} + c_2 dt \frac{dx}{y} \qquad (c_1, c_2 \in \mathbb{C}). \tag{4.3}$$

The coefficients c_1, c_2 are complex numbers depending on s, which we shall discuss in detail in **Step 3**.

Step 3 Let $\gamma : [0, 1] \to \mathbb{P}^1$ be a path such that

$$\gamma(0) = -\sqrt{s-1},\ \gamma(1) = \sqrt{s-1},\ \gamma(a) \in S \setminus \{0\}\ (0 < a < 1),$$

and γ does not pass through the origin and the argument satisfies $0 \le \arg(\gamma(a)) < \pi$. Let $\delta_{1,\gamma(a)} \in H_1(f^{-1}(\gamma(a)), \mathbb{Z})$ be a homology cycle which vanishes at $t = \pm\sqrt{s-1}$. Then, we have a Lefschetz thimble $\Delta_1 \in H_2(X^\circ, \mathbb{Z})$ which is defined to be the fibration over $[0,1]$ with fiber $\delta_{1,\gamma(a)}$. Similarly, letting $\gamma' : [0,1] \to \mathbb{P}^1$ be a path such that

$$\gamma'(0) = -\sqrt{s},\ \gamma'(1) = \sqrt{s},\ \gamma'(a) \in S \setminus \{0\}\ (0 < a < 1),$$

we have a Lefschetz thimble $\Delta_2 \in H_2(X^\circ, \mathbb{Z})$ defined from a vanishing cycle $\delta_{2,\gamma'(a)} \in H_1(f^{-1}(\gamma(a)), \mathbb{Z})$ at $t = \pm\sqrt{s}$. Since $(\omega_C|_U)_{\mathrm{Del}} \in H^2(X^\circ, \mathbb{Z}(1))$ as it is the cycle class of C, one has

$$\int_{\Delta_1} (\omega_C|_U)_{\mathrm{Del}} \in \mathbb{Z}(1), \quad \int_{\Delta_2} (\omega_C|_U)_{\mathrm{Del}} \in \mathbb{Z}(1). \tag{4.4}$$

We compute the both integrals. By the construction of Δ_1, we have

$$\begin{aligned}\int_{\Delta_1} (\omega_C|_U)_{\mathrm{Del}} &= \int_{\Delta_1} c_1 \frac{dt}{t^2}\frac{dx}{y} + c_2 dt \frac{dx}{y} \\ &= c_1 \int_\gamma \frac{dt}{t^2} \int_{\delta_{1,t}} \frac{dx}{y} + c_2 \int_\gamma dt \int_{\delta_{1,t}} \frac{dx}{y}.\end{aligned}$$

One can show

$$\int_{\delta_{1,t}} \frac{dx}{y} = \sqrt{2}\pi\, {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1-s+t^2\right)$$

in the same way as Lemma 3.1. Therefore, we have

$$\int_{\Delta_1} (\omega_C|_U)_{\mathrm{Del}} = \sqrt{2}\pi c_1 \int_\gamma {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1-s+t^2\right)\frac{dt}{t^2} + \sqrt{2}\pi c_2 \int_\gamma {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1-s+t^2\right) dt. \tag{4.5}$$

Lemma 4.1. *Put*

$$I_1(a,s) := \int_\gamma t^a {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1-s+t^2\right) dt$$

for $a \in \mathbb{C} \setminus \{1\}$. *Then*

$$I_1(a,s) = \frac{(-1)^{\frac{a+1}{2}}}{a+1}(1+\mathrm{e}^{i\pi a})(1-s)^{\frac{a+1}{2}} {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; \frac{a+3}{2}; 1-s\right)$$

Proof. Since both side are analytic with respect to a, it is enough to show lemma for $\mathrm{Re}(a) > 0$. Replacing $s - t^2$ with t,

$$\begin{aligned}I_1(a,s) &= \int_1^s \frac{1}{2}(s-t)^{\frac{a-1}{2}} {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1-t\right) dt + \mathrm{e}^{i\pi a}\int_s^1 -\frac{1}{2}(s-t)^{\frac{a-1}{2}} {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1-t\right) dt \\ &= \frac{1+\mathrm{e}^{i\pi a}}{2}\int_1^s (s-t)^{\frac{a-1}{2}} {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1-t\right) dt\end{aligned}$$

Replacing $1 - (1-s)u$ with t,

$$\begin{aligned}I_1(a,s) &= \frac{1+\mathrm{e}^{i\pi a}}{2}\int_0^1 -(1-s)\left(s+(1-s)u-1\right)^{\frac{a-1}{2}} {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; (1-s)u\right) du \\ &= (-1)^{\frac{a+1}{2}}\frac{1+\mathrm{e}^{i\pi a}}{2}(1-s)^{\frac{a+1}{2}}\int_0^1 (1-u)^{\frac{a-1}{2}} {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; (1-s)u\right) du\end{aligned}$$

By [16, 16.5.2],

$$\int_0^1 t^{c-1}(1-t)^{e-c-1}{}_2F_1\left(a,b;c;\lambda t\right)dt = \frac{\Gamma(c)\Gamma(e-c)}{\Gamma(e)}{}_3F_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix};\lambda\right)$$

when $\mathrm{Re}(e) > \mathrm{Re}(c) > 0$. From this formula, one has

$$\begin{aligned} I_1(a,s) &= (-1)^{\frac{a+1}{2}}\frac{1+\mathrm{e}^{i\pi a}}{2}(1-s)^{\frac{a+1}{2}}\frac{\Gamma(1)\Gamma(\frac{a+1}{2})}{\Gamma(\frac{a+3}{2})}{}_3F_2\left(\begin{matrix}\frac14,\frac34,1\\ 1,\frac{a+3}{2}\end{matrix};1-s\right) \\ &= \frac{(-1)^{\frac{a+1}{2}}}{a+1}(1+\mathrm{e}^{i\pi a})(1-s)^{\frac{a+1}{2}}{}_2F_1\left(\frac14,\frac34;\frac{a+3}{2};1-s\right). \end{aligned}$$

□

Applying Lemma 4.1 to (4.5), we have

$$\int_{\Delta_1}(\omega_C|_U)_{\mathrm{Del}} = \frac{2\sqrt{2}\pi i c_1}{\sqrt{1-s}}{}_2F_1\left(\frac14,\frac34;\frac12;1-s\right)+2\sqrt{2}\pi i c_2\sqrt{1-s}\,{}_2F_1\left(\frac14,\frac34;\frac32;1-s\right). \tag{4.6}$$

The second integral in (4.4) can be computed similarly,

$$\int_{\Delta_2}(\omega_C|_U)_{\mathrm{Del}} = -\frac{4\pi i c_1}{\sqrt{s}}{}_2F_1\left(\frac14,\frac34;\frac12;s\right)+4\pi i c_2\sqrt{s}\,{}_2F_1\left(\frac14,\frac34;\frac32;s\right). \tag{4.7}$$

The hypergeometric functions in the above are algebraic functions ([1, 15.1.10, 15.1.14]),

$${}_2F_1\left(\frac14,\frac34;\frac12;s\right) = \frac{\sqrt{1+\sqrt{1-s}}}{\sqrt{2}\sqrt{1-s}}, \qquad {}_2F_1\left(\frac14,\frac34;\frac32;s\right) = \frac{\sqrt{2}}{\sqrt{1+\sqrt{1-s}}}.$$

By (4.6) and (4.7) together with this, (4.4) implies

$$\begin{aligned} &\begin{pmatrix} \frac{\sqrt{\sqrt{s}+1}}{\sqrt{s(1-s)}} & \frac{2\sqrt{1-s}}{\sqrt{\sqrt{s}+1}} \\ -\frac{2\sqrt{\sqrt{1-s}+1}}{\sqrt{2s(1-s)}} & \frac{2\sqrt{2s}}{\sqrt{\sqrt{1-s}+1}} \end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix} = \begin{pmatrix}k_1\\ k_2\end{pmatrix} \quad (k_1,k_2\in\mathbb{Z}) \\ \iff\quad &\begin{pmatrix}c_1\\ c_2\end{pmatrix} = \begin{pmatrix} \frac{s\sqrt{1-s}}{\sqrt{2(\sqrt{1-s}+1)}} & -\frac{(1-s)\sqrt{s}}{2\sqrt{\sqrt{s}+1}} \\ \frac{\sqrt{\sqrt{1-s}+1}}{2\sqrt{2}} & \frac{\sqrt{\sqrt{s}+1}}{4} \end{pmatrix}\begin{pmatrix}k_1\\ k_2\end{pmatrix}. \end{aligned} \tag{4.8}$$

Step 4. In this step, we compute the RHS of (2.16). Let $\gamma'' : [0,1]\to\mathbb{P}^1$ be a path such that

$$\gamma''(0) = \sqrt{s-1},\ \gamma''(1) = \sqrt{s},\quad \gamma''(a)\in S\setminus\{0\}\ (0<a<1).$$

Let ξ be the element (4.2), and $\gamma_\xi = c_Z(\xi)\in H^3_Z(X,\mathbb{Z}(2)) = H_1(Z,\mathbb{Z})\cong\mathbb{Z}$ the homology cycle where c_Z is the cycle map. We note $\gamma_\xi\neq 0$. There is a homology cycle $\delta_{3,\gamma''(a)}\in H_1(f^{-1}(\gamma''(a)),\mathbb{Z})$ which vanishes at $t=\sqrt{s-1}$ and such that $\delta_{3,\gamma''(1)} = \gamma_\xi$. Let Γ be a Lefschetz thimble over γ'' with fiber $\delta_{3,\gamma(a)}$. We have from (4.3)

$$(\text{RHS of (2.16)}) = \int_\Gamma(\omega_C|_U)_{\mathrm{Del}} = c_1\int_\Gamma\frac{dt}{t^2}\frac{dx}{y}+c_2\int_\Gamma dt\frac{dx}{y}.$$

Let us compute

$$J(a,s)=\int_{\Gamma}t^a dt\frac{dx}{y}=\int_{\sqrt{s-1}}^{\sqrt{s}}t^a dt\int_{\delta_{3,\gamma(a)}}\frac{dx}{y}.$$

Firstly, one can show

$$\int_{\delta_{3,\gamma(a)}}\frac{dx}{y}=\sqrt{2}\pi\,{}_2F_1\left(\frac{1}{4},\frac{3}{4};1;1-s+t^2\right)$$

in the same way as Lemma 3.1, and hence

$$J(a,s)=\sqrt{2}\pi\int_{\sqrt{s-1}}^{\sqrt{s}}t^a\,{}_2F_1\left(\frac{1}{4},\frac{3}{4};1;1-s+t^2\right)dt.$$

Replacing $s-t^2$ with t,

$$J(a,s)=\frac{\sqrt{2}\pi}{2}\int_0^1(s-t)^{\frac{a-1}{2}}\,{}_2F_1\left(\frac{1}{4},\frac{3}{4};1;1-t\right)dt.$$

Using binomial theorem,

$$(s-t)^{\frac{a-1}{2}}=s^{\frac{a-1}{2}}(1-s^{-1}t)^{\frac{a-1}{2}}=s^{\frac{a-1}{2}}\sum_{n=0}^{\infty}\binom{\frac{a-1}{2}}{n}(-s^{-1}t)^n=\sum_{n=0}^{\infty}\frac{\left(-\frac{a-1}{2}\right)_n}{n!}s^{\frac{a-1}{2}-n}t^n,$$

we have

$$\begin{aligned}
J(a,s)&=\frac{\sqrt{2}\pi}{2}\sum_{n=0}^{\infty}\frac{\left(-\frac{a-1}{2}\right)_n}{n!}s^{\frac{a-1}{2}-n}\int_0^1 t^n\,{}_2F_1\left(\frac{1}{4},\frac{3}{4};1;1-s+t^2\right)dt\\
&=\frac{\sqrt{2}\pi}{2}\sum_{n=0}^{\infty}\frac{\left(-\frac{a-1}{2}\right)_n}{(n+1)!}s^{\frac{a-1}{2}-n}\,{}_2F_1\left(\frac{1}{4},\frac{3}{4};n+2;1\right)\\
&=\frac{\sqrt{2}\pi}{2}\sum_{n=0}^{\infty}\frac{\left(-\frac{a-1}{2}\right)_n}{(n+1)!}s^{\frac{a-1}{2}-n}\frac{\Gamma(n+1)\Gamma(n+2)}{\Gamma(n+\frac{5}{4})\Gamma(n+\frac{7}{4})}\\
&=\frac{\sqrt{2}\pi}{2}\sum_{n=0}^{\infty}\frac{\left(-\frac{a-1}{2}\right)_n}{n!}s^{\frac{a-1}{2}-n}\frac{(1)_n(1)_n}{\Gamma(\frac{5}{4})\Gamma(\frac{7}{4})\left(\frac{5}{4}\right)_n\left(\frac{7}{4}\right)_n}\\
&=\frac{8}{3}s^{\frac{a-1}{2}}\sum_{n=0}^{\infty}\frac{(1)_n(1)_n\left(-\frac{a-1}{2}\right)_n}{\left(\frac{5}{4}\right)_n\left(\frac{7}{4}\right)_n}\frac{(s^{-1})^n}{n!}\\
&=\frac{8}{3}s^{\frac{a-1}{2}}\,{}_3F_2\left(\begin{matrix}1,1,-\frac{a-1}{2}\\ \frac{5}{4},\frac{7}{4}\end{matrix};\frac{1}{s}\right).
\end{aligned}$$

Summing up the above,

$$\text{(RHS of (2.16))}=\frac{8}{3}c_1s^{-\frac{3}{2}}\,{}_3F_2\left(\begin{matrix}1,1,\frac{3}{2}\\ \frac{5}{4},\frac{7}{4}\end{matrix};\frac{1}{s}\right)+\frac{8}{3}c_2s^{-\frac{1}{2}}\,{}_3F_2\left(\begin{matrix}1,1,\frac{1}{2}\\ \frac{5}{4},\frac{7}{4}\end{matrix};\frac{1}{s}\right). \tag{4.9}$$

Step 5. This is the final step. We apply (2.16) in Theorem 2.5 to the results in **Step 1** and **Step 4**,

$$\log\left(\frac{\sqrt{\sqrt{s+1}-1}}{\sqrt{\sqrt{s+1}+1}}\right)+2\pi im=\frac{8}{3}c_1s^{-\frac{3}{2}}\,{}_3F_2\left(\begin{matrix}1,1,\frac{3}{2}\\ \frac{5}{4},\frac{7}{4}\end{matrix};\frac{1}{s}\right)+\frac{8}{3}c_2s^{-\frac{1}{2}}\,{}_3F_2\left(\begin{matrix}1,1,\frac{1}{2}\\ \frac{5}{4},\frac{7}{4}\end{matrix};\frac{1}{s}\right)$$

where $m \in \mathbb{Z}$. We think s of being a parameter, and change the variable $s = 1/t^4$. By (4.8), we have

$$\begin{aligned}&\log\left(\frac{\sqrt{1+t^2}-t}{\sqrt{1+t^2}+t}\right)+2\pi i m\\ =&\frac{4}{3}\left(-\frac{\mathrm{e}^{3\pi i/4} i t\sqrt{2(1-t^4)}}{\sqrt{\sqrt{1-t^4}-it^2}}{}_3F_2\left(\begin{matrix}1,1,\frac{3}{2}\\ \frac{5}{4},\frac{7}{4}\end{matrix};t^4\right)+\frac{\mathrm{e}^{\pi i/4} t\sqrt{\sqrt{1-t^4}-it^2}}{\sqrt{2}}{}_3F_2\left(\begin{matrix}1,1,\frac{1}{2}\\ \frac{5}{4},\frac{7}{4}\end{matrix};t^4\right)\right)k_1\\ &+\frac{4}{3}\left(\frac{t(1-t^4)}{\sqrt{1+t^2}}{}_3F_2\left(\begin{matrix}1,1,\frac{3}{2}\\ \frac{5}{4},\frac{7}{4}\end{matrix};t^4\right)+\frac{t\sqrt{1+t^2}}{2}{}_3F_2\left(\begin{matrix}1,1,\frac{1}{2}\\ \frac{5}{4},\frac{7}{4}\end{matrix};t^4\right)\right)k_2.\end{aligned}$$

The first few terms of the Taylor expansions at $t = 0$ of each term is as follows,

$$\log\left(\frac{\sqrt{1+t^2}-t}{\sqrt{1+t^2}+t}\right)=2\log(\sqrt{1+t^2}-t)=-2t+\frac{t^3}{3}-\frac{3t^5}{20}+O(t^6)$$

$${}_3F_2\left(\begin{matrix}1,1,\frac{3}{2}\\ \frac{5}{4},\frac{7}{4}\end{matrix};t\right)=1+\frac{24}{35}t+\frac{128}{231}t^2+\frac{1024}{2145}t^3+O(t^4)$$

$${}_3F_2\left(\begin{matrix}1,1,\frac{1}{2}\\ \frac{5}{4},\frac{7}{4}\end{matrix};t\right)=1+\frac{8}{35}t+\frac{128}{1155}t^2+\frac{1024}{15015}t^3+O(t^4)$$

$$\frac{t\sqrt{2(1-t^4)}}{\sqrt{\sqrt{1-t^4}-it^2}}=\sqrt{2}t+\frac{i}{\sqrt{2}}t^3+\frac{5}{4\sqrt{2}}t^5+O(t^6)$$

$$\frac{t\sqrt{\sqrt{1-t^4}-it^2}}{\sqrt{2}}=\frac{1}{\sqrt{2}}t-\frac{i}{2\sqrt{2}}t^3-\frac{1}{8\sqrt{2}}t^5+O(t^6)$$

$$\frac{t(t^4-1)}{\sqrt{t^2+1}}=-t+\frac{1}{2}t^3+\frac{5}{8}t^5+O(t^6)$$

$$\frac{t\sqrt{t^2+1}}{2}=\frac{1}{2}t+\frac{1}{4}t^3-\frac{1}{16}t^5+O(t^6).$$

Comparing the coefficients, we get

$$k_1 = 0, \quad k_2 = -1, \quad m = 0.$$

Hence we have

$$\log(\sqrt{1+t^2}-t)=-\frac{2}{3}\left(\frac{t(1-t^4)}{\sqrt{1+t^2}}{}_3F_2\left(\begin{matrix}1,1,\frac{3}{2}\\ \frac{5}{4},\frac{7}{4}\end{matrix};t^4\right)+\frac{t\sqrt{1+t^2}}{2}{}_3F_2\left(\begin{matrix}1,1,\frac{1}{2}\\ \frac{5}{4},\frac{7}{4}\end{matrix};t^4\right)\right).$$

Substituting t to it, we have

$$\log(\sqrt{1-t^2}-it)=-\frac{2}{3}\left(\frac{it(1-t^4)}{\sqrt{1-t^2}}{}_3F_2\left(\begin{matrix}1,1,\frac{3}{2}\\ \frac{5}{4},\frac{7}{4}\end{matrix};t^4\right)+\frac{it\sqrt{1-t^2}}{2}{}_3F_2\left(\begin{matrix}1,1,\frac{1}{2}\\ \frac{5}{4},\frac{7}{4}\end{matrix};t^4\right)\right).$$

The two equalities implies,

$${}_3F_2\left(\begin{matrix}1,1,\frac{1}{2}\\ \frac{5}{4},\frac{7}{4}\end{matrix};t^4\right)=-\frac{3i\sqrt{1-t^2}}{2t^3}\log(\sqrt{1-t^2}-it)-\frac{3\sqrt{1+t^2}}{2t^3}\log(\sqrt{1+t^2}-t).$$

This completes the proof of Theorem 1.5.

4.2. **The case of** ${}_3F_2\left({1,1,\frac{1}{2} \atop \frac{7}{6},\frac{11}{6}};t^6\right)$**.** We prove Theorem 1.6. Since the outline of the proof is similar to before, we only give a sketch.

We begin with the elliptic fibration $f_0 : X_0 \to \mathbb{P}^1(t_0)$ whose generic fiber is given by a Weierstrass equation $y^2 = 2x^3 - 3x^2 + t_0$. The Picard-Fuchs equation is

$$\left((t_0 - t_0^2)\partial_{t_0}^2 + (1-2t_0)\partial_{t_0} - \frac{5}{36}\right)\left(\frac{dx}{y}\right) = 0, \quad \partial_{t_0} := \nabla_{\frac{d}{dt_0}},$$

and hence the period is the hypergeometric function

$${}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; t_0\right).$$

In the same way as in §4.1, we take a base change $\phi : \mathbb{P}^1(t) \to \mathbb{P}^1(t_0)$ given by $\phi^*(t_0) = s - t^2$, so that one has the elliptic fibration $f : X \to \mathbb{P}^1$ defined by an affine equation

$$f^{-1}(t) = \{y^2 = 2x^3 - 3x^2 + s - t^2\} \quad (s \neq 0, 1).$$

Put $T = \{t = \pm\sqrt{s-1}, \pm\sqrt{s}, \infty\} \subset \mathbb{P}^1(t)$, $S = \mathbb{P}^1 \setminus T$ and $U = f^{-1}(S)$. We construct a divisor C in X as follows. Let $e_j(s)$ $(j = 1,2,3)$ be the solutions of $2x^3 - 3x^2 + s = 0$, which are explicitly given as follows,

$$e_1(s) = \frac{1}{2}\left((\sqrt{1-s} + i\sqrt{s})^{\frac{2}{3}} + (\sqrt{1-s} - i\sqrt{s})^{\frac{2}{3}} + 1\right),$$

$$e_2(s) = \frac{1}{2}\left(-\mathrm{e}^{-\frac{\pi i}{3}}(\sqrt{1-s} + i\sqrt{s})^{\frac{2}{3}} - \mathrm{e}^{\frac{\pi i}{3}}(\sqrt{1-s} - i\sqrt{s})^{\frac{2}{3}} + 1\right),$$

$$e_3(s) = \frac{1}{2}\left(-\mathrm{e}^{\frac{\pi i}{3}}(\sqrt{1-s} + i\sqrt{s})^{\frac{2}{3}} - \mathrm{e}^{-\frac{\pi i}{3}}(\sqrt{1-s} - i\sqrt{s})^{\frac{2}{3}} + 1\right).$$

Let

$$C_j : x = e_j(s), y = it, \quad (j = 1,2,3)$$

be sections of f. We note that $C_1 + C_2 + C_3 - 3(\infty) = \mathrm{div}(y - it)$ is linearly equivalent to zero. Put $Z = f^{-1}(\sqrt{s})$ the multiplicative fiber of Kodaira type I_1. By Zariski's lemma ([10, III,(8.2)]) there exists a fibral divisor F supported in the singular fibers other than Z such that

$$C = C_1 - (\infty) + F$$

is perpendicular to any components of singular fibers. Let

$$\xi = \left(\frac{y - \sqrt{3}x}{y + \sqrt{3}x}, f^{-1}(\sqrt{s})\right) \in H^3_{\mathcal{M},Z}(X, \mathbb{Z}(2)).$$

We then apply Theorem 2.5 to

$$(X, C, \xi).$$

In a similar way to **Step 1** in §4.1, we have

$$(\text{LHS of (2.16)}) = \log\frac{\sqrt{s} - \sqrt{3}e_1(s)}{\sqrt{s} + \sqrt{3}e_1(s)}.$$

In a similar way to **Step 2** and **Step 3** in §4.1, one can show that

$$(\omega_C|_U)_{\mathrm{Del}} = c_1\frac{dt}{t^2}\frac{dx}{y} + c_2 dt\frac{dx}{y}$$

and the constants c_1, c_2 (depending on s) satisfy

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -\frac{2i}{3} \begin{pmatrix} s\sqrt{1-s}\,{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{3}{2};s\right) & (1-s)\sqrt{s}\,{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{3}{2};1-s\right) \\ \sqrt{1-s}\,{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};s\right) & -\sqrt{s}\,{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};1-s\right) \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

for some $k_1, k_2 \in \mathbb{Z}$. The hypergeometric functions in the above are algebraic functions ([1, 15.1.11, 15.1.12, 15.2.25]),

$${}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};s\right) = \frac{1}{2\sqrt{1-s}}\left(\frac{1}{(\sqrt{1-s}+i\sqrt{s})^{\frac{2}{3}}} + (\sqrt{1-s}+i\sqrt{s})^{\frac{2}{3}}\right),$$

$${}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{3}{2};s\right) = \frac{3i}{4\sqrt{s}}\left(\frac{1}{(\sqrt{1-s}+i\sqrt{s})^{\frac{2}{3}}} - (\sqrt{1-s}+i\sqrt{s})^{\frac{2}{3}}\right).$$

The method of computation of (4.9) works as well, so that we have

$$(\text{RHS of (2.16)}) = \frac{18i}{5\sqrt{3}}\left(c_1 s^{-\frac{3}{2}}\,{}_3F_2\left({1,1,\frac{3}{2} \atop \frac{7}{6},\frac{11}{6}};\frac{1}{s}\right) + c_2 s^{-\frac{1}{2}}\,{}_3F_2\left({1,1,\frac{1}{2} \atop \frac{7}{6},\frac{11}{6}};\frac{1}{s}\right)\right).$$

Summing up the above, it follows from Theorem 2.5 that we have

$$\begin{aligned}
&\log\frac{\sqrt{s}-\sqrt{3}e_1(s)}{\sqrt{s}+\sqrt{3}e_1(s)} + 2\pi i m \\
&= \frac{12}{5\sqrt{3}}\frac{\sqrt{1-s}}{\sqrt{s}}\left({}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{3}{2};s\right)F_1\left(\frac{1}{s}\right) + {}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};s\right)F_2\left(\frac{1}{s}\right)\right)k_1 \\
&+ \frac{12}{5\sqrt{3}}\left(\frac{1-s}{s}\,{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{3}{2};1-s\right)F_1\left(\frac{1}{s}\right) - {}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};1-s\right)F_2\left(\frac{1}{s}\right)\right)k_2
\end{aligned}$$

where we put

$$F_1(s) = {}_3F_2\left({1,1,\frac{3}{2} \atop \frac{7}{6},\frac{11}{6}};s\right), \quad F_2(s) = {}_3F_2\left({1,1,\frac{1}{2} \atop \frac{7}{6},\frac{11}{6}};s\right).$$

Thinking s of being a parameter, we change the variable $s = 1/t^6$,

$$\begin{aligned}
&\log\frac{1-\sqrt{3}t^3e(t)}{1+\sqrt{3}t^3e(t)} + 2\pi i m \\
&= -\frac{12i}{5\sqrt{3}}\sqrt{1-t^6}\left({}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{3}{2};\frac{1}{t^6}\right)F_1(t^6) + {}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};\frac{1}{t^6}\right)F_2(t^6)\right)k_1 \\
&+ \frac{12}{5\sqrt{3}}\left((1-t^6)\,{}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{3}{2};1-\frac{1}{t^6}\right)F_1(t^6) + {}_2F_1\left(\frac{1}{6},\frac{5}{6};\frac{1}{2};1-\frac{1}{t^6}\right)F_2(t^6)\right)k_2.
\end{aligned}$$

where

$$e(t) = \frac{1}{2}\left(\frac{t^2}{\left(1+\sqrt{1-t^6}\right)^{\frac{2}{3}}} + \frac{\left(1+\sqrt{1-t^6}\right)^{\frac{2}{3}}}{t^2} - 1\right).$$

By comparing the coefficients in the Taylor expansions at $t = 0$, one concludes

$$k_1 = -1, \quad k_2 = 0, \quad m = 0,$$

and therefore, we have

$$\log\frac{1-\sqrt{3}t^3e(t)}{1+\sqrt{3}t^3e(t)} = \frac{9}{5\sqrt{3}}t^3\sqrt{1-t^6}\left(\frac{t^2}{(1+\sqrt{1-t^6})^{\frac{2}{3}}} - \frac{\left(1+\sqrt{1-t^6}\right)^{\frac{2}{3}}}{t^2}\right)F_1(t^6)$$
$$-\frac{6}{5\sqrt{3}}t^6\left(\frac{t^2}{(1+\sqrt{1-t^6})^{\frac{2}{3}}} + \frac{\left(1+\sqrt{1-t^6}\right)^{\frac{2}{3}}}{t^2}\right)F_2(t^6).$$

Substituting t to $\mathrm{e}^{\pi i/3}t$, we have

$$\log\frac{1-\sqrt{3}t^3e(\mathrm{e}^{\frac{\pi i}{3}}t)}{1+\sqrt{3}t^3e(\mathrm{e}^{\frac{\pi i}{3}}t)} = \frac{9}{5\sqrt{3}}t^3\sqrt{1-t^6}\left(\frac{\mathrm{e}^{\frac{2\pi i}{3}}t^2}{(1+\sqrt{1-t^6})^{\frac{2}{3}}} - \frac{\mathrm{e}^{-\frac{2\pi i}{3}}\left(1+\sqrt{1-t^6}\right)^{\frac{2}{3}}}{t^2}\right)F_1(t^6)$$
$$+\frac{6}{5\sqrt{3}}t^6\left(\frac{\mathrm{e}^{\frac{2\pi i}{3}}t^2}{\left(1+\sqrt{1-t^6}\right)^{\frac{2}{3}}} + \frac{\mathrm{e}^{-\frac{2\pi i}{3}}\left(1+\sqrt{1-t^6}\right)^{\frac{2}{3}}}{t^2}\right)F_2(t^6).$$

Theorem 1.6 is derived from the two equalities.

4.3. **The case of** ${}_3F_2\left({1,1,\frac{1}{2} \atop \frac{4}{3},\frac{5}{3}};t^6\right)$**.** The proof of Theorem 1.4 is also similar to before. We use the elliptic fibration $f_0 : X_0 \to \mathbb{P}^1(t_0)$ defined by a Weierstrass equation

$$f_0^{-1}(t_0) = \{y^2 = x^3 - (3x-4t_0)^2\}$$

whose period is given by the hypergeometric function

$$F\left(\frac{1}{3},\frac{2}{3};\frac{1}{2};t_0\right).$$

Let $f : X \to \mathbb{P}^1$ be the elliptic surface defined by an affine equation

$$y^2 = x^3 - (3x-4s+4t^2)^2 \quad (s \neq 0,1).$$

Let e_j $(j=1,2,3)$ be solutions of $x^3 - 48x + 128\sqrt{s} = 0$. Specifically,

$$e_1 = 4\left(\frac{\mathrm{e}^{-\frac{\pi i}{6}}}{(\sqrt{1-s}+i\sqrt{s})^{\frac{1}{3}}} + \mathrm{e}^{\frac{\pi i}{6}}(\sqrt{1-s}+i\sqrt{s})^{\frac{1}{3}}\right)$$
$$e_2 = 4\left(-\frac{\mathrm{e}^{\frac{\pi i}{6}}}{(\sqrt{1-s}+i\sqrt{s})^{\frac{1}{3}}} - \mathrm{e}^{-\frac{\pi i}{6}}(\sqrt{1-s}+i\sqrt{s})^{\frac{1}{3}}\right)$$
$$e_3 = 4\left(\frac{i}{(\sqrt{1-s}+i\sqrt{s})^{\frac{1}{3}}} - i(\sqrt{1-s}+i\sqrt{s})^{\frac{1}{3}}\right)$$

Then we obtain sections of f,

$$C_i : x = e_i(t+\sqrt{s}),\ y = i(t+\sqrt{s})(4t-3e_i+12\sqrt{s}) \quad (i=1,2,3).$$

Let $Z = f^{-1}(\sqrt{s})$ be the multiplicative fiber, which is of Kodaira type I_3. Let Z_i $(i=1,2,3)$ be the irreducible components, and let Z_1 be the strict transform of the curve $y^2 = x^3 - 9x^2$. Since both of C_1 and (∞) intersect with a single component Z_1, there exists a fibral divisor F supported in the singular fibers other than Z such that

$$C = C_1 - (\infty) + F$$

is perpendicular to any components of singular fibers (Zariski's lemma). Put

$$\xi = \left(\frac{y-3ix}{y+3ix}, Z_1\right) + (f_2, Z_2) + (f_3, Z_3) \in H^3_{\mathcal{M},Z}(X, \mathbb{Z}(2))$$

where f_2, f_3 are taken such that ξ lies in the kernel of (2.2) (there exist such f_i). We apply Theorem 2.5 to (X, C, ξ) . The LHS of (2.16) is expressed as follows

$$(\text{LHS of } (2.16)) = \log\left(1 - \frac{3e_1}{8\sqrt{s}}\right).$$

Moreover we have

$$(\omega_C|_U)_{\mathrm{Del}} = c_1 \frac{dt}{t^2}\frac{dx}{y} + c_2 dt \frac{dx}{y} \qquad (c_1, c_2 \in \mathbb{C})$$

with

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -i \begin{pmatrix} s\sqrt{1-s}\,{}_2F_1\left(\frac13, \frac23; \frac32; s\right) & \frac{(1-s)\sqrt{s}}{\sqrt{3}}\,{}_2F_1\left(\frac13, \frac23; \frac32; 1-s\right) \\ \sqrt{1-s}\,{}_2F_1\left(\frac13, \frac23; \frac12; s\right) & -\frac{\sqrt{s}}{\sqrt{3}}\,{}_2F_1\left(\frac13, \frac23; \frac12; 1-s\right) \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

for some $k_1, k_2 \in \mathbb{Z}$. The hypergeometric functions in the above are algebraic functions ([1, 15.1.11, 15.1.12, 15.2.25]),

$${}_2F_1\left(\frac13, \frac23; \frac12; s\right) = \frac{1}{2\sqrt{1-s}}\left(\frac{1}{(\sqrt{1-s}+i\sqrt{s})^{1/3}} + (\sqrt{1-s}+i\sqrt{s})^{1/3}\right),$$

$${}_2F_1\left(\frac13, \frac23; \frac32; s\right) = \frac{3i}{2\sqrt{s}}\left(\frac{1}{(\sqrt{1-s}+i\sqrt{s})^{1/3}} - (\sqrt{1-s}+i\sqrt{s})^{1/3}\right).$$

The RHS of (2.16) is expressed as follows

$$(\text{RHS of } (2.16)) = \frac{9i}{4\sqrt{3}}\left(c_1 s^{-\frac32}\,{}_3F_2\left(\begin{matrix} 1, 1, \frac32 \\ \frac43, \frac53 \end{matrix}; \frac1s\right) + c_2 s^{-\frac12}\,{}_3F_2\left(\begin{matrix} 1, 1, \frac12 \\ \frac43, \frac53 \end{matrix}; \frac1s\right)\right).$$

In the same way as before, one concludes

$$k_1 = 0, \quad k_2 = -1, \quad m = 0,$$

and hence we have

$$\begin{aligned} \log\left(1 - \frac32 t^3 e(t)\right) &= \frac98 t^3\sqrt{1-t^6}\left(-\frac{\mathrm{e}^{-\frac{2\pi i}{3}} t}{(1+\sqrt{1-t^6})^{\frac13}} + \frac{\mathrm{e}^{\frac{2\pi i}{3}}(1+\sqrt{1-t^6})^{\frac13}}{t}\right) F_1(t^6) \\ &\quad + \frac38 t^3\left(\frac{\mathrm{e}^{-\frac{2\pi i}{3}} t}{(1+\sqrt{1-t^6})^{\frac13}} + \frac{\mathrm{e}^{\frac{2\pi i}{3}}(1+\sqrt{1-t^6})^{\frac13}}{t}\right) F_2(t^6) \end{aligned}$$

where we put

$$e(t) = \frac{\mathrm{e}^{\frac{\pi i}{3}} t}{(1+\sqrt{1-t^6})^{\frac13}} + \frac{\mathrm{e}^{-\frac{\pi i}{3}}(1+\sqrt{1-t^6})^{\frac13}}{t},$$

$$F_1(t) = {}_3F_2\left(\begin{matrix} 1, 1, \frac32 \\ \frac43, \frac53 \end{matrix}; t\right), \quad F_2(t) = {}_3F_2\left(\begin{matrix} 1, 1, \frac12 \\ \frac43, \frac53 \end{matrix}; t\right).$$

Substituting t to $\mathrm{e}^{-\pi i/3} t$, we have two equations, and then derive the formula in Theorem 1.4.

4.4. **Complement.** We have proved the explicit log formulas of ${}_3F_2\left({1,1,q \atop a,b};t\right)$ when $(a,b,q)=(\frac{4}{3},\frac{5}{3},\frac{1}{2}),(\frac{5}{4},\frac{7}{4},\frac{1}{2}),(\frac{7}{6},\frac{11}{6},\frac{1}{2})$. Then one can derive the formulas of

$$ {}_3F_2\left({n_1,n_2,q+n_3, \atop a+n_4,b+n_5};t\right),\quad (n_i\in\mathbb{Z}) $$

automatically. Indeed, they can be obtained by using the following differential operators repeatedly.

$$ (b_1-1){}_3F_2\left({a_1,a_2,a_3 \atop b_1-1,b_2};t\right)=\left(b_1-1+t\frac{d}{dt}\right){}_3F_2\left({a_1,a_2,a_3 \atop b_1,b_2};t\right), $$

$$ a_1\cdot{}_3F_2\left({a_1+1,a_2,a_3 \atop b_1,b_2};t\right)=\left(a_1+t\frac{d}{dt}\right){}_3F_2\left({a_1,a_2,a_3 \atop b_1,b_2};t\right), $$

$$ (a_2-b_1)(a_1-b_1)(a_3-b_1){}_3F_2\left({a_1,a_2,a_3 \atop b_1+1,b_2};t\right)=\theta_1\left({}_3F_2\left({a_1,a_2,a_3 \atop b_1,b_2};t\right)\right), $$

$$ (a_1-b_1)(a_1-b_2){}_3F_2\left({a_1-1,a_2,a_3 \atop b_1,b_2};t\right)=\theta_2\left({}_3F_2\left({a_1,a_2,a_3 \atop b_1,b_2};t\right)\right), $$

where

$$ \begin{aligned} \theta_1 &:= -a_1a_2a_3+(a_2-b_1)(a_1-b_1)(a_3-b_1) \\ &\quad + b_1(b_2+(b_1-a_1-a_2-a_3-1)t)\frac{d}{dt}+b_1(t-t^2)\frac{d^2}{dt^2} \\ \theta_2 &:= (a_1-b_1)(a_1-b_2)-a_2a_3t \\ &\quad +((b_1+b_2-a_1)-(a_2+a_3+1)t)t\frac{d}{dt}+(1-t)t^2\frac{d^2}{dt^2}. \end{aligned} $$

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