博士学位論文

Congruence relations for p-adic hypergeometric functions $\widehat{\mathscr{F}}^{(\sigma)}_{a,\ldots,a}(t)$ and its transformation formula

（p 進超幾何関数の合同関係と変数変換公式）

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Congruence relations for p-adic hypergeometric functions $\widehat{\mathscr{F}}^{(\sigma)}_{a,\ldots,a}(t)$ and its transformation formula

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Abstract

We introduce new kinds of p-adic hypergeometric functions. We show these functions satisfy congruence relations similar to Dwork's p-adic hypergeometric functions, so they are convergent functions. We also show that there is a transformation formula between our new p-adic hypergeometric functions and p-adic hypergeometric functions of logarithmic type defined in [1] in a particular case.

Keywords— p-adic hypergeometric functions, p-adic hypergeometric functions of logarithmic type, congruence relations

MSC Classification Codes: 33E50

1 Introduction

The classical hypergeometric function of one-variable is defined to be the power series

$$ {}_sF_{s-1}\left({a_1,\ldots,a_s \atop b_1,\ldots,b_{s-1}};t\right)=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_s)_k}{(b_1)_k\cdots(b_{s-1})_k}\frac{t^k}{k!} $$

where $(\alpha)_k$ denotes the Pochhammer symbol (cf. [11]). Let $(a_1,...,a_s)\in\mathbb{Z}_p^s$ be a s-tuple of p-adic integers, and consider the series

$$ F_{a_1,\ldots,a_s}(t)={}_sF_{s-1}\left({a_1,\ldots,a_s \atop 1,\ldots,1};t\right)=\sum_{k=0}^{\infty}\frac{(a_1)_k}{k!}\cdots\frac{(a_s)_k}{k!}t^k. $$

This is a formal power series with $\mathbb{Z}_p$-coefficients. In the paper [4], B. Dwork introduced his p-adic hypergeometric functions. Let a' be the Dwork prime of a, which is defined to be $(a+l)/p$ where $l\in\{0,1,...,p-1\}$ is the unique integer such that $a+l\equiv 0 \mod p$. Then Dwork proved his p-adic hypergeometric function $\mathscr{F}^{\mathrm{Dw}}_{a_1,\ldots,a_s}(t)$ (Definition 4.16) satisfies the congruence relations

$$ \mathscr{F}^{\mathrm{Dw}}_{a_1,\ldots,a_s}(t)\equiv\frac{F_{a_1,\ldots,a_s}(t)_{<p^n}}{[F_{a_1',\ldots,a_s'}(t^p)]_{<p^n}} \mod p^n\mathbb{Z}_p[[t]] $$

where for a power series $f(t)=\sum_{n=0}^{\infty} A_n t^n$, we denote $f(t)_{<m} := \sum_{n<m} A_n t^n$ the truncated polynomial. As a consequence, his function is p-adically analytic in the sense of Krasner (i.e. an element of the Tate algebra, [5, 3.1]), and one can define the value at $t=\alpha$ with $|\alpha|_p = 1$. Dwork applied his function to the unit root formula of elliptic curve of Legendre type (e.g. [8, §7]).

Recently, M. Asakura introduced a new function which he calls p-adic hypergeometric functions of logarithmic type. Let $W = W(\overline{\mathbb{F}}_p)$ denote the Witt ring, and K=Frac$W$ its fraction field (e.g. [10, Ch II, §6]). Let $\sigma : W[[t]] \to W[[t]]$ be a p-th Frobenius given by $\sigma(t) = ct^p$ with $c \in 1+pW$:

$$\left(\sum_i a_i t^i\right)^{\sigma} = \sum_i a_i^F c^i t^{ip}$$

where $F : W \to W$ is the Frobenius on the Witt ring. Then we can define Asakura's function $\mathscr{F}^{(\sigma)}_{a_1,\ldots,a_s}(t)$ (see Definition 4.5 or [1, §3.1]). If we write $\mathscr{F}^{(\sigma)}_{a_1,\ldots,a_s}(t) = G_{a_1,\ldots,a_s}(t)/F_{a_1,\ldots,a_s}(t)$, these p-adic hypergeometric functions of logarithmic type satisfy congruence relations

$$\mathscr{F}^{(\sigma)}_{a_1,\ldots,a_s}(t) \equiv \frac{G_{a_1,\ldots,a_s}(t)_{<p^n}}{F_{a_1,\ldots,a_s}(t)_{<p^n}} \mod p^n W[[t]].$$

which are similar to Dwork's. Asakura studied his functions from the viewpoint of p-adic regulators. In particular, some values of $\mathscr{F}^{(\sigma)}_{a_1,\ldots,a_s}(t)$ are expected to be the special values of p-adic L-function of elliptic curves over $\mathbb{Q}$ ([1, §5]).

In this paper, we introduce new p-adic hypergeometric functions which we denote by $\widehat{\mathscr{F}}^{(\sigma)}_{a,\ldots,a}(t)$ (Definition 2.1). Let $a_1 = \cdots = a_s = a$. Then our function is defined as

$$\widehat{\mathscr{F}}^{(\sigma)}_{a,\ldots,a}(t) := \frac{t^{-a}}{F_{a,\ldots,a}(t)} \int_0^t (t^a F_{a,\ldots,a}(t) - (-1)^{se}[t^{a'} F_{a',\ldots,a'}(t)]^{\sigma}) \frac{dt}{t}.$$

Write $\widehat{\mathscr{F}}^{(\sigma)}_{a,\ldots,a}(t) = \widehat{G}^{(\sigma)}_{a,\ldots,a}(t)/F_{a,\ldots,a}(t)$. Our first main result is the following congruence relations (Theorem 3.1)

$$\widehat{\mathscr{F}}^{(\sigma)}_{a,\ldots,a}(t) \equiv \frac{\widehat{G}^{(\sigma)}_{a,\ldots,a}(t)_{<p^n}}{F_{a,\ldots,a}(t)_{<p^n}} \mod p^n W[[t]].$$

The second main result is the transformation formula (=Theorem 4.14). Let $\sigma(t) = ct^p$ and $\widehat{\sigma}(t) = c^{-1}t^p$. We will prove that

$$\mathscr{F}^{(\sigma)}_{a,a}(t) = -\widehat{\mathscr{F}}^{(\widehat{\sigma})}_{a,a}(t^{-1})$$

in case that $a \in \frac{1}{N}\mathbb{Z}$, $0 < a < 1$ and $p > N$ which comes from geometric observation (see next paragraph). We then generally conjecture a formula

$$\mathscr{F}^{(\sigma)}_{a,\cdots,a}(t) = -\widehat{\mathscr{F}}^{(\hat{\sigma})}_{a,\cdots,a}(t^{-1})$$

between $\mathscr{F}^{(\sigma)}_{a,\cdots,a}(t)$ and $\widehat{\mathscr{F}}^{(\hat{\sigma})}_{a,\cdots,a}(t)$, which we refer to as the transformation formula (Conjecture 4.12) and prove our theorem as a special case.

The strategy of the proof of transformation formula is as follows. Let N, A be integers with $N \geq 2$, $p > N$, $1 \leq A < N$ and $\gcd(A, N) = 1$. We consider the fibration $f : Y \to \mathbb{P}^1$ over $K = \mathrm{Frac}(W)$ whose general fiber $f^{-1}(t)$ is the projective nonsingular model of

$$y^N = x^A(1-x)^A(1-(1-t)x)^{N-A},$$

and put $X_0 := f^{-1}(S)$ where $S_K := \mathrm{Spec}K[t, (t-t^2)^{-1}]$.

Let ξ be the K_2-symbol defined in (4.14). Asakura showed that his $\mathscr{F}^{(\sigma)}_{a_1,\ldots,a_s}(t)$ appears in the p-adic regulator of ξ. The key step in our proof is to provide an alternative description of the p-adic regulator by our $\widehat{\mathscr{F}}^{(\sigma)}_{a,\cdots,a}(t)$. A key ingredient is the different description of X via $(z, w, s_0) = (1-x, t_0^{A-N}y, t_0^{-1})$ with

$$X : y^N = x^A(1-x)^A(1-(1-t_0^N)x)^{N-A}$$

and

$$\widehat{X} : w^N = z^A(1-z)^A(1-(1-s_0^N)z)^{N-A}$$

with $t = t_0^N$.

We also conjecture a similar transformation formula for Dwork's hypergeometric functions (Conjecture 4.18). We attach a proof in case $s = 2$, $a \in \frac{1}{N}\mathbb{Z}$, $0 < a < 1$ and $p > N$ which is based on a similar idea to the above (Theorem 4.20).

This paper is organized as follows. In §2, we give the definition of $\widehat{\mathscr{F}}^{(\sigma)}_{a,\ldots,a}(t)$. In §3, we prove the congruence relations. In §4, we give the proof of the transformation formula. We also give a brief review on hypergeometric curves and some results in [1] which are necessary in the proof.

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2 Definition of $\widehat{\mathscr{F}}^{(\sigma)}_{a,...,a}(t)$

Let $W = W(\overline{\mathbb{F}}_p)$ denote the Witt ring, and K=Frac$W$ denote its fraction field. Let $\sigma : W[[t]] \to W[[t]]$ be a p-th Frobenius given by $\sigma(t) = ct^p$ with $c \in 1 + pW$:

$$\left(\sum_i a_i t^i\right)^\sigma = \sum_i a_i^F c^i t^{ip} \tag{2.1}$$

where $F : W \to W$ is the Frobenius on W. Let $a \in \mathbb{Z}_p$. Let a' be the Dwork prime of a, which is defined to be $(a + l)/p$ where $l \in \{0, 1, ..., p - 1\}$ is the unique integer such that $a + l \equiv 0 \mod p$. We denote the i-th Dwork prime by $a^{(i)}$ which is defined to be $(a^{(i-1)})'$ with $a^{(0)} = a$.

Let

$$F_{a,...,a}(t) = \sum_{k=0}^{\infty} \left(\frac{(a)_k}{k!}\right)^s t^k, \quad F_{a',...,a'}(t) = \sum_{k=0}^{\infty} \left(\frac{(a')_k}{k!}\right)^s t^k \tag{2.2}$$

be the hypergeometric series for $a \in \mathbb{Z}_p$, where $(a)_k$ denotes the Pochhammer symbol (i.e. $(\alpha)_k = \alpha(\alpha + 1)\cdots(\alpha + k + 1)$ when $k \neq 0$ and $(\alpha)_0 = 1$).

Put

$$q := \begin{cases} 4 & p = 2 \\ p & p \geq 3. \end{cases}$$

Let $l' \in \{0, 1, ..., q - 1\}$ be the unique integer such that $a + l' \equiv 0 \mod q$.

Put

$$e := l' - \lfloor \frac{l'}{p} \rfloor.$$

Define a power series

$$\begin{aligned} \widehat{G}^{(\sigma)}_{a,...,a}(t) &:= t^{-a} \int_0^t (t^a F_{a,...,a}(t) - (-1)^{se}[t^{a'} F_{a',...,a'}(t)]^\sigma) \frac{dt}{t} \\ &= \sum_{k=0}^{\infty} B_k t^k \end{aligned}$$

for $a \in \mathbb{Z}_p \backslash \mathbb{Z}_{\leq 0}$. Here we think t^α to be an abstract symbol with relations $t^\alpha \cdot t^\beta = t^{\alpha+\beta}$, on which σ acts by $\sigma(t^\alpha) = c^\alpha t^{p\alpha}$, where

$$c^\alpha = (1 + pu)^\alpha := \sum_{i=0}^{\infty} \binom{\alpha}{i} p^i u^i, \quad u \in W.$$

Moreover we think $\int_0^t (-)\frac{dt}{t}$ to be a operator such that

$$\int_0^t t^\alpha \frac{dt}{t} = \frac{t^\alpha}{\alpha}, \quad \alpha \neq 0.$$

Definition 2.1. *Define*

$$\widehat{\mathscr{F}}^{(\sigma)}_{a,\ldots,a}(t) := \frac{\widehat{G}^{(\sigma)}_{a,\ldots,a}(t)}{F_{a,\ldots,a}(t)}, \quad a \in \mathbb{Z}_p \backslash \mathbb{Z}_{\leq 0}.$$

If we write $F_{a,\ldots,a}(t) = \sum A_k t^k$ and $F_{a',\ldots,a'}(t) = \sum A^{(1)}_k t^k$, then we have

$$B_k = \frac{1}{k+a}\left(A_k - (-1)^{se}(A^{(1)}_{\frac{k-l}{p}})c^{\frac{k+a}{p}}\right),$$

where $A^{(1)}_{\frac{m}{p}} = 0$ if $m \not\equiv 0 \mod p$ or $m < 0$. In fact, we have $\widehat{\mathscr{F}}^{(\sigma)}_{a,\ldots,a}(t)$ and $\widehat{G}^{(\sigma)}_{a,\ldots,a}(t)$ are power series with W-coefficients from the following lemma.

Lemma 2.2. *We have*

$$B_k \in W, \quad \forall k \in \mathbb{Z}_{\geq 0}.$$

Proof. If $k \not\equiv l \mod p$, then $k + a \not\equiv 0 \mod p$. Hence we have

$$B_k = \frac{A_k}{k+a} \in W.$$

For $k \equiv l \mod p$, we write $k + a = p^n m$ with $p \nmid m$. Then

$$c^{\frac{k+a}{p}} = c^{p^{n-1}m} \equiv 1 \mod p^n$$

since $c \in 1 + pW$. So if we can show

$$A_k \equiv (-1)^{se} A^{(1)}_{\frac{k-l}{p}} \mod p^n$$

then $B_k \in W$. Indeed, for any p-adic integer $\alpha \in \mathbb{Z}_p$ and $n \in \mathbb{Z}_{\geq 1}$ we define

$$\{\alpha\}_n := \prod_{\substack{1 \leq i \leq n \\ p \nmid (\alpha+i-1)}} (\alpha + i - 1).$$

Using [1, Lemma 3.6], one have

$$\frac{(a)_k}{k!} = \left(\frac{(a')_{\lfloor k/p \rfloor}}{\lfloor k/p \rfloor}!\right)^{-1} \frac{\{a\}_k}{\{1\}_k} = \frac{(a')_{\frac{k-l}{p}}}{(\frac{k-l}{p})!} \frac{\{a\}_k}{\{1\}_k}.$$

Therefore we obtain

$$A_k = A^{(1)}_{\frac{k-l}{p}} \left(\frac{\{a\}_k}{\{1\}_k}\right)^s.$$

Since $k+a \equiv 0 \mod p^n$, we have that $\{a\}_k \equiv \{-k\}_k \mod p^n$. Therefore

$$\begin{aligned}\frac{\{a\}_k}{\{1\}_k} &\equiv \frac{\{-k\}_k}{\{1\}_k} \\ &\equiv \frac{\prod\limits_{\substack{1\le i\le k \\ p\nmid(-k+i-1)}} (-k+i-1)}{\prod\limits_{\substack{1\le i\le k \\ p\nmid i}} i} \\ &\equiv \prod_{\substack{i=1 \\ p\nmid i}}^{k}(-1) = (-1)^{k-\lfloor k/p\rfloor} \mod p^n.\end{aligned}$$

Now we claim $(-1)^{k-\lfloor k/p\rfloor} = (-1)^e$.

• For $p \ge 3$, write $k = l + bp$. Then

$$k - \lfloor k/p\rfloor = l + bp - \lfloor l/p + b\rfloor = e + b(p-1) \equiv e \mod 2.$$

Therefore $(-1)^{k-\lfloor k/p\rfloor} = (-1)^e$.

• For $p = 2$, write $k + a = 2^n m$, $2 \nmid m$.

(1)If $n = 1$, we have $(-1)^{k-\lfloor k/p\rfloor} = (-1)^e$.

(2)If $n \ge 2$, we have $k + a \equiv 0 \mod 4$. Therefore $k \equiv L \mod 4$ where $L \in \{0,1,2,3\}$ is the unique integer such that $a + L \equiv 0 \mod 4$. Write $k = L + 4b$, then

$$k - \lfloor k/2\rfloor = L + 4b - \lfloor L/2\rfloor - 2b \equiv e \mod 2.$$

Again, we obtain $(-1)^{k-\lfloor k/p\rfloor} = (-1)^e$.

Therefore

$$A_k \equiv A^{(1)}_{\frac{k-l}{p}}(-1)^{es} \mod p^n.$$

This completes the proof. $\square$

3 Congruence Relations

For a power series $f(t) = \sum_{n=0}^{\infty} A_n t^n$, we denote $f(t)_{<m} := \sum_{n<m} A_n t^n$ the truncated polynomial. Then we have the following theorem which we call the congruence relations of $\widehat{\mathscr{F}}^{(\sigma)}_{a,\dots,a}(t)$.

Theorem 3.1. *Let $a \in \mathbb{Z}_p \backslash \mathbb{Z}_{\le 0}$ and suppose $c \in 1 + qW$. Then*

$$\widehat{\mathscr{F}}^{(\sigma)}_{a,\dots,a}(t) \equiv \frac{\widehat{G}^{(\sigma)}_{a,\dots,a}(t)_{<p^n}}{F_{a,\dots,a}(t)_{<p^n}} \mod p^n W[[t]]$$

for all $n \in \mathbb{Z}_{\ge 0}$.

Corollary 3.2. *If $a^{(r)} = a$ for some $r > 0$, where $(-)^{(r)}$ denote the r-th Dwork prime, then*

$$\widehat{\mathscr{F}}^{(\sigma)}_{a,\ldots,a}(t) \in W\langle t, t^{-1}, h(t)^{-1}\rangle, \quad h(t) := \prod_{i=0}^{r-1} F_{a^{(i)},\ldots,a^{(i)}}(t)_{<p}$$

is a convergent function, where $W\langle t, t^{-1}, h(t)^{-1}\rangle := \varprojlim_n (W/p^n[t, t^{-1}, h(t)^{-1}])$.

Proof. It suffices to show that modulo p^n

$$\frac{1}{F_{a,\cdots,a}(t)_{<p^n}}$$

lies in $W/p^n[t, t^{-1}, h(t)^{-1}]$. From Dwork congruence [4, p.37, Theorem 2], we have

$$F_{a,\cdots,a}(t)_{<p} \equiv \frac{F_{a,\cdots,a}(t)_{<p^n}}{\Big[F_{a',\cdots,a'}(t^p)\Big]_{<p^n}} \mod p\mathbb{Z}_p[[t]].$$

Thus one has

$$F_{a,\cdots,a}(t)_{<p^n} \equiv F_{a,\cdots,a}(t)_{<p}\Big[F_{a',\cdots,a'}(t^p)\Big]_{<p^n} \mod p\mathbb{Z}_p[[t]].$$

Using Dwork congruence again, we obtain

$$\begin{aligned} &F_{a,\cdots,a}(t)_{<p}\Big[F_{a',\cdots,a'}(t^p)\Big]_{<p^n} \\ \equiv &F_{a,\cdots,a}(t)_{<p}\Big[F_{a',\cdots,a'}(t^p)\Big]_{<p^2}\Big[F_{a^{(2)},\cdots,a^{(2)}}(t^{p^2})\Big]_{<p^n} \mod p\mathbb{Z}_p[[t]]. \end{aligned}$$

So by applying Dwork congruence inductively, we have $F_{a,\cdots,a}(t)_{<p^n}$ is congruent to

$$F_{a,\cdots,a}(t)_{<p}\cdots\Big[F_{a^{(i)},\cdots,a^{(i)}}(t^{p^i})\Big]_{<p^{i+1}}\cdots\Big[F_{a^{(n-1)},\cdots,a^{(n-1)}}(t^{p^{n-1}})\Big]_{<p^n}$$

modulo $p\mathbb{Z}_p[[t]]$. Since both sides of the equation are polynomials, $p\mathbb{Z}_p[[t]]$ can be replaced by $p\mathbb{Z}_p[t]$. Now we use the fact that for any $F(x) \in \mathbb{Z}_p[[t]]$, one has

$$F(x)_{<p^{n-1}}\big|_{x=t^p} \equiv \Big(F(x)_{<p^{n-1}}\big|_{x=t}\Big)^p = \Big(F(t)_{<p^{n-1}}\Big)^p \mod p\mathbb{Z}_p[t].$$

Then by induction, we have

$$\Big[F_{a^{(i)},\cdots,a^{(i)}}(t^{p^i})\Big]_{<p^{i+1}} \equiv \Big[F_{a^{(i)},\cdots,a^{(i)}}(t)_{<p}\Big]^{p^i} \mod p\mathbb{Z}_p[t]$$

Therefore we have $F_{a,\cdots,a}(t)_{<p^n}$ is congruent to

$$F_{a,\cdots,a}(t)_{<p}\cdots\Big[F_{a^{(i)},\cdots,a^{(i)}}(t)_{<p}\Big]^{p^i}\cdots\Big[F_{a^{(n-1)},\cdots,a^{(n-1)}}(t)_{<p}\Big]^{p^{n-1}} \mod p\mathbb{Z}_p[t].$$

So we can write

$$
\begin{aligned}
&F_{a,\cdots,a}(t)_{<p^n}\\
=&F_{a,\cdots,a}(t)_{<p}\cdots\Big[F_{a^{(i)},\cdots,a^{(i)}}(t)_{<p}\Big]^{p^i}\cdots\Big[F_{a^{(n-1)},\cdots,a^{(n-1)}}(t)_{<p}\Big]^{p^{n-1}}+pg(t)
\end{aligned}
$$

for some $g(t)\in\mathbb{Z}_p[t]$. Then considering $(F_{a,\cdots,a}(t)_{<p^n})^{-1}$ modulo p^n, we see that it lies in $W/p^n[t,t^{-1},h(t)^{-1}]$. □

3.1 Proof of Congruence Relations: Lipschitz Functions and Congruence Relations for B_k/A_k

Here we introduce a notion called Lipschitz functions.

Definition 3.3. *Let W be the Witt ring of $\overline{\mathbb{F}}_p$. A function $f:\mathbb{Z}_{\geq 0}\to Frac(W)$ is called Lipschitz if it takes values in W and $p^s|(n-m)$ implies $p^s|(f(n)-f(m))$ for any $n,m,s\in\mathbb{Z}_{\geq 0}$.*

Remark 3.4. *Generally, one can define Lipschitz functions as functions from X to K where K denote a complete extension of $\mathbb{Q}_p$ and $X\subseteq K$ is a subset with no isolated point and satisfy*

$$|f(x)-f(y)|_p\leq M|x-y|_p$$

for some constant M and $x,y\in X$ (See [9, §5.1]). In our case, $M=1$.

In this section, we fix $a\in\mathbb{Z}_p\backslash\mathbb{Z}_{\leq 0}$. The main result of this section is the following lemma.

Lemma 3.5. *The function defined by $k\mapsto B_k/A_k$ is Lipschitz.*

Before proving Lemma 3.5, we need some preliminary lemmas.

Lemma 3.6. *Polynomial functions are Lipschitz. The sums and products of Lipschitz functions are Lipschitz.*

Lemma 3.7. *Lemma 3.5 can be reduced to the case $\sigma(t)=t^p$, i.e., $c=1$.*

Proof. Suppose the function defined by $n\mapsto B_n^\circ/A_n$ is Lipschitz. Let B_n° be the coefficients of $\widehat{G}^{(\sigma)}_{a,\cdots,a}(t)$ with respect to $c=1$. Then

$$n\mapsto\frac{A^{(1)}_{\frac{n-l}{p}}}{A_n}=(-1)^{se+1}\left((n+a)\frac{B_n^\circ}{A_n}-1\right)$$

is also Lipschitz by Lemma 3.6. Observe that

$$\frac{B_n}{A_n} = \frac{B_n^\circ}{A_n} + (-1)^{se+1}\left(\frac{c^{\frac{n+a}{p}} - 1}{n+a}\right)\frac{A^{(1)}_{\frac{n-l}{p}}}{A_n}.$$

So to prove $n \mapsto B_n/A_n$ is Lipschitz, it suffices to prove $n \mapsto \frac{c^{n/p}-1}{n}$ is Lipschitz(here the value is 0 when $p \nmid n$). Write $c = 1 + kq$ for some $k \in W$. Then for those n with $p|n$, we have

$$\frac{c^{n/p} - 1}{n} = \frac{(1+kq)^{n/p} - 1}{n} = \frac{1}{n}\sum_{i=1}^{\infty}\binom{n/p}{i}k^i q^i.$$

So to prove $n \mapsto \frac{c^{n/p}-1}{n}$ is Lipschitz, it suffices to show that

$$\frac{1}{n_1}\binom{n_1/p}{i}k^i q^i \equiv \frac{1}{n_2}\binom{n_2/p}{i}k^i q^i \mod p^m$$

for $n_1 \equiv n_2 \mod p^m$. For $i = 1$, they are equal. Suppose $i \geq 2$, we have

$$\frac{1}{n_1}\binom{n_1/p}{i}k^i q^i = \left(\frac{n_1}{p} - 1\right)\cdots\left(\frac{n_1}{p} - i + 1\right)\frac{k^i q^i}{i!p}.$$

By [13, §5.1 p.49], we know

$$v_p(i!) < \frac{i}{p-1}.$$

Thus we have

$$v_p\left(\frac{k^i q^i}{i!p}\right) \geq 1.$$

Therefore

$$\begin{aligned}\left(\frac{n_1}{p} - 1\right)\cdots\left(\frac{n_1}{p} - i + 1\right)\frac{k^i q^i}{i!p} &\equiv \left(\frac{n_2}{p} - 1\right)\cdots\left(\frac{n_2}{p} - i + 1\right)\frac{k^i q^i}{i!p} \mod p^m \\ &= \frac{1}{n_2}\binom{n_2/p}{i}k^i q^i.\end{aligned}$$

Therefore, the result follows. □

Lemma 3.8. *Assuming $\sigma(t) = t^p$, lemma 3.5 can be reduced to $s = 1$.*

Proof. Assume that $s = 1$, $\sigma(t) = t^p$ and suppose we have proved that $n \mapsto B_n/A_n$ is Lipschitz. Then $n \mapsto A^{(1)}_{\frac{n-l}{p}}/A_n$ is also Lipschitz (see the proof of Lemma 3.7). Now for arbitraty $s \in \mathbb{Z}_{>0}$, we have the function

$$
\begin{aligned}
n \mapsto & \frac{1}{n+a}\left(\frac{A_n^s - ((-1)^e A^{(1)}_{\frac{n-l}{p}})^s}{A_n^s}\right) \\
&= \frac{1}{n+a}\left(\frac{A_n - (-1)^e A^{(1)}_{\frac{n-l}{p}}}{A_n}\right)\sum_{j=0}^{s-1}\left(\frac{(-1)^e A^{(1)}_{\frac{n-l}{p}}}{A_n}\right)^j \\
&= \frac{B_n}{A_n}\sum_{j=0}^{s-1}\left(\frac{(-1)^e A^{(1)}_{\frac{n-l}{p}}}{A_n}\right)^j
\end{aligned}
$$

is also Lipschitz by Lemma 3.6. □

By Lemma 3.7 and Lemma 3.8, from now on, we assume that $\sigma(t) = t^p$ and $s = 1$.

Lemma 3.9. *If $a + l = cp^n$ with $p \nmid c$, then*

$$
\frac{B_l}{A_l} \equiv \psi_p(a+l) - \psi_p(1+l) \equiv -(\psi_p(1+l) + \gamma_p) \mod p^n,
$$

where $\psi_p(z)$ is the p-adic digamma function and γ_p is the p-adic Euler constant ([1, §2.2]).

Proof. Case I : $p \neq 2$

First, if $l = 0$, then $e = l = 0$. Therefore we have

$$
\frac{B_l}{A_l} = \frac{B_0}{A_0} = \frac{1}{a}(1 - (-1)^e) = 0
$$

and

$$
\psi_p(a+l) - \psi_p(1+l) \equiv \psi_p(0) - \psi_p(1) = 0 \mod p^n.
$$

If $l \neq 0$, then

$$
\begin{aligned}
\frac{B_l}{A_l} &= \frac{1}{a+l}\left(1 - (-1)^e \frac{1}{A_l}\right) \\
&= \frac{1}{cp^n}\left(1 - (-1)^e \frac{l!}{(a)_l}\right).
\end{aligned}
$$

Arranging the formula above and using $a+l=cp^n$, we obtain

$$\begin{aligned}(-1)^e\Big(1-cp^n\frac{B_l}{A_l}\Big) &= (-1)^l\frac{1\cdot 2\cdots l}{(l-cp^n)\cdots(1-cp^n)}\\ &= (-1)^l\prod_{1\le k\le l}\left(\frac{1}{1-cp^n/k}\right)\\ &\equiv (-1)^l\left(1+\sum_{1\le k\le l}\frac{cp^n}{k}\right) \mod p^{2n}\\ &\equiv (-1)^l\left(1+cp^n\sum_{1\le k\le l}\frac{1}{k}\right) \mod p^{2n}\\ &= (-1)^l\Big(1+cp^n(\psi_p(1+l)+\gamma_p)\Big).\end{aligned}$$

Therefore, we have

$$1-cp^n\frac{B_l}{A_l}\equiv(-1)^{(e-l)}\Big(1+cp^n(\psi_p(1+l)+\gamma_p)\Big) \mod p^{2n}.$$

Since we assume p is odd, e is equal to l. Hence

$$\frac{B_l}{A_l}\equiv-(\psi_p(1+l)+\gamma_p)\equiv\psi_p(a+l)-\psi_p(1+l) \mod p^n$$

from the formula in [1, (2.13) and Theorem 2.4].

Case II : $p=2$

Again, we have

$$\frac{B_l}{A_l}=\frac{1}{a+l}\Big(1-(-1)^e\frac{1}{A_l}\Big).$$

(1) For $a=2^nc$ with $2\not|\,c$, we have $l=0$. Hence

$$\frac{B_l}{A_l}=\frac{B_0}{A_0}=\frac{1}{a}(1-(-1)^e)=\frac{1}{2^nc}(1-(-1)^e).$$

If $n\ge 2$, then $l'=0$ and $e=0$. Therefore, we obtain

$$\frac{B_0}{A_0}=0,\quad \psi_p(a)-\psi_p(1)\equiv 0 \mod p^n$$

by [1, (2.13) Case II].

If $n=1$, then $l'=2$ and $e=1$. This implies

$$\frac{B_0}{A_0}=\frac{1}{2c}(1-(-1))\equiv 1 \mod 2$$

and

$$\psi_p(a) - \psi_p(1) = \psi_p(2c) - \psi_p(0) \equiv c \equiv 1 \mod 2.$$

Hence, in both cases, we have

$$\frac{B_l}{A_l} \equiv \psi_p(a+l) - \psi_p(1+l) \mod p^n.$$

(2) For $a+1 = 2^n c$ with $2 \not| c$, we have $l = 1$. This implies

$$\frac{B_l}{A_l} = \frac{B_1}{A_1} = \frac{1}{a+1}(1 - (-1)^e \frac{1}{A_1}) = \frac{1}{2^n c}(1 - (-1)^e \frac{1}{a}).$$

If $n \geq 2$, then $l' = 1$ and $e = 1$. We obtain

$$\begin{aligned} 2^n c \frac{B_1}{A_1} &= (1 + \frac{1}{a}) \\ &= (1 - \frac{1}{1 - 2^n c}) \\ &\equiv 1 - (1 + 2^n c) = -2^n c \mod 2^{2n}, \end{aligned}$$

which implies

$$\frac{B_1}{A_1} \equiv -1 \mod 2^n.$$

Also, we have

$$\psi_p(a+1) - \psi_p(2) \equiv \psi_p(0) - \psi_p(2) \equiv -1 \mod 2^n.$$

If $n = 1$, then $l' = 3$ and $e = 2$. We have

$$\begin{aligned} 2c \frac{B_1}{A_1} &= 1 + \left(\frac{1}{1 - 2c} \right) \\ &\equiv 1 + (1 + 2c) \mod 4 \\ &\equiv 0 \mod 4. \end{aligned}$$

Hence, we get

$$\frac{B_1}{A_1} \equiv 0 \mod 2.$$

On the other hand, we have

$$\begin{aligned} \psi_p(a+1) - \psi_p(2) &\equiv \psi_p(2c) - \psi_p(2) \\ &\equiv c - 1 \\ &\equiv 0 \mod 2. \end{aligned}$$

□

Lemma 3.10. *If $k = l + bp^m$ with $p \not| b$, then*

$$\frac{B_k}{A_k} \equiv \frac{B_l}{A_l} \mod p^m.$$

Proof. Again, we write $a + l = cp^n$ with $p \not| c$.

$\boxed{\text{Case I} : m \neq n}$

We have

$$\begin{aligned}
&1 - (a + l + bp^m)\frac{B_{l+bp^m}}{A_{l+bp^m}} \\
=&(-1)^e \frac{A^{(1)}_{bp^m-1}}{A_{l+bp^m}} \\
\overset{(*)}{\equiv}&(-1)^e \frac{\{1\}_{l+bp^m}}{\{a\}_{l+bp^m}} \\
=&(-1)^e \frac{\{1\}_l}{\{a\}_l} \cdot \frac{\{1+l\}_{bp^m}}{\{1\}_{bp^m}} \frac{\{1\}_{bp^m}}{\{a+l\}_{bp^m}} \\
\overset{(**)}{\equiv}&(-1)^e \frac{1}{A_l}\Big(1 - bp^m(\psi_p(a+l) - \psi_p(1+l))\Big) \mod p^{2m} \\
=&\Big(1 - (a+l)\frac{B_l}{A_l}\Big)\Big(1 - bp^m(\psi_p(a+l) - \psi_p(1+l))\Big) \mod p^{2m}
\end{aligned}$$

where $(*)$ and $(**)$ follow from [1, Lemma 3.6] and [1, Lemma 3.8], respectively. By arranging the equation above, we obtain

$$1 - (a + l + bp^m)\frac{B_{l+bp^m}}{A_{l+bp^m}} \equiv 1 - (a+l)\frac{B_l}{A_l} - bp^m(\psi_p(a+l) - \psi_p(1+l))$$

modulo p^{2m} when $n > m$ (modulo p^{n+m} when $n < m$). Hence we obtain

$$(a + l + bp^m)\Big(\frac{B_l}{A_l} - \frac{B_{l+bp^m}}{A_{l+bp^m}}\Big) \equiv bp^m\left(\frac{B_l}{A_l} - (\psi_p(a+l) - \psi_p(1+l))\right).$$

Therefore, by Lemma 3.9, we have

$$\begin{aligned}
(a + l + bp^m)\Big(\frac{B_l}{A_l} - \frac{B_{l+bp^m}}{A_{l+bp^m}}\Big) \equiv 0 \bmod p^{2m}, &\text{when } n > m \\
(\bmod p^{n+m}, &\text{when } n < m).
\end{aligned}$$

$$\Longrightarrow \frac{B_{l+bp^m}}{A_{l+bp^m}} \equiv \frac{B_l}{A_l} \equiv 0 \mod p^m$$

in both cases since $\mathrm{ord}_p(a + l + bp^m)$ is m if $m < n$ (n if $n < m$).

$\boxed{\text{Case II} : m = n}$

We write $a+l=cp^m$, $k=l+bp^m$ and $a+k=(b+c)p^m=dp^{m'+m}$ with $p\not| d$. Here we can assume $m'\geq 1$; otherwise, use the method in Case I. Then

$$\frac{B_{l+bp^m}}{A_{l+bp^m}}-\frac{B_l}{A_l}=\frac{c[1-(-1)^e\frac{A^{(1)}_{bp^{m-1}}}{A_{bp^m+l}}]-dp^{m'}[1-(-1)^e\frac{1}{A_l}]}{cdp^{m+m'}}.$$

We claim that the numerator

$$c[1-(-1)^e\frac{A^{(1)}_{bp^{m-1}}}{A_{bp^m+l}}]-dp^{m'}[1-(-1)^e\frac{1}{A_l}]\equiv 0 \mod p^{2m+m'}.$$

(1) p is odd or $m\geq 2$.

First, we calculate $A^{(1)}_{bp^{m-1}}/A_{bp^m+l}$. We have that

$$\frac{A^{(1)}_{bp^{m-1}}}{A_{bp^m+l}}=\frac{\{1\}_{l+bp^m}}{\{a\}_{l+bp^m}}=\frac{\{1\}_l}{\{a\}_l}\frac{\{1+l\}_{bp^m}}{\{a+l\}_{bp^m}}=\frac{1}{A_l}\frac{\{1+l\}_{bp^m}}{\{a+l\}_{bp^m}}$$

and

$$\begin{aligned}
&\frac{\{1+l\}_{bp^m}}{\{a+l\}_{bp^m}}\\
=&\frac{\prod_{\substack{1+l\leq i\leq bp^m+l\\ p\not| i}} i}{\prod_{\substack{1\leq i\leq bp^m\\ p\not| i}}(cp^m+i)}\\
=&\frac{\prod_{\substack{1\leq i\leq bp^m\\ p\not| i}} i\cdot\prod_{\substack{bp^m\leq i\leq bp^m+l\\ p\not| i}} i}{\prod_{\substack{1\leq i\leq l\\ p\not| i}} i}\cdot\frac{1}{\prod_{\substack{1\leq i\leq bp^m\\ p\not| i}}(cp^m+i)}\\
=&\prod_{\substack{1\leq i\leq bp^m\\ p\not| i}}\left(\frac{1}{1+cp^m/i}\right)\frac{\prod_{1\leq i\leq l}(dp^{m+m'}-a-l+i)}{\prod_{1\leq i\leq l} i}.
\end{aligned}$$

Since p is odd, or $m\geq 2$ and $p\not| i$, we have that

$$\prod_{\substack{1\leq i\leq bp^m\\ p\not| i}}\left(\frac{1}{1+cp^m/i}\right)=\prod_{\substack{1\leq i<\frac{bp^m}{2}\\ p\not| i}}\left(\frac{1}{1+cp^m/i}\right)\left(\frac{1}{1+cp^m/(bp^m-i)}\right)$$

where $bp^m-i\neq i$. Since

$$\prod_{\substack{1\le i<\frac{bp^m}{2}\\ p\nmid i}} \Big(\frac{1}{1+cp^m/i}\Big)\Big(\frac{1}{1+cp^m/(bp^m-i)}\Big) = \prod_i \frac{1}{1+\frac{cp^m\cdot p^m(b+c)}{i(bp^m-i)}} \equiv 1$$

modulo $p^{2m+m'}$, we obtain

$$\begin{aligned}
\frac{\{1+l\}_{bp^m}}{\{a+l\}_{bp^m}} &\equiv \frac{\prod\limits_{1\le i\le l}(dp^{m+m'}-a-l+i)}{\prod\limits_{1\le i\le l} i} \mod p^{2m+m'}\\
&= \frac{(-1)^l \prod\limits_{0\le i\le l-1}(a+i-dp^{m+m'})}{l!}\\
&\equiv \frac{(-1)^l}{l!}\Big[\prod_{i=0}^{l-1}(a+i) + (-dp^{m+m'})\sum_{i=0}^{l-1}(a+i)\Big]\\
&\equiv (-1)^l\Big[(-dp^{m+m'})\frac{la+l(l-1)/2}{l!} + \frac{(a)_l}{l!}\Big] \mod p^{2m+m'}.
\end{aligned}$$

Multiplying $\frac{1}{A_l}$ on both sides, we get

$$\begin{aligned}
\frac{A^{(1)}_{bp^{m-1}}}{A_{bp^m+l}} &\equiv \frac{(-1)^l}{A_l}\Big[A_l + (-dp^{m+m'})\frac{la+l(l-1)/2}{l!}\Big] \mod p^{2m+m'}\\
&= (-1)^l\Big[1 + \frac{l!}{(a)_l}(-dp^{m+m'})\frac{la+l(l-1)/2}{l!}\Big]\\
&\equiv (-1)^l\Big[1 + (-1)^l(-dp^{m+m'})\frac{la+l(l-1)/2}{l!}\Big] \mod p^{2m+m'}
\end{aligned}$$

since $\frac{l!}{(a)_l} \equiv (-1)^l \mod p^m$ (use $a \equiv -l \mod p^m$).

For $1/A_l$, since

$$\frac{(a)_l}{l!} \equiv \frac{-cp^m(l+1)l/2 + (-1)^l l!}{l!} = \frac{-cp^m(l+1)}{2(l-1)!} + (-1)^l \mod p^{2m},$$

we have

$$\frac{1}{A_l} = \frac{l!}{(a)_l} \equiv \frac{(-1)^l}{1+(-1)^{l+1}\frac{cp^m(l+1)}{2(l-1)!}} \equiv (-1)^l + \frac{cp^m(l+1)}{2(l-1)!} \mod p^{2m}.$$

Hence the numerator

$$\begin{aligned}
&c[1-(-1)^e\frac{A^{(1)}_{bp^{m-1}}}{A_{bp^m+l}}]-dp^{m'}[1-(-1)^e\frac{1}{A_l}]\\
=&-b+(-1)^e\left(\frac{dp^{m'}}{A_l}-c\frac{A^{(1)}_{bp^{m-1}}}{A_{bp^m+l}}\right)\\
\equiv&-b+(-1)^e\left((-1)^l b+dp^{m'}\frac{cp^m(l+1)}{2[(l-1)!]}+c(dp^{m+m'})\frac{2la+l(l-1)}{2(l!)}\right)\\
\equiv&-b+(-1)^e\left((-1)^l b+\frac{dp^{m+m'}c}{2(l!)}(2l(a+l))\right)\\
\equiv&b((-1)+(-1)^{(e+l)})\mod p^{2m+m'}.
\end{aligned}$$

Since p is odd or $p=2(m\geq 2)$, we have $e=l$. Hence $(-1)+(-1)^{(e+l)}=0$. So we have the numerator is congruent to 0 modulo $p^{2m+m'}$. And this implies

$$\frac{B_k}{A_k}\equiv\frac{B_l}{A_l}\mod p^m.$$

(2) $p=2$ and $m=1$.

Write $k=l+2b, a+l=2c, a+k=2(b+c)=2^{n+1}d$ with $2\nmid bc$. Then

$$\begin{aligned}
\frac{A^{(1)}_{b2^{m-1}}}{A_{b2^m}}&=\frac{1}{A_l}\frac{\{1+l\}_{b2^m}}{\{a+l\}_{b2^m}}\\
&=\frac{1}{A_l}\left[\prod_{\substack{1\leq i\leq b2^m\\ 2\nmid i}}\frac{1}{1+c2^m/i}\right]\left[\frac{(-1)^l(a-d2^{m+n})_l}{l!}\right]\\
&\equiv\frac{1}{A_l}(\frac{1}{1+2c/b})(-1)^l\left(\frac{(a-d2^{m+n})_l}{l!}\right)\mod 2^{2+n}.
\end{aligned}$$

(Note: Only the term $\left(\frac{1}{1+2c/b}\right)$ is different from case (1)).

Again, we have

$$\begin{aligned}
&(-b)+(-1)^e\left(\frac{d2^n}{A_l}-c\frac{A_b^{(1)}}{A_{2b}}\right)\\
\equiv&(-b)+(-1)^e\Bigg[(-1)^l d2^n + d2^n\frac{c2(l+1)}{2(l-1)!}-c(\frac{b}{b+2c})(-1)^l-\\
&c(\frac{b}{b+2c})(-d2^{1+n})\frac{2la+l(l-1)}{2(l!)}\Bigg] \mod 4\cdot 2^n\\
\equiv&(-b)+(-1)^e\Bigg[(-1)^l d2^n - c(\frac{b}{b+2c})(-1)^l + d2^n\frac{c2(l+1)}{2(l-1)!}\\
&+c(d2^{1+n})\frac{2la+l(l-1)}{2(l!)}\Bigg] \mod 4\cdot 2^n \text{ (since } \frac{b}{b+2c}\equiv 1 \mod 2)\\
\equiv&(-b)+(-1)^e\Bigg[(-1)^l d2^n - c(\frac{b}{b+2c})(-1)^l\Bigg]\\
=&(-b)+(-1)^{(e+l)}\Bigg[d2^n - c(\frac{b}{b+2c})\Bigg]. \quad (*)
\end{aligned}$$

Observe that $e+l$ is odd. This is because if $l=0$, then $l'=2$ and $e=1$. If $l=1$, then $l'=3$ and $e=2$. Hence, we have

$$\begin{aligned}
(*) &= (-b)-\left(b+c-\frac{bc}{b+2c}\right)\\
&=\frac{(-2)(b+c)^2}{(b+2c)}\\
&\equiv 0 \mod 4\cdot 2^n.
\end{aligned}$$

Therefore, we have

$$\frac{B_k}{A_k}\equiv\frac{B_l}{A_l} \mod 2.$$

□

Now we can start the proof of Lemma 3.5 with the assumption $\sigma(t)=t^p$ and $s=1$.

(proof of Lemma 3.5). If $k\not\equiv l \mod p$, then

$$\frac{B_k}{A_k}=\frac{1}{k+a}.$$

So the result follows. Suppose $k\equiv l \mod p$. We write $a+l=cp^n, k=l+bp^m$ with $p\not| bc$. It is enough to prove the lemma for the case $k'=k+p^{m'}$ and for any $m'\in\mathbb{N}$.

Case I: $m'\leq m$

This follows from Lemma 3.10 since we have $k\equiv k'\equiv l \mod p^{m'}$.

Case II: $m' > m$ and $n \neq m$ or $n = m$ with $\mathrm{ord}_p(k+a) = m$

We have

$$
\begin{aligned}
&1-(k'+a)\frac{B_{k'}}{A'_k}\\
&=(-1)^e\frac{A^{(1)}_{\lfloor\frac{k'}{p}\rfloor}}{A_{k'}}\\
&=(-1)^e\frac{\{1\}_{l+bp^m+p^{m'}}}{\{a\}_{l+bp^m+p^{m'}}}\\
&=(-1)^e\frac{\{1\}_{l+bp^m}}{\{a\}_{l+bp^m}}\frac{\{1+l+bp^m\}_{p^{m'}}}{\{a+l+bp^m\}_{p^{m'}}}\\
&=(-1)^e\Big(\frac{A^{(1)}_{bp^{m-1}}}{A_{l+bp^m}}\Big)\cdot\Big(\frac{\{1+l+bp^m\}_{p^{m'}}}{\{1\}_{p^{m'}}}\frac{\{1\}_{p^{m'}}}{\{a+l+bp^m\}_{p^{m'}}}\Big)\\
&\equiv\Big(1-(k+a)\frac{B_k}{A_k}\Big)\Big(1+p^{m'}(\psi_p(1+l+bp^m)+\gamma_p)\Big)\cdot\\
&\quad\Big(1-p^{m'}(\psi_p(a+l+bp^m)+\gamma_p)\Big)\mod p^{2m'}\\
&\equiv\Big(1-(k+a)\frac{B_k}{A_k}\Big)\Big(1+p^{m'}(\psi_p(1+l+bp^m)-\psi_p(a+l+bp^m))\Big)\\
&\overset{(*)}{\equiv}\Big(1-(k+a)\frac{B_k}{A_k}\Big)\Big(1-p^{m'}\frac{B_l}{A_l}\Big)\mod p^{m+m'}
\end{aligned}
$$

where $(*)$ follows from [1, (2.13)] and Lemma 3.9 when $p \neq 2$ or $m \geq 2$. For $p = 2, m = 1$, $(*)$ follows from

$$\psi_p(1+l+2b)-\psi_p(a+l+2b)\equiv\psi_p(1+l)-\psi_p(a+l)\mod 2.$$

Therefore one has

$$
\begin{aligned}
&1-(k+a+p^{m'})\frac{B_{k'}}{A_{k'}}\\
\equiv&1-(k+a)\frac{B_k}{A_k}-p^{m'}\frac{B_l}{A_l}+p^{m\prime}(k+a)\frac{B_k}{A_k}\frac{B_l}{A_l}\mod p^{m+m'}\\
\equiv&1-(k+a)\frac{B_k}{A_k}-p^{m'}\frac{B_l}{A_l}\mod p^{n^*+m'}(n^*:=min\{n,m\}).
\end{aligned}
$$

We get

$$(k+a)\Big(\frac{B_k}{A_k}-\frac{B_{k'}}{A_{k'}}\Big)\equiv p^{m'}\Big(\frac{B_{k'}}{A_{k'}}-\frac{B_l}{A_l}\Big)\equiv 0\mod p^{n^*+m'}.$$

By assumption in Case II, we have $\mathrm{ord}_p(k+a) = n^*$. Hence we obtain our result

$$\frac{B_k}{A_k}\equiv\frac{B_{k'}}{A_{k'}}\mod p^{m'}.$$

Case III: $m' > m$ and $n = m$ with $\text{ord}_p(k+a) > m$

In this case, we write $a + l = cp^m, k = l + bp^m, k + a = (b+c)p^m = dp^{n+m}$ and $k' = k + p^{m'}$ with $m \geq 1, n \geq 1, p \not| bcd$. Then

$$\frac{B_{k'}}{A_{k'}} - \frac{B_k}{A_k} = \frac{-p^{m'} + (-1)^e \Big[(dp^{m+n} + p^{m'})\frac{A^{(1)}_{bp^{m-1}}}{A_{l+bp^m}} - (dp^{m+n})\frac{A^{(1)}_{bp^{m-1}+p^{m'-1}}}{A_{l+bp^m+p^{m'}}}\Big]}{(dp^{m+n} + p^{m'})dp^{m+n}}$$

We claim that the numerator is congruent to 0 modulo $(dp^{m+n} + p^{m'})p^{m+n} \cdot p^{m'}$. For simplicity, we denote the numerator by $(*)$. First, we consider

(1) For $m' \leq m + n$, we have that

$$\frac{A^{(1)}_{bp^{m-1}+p^{m'-1}}}{A_{l+bp^m+p^{m'}}} = \frac{\{1\}_{l+bp^m+p^{m'}}}{\{a\}_{l+bp^m+p^{m'}}}$$

and

$$\begin{aligned}
&\frac{\{1\}_{l+bp^m+p^{m'}}}{\{a\}_{l+bp^m+p^{m'}}} \\
=&\frac{\prod\limits_{\substack{1\leq i\leq l+bp^m+p^{m'}\\ p\not| i}} i}{\prod\limits_{\substack{1\leq i\leq l+bp^m+p^{m'}\\ p\not| i}} a+l+bp^m+p^{m'}-i} \\
=&\Bigg(\prod_{\substack{1\leq i\leq l+bp^m+p^{m'}\\ p\not| i}} \frac{1}{1-(dp^{m+n}+p^{m'})/i}\Bigg)(-1)^{l+bp^m+p^{m'}-\lfloor\frac{l+bp^m+p^{m'}}{p}\rfloor}
\end{aligned}$$

This is congruent to

$$\Bigg(1 + (dp^{m+n} + p^{m'}) \sum_{\substack{1\leq i\leq l+bp^m+p^{m'}\\ p\not| i}} \frac{1}{i}\Bigg)(-1)^{l+bp^{m-1}(p-1)}$$

modulo $(dp^{m+n} + p^{m'})p^{m'}$ since $m' \leq m + n$ and $p^{m'-1}(p-1) \equiv 0 \mod 2$. It equals

$$\Big[1 + (dp^{n+m} + p^{m'})(\psi_p(1 + l + bp^m + p^{m'}) + \gamma_p)\Big](-1)^{l+bp^{m-1}(p-1)}.$$

Therefore we have

$$\frac{A^{(1)}_{bp^{m-1}+p^{m'-1}}}{A_{l+bp^m+p^{m'}}} \equiv \Big[1{+}(dp^{n+m}{+}p^{m'})(\psi_p(1{+}l{+}bp^m{+}p^{m'}){+}\gamma_p)\Big](-1)^{l+bp^{m-1}(p-1)}$$

mod $(dp^{m+n}+p^{m'})p^{m'}$.

Similarly, we can derive

$$\frac{A^{(1)}_{bp^{m-1}}}{A_{l+bp^m}} \equiv \Big[1+dp^{n+m}(\psi_p(1+l+bp^m)+\gamma_p)\Big](-1)^{l+bp^{m-1}(p-1)} \mod (dp^{m+n})p^{m'}.$$

Hence, $(*)$ is congruent to

$$-p^{m'}+(-1)^{(e+l+bp^{m-1}(p-1))}\cdot\Bigg(p^{m'}+\Big(dp^{m+n}+p^{m'}\Big)dp^{m+n}\Big(\psi_p(1+l+bp^m) \\ -\psi_p(1+l+bp^m+p^{m'})\Big)\Bigg)$$

modulo $p^{m+n+m'}(dp^{m+n}+p^{m'})$.

Furthermore, since $m'\geq 2$ we have

$$\psi_p(1+l+bp^m)\equiv\psi_p(1+l+bp^m+p^{m'}) \mod p^{m'}.$$

Therefore, $(*)$ is congruent to

$$p^{m'}\Bigg(1-(-1)^{(e+l+bp^{m-1}(p-1))}\Bigg).$$

Now we discuss case by case.

• If p is odd, then $(-1)^{(e+l+bp^{m-1}(p-1))}=1$.

• If $p=2, m\geq 2$, then $(-1)^{(e+l+bp^{m-1}(p-1))}=(-1)^{(e+l)}=1$ ($e=l$ since $m\geq 2$).

• If $p=2, m=1$, then $(-1)^{(e+l+b)}=1$ since it can again be divided into the following two cases:

$$\begin{cases} l=0, l'=2, e=1, b\equiv 1 \mod 2 \\ l=1, l'=3, e=2, b\equiv 1 \mod 2. \end{cases}$$

Hence, $(*)$ is congruent to 0 modulo $p^{m+n+m'}(dp^{m+n}+p^{m'})$, and this implies

$$\frac{B_k}{A_k}\equiv\frac{B_{k'}}{A_{k'}} \mod p^{m'}.$$

We obtain the first case.

(2) For $m' > m+n$, we have

$$\begin{aligned}
&\frac{A^{(1)}_{bp^{m-1}+p^{m'-1}}}{A_{l+bp^m+p^{m'}}} \\
=&\frac{\{1\}_{l+bp^m+p^{m'}}}{\{a\}_{l+bp^m+p^{m'}}} \\
=&\frac{A^{(1)}_{bp^{m-1}}}{A_{l+bp^m}}\left(\frac{\{1+l+bp^m\}_{p^{m'}}}{\{a+l+bp^m\}_{p^{m'}}}\right) \\
=&\frac{A^{(1)}_{bp^{m-1}}}{A_{l+bp^m}}\left(\frac{\{1+l+bp^m\}_{p^{m'}}}{\{1\}_{p^{m'}}}\frac{\{1\}_{p^{m'}}}{\{a+l+bp^m\}_{p^{m'}}}\right) \\
\equiv&\frac{A^{(1)}_{bp^{m-1}}}{A_{l+bp^m}}\left(1+p^{m'}\left(\psi_p(1+l+bp^m)+\gamma_p\right)\right)\left(1-p^{m'}\left(\psi_p(a+l+bp^m)+\gamma_p\right)\right)
\end{aligned}$$

modulo $p^{2m'}$. Since $m' > m+n$ and $\psi_p(a+l+bp^m)+\gamma_p \equiv 0 \mod dp^{m+n}$, the formula above is congruent to

$$\frac{A^{(1)}_{bp^{m-1}}}{A_{l+bp^m}}\left(1+p^{m'}\left(\psi_p(1+l+bp^m)+\gamma_p\right)\right)$$

modulo $(dp^{m+n}+p^{m'})p^{m'}$.

Substituting this result into $(*)$, we have

$$(*) \equiv -p^{m'} + (-1)^e \frac{A^{(1)}_{bp^{m-1}}}{A_{l+bp^m}} p^{m'}\left(1-dp^{m+n}\left(\psi_p(1+l+bp^m)+\gamma_p\right)\right)$$

modulo $(dp^{m+n}+p^{m'})p^{m'}$.

Now we calculate the term $A^{(1)}_{bp^{m-1}}/A_{l+bp^m}$.

Since we have

$$
\begin{aligned}
&\frac{A^{(1)}_{bp^{m-1}}}{A_{l+bp^m}}\\
=&\frac{\{1\}_{l+bp^m}}{\{a\}_{l+bp^m}}\\
=&\prod_{\substack{1\le i\le l+bp^m\\ p\nmid i}}\left(\frac{i}{a+l+bp^m-i}\right)\\
=&\left(\prod_{\substack{1\le i\le l+bp^m\\ p\nmid i}}\frac{1}{1-dp^{m+n}/i}\right)(-1)^{l+bp^{m-1}(p-1)}\\
=&\left(\prod_{\substack{1\le i\le l+bp^m\\ p\nmid i}}\frac{1}{1-dp^{m+n}/i}\right)(-1)^{e}\quad\left((-1)^{l+bp^{m-1}(p-1)}=(-1)^e\ \forall p\right)\\
\equiv&(-1)^e\left(1+dp^{m+n}\left(\psi_p(1+l+bp^m)+\gamma_p\right)\right)\mod (p^{m+n})^2,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
(*)\equiv&-p^{m'}+p^{m'}\left(1+dp^{m+n}\left(\psi_p(1+l+bp^m)+\gamma_p\right)\right)\Bigg(1-\\
&dp^{m+n}\left(\psi_p(1+l+bp^m)+\gamma_p\right)\Bigg)\\
\equiv&0\mod (dp^{m+n}+p^{m'})p^{m+n}p^{m'}.
\end{aligned}
$$

Hence, again, we obtain

$$\frac{B_k}{A_k}\equiv\frac{B_{k'}}{A_{k'}}\mod p^{m'}.$$

Now we prove that $B_k/A_k\in W$. For $k\not\equiv l\mod p$, $B_k/A_k=1/(k+a)\in W$. For $k\equiv l\mod p$, it follows from $A_l\in\mathbb{Z}_p^\times$ and

$$\frac{B_k}{A_k}\equiv\frac{B_l}{A_l}\mod p.$$

□

3.2 Proof of Congruence Relations: End of the Proof

Here we follow the method used in the paper [1, §3.5] where M. Asakura prove the congruence relations of p-adic hypergeometric functions of logarithmic type.

Here we prove it in a more general setting.

Lemma 3.11. *Let $a \in \mathbb{Z}_p \backslash \mathbb{Z}_{\leq 0}$ and let $B_n \in \mathbb{Z}_p$ for $n \in \mathbb{Z}_{\geq 0}$. Suppose the map $n \mapsto B_n/A_n$ is a Lipschitz function. Put $B(t) := \sum_{n=0}^{\infty} B_n t^n$. Then there are Dwork's congruences for $B(t)/F_{a,\cdots,a}(t)$; that is*

$$\frac{B(t)}{F_{a,\cdots,a}(t)} \equiv \frac{B(t)_{<p^n}}{F_{a,\cdots,a}(t)_{<p^n}} \mod p^n W[[t]]$$

for all $n \in \mathbb{Z}_{\geq 0}$.

If we can prove this lemma, then by Lemma 3.5 we obtain the congruence relations for our hypergeometric functions.

Put $S_m := \sum_{i+j=m} A_{i+p^n} B_j - A_i B_{j+p^n}$ for $m \in \mathbb{Z}_{\geq 0}$. We claim that $S_m \equiv 0 \mod p^n$ for $n \in \mathbb{Z}_{\geq}$ since Lemma 3.11 is equivalent to for any $n, m \in \mathbb{Z}_{\geq 0}$, $S_m \equiv 0 \mod p^n$. First, we need the following lemmas.

Lemma 3.12. *We have*

$$S_m \equiv \sum_{i+j=m} (A_{i+p^n} A_j - A_i A_{j+p^n}) \frac{B_j}{A_j} \mod p^n.$$

Proof. We compute

$$\begin{aligned} S_m &= \sum_{i+j=m} A_{i+p^n} B_j - A_i A_{j+p^n} \frac{B_{j+p^n}}{A_{j+p^n}} \\ &\equiv \sum_{i+j=m} A_{i+p^n} B_j - A_i A_{j+p^n} \frac{B_j}{A_j} \mod p^n \quad \text{(Lipschitz)} \\ &= \sum_{i+j=m} (A_{i+p^n} A_j - A_i A_{j+p^n}) \frac{B_j}{A_j}. \end{aligned}$$

□

Lemma 3.13. *We have*

$$S_m \equiv \sum_{i+j=m} (A^{(1)}_{\lfloor j/p \rfloor} A^{(1)}_{\lfloor i/p \rfloor + p^{n-1}} - A^{(1)}_{\lfloor i/p \rfloor} A^{(1)}_{\lfloor j/p \rfloor + p^{n-1}}) \frac{A_i}{A^{(1)}_{\lfloor i/p \rfloor}} \frac{A_j}{A^{(1)}_{\lfloor j/p \rfloor}} \frac{B_j}{A_j} \mod p^n.$$

Proof. By Lemma 3.12, we know that

$$\begin{aligned} S_m &\equiv \sum_{i+j=m} (A_{i+p^n} A_j - A_i A_{j+p^n}) \frac{B_j}{A_j} \mod p^n \\ &= \sum_{i+j=m} \left(\frac{A_{i+p^n} A^{(1)}_{\lfloor i/p \rfloor} A^{(1)}_{\lfloor j/p \rfloor}}{A_i} - \frac{A_{j+p^n} A^{(1)}_{\lfloor i/p \rfloor} A^{(1)}_{\lfloor j/p \rfloor}}{A_j} \right) \frac{A_i}{A^{(1)}_{\lfloor i/p \rfloor}} \frac{A_j}{A^{(1)}_{\lfloor j/p \rfloor}} \frac{B_j}{A_j}. \end{aligned}$$

By [1, Lemma 3.8], we have

$$\frac{A_{i+p^n}A^{(1)}_{\lfloor i/p \rfloor}}{A_i} \equiv A^{(1)}_{\lfloor i/p \rfloor + p^{n-1}}, \quad \frac{A_{j+p^n}A^{(1)}_{\lfloor j/p \rfloor}}{A_j} \equiv A^{(1)}_{\lfloor j/p \rfloor + p^{n-1}} \mod p^n.$$

Since $A_i/A^{(1)}_{\lfloor i/p \rfloor}, A_j/A^{(1)}_{\lfloor j/p \rfloor}$ and B_j/A_j are all in W ([1, Lemma 3.8] and assumption), we have S_m is congruent to

$$\sum_{i+j=m} (A^{(1)}_{\lfloor j/p \rfloor} A^{(1)}_{\lfloor i/p \rfloor + p^{n-1}} - A^{(1)}_{\lfloor i/p \rfloor} A^{(1)}_{\lfloor j/p \rfloor + p^{n-1}}) \frac{A_i}{A^{(1)}_{\lfloor i/p \rfloor}} \frac{A_j}{A^{(1)}_{\lfloor j/p \rfloor}} \frac{B_j}{A_j} \mod p^n.$$

□

Now we start the proof of Lemma 3.11.

(proof of Lemma 3.11). Put

$$q_i := \frac{A_i}{A^{(1)}_{\lfloor i/p \rfloor}}, \quad A(i,j) := A^{(1)}_i A^{(1)}_j, \quad A^*(i,j) := A(j, i+p^{n-1}) - A(i, j+p^{n-1})$$

$$B(i,j) := A^*(\lfloor i/p \rfloor, \lfloor j/p \rfloor).$$

Put $m = pt + l$ with $l \in \{0, 1, \cdots, p-1\}$. Then we have

$$\begin{aligned}
S_m &\equiv \sum_{i+j=m} B(i,j) q_i q_j \frac{B_j}{A_j} \mod p^n \\
&= \sum_{i=0}^{p-1} \sum_{k=0}^{\lfloor (m-i)/p \rfloor} B(i+kp, m-(i+kp)) q_{i+kp} q_{m-(i+kp)} \frac{B_{m-(i+kp)}}{A_{m-(i+kp)}} \\
&= \sum_{k=0}^{t} \sum_{i=0}^{l} B(i+kp, m-(i+kp)) q_{i+kp} q_{m-(i+kp)} \frac{B_{m-(i+kp)}}{A_{m-(i+kp)}} \\
&\quad + \sum_{k=0}^{t-1} \sum_{i=l+1}^{p-1} B(i+kp, m-(i+kp)) q_{i+kp} q_{m-(i+kp)} \frac{B_{m-(i+kp)}}{A_{m-(i+kp)}} \\
&= \sum_{k=0}^{t} A^*(k, t-k) \overbrace{\left(\sum_{i=0}^{l} q_{i+kp} q_{m-(i+kp)} \frac{B_{m-(i+kp)}}{A_{m-(i+kp)}} \right)}^{P_k} \\
&\quad + \sum_{k=0}^{t-1} A^*(k, t-k-1) \underbrace{\left(\sum_{i=l+1}^{p-1} q_{i+kp} q_{m-(i+kp)} \frac{B_{m-(i+kp)}}{A_{m-(i+kp)}} \right)}_{Q_k}.
\end{aligned}$$

We show that the first term vanishes modulo p^n.

It follows from the assumption and [1, Lemma 3.7] that $B_k/A_k, q_k \in W$ for all $k \in \mathbb{Z}_{\geq 0}$ and

$$k \equiv k' \mod p^i \quad \Longrightarrow \frac{B_k}{A_k} \equiv \frac{B_{k'}}{A_{k'}}, q_k \equiv q_{k'} \mod p^i.$$

Therefore, we have

$$k \equiv k' \mod p^i \quad \Longrightarrow \quad P_k \equiv P_{k'} \mod p^{i+1}. \tag{3.1}$$

Then one can write

$$\sum_{k=0}^{s} A^*(k, s-k) P_k \equiv \sum_{i=0}^{p^{n-1}-1} P_i \overbrace{\left(\sum_{k \equiv i \text{ mod } p^{n-1}} A^*(k, s-k) \right)}^{(*)} \mod p^n.$$

Let us recall Lemma 3.12 in [1].

Lemma 3.14. *For all $m, k, s \in \mathbb{Z}_{\geq 0}$ and $0 \leq l \leq n$, then*

$$\sum_{\substack{i+j=m \\ i \equiv k \text{ mod } p^{n-l}}} A_i A_{j+p^{n-1}} - A_j A_{i+p^{n-1}} \equiv 0 \mod p^l.$$

Using this lemma, we obtain $(*)$ is 0 modulo p. Hence, by (3.1) again, we can write

$$\sum_{k=0}^{s} A^*(k, s-k) P_k \equiv \sum_{i=0}^{p^{n-2}-1} P_i \overbrace{\left(\sum_{k \equiv i \text{ mod } p^{n-2}} A^*(k, s-k) \right)}^{(**)} \mod p^n.$$

It follows from Lemma 3.9 that $(**)$ is 0 modulo p^2, so

$$\sum_{k=0}^{s} A^*(k, s-k) P_k \equiv \sum_{i=0}^{p^{n-3}-1} P_i \left(\sum_{k \equiv i \text{ mod } p^{n-3}} A^*(k, s-k) \right) \mod p^n.$$

Continuing the same discussion, we have

$$\sum_{k=0}^{s} A^*(k, s-k) P_k \equiv \sum_{k=0}^{s} A^*(k, s-k) P_0 = 0 \mod p^n.$$

Similarly, we can show the vanishing of the second term

$$\sum_{k=0}^{s-1} A^*(k, s-k) Q_k \equiv 0 \mod p^n.$$

Hence $S_m \equiv 0 \mod p^n$. This is the end of the proof. □

4 Transformation Formulas

In this section, we will introduce two conjectures and prove these conjectures in a particular case. The first one is "Transformation Formulas between p-adic hypergeometric functions $\widehat{\mathscr{F}}^{(\sigma)}_{a,\ldots,a}(t)$ and p-adic hypergeometric functions of logarithmic type" which will be discussed in Section 4.3. The second one is "Transformation formula on Dwork's p-adic Hypergeometric Functions" which is discussed in Section 4.4.

4.1 Hypergeometric Curves and Hypergeometric Curves of Gauss Type

The main reference of this section is [1, §4.1, §4.2, §4.6]

4.1.1 Hypergeometric Curves

Let $W = W(\overline{\mathbb{F}}_p)$ be the Witt ring of $\overline{\mathbb{F}}_p$ and $R = W[t, (t-t^2)^{-1}]$. Let $N \geq 2$ be an integer and a prime $p > N$. We denote by $\mathbb{P}^1_R(Z_0, Z_1)$ the projective line over R with homogeneous coordinate (Z_0, Z_1). Then we define U to be

$$\mathrm{Spec}R[u,v]/((1-u^N)(1-v^N)-t) \subset Y := \overline{U}$$

where Y is the closure in $\mathbb{P}^1_R \times \mathbb{P}^1_R$. It is called a *hypergeometric curve* over R. The morphism $Y \to \mathrm{Spec}R$ is smooth and projective with connected fibers of relative dimension one and the genus of a geometric fiber is $(N-1)^2$.

Lemma 4.1. *There is a morphism*

$$\overline{f} : \overline{Y} \to \mathbb{P}^1_W = \mathbb{P}^1_W(T_0, T_1)$$

of smooth projective W-schemes satisfying

(1) *Let $S := \mathrm{Spec}R = \mathrm{Spec}W[t,(t-t^2)^{-1}] \subset \mathbb{P}^1_W$ with $t := T_1/T_0$. Then $Y = \overline{f}^{-1}(S) \to S$ is the hypergeometric curve.*

(2) *The morphism $\overline{f}$ has a semistable reduction at $t = 0$. The fiber $D := \overline{f}^{-1}(t=0)$ is a relative simple NCD, and the multiplicities of the components are one.*

Proof. This is [1, Lemma 4.1]. □

Let $K = \mathrm{Frac}W(\overline{\mathbb{F}}_p)$. For a W-scheme Z and a W-algebra R, we write $Z_R = Z \times_W R$. The group $\mu_N(K) \times \mu_N(K)$ acts on Y in the following way

$$[\zeta, \nu] \cdot (x, y, t) = (\zeta x, \nu y, t), \quad (\zeta, \nu) \in \mu_N \times \mu_N.$$

One has the eigen decomposition

$$H^1_{\mathrm{dR}}(Y_K/S_K) = \bigoplus_{i=1}^{N-1}\bigoplus_{j=1}^{N-1} H^1_{\mathrm{dR}}(Y_K/S_K)(i,j)$$

where $H^1_{\mathrm{dR}}(Y_K/S_K)(i,j)$ denotes the subspace on which (ζ,ν) acts by multiplication by $\zeta^i\nu^j$ for all (ζ,ν). Then each eigenspace $H^1_{\mathrm{dR}}(Y_K/S_K)(i,j)$ is free of rank 2 over $\mathcal{O}(S_K)$. Put

$$a_i := 1 - \frac{i}{N}, \quad b_j := 1 - \frac{j}{N}, \tag{4.1}$$

$$\omega_{i,j} := N\frac{x^{i-1}j^{j-N}}{1-x^N}dx = -N\frac{x^{i-N}y^{j-1}}{1-y^N}dy, \tag{4.2}$$

and

$$\eta_{i,j} := \frac{1}{x^N - 1 + t}\omega_{i,j} = Nt^{-1}x^{i-N}y^{j-N-1}dy \tag{4.3}$$

for integers i, j such that $1 \le i, j \le N-1$. Then they forms a $\mathcal{O}(S_K)$-free basis of $H^1_{\mathrm{dR}}(Y_K/S_K)(i,j)$ ([2, Lemma 2.2]).

Let

$$F_{a,b}(t) = \sum_{i=0}^{\infty}\frac{(a)_i}{i!}\frac{(b)_i}{i!}t^i \in K[[t]]$$

be the hypergeometric series. Put

$$\tilde{\omega}_{i,j} := \frac{1}{F_{a_i,b_j}(t)}\omega_{i,j}, \quad \tilde{\eta}_{i,j} := -t(1-t)^{a_i+b_j}(F'_{a_i,b_j}(t)\omega_{i,j} + b_jF_{a_i,b_j}(t)\eta_{i,j}). \tag{4.4}$$

Lemma 4.2. *Let $\nabla_{i,j}$ be the connection on the eigen component $H_{i,j} := K((t)) \otimes_{\mathcal{O}_S} H^1_{\mathrm{dR}}(Y_K/S_K)(i,j)$. Then $\mathrm{Ker}\nabla_{i,j} = K\widetilde{\eta}_{i,j}$. Moreover let $\overline{\nabla}_{i,j}$ be the connection on $H_{i,j}/K((t))\widetilde{\eta}_{i,j}$ induced from $\nabla_{i,j}$. Then $\mathrm{Ker}\overline{\nabla}_{i,j} = K\widetilde{\omega}_{i,j}$.*

Proof. See [1, Proposition 4.3, Corollary 4.4]. □

Let $(i,j) = (q/r, q'/r') \in \mathbb{Q}^2$ such that $\gcd(r,N) = \gcd(r',N) = 1$ and q, q' are not divided by N. Then we define

$$H^1_{\mathrm{dR}}(Y_K/S_K)(i,j) = H^1_{\mathrm{dR}}(Y_K/S_K)(i_0,j_0), \quad \omega_{i,j} = \omega_{i_0,j_0}, \cdots, \widetilde{\eta}_{i,j} = \widetilde{\eta}_{i_0,j_0} \tag{4.5}$$

where i_0, j_0 are the unique integers such that $i_0 \equiv i \mod N$, $j_0 \equiv j \mod N$ and $1 \le i_0, j_0 < N$.

4.1.2 Hypergeometric Curves of Gauss Type

Let A, B be integers such that $1 \leq A, B < N$ and $\gcd(A,N)$=$\gcd(B,N)$=1.

Let $f : Y \to \mathbb{P}^1$ be a fibration over K ($K = \mathrm{Frac}W(\overline{\mathbb{F}}_p)$) whose general fiber $Y_t = f^{-1}(t)$ is the projective nonsingular curve associated to the affine curve

$$y^N = x^A(1-x)^B(1-(1-t)x)^{N-B}.$$

We call X a hypergeometric curve of Gauss type. This is a fibration of curves of genus $N-1$, smooth outside $t = 0, 1, \infty$. Put $S_K := \mathrm{Spec}K[t, (t-t^2)^{-1}]$ and $X_0 := f^{-1}(S_K)$.

Let $[\zeta] : X_0 \to X_0$ be the automorphism given by

$$[\zeta](x, y, t) = (x, \zeta^{-1}y, t)$$

for any N-th root of unity $\zeta \in \mu_N = \mu_N(K)$. We denote

$$H^1_{\mathrm{dR}}(X_0/S_K)(n) := \{x \in H^1_{\mathrm{dR}}(X_0/S_K) \mid [\zeta]x = \zeta^n x, \forall \zeta \in \mu_N\}.$$

Then one has the eigen decomposition

$$H^1_{\mathrm{dR}}(X_0/S_K) = \bigoplus_{n=1}^{N-1} H^1_{\mathrm{dR}}(X_0/S_K)(n)$$

of $\mathscr{O}(S_K)$-module and each eigen space is free of rank 2.

A basis of $H^1_{\mathrm{dR}}(X_0/S_K)(n)$ ([2, Lemma 2.5]) is given by

$$\omega_n := x^{A_n}(1-x)^{B_n}(1-(1-t)x)^{n-1-B_n}\frac{dx}{y^n}, \quad \eta_n := \frac{x}{1-(1-t)x}\omega_n$$

where

$$A_n := \lfloor \frac{nA}{N} \rfloor, \quad B_n := \lfloor \frac{nB}{N} \rfloor.$$

Let

$$a_n := \left\{ \frac{-nA}{N} \right\}, b_n := \left\{ \frac{-nB}{N} \right\}$$

where $\{x\} := x - \lfloor x \rfloor$. Put

$$\widetilde{\omega}_n := \frac{1}{F_{a_n,b_n}(t)}\omega_n, \quad \widetilde{\eta}_n := -t(1-t)^{a_n+b_n}(F'_{a_n,b_n}(t)\omega_n + b_j F_{a_n,b_n}(t)\eta_n). \tag{4.6}$$

which form a $K((t))$-basis of $K((t)) \otimes H^1_{\mathrm{dR}}(X_0/S_K)$.

Let Y/S be the hypergeometric curve in §4.1.1. Then we have the following lemma.

Lemma 4.3. ([*1*, §4.6])*Let* $g_{A,B}$ *be the automorphism of* Y_K/S_K *given by* $g_{A,B} : (u, v) \mapsto (\zeta_N^B u, \zeta_N^{-A} v)$ *for a fixed primitive* N*-root of unity in* $\mu_N(K)$*. Let* $G = \langle g_{A,B} \rangle \subset \mathrm{Aut}(Y_K/S_K)$ *be the cyclic group of order* N*. Then the quotient* $Y_G := Y_K/G$ *by* G *is isomorphic to the curve* $X : y^N = x^A(1 - x)^B(1 - (1 - t)x)^{N-B}$*. The quotient map* $\rho : Y_K \to X$ *is given by*

$$(u, v) \longmapsto (x, y) = (u^{-N}, u^{-A}(1 - u^{-N})v^{N-B})$$

which is a S*-morphism (i.e.* $t \mapsto t$*).*

Put

$$\widehat{\omega}_n = \sum_{g\in G} g^*\omega_{nA,nB}, \quad \widehat{\eta}_n = \sum_{g\in G} g^*\eta_{nA,nB}.$$

Lemma 4.4. *Using the notation above, we have*

$$\omega_{nA,nB} = \rho^*(\omega_n), \quad \eta_{nA,nB} = \rho^*(\eta_n) \tag{4.7}$$

$$\widetilde{\omega}_{nA,nB} = \rho^*(\widetilde{\omega}_n), \quad \widetilde{\eta}_{nA,nB} = \rho^*(\widetilde{\eta}_n) \tag{4.8}$$

$$\widehat{\omega}_n = N\rho^*(\omega_n), \quad \widehat{\eta}_n = N\rho^*(\eta_n). \tag{4.9}$$

Then we see that the pull-back ρ^* satisfies

$$\rho^*(H^1_{\mathrm{dR}}(X_0/S_K)(n)) = H^1_{\mathrm{dR}}(Y_K/S_K)(nA, nB), \quad 0 < n < N \tag{4.10}$$

and the push-forward ρ_* satisfies

$$\rho_*(H^1_{\mathrm{dR}}(Y_K/S_K)(i, j)) = \begin{cases} H^1_{\mathrm{dR}}(X_0/S_K)(n) & (i, j) \equiv (nA, nB) \mod N \\ 0 & \text{otherwise} \end{cases} \tag{4.11}$$

for $0 < i, j < N$. One has

$$\rho_*(\omega_{nA,nB}), \quad \rho_*(\eta_{nA,nB}) \tag{4.12}$$

form a basis of $H^1_{\mathrm{dR}}(X_0/S_K)(n)$ and

$$\rho^*\rho_*(\omega_{nA,nB}) = N\omega_{nA,nB}.$$

from (4.10), (4.11) and $\rho_*\rho^* = N$.

4.2 Review of paper [1]

In this part, we will give a brief review of what we need in the paper [1]. First, let us recall the definition of p-adic hypergeometric functions of logarithmic type.

Definition 4.5 (**p-adic hypergeometric functions of logarithmic type**). *Let $s \geq 1$ be a positive integer. Let $(a_1, \cdots, a_s) \in \mathbb{Z}_p$ and $(a_1', \cdots, a_s')$ where a_i' denotes the Dwork prime. Let $\sigma : W[[t]] \to W[[t]]$ be the p-th Frobenius endomorphism given by $\sigma(t) = ct^p$ with $c \in 1 + pW$. Then we define the p-adic hypergeometric functions of logarithmic type $\mathscr{F}^{(\sigma)}_{a_1,\cdots,a_s}(t)$ to be*

$$\frac{1}{F_{a_1,\cdots,a_s}(t)}\Bigg[\psi_p(a_1) + \cdots + \psi_p(a_s) + s\gamma_p - p^{-1}\log(c) + \int_0^t (F_{a_1,\cdots,a_s}(t) - F_{a_1',\cdots,a_s'}(t^\sigma))\frac{dt}{t}\Bigg]$$

where $\psi_p(z)$ is the p-adic digamma function([1, §2.2]), γ_p is the p-adic Euler constant ([1, §2.2]) and $\log(z)$ is the Iwasawa logarithmic function.

Definition 4.6 (cf. [6, 3.1.1]). *Let $W = W(\overline{k})$ where $\overline{k}$ is an algebraic closed field with $\mathrm{char}\overline{k} > 0$. Define*

$$\begin{aligned} W\langle t_1, \cdots, t_n\rangle &:= \varprojlim_n W/p^n[t_1, \cdots, t_n] \\ &= \left\{ \sum_I a_I t^I \in W[[t_1, \cdots, t_n]] \;\middle|\; |a_I|_p \to 0 \textit{ as } |I| \to \infty \right\} \end{aligned}$$

$$W[t_1, \cdots, t_n]^\dagger := \left\{ \sum_I a_I t^I \in W[[t_1, \cdots, t_n]] \;\middle|\; \exists r > 1 \textit{ such that } |a_I|_p r^{|I|} \to 0 \textit{ as } |I| \to \infty \right\}.$$

Furthermore, if

$$A = W[t_1, \cdots, t_n]/(f_1, \cdots, f_r),$$

we define the weak completion of A to be

$$A^\dagger := W[t_1, \cdots, t_n]^\dagger/(f_1, \cdots, f_r).$$

Definition 4.7. *Let $F : W \to W$ be the p-th Frobenius. Then $\sigma : A^\dagger \to A^\dagger$ is called a p-th Frobenius if*

- *σ is F-linear ($\sigma(\alpha x) = F(\alpha)\sigma(x), \alpha \in W, x \in A^\dagger$)*
- *$\sigma \mod p$ on $A^\dagger/pA^\dagger \simeq A/pA$ is given by $x \to x^p$.*

Let σ be a p-th Frobenius on $W[t, (t - t^2)^{-1}]^\dagger$ which extends on $K[t, (t - t^2)^{-1}]^\dagger := K \otimes W[t, (t - t^2)^{-1}]^\dagger$. Write $X_{\overline{\mathbb{F}}_p} := X_W \times_W \overline{\mathbb{F}}_p$ and $S_{\overline{\mathbb{F}}_p} := S_W \times_W \overline{\mathbb{F}}_p$. Then the *rigid cohomology* groups

$$H^\bullet_{\mathrm{rig}}(X_{\overline{\mathbb{F}}_p}/S_{\overline{\mathbb{F}}_p})$$

are defined. We refer the book [7] for the theory of rigid cohomology.

The required properties are the following.

- The cohomology group $H^\bullet_{\mathrm{rig}}(X_{\overline{\mathbb{F}}_p}/S_{\overline{\mathbb{F}}_p})$ is a finitely generated $\mathscr{O}(S_K)^\dagger$-module.
- (Frobenius) The p-th Frobenius Φ on $H^\bullet_{\mathrm{rig}}(X_{\overline{\mathbb{F}}_p}/S_{\overline{\mathbb{F}}_p})$(depending on σ) is defined. This is a σ-linear endomorphism:

$$\Phi(f(t)x) = \sigma(f(t))\Phi(x), \text{for } x \in H^\bullet_{\mathrm{rig}}(X_{\overline{\mathbb{F}}_p}/S_{\overline{\mathbb{F}}_p}), f(t) \in \mathscr{O}(S_K)^\dagger.$$

- (Comparison) There is the comparison isomorphism with algebraic de Rham cohomology,

$$c : H^\bullet_{\mathrm{rig}}(X_{\overline{\mathbb{F}}_p}/S_{\overline{\mathbb{F}}_p}) \cong H^\bullet_{\mathrm{dR}}(X_0/S_K) \otimes_{\mathscr{O}(S_K)} \mathscr{O}(S_K)^\dagger.$$

Theorem 4.8. *Let σ be a p-Frobenius on $\mathscr{O}(S_K)^\dagger$ such that $\sigma(t) = ct^p$ with $c \in 1 + pW$. Then there exists an exact sequence*

$$0 \longrightarrow \mathscr{O}(S_K)^\dagger \otimes_{\mathscr{O}(S_K)} H^1_{\mathrm{dR}}(X_0/S_K) \longrightarrow \mathscr{O}(S_K)^\dagger \otimes_{\mathscr{O}(S_K)} M_\xi(X_0/S_K) \longrightarrow \mathscr{O}(S_K)^\dagger \longrightarrow 0$$

endowed with

- *Frobenius Φ-action which is σ-linear*
- $\mathrm{Fil}^i \subseteq M_\xi(X_0/S_K)$ (Hodge filtration) *with*

$$\mathscr{O}(S_K)^\dagger \otimes_{\mathscr{O}(S_K)} \mathrm{Fil}^0 M_\xi(X_0/S_K) \xrightarrow{\sim} \mathscr{O}(S_K)^\dagger$$

  In particular, from this isomorphism, there exists a unique lifting e_ξ in $\mathscr{O}(S_K)^\dagger \otimes_{\mathscr{O}(S_K)} \mathrm{Fil}^0 M_\xi(X_0/S_K)$ of $1 \in \mathscr{O}(S_K)^\dagger$.

Proof. Recall $U = \mathrm{Spec} R[u,v]/((1-u^N)(1-v^N)-t) \subset Y := \overline{U}$ where Y is the closure in $\mathbb{P}^1_R \times \mathbb{P}^1_R$. For $(\nu_1, \nu_2) \in \mu_N(W) \times \mu_N(W)$, put

$$\xi(\nu_1, \nu_2) = \left\{ \frac{u-1}{u-\nu_2}, \frac{v-1}{v-\nu_2} \right\} \in K_2^M(\mathcal{O}(U)).$$

Then according to [3, §2.6], we have the 1-extension

$$0 \longrightarrow H^1(Y/S)(2) \longrightarrow M_{\xi(\nu_1,\nu_2)}(Y/S) \longrightarrow \mathcal{O}_S \longrightarrow 0 \tag{4.13}$$

in the category of Fil-F-MIC(S, σ).

Let $\xi := \sum_{i=0}^{N-1} (g^i_{A,B})^* \xi(\nu_1, \nu_2) \in K_2^M(\mathcal{O}(U))$ which is fixed under the action of G, so that G acts on $M_\xi(Y/S)$. Taking the fixed part of (4.13) by $\langle g_{A,B} \rangle$, we have a 1-extension

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(Y/S)^G(2) & \longrightarrow & M_\xi(Y/S)^G & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0 \\ & & \| & & \| & & & & \\ & & H & & M_\xi & & & & \end{array}$$

By Lemma 4.3, it follows

- $H_{\mathrm{dR}} \simeq H^1_{\mathrm{dR}}(X_0/S_K)$
- $H_{\mathrm{rig}} \simeq \mathcal{O}(S_K)^\dagger \otimes H^1_{\mathrm{dR}}(X_0/S_K)$
- $\mathrm{Fil}^i H_{\mathrm{dR}} = \mathrm{Fil}^{i+2} H^1_{\mathrm{dR}}(X_0/S_K)$ where $\mathrm{Fil}^\bullet$ in the right denotes the Hodge filtration. In particular, $\mathrm{Fil}^0 H_{\mathrm{dR}} = 0$.

Therefore there is a unique element $e_\xi \in \mathrm{Fil}^0 M_\xi$ which is a lifting of $1 \in \mathcal{O}(S_K)$. □

Let $1 \le n \le N-1$ be an integer and A, B be integers such that $1 \le A, B < N$ and $\gcd(A,N)$=$\gcd(B,N)$=1.

Put

$$a_n := \left\{ \frac{-nA}{N} \right\}, \quad b_n := \left\{ \frac{-nB}{N} \right\}$$

where $\{x\} := x - \lfloor x \rfloor$ denotes the fractional part. Let

$$F_n(t) := \sum_{i=0}^{\infty} \frac{(a_n)_i}{i!} \frac{(b_n)_i}{i!} t^i \in \mathbb{Z}_p[[t]]$$

be the hypergeometric power series. Put

$$e^{\mathrm{unit}}_{i,j} := (1-t)^{-a_i - b_j} F_{a_i, b_j}(t)^{-1} \widetilde{\eta}_{i,j}.$$

Using the group G in Lemma 4.3, we define

$$\widehat{\omega}_n = \sum_{g \in G} g^* \omega_{nA,nB}, \quad e^{\mathrm{unit}}_n := \sum_{g \in G} g^* e^{\mathrm{unit}}_{nA,nB}.$$

Let e_ξ be the unique lifting of 1 in the above theorem.

Lemma 4.9.

$$\nabla(e_\xi) = -\sum_{n=1}^{N-1} \frac{(1-\nu_1^{-nA})(1-\nu_2^{-nB})}{N^2} \frac{dt}{t} \widehat{\omega}_n.$$

Proof. Let $\xi := \sum_{i=0}^{N-1} g^i_{A,B} \xi(\nu_1, \nu_2)$ with

$$\xi(\nu_1, \nu_2) = \left\{ \frac{x-1}{x-\nu_1}, \frac{y-1}{y-\nu_2} \right\}. \tag{4.14}$$

Following from the compatibility of the connection with respect to the pull-back, we have

$$\nabla(e_\xi) = \sum_{i=0}^{N-1} (g^*_{A,B})^i \nabla(e_{\xi(\nu_1,\nu_2)}).$$

Then by [1, §4.4 (4.25)], we have

$$\nabla(e_{\xi(\nu_1,\nu_2)}) = -d\log(\xi(\nu_1,\nu_2)) = -\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\frac{(1-\nu_1^{-i})(1-\nu_2^{-j})}{N^2}\frac{dt}{t}\omega_{i,j}.$$

So we have

$$\begin{aligned}\nabla(e_\xi) &= \sum_{i=0}^{N-1}(g_{A,B}^*)^i\nabla(e_{\xi(\nu_1,\nu_2)})\\ &= -\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\frac{(1-\nu_1^{-i})(1-\nu_2^{-j})}{N^2}\frac{dt}{t}\sum_{g\in G}g^*\omega_{i,j}\\ &= -\sum_{n=1}^{N-1}\frac{(1-\nu_1^{-nA})(1-\nu_2^{-nB})}{N^2}\frac{dt}{t}\sum_{g\in G}g^*\omega_{nA,nB}\\ &= -\sum_{n=1}^{N-1}\frac{(1-\nu_1^{-nA})(1-\nu_2^{-nB})}{N^2}\frac{dt}{t}\widehat{\omega}_n.\end{aligned}$$

□

Lemma 4.10. *Assume σ is given by $\sigma(t) = ct^p$ with $c \in 1+pW$. Let $h(t) = \prod_{m=0}^{s} F_{a_n^{(m)},b_n^{(m)}}(t)_{<p}$ where s is the minimal integer such that $(a_n^{(s+1)}, b_n^{(s+1)}) = (a_n, b_n)$ for all $n \in \{1, 2, \cdots, N-1\}$. Then*

$$e_\xi - \Phi(e_\xi) \equiv -\sum_{n=1}^{N-1}\frac{(1-\nu_1^{-nA})(1-\nu_2^{-nB})}{N^2}\mathscr{F}^{(\sigma)}_{a_n,b_n}(t)\widehat{\omega}_n$$

modulo $\sum_{n=1}^{N-1} K\langle t, (t-t^2)^{-1}, h(t)^{-1}\rangle e_n^{\mathrm{unit}}$.

Proof. See [1, Theorem 4.19]. □

4.3 Transformation Formula

Let $a_i \in \mathbb{Z}_p (0 \le i \le r-1)$ and put $h(t) := \prod_{i=0}^{r-1} F_{a_i,\cdots,a_i}(t)_{<p}$, where $F_{a_i,\cdots,a_i}(t)$ is hypergeometric power series. Then there is an involution

$$\omega : W\langle t, t^{-1}, h(t)^{-1}\rangle \longrightarrow W\langle t, t^{-1}, h(t)^{-1}\rangle, \quad \omega(f(t)) = f(t^{-1}).$$

This follows from the following proposition.

Proposition 4.11. (1) *Let $a \in \mathbb{Z}_p$. Let $F(t) := F_{a,\cdots,a}(t)_{<p} \mod p$. Then $F(t) = (-1)^{ls}t^l F(t^{-1})$ in $\mathbb{F}_p[t]$ where l is the degree of $F(t)$ which equals the unique integer in $\{0, 1, \cdots, p-1\}$ such that $a + l \equiv 0 \mod p$.*

(2) *Let* $a_i \in \mathbb{Z}_p (0 \le i \le r-1)$ *and put* $h(t) := \prod_{i=0}^{r-1} F_{a_i,\cdots,a_i}(t)_{<p}$. *Then there is a ring homomorphism*

$$\omega_n : W/p^n[t, t^{-1}, h(t)^{-1}] \to W/p^n[t, t^{-1}, h(t)^{-1}], \quad f(t) \mapsto f(t^{-1}).$$

(3) *There is an involution*

$$\omega : W\langle t, t^{-1}, h(t)^{-1}\rangle \longrightarrow W\langle t, t^{-1}, h(t)^{-1}\rangle, \quad \omega(f(t)) = f(t^{-1}).$$

Proof. (1)Write

$$F(t) = \sum_{i=0}^{l} \left(\frac{(a)_i}{i!}\right)^s t^i.$$

Since $F(t) = F_{a,\cdots,a}(t)_{<p} \mod p$, we have $(a)_l \not\equiv 0 \mod p$ and $(a)_{l+1} \equiv 0 \mod p$. That is, we have $a + l \equiv 0 \mod p$.

If $l = i + j$, then

$$\frac{(a)_i}{i!} \equiv \frac{(-l)_i}{i!} = (-1)^i \binom{l}{i} = (-1)^i \binom{l}{j} = (-1)^l \frac{(-l)_j}{j!} \equiv (-1)^l \frac{(a)_j}{j!} \mod p.$$

Therefore, we have

$$\left(\frac{(a)_i}{i!}\right)^s \equiv (-1)^{ls} \left(\frac{(a)_j}{j!}\right)^s \mod p.$$

This implies

$$t^l F(t^{-1}) = (-1)^{ls} F(t).$$

(2) Observe that it is enough to show that

$$h(t^{-1}) \in (W/p^n[t, t^{-1}, h(t)^{-1}])^\times.$$

An element is a unit in $W\langle t, t^{-1}, h(t)^{-1}\rangle$ if and only if it is a unit modulo $pW[t, t^{-1}, h(t)^{-1}]$. So if we can prove $h(t^{-1})$ is a unit modulo $pW[t, t^{-1}, h(t)^{-1}]$, then we are done. Put

$$F_i(t) := F_{a_i,\cdots,a_i}(t)_{<p} \mod p.$$

From (1), we have

$$F_i(t) = \pm t^{l_i} F_i(t^{-1})$$

where l_i is the degree of $F_i(t)$. Hence we have

$$F_i(t^{-1})^{-1} = \frac{\pm t^{l_i}}{F_i(t)}$$

in $W/p[t, t^{-1}, h(t)^{-1}]$, i.e., $F_i(t)$ is a unit in $W/p[t, t^{-1}, h(t)^{-1}]$. Since $h(t)$ is the product of $F_i(t)$, it is also a unit.

(3) ω is defined by using ω_n in (2) in the following way:

$$\omega : W\langle t, t^{-1}, h(t)^{-1}\rangle \longrightarrow W\langle t, t^{-1}, h(t)^{-1}\rangle, \quad (f_n(t)) \mapsto (\omega_n f_n(t)).$$

By (2), ω is well-defined and also it is an involution. $\square$

Conjecture 4.12 (**Transformation Formula between $\mathscr{F}^{(\sigma)}_{a,\cdots,a}(t)$ and $\widehat{\mathscr{F}}^{(\widehat{\sigma})}_{a,\cdots,a}(t)$**)**.** *Let $\sigma(t) = ct^p$ and $\widehat{\sigma}(t) = c^{-1}t^p$. Let $a \in \mathbb{Z}_p\backslash\mathbb{Z}_{\leq 0}$ and the rth Dwork prime $a^{(r)} = a$ for some $r > 0$. Put $h(t) := \prod_{i=0}^{r-1} F_{a^{(i)},\cdots,a^{(i)}}(t)_{<p}$, then*

$$\mathscr{F}^{(\sigma)}_{a,\cdots,a}(t) = -\widehat{\mathscr{F}}^{(\widehat{\sigma})}_{a,\cdots,a}(t^{-1})$$

in the ring $W\langle t, t^{-1}, h(t)^{-1}\rangle$ where $\widehat{\mathscr{F}}^{(\widehat{\sigma})}_{a,\cdots,a}(t^{-1})$ is defined as $\omega(\widehat{\mathscr{F}}^{(\widehat{\sigma})}_{a,\cdots,a}(t))$ and $\mathscr{F}^{(\sigma)}_{a,\cdots,a}(t)$ is the p-adic hypergeometric functions of logarithmic type.

Before moving to the next theorem, we prove, as an example, that this conjecture is true modulo p.

Example 4.13. By Theorem 3.1 and [1, Theorem 3.3], we know that

$$\mathscr{F}^{(\sigma)}_{a,\cdots,a}(t) \equiv \frac{G^{(\sigma)}_{a,\cdots,a}(t)_{<p}}{F_{a,\cdots,a}(t)_{<p}}, \quad \widehat{\mathscr{F}}^{(\widehat{\sigma})}_{a,\cdots,a}(t^{-1}) \equiv \left.\frac{\widehat{G}^{(\widehat{\sigma})}_{a,\cdots,a}(t)_{<p}}{F_{a,\cdots,a}(t)_{<p}}\right|_{t^{-1}} \mod pW[[t]].$$

Let $F(t) = F_{a,\cdots,a}(t)_{<p}$, $G(t) \equiv G^{(\sigma)}_{a,\cdots,a}(t)_{<p}$ and $\widehat{G}(t) \equiv \widehat{G}^{(\widehat{\sigma})}_{a,\cdots,a}(t)_{<p} \mod p$.

Then by Proposition 4.11, one has

$$\frac{G(t)}{F(t)} + \frac{\widehat{G}(t^{-1})}{F(t^{-1})} = \frac{G(t)}{(-1)^{ls}t^l F(t^{-1})} + \frac{\widehat{G}(t^{-1})}{F(t^{-1})} = \frac{G(t) + (-1)^{ls}t^l\widehat{G}(t^{-1})}{(-1)^{ls}t^l F(t^{-1})}.$$

We claim that $G(t) + (-1)^{ls}t^l\widehat{G}(t^{-1}) \equiv 0 \mod p$. Indeed, since

$$B_k = A_k\frac{B_k}{A_k}, \quad \widehat{B}_k = A_k\frac{\widehat{B}_k}{A_k} \equiv 0 \mod p$$

when $l+1 \leq k \leq p-1$, we have $G(t) = \sum_{i=0}^{l} B_k t^k$ and $\widehat{G}(t) = \sum_{i=0}^{l} \widehat{B}_k t^k$. Thus

$$G(t)+(-1)^{ls}t^l\widehat{G}(t^{-1}) = \sum_{i=0}^{l} B_k t^k + (-1)^{ls}\sum_{i=0}^{l}\widehat{B}_{l-k}t^k = \sum_{i=0}^{l}\Big(B_k+(-1)^{ls}\widehat{B}_{l-k}\Big)t^k.$$

Then for $1 \leq k \leq l$, we have

$$B_k+(-1)^{ls}\widehat{B}_{l-k} = \frac{A_k}{k}+(-1)^{ls}\frac{A_{l-k}}{a+l-k} \equiv \frac{1}{k}\Big(A_k-(-1)^{ls}A_{l-k}\Big) \equiv 0 \mod p.$$

When $k = 0$, we have

$$\begin{aligned}
&B_0 + (-1)^{ls}\widehat{B}_l \\
=&s(\psi_p(a)+\gamma_p) - p^{-1}\log(c) - (-1)^{ls}\frac{1}{a+l}\Big(A_l - c^{\frac{l+a}{p}}(-1)^{se}\Big) \\
=&s(\psi_p(a)+\gamma_p) - (-1)^{ls}\frac{A_l}{a+l} - p^{-1}\log(c) + (-1)^{s(e+l)}\frac{c^{\frac{l+a}{p}}}{a+l}.
\end{aligned}$$

Write $c = 1 + pz$ and $a + l = dp^n$ with $p \nmid d$. Then

$$-p^{-1}\log(c) + (-1)^{s(e+l)} \frac{c^{\frac{l+a}{p}}}{a+l} \equiv (-1)^{s(e+l)} \frac{1}{a+l} \mod p.$$

So we obtain

$$\begin{aligned} & B_0 + (-1)^{ls} \widehat{B}_l \\ \equiv & s(\psi_p(a) + \gamma_p) + (-1)^{ls} \frac{A_l - (-1)^{se}}{a+l} \\ \overset{(*)}{\equiv} & s(\psi_p(-l) + \gamma_p) - s(\psi_p(1+l) + \gamma_p) \mod p \end{aligned}$$

where $(*)$ follows from [1, (2.13)] and imitate the proof of Lemma 3.9. Since $\psi_p(-l) = \psi_p(1+l)$ (cf. [1, Theorem 2.4 (2)]), we obtain

$$B_0 + (-1)^{ls} \widehat{B}_l \equiv 0 \mod p.$$

Recall $a_n = \{-nA/N\}$ in Section 4.2. Let $r \in \mathbb{Z}_{\geq 1}$ be a number such that $a_n^{(r)} = a_n$. We will prove a special case of Conjecture 4.9 in this section.

Theorem 4.14. *Let $\sigma(t) = ct^p$ and $\widehat{\sigma}(t) = c^{-1}t^p$. Put $h(t) = \prod_{i=0}^{r-1} F_{a_n^{(i)}, a_n^{(i)}}(t)_{<p}$. Then*

$$\mathscr{F}^{(\sigma)}_{a_n,a_n}(t) = -\widehat{\mathscr{F}}^{(\widehat{\sigma})}_{a_n,a_n}(t^{-1})$$

in the ring $W\langle t, t^{-1}, h(t)^{-1}\rangle$ where $\mathscr{F}^{(\sigma)}_{a_n,a_n}(t)$ is the p-adic hypergeometric functions of logarithmic type.

Before giving the proof, we need some settings. First, we assume $A = B$ and let $t_0^N = t$. Then we have two descriptions of the quotient curve Y_G/S_K (Lemma 4.3):

(i) $Y_G \simeq X$ with coordinates (x, y, t_0),

$$X : y^N = x^A(1-x)^A(1-(1-t_0^N)x)^{N-A}$$

(ii) $Y_G \simeq \widehat{X}$ with coordinates (z, w, s_0),

$$\widehat{X} : w^N = z^A(1-z)^A(1-(1-s_0^N)z)^{N-A}$$

where $z = 1 - x, w = t_0^{A-N}y, s_0 = t_0^{-1}$.

Lemma 4.15. *Assume $\widehat{\sigma}$ is given by $\widehat{\sigma}(t) = c^{-1}t^p$ with $c \in 1 + pW$. Let $h(t) = \prod_{m=0}^{s} F_{a_n^{(m)}, b_n^{(m)}}(t)_{<p}$ where s is the minimal integer such that $(a_n^{(s+1)}, b_n^{(s+1)}) = (a_n, b_n)$ for all $n \in \{1, 2, \cdots, N-1\}$. Then*

$$e_\xi - \Phi(e_\xi) \equiv \sum_{n=1}^{N-1} \frac{(1-\nu_1^{-nA})(1-\nu_2^{-nB})}{N^2} \widehat{\mathscr{F}}^{(\widehat{\sigma})}_{a_n,b_n}(s_0^N)\widehat{\omega}_n$$

modulo $\sum_{n=1}^{N-1} K\langle t, (t-t^2)^{-1}, h(t)^{-1}\rangle e_n^{\mathrm{unit}}$.

Proof. We consider the quotient map (Lemma 4.3) $Y_G \simeq \widehat{X}$ in coordinate (z, w, s_0). From the coordinate (z, w, s_0), let $e_\xi^{\widehat{X}}$ be the unique lifting in Theorem 4.8. We write $\widehat{e}_\xi$ for $e_\xi^{\widehat{X}}$ via the change of coordinate $(z, w, s_0) = (1-x, t_0^{A-N}y, t_0^{-1})$, and $\widehat{\Phi}$ is the Frobenius action which is $\widehat{\sigma}$-linear. Write

$$\widehat{e}_\xi - \widehat{\Phi}(\widehat{e}_\xi) = \sum_{n=1}^{N-1} \frac{(1-\nu_1^{-nA})(1-\nu_2^{-nB})}{N} \widehat{E}_1^{(n)}(t_0)\widetilde{\omega}_n + \widehat{E}_2^{(n)}(t_0)\widetilde{\eta}_n \tag{4.15}$$

in $K((t)) \otimes H^1_{\mathrm{dR}}(X/S_K)$. We apply the Gauss-Manin connection ∇ on (4.15). Since $\nabla\widehat{\Phi} = \widehat{\Phi}\nabla$ and

$$\nabla(\widehat{e}_\xi) = -\sum_{n=1}^{N-1} \frac{(1-\nu_1^{-nA})(1-\nu_2^{-nA})}{N^2} \frac{dt}{t} t^{a_n}(N\omega_n),$$

from Lemma 4.9 and change of coordinate $(z, w, s_0) = (1-x, t_0^{A-N}y, t_0^{-1})$, we have

$$\begin{aligned}
&-\sum_{n=1}^{N-1} \frac{(1-\nu_1^{-nA})(1-\nu_2^{-nA})}{N}(1-\widehat{\Phi})\left(t^{a_n}F_n(t)\frac{dt}{t}\wedge\widetilde{\omega}_n\right) \\
&=\sum_{n=1}^{N-1} \frac{(1-\nu_1^{-nA})(1-\nu_2^{-nA})}{N}\nabla(\widehat{E}_1^{(n)}(t_0)\widetilde{\omega}_n + \widehat{E}_2^{(n)}(t_0)\widetilde{\eta}_n).
\end{aligned} \tag{4.16}$$

By [1, Proposition 4.7], we have $\widehat{\Phi}(\widetilde{\omega}_m) \equiv p^{-1}\widetilde{\omega}_n \mod K((t_0))\widetilde{\eta}_n$ where m is the unique integer in $\{1, 2, ..., N-1\}$ such that $pm \equiv n \mod N$.

Therefore

$$\text{LHS of (4.16)} \equiv -\sum_{n=1}^{N-1} \frac{(1-\nu_1^{-nA})(1-\nu_2^{-nA})}{N}\left[t^{a_n}F_n(t) - \left(t^{a_m}F_m(t)\right)^{\widehat{\sigma}}\right]\frac{dt}{t}\wedge\widetilde{\omega}_n.$$

On the other hand, it follows from [1, Proposition 4.3], we have

$$\text{RHS of (4.16)} \equiv \sum_{n=1}^{N-1} \frac{(1-\nu_1^{-nA})(1-\nu_2^{-nA})}{N}\frac{d\widehat{E}_1^{(n)}}{dt}dt\wedge\widetilde{\omega}_n \mod K((t_0))\widetilde{\eta}_n.$$

Thus

$$\frac{d\widehat{E}_1^{(n)}}{dt} = -\frac{t^{a_n}F_n(t) - \left(t^{a_m}F_m(t)\right)^{\widehat{\sigma}}}{t}.$$

Namely, we have

$$-\widehat{E}_1^{(n)}(t_0) = C + \int_0^t t^{a_n}F_n(t) - \left(t^{a_m}F_m(t)\right)^{\widehat{\sigma}}\frac{dt}{t}$$

for some constant $C \in K$ and with $t = t_0^N$. We claim that this constant C is 0. Indeed, since $\widehat{E}_1^{(n)}(t_0)/F_n(t_0^N)$ is an overconvergent function, $\widehat{E}_1^{(n)}(t_0)t_0^{-Na_n}/F_n(t_0^N)$ is also overconvergent.

If $C = 0$, then $\widehat{E}_1^{(n)}(t_0)t_0^{-Na_n}/F_n(t_0^N) = \widehat{\mathscr{F}}^{(\widehat{\sigma})}_{a_n,a_n}(t)$ is a convergent function by Corollary 3.2. If there is another C' such that $\widehat{E}_1^{(n)}(t_0)t_0^{-Na_n}/F_n(t_0^N)$ is a convergent function, then after subtraction, we have

$$\frac{C' t_0^{-Na_n}}{F_n(t_0^N)} \in K\langle t_0, (t_0 - t_0^2)^{-1}, h(t_0^N)^{-1}\rangle.$$

This is a contradiction (as it is shown in the proof of [1, proof in Theorem 4.9]). So C must be 0. Therefore we have

$$\widehat{e}_\xi - \widehat{\Phi}(\widehat{e}_\xi) = \sum_{n=1}^{N-1} \frac{(1-\nu_1^{-nA})(1-\nu_2^{-nB})}{N}[-\widehat{\mathscr{F}}^{(\widehat{\sigma})}_{a_n,a_n}(t)\cdot t^{a_n}]\omega_n \mod \widetilde{\eta}_n.$$

Now we write everything back to coordinate (z, w, s_0). Since Ker∇ is generated by $\{\widetilde{\eta}_n\}$ over K (Lemma 4.2) and Ker∇ contains in Ker∇ via change of coordinate, we have

$$e_\xi^{\widehat{X}} - \Phi(e_\xi^{\widehat{X}}) = \sum_{n=1}^{N-1} \frac{(1-\nu_1^{-nA})(1-\nu_2^{-nB})}{N^2}[\widehat{\mathscr{F}}^{(\widehat{\sigma})}_{a_n,a_n}(s_0^{-N})]N\omega_n \mod \widetilde{\eta}_n.$$

in coordinate (z, w, s_0). Therefore from Lemma 4.4, we have

$$e_\xi - \Phi(e_\xi) \equiv \sum_{n=1}^{N-1} \frac{(1-\nu_1^{-nA})(1-\nu_2^{-nB})}{N^2}\widehat{\mathscr{F}}^{(\widehat{\sigma})}_{a_n,b_n}(t^{-1})\widehat{\omega}_n$$

modulo $\sum_{n=1}^{N-1} K\langle t, (t-t^2)^{-1}, h(t)^{-1}\rangle e_n^{\text{unit}}$. This completes the proof. □

Using these lemmas, we can prove Theorem 4.14.

(proof of Theorem 4.14). By Lemma 4.10, we know

$$e_\xi - \Phi(e_\xi) \equiv \sum_{n=1}^{N-1} \frac{(1-\nu_1^{-nA})(1-\nu_2^{-nA})}{N^2}\big[-\mathscr{F}^{(\sigma)}_{a_n,a_n}(t)\big]\widehat{\omega}_n \tag{4.17}$$

modulo $\sum_{n=1}^{N-1} K\langle t, (t-t^2)^{-1}, h(t)^{-1}\rangle e_n^{\text{unit}}$.

Also, by Lemma 4.15, we have

$$e_\xi - \Phi(e_\xi) \equiv \sum_{n=1}^{N-1} \frac{(1-\nu_1^{-nA})(1-\nu_2^{-nA})}{N^2}\big[\widehat{\mathscr{F}}^{(\widehat{\sigma})}_{a_n,a_n}(t^{-1})\big]\widehat{\omega}_n \tag{4.18}$$

modulo $\sum_{n=1}^{N-1} K\langle t, (t-t^2)^{-1}, h(t)^{-1}\rangle e_n^{\text{unit}}$.

Comparing equations (4.17) and (4.18), we have

$$-\widehat{\mathscr{F}}^{(\widehat{\sigma})}_{a_n,a_n}(t) = \mathscr{F}^{(\sigma)}_{a_n,a_n}(t^{-1}).$$

This completes the proof. □

4.4 Transformation formula on Dwork's p-adic Hypergeometric Functions

In this section, the settings are the same as in Section 4.3. First, let us recall the definition of Dwork's p-adic hypergeometric functions.

Definition 4.16 (**Dwork's p-adic hypergeometric functions**). *Let $s \geq 1$ be an integer. For $(a_1, \cdots, a_s) \in \mathbb{Z}_p^s$, Dwork's p-adic hypergeometric function is defined as*

$$\mathscr{F}^{\mathrm{Dw}}_{a_1,\cdots,a_s}(t) := F_{a_1,\cdots,a_s}(t)/F_{a'_1,\cdots,a'_s}(t^p)$$

where $F_{a_1,\cdots,a_s}(t)$ and $F_{a'_1,\cdots,a'_s}(t)$ are hypergeometric power series.

Remark 4.17. *We have Dwork's p-adic hypergeometric function belongs to $W\langle t, f(t)^{-1}\rangle$ where*

$$f(t) := \prod_{i=0}^{N} F_{a_1^{(i)},\cdots,a_s^{(i)}}(t)_{<p}$$

and N is an integer such that

$$\{\overline{F_{a_1^{(i)},\cdots,a_s^{(i)}}(t)_{<p}} \mid i \in \mathbb{Z}_{\geq 0}\} = \{\overline{F_{a_1^{(i)},\cdots,a_s^{(i)}}(t)_{<p}} \mid i = 0, 1, \cdots, N\}$$

with $\overline{f(t)} = f(t) \mod p$ (consequence of Dwork's congruence).

Conjecture 4.18 (**Transformation formula on Dwork's p-adic Hypergeometric Functions**). *Suppose $a_1 = \cdots = a_s = a$. Let l is the unique integer in $\{0, 1, \cdots, p-1\}$ such that $a + l \equiv 0 \mod p$. Then*

$$\mathscr{F}^{\mathrm{Dw}}_{a,\cdots,a}(t) = ((-1)^s t)^l \mathscr{F}^{\mathrm{Dw}}_{a,\cdots,a}(t^{-1}).$$

Remark 4.19. *From Proposition 4.11, we have Conjecture 4.18 holds modulo p.*

We have the special case of Conjecture 4.18.

Theorem 4.20. *Let $a \in \frac{1}{N}\mathbb{Z}, 0 < a < 1$ and $p > N$. We have*

$$\mathscr{F}^{\mathrm{Dw}}_{a,a}(t) = t^l \mathscr{F}^{\mathrm{Dw}}_{a,a}(t^{-1}),$$

where l is the unique integer in $\{0, 1, \cdots, p-1\}$ such that $a + l \equiv 0 \mod p$.

Proof. Let $\sigma : W[[t]] \to W[[t]]$ be the p-th Frobenius given by $\sigma(t) = t^p$, and let Φ be the p-th Frobenius induced by σ on $K((t_0)) \otimes H^1_{\mathrm{dR}}(Y/S)$ where Y/S is the hypergeometric curve in §4.1.1. By [1, Proposition 4.7], the p-th Frobenius Φ induces a map with

$$\Phi(\widetilde{\omega}_{p^{-1}nA,p^{-1}nA}) \equiv p^{-1}\widetilde{\omega}_{nA,nA} \mod K((t))\widetilde{\eta}_{nA,nA}.$$

Let $m \equiv p^{-1}nA \mod N$ with $1 \leq m \leq N-1$ and $k \equiv nA \mod N$ with $1 \leq n \leq N-1$. Then we have

$$\Phi(\omega_{p^{-1}nA,p^{-1}nA}) \equiv p^{-1}\frac{F_m(t^\sigma)}{F_k(t)}\omega_{nA,nA} = p^{-1}\mathscr{F}^{\mathrm{Dw}}_{a_n,a_n}(t)^{-1}\omega_{nA,nA} \tag{4.19}$$

where $a_n = \{-nA/N\}$.

Therefore we obtain

$$\Phi(\widehat{\omega}_m) \equiv p^{-1}\mathscr{F}^{\mathrm{Dw}}_{a_n,a_n}(t)^{-1}\widehat{\omega}_n. \tag{4.20}$$

Recall that we have two description of Y/G. From the coordinate (z, w, s_0), one has

$$\Phi(\widetilde{\omega}_m) \equiv p^{-1}\mathscr{F}^{\mathrm{Dw}}_{a_n,a_n}(s_0^N)^{-1}\widetilde{\omega}_n. \tag{4.21}$$

By using $(z, w, s_0) = (1-x, t_0^{A-N}y, t_0^{-1})$, we obtain

$$\Phi(-t^{a_m}\widetilde{\omega}_m) \equiv p^{-1}\mathscr{F}^{\mathrm{Dw}}_{a_n,a_n}(t^{-1})^{-1}(-t^{a_n})\widetilde{\omega}_n. \tag{4.22}$$

On the other hand, we have

$$\Phi(-t^{a_m}\widetilde{\omega}_m) \equiv (-t^{pa_m})p^{-1}\mathscr{F}^{\mathrm{Dw}}_{a_n,a_n}(t)^{-1}\widetilde{\omega}_n \tag{4.23}$$

since $\Phi(t) = t^p$ and (4.20). Then comparing (4.22) and (4.23), we obtain the result. □

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