Mean-field bound on the 1-arm exponent for Ising ferromagnets in high dimensions

Satoshi Handa*, Markus Heydenreich†, Akira Sakai‡

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Abstract

The 1-arm exponent ρ for the ferromagnetic Ising model on $\mathbb{Z}^d$ is the critical exponent that describes how fast the critical 1-spin expectation at the center of the ball of radius r surrounded by plus spins decays in powers of r. Suppose that the spin-spin coupling J is translation-invariant, $\mathbb{Z}^d$-symmetric and finite-range. Using the random-current representation and assuming the anomalous dimension $\eta = 0$, we show that the optimal mean-field bound $\rho \le 1$ holds for all dimensions $d > 4$. This significantly improves a bound previously obtained by a hyperscaling inequality.

We dedicate this work to Chuck Newman on the occasion of his 70th birthday.

A personal note. I (AS) met Chuck for the first time when I visited New York for a month in summer 2004. I knew him before, not in person, but for his influential papers on phase transitions and critical behavior of percolation and the Ising model. So, naturally, I felt awful to speak with him (partly because of my poor language skills) and to give a presentation on my work back then. However, he was friendly and positive about my presentation, which he may no longer remember. Since then, I became a big fan of his. There must be many researchers like me who were greatly encouraged by Chuck, and I am sure there will be many more.

The topic of my presentation back in summer 2004 was about the critical exponent ρ for the percolation 1-arm probability in high dimensions [25]. The 1-arm probability is the probability that the center of the ball of radius r is connected to its surface by a path of occupied bonds. Compared with most of the other critical exponents, such as β, γ, η and δ, the 1-arm exponent ρ is harder to investigate, not because we still do not know the continuity of the critical percolation probability (= the $r \uparrow \infty$ limit of the critical 1-arm probability), but because ρ is associated with a finite-volume quantity and therefore deals with boundary effects. Even in high dimensions, it is difficult to identify the mean-field value of ρ. However, by the second-moment method [25], it is rather easy to show the one-sided inequality $\rho \le 2$, if ρ exists and $\eta = 0$. The latter assumption is known to hold in high dimensions, thanks to the lace-expansion results [13, 14]. Then, in [22], Kozma and Nachmias finally proved the equality $\rho = 2$ by assuming $\eta = 0$ and using a sophisticated inductive argument.

1 Introduction

We consider the ferromagnetic Ising model at its critical temperature $T = T_c$, and study the 1-spin expectation $\langle \sigma_o \rangle_r^+$ at the center of a ball of radius r surrounded by plus spins. The

*Graduate School of Mathematics, Hokkaido University, Japan. handa@math.sci.hokudai.ac.jp

†Mathematisches Institut, Ludwig-Maximilians-Universität München, Germany. m.heydenreich@lmu.de

‡Faculty of Science, Hokkaido University, Japan. sakai@math.sci.hokudai.ac.jp, https://orcid.org/0000-0003-0943-7842

decreasing limit of $\langle\sigma_o\rangle_r^+$ as $r \uparrow \infty$ is the spontaneous magnetization. Recently, Aizenman, Duminil-Copin and Sidoravicius [4] showed that, if the spin-spin coupling satisfies a strong symmetry condition called reflection-positivity, then the spontaneous magnetization is a continuous function of temperature in all dimensions $d > 2$, in particular $\lim_{r\uparrow\infty}\langle\sigma_o\rangle_r^+ = 0$ at criticality. The present paper gives quantitative bounds on the rate of convergence. The nearest-neighbor model is an example that satisfies reflection-positivity. Also, its spontaneous magnetization on $\mathbb{Z}^2$ is known to be zero at criticality [34]. However, in general, finite-range models do not satisfy reflection-positivity, and therefore we cannot automatically justify continuity of the spontaneous magnetization for, e.g., the next-nearest-neighbor model. Fortunately, by using the lace expansion [7, 26], we can avoid assuming reflection-positivity to ensure $\eta = 0$ (as well as $\beta = 1/2$, $\gamma = 1$, $\delta = 3$) and $\lim_{r\uparrow\infty}\langle\sigma_o\rangle_r^+ = 0$ at criticality in dimensions $d > 4$ if the support of J is large enough.

In this paper, we prove that it does not decay very fast whenever $d > 4$; in this case we prove $\langle\sigma_o\rangle_r^+ \geq r^{-1+o(1)}$. The proof relies on the random-current representation, which is a sophisticated version of the high-temperature expansion. It was initiated in [11] to show the GHS inequality. Then, in 1980's, Aizenman revived it to show that the bubble condition (i.e., square-summability of the critical 2-spin expectation) is a sufficient condition for the mean-field behavior [1, 3, 5]. It is also used in [4, 26, 27] to obtain many useful results for the Ising and φ^4 models. In combination with the second-moment method, we prove a correlation inequality that involves $\langle\sigma_o\rangle_r^+$ and free-boundary 2-spin expectations. Then, by using this correlation inequality, we derive the desired result.

First, we provide the precise definition of the model.

1.1 The model

First we define the Ising model on V_R, which is the d-dimensional ball of radius $R > 0$:

$$V_R = \{v \in \mathbb{Z}^d : |v| \leq R\}. \tag{1.1}$$

It is convenient to use the Euclidean distance $|\cdot|$ here, but our results hold for any norm on the lattice $\mathbb{Z}^d$. We define the Hamiltonian for a spin configuration $\boldsymbol{\sigma} \equiv \{\sigma_v\}_{v\in V_R} \in \{\pm1\}^{V_R}$ as

$$H_{r,R}^h(\boldsymbol{\sigma}) = -\sum_{\{u,v\}\subset V_R} J_{u,v}\sigma_u\sigma_v - h\sum_{v\in\partial V_r}\sigma_v, \tag{1.2}$$

where $J_{u,v} \geq 0$ is a translation-invariant, $\mathbb{Z}^d$-symmetric and finite-range coupling, h is the strength of the external magnetic field, and ∂V_r ($r < R$) is the boundary of V_r:

$$\partial V_r = \{v \in V_R \setminus V_r : \exists u \in V_r \text{ such that } J_{u,v} > 0\}. \tag{1.3}$$

We note that it is crucial to impose the external magnetic field only on ∂V_r. Due to this slightly unusual setup, we will eventually be able to derive an essential correlation inequality that differs from the one for percolation.

The thermal expectation of a function f on spin configurations at the critical temperature T_c is given by

$$\langle f\rangle_{r,R}^h = \frac{1}{2^{|V_R|}}\sum_{\boldsymbol{\sigma}\in\{\pm1\}^{V_R}} f(\boldsymbol{\sigma})\,\frac{e^{-H_{r,R}^h(\boldsymbol{\sigma})/T_\mathrm{c}}}{Z_{r,R}^h}, \qquad Z_{r,R}^h = \frac{1}{2^{|V_R|}}\sum_{\boldsymbol{\sigma}\in\{\pm1\}^{V_R}} e^{-H_{r,R}^h(\boldsymbol{\sigma})/T_\mathrm{c}}. \tag{1.4}$$

The major quantities to be investigated are the 1-spin and 2-spin expectations. Since they are increasing in h by Griffiths' inequality [10], we simply denote their limits by

$$\langle\sigma_x\rangle_r^+ = \lim_{h\uparrow\infty}\langle\sigma_x\rangle_{r,R}^h \qquad [x\in V_r\cup\partial V_r], \tag{1.5}$$

$$\langle\sigma_x\sigma_y\rangle_R = \lim_{h\downarrow 0}\langle\sigma_x\sigma_y\rangle_{r,R}^h \qquad [x,y\in V_R]. \tag{1.6}$$

Since $\langle\sigma_x\sigma_y\rangle_R$ is also increasing in R by Griffiths' inequality, we denote its limit by

$$\langle\sigma_x\sigma_y\rangle = \lim_{R\uparrow\infty}\langle\sigma_x\sigma_y\rangle_R. \tag{1.7}$$

In the following statement (as well as later in the proofs) we use the notation $f\asymp g$ to mean that the ratio f/g is bounded away from zero and infinity (in the prescribed limit). One assumption that we shall make throughout is the mean-field decay for the critical 2-spin expectation (or often called two-point function)

$$\langle\sigma_o\sigma_x\rangle \asymp |x|^{2-d} \quad \text{as } |x|\uparrow\infty. \tag{1.8}$$

A sharp asymptotic expression that implies (1.8) is proven by the lace expansion for a fairly general class of J, whenever the support of J is sufficiently large [26]. We note that reflection positivity has not succeeded in providing the above two-sided x-space bound; only one exception is the nearest-neighbor model, for which a one-sided x-space bound is proven [31]. In dimensions $d<4$, the exponent on the right-hand side may change. An exact solution for $d=2$ was identified by Wu et al. [33], which implies $\langle\sigma_o\sigma_x\rangle\asymp|x|^{-1/4}$ as $|x|\uparrow\infty$.

1.2 The main result

We are investigating the 1-arm exponent for the Ising model at criticality, informally described as $\langle\sigma_o\rangle_r^+\approx r^{-\rho}$ as $r\uparrow\infty$. In order to make the symbol $\approx$ precise, we give the formal definition

$$\rho = -\liminf_{r\to\infty}\frac{\log\langle\sigma_o\rangle_r^+}{\log r}. \tag{1.9}$$

A more conventional way of defining ρ is by letting $\langle\sigma_o\rangle_r^+\asymp r^{-\rho}$ as $r\uparrow\infty$, which was used to define the percolation 1-arm exponent [22, 25]. However, the latter definition does not necessarily guarantee the existence of ρ. To avoid this existence issue, we adopt the former definition (1.9).

Our main result is the one-sided bound $\rho\le 1$ in the mean-field regime, i.e., when $d>4$ and (1.8) holds. Folklore of statistical physics predicts that (1.9) is actually a limit. The use of limit inferior is somewhat arbitrary (lim sup would be also possible), but this choice gives the strongest result.

Theorem 1.1. *For the ferromagnetic Ising model on $\mathbb{Z}^d$, $d>4$, defined by a translation-invariant, $\mathbb{Z}^d$-symmetric and finite-range spin-spin coupling satisfying (1.8),*

$$\liminf_{r\to\infty} r^{1+\varepsilon}\langle\sigma_o\rangle_r^+ = \infty \tag{1.10}$$

whenever $\varepsilon>0$. Consequently, the critical exponent ρ defined in (1.9) satisfies $\rho\le 1$.

Tasaki [32] proved that $\langle\sigma_o\rangle^+_{|x|/3} \ge \sqrt{\langle\sigma_o\sigma_x\rangle}$ holds at any temperature for sufficiently large $|x|$ (so that $|x|/3$ is larger than the range of the spin-spin coupling). In dimensions $d > 4$, this implies the hyperscaling inequality $\rho \le (d-2)/2$, and our bound in (1.10) improves on Tasaki's result.

It is a challenge now to prove

$$\limsup_{r\to\infty} r^{1-\varepsilon}\langle\sigma_o\rangle^+_r = 0 \tag{1.11}$$

for any $\varepsilon > 0$, which implies readily (together with our theorem) that (1.9) is actually a limit and $\rho = 1$.

Our proof of $\rho \le 1$ uses (1.8), which requires $d > 4$ (and the support of J to be large), even though the result is believed to be true for all dimensions $d \ge 2$. The aforementioned correlation inequality $\langle\sigma_o\rangle^+_{|x|/3} \ge \sqrt{\langle\sigma_o\sigma_x\rangle}$ combined with the exact solution for $d = 2$ [33] and numerical predictions for $d = 3, 4$ supports this belief. This is why we call $\rho \le 1$ the optimal mean-field bound.

Another key ingredient for the proof of $\rho \le 1$ is the random-current representation, which provides a translation between spin correlations and percolation-like connectivity events. Then, by applying the second-moment method to the connectivity events as explained below for percolation, we can derive a crucial correlation inequality (cf. (2.7)) that relates 1-spin and 2-spin expectations. To explain what the second-moment method is and to compare the resulting correlation inequalities for the two models, we spend the next subsection to explain the derivation of the mean-field bound on the percolation 1-arm exponent, i.e., $\rho \le 2$ for $d > 6$.

1.3 Derivation of the mean-field bound for percolation

We consider the following bond percolation on $\mathbb{Z}^d$. Each bond $\{u, v\} \subset \mathbb{Z}^d$ is either occupied or vacant with probability $pJ_{u,v}$ or $1 - pJ_{u,v}$, independently of the other bonds, where $p \ge 0$ is the percolation parameter. The 2-point function $G_p(x, y)$ is the probability that x is connected to y by a path of occupied bonds ($G_p(x, x) = 1$ by convention). It is well-known that, for any $d \ge 2$, there is a nontrivial critical point p_c such that the susceptibility $\sum_x G_p(o, x)$ is finite if and only if $p < p_c$ [6]. The 1-arm probability θ_r, which is the probability that the center of the ball of radius r is connected to its surface by a path of occupied bonds, also exhibits a phase transition at p_c [2]: $\theta(p) \equiv \lim_{r\uparrow\infty} \theta_r = 0$ if $p < p_c$ and $\theta(p) > 0$ if $p > p_c$. Although the continuity $\theta(p_c) = 0$ has not yet been proven in full generality, it is shown by the lace expansion [9, 15] that, if $d > 6$ and the support of J is sufficiently large, then $\theta(p_c) = 0$ and $G_{p_c}(o, x) \asymp |x|^{2-d}$ as $|x| \uparrow \infty$. This Newtonian behavior of G_{p_c} is believed not to hold in lower dimensions.

Fix $p = p_c$ and define the percolation 1-arm exponent ρ by letting $\theta_r \asymp r^{-\rho}$ as $r \uparrow \infty$. Since $\theta_{|x|/3} \ge \sqrt{G_p(o, x)}$ if $|x| \gg 1$ [32], it is known that the same hyperscaling inequality $\rho \le (d-2)/2$ holds for all dimensions $d > 6$. In [25], we were able to improve this to the optimal mean-field bound $\rho \le 2$ for $d > 6$ by using the second-moment method, which we explain now. Let X_r be the random number of vertices on ∂V_r that are connected to the origin o. We note that X_r can be positive only when o is connected to ∂V_r. Then, by the Schwarz inequality,

$$\mathbb{E}_p[X_r]^2 = \mathbb{E}_p\Big[X_r \mathbb{1}_{\{o \text{ is connected to } \partial V_r\}}\Big]^2 \le \underbrace{\mathbb{E}_p\Big[\mathbb{1}_{\{o \text{ is connected to } \partial V_r\}}\Big]}_{=\theta_r} \mathbb{E}_p[X_r^2], \tag{1.12}$$

which implies $\theta_r \geq \mathbb{E}_p[X_r]^2/\mathbb{E}_p[X_r^2]$. Notice that $\mathbb{E}_p[X_r] = \sum_{x\in\partial V_r} G_p(o,x)$ and that, by the tree-graph inequality [6],

$$\mathbb{E}_p[X_r^2] = \sum_{x,y\in\partial V_r} \mathbb{P}_p(o \text{ is connected to } x, y) \leq \sum_{\substack{u\in\mathbb{Z}^d\\ x,y\in\partial V_r}} G_p(o,u)\, G_p(u,x)\, G_p(u,y). \tag{1.13}$$

As a result, we arrive at the correlation inequality

$$\theta_r \geq \frac{\Big(\sum_{x\in\partial V_r} G_p(o,x)\Big)^2}{\sum_{\substack{u\in\mathbb{Z}^d\\ x,y\in\partial V_r}} G_p(o,u)\, G_p(u,x)\, G_p(u,y)}. \tag{1.14}$$

Using $G_{p_c}(x,y) \asymp |||x-y|||^{2-d}$, where $|||\cdot||| = |\cdot| \vee 1$ is to avoid singularity around zero, we can readily show that the right-hand side of the above inequality is bounded from below by a multiple of r^{-2}, resulting in $\rho \leq 2$ for $d > 6$.

In order to prove the opposite inequality $\rho \geq 2$ for $d > 6$ to conclude the equality, Kozma and Nachmias [22] use another correlation inequality that involves not only θ_r and G_p but also the mean-field cluster-size distribution. The Ising cluster-size distribution under the random-current representation is not available yet, and we are currently heading in that direction.

1.4 Further discussion

1. *On trees.* The Ising model on trees, also known as the Ising model on the Bethe lattice, is rigorously studied since the 1970's [21, 24]. In contrast to amenable graphs, the phase transition on trees can appear even when there is a non-zero homogeneous random field, cf. [20]. One line of research considers critical-field Ising models under the influence of an inhomogeneous external field, for which we refer to the discussion in [8]. The absence of loops in the underlying graph makes it easier to analyze, and a number of critical exponents are known to take on their mean-field values, see [28, Section 4.2].

   For the Ising model on a regular tree, it is shown [18] that

   $$\langle\sigma_o\rangle_r^+ \asymp r^{-1/2} \quad \text{as } r \uparrow \infty, \tag{1.15}$$

   where, instead of the ball V_r, we are using the subtree of depth r from the root (with the plus-boundary condition). This proves $\rho = 1/2$ on trees. The discrepancy to the high-dimensional setting can be resolved by adjusting the notion of distance in the tree, that is, one should rather work with the metric $\mathrm{dist}(o,x) := \sqrt{\mathrm{depth}(x)}$ incorporating spatial effects when embedding the tree into the lattice $\mathbb{Z}^d$. With this notion of distance, we get the mean-field value $\rho = 1$. The same situation occurs for percolation, where we refer to [12, 16] and for a discussion of this issue.

2. *Long-range models.* In our result, we assumed that the spin-spin coupling J is finite-range, that is, there is an $M > 0$ such that $J_{o,x} = 0$ whenever $|x| > M$. We believe that $\rho = 1$ is true even for infinite-range couplings with sufficiently fast decaying tails, although the boundary ∂V_r on which the external magnetic field is imposed under the current setting is

no longer bounded. The situation may change when we consider couplings with regularly-varying tails, and we focus now on the situation when

$$J_{o,x} \asymp |x|^{-d-\alpha} \quad \text{as } |x| \uparrow \infty, \tag{1.16}$$

for some $\alpha > 0$. In our earlier work [7], we show that, under a suitable spread-out condition, the critical 2-spin expectation scales as

$$\langle \sigma_o \sigma_x \rangle \asymp |x|^{\alpha \wedge 2-d} \quad \text{as } |x| \uparrow \infty, \tag{1.17}$$

in contrast to (1.8). In particular, there is a crossover at $\alpha = 2$ between a "finite-range regime" and a "long-range regime".

For the critical exponent ρ, it is tempting to believe that this crossover happens for $\alpha = 4$. The reason for this is again a comparable result for percolation: Hulshof [19] proved that, if $G_{p_c}(o,x) \asymp |x|^{\alpha\wedge 2-d}$ as $|x| \uparrow \infty$, then the critical 1-arm probability scales as $\theta_r \asymp r^{-(\alpha\wedge 4)/2}$ as $r \uparrow \infty$. In view of Hulshof's result, it is plausible that, for the long-range Ising model with couplings like in (1.16), it is the case that $\rho = (\alpha \wedge 4)/4$.

3. *The (1-component) φ^4 model.* This spin model is considered to be in the same universality class as Ising ferromagnets [1]. It can be constructed as an $N \uparrow \infty$ limit of a properly coupled N ferromagnetic Ising systems [30], and therefore we can apply the random-current representation for the Ising model. By virtue of this representation, we can use the lace expansion to show that the critical 2-spin expectation satisfies (1.8) for a large class of short-range models [27]. It is natural to be interested in the critical 1-spin expectation similar to $\langle \sigma_o \rangle_r^+$ for the Ising model. However, since the φ^4 spin is an unbounded variable, we cannot simply take $h \uparrow \infty$ to define the 1-spin expectation under the "plus-boundary" condition. Once it is defined appropriately, we believe that its 1-arm exponent also satisfies the mean-field bound $\rho \le 1$ for $d > 4$.

From the next section, we begin the proof of the main theorem. In Section 2.1, we introduce notation and definitions associated with the random-current representation. In Section 2.2, we use the random-current representation and the second-moment method to derive a key correlation inequality. Finally, in Section 2.3, we use the obtained correlation inequality and (1.8) to conclude that $\rho \le 1$ for $d > 4$.

2 Proof of the results

2.1 The random-current representation

A current configuration $\boldsymbol{n} \equiv \{n_b\}$ is a set of nonnegative integers on bonds $b \in B_R \equiv \big\{\{u,v\} \subset V_R : J_{u,v} > 0\big\}$ or $b \in G_r \equiv \big\{\{v,g\} : v \in \partial V_r\big\}$, where g is an imaginary ghost site. Given a current configuration $\boldsymbol{n}$, we define the source set $\partial \boldsymbol{n}$ as

$$\partial \boldsymbol{n} = \bigg\{ v \in V_R \cup \{g\} : \sum_{b \ni v} n_b \text{ is odd} \bigg\}, \tag{2.1}$$

and the weight functions $w_{r,R}^h(\boldsymbol{n})$ and $w_R(\boldsymbol{n})$ as

$$w_{r,R}^h(\boldsymbol{n}) = \prod_{b \in B_R} \frac{(J_b/T_c)^{n_b}}{n_b!} \prod_{b' \in G_r} \frac{(h/T_c)^{n_{b'}}}{n_{b'}!}, \qquad w_R(\boldsymbol{n}) = w_{r,R}^0(\boldsymbol{n}). \tag{2.2}$$

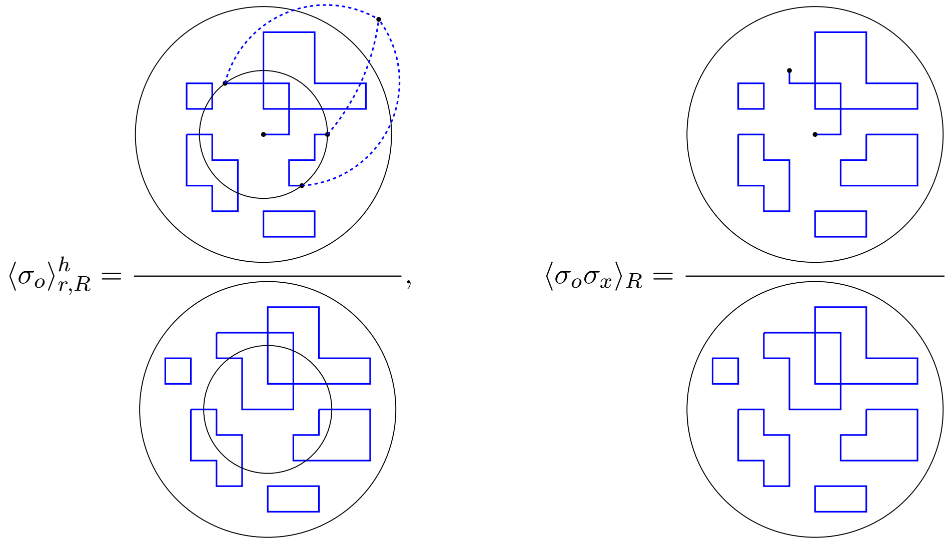

Figure 1: The random-current representation for $\langle \sigma_o \rangle^h_{r,R}$ and $\langle \sigma_o \sigma_x \rangle_R$. The bonds with even current are all omitted. The vertex connected by dashed line segments is the ghost site g.

Then, we obtain the following random-current representation (cf. Figure 1):

$$Z^h_{r,R} = \sum_{\partial \boldsymbol{n} = \varnothing} w^h_{r,R}(\boldsymbol{n}), \qquad\qquad Z_R = \sum_{\partial \boldsymbol{n} = \varnothing} w_R(\boldsymbol{n}), \tag{2.3}$$

and for $x, y \in V_R$,

$$\langle \sigma_x \rangle^h_{r,R} = \sum_{\partial \boldsymbol{n} = \{x,g\}} \frac{w^h_{r,R}(\boldsymbol{n})}{Z^h_{r,R}}, \qquad\qquad \langle \sigma_x \sigma_y \rangle_R = \sum_{\partial \boldsymbol{n} = \{x\} \triangle \{y\}} \frac{w_R(\boldsymbol{n})}{Z_R}. \tag{2.4}$$

Given a current configuration $\boldsymbol{n} = \{n_b\}$, we say that x is $\boldsymbol{n}$-connected to y, denoted $x \underset{\boldsymbol{n}}{\longleftrightarrow} y$ if either $x = y \in V_R \cup \{g\}$ or there is a path from x to y consisting of bonds $b \in B_R \cup G_r$ with $n_b > 0$. For $A \subset V_R \cup \{g\}$, we also say that x is $\boldsymbol{n}$-connected to y in A, denoted $x \underset{\boldsymbol{n}}{\longleftrightarrow} y$ in A, if either $x = y \in A$ or there is a path from x to y consisting of bonds $b \subset A$ with $n_b > 0$.

Given a subset $A \subset V_R$, we define

$$W_A(\boldsymbol{m}) = \prod_{b \in A} \frac{(J_b/T_{\mathrm{c}})^{m_b}}{m_b!}, \qquad\qquad \mathcal{Z}_A = \sum_{\partial \boldsymbol{m} = \varnothing} W_A(\boldsymbol{m}). \tag{2.5}$$

The most important feature of the random-current representation is the so-called source-switching lemma (e.g., [26, Lemma 2.3]). We state the version we use the most in this paper as below. This is an immediate consequence from the source-switching lemma.

Lemma 2.1 (Consequence from the source-switching lemma, [26])**.** *For any subsets $A \subset V_R$ and $B \subset V_R \cup \{g\}$, any $x, y \in V_R$ and any function f on current configurations,*

$$\sum_{\substack{\partial \boldsymbol{n} = B \\ \partial \boldsymbol{m} = \varnothing}} w^h_{r,R}(\boldsymbol{n})\, W_A(\boldsymbol{m}) \mathbb{1}\{x \underset{\boldsymbol{n}+\boldsymbol{m}}{\longleftrightarrow} y \text{ in } A\}\, f(\boldsymbol{n}+\boldsymbol{m}) = \sum_{\substack{\partial \boldsymbol{n} = B \triangle \{x\} \triangle \{y\} \\ \partial \boldsymbol{m} = \{x\} \triangle \{y\}}} w^h_{r,R}(\boldsymbol{n})\, W_A(\boldsymbol{m})\, f(\boldsymbol{n}+\boldsymbol{m}). \tag{2.6}$$

For a proof, we refer to [26, Lemma 2.3].

2.2 A correlation inequality

The main technical vehicle in the proof of Theorem 1.1 is the following correlation inequality that relates $\langle\sigma_o\rangle_r^+$ to the sum of 2-spin expectations.

Proposition 2.2. *For the ferromagnetic Ising model,*

$$\langle\sigma_o\rangle_r^+ \geq \frac{\left(\sum_{x\in\partial V_r}\langle\sigma_o\sigma_x\rangle\right)^2}{\sum_{x,y\in\partial V_r}\langle\sigma_o\sigma_x\rangle\langle\sigma_x\sigma_y\rangle + \sum_{\substack{u\in\mathbb{Z}^d\\ x,y\in\partial V_r}}\langle\sigma_o\sigma_u\rangle\langle\sigma_u\sigma_x\rangle\langle\sigma_u\sigma_y\rangle\langle\sigma_o\rangle^+_{\mathrm{dist}(u,\partial V_r)}}. \tag{2.7}$$

Compare this with the correlation inequality (1.14) for percolation. The extra factor in the denominator of (2.7), $\langle\sigma_o\rangle^+_{\mathrm{dist}(u,\partial V_r)}$, will eventually be the key to obtain the optimal mean-field bound on the Ising 1-arm exponent.

Proof of Proposition 2.2. The proof is carried out in four steps.

Step 1: The second-moment method. Let

$$X_r(\boldsymbol{n}) = \sum_{x\in\partial V_r}\mathbb{1}\{o \underset{\boldsymbol{n}}{\longleftrightarrow} x \text{ in } V_R\}. \tag{2.8}$$

Then, by the Schwarz inequality, we obtain

$$\begin{aligned}\sum_{\substack{\partial\boldsymbol{n}=\{o,g\}\\ \partial\boldsymbol{m}=\varnothing}}\frac{w^h_{r,R}(\boldsymbol{n})}{Z^h_{r,R}}\frac{w_R(\boldsymbol{m})}{Z_R}X_r(\boldsymbol{n}+\boldsymbol{m}) &\leq \Bigg(\underbrace{\sum_{\partial\boldsymbol{n}=\{o,g\}}\frac{w^h_{r,R}(\boldsymbol{n})}{Z^h_{r,R}}}_{=\langle\sigma_o\rangle^h_{r,R}}\underbrace{\sum_{\partial\boldsymbol{m}=\varnothing}\frac{w_R(\boldsymbol{m})}{Z_R}}_{=1}\Bigg)^{1/2}\\ &\quad\times\Bigg(\sum_{\substack{\partial\boldsymbol{n}=\{o,g\}\\ \partial\boldsymbol{m}=\varnothing}}\frac{w^h_{r,R}(\boldsymbol{n})}{Z^h_{r,R}}\frac{w_R(\boldsymbol{m})}{Z_R}X_r(\boldsymbol{n}+\boldsymbol{m})^2\Bigg)^{1/2}.\end{aligned} \tag{2.9}$$

By Lemma 2.1, we can rewrite the left-hand side as

$$\begin{aligned}\sum_{x\in\partial V_r}\sum_{\substack{\partial\boldsymbol{n}=\{o,g\}\\ \partial\boldsymbol{m}=\varnothing}}\frac{w_R(\boldsymbol{n})}{Z_{r,R}}\frac{w_R(\boldsymbol{m})}{Z_R}\mathbb{1}\{o \underset{\boldsymbol{n}+\boldsymbol{m}}{\longleftrightarrow} x \text{ in } V_R\} &= \sum_{x\in\partial V_r}\sum_{\substack{\partial\boldsymbol{n}=\{x,g\}\\ \partial\boldsymbol{m}=\{o,x\}}}\frac{w^h_{r,R}(\boldsymbol{n})}{Z^h_{r,R}}\frac{w_R(\boldsymbol{m})}{Z_R}\\ &= \sum_{x\in\partial V_r}\langle\sigma_x\rangle^h_{r,R}\langle\sigma_o\sigma_x\rangle_R.\end{aligned} \tag{2.10}$$

As a result, we obtain

$$\langle\sigma_o\rangle^h_{r,R} \geq \frac{\left(\sum_{x\in\partial V_r}\langle\sigma_x\rangle^h_{r,R}\langle\sigma_o\sigma_x\rangle_R\right)^2}{\sum_{\substack{\partial\boldsymbol{n}=\{o,g\}\\ \partial\boldsymbol{m}=\varnothing}}\frac{w^h_{r,R}(\boldsymbol{n})}{Z^h_{r,R}}\frac{w_R(\boldsymbol{m})}{Z_R}X_r(\boldsymbol{n}+\boldsymbol{m})^2}. \tag{2.11}$$

Step 2: Switching sources. Next, we investigate the denominator of the right-hand side of (2.11), which equals

$$\sum_{x,y\in\partial V_r}\sum_{\substack{\partial\boldsymbol{n}=\{o,g\}\\ \partial\boldsymbol{m}=\varnothing}}\frac{w^h_{r,R}(\boldsymbol{n})}{Z^h_{r,R}}\,\frac{w_R(\boldsymbol{m})}{Z_R}\,\mathbb{1}\{o\underset{\boldsymbol{n}+\boldsymbol{m}}{\longleftrightarrow}x,y\text{ in }V_R\}. \tag{2.12}$$

The contribution from the summands $x=y$ may be rewritten as in (2.10). Similarly, for the case of $x\neq y$, we use Lemma 2.1 to obtain

$$\begin{aligned}
&\sum_{\substack{x,y\in\partial V_r\\(x\neq y)}}\sum_{\substack{\partial\boldsymbol{n}=\{o,g\}\\ \partial\boldsymbol{m}=\varnothing}}\frac{w^h_{r,R}(\boldsymbol{n})}{Z^h_{r,R}}\,\frac{w_R(\boldsymbol{m})}{Z_R}\,\mathbb{1}\{o\underset{\boldsymbol{n}+\boldsymbol{m}}{\longleftrightarrow}x,y\text{ in }V_R\}\\
&=\sum_{\substack{x,y\in\partial V_r\\(x\neq y)}}\sum_{\substack{\partial\boldsymbol{n}=\{x,g\}\\ \partial\boldsymbol{m}=\{o,x\}}}\frac{w^h_{r,R}(\boldsymbol{n})}{Z^h_{r,R}}\,\frac{w_R(\boldsymbol{m})}{Z_R}\,\mathbb{1}\{o\underset{\boldsymbol{n}+\boldsymbol{m}}{\longleftrightarrow}y\text{ in }V_R\}.
\end{aligned} \tag{2.13}$$

For the event $o\underset{\boldsymbol{n}+\boldsymbol{m}}{\longleftrightarrow}y$ in V_R to occur under the source constraint $\partial\boldsymbol{n}=\{x,g\}$, $\partial\boldsymbol{m}=\{o,x\}$, either one of the following must be the case:

(i) $o\underset{\boldsymbol{m}}{\longleftrightarrow}y$.

(ii) $o\underset{\boldsymbol{m}}{\not\longleftrightarrow}y$ and $\exists u\in\overline{\mathcal{C}_{\boldsymbol{m}}(o)}\equiv\{v\in V_R:o\underset{\boldsymbol{m}}{\longleftrightarrow}v\}$ that is $(\boldsymbol{n}+\boldsymbol{m}')$-connected to y in $V_R\setminus\overline{\mathcal{C}_{\boldsymbol{m}}(o)}$, where $\boldsymbol{m}'$ is the restriction of $\boldsymbol{m}$ on bonds $b\subset V_R\setminus\overline{\mathcal{C}_{\boldsymbol{m}}(o)}$.

Case (i) is easy; by Lemma 2.1, the contribution to (2.13) is bounded as

$$\begin{aligned}
&\sum_{\substack{x,y\in\partial V_r\\(x\neq y)}}\underbrace{\sum_{\partial\boldsymbol{n}=\{x,g\}}\frac{w^h_{r,R}(\boldsymbol{n})}{Z^h_{r,R}}}_{=\langle\sigma_x\rangle^h_{r,R}\leq 1}\sum_{\partial\boldsymbol{m}=\{o,x\}}\frac{w_R(\boldsymbol{m})}{Z_R}\,\mathbb{1}\{o\underset{\boldsymbol{m}}{\longleftrightarrow}y\}\\
&\leq\sum_{\substack{x,y\in\partial V_r\\(x\neq y)}}\sum_{\substack{\partial\boldsymbol{m}=\{o,x\}\\ \partial\boldsymbol{l}=\varnothing}}\frac{w_R(\boldsymbol{m})}{Z_R}\,\frac{w_R(\boldsymbol{l})}{Z_R}\,\mathbb{1}\{o\underset{\boldsymbol{m}+\boldsymbol{l}}{\longleftrightarrow}y\}\\
&=\sum_{\substack{x,y\in\partial V_r\\(x\neq y)}}\underbrace{\sum_{\partial\boldsymbol{m}=\{x,y\}}\frac{w_R(\boldsymbol{m})}{Z_R}}_{=\langle\sigma_x\sigma_y\rangle_R}\;\underbrace{\sum_{\partial\boldsymbol{l}=\{o,y\}}\frac{w_R(\boldsymbol{l})}{Z_R}}_{=\langle\sigma_o\sigma_y\rangle_R}.
\end{aligned} \tag{2.14}$$

Step 3: Conditioning on clusters. Case (ii) is a bit harder and needs extra care. Here we use the conditioning-on-clusters argument. First, by conditioning on $\overline{\mathcal{C}_{\boldsymbol{m}}(o)}$, we can rewrite the contribution to (2.13) from case (ii) as

$$\sum_{\substack{u\in V_R\\x,y\in\partial V_r\\(x\neq y)}}\sum_{\substack{A\subset V_R\\(o,u,x\in A)}}\sum_{\substack{\partial\boldsymbol{n}=\{x,g\}\\ \partial\boldsymbol{m}=\{o,x\}}}\frac{w^h_{r,R}(\boldsymbol{n})}{Z^h_{r,R}}\,\frac{w_R(\boldsymbol{m})}{Z_R}\,\mathbb{1}\{\overline{\mathcal{C}_{\boldsymbol{m}}(o)}=A\}\,\mathbb{1}\{u\underset{\boldsymbol{n}+\boldsymbol{m}'}{\longleftrightarrow}y\text{ in }V_R\setminus A\}. \tag{2.15}$$

Then, the sum over the current configurations in (2.15) can be rewritten as

$$\sum_{\partial\boldsymbol{m}=\{o,x\}} \frac{W_A(\boldsymbol{m})\,\mathcal{Z}_{V_R\setminus A}}{Z_R}\,\mathbb{1}\{\overline{\mathcal{C}_{\boldsymbol{m}}(o)}=A\} \sum_{\substack{\partial\boldsymbol{n}=\{x,g\}\\ \partial\boldsymbol{m}'=\varnothing}} \frac{w^h_{r,R}(\boldsymbol{n})}{Z^h_{r,R}}\,\frac{W_{V_R\setminus A}(\boldsymbol{m}')}{\mathcal{Z}_{V_R\setminus A}}\,\mathbb{1}\{u \underset{\boldsymbol{n}+\boldsymbol{m}'}{\longleftrightarrow} y \text{ in } V_R\setminus A\}. \tag{2.16}$$

Now, by using Lemma 2.1, the above expression is equal to

$$\sum_{\partial\boldsymbol{m}=\{o,x\}} \frac{W_A(\boldsymbol{m})\,\mathcal{Z}_{V_R\setminus A}}{Z_R}\,\mathbb{1}\{\overline{\mathcal{C}_{\boldsymbol{m}}(o)}=A\} \underbrace{\sum_{\partial\boldsymbol{n}=\{u,x,y,g\}} \frac{w^h_{r,R}(\boldsymbol{n})}{Z^h_{r,R}}}_{=\langle\sigma_u\sigma_x\sigma_y\rangle^h_{r,R}} \underbrace{\sum_{\partial\boldsymbol{m}'=\{u,y\}} \frac{W_{V_R\setminus A}(\boldsymbol{m}')}{\mathcal{Z}_{V_R\setminus A}}}_{=\langle\sigma_u\sigma_y\rangle_{V_R\setminus A}}, \tag{2.17}$$

where $\langle\sigma_u\sigma_y\rangle_{V_R\setminus A}$ is the 2-spin expectation on the vertex set $V_R\setminus A$ under the free-boundary condition, and is bounded by $\langle\sigma_u\sigma_y\rangle_R$ due to monotonicity. As a result, we obtain

$$\begin{aligned}
(2.15) &\le \sum_{\substack{u\in V_R\\ x,y\in\partial V_r\\ (x\neq y)}} \langle\sigma_u\sigma_x\sigma_y\rangle^h_{r,R}\,\langle\sigma_u\sigma_y\rangle_R \sum_{\substack{A\subset V_R\\ (o,u,x\in A)}} \sum_{\partial\boldsymbol{m}=\{o,x\}} \frac{W_A(\boldsymbol{m})\,\mathcal{Z}_{V_R\setminus A}}{Z_R}\,\mathbb{1}\{\overline{\mathcal{C}_{\boldsymbol{m}}(o)}=A\}\\
&= \sum_{\substack{u\in V_R\\ x,y\in\partial V_r\\ (x\neq y)}} \langle\sigma_u\sigma_x\sigma_y\rangle^h_{r,R}\,\langle\sigma_u\sigma_y\rangle_R \sum_{\partial\boldsymbol{m}=\{o,x\}} \frac{w_R(\boldsymbol{m})}{Z_R}\,\mathbb{1}\{o \underset{\boldsymbol{m}}{\longleftrightarrow} u\}\\
&\le \sum_{\substack{u\in V_R\\ x,y\in\partial V_r\\ (x\neq y)}} \langle\sigma_u\sigma_x\sigma_y\rangle^h_{r,R}\,\langle\sigma_u\sigma_y\rangle_R \sum_{\substack{\partial\boldsymbol{m}=\{o,x\}\\ \partial\boldsymbol{l}=\varnothing}} \frac{w_R(\boldsymbol{m})}{Z_R}\,\frac{w_R(\boldsymbol{l})}{Z_R}\,\mathbb{1}\{o \underset{\boldsymbol{m}+\boldsymbol{l}}{\longleftrightarrow} u\}\\
&= \sum_{\substack{u\in V_R\\ x,y\in\partial V_r\\ (x\neq y)}} \langle\sigma_u\sigma_x\sigma_y\rangle^h_{r,R}\,\langle\sigma_u\sigma_y\rangle_R \underbrace{\sum_{\partial\boldsymbol{m}=\{u,x\}} \frac{w_R(\boldsymbol{m})}{Z_R}}_{=\langle\sigma_u\sigma_x\rangle_R} \underbrace{\sum_{\partial\boldsymbol{l}=\{o,u\}} \frac{w_R(\boldsymbol{l})}{Z_R}}_{=\langle\sigma_o\sigma_u\rangle_R},
\end{aligned} \tag{2.18}$$

where, in the last line, we have used Lemma 2.1 again.

Step 4: Conclusion. Summarizing (2.11), (2.14) and (2.18), we arrive at

$$\langle\sigma_o\rangle^h_{r,R} \ge \frac{\left(\displaystyle\sum_{x\in\partial V_r}\langle\sigma_x\rangle^h_{r,R}\,\langle\sigma_o\sigma_x\rangle_R\right)^2}{\displaystyle\sum_{x,y\in\partial V_r}\langle\sigma_o\sigma_x\rangle_R\,\langle\sigma_x\sigma_y\rangle_R + \sum_{\substack{u\in V_R\\ x,y\in\partial V_r}}\langle\sigma_o\sigma_u\rangle_R\,\langle\sigma_u\sigma_x\rangle_R\,\langle\sigma_u\sigma_y\rangle_R\,\langle\sigma_u\sigma_x\sigma_y\rangle^h_{r,R}}. \tag{2.19}$$

Now we take $h\uparrow\infty$ in both sides. In this limit, the spins on ∂V_r take on $+1$. Moreover, by Griffiths' inequality, we have $\lim_{h\uparrow\infty}\langle\sigma_u\sigma_x\sigma_y\rangle^h_{r,R} = \langle\sigma_u\rangle^\infty_{r,R} \le \langle\sigma_o\rangle^+_{\mathrm{dist}(u,\partial V_r)}$. Therefore,

$$\langle\sigma_o\rangle^+_r \ge \frac{\left(\displaystyle\sum_{x\in\partial V_r}\langle\sigma_o\sigma_x\rangle_R\right)^2}{\displaystyle\sum_{x,y\in\partial V_r}\langle\sigma_o\sigma_x\rangle_R\,\langle\sigma_x\sigma_y\rangle_R + \sum_{\substack{u\in V_R\\ x,y\in\partial V_r}}\langle\sigma_o\sigma_u\rangle_R\,\langle\sigma_u\sigma_x\rangle_R\,\langle\sigma_u\sigma_y\rangle_R\,\langle\sigma_o\rangle^+_{\mathrm{dist}(u,\partial V_r)}}. \tag{2.20}$$

Taking $R \uparrow \infty$, we finally obtain (2.7). ■

2.3 Proof of the main theorem

Proof of Theorem 1.1. We proceed indirectly and assume, by contradiction, that (1.10) is false. Then there exist a constant $K > 0$ and a monotone sequence $(r_k)_{k\in\mathbb{N}}$ diverging to ∞ such that

$$\langle \sigma_o \rangle_{r_k}^+ \le K r_k^{-(1+\varepsilon)} \tag{2.21}$$

whenever k is large enough.

We are starting from Proposition 2.2. We estimate every term in the numerator and the denominator of (2.7) using (1.8). Firstly, the numerator of (2.7) is of the order r^2 since

$$\sum_{x\in\partial V_r} \langle \sigma_o \sigma_x \rangle \asymp \sum_{x\in\partial V_r} |x|^{2-d} \asymp r^{d-1} r^{2-d} = r. \tag{2.22}$$

Secondly, the first term in the denominator is of order $O(r^2)$ since

$$\sum_{x,y\in\partial V_r} \langle \sigma_o \sigma_x \rangle \langle \sigma_x \sigma_y \rangle \asymp \sum_{x\in\partial V_r} |||x|||^{2-d} \sum_{y\in\partial V_r} |||x-y|||^{2-d} \asymp r^{d-1} r^{2-d} r = r^2, \tag{2.23}$$

where we have used $|||\cdot||| = |\cdot| \vee 1$ (cf. below (1.14)). The second term in the denominator is the dominant one. To this end, we fix a sequence r_k satisfying (2.21). We split the sum over u into three cases: (i) $|u| < r_k/2$, (ii) $r_k/2 \le |u| < 3r_k/2$, (iii) $3r_k/2 \le |u|$, and show that it is $O({r_k}^3)$ for any $\varepsilon > 0$.

Case (i):

$$\begin{aligned}
&\sum_{\substack{|u|<r_k/2\\ x,y\in\partial V_{r_k}}} |||u|||^{2-d} |||u-x|||^{2-d} |||u-y|||^{2-d} \big|\big|\big| r_k - |u| \big|\big|\big|^{-(1+\varepsilon)} \\
&\quad \asymp r_k^{2(2-d)-(1+\varepsilon)} \underbrace{\sum_{x,y\in\partial V_{r_k}} \sum_{|u|<r_k/2} |||u|||^{2-d}}_{\asymp r_k^{2(d-1)+2}} \\
&\quad \asymp r_k^{3-\varepsilon}.
\end{aligned} \tag{2.24}$$

Case (ii):

$$\begin{aligned}
&\sum_{\substack{r_k/2\le|u|<3r_k/2\\ x,y\in\partial V_{r_k}}} |u|^{2-d} |||u-x|||^{2-d} |||u-y|||^{2-d} \big|\big|\big| r_k - |u| \big|\big|\big|^{-(1+\varepsilon)} \\
&\asymp r_k^{2-d} \sum_{r_k/2\le|u|<3r_k/2} \big|\big|\big| r_k - |u| \big|\big|\big|^{-(1+\varepsilon)} \underbrace{\sum_{x\in\partial V_{r_k}} |||u-x|||^{2-d} \sum_{y\in\partial V_{r_k}} |||u-y|||^{2-d}}_{\asymp r_k^2} \\
&\asymp r_k^{4-d} \int_{r_k/2}^{3r_k/2} |||r_k - l|||^{-(1+\varepsilon)} l^{d-1} dl \\
&\asymp r_k^3 \int_0^{r_k/2} |||l|||^{-(1+\varepsilon)} dl \ \asymp {r_k}^3.
\end{aligned} \tag{2.25}$$

Case (iii): By the Schwarz inequality,

$$
\begin{aligned}
&\sum_{\substack{|u|\geq 3r_k/2\\ x,y\in\partial V_{r_k}}} |u|^{2-d}|u-x|^{2-d}|u-y|^{2-d}(|u|-r_k)^{-(1+\varepsilon)}\\
&\quad\asymp r_k^{2-d-(1+\varepsilon)} \underbrace{\sum_{x,y\in\partial V_{r_k}} \sqrt{\sum_{|u|\geq 3r_k/2}|u-x|^{4-2d}}\sqrt{\sum_{|u|\geq 3r_k/2}|u-y|^{4-2d}}}_{\asymp r_k^{2(d-1)+4-d}}\\
&\quad\asymp r_k^{3-\varepsilon}.
\end{aligned}
\tag{2.26}
$$

Plugging these estimates into the bound of Proposition 2.2 obtains

$$
\langle\sigma_o\rangle_{r_k}^{+} \geq C\frac{{r_k}^2}{{r_k}^2+{r_k}^3} \asymp r_k^{-1} \tag{2.27}
$$

for any large k and for some C (independent of k), which contradicts (2.21), and (1.10) follows.

The claim $\rho \leq 1$ follows straightforwardly: Suppose $\rho > 1+\varepsilon$ for some $\varepsilon > 0$, then there exists a sequence $(r_k)_{k\in\mathbb{N}}$ such that

$$
\frac{\log\langle\sigma_o\rangle_{r_k}^{+}}{\log r_k} < -1-\varepsilon,
$$

and this contradicts (1.10). ■

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References

[1] M. Aizenman. Geometric analysis of ϕ^4 fields and Ising models. *Commun. Math. Phys.* **86**: 1–48 (1982).

[2] M. Aizenman and D.J. Barsky. Sharpness of the phase transition in percolation models. *Commun. Math. Phys.* **108**: 489–526 (1987).

[3] M. Aizenman, D.J. Barsky and R. Fernández. The phase transition in a general class of Ising-type models is sharp. *J. Stat. Phys.* **47**: 343–374 (1987).

[4] M. Aizenman, H. Duminil-Copin and V. Sidoravicius. Random currents and continuity of Ising model's spontaneous magnetization. *Commun. Math. Phys.* **334**: 719–742 (2015).

[5] M. Aizenman and R. Fernández. On the critical behavior of the magnetization in high-dimensional Ising models. *J. Stat. Phys.* **44**: 393–454 (1986).

[6] M. Aizenman and C.M. Newman. Tree graph inequalities and critical behavior in percolation models. *J. Stat. Phys.* **36**: 107–143 (1984).

[7] L.-C. Chen and A. Sakai. Critical two-point functions for long-range statistical-mechanical models in high dimensions. *Ann. Probab.* **43**: 639–681 (2015).

[8] R. Bissacot, E.O. Endo and A.C.D. van Enter. Stability of the phase transition of critical-field Ising model on Cayley trees under inhomogeneous external fields. *Stoch. Proc. Appl.* **127**: 4126–4138 (2017).

[9] R. Fitzner and R. van der Hofstad. Mean-field behavior for nearest-neighbor percolation in $d > 10$. *Electron. J. Probab.* 22: No. 43, 1–65 (2017). An extended version is available at arXiv:1506.07977.

[10] J. Ginibre. General formulation of Griffiths' inequalities. *Commun. Math. Phys.* **16**: 310–328 (1970).

[11] R.B. Griffiths, C.A. Hurst and S. Sherman. Concavity of magnetization of an Ising ferromagnet in a positive external field. *J. Math. Phys.* **11**: 790–795 (1970).

[12] G.R. Grimmett. *Percolation*, 2nd ed., Springer, Berlin (1999).

[13] T. Hara. Decay of correlations in nearest-neighbour self-avoiding walk, percolation, lattice trees and animals. *Ann. Probab.* **36**: 530–593 (2008).

[14] T. Hara, R. van der Hofstad and G. Slade. Critical two-point functions and the lace expansion for spread-out high-dimensional percolation and related models. *Ann. Probab.* **31**: 349–408 (2003).

[15] T. Hara and G. Slade. Mean-field critical behaviour for percolation in high dimensions. *Commun. Math. Phys.* **128**: 333–391 (1990).

[16] M. Heydenreich and R. van der Hofstad. *Progress in High-Dimensional Percolation and Random Graphs*, Springer International Publishing Switzerland (2017).

[17] M. Heydenreich, R. van der Hofstad and A. Sakai. Mean-field behavior for long- and finite range Ising model, percolation and self-avoiding walk. *J. Stat. Phys.* **132**: 1001–1049 (2008).

[18] M. Heydenreich and L. Kolesnikov. The critical 1-arm exponent for the ferromagnetic Ising model on the Bethe lattice. *J. Math. Phys.* **59**: 043301 (2018).

[19] T. Hulshof. The one-arm exponent for mean-field long-range percolation. *Electron. J. Probab.* **20**: No. 115, 1–26 (2015).

[20] J. Jonasson and J.E. Steif. Amenability and phase transition in the ising model. *J. Theor. Probab.* **12**: 549–559 (1999).

[21] S. Katsura and M. Takizawa. Bethe lattice and the Bethe approximation. *Progr. Theor. Phys.* **51**: 82–98 (1974).

[22] G. Kozma and A. Nachmias. Arm exponents in high dimensional percolation. *J. Amer. Math. Soc.* **24**: 375–409 (2011).

[23] E.H. Lieb. A refinement of Simon's correlation inequality. *Commun. Math. Phys.* **77**: 127–135 (1980).

[24] C.J. Preston. *Gibbs States on Countable Sets*, Cambridge University Press, London (1974).

[25] A. Sakai. Mean-field behavior for the survival probability and the percolation point-to-surface connectivity. *J. Stat. Phys.* **117**: 111–130 (2004).

[26] A. Sakai. Lace expansion for the Ising model. *Commun. Math. Phys.* **272**: 283–344 (2007).

[27] A. Sakai. Application of the lace expansion to the φ^4 model. *Commun. Math. Phys.* **336**: 619–648 (2015).

[28] R.H. Schonmann. Multiplicity of phase transitions and mean-field criticality on highly non-amenable graphs. *Commun. Math. Phys.* **219**: 271–322 (2001).

[29] B. Simon. Correlation inequalities and the decay of correlations in ferromagnets. *Commun. Math. Phys.* **77**: 111–126 (1980).

[30] B. Simon and R.B. Griffiths. The $(\phi^4)_2$ field theory as a classical Ising model. *Commun. Math. Phys.* **33**: 145–164 (1973).

[31] A.D. Sokal. An alternate constructive approach to the φ^4_3 quantum field theory, and a possible destructive approach to φ^4_4. *Ann. Inst. Henri Poincaré Phys. Théorique* **37**: 317–398 (1982).

[32] H. Tasaki. Hyperscaling inequalities for percolation. *Commun. Math. Phys.* **113**: 49–65 (1987).

[33] T.T. Wu, B.M. McCoy, C.A. Tracy and E. Barouch. Spin-spin correlation functions for the two-dimensional Ising model: Exact theory in the scaling region. *Phys. Rev. B* **13**: 316–374 (1976).

[34] C.N. Yang. The spontaneous magnetization of a two-dimensional Ising model. *Phys. Rev.* **85**: 808–816 (1952).