RIMS Kôkyûroku Bessatsu
B79 (2020), 51–62

Crossover phenomena in the critical behavior for long-range models with power-law couplings

By

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Abstract

This is a short review of the two papers [9, 10] on the x-space asymptotics of the critical two-point function $G_{p_c}(x)$ for the long-range models of self-avoiding walk, percolation and the Ising model on $\mathbb{Z}^d$, defined by the translation-invariant power-law step-distribution/coupling $D(x) \propto |x|^{-d-\alpha}$ for some $\alpha > 0$. Let $S_1(x)$ be the random-walk Green function generated by D. We have shown that

- $S_1(x)$ changes its asymptotic behavior from Newton ($\alpha > 2$) to Riesz ($\alpha < 2$), with log correction at $\alpha = 2$;
- $G_{p_c}(x) \sim \frac{A}{p_c} S_1(x)$ as $|x| \to \infty$ in dimensions higher than (or equal to, if $\alpha = 2$) the upper critical dimension d_c (with sufficiently large spread-out parameter L). The model-dependent A and d_c exhibit crossover at $\alpha = 2$.

The keys to the proof are (i) detailed analysis on the underlying random walk to derive sharp asymptotics of S_1, (ii) bounds on convolutions of power functions (with log corrections, if $\alpha = 2$) to optimally control the lace-expansion coefficients $\pi_p^{(n)}$, and (iii) probabilistic interpretation (valid only when $\alpha \le 2$) of the convolution of D and a function Π_p of the alternating series $\sum_{n=0}^{\infty}(-1)^n \pi_p^{(n)}$. We outline the proof, emphasizing the above key elements for percolation in particular.

§ 1. Introduction and the main results

Since the dawn of research on phase transitions and critical behavior, it has been standard to investigate short-range models, among which the nearest-neighbor model on $\mathbb{Z}^d$ is the most popular. Thanks to intensive studies for more than half a century,

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Received December 26, 2018. Revised October 15, 2019.
2010 Mathematics Subject Classification(s): 60K35, 82B20, 82B27, 82B41, 82B43.
Key Words: *Random walk, self-avoiding walk, percolation, Ising model, power-law coupling, critical two-point function, critical dimension, crossover phenomena, lace expansion.*
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nearest-neighbor bond percolation is now known to exhibit a phase transition for all $d \geq 2$ and mean-field behavior (i.e., the critical two-point function $G_{p_c}(x)$ decays as $|x|^{2-\eta_{\text{short}}-d}$ with the mean-field value $\eta_{\text{short}} = 0$) for all $d \geq 11$ [11, 12]. Believing in universality, we expect the mean-field behavior for all dimensions above the upper-critical dimension $d_{\text{short}} = 6$ for short-range percolation [15].

Recently, long-range random walk and statistical-mechanical models defined by the power-law step-distribution/coupling $D(x) \propto |x|^{-d-\alpha}$, $\alpha > 0$, have regained popularity, due to unconventional macroscopic behavior [4,6–10,18,19]. Among those references, an infrared bound and mean-field behavior are proven for long-range oriented percolation (OP for short) on $\mathbb{Z}^{d>d_c} \times \mathbb{Z}_+$ [6,7] and for long-range models of percolation, self-avoiding walk (SAW for short) and the Ising model on $\mathbb{Z}^{d>d_c}$ [18], where

$$d_c = (\alpha \wedge 2) \times \begin{cases} 2 & \text{[OP, SAW \& Ising]}, \\ 3 & \text{[percolation]}. \end{cases} \tag{1.1}$$

Also, an asymptotic expression of the gyration radius for long-range models of SAW and OP for $d > d_c$ are proven in [8]. In physics, Brezin, Parisi and Ricci-Tersenghi [4] conjectured that $G_{p_c}(x)$ would decay as $|x|^{\alpha \wedge (2-\eta_{\text{short}})-d}$ if $\alpha \neq 2 - \eta_{\text{short}}$, and as $|x|^{\alpha-d}/\log|x|$ if $\alpha = 2 - \eta_{\text{short}}$. We have shown in [9, 10] that the conjectured behavior holds true for $d > d_{\text{short}}$ ($= d_c$ with $\alpha = \infty$), because $\eta_{\text{short}} = 0$, with sufficiently large spread-out parameter L [13, 14, 23]. In fact, the obtained results are much stronger, as summarized as follows.

Theorem 1.1 (Proposition 2.1 of [9] and Theorem 1.3 of [10]). *Let $\alpha > 0$, $L \geq 1$ and $D(x) \asymp \frac{1}{L^d}(\frac{|x|}{L} \vee 1)^{-d-\alpha}$, i.e.,*

$$^\exists c > 0,\ ^\forall x \in \mathbb{Z}^d,\ ^\forall L \in [1, \infty):\ c \leq \frac{D(x)}{\frac{1}{L^d}(\frac{|x|}{L} \vee 1)^{-d-\alpha}} \leq \frac{1}{c}. \tag{1.2}$$

Let

$$\gamma_\alpha = \frac{\Gamma(\frac{d-\alpha\wedge 2}{2})}{2^{\alpha\wedge 2}\pi^{d/2}\Gamma(\frac{\alpha\wedge 2}{2})}, \qquad v_\alpha = \begin{cases} \lim_{|k|\to 0} \dfrac{1-\hat{D}(k)}{|k|^{\alpha\wedge 2}} & [\alpha \neq 2], \\ \lim_{|k|\to 0} \dfrac{1-\hat{D}(k)}{|k|^2 \log(1/|k|)} & [\alpha = 2], \end{cases} \tag{1.3}$$

where $\hat{D}(k) = \sum_{x\in\mathbb{Z}^d} e^{ik\cdot x} D(x)$. Then, for all $d > \alpha \wedge 2$, the random-walk Green function $S_1(x)$ generated by the step distribution D exhibits the following asymptotic behavior: there is an $\epsilon > 0$ such that, as $|x| \to \infty$,

$$S_1(x) = \frac{\gamma_\alpha/v_\alpha}{|x|^{d-\alpha\wedge 2}} \times \begin{cases} \left(1 + \dfrac{O(L^\epsilon)}{|x|^\epsilon}\right) & [\alpha \neq 2], \\ \dfrac{1}{\log|x|}\left(1 + \dfrac{O(1)}{(\log|x|)^\epsilon}\right) & [\alpha = 2], \end{cases} \tag{1.4}$$

where the $O(1)$ term is independent of L.

Theorem 1.2 (Theorem 1.2 of [9] and Theorem 1.6 of [10])**.** *Let D be the same as in Theorem 1.1 and recall the definition (1.1) of d_c. Suppose $d > d_c$ for $\alpha \neq 2$ or that $d \geq d_c$ for $\alpha = 2$. For $\alpha > 2$, we also assume a bound on the "derivative" of D (see the last part of Section 3). Then, there is an $L_0(d) < \infty$ such that, for any $L \geq L_0$, there are $A = 1 + O(L^{-2})\mathbb{1}_{\{\alpha>2\}}$ and $\epsilon > 0$ such that, as $|x| \to \infty$,*

$$G_{p_c}(x) = \frac{A}{p_c} \frac{\gamma_\alpha / v_\alpha}{|x|^{d-\alpha\wedge 2}} \times \begin{cases} \left(1 + \dfrac{O(L^\epsilon)}{|x|^\epsilon}\right) & [\alpha \neq 2], \\ \dfrac{1}{\log|x|}\left(1 + \dfrac{O(1)}{(\log|x|)^\epsilon}\right) & [\alpha = 2], \end{cases} \tag{1.5}$$

where the $O(1)$ term is independent of L.

In short, the critical two-point function $G_{p_c}(x)$ exhibits the same asymptotic behavior as $S_1(x)$, modulo multiplication of the model-dependent constant A/p_c, for all $d > d_c$ (with large spread-out parameter L) and, most interestingly, for $d = d_c$ when $\alpha = 2$. For $d \in (d_c, d_{\text{short}})$, which is not empty for $\alpha < 2$ and in which η_{short} is believed to be nonzero, Theorem 1.2 claims that $G_{p_c}(x)$ decays as $|x|^{\alpha-d}$, not as $|x|^{2-\eta_{\text{short}}-d}$. This power-law behavior has been extended even below d_c by Lohmann, Slade and Wallace [19] using a rigorous version of the ε-expansion.

§ 2. Key ideas for the proof of Theorem 1.1

Let D^{*n} be the n-fold convolution of D (i.e., the n-step distribution) and denote by S_q the random-walk Green function generated by D with survival rate $q \in [0,1]$:

$$D^{*n}(x) = (D^{*(n-1)} * D)(x) \equiv \sum_y D^{*(n-1)}(y)\, D(x-y), \tag{2.1}$$

$$S_q(x) = \sum_{n=0}^{\infty} q^n D^{*n}(x). \tag{2.2}$$

Let

$$|||x|||_r = \frac{\pi}{2}(|x| \vee r) \qquad [x \in \mathbb{R}^d,\ 1 \leq r < \infty], \tag{2.3}$$

where $|\cdot|$ is the Euclidean norm. Suppose that, as explained in (1.2), $D(x)$ decays as

$$D(x) \asymp L^{-d} |||\tfrac{x}{L}|||_1^{-d-\alpha} \equiv L^\alpha |||x|||_L^{-d-\alpha}. \tag{2.4}$$

An example of D is the following compound zeta distribution [9]:

$$D(x)=\sum_{t\in\mathbb{N}}U_L^{*t}(x)\,\frac{t^{-1-\alpha/2}}{\zeta(1+\alpha/2)}\qquad[x\in\mathbb{Z}^d], \tag{2.5}$$

where U_L is the uniform distribution over the d-dimensional box of side-length $2L$.

The step distribution D in (2.4) satisfies the following properties (D1)–(D3) that are essential to the proof of (1.4).

(D1) k-space bounds [6, Proposition 1.1] (and [10, Assumption 1.1]): ${}^\exists\Delta=\Delta(L)\in(0,1)$ such that

$$1-\hat{D}(k)\begin{cases}<2-\Delta & [{}^\forall k\in[-\pi,\pi]^d],\\ >\Delta & [|k|>1/L],\end{cases} \tag{2.6}$$

and for $|k|\le 1/L$,

$$1-\hat{D}(k)\asymp(L|k|)^{\alpha\wedge2}\times\begin{cases}1 & [\alpha\ne2],\\ \log\frac{\pi}{2L|k|} & [\alpha=2].\end{cases} \tag{2.7}$$

(D2) k-space asymptotics [8, Lemma A.1] (and [10, Assumption 1.1]): ${}^\exists\epsilon>0$ such that, as $|k|\to0$,

$$1-\hat{D}(k)=v_\alpha|k|^{\alpha\wedge2}\times\begin{cases}\big(1+O(L^\epsilon|k|^\epsilon)\big) & [\alpha\ne2],\\ \big(\log\frac{1}{L|k|}+O(1)\big) & [\alpha=2],\end{cases} \tag{2.8}$$

where the constant in the $O(1)$ term is independent of L.

(D3) x-space bounds [9, (1.19)–(1.21)] (and [10, Assumption 1.2]): ${}^\forall n\in\mathbb{N}$ and ${}^\forall x\in\mathbb{Z}^d$,

$$\|D^{*n}\|_\infty\le O(L^{-d})\times\begin{cases}n^{-d/(\alpha\wedge2)} & [\alpha\ne2],\\ (n\log\frac{\pi n}{2})^{-d/2} & [\alpha=2],\end{cases} \tag{2.9}$$

$$D^{*n}(x)\le n\frac{O(L^{\alpha\wedge2})}{\|\!|x|\!\|_L^{d+\alpha\wedge2}}\times\begin{cases}1 & [\alpha\ne2],\\ \log\|\!|\frac{x}{L}|\!\|_1 & [\alpha=2].\end{cases} \tag{2.10}$$

For example, to show (2.7) for $|k|\le1/L$, we first split the sum as

$$1-\hat{D}(k)\asymp L^\alpha\sum_x\|\!|x|\!\|_L^{-d-\alpha}(1-\cos k\cdot x)\,\Big(\mathbb{1}_{\{|x|<L\}}+\mathbb{1}_{\{L\le|x|\le\frac{\pi}{2|k|}\}}+\mathbb{1}_{\{|x|>\frac{\pi}{2|k|}\}}\Big). \tag{2.11}$$

It is easy to see that the contributions from the first and third indicators are $O(L^2|k|^2)$ and $O(L^\alpha|k|^\alpha)$, respectively. The contribution from the second indicator is the main term since

$$
\begin{aligned}
L^\alpha \sum_{L\le|x|\le\frac{\pi}{2|k|}} \|x\|_L^{-d-\alpha}(1-\cos k\cdot x) &\asymp L^\alpha|k|^2 \sum_{L\le|x|\le\frac{\pi}{2|k|}} |x|^{-d-\alpha+2} \\
&\asymp \begin{cases} (L|k|)^{\alpha\wedge 2} & [\alpha\ne 2], \\ (L|k|)^2\log\frac{\pi}{2L|k|} & [\alpha=2]. \end{cases} \qquad (2.12)
\end{aligned}
$$

To prove (1.4), we first rewrite $S_1(x)$ for the transient case $d>\alpha\wedge 2$ as

$$
\begin{aligned}
S_1(x) = \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^dk}{(2\pi)^d}\,\frac{e^{-ik\cdot x}}{1-\hat D(k)} &= \int_0^\infty \mathrm{d}t \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^dk}{(2\pi)^d}\, e^{-ik\cdot x-t(1-\hat D(k))} \\
&= \int_0^\infty \mathrm{d}t \int_{|k|\le R} \frac{\mathrm{d}^dk}{(2\pi)^d}\, e^{-ik\cdot x-t(1-\hat D(k))} + E_1, \qquad (2.13)
\end{aligned}
$$

where R is arbitrary for the moment. Then, by replacing $1-\hat D(k)$ by its limit (2.8), we can further rewrite $S_1(x)$ for $\alpha\ne 2$ as

$$
S_1(x) = \int_0^\infty \mathrm{d}t \int_{\mathbb{R}^d} \frac{\mathrm{d}^dk}{(2\pi)^d}\, e^{-ik\cdot x-v_\alpha t|k|^{\alpha\wedge 2}} + E_1 + E_2, \qquad (2.14)
$$

and for $\alpha=2$ as

$$
S_1(x) = \int_0^\infty \mathrm{d}t \int_{\mathbb{R}^d} \frac{\mathrm{d}^dk}{(2\pi)^d}\, e^{-ik\cdot x-v_2 t|k|^2\log\frac{1}{L|k|}} + E_1 + E_2. \qquad (2.15)
$$

Since

$$
\int_0^\infty \mathrm{d}t\, e^{-v_\alpha t|k|^{\alpha\wedge 2}} = \frac{1}{v_\alpha|k|^{\alpha\wedge 2}} = \frac{1}{v_\alpha\Gamma(\frac{\alpha\wedge 2}{2})}\int_0^\infty \frac{\mathrm{d}t}{t}\, t^{(\alpha\wedge 2)/2} e^{-t|k|^2}, \qquad (2.16)
$$

we readily obtain for $\alpha\ne 2$ that

(2.17)

$$
S_1(x)-E_1-E_2 = \frac{1}{v_\alpha\Gamma(\frac{\alpha\wedge 2}{2})}\int_0^\infty \frac{\mathrm{d}t}{t}\, t^{(\alpha\wedge 2)/2} \underbrace{\int_{\mathbb{R}^d} \frac{\mathrm{d}^dk}{(2\pi)^d}\, e^{-ik\cdot x-t|k|^2}}_{=(4\pi t)^{-d/2}\exp(-|x|^2/(4t))} = \frac{\gamma_\alpha/v_\alpha}{|x|^{d-\alpha\wedge 2}}.
$$

Using the k-space and x-space bounds (D1) and (D3) and choosing R accordingly (as in [9, (2.20)]), we can show that E_1+E_2 is the error term in (1.4). See [9, Section 2.1] for more details.

For $\alpha = 2$, we change variables as $\xi = x/|x|$, $\kappa = |x|k$ and $\tau = \frac{v_2 t}{|x|^2}\log\frac{|x|}{L}$ to obtain

$$\begin{aligned}
S_1(x) - E_1 - E_2 &= |x|^{-d}\int_0^\infty \mathrm{d}t \int_{\mathbb{R}^d}\frac{\mathrm{d}^d\kappa}{(2\pi)^d}\exp\bigg(-i\kappa\cdot\xi - \frac{v_2 t|\kappa|^2}{|x|^2}\log\frac{|x|}{L|\kappa|}\bigg)\\
&= \frac{|x|^{2-d}}{v_2\log\frac{|x|}{L}}\int_0^\infty \mathrm{d}\tau \int_{\mathbb{R}^d}\frac{\mathrm{d}^d\kappa}{(2\pi)^d}\exp\Bigg(-i\kappa\cdot\xi - \tau|\kappa|^2\frac{\log\frac{|x|}{L|\kappa|}}{\log\frac{|x|}{L}}\Bigg)\\
&= \frac{|x|^{2-d}}{v_2\log\frac{|x|}{L}}\underbrace{\int_0^\infty \mathrm{d}\tau \int_{\mathbb{R}^d}\frac{\mathrm{d}^d\kappa}{(2\pi)^d}\, e^{-i\kappa\cdot\xi-\tau|\kappa|^2}}_{=\gamma_2} + E_3. \qquad (2.18)
\end{aligned}$$

Again, by using the k-space and x-space bounds on D and choosing R accordingly (as in [10, (2.5)]), we can show that $E_1 + E_2 + E_3$ is the error term in (1.4). See [10, Section 2.1] for more details. This completes the sketch proof of Theorem 1.1.

§ 3. Key ideas for the proof of Theorem 1.2

The proof of Theorem 1.2 is based on the lace expansion, which is one of the few methods to prove mean-field results mathematically rigorously. Since its invention by Brydges and Spencer for weakly SAW [5], the method has been extended to strictly SAW [17], oriented/unoriented percolation [15,21], lattice trees and lattice animals [16], the contact process [22], the Ising and φ^4 models [23, 24].

The lace expansion yields a formal recursion equation for the two-point function $G_p(x)$, which is similar to the recursion equation for the random-walk Green function $S_p(x)$. For (strictly) SAW, $G_p(x)$ is defined as

$$G_p(x) = \sum_{\omega:o\to x} p^{|\omega|}\prod_{j=1}^{|\omega|} D(\omega_j - \omega_{j-1})\prod_{s<t}(1-\delta_{\omega_s,\omega_t}), \qquad (3.1)$$

where the sum is over the paths ω from o to x. The contribution from the zero-step walk is regarded as $\delta_{o,x}$. The last product over s, t is either 0 or 1 depending on whether or not ω intersects to itself.

For Bernoulli bond percolation, in which each bond $\{u, v\}$ is occupied with probability $pD(v-u)$ independently of the other bonds, the two-point function is defined as

$$G_p(x) = \mathbb{P}_p(o \longleftrightarrow x), \qquad (3.2)$$

where $\mathbb{P}_p$ is the induced law from the above bond-occupation probability ($p(1 - D(o))$ is the expected number of occupied bonds per vertex), and $\{o \longleftrightarrow x\}$ is the event that either $x = o$ or there is a self-avoiding path of occupied bonds from o to x.

For the Ising model, see, e.g., [10, Section 1.2.4].

Due to monotonicity in p and subadditivity in self-avoiding paths, the critical point p_c is characterized by the divergence of the susceptibility χ_p for all models, as follows:

$$\chi_p = \sum_x G_p(x), \qquad\qquad p_c = \sup\{p \geq 0 : \chi_p < \infty\}. \tag{3.3}$$

The proof of Theorem 1.2 consists of the following two steps:

Step 1: Prove that $G_p(x)$ is bounded by $2\lambda \|\!|x|\!\|_L^{\alpha\wedge 2-d}$ if $\alpha \neq 2$ and by $2\lambda \|\!|x|\!\|_L^{2-d}/\log \|\!|\frac{x}{L}|\!\|_1$ if $\alpha = 2$, uniformly in $x \in \mathbb{Z}^d$ and $p < p_c$, where

$$\lambda = \begin{cases} \sup\limits_{x \neq o} S_1(x) \|\!|x|\!\|_L^{d-\alpha\wedge 2} & [\alpha \neq 2], \\ \sup\limits_{x \neq o} S_1(x) \|\!|x|\!\|_L^{d-2} \log \|\!|\frac{x}{L}|\!\|_1 & [\alpha = 2], \end{cases} \tag{3.4}$$

which is of order $L^{-\alpha\wedge 2}$, by Theorem 1.1.

Step 2: Use the lace expansion as a recursion equation for $G_{p_c}(x)$ to derive its asymptotic expression.

To complete **Step 2** is rather straightforward as soon as **Step 1** is completed; see [9, Section 3.3] for $\alpha \neq 2$ and [10, Section 3.5] for $\alpha = 2$. To complete **Step 1**, it suffices to show that g_p, defined as

$$g_p = \begin{cases} p \vee \sup\limits_{x \neq o} \dfrac{G_p(x)}{\lambda \|\!|x|\!\|_L^{\alpha\wedge 2-d}} & [\alpha \neq 2], \\ p \vee \sup\limits_{x \neq o} \dfrac{G_p(x)}{\lambda \|\!|x|\!\|_L^{2-d}/\log \|\!|\frac{x}{L}|\!\|_1} & [\alpha = 2], \end{cases} \tag{3.5}$$

satisfies the following three properties:

(S1.1) $g_1 \leq 1$.

(S1.2) g_p is continuous (and nondecreasing) in $p \in [1, p_c)$.

(S1.3) $g_p \leq 3$ implies $g_p \leq 2$ for every $p \in (1, p_c)$, if $\lambda \ll 1$.

The third property implies that there is a prohibited region in the p–g_p plane. Therefore, g_p is either ≤ 2 or > 3, as long as $p \in (1, p_c)$. However, due to the continuity (S1.2) with the initial condition (S1.1), the possibility of $g_p > 3$ is eliminated. This completes **Step 1**.

(S1.1)–(S1.2) are not so difficult, due to [10, Propositions 3.1–3.3]. To show (S1.3), we use the lace expansion, which is formally written as

$$G_p(x) = \mathit{\Pi}_p(x) + (\mathit{\Pi}_p * pD * G_p)(x), \tag{3.6}$$

where (cf., [10, Section 3.1])

$$\Pi_p(x) = \begin{cases} \delta_{o,x} + \sum_{n=1}^{\infty} \big(-pD(o)\delta + \pi_p \big)^{*n}(x) & \text{[SAW]}, \\ \pi_p(x) + \sum_{n=1}^{\infty} \big(-pD(o) \big)^n \pi_p^{*(n+1)}(x) & \text{[Ising \& percolation]}. \end{cases} \tag{3.7}$$

Here, π_p is the alternating series of the nonnegative lace-expansion coefficients $\{\pi_p^{(n)}\}_{n=0}^{\infty}$ ($\pi_p^{(0)} \equiv 0$ for SAW):

$$\pi_p(x) = \sum_{n=0}^{n} (-1)^n \pi_p^{(n)}(x). \tag{3.8}$$

The proof of Item (S1.3) goes as follows.

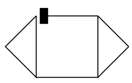
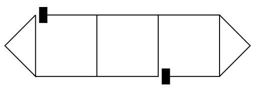
(i) Bound $\pi_p^{(n)}$ in terms of G_p by using correlation inequalities, such as the BK inequality for percolation [3].

(ii) Derive an optimal x-space bound on Π_p in (3.7) by applying the hypothesis $g_p \le 3$ to the bounds on $\pi_p^{(n)}$ obtained in (i) and using convolution bounds (see below) on power functions, with log corrections for $\alpha = 2$.

(iii) Prove the improved bound $g_p \le 2$ by applying the bound on Π_p obtained in (ii) to (3.6).

From now on, we restrict our attention to percolation. By the BK inequality, the first few terms are bounded as

(3.9)

$\pi_p^{(0)}(x) \le G_p(x)^2$, $\pi_p^{(1)}(x) \le o$  x, $\pi_p^{(2)}(x) \le o$  $x + \cdots$,

where each line segment represents G_p, small filled rectangles are pD and unlabeled vertices are summed over $\mathbb{Z}^d$. For more explanation on those diagrammatic expressions, we refer to the original paper [15]. Then, we use $g_p \le 3$ and the following convolution bounds:

Lemma 3.1 (Lemma 3.5 of [10]). *For $a_1 \ge b_1 > 0$ with $a_1 + b_1 \ge d$, and for $a_2, b_2 \ge 0$ with $a_2 \ge b_2$ when $a_1 = b_1$, there is an L-independent constant $C =$*

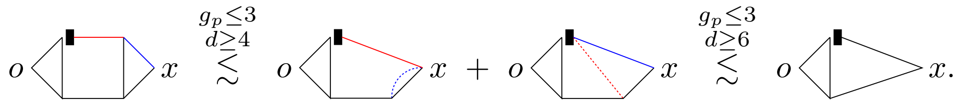
$C(d,a_1,a_2,b_1,b_2)<\infty$ *such that*

(3.10)

$$\sum_{y\in\mathbb{Z}^d}\frac{\||x-y|\|_L^{-a_1}}{(\log\||\frac{x-y}{L}|\|_1)^{a_2}}\frac{\||y|\|_L^{-b_1}}{(\log\||\frac{y}{L}|\|_1)^{b_2}}$$
$$\le\frac{C\,\||x|\|_L^{-b_1}}{(\log\||\frac{x}{L}|\|_1)^{b_2}}\times\begin{cases}L^{d-a_1} & [a_1>d],\\ \log\log\||\frac{x}{L}|\|_1 & [a_1=d,\ a_2=1],\\ (\log\||\frac{x}{L}|\|_1)^{0\vee(1-a_2)} & [a_1=d,\ a_2\neq 1],\\ \||x|\|_L^{d-a_1} & [a_1<d,\ a_1+b_1>d],\\ \||x|\|_L^{b_1}(\log\||\frac{x}{L}|\|_1)^{0\vee(1-a_2)} & [a_1<d,\ a_1+b_1=d,\ a_2+b_2>1].\end{cases}$$

Take $\pi_p^{(1)}(x)$ for $\alpha=2$, for example. By repeated applications of the above convolution bounds, we can reduce the number of vertices (and line segments) one by one, as depicted as follows:

(3.11)



Explanation of the above inequality. Let v be the unlabeled top-right vertex in the leftmost figure at which three line segments (each in red, blue and black) meet, and let y,z be the other end vertices of the horizontal (in red) and vertical (in black) line segments, respectively. In the first inequality, we use (3.10) between the vertical line segment and one of the other two line segments, depending on whether $|x-v|\ge|y-v|$ or $|x-v|\le|y-v|$. If $|x-v|\le|y-v|$, then $|x-y|\le|x-v|+|y-v|\le 2|y-v|$ and therefore

$$\sum_{v:|x-v|\le|y-v|}\frac{\||x-v|\|_L^{2-d}}{\log\||\frac{x-v}{L}|\|_1}\frac{\||y-v|\|_L^{2-d}}{\log\||\frac{y-v}{L}|\|_1}\frac{\||z-v|\|_L^{2-d}}{\log\||\frac{z-v}{L}|\|_1}$$

(3.12)
$$\le\frac{\||\frac{x-y}{2}|\|_L^{2-d}}{\log\||\frac{x-y}{2L}|\|_1}\sum_v\frac{\||x-v|\|_L^{2-d}}{\log\||\frac{x-v}{L}|\|_1}\frac{\||z-v|\|_L^{2-d}}{\log\||\frac{z-v}{L}|\|_1}\overset{d\ge4}{\le}{}^{\exists}C'\frac{\||x-y|\|_L^{2-d}}{\log\||\frac{x-y}{L}|\|_1}\underbrace{\frac{\||x-z|\|_L^{4-d}}{\log\||\frac{x-z}{L}|\|_1}}_{\text{blue-dotted}},$$

which is depicted as the left figure in the middle expression in (3.11). Then, by gathering all line segments meeting at z (denote the other end vertex of the horizontal line segment by u) and using (3.10) again, we obtain

(3.13)
$$\sum_z\frac{\||x-z|\|_L^{(4-d)+(2-d)}}{(\log\||\frac{x-z}{L}|\|_1)^2}\frac{\||u-z|\|_L^{2-d}}{\log\||\frac{u-z}{L}|\|_1}\overset{d\ge6}{\le}C\frac{\||x-u|\|_L^{2-d}}{\log\||\frac{x-u}{L}|\|_1},$$

which yields the rightmost figure of (3.11). We should emphasize that the above bound holds even at $d_c = 6$, because of the log-squared term in the denominator. This is one of the reasons why the mean-field results[1] hold for $d \geq d_c$ (including equality) when $\alpha = 2$.

The other case $|x - v| \geq |y - v|$ can be evaluated similarly, and we refrain from showing it here. □

Applying the same analysis to the other $\pi_p^{(n)}$ and using (3.7)–(3.8), we can get (cf., [9, (3.4)] and [10, (3.29)])

$$|\Pi_p(x) - \delta_{o,x}| \leq O(L^{-d})\delta_{o,x} + O(\lambda^2) \times \begin{cases} \|\!|x|\!\|_L^{(\alpha\wedge 2-d)\ell} & [\alpha \neq 2], \\ (\|\!|x|\!\|_L^{2-d}/\log \|\!|\frac{x}{L}|\!\|_1)^\ell & [\alpha = 2], \end{cases} \tag{3.15}$$

where

$$\ell = \begin{cases} 2 & \text{[percolation]}, \\ 3 & \text{[SAW \& Ising]}. \end{cases} \tag{3.16}$$

Notice from (3.15) that, if $\alpha < 2$ and $d > d_c$ or if $\alpha = 2$ and $d \geq d_c$, then $\Pi_p * D$ in (3.6) can be treated, after normalization, as a probability distribution. For $\alpha = 2$, for example, there are finite constants c, c', c'' such that

$$\begin{aligned} (\Pi_p * D)(x) &\overset{(3.15)}{\geq} (1 - cL^{-d})D(x) - c'\lambda^2 \sum_y \frac{\|\!|y|\!\|_L^{\ell(2-d)}}{(\log \|\!|\frac{y}{L}|\!\|_1)^\ell} D(x-y) \\ &\overset{\text{Lemma 3.1}}{\geq} (1 - cL^{-d} - c''\lambda^3)D(x), \end{aligned} \tag{3.17}$$

which is positive for all x, if $\lambda \ll 1$. Therefore,

$$\mathcal{D}(x) = \frac{(\Pi_p * D)(x)}{\hat{\Pi}_p(0)} \tag{3.18}$$

is a probability distribution that satisfies all the properties in (D1)–(D3), and its Green function $\sum_{n=0}^\infty \mathcal{D}^{*n}(x)$ is bounded by $(1 + O(\lambda^3))S_1(x)$ for every x (see [10, Section 3.2]

[1] The bubble condition $G_{p_c}^{*2}(o) < \infty$ for SAW/the Ising model and the triangle condition $G_{p_c}^{*3}(o) < \infty$ for percolation are sufficient conditions for the susceptibility χ_p and other observables to exhibit their mean-field behavior. The log correction for $\alpha = 2$ is the key to extend the mean-field results down to $d = d_c$ since, for example, the tail of the sum in the triangle condition can be estimated, for any $R > 1$, as

$$\sum_{x:|x|>R} G_{p_c}(x)\, G_{p_c}^{*2}(x) \overset{d\geq 4}{\lesssim} \int_R^\infty \frac{\mathrm{d}r}{r} \frac{r^{6-d}}{(\log r)^2} \overset{d\geq 6}{<} \infty. \tag{3.14}$$

for more details). By (3.15) and Lemma 3.1, we obtain that, for $x \neq o$,

$$G_p(x) \leq \big(1+O(\lambda^3)\big)(\Pi_p * S_1)(x) \leq \big(1+O(\lambda^3)\big)S_1(x) + O(\lambda^4)\frac{|||x|||_L^{2-d}}{\log |||\frac{x}{L}|||_1}$$

$$\stackrel{\lambda \ll 1}{\leq} 2\lambda \frac{|||x|||_L^{2-d}}{\log |||\frac{x}{L}|||_1}, \tag{3.19}$$

as required. This completes all the steps (i)–(iii) for $\alpha \leq 2$.

If $\alpha > 2$, then we can no longer interpret $\Pi_p * D$ as a probability distribution, because the second term in (3.17) decays slower than D; this is why the model-dependent multiplicative constant A in (1.5) is reduced to 1 only when $\alpha \leq 2$. To overcome this difficulty for $\alpha > 2$, we assume that the "derivative" of the n-step distribution D^{*n} obeys the following bound: for $|y| \leq \frac{1}{3}|x|$,

$$\left| D^{*n}(x) - \frac{D^{*n}(x+y) + D^{*n}(x-y)}{2} \right| \leq n\, \frac{O(L^{\alpha\wedge 2})\, |||y|||_L^2}{|||x|||_L^{d+\alpha\wedge 2+2}}. \tag{3.20}$$

We have shown in [9] that the compound zeta distribution (2.5) for $\alpha \neq 2$ satisfies the above assumption. See [9, Appendix] for more details.

Acknowledgements

This work was supported in part by JSPS KAKENHI Grant Number 18K03406, and in part by JST CREST Grant Number JP22180021. I would like to thank Lung-Chi Chen for the long-time collaboration [6–10] on the long-range models defined by the power-law decaying step-distribution/coupling. I would also like to thank the organizers of 17$^{\text{th}}$ International Symposium "Stochastic Analysis on Large-scale Interacting Systems" at RIMS, Kyoto University, during November 5–8, 2018, for the opportunity to speak on the topic of this article. Finally, I would like to express my gratitude to the referee for positive feedback and valuable suggestions to improve presentation of this paper.

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