Proceedings of 47th Sapporo Symposium on Partial Differential Equations

Edited by
S.-I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, H. Kubo, H. Kuroda,
Y. Liu, T. Ozawa, T. Sakajo, and K. Tsutaya

Series #183, 2022

HOKKAIDO UNIVERSITY
TECHNICAL REPORT SERIES IN MATHEMATICS

#164 S. -I. Ei, Y. Giga, S. Jimbo, H. Kubo, T. Ozawa, T. Sakajo, H. Takaoka, Y. Tonegawa and K. Tsutaya, Proceedings of the 40th Sapporo Symposium on Partial Differential Equations, 122 pages. 2015.

#165 A. Tsuchida, Y. Aikawa, K. Asahara, M. Abe, Y. Kabata, H. Saito, F. Nakamura and S. Honda, 第 12 回数学総合若手研究集会, 373 pages. 2016.

#166 S. -I. Ei, Y. Giga, S. Jimbo, H. Kubo, T. Ozawa, T. Sakajo, H. Takaoka, Y. Tonegawa and K. Tsutaya, Proceedings of the 41st Sapporo Symposium on Partial Differential Equations, 110 pages. 2016.

#167 Kubo, Hideo and Ozawa, Tohru and Takamura, Hiroyuki, Mathematical Analysis for Stability in Nonlinear Dynamics - in honor of Professor Vladimir Georgiev on his 60th birthday -, 79 pages. 2016.

#168 F. Nakamura, Y. Aikawa, K. Asahara, M. Abe, D. Komori, H. Saito, S. Handa, K. Fujisawa, S. Honda, A. Rodríguez Mulet, 第 13 回数学総合若手研究集会, 501 pages. 2017.

#169 N. Tanaka (K. Kiyohara, T. Morimoto, K. Yamaguchi, Eds.), Geometric Theory of Systems of Ordinary Differential Equations I, 177 pages. 2017.

#170 N. Tanaka (K. Kiyohara, T. Morimoto, K. Yamaguchi, Eds.), Geometric Theory of Systems of Ordinary Differential Equations II, 519 pages. 2017.

#171 S. -I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, H. Kubo, T. Ozawa, T. Sakajo, Y. Tonegawa, and K. Tsutaya, Proceedings of the 42nd Sapporo Symposium on Partial Differential Equations ‐ In memory of Professor Taira Shirota ‐ , 102 pages. 2017.

#172 Shigeru Kuroda, Takao Namiki, Akihiro Yamaguchi, Yutaka Yamaguchi, 複雑系数理の新展開：津田一郎教授退職記念研究集会報告集, 69 pages. 2018.

#173 S. Handa, D. Komori, K. Fujisawa, A. Rodriguez Mulet, M. Aoki, Y. kamijima, I. Fukuda, T. Yabu, 第 14 回数学総合若手研究集会, 580 pages. 2018.

#174 Yoshikazu Giga, Hideo Kubo, Tohru Ozawa, The 11th Mathematical Society of Japan - Seasonal Institute - The Role of Metrics in the Theory of Partial Differential Equations ‐ , 160 pages. 2018.

#175 S. -I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, H. Kubo, H. Kuroda, Proceedings of 43rd Sapporo Symposium on Partial Differential Equations, 99 pages. 2018.

#176 I. Fukuda, M. Aoki, Y. Ueda, Y. Kamijima, T. Yabu, T. Niimura, N. Satoh, H. Toyokawa, K. Matsuzaka, S. Yamagata, K. Yoshida, 第 15 回数学総合若手研究集会, 657 pages. 2019.

#177 S. -I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, H. Kubo, H. Kuroda, T. Ozawa, T. Sakajo, and K. Tsutaya, Proceedings of 44th Sapporo Symposium on Partial Differential Equations, 115 pages. 2019.

#178 H. Toyokawa, Y. Otsuto, N. Satoh, K. Matsuzaka, S. Yamagata, K. Yoshida, R. Ashikaga, S. Okida, M. Oyama, K. Kuwata K. Misu, H. Ishii, T. Sukekawa, 第 16 回数学総合若手研究集会, 692 pages. 2020.

#179 S. -I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, H. Kubo, H. Kuroda, T. Ozawa, T. Sakajo, K. Tsutaya, Proceedings of 45th Sapporo Symposium on Partial Differential Equations, 106 pages. 2020.

#180 K. Misu, S. Okida, R. Ashikaga, M. Oyama, K. Kuwata, H. Ishii, I. Kudo, T. Sukekawa, M. Seki, A. Hasegawa, K. Hatano, 第 17 回数学総合若手研究集会, 595 pages. 2021.

#181 S.-I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, H. Kubo, H. Kuroda, Y. Liu, T. Ozawa, T. Sakajo, and K. Tsutaya, Proceedings of 46th Sapporo Symposium on Partial Differential Equations, 116 pages. 2021.

#182 Tsubasa Sukekawa(Representative organizer), Hiroshi Ishii, Isamu Kudo, Motoki Seki, Aoi Hasegawa, Kohei Hatano, Keisuke Abiko, Yu Tajima, Katsunori Fujie, Shinpei Makida, 第 18 回数学総合若手研究集会, 862 pages. 2022.

Proceedings of 47th Sapporo Symposium on Partial Differential Equations

Edited by
S.-I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, H. Kubo, H. Kuroda,
Y. Liu, T. Ozawa, T. Sakajo, and K. Tsutaya

Sapporo, 2022

Partially supported by Grant-in-Aid for Scientific Research, the Japan Society for the Promotion of Science.

日本学術振興会科学研究費補助金（基盤研究 B 課題番号 19H01795）
日本学術振興会科学研究費補助金（基盤研究 B 課題番号 22H01129）

Preface

This volume is intended as the proceedings of Sapporo Symposium on Partial Differential Equations, held on August 8 through August 10 in 2022 at Faculty of Science, Hokkaido University.

Sapporo Symposium on PDE has been held annually to present the latest developments on PDE with a broad spectrum of interests not limited to the methods of a particular school. Late Professor Taira Shirota started the symposium more than 40 years ago. Late Professor Kôji Kubota and late Professor Rentaro Agemi made a large contribution to its organization for many years.

We always thank their significant contribution to the progress of the Sapporo Symposium on PDE.

S.-I. Ei (Hokkaido University)
Y. Giga (The University of Tokyo)
N. Hamamuki (Hokkaido University)
S. Jimbo (Hokkaido University)
H. Kubo (Hokkaido University)
H. Kuroda (Hokkaido University)
Y. Liu (Hokkaido University)
T. Ozawa (Waseda University)
T. Sakajo (Kyoto University)
K. Tsutaya (Hirosaki University)

Contents

The 47th Sapporo Symposium on Partial Differential Equations

第 47 回偏微分方程式論札幌シンポジウム

Period	August 8, 2022 – August 10, 2022
Organizers	Hideo Kubo, Yikan Liu
Program Committee	Shin-Ichiro Ei, Yoshikazu Giga, Nao Hamamuki, Shuichi Jimbo, Hideo Kubo, Hirotoshi Kuroda, Yikan Liu, Tohru Ozawa, Takashi Sakajo, Kimitoshi Tsutaya
URL	https://www.math.sci.hokudai.ac.jp/sympo/sapporo/program220808.html

<u>Aug. 8, 2022 (Monday)</u>

13:20-13:30	Opening
13:30-14:30	Daniel Hauer（The University of Sydney） The Dirichlet-to-Neumann map of nonlinear diffusion operators – theory & application
14:40-15:20	菅 徹（大阪公立大学）Toru Kan（Osaka Metropolitan University） On solvability of an initial value problem for a superlinear heat equation
15:20-15:40	＊
15:40-16:40	Sun-Sig Byun (Seoul National University) Calderón-Zygmund type estimates for nonlinear elliptic and parabolic equations with matrix weights
16:50-17:20	古川 賢（理化学研究所）Ken Furukawa (RIKEN) Mathematical justification of the hydrostatic approximation of the primitive equations in anisotropic spaces

<u>August 9, 2022 (Tuesday)</u>

10:00-11:00	中村 玄（北海道大学）Gen Nakamura (Hokkaido University) Forward and inverse analyses for anisotropic elastic and viscoelastic systems
11:10-11:40	Maolin Zhou (Nankai University) Type II Blow-Up and Asymptotic Profile of Keller-Segel Model
13:30-14:30	Martin Wechselberger (The University of Sydney) A geometric singular perturbation analysis of regularised reaction-nonlinear diffusion models including shocks

14:40-15:10	祐川 翼（北海道大学）Tsubasa Sukekawa (Hokkaido University) Stability analysis for 2-mode solutions for mass conserved reaction diffusion system
15:10-15:30	＊
15:30-16:10	本多 正平（東北大学）Shouhei Honda (Tohoku University) PDEs, Geometric measure theory and Ricci curvature on nonsmooth spaces
16:20-16:50	Xiaodan Zhou (Okinawa Institute of Science and Technology Graduate University) Horizontally quasiconvex envelope in the Heisenberg group
16:50-18:00	Free discussion with speakers

August 10, 2022 (Wednesday)

10:00-11:00	Todd Oliynyk (Monash University) Stable big bang singularity formation in general relativity
11:10-11:50	前田 昌也（千葉大学）Masaya Maeda (Chiba University) Small energy stabilization of 1D Klein-Gordon equation with potential
11:50-12:00	Closing

(*: breaks/discussions)

The Dirichlet-to-Neumann map of nonlinear diffusion operators – theory & application

Daniel Hauer

School of Mathematics and Statistics, The University of Sydney, NSW 2006, Australia

Keywords: Elliptic & parabolic problems, regularity of solutions, nonlocal operator, p-Laplace type operators, Asymptotic behavior
2022 MSC: 35J92, 35K92, 35B65, 35B40

My talk is based on the following content and provides a summary of the main results.

1. Motivation - Physical Background

Let $\sigma(x)$ be a positive function that models the electrical *conductivity* of a connected body Ω with smooth boundary $\partial\Omega$. If $u_h : \overline{\Omega} \to \mathbb{R}$ is the electric *potential* corresponding to a given *voltage* h on the boundary $\partial\Omega$ of Ω, then by Ohm's law, the negative *current flux* $-J$ is proportional to the *electric field* $\sigma\nabla u_h$ w.r.t. the conductivity σ; that is, one has

$$-J = \sigma\,\nabla u_h. \tag{1.1}$$

If we assume that there are no sources or sinks of electricity in Ω, then the law of conservation of energy says that

$$\operatorname{div}(J) = 0 \qquad \text{in } \Omega. \tag{1.2}$$

Inserting J given by (1.1) into (1.2) shows that the potential u_h is a solution of the *Dirichlet problem*

$$\begin{cases} -\nabla\cdot\Big(\sigma\,\nabla u_h\Big) = 0 & \text{in } \Omega,\\ \qquad\qquad u_h = h & \text{on } \partial\Omega. \end{cases}$$

It is worth noting that under sufficient regularity assumptions on the conductivity σ and the voltage h, one knows that the solution u_h of this boundary-value problem is unique.

However, there are materials of Ω that are *non-Ohmic* and hence, the current flux J does not obey the *linear* law given by (1.1). One possibility is to assume that the current flux J satisfies a *nonlinear power law* (see, e.g., [10])

$$-J = \sigma\,|\nabla u_h|^{p-2} \tag{1.3}$$

Email address: daniel.hauer@sydney.edu.au (Daniel Hauer)
URL: http://sydney.edu.au/science/people/daniel.hauer.php (Daniel Hauer)

for some exponent $p \in \mathbb{R}$. Then, the power law (1.3) leads to the *nonlinear diffusion equation*

$$- \nabla \cdot \Big(\sigma \, |\nabla u_h|^{p-2} \nabla u_h \Big) = 0. \tag{1.4}$$

In the case of normalized conductivity $\sigma \equiv 1$, equation (1.4) reduces to the celebrated *p-Laplace equation*

$$- \Delta_p u := -\nabla \cdot \Big(|\nabla u|^{p-2} \nabla u \Big) = -|\nabla u|^{p-4} \Big[|\nabla u|^2 \Delta u + (p-2) \sum_{i,j=1}^{d} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} \Big] = 0, \tag{1.5}$$

which for $p = 2$ becomes the famous linear *Laplace equation*

$$-\Delta u := -\sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = 0.$$

In electricity, the power law (1.3) appears, for example, in certain polycrystalline materials near the superconducting-normal transition (see, e.g., [15, 23]). Further, it may be used to approximate more complicated non-Ohmic problems; for example, for $p = 1$, one finds the stationary equation of the *total variational flow* [4, 34]

$$-\nabla \cdot \Big(\frac{\nabla u}{|\nabla u|} \Big) = 0,$$

(see Section 5 for more details), or in the limit case "$p = \infty$", one arrives to the famous ∞-*Laplace equation* [20]

$$- \sum_{i,j=1}^{d} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0.$$

The quasi-linear 2^{nd}-order partial differential equation (1.4) appears in many other diffusion phenomena including nonlinear dielectrics [15, 27, 26] and plastic moulding [9]. In particular, (1.4) occurs to model electro-rheological [43] and thermorheological fluids [5], non-Newtonian fluids such as plasma or glue, viscous flows in glaciology [28], some plasticity phenomena [11], in image processing [38] and conformal geometry [40]. We refer, for example, to the PhD thesis [14] by Brander for further references.

Physicists study physical phenomena through observing multiple experiments with various boundary data h on $\partial\Omega$ and collect data through measurements, for example, of the *current flux*

$$\sigma \, |\nabla u_h|^{p-2} \nabla u_h \cdot \vec{\nu} \qquad \text{at the boundary } \partial\Omega$$

of a solution u_h of the *Dirichlet problem*

$$\begin{cases} -\nabla \cdot \Big(\sigma \, |\nabla u_h|^{p-2} \nabla u_h \Big) = 0 & \text{in } \Omega, \\ u_h = h & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

Here, we denote by $\vec{\nu}$ the outward pointing unit normal vector at the boundary $\partial\Omega$. This type of research of physical phenomenon leads naturally to the following open problem (cf. [17] for $p = 2$ and [44, 13] for $p \neq 2$).

Problem 1 (Calderón's inverse conductivity problem)**.** Determine the conductivity function $\sigma(x)$ from the knowledge of the current measurements $\sigma\,|\nabla u_h|^{p-2}\nabla u_h\cdot\vec{\nu}$ at the boundary $\partial\Omega$ induced by the voltages h on $\partial\Omega$. In other words, recover σ from the knowledge of the *Dirichlet-to-Neumann map*

$$\Lambda_\sigma : h \mapsto \Lambda_\sigma := \sigma\,|\nabla u_h|^{p-2}\nabla u_h\cdot\vec{\nu}_{|\partial\Omega}, \tag{1.7}$$

where u_h is a solution of the Dirichlet problem (1.6).

In order to be able to attack Calderón's inverse conductivity problem, it is crucial to know the analytic properties of the Dirichlet-to-Neumann map Λ_σ. In this talk, my aim is to illustrate a few of the many properties and applications of the Dirichlet-to-Neumann map Λ_σ associated with the *weighted p-Laplace operator*

$$\Delta_{p,\sigma}u := \nabla\cdot\Big(\sigma\,|\nabla u|^{p-2}\nabla u\Big).$$

2. The Dirichlet-to-Neumann map - An Analyst's Perspective

Let Ω be a bounded domain in $\mathbb{R}^d$, $d\geq 1$, which in dimension $d\geq 2$ has at least a continuously differentiable boundary $\partial\Omega$, and let $1<p<\infty$. For simplicity, we first discuss the Dirichlet-to-Neumann map Λ_σ given by (1.7) in the case of *normalized conductivity* $\sigma\equiv 1$ and later discuss results for more general conductivity functions σ. In addition, we write Λ instead of Λ_1.

Besides the physically motivated inverse problem (Problem 1), the Dirichlet-to-Neumann map Λ also appears in the mathematical notion of p-capacity and therefore is also named *interior capacity operator* [21, §II.5.1] or, *Neumann operator* (see, for example, [45, p. 41]).

The Dirichlet-to-Neumann map Λ is constructed in two crucial steps, which we intend to discuss now in more detail.

2.1. Step 1 - The Dirichlet problem

As a beginning point, it is worth noting that for $p\neq 2$, solutions u of the p-Laplace equation (1.5) are merely differentiable with a Hölder-continuous gradient ∇u; in other words, u merely belongs to the class $C^{1,\alpha}$. Thus, we cannot expect to find solutions of the Dirichlet problem

$$\begin{cases} -\Delta_p u_h = 0 & \text{in } \Omega,\\ \quad\;\; u_h = h & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

in the sense of *classical* solutions $u_h\in C^2(\Omega)$, and rather use the notion of *weak solutions* or *viscosity/Perron solutions*. For our purposes, it is sufficient to employ the notion of weak solutions u belonging to the function space

$$W^{1,p}(\Omega) := \Big\{u\in L^p(\Omega)\,\Big|\,\exists\,\vec{g}\in L^p(\Omega)^d \text{ s.t. } \int_\Omega u\,\frac{\partial\xi}{\partial x_i} = -\int_\Omega \vec{g}_i\xi\;\forall\,\xi\in C^1_c(\Omega)\Big\},$$

called the first *Sobolev space*. Here, $L^p(\Omega)$ refers to the *Lebesgue space* of all *a.e.-equivalent* classes of measurable functions u with finite L^p-norm

$$\|u\|_p = \left[\int_\Omega |u|^p \mathrm{d}x\right]^{\frac{1}{p}}.$$

Since we assumed that Ω has at least a continuously differentiable boundary $\partial\Omega$, the mapping $u \mapsto u_{|\partial\Omega}$ from $C^{0,1}(\overline{\Omega})$ to $C^{0,1}(\partial\Omega)$ can be extended uniquely and continuously to a mapping

$$\mathcal{Tr} : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

called the *trace operator* (see, e.g., [42]), which admits the following properties.

(1.) The kernel $ker(\mathcal{Tr}) := \{u \in W^{1,p}(\Omega) \mid \mathcal{Tr}(u) = 0\}$ of $\mathcal{Tr}$ is isomorphic with the space $W_0^{1,p}(\Omega)$, which is the $W^{1,p}(\Omega)$-closure of the set of test functions $C_c^\infty(\Omega)$.

(2.) The range $Rg(\mathcal{Tr}) := \{\mathcal{Tr}(u) \mid u \in W^{1,p}(\Omega)\}$ of $\mathcal{Tr}$ is isomorphic to the *fractional* Sobolev (-Slobodečki) space $W^{1-1/p,p}(\partial\Omega)$, which is the linear subspace of all $\varphi \in L^p(\partial\Omega)$ with finite semi-norm (see [42, Section 3.8])

$$[f]_p^p := \int_{\partial\Omega}\int_{\partial\Omega} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{d-2+p}}\,\mathrm{d}x\,\mathrm{d}y.$$

(3.) The trace operator $\mathcal{Tr}$ has a linear bounded right inverse (see [42, Théorème 5.7])

$$Z : W^{1-1/p,p}(\partial\Omega) \to W^{1,p}(\Omega). \tag{2.2}$$

Now, for any boundary value $h \in W^{1-1/p,p}(\partial\Omega)$, let $H \in W^{1,p}(\Omega)$ be such that $\mathcal{Tr}(H) = h$ and set $\mathcal{K}_H := H + W_0^{1,p}(\Omega)$. We consider the energy functional $\mathcal{E} : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{E}(u) = \tfrac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x \tag{2.3}$$

for every $u \in W^{1,p}(\Omega)$. The functional $\mathcal{E}$ is continuously differentiable and the (Fréchet-)derivative $\mathcal{E}' : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))'$ is given by

$$\langle \mathcal{E}'(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\,dx \tag{2.4}$$

for every u, $v \in W^{1,p}(\Omega)$. Further, since $\mathcal{E}$ is strictly convex and by Poincaré's inequality, the restriction $\mathcal{E}_{|\mathcal{K}_H}$ of $\mathcal{E}$ on the affine closed set $\mathcal{K}_H = H + W_0^{1,p}(\Omega)$ is *coercive*; that is,

$$\lim_{\substack{\|u\|_{W^{1,p}(\Omega)}\to\infty \\ u\in\mathcal{K}_H}} \mathcal{E}(u) = \infty,$$

the classical theory of calculus of variation [47, Theorem 2.E] implies that there is a unique solution $u_h \in W^{1,p}(\Omega)$ to the minimization problem:

$$\min\left\{\tfrac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x \;\middle|\; u \in W^{1,p}(\Omega) \text{ with } u - H \in W_0^{1,p}(\Omega)\right\}. \tag{2.5}$$

Moreover, $u_h \in W^{1,p}(\Omega)$ is a minimizer of (2.5) if and only if $u_h - H \in W_0^{1,p}(\Omega)$ and u_h satisfies $\mathcal{E}'(u) = 0$ (see [47, Theorem 2.E]) or, equivalently,

$$\int_\Omega |\nabla u_h|^{p-2}\nabla u_h \nabla v\,dx = 0 \qquad \text{for every } v \in W_0^{1,p}(\Omega). \tag{2.6}$$

From this, the definition of a *weak* solution to Dirichlet problem (2.1) follows naturally.

Definition 2.1. For given boundary value $h \in W^{1-1/p,p}(\partial\Omega)$, we call a function $u_h \in W^{1,p}(\Omega)$ a *weak* solution of Dirichlet problem (2.1) on Ω if $u_h - Z\varphi \in W_0^{1,p}(\Omega)$ and u_h satisfies (2.6).

2.2. Step 2 - The Neumann part

For every boundary value $h \in W^{1-1/p,p}(\partial\Omega)$, let u_h be the unique weak solution of Dirichlet problem (2.1). Then we set

$$P(h) := u_h. \tag{2.7}$$

The mapping P admits several important properties, which we want to summarize in our next proposition.

Proposition 2.2 ([29]). *The following statements hold.*

(1.) *P defined by* (2.7) *is a well-defined, continuous, homogeneous, and injective mapping*

$$P : W^{1-1/p,p}(\partial\Omega) \to W^{1,p}(\Omega).$$

(2.) *Let h_1, $h_2 \in W^{1-1/p,p}(\partial\Omega)$, $H \in W^{1,p}(\Omega)$ with trace $\mathcal{T}r(H) = h_2$, and $\lambda \in \mathbb{R}$. Then,*

$$\tfrac{1}{p}\int_\Omega |\nabla P(\lambda h_1 + h_2)| \,\mathrm{d}x \leq \tfrac{1}{p}\int_\Omega |\lambda \nabla P(h_1) + \nabla H| \,\mathrm{d}x.$$

(3.) *Let $v \in W^{1-1/p,p}(\partial\Omega)$. Then for every $V \in W^{1,p}(\Omega)$ with trace $\mathcal{T}r(V) = v$, there exists a unique $V_0 \in W^{1,p}_0(\Omega)$ such that $P(v) = V_0 + V$.*

Now, if the boundary $\partial\Omega$ is smooth enough (for example, $C^{1,\alpha}$, see [39]) and if the boundary data $h \in C^{1,\alpha}(\partial\Omega)$, then the weak solution $P(h)$ of Dirichlet problem (2.1) is of the class $C^{1,\alpha}(\overline{\Omega})$ and hence the *co-normal derivative*

$$[\Lambda h](x) := |\nabla P(h)(x)|^{p-2} \tfrac{\partial P(h)}{\partial \nu}(x) \qquad \text{exists at every } x \in \partial\Omega. \tag{2.8}$$

On the other hand, we would like to be able to define the Dirichlet-to-Neumann map Λ for boundary data h with less regularity. To see how this can be achieved, we first multiply (2.8) by $P(v)$ for some $v \in C^{1,\alpha}(\partial\Omega)$ with respect to the inner product on $L^2(\partial\Omega)$. Then by Green's formula, one obtains that

$$\int_{\partial\Omega} \Lambda h \, v \,\mathrm{d}\mathcal{H} = \int_\Omega |\nabla P(h)|^{p-2} \nabla P(h) \nabla P(v) \,\mathrm{d}x.$$

Even if $P(h)$ and $P(v)$ merely belong to $W^{1,p}(\Omega)$, the integral on the right-hand side of this equation exists. Thus, we could use this integral to define the Dirichlet-to-Neumann operator Λ as a mapping from $W^{1-1/p,p}(\partial\Omega)$ into the dual space $W^{-(1-1/p),p'}(\partial\Omega) := (W^{1-1/p,p}(\partial\Omega))^*$. But, in general, P is not linear. Thus, it is *a priori* not clear whether the functional

$$v \mapsto \int_\Omega |\nabla P(h)|^{p-2} \nabla P(h) \nabla P(v) \,\mathrm{d}x \tag{2.9}$$

is a bounded linear functional on $W^{1-1/p,p}(\partial\Omega)$. However, according to (3) of Proposition 2.2, for every $v \in W^{1-1/p,p}(\partial\Omega)$, there is a unique $V_0 \in W^{1,p}_0(\Omega)$ such that $P(v) = V_0 + Zv$ and so,

$$\begin{aligned}
&\int_\Omega |\nabla P(h)|^{p-2} \nabla P(h) \nabla P(v) \,\mathrm{d}x \\
&\quad = \int_\Omega |\nabla P(h)|^{p-2} \nabla P(h) \nabla V_0 \,\mathrm{d}x + \int_\Omega |\nabla P(h)|^{p-2} \nabla P(h) \nabla Z v \,\mathrm{d}x \\
&\quad = \int_\Omega |\nabla P(h)|^{p-2} \nabla P(h) \nabla Z v \,\mathrm{d}x
\end{aligned}$$

for every h, $v \in W^{1-1/p,p}(\partial\Omega)$. This shows that the functional (2.9) is linear. Moreover, by Hölder's inequality, one sees that

$$|\langle \Lambda h, v\rangle| \leq \|\nabla P(h)\|_p^{p-1}\|\nabla Zv\|_p,$$

from where we can deduce that $\Lambda \in W^{-(1-1/p),p'}(\partial\Omega)$. This outline justifies our next definition and shows its consistency with the case of smooth functions.

Definition 2.3. The mapping $\Lambda : W^{1-1/p,p}(\partial\Omega) \to W^{-(1-1/p),p'}(\partial\Omega)$ defined by

$$\langle \Lambda h, v\rangle = \int_\Omega |\nabla P(h)|^{p-2}\nabla P(h)\nabla Zv \,\mathrm{d}x \tag{2.10}$$

for every h, $v \in W^{1-1/p,p}(\partial\Omega)$ is called the *Dirichlet-to-Neumann map* associated with the p-Laplace operator Δ_p.

The preceding definition provides a first rigorous mathematical realization of the Dirichlet-to-Neumann map Λ for weak solutions of Dirichlet problem (2.1) and appeared first in [3] and later in [1] (see also [29].

One disadvantage of this definition is that the dual space $W^{-(1-1/p),p'}(\partial\Omega)$ contains distributions which might not be *regular* in the sense of L^1_{loc}-functions. Thus, the realization of the Dirichlet-to-Neumann map Λ with values in $W^{-(1-1/p),p'}(\partial\Omega)$ might not always be suitable.

An alternative realization of the *Dirichlet-to-Neumann map* can be obtained by using the notion of *sub-differential operators* of convex functionals defined on a Hilbert space. For every $\varphi \in L^2(\partial\Omega)$, let

$$\mathcal{F}(\varphi) := \begin{cases} \frac{1}{p}\displaystyle\int_\Omega |\nabla P(h)|^p \,\mathrm{d}x, & \text{if } h \in W^{1-1/p,p}(\partial\Omega) \cap L^2(\partial\Omega), \\ +\infty, & \text{if otherwise.} \end{cases} \tag{2.11}$$

Then, by Proposition 2.2, $\mathcal{F}$ is convex and proper. Moreover, $\mathcal{F}$ is lower-semicontinuous and densely defined in $L^2(\partial\Omega)$ (see [29, Lemma 3.13]). The sub-differential operator $\partial_{L^2}\mathcal{F}$ of $\mathcal{F}$ is a single-valued mapping on $L^2(\partial\Omega)$ and can be characterized by

$$\begin{aligned} &D(\partial_{L^2}\mathcal{F}) = \Big\{ h \in W^{1-1/p,p}(\partial\Omega) \cap L^2(\partial\Omega) \,\Big|\, \exists\, g_h \in L^2(\partial\Omega) \text{ s.t. } g_h = \Lambda h \Big\} \\ &\partial_{L^2}\mathcal{F}(h) = g_h. \end{aligned}$$

In particular, for the realization $\partial_{L^2}\mathcal{F}$ in $L^2(\partial\Omega)$ of the Dirichlet-to-Neumann map Λ, one has that

$$\int_{\partial\Omega} \partial_{L^2}\mathcal{F}(h)\, v \,\mathrm{d}\mathcal{H} = \int_\Omega |\nabla P(h)|^{p-2}\nabla P(h)\nabla Zv \,\mathrm{d}x$$

for every $h \in W^{1-1/p,p}(\partial\Omega) \cap L^2(\partial\Omega)$.

It is worth noting that the realization of the Dirichlet-to-Neumann map Λ in $L^2(\partial\Omega)$ has been first used in [22] (and see also [37]).

2.3. The Dirichlet-to-Neumann map is a nonlocal operator

The Dirichlet-to-Neumann operator Λ is a *nonlocal* operator in the sense that for a given boundary function h, the values $[\Lambda h](x)$ for x ranging in a relatively open neighborhood Γ_{x_0} of a given boundary point $x_0 \in \partial\Omega$ do not only depend on the values $h(x)$ for $x \in \Gamma_{x_0}$, but on the values of h along the whole boundary $\partial\Omega$. To be more precise, suppose the boundary $\partial\Omega$ is of class $C^{2,\beta}$ for a $\beta \in (0,1)$ and let $\partial\Omega = \Gamma_1 \dot{\cup} \Gamma_2$ with Γ_1 relatively open and nonempty. Further, suppose the boundary data $h \in C^{1,\alpha}(\partial\Omega)$ satisfies $h > 0$ on Γ_2 and $h = 0$ on Γ_1. By the weak maximum principle, $P(h)$ is positive in $\overline{\Omega}$ and attains its minimum on the boundary $\partial\Omega$. But since the boundary $\partial\Omega \in C^{2,\beta}$, at every boundary point $x \in \partial\Omega$ the inner ball condition holds. Thus Hopf's boundary-point lemma [46] yields that

$$\Lambda h(x) = |\nabla P(h)(x)|^{p-2} \nabla P(h) \cdot \vec{\nu}(x) > 0 \qquad \text{at every } x \in \Gamma_1,$$

which is in contrast to the condition $h = 0$ on Γ_1 satisfied by the boundary data h.

3. Elliptic & parabolic problems involving the Dirichlet-to-Neumann map

In the past much research has been done to investigate the *linear* Dirichlet-to-Neumann map Λ, that is, when $p = 2$ (cf., for instance, [21, Proposition 1 in §II.5.1], [6, 24]). But one finds only a few results in the literature about elliptic boundary-value problems involving the nonlinear Dirichlet-to-Neumann map Λ. It was first shown (see [1, 2]) that well-posedness in the sense of *entropy solutions* holds for elliptic problems involving Λ with non-homogeneous boundary data in $L^1(\partial\Omega)$. More than 10 years later, it was shown in [29] that well-posedness of such problems also holds in the sense of weak solutions (which is a much simpler notion). In particular, the author established Hölder-continuity of weak solutions h to the *Poisson problem*

$$\Lambda h_f = f \qquad \text{on } \partial\Omega. \tag{3.1}$$

It is worth noting that for given $f \in W_m^{-(1-1/p),p'}(\partial\Omega)$, the Poisson problem (3.1) is nothing less than the inhomogeneous *Neumann problem*

$$\begin{cases} -\Delta_p P(h) = 0 & \text{in } \Omega, \\ |\nabla P(h)|^{p-2} \nabla P(h) \cdot \vec{\nu} = f & \text{on } \partial\Omega, \end{cases}$$

and the resolution map $\Lambda^{-1} : W_m^{-(1-1/p),p'}(\partial\Omega) \to W_m^{1-1/p,p}(\partial\Omega)$ assigning $f \mapsto h_f$ is the *Neumann-to-Dirichlet map*. Here, $W_m^{1-1/p,p}(\partial\Omega)$ denotes the subspace of all functions $h \in W^{1-1/p,p}(\partial\Omega)$ with zero mean $\int_{\partial\Omega} h \,\mathrm{d}\mathcal{H} = 0$, and we write $W_m^{-(1-1/p),p'}(\partial\Omega)$ for its dual space.

Theorem 3.1 ([29])**.** *The following assertions hold.*

(1.) *For every $f \in W_m^{-(1-1/p),p'}(\partial\Omega)$, there is a unique weak solution $h_f \in W_m^{1-1/p,p}(\partial\Omega)$ of Poisson problem (3.1). Moreover, the Neumann-to-Dirichlet map $f \mapsto h_f$ is a continuous mapping $\Lambda^{-1} : W_m^{-(1-1/p),p'}(\partial\Omega) \to W_m^{1-1/p,p}(\partial\Omega)$.*

(2.) *Let $q = \frac{d-1}{p-1-\varepsilon}$ for some $\varepsilon \in (0,1)$ if $p \le d$ and $q = 1$ if $p > d$. Further, let $f \in L^q(\partial\Omega)$ have zero mean $\int_{\partial\Omega} f \,\mathrm{d}\mathcal{H} = 0$. Then there are $\alpha \in (0,1)$ and $c_\alpha \ge 0$ such that every weak solution $h_f \in W^{1-1/p,p}(\partial\Omega)$ of (3.1) belongs to $C^{0,\alpha}(\partial\Omega)$ and satisfies*

$$\|h_f\|_{C^{0,\alpha}(\partial\Omega)} \le c_\alpha \left(\|f\|_{L^q(\partial\Omega)}^{\frac{1}{p-1}} + \|P(h)\|_{L^p(\Omega)} \right) + c_\alpha.$$

The first-order evolution problem

$$\partial_t h + \Lambda h = 0 \qquad \text{on } \partial\Omega \times (0,\infty) \tag{3.2}$$

governed by the Dirichlet-to-Neumann map Λ is equivalent to the *elliptic-parabolic* boundary value problem

$$\begin{cases} -\Delta_p P(h) = 0 & \text{in } \Omega \times (0,\infty), \\ \partial_t h + |\nabla P(h)|^{p-2}\nabla P(h)\cdot \vec{\nu} = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases}$$

To the best of our knowledge, the first result regarding the evolution problem (3.2) goes back to the paper [22] by Díaz and Jiménez [37]. Results about existence and uniqueness of entropy solutions of elliptic and parabolic equations involvingΛ with boundary data in $L^1(\partial\Omega)$ have been first announced in [2] and later established in the thesis [3]. We could complement the existing literature by establishing Hölder-regularity of weak solutions to (3.2) and by establishing their large time asymptotic behavior.

In the following, a solution h of (3.2) is said to be *strong* provided the time-derivative $\partial_t h(t)$ exists in some Banach space X for a.e. $t > 0$.

Theorem 3.2 ([29, 19])**.** *Let $1 \le q \le \infty$ and $\partial\Omega \in C^{2,\alpha}$. Then, the following statements hold.*

(1.) *For every $h_0 \in L^q(\partial\Omega)$ or $h_0 \in C(\partial\Omega)$, there is a unique strong solution*

$$h \in C((0,\infty); W^{1-1/p,p}(\partial\Omega)) \cap W^{1,\infty}([\delta,\infty); L^q(\partial\Omega)) \cap C([0,\infty); L^q(\partial\Omega))$$

$\delta > 0$, of the evolution equation (3.2) with initial datum $h(0) = h_0$ satisfying "conservation of mass"

$$\int_{\partial\Omega} h(t)\,\mathrm{d}\mathcal{H} = \int_{\partial\Omega} h_0\,\mathrm{d}\mathcal{H} \qquad \text{for every } t \ge 0,$$

and there is a constant $C > 0$ such that the right-hand side time-derivative $\partial_{t^+} h$ satisfies

$$\|\partial_{t^+} h(t)\|_{L^q(\partial\Omega)} \le C\,\frac{\|h_0\|_{L^q(\partial\Omega)}}{t} \qquad \text{for every } t > 0. \tag{3.3}$$

In particular, the negative Dirichlet-to-Neumann map $-\Lambda$ generates a C_0-semigroup $\{e^{t\Lambda}\}_{t\ge 0}$ of contractions on $L^q(\partial\Omega)$ (provided $q \neq \infty$) and on $C(\partial\Omega)$.

(2.) *Let $1 < p < d$ and choose $q_0 \ge p$ minimal such that $((d-1)/(d-p)-1)q_0 + p - 2 > 0$. Then for every $1 \le q \le (d-1)q_0/(d-p)$ satisfying $q > (2-p)(d-1)/(p-1)$, one has that*

$$\|e^{t\Lambda}h_0 - \overline{h_0}\|_\infty \lesssim t^{-\delta_q}\,\|h_0 - \overline{h_0}\|_q^{\gamma_q} \tag{3.4}$$

for every $t > 0$ and $h_0 \in L^q(\partial\Omega)$, where the exponents α_q and γ_q are given by

$$\alpha_q = \frac{\alpha^*}{1-\gamma^*\left(1-\frac{q(d-p)}{(d-1)q_0}\right)}, \quad \gamma_q = \frac{\gamma^*\,\frac{q(d-p)}{(d-1)q_0}}{1-\gamma^*\left(1-\frac{q(d-p)}{(d-1)q_0}\right)},$$
$$\alpha^* = \frac{d-p}{(p-1)q_0 + (d-p)(p-2)}, \quad \gamma^* = \frac{(p-1)q_0}{(p-1)q_0 + (d-p)(p-2)}.$$

Similar L^q-L^∞ regularity estimates hold in the case $p = d \ge 2$ and for $p > d$.

(3.) *For* $(2\vee\frac{d-1}{p-1-\varepsilon})\le q\le\infty$ *with some* $\varepsilon\in(0,1)$ *if* $p\le d$ *and for* $2\le q\le\infty$ *if* $p>d$*, there are* $\alpha\in(0,1)$ *and* $c_\alpha>0$ *such that for every* $h_0\in L^q(\partial\Omega)$*, the strong solution* h *of* (3.2) *with initial datum* $h(0)=h_0$ *satisfies*

$$\|h(t)\|_{C^{0,\alpha}(\partial\Omega)}\le c_\alpha\left[\left(\frac{\|h_0\|_{L^q(\partial\Omega)}}{t}\right)^{\frac{1}{p-1}}+\Phi(h(t))\right]+c_\alpha$$

for every $t>0$*, where the function* $t\mapsto\Phi(h(t)):=\mathcal{F}(h(t))^{1/p}+\|h(t)\|_{L^2(\partial\Omega)}$ *is continuous and decreasing on* $(0,\infty)$*.*

(4.) *For every* $h_0\in L^q(\partial\Omega)$*, one has*

$$\lim_{t\to+\infty}h(t)=\overline{h_0}:=\tfrac{1}{\mathcal{H}(\partial\Omega)}\int_{\partial\Omega}h_0\,\mathrm{d}\mathcal{H}\qquad in\ L^q(\partial\Omega). \tag{3.5}$$

Moreover, if $(2\vee\frac{d-1}{p-1-\varepsilon})\le q\le\infty$ *for some* $\varepsilon\in(0,1)$ *provided* $p\le d$*, or if* $2\le q\le\infty$ *provided* $p>d$*, then* (3.5) *holds, in particular, in* $C(\partial\Omega)$ *for every* $h_0\in L^q(\partial\Omega)$*.*

It is worth noting that the two inequalities (3.3) and (3.4) describe an immediate *smoothing effect*. Inequality (3.3) follows either from the fact that the Dirichlet-to-Neumann map Λ associated with the p-Laplace operator generates an analytic semigroup $\{e^{t\Lambda}\}_{t\ge0}$ on $L^q(\partial\Omega)$ if $p=2$ or if $p\neq2$, from the property that Λ is *homogeneous of order* $p-1$ (see [12]), that is, $\Lambda(th)=t^{p-1}\Lambda(h)$ for every $t>0$ and $h\in D(\Lambda)$. The same regularity result still holds for solutions of the equation

$$\partial_t h+\Lambda h+f(h)=0\qquad\text{on }\partial\Omega\times(0,\infty)$$

provided the nonlinearity f is globally Lipschitz continuous (see [30]). If the perturbation f has merely *sublinear growth*, then one still obtains existence of strong solutions but uniqueness can fail. We refer the interested reader to [8] for more details on this.

4. More general conductivity functions

At the current point of knowledge, most of the results mentioned above also hold for the Dirichlet-to-Neumann map Λ_σ associated with the weighted p-Laplace operator $\Delta_{p,\sigma}$ provided the conductivity function σ satisfies at least one of the following conditions:

(i.) $\sigma\in L^\infty(\overline{\Omega})$ and there is a constant $\sigma_0>0$ such that $\sigma\ge\sigma_0$ a.e. on Ω,

(ii.) σ belongs to the class $\mathcal{A}_p$ of *p-Muckenhoupt weights*, or, slightly more generally, is *p-admissible* (cf., [36]).

5. A limiting case: The Dirichlet-to-Neumann map associated with the 1-Laplace operator

The goal of this section is to briefly mention the Dirichlet-to-Neumann map Λ in the *limit case* $p=1$. In this case, the operator associated with Λ is the 1-*Laplace operator*

$$\Delta_1 u:=\operatorname{div}\left(\frac{Du}{|Du|}\right).$$

It is well-known that for every boundary datum $h \in L^1(\partial\Omega)$, there exists a *weak* solution u_h of the Dirichlet problem (see [41])

$$\begin{cases} -\Delta_1 u = 0 & \text{in } \Omega, \\ \quad\; u = h & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

However, difficulties arise in deriving properties of the Dirichlet-to-Neumann map Λ, for instance, from the fact that the notion of weak solutions u_h of (5.1) merely requires that the Dirichlet boundary condition $u = h$ on $\partial\Omega$in a *very weak* sense as we can see as follows.

Definition 5.1. For given $h \in L^1(\partial\Omega)$, we call a function $u \in BV(\Omega)$ a *weak solution* of Dirichlet problem (5.1) if there is a vector field $\mathbf{z}_h \in L^\infty(\Omega;\mathbb{R}^d)$ generalizing $Du/|Du|$ through the three conditions

$$\|\mathbf{z}_h\|_\infty \le 1, \tag{5.2}$$

$$-\operatorname{div}(\mathbf{z}_h) = 0 \qquad \text{in } \mathcal{D}'(\Omega), \text{ and} \tag{5.3}$$

$$(\mathbf{z}_h, Du) = |Du| \qquad \text{as Radon measures} \tag{5.4}$$

and the *weak trace* $[\mathbf{z}_h,\nu]$ on $\partial\Omega$ of the generalized co-normal derivative $\mathbf{z}_h \cdot \nu$ satisfies

$$[\mathbf{z}_h,\nu] \in \operatorname{sign}(h - \mathcal{T}r(u)) \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega. \tag{5.5}$$

To emphasize the dependence of a weak solution u of Dirichlet problem (5.1) on the boundary data h, we often write u_h instead of u.

It was shown in [41] (see also [34]) that for this notion of solutions, the Dirichlet problem (5.1) might have infinitely many solutions u_h. In addition, for each weak solutions u_h of (5.1), there might be infinitely many vector fields $\mathbf{z}_h \in L^\infty(\Omega;\mathbb{R}^d)$ satisfying (5.2)-(5.5) with u_h, and if $\hat{\mathbf{z}}_h$ is a second vector field satisfying (5.2)-(5.5) for another weak solution $\hat{u}_h$ of (5.1) then $\mathbf{z}_h$ also satisfies (5.4)-(5.5) with $\hat{u}_h$ and $\hat{\mathbf{z}}_h$ satisfies (5.4)-(5.5) with u_h (see [34]). Thus, for given boundary data $h \in L^1(\partial\Omega)$, we introduce the set of *divergence-free* vector fields

$$\mathcal{Z}_h := \left\{ \mathbf{z}_h \in L^\infty(\Omega;\mathbb{R}^d) \;\middle|\; \begin{array}{l} \mathbf{z}_h \text{ satisfying (5.2)-(5.5) for a weak} \\ \text{solution } u_h \text{ of Dirichlet problem (5.1)} \end{array} \right\} \tag{5.6}$$

Note, due to the existence theory of weak solutions of Dirichlet problem (5.1), the set $\mathcal{Z}_h$ is non-empty for every $h \in L^1(\partial\Omega)$. But since the corresponding vector fields $\mathbf{z}_h$ might not be unique, the Dirichlet-to-Neumann operator Λ associated with the 1-Laplace operator might be a multi-valued operator.

In the following, let $\overline{B}_{L^\infty(\partial\Omega)}$ denote the closed unit ball of $L^\infty(\partial\Omega)$ centered at $h = 0$.

Definition 5.2. We call the operator Λ defined by

$$\Lambda = \left\{ (h,g) \in L^1(\partial\Omega) \times \overline{B}_{L^\infty(\partial\Omega)} \;\middle|\; \begin{array}{l} \exists\, \mathbf{z}_h \in \mathcal{Z}_h \text{ satisfying} \\ [\mathbf{z}_h,\nu] = g \;\, \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega \end{array} \right\}$$

the *Dirichlet-to-Neumann operator in $L^1(\partial\Omega)$ associated with the* 1*-Laplace operator* Δ_1.

In our next theorem, we provide a useful characterization of Λ. In the following, we write $L^\infty_\sigma(\partial\Omega)$ for the space $L^\infty(\partial\Omega)$ equipped with the weak*-topology $\sigma(L^\infty(\partial\Omega), L^1(\partial\Omega))$.

Theorem 5.3 ([34]). *For given $h \in L^1(\partial\Omega)$, let $\mathcal{Z}_h$ be the set defined in (5.6). Then the Dirichlet-to-Neumann map Λ in $L^1(\partial\Omega)$ associated with the 1-Laplace operator Δ_1 admits the following properties.*

(1.) *The map Λ has the effective domain*

$$D(\Lambda) = L^1(\partial\Omega)$$

and $-\Lambda$ generates a C_0-semigroup $\{e^{t\Lambda}\}_{t\ge 0}$ of order-preserving contractions on $L^1(\partial\Omega)$.

(2.) *Λ is homogeneous of order zero;*

(3.) *Λ is closed in $L^1(\partial\Omega) \times L^\infty_\sigma(\partial\Omega)$;*

(4.) *Λ can be characterized by*

$$\Lambda = \partial_{L^1\times L^\infty(\partial\Omega)}\varphi \tag{5.7}$$

for the sub-differential operator $\partial_{L^1\times L^\infty(\partial\Omega)}\varphi$ in $L^1\times L^\infty(\partial\Omega)$ of the convex, even, homogeneous of order one, and continuous functional $\varphi : L^1(\partial\Omega) \to [0,\infty)$ defined by

$$\varphi(h) = \int_{\partial\Omega} [\mathbf{z}_h, \nu]\, h \,\mathrm{d}\mathcal{H}^{d-1} \tag{5.8}$$

for every $h \in L^1(\partial\Omega)$ and $\mathbf{z}_h \in \mathcal{Z}_h$. In particular, the value of the integral on the right-hand side in (5.8) does not change among all vector fields $\mathbf{z}_h \in \mathcal{Z}_h$.

As an application of Theorem 5.3, we can prove well-posedness, and regularity properties of strong solutions h in $L^q(\partial\Omega)$ (see [34] and also [33, 30]) of the *elliptic-parabolic boundary value problem*

$$\begin{cases} \lambda h - \operatorname{div}\left(\dfrac{Du_h}{|Du_h|}\right) = 0 & \text{in } \Omega\times(0,T),\\ u_h = h & \text{on } \partial\Omega\times(0,T),\\ \partial_t h + \dfrac{Du_h}{|Du_h|}\cdot\nu \ni g & \text{on } \partial\Omega\times(0,T),\\ h = h_0 & \text{on } \partial\Omega\times\{t=0\}, \end{cases}$$

where $f : \partial\Omega\times\mathbb{R}\to\mathbb{R}$ is a Lipschitz-continuous Carathéodory function. In particular, we have the following long-time stability result.

Theorem 5.4 ([34]). *Let $1 \le q \le \infty$. Then the following statements hold.*

(1.) *(Energy decreasing) For every $h_0 \in L^1(\partial\Omega)$, the energy functional φ given by (5.8) is monotonically decreasing along the trajectory*

$$\{e^{-t(\Lambda+f)}h_0 \,|\, t \ge 0\}.$$

In particular, one has that

$$\varphi_\infty := \lim_{n\to\infty} \varphi(e^{-t(\Lambda+f)}h_0) \qquad \textit{exists.}$$

(2.) *(Conservation of mass) If $f \equiv 0$, then one has that*

$$\int_{\partial\Omega} e^{-t\Lambda} h_0 \, \mathrm{d}\mathcal{H}^{d-1} = \overline{h}_0 := \tfrac{1}{\mathcal{H}^{d-1}(\partial\Omega)} \int_{\partial\Omega} h_0 \, \mathrm{d}\mathcal{H}^{d-1} \qquad \text{for all } t \geq 0$$

and all $h_0 \in L^1(\partial\Omega)$.

(3.) *(Long-time stability in $L^q(\partial\Omega)$) If $f \equiv 0$, then for every $h_0 \in L^q(\partial\Omega)$ and $q < \infty$, then one has that*

$$\lim_{t\to\infty} e^{-t\Lambda} h_0 = \overline{h}_0 \qquad \text{in } L^q(\partial\Omega)$$

and $\varphi_\infty = \varphi(\overline{h}_0) = 0$.

(4.) *(Entropy-Transport inequality) If $F \equiv 0$, then there is a $C > 0$ such that*

$$\|e^{-t\Lambda} h_0 - \overline{h_0}\|_1 \leq C \, \varphi(e^{-t\Lambda} h_0) \qquad \text{for all } t > 0.$$

(5.) *For every $h_0 \in L^2(\partial\Omega)$, one has that*

$$\varphi(e^{-t\Lambda} h_0) \leq 2 \frac{\|h_0\|_2^2}{t} \qquad \text{for all } t > 0.$$

It is worth noting that the semigroup $\{e^{t\Lambda}\}_{t\geq 0}$ does not admit an L^q-L^∞-regularization effect since the trace operator $\mathcal{T}r$ only maps $BV(\Omega)$ into $L^1(\partial\Omega)$. Moreover, we believe that the following conjecture holds.

Conjecture. For every $h_0 \in L^1(\partial\Omega)$, the trajectory $t \mapsto e^{-t\Lambda} h_0 - \overline{h_0}$ becomes extinct in finite time.

6. Other applications

The list of known applications or occurrences of the Dirichlet-to-Neumann map is very long. Indeed, it would be impossible to mention all. But among those, it is worth mentioning the result by Caffarelli and Silvestre [16], which is known as the so-called *extension technique*. In this paper, they showed that the fractional Laplace operator $\psi(-\Delta) = (-\Delta)^s$ on $\mathbb{R}^d$, $d \geq 1$, for $\psi(r) = r^s$, $r \geq 0$, and $0 < s < 1$, defined as the singular integral operator

$$(-\Delta)^s h(x) = C_{s,d} \, \text{C.V.} \int_{\mathbb{R}^d} \frac{h(x) - h(\xi)}{|x - \xi|^{d+2s}} \mathrm{d}\xi,$$

$h \in C_c^\infty(\mathbb{R}^d)$, can be characterized by

$$(-\Delta)^s h = C_s \, \Lambda h$$

for every $h \in C_c^\infty(\mathbb{R}^d)$, where Λ is the *Dirichlet-to-Neumann map*

$$h_{|\mathbb{R}^d} \mapsto \Lambda h := \lim_{y\to 0^+} -y^{1-2s} \frac{\partial u_h}{\partial y}(.,y)_{|\mathbb{R}^d} \tag{6.1}$$

corresponding to the *incomplete Dirichlet/extension problem*

$$\begin{cases} -\Delta u_h - \frac{1-2s}{y}\frac{\partial u_h}{\partial y} - \frac{\partial^2 u_h}{\partial y^2} = 0 & \text{on } \mathbb{R}^{d+1}_+, \\ \hfill u_h = h & \text{on } \partial\mathbb{R}^{d+1}_+ = \mathbb{R}^d, \end{cases}$$

associated with the negative *Bessel operator*

$$\mathcal{A} := -\left(\Delta + \frac{1-2s}{y}\frac{\partial}{\partial y} + \frac{\partial^2}{\partial y^2}\right) \qquad \text{on the half space } \mathbb{R}^{d+1}_+ := \mathbb{R}^d \times (0,\infty).$$

This result established an important link between non-local and local diffusion problems and has been generalized in various directions (see, for example, [25, 31, 32]. But the Dirichlet-to-Neumann map Λ given by (6.1) occurs naturally in *interpolation theory* (see [7]). This result by Caffarelli and Silvestre triggered various new research directions, and, in particular, laid the mathematical foundation for defining *fractional powers* A^s of general monotone operators A in Hilbert spaces H (see [35]). Further, it lead to a refinement of the classical definition of the *sub-differential* operator $\partial\mathcal{E}$ in Hilbert spaces to sub-differentials of so-called *j-elliptic functionals* [18].

[1] K. Ammar, F. Andreu, and J. Toledo. Quasi-linear elliptic problems in L^1 with non homogeneous boundary conditions. *Rend. Mat. Appl. (7)*, 26(3-4):291–314, 2006.

[2] K. Ammar. Solutions entropiques des problèmes non locaux. *C. R. Math. Acad. Sci. Paris*, 334(9):751–756, 2002.

[3] K. Ammar. *Solutions entropiques et renormalisées de quelques E. D. P. non linéaires dans L^1*, volume 2003/14 of *Prépublication de l'Institut de Recherche Mathématique Avancée [Prepublication of the Institute of Advanced Mathematical Research]*. Université Louis Pasteur, Département de Mathématique, Institut de Recherche Mathématique Avancée, Strasbourg, 2003.

[4] F. Andreu-Vaillo, J. M. Mazón, J. D. Rossi, and J. J. Toledo-Melero. *Nonlocal diffusion problems*, volume 165 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI; Real Sociedad Matemática Española, Madrid, 2010.

[5] S. N. Antontsev and J. F. Rodrigues. On stationary thermo-rheological viscous flows. *Ann. Univ. Ferrara Sez. VII Sci. Mat.*, 52(1):19–36, 2006.

[6] W. Arendt and A. F. M. ter Elst. The Dirichlet-to-Neumann operator on rough domains. *J. Differential Equations*, 251(8):2100–2124, 2011.

[7] W. Arendt, A. F. M. ter Elst, and M. Warma. Fractional powers of sectorial operators via the Dirichlet-to-Neumann operator. *Comm. Partial Differential Equations*, 43(1):1–24, 2018.

[8] W. Arendt and D. Hauer. Maximal L^2-regularity in nonlinear gradient systems and perturbations of sublinear growth. *Pure Appl. Anal.*, 2(1):23–34, 2020.

[9] G. Aronsson. On p-harmonic functions, convex duality and an asymptotic formula for injection mould filling. *European J. Appl. Math.*, 7(5):417–437, 1996.

[10] G. Aronsson and U. Janfalk. On Hele-Shaw flow of power-law fluids. *European J. Appl. Math.*, 3(4):343–366, 1992.

[11] C. Atkinson and C. R. Champion. Some boundary-value problems for the equation $\nabla\cdot(\mid\nabla\varphi\mid^N \nabla\varphi) = 0$. *Quart. J. Mech. Appl. Math.*, 37(3):401–419, 1984.

[12] P. Bénilan and M. G. Crandall. Regularizing effects of homogeneous evolution equations. In *Contributions to analysis and geometry (Baltimore, Md., 1980)*, pages 23–39. Johns Hopkins Univ. Press, Baltimore, Md., 1981.

[13] T. Brander. Calderón problem for the p-Laplacian: first order derivative of conductivity on the boundary. *Proc. Amer. Math. Soc.*, 144(1):177–189, 2016.

[14] T. Brander. *Calderón's problem for p-Laplace type equations*, volume 155 of *Rep., Univ. Jyväskylä, Dep. Math. Stat.* Jyväskylä: University of Jyväskylä, Department of Mathematics and Statistics (Diss.), 2016.

[15] P. R. Bueno, J. A. Varela, and E. Longo. Sno2, zno and related polycrystalline compound semiconductors: An overview and review on the voltage-dependent resistance (non-ohmic) feature. *Journal of the European Ceramic Society*, 28(3):505 – 529, 2008.

[16] L. Caffarelli and L. Silvestre. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, 32(7-9):1245–1260, 2007.

[17] Alberto-P. Calderón. On an inverse boundary value problem. In *Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980)*, pages 65–73. Soc. Brasil. Mat., Rio de Janeiro, 1980.

[18] R. Chill, D. Hauer, and J. Kennedy. Nonlinear semigroups generated by j-elliptic functionals. *J. Math. Pures Appl. (9)*, 105(3):415–450, 2016.

[19] T. Coulhon and D. Hauer. Functional inequalities and regularising properties of nonlinear semigroups - Theory and Applications. Accepted for publication in the Springer Serie SMAI - Mathématiques et Applications, 2021.

[20] M. G. Crandall. A visit with the ∞-Laplace equation. In *Calculus of variations and nonlinear partial differential equations*, volume 1927 of *Lecture Notes in Math.*, pages 75–122. Springer, Berlin, 2008.

[21] R. Dautray and J.-L. Lions. *Mathematical analysis and numerical methods for science and technology. Vol. 1.* Springer-Verlag, Berlin, 1990.

[22] J. Ildefonso Díaz and R. F. Jiménez. Application of nonlinear semigroup theory to a pseudodifferential operator. In *Proceedings of the seventh congress on differential equations and applications (Granada, 1984)*, pages 137–142. Univ. Granada, Granada, 1985.

[23] M.A. Dubson, S.T. Herbert, J.J. Calabrese, D.C. Harris, B.R. Patton, and J.C. Garland. Non-ohmic dissipative regime in the superconducting transition of polycrystalline y1ba2cu3ox. *Physical Review Letters*, 60(11):1061 – 1064, 1988.

[24] K.-J. Engel. The Laplacian on $C(\overline{\Omega})$ with generalized Wentzell boundary conditions. *Arch. Math. (Basel)*, 81(5):548–558, 2003.

[25] J. E. Galé, P. J. Miana, and P. R. Stinga. Extension problem and fractional operators: semigroups and wave equations. *J. Evol. Equ.*, 13(2):343–368, 2013.

[26] A. Garroni, V. Nesi, and M. Ponsiglione. Dielectric breakdown: optimal bounds. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 457(2014):2317–2335, 2001.

[27] A. Garroni and R. V. Kohn. Some three-dimensional problems related to dielectric breakdown and polycrystal plasticity. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 459(2038):2613 – 2625, 2003.

[28] R. Glowinski and J. Rappaz. Approximation of a nonlinear elliptic problem arising in a non-Newtonian fluid flow model in glaciology. *M2AN Math. Model. Numer. Anal.*, 37(1):175–186, 2003.

[29] D. Hauer. The p-Dirichlet-to-Neumann operator with applications to elliptic and parabolic problems. *J. Differential Equations*, 259(8):3615–3655, 2015.

[30] D. Hauer. Regularizing effect of homogeneous evolution equations with perturbation. *Nonlinear Anal.*, 206:Paper No. 112245, 34, 2021.

[31] D. Hauer and D. Lee. Functional calculus via the extension technique: a first hitting time approach, 2021. URL `https://arxiv.org/abs/2101.11305`.

[32] D. Hauer and D. Lee. Functional calculus via the extension technique: a Sturm-Liouville approach in the Hilbert space case, 2022. To be submitted.

[33] D. Hauer and J. M. Mazón. Regularizing effects of homogeneous evolution equations: the case of homogeneity order zero. *J. Evol. Equ.*, 19(4):965–996, 2019.

[34] D. Hauer and J. M. Mazón. The Dirichlet-to-Neumann operator associated with the 1-Laplacian and evolution problems. *Calc. Var. Partial Differential Equations*, 61(1):Paper No. 37, 50, 2022.

[35] D. Hauer, Y. He, and D. Liu. Fractional powers of monotone operators in Hilbert spaces. *Adv. Nonlinear Stud.*, 19(4):717–755, 2019.

[36] J. Heinonen, T. Kilpeläinen, and O. Martio. *Nonlinear potential theory of degenerate elliptic equations.* Dover Publications, Inc., Mineola, NY, 2006.

[37] R. F. Jiménez. Nonlinear evolution problems on manifolds without boundary. *Mat. Apl. Comput.*, 10(1):3–18, 1991.

[38] A. Kuijper. p-Laplacian driven image processing. volume 5, page V257–V260, 2007.

[39] G. M. Lieberman. Boundary regularity for solutions of degenerate elliptic equations. *Nonlinear Anal.*, 12(11):1203–1219, 1988.

[40] T. Liimatainen and M. Salo. n-harmonic coordinates and the regularity of conformal mappings. *Math. Res. Lett.*, 21(2):341–361, 2014.

[41] J. M. Mazón, J. D. Rossi, and S. Segura de León. Functions of least gradient and 1-harmonic functions. *Indiana Univ. Math. J.*, 63(4):1067–1084, 2014.

[42] J. Nečas. *Les méthodes directes en théorie des équations elliptiques.* Masson et Cie, Éditeurs, Paris; Academia, Éditeurs, Prague, 1967.

[43] M. Růžička. *Electrorheological fluids: modeling and mathematical theory*, volume 1748 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 2000.

[44] M. Salo and X. Zhong. An inverse problem for the p-Laplacian: boundary determination. *SIAM J. Math. Anal.*, 44(4):2474–2495, 2012.

[45] M. E. Taylor. *Partial differential equations. II*, volume 116 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 1996.

[46] J. L. Vázquez. A strong maximum principle for some quasilinear elliptic equations. *Appl. Math. Optim.*, 12(3):191–202, 1984.

[47] E. Zeidler. *Applied functional analysis*, volume 109 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 1995.

On solvability of an initial value problem for a superlinear heat equation

Toru Kan
Department of Mathematics, Osaka Metropolitan University

1 Introduction

We discuss the initial value problem for the nonlinear heat equation

$$\begin{cases} \partial_t u = \Delta u + u^p & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(\cdot,0) = \mu & \text{on } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $p > 1$, $N \geq 1$ and μ is a given positive Radon measure. Our concern is conditions on μ for a nonnegative solution to exist.

1.1 Known facts

In the case $p < p_F := 1 + 2/N$, a condition for local existence is explicitly given. More precisely, it was shown in [3, 1] that problem (1.1) has a nonnegative local-in-time solution if and only if

$$\sup_{B\in\mathcal{B},\mathrm{rad}(B)=1} \mu(B) < \infty,$$

where $\mathcal{B}$ is the set of all balls in $\mathbb{R}^N$ and $\mathrm{rad}(B)$ denotes the radius of $B \in \mathcal{B}$ (we regard the empty set as a ball with radius zero: $\emptyset \in \mathcal{B}$ and $\mathrm{rad}(\emptyset) = 0$).

Let us turn to the case $p \geq p_F$. A necessary condition was obtained by [3, 1]: if (1.1) has a nonnegative local-in-time solution, then

$$\begin{aligned} &\sup_{B\in\mathcal{B},0<\mathrm{rad}(B)\ll 1} \left[\left(\log \frac{1}{\mathrm{rad}(B)} \right)^{\frac{N}{2}} \mu(B) \right] < \infty \quad \text{if } p = p_F, \\ &\sup_{B\in\mathcal{B},0<\mathrm{rad}(B)\ll 1} \left[\mathrm{rad}(B)^{-N+\frac{2}{p-1}} \mu(B) \right] < \infty \qquad \text{if } p > p_F. \end{aligned} \tag{1.2}$$

These conditions are, unfortunately, not sufficient. Indeed, there is a positive Radon measure μ satisfying (1.2) such that (1.1) has no nonnegative local-in-time solution (see [3, 6]). It is known (see [5, 2, 6]) that the existence of a local-in-time solution is guaranteed under the stronger condition

$$\sup_{B\in\mathcal{B},\mathrm{rad}(B)>0} \left[\mathrm{rad}(B)^{-N+\frac{2}{p-1}} \left(\log \left(e + \frac{1}{\mathrm{rad}(B)} \right) \right)^{\frac{1}{p-1}+\varepsilon} \mu(B) \right] < \infty, \tag{1.3}$$

where $\varepsilon > 0$ is an arbitrary constant. Furthermore, if $p > p_F$ the left-hand side of (1.3) is small enough, then (1.1) has a nonnegative global-in-time solution.

The equation in problem (1.1) is invariant under the transformation

$$u(x,t) \mapsto u_\lambda(x,t) := \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t), \quad \lambda > 0.$$

The corresponding transformation for the initial value is given by

$$\mu(E) \mapsto \lambda^{-N+\frac{2}{p-1}} \mu(\lambda E), \quad \lambda E := \{\lambda x; x \in E\}, \tag{1.4}$$

which is due to the formal computation

$$\int_E u_\lambda(x,t)dx = \lambda^{-N+\frac{2}{p-1}} \int_{\lambda E} u(y, \lambda^2 t)dy \xrightarrow{t\to 0} \lambda^{-N+\frac{2}{p-1}} \mu(\lambda E).$$

One of natural spaces where initial values live is a space the norm of which is invariant under the above transformation. For $q = N(p-1)/2$, the space $L^q(\mathbb{R}^N)$ is an example of such spaces, and it is well known that (1.1) has a global-in-time solution if $p > p_F$ and the L^q norm of the initial value is small enough (see for instance [7, 4]). This space, of course, does not contain a Radon measure which is not absolutely continuous with respect to the Lebesgue measure. The purpose of this study is to give an invariant norm for Radon measures such that (1.1) has a nonnegative global-in-time solution if the norm of the initial value is small enough (note that the norm given by the left-hand side of (1.3) is not invariant).

1.2 Main result

To state our main result, we introduce some definitions and notation. For a positive Radon measure μ, we define

$$S_t[d\mu](x) := \int_{\mathbb{R}^N} G(x-y,t)d\mu(y),$$

where $G(x,t)$ stands for the heat kernel on $\mathbb{R}^N$:

$$G(x,t) := (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}.$$

If μ is given by $d\mu = hdx$ for some nonnegative function $h \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, then we simply write $S_t[h]$ for $S_t[d\mu]$. By a nonnegative solution of (1.1), we mean a nonnegative measurable function u satisfying

$$\infty > u(x,t) = S_t[d\mu](x) + \int_0^t S_{t-s}[u(\cdot,s)^p](x)ds \quad \text{for } (x,t) \in \mathbb{R}^N \times (0,\infty).$$

For $\alpha \in (0, N]$, let $\mathcal{H}^\alpha_\infty$ denote the α-dimensional Hausdorff content defined by

$$\mathcal{H}^\alpha_\infty(E) := \inf \left\{ \sum_{j=1}^\infty \mathrm{rad}(B_j)^\alpha; \{B_j\}_{j=1}^\infty \subset \mathcal{B}, E \subset \bigcup_{j=1}^\infty B_j \right\}.$$

Let $\mathcal{M}$ denote the set of Radon measures on $\mathbb{R}^N$. For $\alpha \in (0, N]$ and $\beta \in (0, 1]$, we define

$$X^\beta_\alpha := \left\{ \mu \in \mathcal{M}; \|\mu\|_{X^\beta_\alpha} < \infty \right\}, \qquad \|\mu\|_{X^\beta_\alpha} := \sup_{0<\mathcal{H}^\alpha_\infty(E)<\infty} \frac{|\mu|(E)}{\mathcal{H}^\alpha_\infty(E)^\beta},$$

where $|\mu|$ is the variation of μ. Then one can check that $\|\cdot\|_{X^\beta_\alpha}$ defines a norm on X^β_α. We remark that this norm is invariant under the transformation (1.4) if $p > p_F$ and $\alpha\beta = N - 2/(p-1)$, since $\mathcal{H}^\alpha_\infty(\lambda E) = \lambda^\alpha \mathcal{H}^\alpha_\infty(E)$ for $\lambda > 0$. Furthermore, as shown in the next section, the space X^β_α contains a positive Radon measure which is not absolutely continuous with respect to the Lebesgue measure.

Our main result is stated as follows.

Theorem 1. *Let $p > p_F$ and let $\alpha \in (0, N]$ and $\beta \in (0, 1)$ satisfy $\alpha\beta = N - 2/(p-1)$. If $\mu \in X^\beta_\alpha$ and $\|\mu\|_{X^\beta_\alpha}$ is small enough, then* (1.1) *has a nonnegative global-in-time solution.*

Remark 2. By the definition of $\mathcal{H}^\alpha_\infty$, we see that $\|\mu\|_{X^1_\alpha}$ coincides with the left-hand side of (1.2) if $p > p_F$ and $\alpha = N - 2/(p-1)$. Since we know that condition (1.2) does not imply the existence of a solution, the case $\beta = 1$ must be excluded in the theorem. Moreover, we find that $X^\beta_N = L^{q,\infty}(\mathbb{R}^N)$ for $q = 1/(1-\beta)$. This is due to the fact that $\mathcal{H}^N_\infty$ coincides with the Lebesgue measure.

2 Characterization of the space X^β_α

We begin with an example of measures contained in X^β_α. Let $\mathcal{H}^\alpha(E)$ denote the Hausdorff measure on $\mathbb{R}^N$:

$$\mathcal{H}^\alpha(E) := \sup_{\delta>0} \mathcal{H}^\alpha_\delta(E),$$

$$\mathcal{H}^\alpha_\delta(E) := \inf \left\{ \sum_{j=1}^\infty \mathrm{rad}(B_j)^\alpha; \{B_j\}_{j=1}^\infty \subset \mathcal{B}, E \subset \bigcup_{j=1}^\infty B_j, \mathrm{rad}(B_j) < \delta \right\}.$$

We consider a Borel set $E_0 \subset \mathbb{R}^N$ such that

$$\mathcal{H}^\alpha(E_0) > 0, \qquad C_0 := \sup_{B \in \mathcal{B}, \mathrm{rad}(B)>0} \frac{\mathcal{H}^\alpha(E_0 \cap B)}{\mathrm{rad}(B)^\alpha} < \infty.$$

We note that the latter condition implies $\mathcal{H}^\alpha(E_0 \cap E) \leq C_0 \mathcal{H}^\alpha_\infty(E)$, since for any covering $\{B_j\} \subset \mathcal{B}$ of E, we have

$$\mathcal{H}^\alpha(E_0 \cap E) \leq \sum_j \mathcal{H}^\alpha(E_0 \cap B_j) \leq C_0 \sum_j \mathrm{rad}(B_j)^\alpha.$$

Let $q \in (1, \infty)$. We take a positive function $g \in L^{q,\infty}(\mathbb{R}^N, d\mathcal{H}^\alpha \lfloor_{E_0})$ and define μ_0 by $d\mu_0 = g d\mathcal{H}^\alpha \lfloor_{E_0}$. Then we have $\mu_0 \in X^\beta_\alpha$ and $\|\mu_0\|_{X^\beta_\alpha} \leq C_0 \|g\|_{L^{q,\infty}}$ for $\beta = 1 - 1/q$, since

$$\mu_0(E) = \int_E g\, d\mathcal{H}^\alpha \lfloor_{E_0} \leq \|g\|_{L^{q,\infty}} \mathcal{H}^\alpha(E_0 \cap E)^{1-\frac{1}{q}} \leq C_0 \|g\|_{L^{q,\infty}} \mathcal{H}^\alpha_\infty(E)^{1-\frac{1}{q}}.$$

Here, for a measure ν on $\mathbb{R}^N$, the norm on $L^{q,\infty}(\mathbb{R}^N, d\nu)$ is defined by

$$\|h\|_{L^{q,\infty}} := \sup_{0<\nu(E)<\infty} \frac{1}{\nu(E)^{1-1/q}} \int_E |h| d\nu.$$

We remark that μ_0 is not absolutely continuous with respect to the Lebesgue measure.

The above example is generalized as follows. We denote by Y_α the set of all positive Radon measures ν on $\mathbb{R}^N$ satisfying

$$\|\nu\|_{Y_\alpha} := \sup_{B \in \mathcal{B}, \mathrm{rad}(B)>0} \frac{\nu(B)}{\mathrm{rad}(B)^\alpha} < \infty$$

(although this space is known as a Morrey space, we adopt the different notation since the definition is slightly different). We take $\nu \in Y_\alpha$ and $g \in L^{q,\infty}(\mathbb{R}^N, d\nu)$. Then, by the same argument as above, we see that μ defined by $d\mu = g d\nu$ belongs to X^β_α for $\beta = 1 - 1/q$ and satisfies $\|\mu\|_{X^\beta_\alpha} \leq \|\nu\|_{Y_\alpha} \|g\|_{L^{q,\infty}}$.

We can show that the converse of the above fact is true: any $\mu \in X^\beta_\alpha$ is written as $d\mu = g d\nu$ for some $\nu \in Y_\alpha$ and $g \in L^{q,\infty}(\mathbb{R}^N, d\nu)$.

Proposition 3. *Let $\alpha \in (0, N]$ and $\beta \in (0, 1)$. Then for any $\mu \in X^\beta_\alpha$, there exist $\nu \in Y_\alpha$ and $g \in L^{q,\infty}(\mathbb{R}^N, d\nu)$ with $q = 1/(1-\beta)$ such that $d\mu = g d\nu$, $\|\nu\|_{Y_\alpha} \leq 1$ and $\|g\|_{L^{q,\infty}} \leq C \|\mu\|_{X^\beta_\alpha}$ for some constant $C > 0$.*

This proposition is shown by applying the following lemma.

Lemma 4. *Suppose that set functions ζ and ξ defined on the Borel σ-algebra on $\mathbb{R}^N$ satisfy the following.*

(i) $\zeta(\emptyset) = \xi(\emptyset) = 0$.

(ii) ζ and ξ are monotone: for any Borel sets E_1, E_2 with $E_1 \subset E_2$,

$$\zeta(E_1) \leq \zeta(E_2), \qquad \xi(E_1) \leq \xi(E_2).$$

(iii) ζ *is supermodular and* ξ *is submodular: for any Borel sets* E_1 *and* E_2*,*

$$\zeta(E_1 \cup E_2) + \zeta(E_1 \cap E_2) \geq \zeta(E_1) + \zeta(E_2),$$
$$\xi(E_1 \cup E_2) + \xi(E_1 \cap E_2) \leq \xi(E_1) + \xi(E_2).$$

(iv) For any Borel set E*,*

$$\zeta(E) = \sup\{\zeta(K); K \subset E, K \textit{ is compact}\},$$
$$\xi(E) = \inf\{\xi(O); O \supset E, O \textit{ is open}\}.$$

(v) $0 \leq \zeta \leq \xi$.

Then there exists a positive Radon measure ν *such that* $\zeta \leq \nu \leq \xi$.

The existence of ν and g is shown in the following way. Let $\mu \in X_\alpha^\beta$. Since the assertion clearly holds if $\mu = 0$, we only consider the case $\mu \neq 0$. By definition, we have

$$\left(\frac{|\mu|}{\|\mu\|_{X_\alpha^\beta}} \right)^{1/\beta} \leq \mathcal{H}_\infty^\alpha.$$

One can take a monotone submodular set function ξ satisfying $c\mathcal{H}_\infty^\alpha \leq \xi \leq \mathcal{H}_\infty^\alpha$ and $\xi(E) = \inf\{\xi(O); O \supset E, O \text{ is open}\}$, where $c > 0$ is a constant. Moreover, from the condition $\beta \in (0,1)$, we see that $\zeta := c(|\mu|/\|\mu\|_{X_\alpha^\beta})^{1/\beta}$ is a monotone supermodular set function. Applying Lemma 4, we obtain a positive Radon measure satisfying $c(|\mu|/\|\mu\|_{X_\alpha^\beta})^{1/\beta} \leq \nu \leq \mathcal{H}_\infty^\alpha$. By the second inequality, we have $\nu \in Y_\alpha$ and $\|\nu\|_{Y_\alpha} \leq 1$. From the first inequality, we deduce that $|\mu|$ is absolutely continuous with respect to ν, and therefore the Radon-Nikodym theorem shows that $d\mu = g d\nu$ for some $g \in L^1_{\mathrm{loc}}(\mathbb{R}^N, d\nu)$. We again use the first inequality to obtain $g \in L^{q,\infty}(\mathbb{R}^N, d\nu)$ for $q = 1/(1-\beta)$ and $\|g\|_{L^{q,\infty}} \leq c^{-\beta}\|\mu\|_{X_\alpha^\beta}$.

3 Existence of a solution

The proof of Theorem 1 is done by monotone methods. We say that a nonnegative measurable function $\overline{u}$ is a supersolution of (1.1) if

$$\infty > \overline{u}(x,t) \geq S_t[d\mu](x) + \int_0^t S_{t-s}[u(\cdot,s)^p](x)ds \quad \text{for } (x,t) \in \mathbb{R}^N \times (0,\infty).$$

It is well known that the existence of a supersolution implies the existence of a solution.

Lemma 5. *Assume that* $\overline{u}$ *is a supersolution of* (1.1)*. Then there exists a nonnegative solution* u *of* (1.1) *satisfying* $u \leq \overline{u}$.

Theorem 1 is then shown by the following lemma.

Lemma 6. *Let $\alpha \in (0, N]$ and $\beta \in (0, 1)$, and assume that $\mu \in X_\alpha^\beta$ is positive. Take a positive Radon measure $\nu \in Y_\alpha$ and a nonnegative function $g \in L^{q,\infty}(\mathbb{R}^N, d\nu)$ obtained in Proposition 3. Fix $r \in (1, q)$ and define $\overline{u}$ by*

$$\overline{u}(x,t) := t^{-\gamma} S_t[g^r d\nu](x)^{\frac{1}{r}}, \quad \gamma := \frac{1}{2}(N-\alpha)\left(1 - \frac{1}{r}\right).$$

Then $\overline{u}$ is a supersolution of (1.1).

References

[1] D. Andreucci, E. DiBenedetto, *On the Cauchy problem and initial traces for a class of evolution equations with strongly nonlinear sources*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **18** (3) (1991) 363–441.

[2] D. Andreucci, *Degenerate parabolic equations with initial data measures*, Trans. Amer. Math. Soc. **349** (10) (1997) 3911–3923.

[3] P. Baras, M. Pierre, *Critère d'existence de solutions positives pour des équations semi-linéaires non monotones*, Ann. Inst. H. Poincaré Anal. Non Linéaire **2** (3) (1985) 185–212.

[4] Y. Giga, *Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system*, J. Differential Equations **62** (2) (1986) 186–212.

[5] Y. Niwa, *Semi-linear heat equations with measures as initial data*, Thesis, The University of Tokyo, 1986.

[6] J. Takahashi, *Solvability of a semilinear parabolic equation with measures as initial data*, Geometric Properties for Parabolic and Elliptic PDE's, 257–276, Springer Proc. Math. Stat. 176, Springer, Switzerland, 2016.

[7] F. B. Weissler, *Local existence and nonexistence for semilinear parabolic equations in L^p*, Indiana Univ. Math. J. **29** (1) (1980) 79–102.

CALDERÓN-ZYGMUND TYPE ESTIMATES FOR NONLINEAR ELLIPTIC AND PARABOLIC EQUATIONS WITH MATRIX WEIGHTS

SUN-SIG BYUN

ABSTRACT. A wider class of nonlinear elliptic and parabolic equations with degenerate/singular matrix weights is studied for Calderón-Zygmund type estimates of weak solutions in the setting of weighted Sobolev spaces. Minimal regularity assumptions on the associated nonlinearities as well as weights are investigated for such classical estimates.

1. INTRODUCTION

The problem under consideration is

$$\begin{cases} \operatorname{div}(\mathbb{M}(x)a(x,\mathbb{M}(x)Du)) &= \operatorname{div}\left(\mathbb{M}^2(x)F(x)\right) & \text{in} & \Omega \\ u &= 0 & \text{on} & \partial\Omega. \end{cases} \tag{1.1}$$

Here $F = F(x) = (f_1, \cdots, f_n) \in L^2(\Omega;\mathbb{R}^n)$ $(n \geq 2)$ is given, and the unknown is $u = u(x) : \overline{\Omega} \mapsto \mathbb{R}$. We first introduce the structural assumptions on the domain Ω and the nonlinearity $a = a(x,\xi)$. Ω is a bounded domain in $\mathbb{R}^n$ with non-smooth boundary $\partial\Omega$. The Carathéodory vector-valued function $a = a(x,\xi) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies that

$$|a(x,\xi)| + |D_\xi a(x,\xi)||\xi| \leq L|\xi|,\ \nu|\eta|^2 \leq D_\xi a(x,\xi)\eta\cdot\eta \tag{1.2}$$

for all $x,\xi,\eta \in \mathbb{R}^n$ and some constants $0 < \nu \leq L < \infty$.

The main feature of the present note is concerned with the function $\mathbb{M} = \mathbb{M}(x) : \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$. It is a symmetric and almost everywhere positive definite $n \times n$ matrix whose main structure assumption is a uniformly A_1 type condition. More precisely, we assume that

$$\sup_{x\in\mathbb{R}^n} |\mathbb{M}(x)| \left|\mathbb{M}^{-1}(x)\right| \leq \Lambda \tag{1.3}$$

for some positive constant Λ. Needless to say, we need to ensure that the resulting scalar weight

$$\omega^2(x) = |\mathbb{M}(x)|^2 \tag{1.4}$$

is in an A_2-Muckenhoupt class for the regularity to be valid in the literature, which is clearer later in the context. Temporarily assuming that all the associated data for the problem (1.1) are good enough, partly thanks to a suitable approximation argument, we multiply (1.1) by u and use integration by part alongside (1.2) and Holder's inequality, to discover

$$\int_\Omega |\mathbb{M}Du|^2 dx \leq c\int_\Omega |\mathbb{M}F|^2 dx$$

for some positive constant $c = c(\nu, L)$. But then a direct computation leads to

$$\begin{aligned} |\mathbb{M}Du| &\leq |\mathbb{M}||Du| = |\mathbb{M}||\mathbb{M}^{-1}\mathbb{M}Du| \\ &\leq |\mathbb{M}||\mathbb{M}^{-1}||\mathbb{M}Du| \leq \Lambda|\mathbb{M}Du|, \end{aligned}$$

where we have used (1.3) for the last inequality. Therefore, we see that $|\mathbb{M}(x)Du(x)|$ is equivalent to $|\mathbb{M}(x)||Du(x)|$ for almost every $x \in \Omega$, which we denote by

$$|\mathbb{M}(x)Du(x)| \approx |\mathbb{M}(x)||Du(x)|\,(x\in\Omega) \tag{1.5}$$

2010 *Mathematics Subject Classification.* Primary: 35B65; Secondary: 35J70, 35J75.
Key words and phrases. Matrix weight, Gradient estimate, Measurable nonlinearity, BMO, Reifenberg domain.
This note was based on the joint work with Yumi Cho and Ho-Sik Lee.

with the convention notation "≈", where u is a solution under consideration in the context. This leads to

$$\int_\Omega |Du|^2\omega^2(x)dx \le c\int_\Omega |F|^2\omega^2(x)dx \tag{1.6}$$

for some positive constant $c = c(\nu, L, \Lambda)$. This is the standard energy estimate for the problem (1.1) in the setting of the weighted Sobolev space $H_0^1(\Omega, \omega^2)$, to which we will return later. Here comes a natural question as to the validity of the extension of the L^2-estimate (1.6) to the L^p-estimate like

$$\int_\Omega |Du|^p\omega^p(x)dx \le c\int_\Omega |F|^p\omega^p(x)dx \tag{1.7}$$

for every $p \in [2, \infty)$ and some positive constant c independent of u and F. This is the so called Calderón-Zymund type estimates in the setting of weighted Lebesgue spaces. Indeed, there has been a rich literature, even for the constant weighted case that $\mathbb{M}$ is a constant matrix, see [1, 4, 5, 6, 10, 16] and references therein.

The main purpose of the present work is to establish an optimal Calderón-Zygmund theory for the degenerate/singular problem (1.1) with Calderón-Zygmund type estimates like (1.7) by finding minimal regularity assumptions additionally imposed on the variable matrix weight $\mathbb{M}$ beside (1.3), as well as on the nonlinearity $a = a(x, \xi)$ and the boundary of the domain $\partial\Omega$, adding to the structure assumptions like (1.2). Here for the sake of simplicity, we only treat local estimates, as the relevant estimates near the boundary can be handled in a similar way available for the constant matrix weight as in [4, 5, 6].

2. Preliminaries and results

We recall the given matrix weight $\mathbb{M}$ and its scalar weight $\omega = |\mathbb{M}|$. Through this note let $p \ge 2$ be an arbitrarily given number. Hereafter we denote by c to mean a universal constant which means the precise value can be computed in terms of known quantities such as n, ν, L, Λ and p.

We start this section with a further discuss regarding the Muckenhoupt weight $\omega(x) = |\mathbb{M}(x)|$. For a given $p \in [2, \infty)$, ω^p is said to be in the Muckenhoupt class A_p, denoted by $\omega^p \in A_p$, if

$$[\omega^p]_p^{\frac{1}{p}} = \sup_B \left(\frac{1}{|B|}\int_B \omega^p\right)^{\frac{1}{p}} \left(\frac{1}{|B|}\int_B \omega^{-\frac{p}{p-1}}\right)^{\frac{p-1}{p}} < +\infty,$$

where B is any choice of balls in $\mathbb{R}^n$. The relevant weighted Lebesgue space is

$$L^p(\Omega, \omega^p) = \{f : \Omega \to \mathbb{R} : f\omega \in L^p(\Omega)\},$$

which is a Banach space with the norm

$$\|f\|_{L^p(\Omega,\omega^p)} = \|f\omega\|_{L^p(\Omega)}.$$

We next discuss some relationship between BMO and Muckenhoupt weight. A locally integral function f on $\mathbb{R}^n$ is in the space BMO, if

$$[f]_{BMO} = \sup_B \frac{1}{|B|}\int_B |f(x) - f_B|dx < +\infty,$$

where

$$f_B = \frac{1}{|B|}\int_B f$$

is the average value of f on the ball B. We then have the following lemma from [3].

Lemma 2.1. *Suppose there exists a constant $c = c(n) > 0$ such that $[\log \mathbb{M}]_{BMO} \le c\frac{1}{p}$. Then we have $\omega^p \in A_p$.*

Thanks to this lemma, we as well assume that $\log \mathbb{M} \in BMO$ with $[\log \mathbb{M}]_{BMO}$ small enough to ensure that $\omega^p \in A_p$. The primary assumption on the nonhomogeneous term F for the problem (1.1) is

$$\mathbb{M}F \in L^p(\Omega). \tag{2.1}$$

Then note that $F \in L^p(\Omega, \omega^p)$ with $\omega^p \in A_p$.

We next describe Logarithmic means of $\mathbb{M}$ and $\omega(x) = |\mathbb{M}(x)|$. The integral means of ω and $\mathbb{M}(x)$ are defined by

$$\overline{\omega}_B = \exp\left(\log \omega_B\right), \ \overline{\mathbb{M}}_B = \exp\left(\log \mathbb{M}_B\right), \tag{2.2}$$

respectively. The following lemma is essentially used later in the comparison estimates.

Lemma 2.2. [3] *Let $1 \le s < \infty$. Then there exists a constant $c = c(s, \nu, \nu, \Lambda)$ such that*

$$\left(\frac{1}{|B|}\int_B \left(\frac{|\mathbb{M} - \overline{\mathbb{M}}_B|}{|\overline{\mathbb{M}}_B|}\right)^s dx\right)^{\frac{1}{s}} \le c[\log \mathbb{M}]_{BMO}$$

and

$$\left(\frac{1}{|B|}\int_B \left(\frac{|\omega - \overline{\omega}_B|}{|\overline{\omega}_B|}\right)^s dx\right)^{\frac{1}{s}} \le c[\log \omega]_{BMO}.$$

We now introduce the Weighted Sobolev space $H^1(\Omega, \omega^2)$ which is a Hilbert space consisting of locally integrable, weakly differentiable functions $f : \Omega \mapsto \mathbb{R}$ for which

$$\|f\|_{H^1(\Omega,\omega^2)} = \left(\int_\Omega |f|^2\omega^2(x)dx\right)^{\frac{1}{2}} + \left(\int_\Omega |Df|^2\omega^2(x)dx\right)^{\frac{1}{2}} < \infty.$$

$H^1_0(\Omega, \omega^2)$ is the closure of $C^\infty_c(\Omega)$ in $H^1(\Omega, \omega^2)$ with

$$\|f\|_{H^1_0(\Omega,\omega^2)} \approx \left(\int_\Omega |Df|^2\omega^2\right)^{\frac{1}{2}} \approx \left(\int_\Omega |\mathbb{M}Df|^2\right)^{\frac{1}{2}}.$$

We refer to [3, 13, 14] for a further discussion on weighted Sobolev spaces.

We next discuss existence and uniqueness for the problem (1.1). As usual, solution is defined in the weak sense.

Definition 2.3. $u \in H^1_0(\Omega, \omega^2)$ is a weak solution to (1.1) if

$$\int_\Omega a(x, \mathbb{M}Du) \cdot \mathbb{M}D\varphi = \int_\Omega \mathbb{M}F \cdot \mathbb{M}D\varphi$$

for every $\varphi \in H^1_0(\Omega, \omega^2)$.

We apply the method of Browder and Minty in $H^1_0(\Omega, \omega^2)$ to achieve the existence and uniqueness for the problem (1.1), see [18].

Lemma 2.4. *There exists a unique weak solution $u \in H^1_0(\Omega, \omega^2)$ to (1.1) with the estimate*

$$\int_\Omega |Du|^2\omega^2 \approx \int_\Omega |\mathbb{M}Du|^2 \le c\int_\Omega |\mathbb{M}F|^2 \approx \int_\Omega |F|^2\omega^2$$

for some positive constant $c = c(\nu, L, \Lambda, n)$.

As mentioned earlier in Introduction, this note is concerned with optimal weighted $W^{1,p}$-regularity. This means that we want as to find reasonable answers to what would be minimal conditions on the mapping $x \mapsto a(x, \xi)$, $\partial\Omega$ and $\mathbb{M}$ under which

$$\mathbb{M}F \in L^p(\Omega) \Longrightarrow \mathbb{M}Du \in L^p(\Omega) \qquad (2 \le p < \infty) \tag{2.3}$$

with the Calderón-Zygmund type estimate (1.7). The classical case $p = 2$ follows from Lemma 2.4 and we always assume $2 < p < \infty$ hereafter.

We now introduce known regularity results regarding degenerate/singular elliptic and parabolic equations associated to Muckenhoupt weights. When $\mathbb{M}(x) = x_1^\alpha I_n$ and $a(x, \xi) = A(x)\xi$, Dong and Phan in [11, 12, 17] extensively studies for α in a suitable range of $\mathbb{R}$ under a small BMO condition on the mapping $x' \mapsto A(x_1, x')$, uniformly in x_1. The case when $a(x, \xi) = A(x)\xi$ is treated by Cao, Mengesha and Phan in [8] for under a small BMO condition on $A(x)$. When $a(x, \xi) = |\mathbb{M}(x)\xi|^{p-2}\mathbb{M}(x)\xi$, interior $W^{1,q}$-regularity for $p < q < \infty$ is obtained by Balci, Diening, Giova and di Napoli in [3] under a small BMO condition on $\ln \mathbb{M}$. Boundary $W^{1,q}$-regularity for $p < q < \infty$ is achieved by Balci, Byun, Diening and Lee in [2] on a Lipschitz domain with small local Lipschitz constant. The present work is a natural outgrowth of the earlier regularity results in [4, 6] for the uniformly elliptic problem when $\mathbb{M}(x) = I_n$ from (1.1) under a small BMO condition on the mapping $x' \mapsto \sup_{\xi\in\mathbb{R}^n} \frac{a(x_1,x',\xi)}{|\xi|}$, uniformly in x_1. Here we want to complement those mentioned works both in [4, 6] to measurable nonlinearities and in [2, 3] to matrix weights, respectively.

Motivated from the regularity assumptions as in [2, 3, 4], very roughly speaking our regularity assumption is that $\log\mathbb{M}$ has a small BMO. On the other hand, $x' \mapsto \sup_{\xi\in\mathbb{R}^n} \frac{a(x_1,x',\xi)}{|\xi|}$ has a small BMO, uniformly in x_1. More precisely, we are able to announce that one can find a small constant $\delta = \delta(p, n, \nu, L)$ such that if

$$\sup_{x_1\in\mathbb{R}} [\beta(x_1,\cdot)]_{BMO} + [\log\mathbb{M}]_{BMO} \le \delta \tag{2.4}$$

for such a constant δ, then the implication (2.3) holds true with the desired regularity estimate (1.7), where

$$\beta(x_1, x') = \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} \frac{|a(x_1, x', \xi)|}{|\xi|} \tag{2.5}$$

is a uniformly bounded function.

We now state the main theorem.

Theorem 2.5. *Assume* (1.1) *and* (1.2)*. Suppose* $|\mathbb{M}F| \in L^p(\Omega)$ *for some* $p \ge 2$*. Then there exists a small* $\delta = \delta(p, n, \nu, L, \Lambda)$ *such that if* (2.4) *holds, then the weak solution* $u \in H_0^1(\Omega, \omega^2)$ *of* (1.1) *satisfies* $|\mathbb{M}Du| \in L^p_{loc}(\Omega)$*.*

We would like to point out that in this note the global estimate is not mentioned for the purpose of mainly presenting the issues naturally arising in the presence of the matrix weight $\mathbb{M}(x)$. Indeed, once a desired local estimate is established, one can revisit the earlier papers including [4, 5, 6] to deal with the δ-Reifenberg flatness as a minimal geometric assumption on $\partial\Omega$, eventually deriving the global Calderón-Zygmund type estimate (1.7). We refer to [9, 15, 19] for a detailed discussion on Reifenberg flat domains.

3. Elliptic equations with matrix weights and measurable nonlinearities

This section is devoted to proving Theorem 2.5. For a point $y = (y_1, y') \in \mathbb{R}\times\mathbb{R}^n$ and $r > 0$, we denote

$$B'_r(y') = \{x' \in \mathbb{R}^{n-1} : |x' - y'| < r\}$$

and

$$Q_r(y) = (y_1 - r, y_1 + r) \times B'_r(y').$$

We now provide a scaling invariance property for the problem (1.1).

Lemma 3.1. *Let* $\lambda \ge 1$ *and* $0 < r \le 1$*. Suppose* u *is a weak solution of* (1.1) *in* Q_r*. Define*

$$\widetilde{a}(x,\xi) = \frac{a(rx, \lambda\xi)}{\lambda},\ \widetilde{\mathbb{M}}(x) = \mathbb{M}(rx),\ \widetilde{F}(x) = \frac{F(rx)}{\lambda},\ \widetilde{u}(x) = \frac{u(rx)}{\lambda r}$$

for $x \in Q_1$*. Then* $\widetilde{u}$ *is a weak solution of*

$$div\left(\widetilde{\mathbb{M}}\ \widetilde{a}(x, \widetilde{\mathbb{M}}D\widetilde{u})\right) = div\left(\widetilde{\mathbb{M}}^2 \tilde{F}\right) \ \text{ in } Q_1$$

with the resulting structure assumptions unchanged. The converse is also true.

We next record here monotonicity property and growth condition of the nonlinearity associated to the matrix weight $\mathbb{M}$. Because of this presence of $\mathbb{M}$, the problem (1.1) falls in the realm of the degenerate/singular problems. Recalling (1.2), we write

$$a_{\mathbb{M}}(x,\xi) = \mathbb{M}a(x, \mathbb{M}\xi) \qquad (x, \xi \in \mathbb{R}^n)\,.$$

Then we have

- $(a_{\mathbb{M}}(x,\xi) - a_{\mathbb{M}}(x,\eta)) \cdot (\xi - \eta) \ge c(\nu, n)\, |\mathbb{M}\xi - \mathbb{M}\eta|$;
- $|a_{\mathbb{M}}(x,\xi)| + |D_\xi a_{\mathbb{M}}(x,\xi)||\xi| \le L|\mathbb{M}\xi|$;
- $\nu|\mathbb{M}\eta|^2 \le D_\xi a_{\mathbb{M}}(x,\xi)\eta\cdot\eta$.

Hence if $u \in H^1(Q_1, \omega^2)$ is a weak solution to

$$\operatorname{div}(\mathbb{M}a(x, \mathbb{M}Du)) = \operatorname{div}\left(\mathbb{M}^2 F\right) \text{ in } Q_1,$$

then it is also a weak solution to

$$\operatorname{div}(a_{\mathbb{M}}(x, Du)) = \operatorname{div}\left(\mathbb{M}^2 F\right) \text{ in } Q_1.$$

For the sake of simplicity, we write

$$\overline{\mathbb{M}}_{Q_r(y)} = \exp \frac{1}{|Q_r|} \int_{Q_r(y)} \log \mathbb{M}$$

and

$$a_{B'_r(y')}(x_1, \xi) = \frac{1}{|B'_r|} \int_{B'_r(y')} a(x_1, x', \xi) dx'$$

to consider

$$a_{\overline{\mathbb{M}}_{Q_r(y)}}(x_1, \xi) = \overline{\mathbb{M}}_{Q_r(y)} a_{B'_r(y')}\left(x_1, \overline{\mathbb{M}}_{Q_r(y)} \xi\right).$$

Our approach to proving Theorem 2.5 is based on a method of approximations used as in [7]. This involves

- a regularity result for a fixed problem $\mathcal{A}_0[v] = 0$;
- a local estimate of a solution to the given problem $\mathcal{A}[u] = 0$ by comparison with a solution to the limiting problem $\mathcal{A}_0[v] = 0$;
- a real variable argument coming from maximal function and Calderón-Zygmund decomposition/Vitali type covering.

With the same spirit as in [1, 5, 7], we are going to use perturbation results along with comparison estimates, in a normalized form thanks to Lemma 3.1. We restate

- a Lipschitz regularity result for a fixed problem
$$\operatorname{div}\left(a_{\overline{\mathbb{M}}_{Q_1}}(x_1, Dv)\right) = 0 \text{ in } Q_1;$$
- a local estimate of a solution to our problem
$$\operatorname{div}(a_{\mathbb{M}}(x, Du)) = \operatorname{div}\left(\mathbb{M}^2 F\right) \text{ in } Q_2.$$
by comparison with a solution to the limiting problem
$$\operatorname{div}\left(a_{\overline{\mathbb{M}}_{Q_1}}(x_1, Dv)\right) = 0;$$
- a real argument using maximal function and a Vitali type covering.

We now present Lipschitz regularity for the limiting problem.

Lemma 3.2. [4] *Let $v \in H^1(Q_2, \overline{\omega^2})$ be a weak solution to*

$$div\left(a_{\overline{\mathbb{M}}_{Q_2}}(x_1, Dv)\right) = 0 \text{ in } Q_2.$$

Then

$$\left\|\overline{\mathbb{M}}_{Q_2} Dv\right\|^2_{L^\infty(Q_1)} \leq c \frac{1}{|Q_2|} \int_{Q_2} |\overline{\mathbb{M}}_{Q_2} Dv|^2$$

for some positive constant $c = c(\nu, L, n, \Lambda)$.

We are now ready to make comparison estimates. We first recall that c denotes a generic constant that varies line by line, but can be computed in terms of known data such as n, ν, L, Λ, p.

Let $u \in H^1(Q_6, \omega^2)$ be a weak solution to

$$\operatorname{div}(a_{\mathbb{M}}(x, Du)) = \operatorname{div}\left(\mathbb{M}^2 F\right) \text{ in } Q_6 \tag{3.1}$$

with

$$\frac{1}{|Q_6|} \int_{Q_6} |\mathbb{M} Du|^2 \leq 1, \quad \frac{1}{|Q_6|} \int_{Q_6} |\mathbb{M} F|^2 \leq \delta^2,$$

where $\delta > 0$ is to be determined later. Let $w \in H^1(Q_6, \omega^2)$ be the weak solution to

$$\begin{cases} \operatorname{div}(a_{\mathbb{M}}(x, Dw)) = 0 & \text{in} \quad Q_6 \\ w = u & \text{on} \quad \partial Q_6. \end{cases} \tag{3.2}$$

We take $u - w \in H_0^1(Q_6, \omega^2)$ to (3.1) and (3.2) as a test function to find

$$\frac{1}{|Q_6|}\int_{Q_6} |\mathbb{M}Dw|^2 \le c, \ \frac{1}{|Q_6|}\int_{Q_6} |\mathbb{M}Du - \mathbb{M}Dw|^2 \le c\delta^2.$$

We also see that there exists a generic constant $\sigma_0 = \sigma_0(\nu, L, n, \Lambda) > 0$ such that

$$\frac{1}{|Q_5|}\int_{Q_5} |\mathbb{M}Dw|^{2+\sigma_0} \le c.$$

This is a higher integrability result for the gradient which is now well understood in the literature, see [2, 3].

Let $h \in H^1(Q_5, \omega^2)$ the weak solution to

$$\begin{cases} \operatorname{div}(a_{\mathbb{M}}(x_1, Dh)) = 0 & \text{in} \quad Q_5 \\ h = w & \text{on} \quad \partial Q_5 \end{cases} \tag{3.3}$$

with

$$\frac{1}{|Q_5|}\int_{Q_5} \left|\beta(x) - \beta(x_1, \cdot)_{B_5'}\right|^2 \le \delta^2,$$

where

$$\beta(x) = \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{|a(x, \xi)|}{|\xi|}$$

and $\beta(x_1, \cdot)_{B_5'}$ is the integral average of the mapping $x' \mapsto \beta(x_1, x')$ over B_5'.

Taking $w - h \in H_0^1(Q_5, \omega^2)$ to (3.2) and (3.3) as a test function, we discover that

$$\frac{1}{|Q_5|}\int_{Q_5} |\mathbb{M}Dh|^2 \le c, \ \frac{1}{|Q_5|}\int_{Q_5} |\mathbb{M}Dw - \mathbb{M}Dh|^2 \le \delta^2.$$

Consider a weak solution $v \in H^1(Q_4, \overline{\omega}^2)$ to

$$\operatorname{div}\left(a_{\overline{\mathbb{M}}_{Q_4}}(x_1, Dv)\right) = 0 \text{ in } Q_4 \tag{3.4}$$

under an extra assumption as

$$\frac{1}{|Q_4|}\int_{Q_4} \left|\log \mathbb{M} - \overline{\log \mathbb{M}}_{Q_4}\right| \le \delta.$$

Then we find

- $h \in H^1(Q_4, \overline{\omega}^2)$;
- $v \in H^1(Q_4, \omega^2)$;
- $$\frac{1}{|Q_4|}\int_{Q_4} \left|\frac{M - \overline{M}_{Q_4}}{\overline{M}_{Q_4}}\right| \le c\frac{1}{|Q_4|}\int_{Q_4} \left|\log \mathbb{M} - \overline{\log \mathbb{M}}_{Q_4}\right| \le c\delta.$$

We then find the following weak compactness result. The proof is based on weak compactness in the weighted Sobolev spaces, comparison estimates and Lemma 3.2.

Lemma 3.3. *For any $\epsilon > 0$, there exist $\delta(\epsilon) > 0$ and a weak solution $v \in H^1(Q_4, \overline{\omega}^2)$ to (3.4) such that if*

$$\frac{1}{|Q_4|}\int_{Q_4} \left|\log \mathbb{M} - \overline{\log \mathbb{M}}_{Q_4}\right| \le \delta,$$

then

$$\frac{1}{|Q_2|}\int_{Q_2} \left|\mathbb{M}Du - \overline{\mathbb{M}}_{Q_2}Dv\right|^2 \le \epsilon^2$$

and

$$\sup_{Q_1} \left|\overline{\mathbb{M}}_{Q_1}Dv\right| \le N_0$$

for some generic constant N_0.

We now restate the main result as follows.

Theorem 3.4. *Suppose* $\mathbb{M}F \in L^p(Q_2)$ *for some* $p > 2$. *Let* $u \in H^1(Q_2, \omega^2)$ *be a weak solution to*

$$div\,(\mathbb{M}a(x, \mathbb{M}Du)) = div\left(\mathbb{M}^2 F\right) \;\; in\ Q_2.$$

Then there exists $\delta = \delta(p, \nu, L, n, \Lambda) > 0$ *such that if*

$$\sup_{x_1 \in \mathbb{R}} [\beta(x_1, \cdot)]_{BMO} + [\log \mathbb{M}]_{BMO} \leq \delta,$$

then

$$\mathbb{M}Du \in L^p(Q_1)$$

and we have the estimate

$$\int_{Q_1} |Du|^p \omega^p dx \leq c \left(\int_{Q_2} |u|^p \omega^p dx + \int_{Q_2} |F|^p \omega^p dx \right)$$

for some positive constant $c = c(n, \nu, L, \Lambda, p)$.

Proof. With some universal constant λ_0 and a parameter $\lambda > 1$, a reasonable expectation of an inductive estimate could be

$$|\{|\mathbb{M}Du| > \lambda_0\}| \leq \epsilon\left(|\{|\mathbb{M}Du| > 1\}| + |\{|\mathbb{M}Du| > \delta\}|\right),$$

which can be scaled to

$$|\{|\mathbb{M}Du| > \lambda\lambda_0\}| \leq \epsilon\left(|\{|\mathbb{M}Du| > \lambda\}| + |\{|\mathbb{M}Du| > \lambda\delta\}|\right).$$

More precisely, we find some generic constant N_1 for which

$$\begin{aligned} \left|\{x \in Q_1 : \mathcal{M}(|\mathbb{M}Du|^2) > \left(N_1^2\right)^k\}\right| \;\; &\leq \;\; \sum_{i=1}^{k} \epsilon^i \left|\{x \in Q_2 : \mathcal{M}(|\mathbb{M}F|^2) > \delta^2 \left(N_1^2\right)^{k-1}\}\right| \\ &\quad + \epsilon^k \left|\{x \in Q_2 : \mathcal{M}(|\mathbb{M}Du|^2) > 1\}\right|, \end{aligned}$$

which implies

$$\begin{aligned} \sum_{k=0}^{\infty} \left(N_1^2\right)^{\frac{p}{2}k} \left|\{x \in Q_1 : \mathcal{M}(|\mathbb{M}Du|^2) > \left(N_1^2\right)^k\}\right| \;\; &\leq \;\; c\|\mathcal{M}F\|^p_{L^p(Q_2)} \sum_{k=0}^{\infty} \left(N_1^p \epsilon\right)^k \\ &\quad + \sum_{k=0}^{\infty} \left(N_1^p \epsilon\right)^k. \end{aligned}$$

Take ϵ such that

$$0 < N_1^p \epsilon < 1.$$

We then determine δ from Lemma 3.3 for which

$$\mathcal{M}(|\mathbb{M}Du|^2) \in L^{\frac{p}{2}}(Q_1).$$

Theorem 3.4 now follows from the basic properties of the Hardy-Littlewood maximal operator. $\square$

We end this section by clearly pointing out that the argument used here can be applied to a wider class of degenerate/singular elliptic and parabolic equations, for instances,

- p-Laplacian type for $1 < p < \infty$:

$$\operatorname{div}\left(\mathbb{M}(x) a(x, \mathbb{M}(x) Du)\right) = \operatorname{div}\left(|\mathbb{M}(x) F(x)|^{p-2}\, \mathbb{M}^2(x) F(x)\right).$$

- Parabolic equations with matrix weight independent of time:

$$|\mathbb{M}(x)|^2 u_t - \operatorname{div}\left(\mathbb{M}(x) a(x, t, \mathbb{M}(x) D_x u)\right) = \operatorname{div}\left(\mathbb{M}^2(x) F(x, t)\right).$$

References

[1] E. Acerbi and G. Mingione, *Gradient estimates for a class of parabolic systems*, Duke Math. J. **136** (2007), no. 2, 285–320.

[2] A. K. Balci, S. Byun, L. Diening, H. Lee,*Global maximal regularity for equations with degenerate weights* (2022), arXiv:2201.03524.

[3] A. K. Balci, L. Diening, R. Giova, and A. P. di Napoli, *Elliptic equations with degenerate weights*, SIAM Journal on Mathematical Analysis 54(2) (2022), 2373-2412.

[4] S. Byun and Y. Kim, *Elliptic equations with measurable nonlinearities in nonsmooth domains*, Adv. Math. **288** (2016), 152–200.

[5] S. Byun and L. Wang, *Elliptic equations with BMO coefficients in Reifenberg domains*, Comm. Pure Appl. Math. **57** (2004), no. 10, 1283–1310.

[6] ———, *Elliptic equations with measurable coefficients in Reifenberg domains*, Adv. Math. **225** (2010), no. 5, 2648–2673.

[7] L. Caffarelli and I. Peral, *On $W^{1,p}$ estimates for elliptic equations in divergence form*, Comm. Pure Appl. Math. **51** (1998), no. 1, 1–21.

[8] D. Cao, T. Mengesha, and T. Phan, *Weighted-$W^{1,p}$ estimates for weak solutions of degenerate and singular elliptic equations*, Indiana Univ. Math. J. **67** (2018), no. 6, 2225–2277.

[9] G. David and T. Toro, *Reifenberg parameterizations for sets with holes*, Mem. Amer. Math. Soc. **215** (2012), no. 1012, vi+102. vol. 2017, Springer, Heidelberg, 2011.

[10] H. Dong and D. Kim, *Elliptic equations in divergence form with partially BMO coefficients*, Arch. Ration. Mech. Anal. **196** (2010), no. 1, 25–70.

[11] H. Dong and T. Phan, *Parabolic and elliptic equations with singular or degenerate coefficients: the Dirichlet problem*, Trans. Amer. Math. Soc. **374** (2021), no. 9, 6611–6647.

[12] H. Dong and T. Phan, *Weighted mixed-norm L_p-estimates for elliptic and parabolic equations in non-divergence form with singular coefficients*, Rev. Mat. Iberoam. **37** (2021), no. 4, 1413–1440.

[13] I. Drelichman and R. G. Durán, *Improved Poincaré inequalities with weights*, J. Math. Anal. Appl. **347** (2008), no. 1, 286–293.

[14] E. B. Fabes, C. E. Kenig, and R. P. Serapioni, *The local regularity of solutions of degenerate elliptic equations*, Comm. Partial Differential Equations **7** (1982), no. 1, 77–116.

[15] C. E. Kenig and T. Toro, *Free boundary regularity for harmonic measures and Poisson kernels*, Ann. of Math. (2) **150** (1999), no. 2, 369–454.

[16] G. Mingione, *Nonlinear aspects of Calderón-Zygmund theory*, Jahresber. Dtsch. Math.-Ver. **112**, (2010), no. 3,159–191.

[17] T. Phan, *Weighted Calderón-Zygmund estimates for weak solutions of quasi-linear degenerate elliptic equations*, Potential Anal. **52** (2020), no. 3, 393–425.

[18] R. E. Showalter, *Monotone operators in Banach space and nonlinear partial differential equations*, Mathematical Surveys and Monographs, vol. 49, American Mathematical Society, Providence, RI, 1997.

[19] T. Toro, *Doubling and flatness: geometry of measures*, Notices Amer. Math. Soc. **44** (1997), no. 9, 1087–1094.

Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University, Seoul 08826, Korea

E-mail address: byun@snu.ac.kr

Mathematical Justification of the Hydrostatic Approximation of the Primitive Equations in Anisotropic Spaces

Ken Furukawa (RIKEN)

1 Introduction

The primitive equations are

$$
\begin{array}{rcll}
\partial_t v - \Delta v + v \cdot \nabla_H v + w \partial_3 v + \nabla_H \pi & = & 0 & \text{in} \quad \mathbb{T}^3 \times (0,T), \\
\partial_3 \pi & = & 0 & \text{in} \quad \mathbb{T}^3 \times (0,T), \\
\operatorname{div}_H v + \partial_3 w & = & 0 & \text{in} \quad \mathbb{T}^3 \times (0,T), \\
v(0) & = & v_0 & \text{in} \quad \mathbb{T}^3,
\end{array}
\tag{1.1}
$$

where $u = (v, w) \in \mathbb{R}^2 \times \mathbb{R}$ is a vector field, π is a scalar function, $\nabla_H = (\partial_1, \partial_2)^T$ is the horizontal gradient, $\operatorname{div}_H = \nabla_H \cdot$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is the flat torus, and $T > 0$. We impose the periodic boundary conditions on the lateral boundary and assume v, w, π are even, odd, and odd with respect to x_3, respectively, as compatibility conditions. The second equation in (1.1) is called the hydrostatic approximation, which is the main topic of this talk. The primitive equations describe the motion of the fluid filled in a thin domain, *e.g.* the ocean and the atmosphere. Since in (1.1) there is no evolution equation for w, we determine w via the divergence-free condition. Integrating the divergence-free condition over $(-\pi, x_3)$ and using the boundary condition $w|_{x_3=\pm\pi} = 0$, we find that w satisfies

$$
w(t, x', x_3) = - \int_{-\pi}^{x_3} \operatorname{div}_H v(t, x', z) dz, \quad x = (x', x_3) \in \mathbb{T}^2 \times \mathbb{T}, \, t > 0. \tag{1.2}
$$

The global well-posedness of the primitive equations in H^1 is established by Cao and Titi [2]. Hieber and Kashiwabara [10] extended this result to L^p-framework. Giga *et al*. [5] proved the global well-posedness in L^p-L^q maximal regularity class. They also proved the global well-posedness in $L^\infty_H L^1_{x_3}$-based anisotropic space [8], where

$$L^\infty_H L^q_{x_3}(\mathbb{T}^3) := \left\{ f \in L^\infty(\mathbb{T}^2; L^q(\mathbb{T})) \,:\, \|f\|_{L^\infty_H L^q_{x_3}(\mathbb{T}^3)} < \infty \right\}$$
$$\|f\|_{L^\infty_H L^q_{x_3}(\mathbb{T}^3)} := \sup_{x' \in \mathbb{T}^2} \left(\int_{\mathbb{T}} |f(x', x_3)|^q \, dx_3 \right)^{1/q},$$

and the space of $f \in L^\infty_H L^q_{x_3}(\mathbb{T}^3)$ which is continuous with respect to x' in $L^q(\mathbb{T})$ is denoted by $C_H L^q_{x_3}(\mathbb{T}^3)$. The result can be extended for $L^\infty_H L^q_{x_3}$ using the same way as [8]. The anisotropic space $L^\infty_t L^\infty_H L^1_{x_3}$ is an scale-invariant under the natural scaling

$$u_\lambda(x', x_3, t) = \lambda u(\lambda x', \lambda x_3, \lambda^2 t).$$

Giga *et al.* proved the well-posedness using mild solution formulation and the Fujita-Kato approach.

The primitive equation is formally derived from the Navier-Stokes equations with anisotropic viscosity in the thin domain $\Omega_\varepsilon = \mathbb{T}^2 \times \varepsilon\mathbb{T}$ such that

$$\begin{aligned} \partial_t u - (\Delta_H + \varepsilon^2 \partial_3^2) u + u \cdot \nabla u + \nabla \pi &= 0 \quad \text{in} \quad \Omega_\varepsilon \times (0, T) \\ \operatorname{div} u &= 0 \quad \text{in} \quad \Omega_\varepsilon \times (0, T) \end{aligned}$$

We apply the scaling to (u, π) such as

$$\begin{aligned} u_\varepsilon &:= (v_\varepsilon, w_\varepsilon), \\ v_\varepsilon(x, y, z, t) &:= v(x, y, \varepsilon z, t), \\ w_\varepsilon(x, y, z, t), &:= w(x, y, \varepsilon z, t)/\varepsilon, \\ p_\varepsilon(x, y, z, t), &:= p(x, y, \varepsilon z, t), \end{aligned}$$

then we obtain the scaled Navier-Stokes equations

$$\begin{array}{rcll} \partial_t v_\varepsilon - \Delta v_\varepsilon + u_\varepsilon \cdot \nabla v_\varepsilon + \nabla_H \pi_\varepsilon &=& 0 & \text{in} \quad \mathbb{T}^3 \times (0, T), \\ \varepsilon \left(\partial_t w_\varepsilon - \Delta w_\varepsilon + u_\varepsilon \cdot \nabla w_\varepsilon \right) + \partial_3 \pi_\varepsilon / \varepsilon &=& 0 & \text{in} \quad \mathbb{T}^3 \times (0, T), \\ \operatorname{div} u &=& 0 & \text{in} \quad \mathbb{T}^3 \times (0, T), \\ u_\varepsilon(0) &=& u_0 & \text{in} \quad \mathbb{T}^3. \end{array} \tag{1.3}$$

Note that the domain is independent of ε. If we take $\varepsilon \to 0$, then we formally get (1.1). Our aim is to justify this naive derivation to show $u_\varepsilon \to u$ in some topologies.

Azérad and Guillén [1] prove the above convergence in L^2 space. They showed the weak convergence with no algebraic convergence rate for ε. Li and Titi [11] showed the strong convergence in the energy space with $O(\varepsilon)$. The authors together with Giga *et al.* [3] extended their result to L^p-L^q base maximal regularity class. We obtain

Theorem 1.1 ([4]). *Let $T > 0$, $q \geq 1$, and $u_0 = (v_0, w_0) \in C_H L^q_{x_3}(\mathbb{T}^3)$ satisfy* $\operatorname{div} u_0 = 0$ *and $\nabla_H v_0 \in C_H L^q_{x_3}(\mathbb{T}^3)$. Let $u \in C_t C_H L^q_{x_3}(\mathbb{T}^3 \times [0,T))$ be a solution to (1.1) with initial data u_0. Then there exists $\varepsilon_0 > 0$, if $\varepsilon < \varepsilon_0$, the difference between the solutions to (1.1) and (1.3) satisfies*

$$\begin{aligned}
&\sup_{0<t<T} \|v - v_\varepsilon\|_{L^\infty_H L^q_{x_3}(\mathbb{T}^3)} + \sup_{0<t<T} t^{q/2} \|\nabla v - v_\varepsilon\|_{L^\infty_H L^q_{x_3}(\mathbb{T}^3)} \\
&+ \sup_{0<t<T} \|\varepsilon(w - w_\varepsilon)\|_{L^\infty_H L^q_{x_3}(\mathbb{T}^3)} + \sup_{0<t<T} t^{q/2} \|\varepsilon \nabla(w - w_\varepsilon)\|_{L^\infty_H L^q_{x_3}(\mathbb{T}^3)} \\
&\leq C\varepsilon,
\end{aligned} \tag{1.4}$$

where C is independent of ε.

2 Strategy

The proof is based on the Fujita-Kato iteration for the mild solution. The difference $(V_\varepsilon, W_\varepsilon) = (v - v_\varepsilon, w - w_\varepsilon)$ and $\Pi_\varepsilon = (\pi - \pi_\varepsilon)$ satisfy

$$\begin{cases}
\partial_t V_\varepsilon - \Delta V_\varepsilon + \nabla_H \Pi_\varepsilon &= F_H(U_\varepsilon, u), \\
\partial_t(\varepsilon W_\varepsilon) - \Delta(\varepsilon W_\varepsilon) + \frac{\partial_3}{\varepsilon}\Pi_\varepsilon &= \varepsilon F_3(U_\varepsilon, u) + \varepsilon F(u), \\
\operatorname{div}_\varepsilon(V_\varepsilon, \varepsilon W_\varepsilon) &= 0, \\
U_\varepsilon &= 0,
\end{cases} \tag{2.1}$$

Note that the initial data of u and u_ε have to coincide. This implies $U_\varepsilon = 0$. where $\operatorname{div}_\varepsilon = \nabla_\varepsilon \cdot = (\partial_1, \partial_2, \partial_3/\varepsilon)^T \cdot$ and

- $F_H(U_\varepsilon, u) = -\left(U_\varepsilon \cdot \nabla W_\varepsilon + u \cdot \nabla V_\varepsilon + U_\varepsilon \cdot \nabla v\right)$,
- $F_3(U_\varepsilon, u) = -\left(U_\varepsilon \cdot \nabla W_\varepsilon + u \cdot \nabla W_\varepsilon + U_\varepsilon \cdot \nabla w\right)$,
- $F(u) = -\left(\partial_t w - \Delta w + u \cdot \nabla w\right)$.

Let $-\Delta_\varepsilon = \nabla_\varepsilon \cdot \nabla_\varepsilon$ be the anisotropic Laplace operator and $\mathbb{P}_\varepsilon = I + \nabla_\varepsilon(-\Delta_\varepsilon)\operatorname{div}_\varepsilon$ be the anisotropic Helmholtz projection. The mild solution to (2.1) satisfies

$$\begin{aligned}
&\begin{pmatrix} V_\varepsilon(t) \\ \varepsilon W_\varepsilon(t) \end{pmatrix} \\
&= \int_0^t e^{(t-s)\Delta} \mathbb{P}_\varepsilon \begin{pmatrix} F_H(U_\varepsilon(s), u(s)) \\ \varepsilon F_3(U_\varepsilon(s), u(s)) \end{pmatrix} ds \\
&+ \varepsilon \int_0^t e^{(t-s)\Delta} \mathbb{P}_\varepsilon \begin{pmatrix} 0 \\ \tilde{F}(v(s), w(s)) \end{pmatrix} ds \\
&=: I_1 + I_2
\end{aligned} \tag{2.2}$$

It is enough to show (1.4) for U_ε but there are some problems;

- ε-independent estimate for the operators $e^{t\Delta}$ and $e^{t\Delta}\mathbb{P}_\varepsilon\partial_j$ $(j = 1, 2, 3)$ in $L^\infty_H L^q_{x_3}(\mathbb{T}^3)$,
- Regularity loss from $\partial_t w - \Delta w$ in $F(u)$.

Although some non-linear estimates are used in the proof, the estimates can be obtained by the same way as [8]. It is well-known that in non-anisotropic case it holds

$$\begin{aligned}
&\|\partial_x^k e^{t\Delta}\|_{L^p(\mathbb{R}^d)\to L^q(\mathbb{R}^d)} \le Ct^{-\frac{k}{2}-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})},\\
&\|\partial_x^k e^{t\Delta}\mathbb{P}_\varepsilon\partial_j\|_{L^p(\mathbb{R}^d)\to L^q(\mathbb{R}^d)} \le Ct^{-\frac{k+1}{2}-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})},
\end{aligned}$$

for $k \in \mathbb{Z}_{\ge 0}$ and $1 \le p \le q \le \infty$.

In $L^\infty_H L^q_{x_3}(\mathbb{T}^3)$-case, we analogously see integrability improvement for x_3-direction (corresponds to $d = 1$), *i.e.*

$$\begin{aligned}
&\|\partial_x^k e^{t\Delta}\|_{L^\infty_H L^p_{x_3}(\mathbb{T}^3)\to L^\infty_H L^q_{x_3}(\mathbb{T}^3)} \le Ct^{-\frac{k}{2}-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})},\\
&\|\partial_x^k e^{t\Delta}\mathbb{P}_1\partial_j\|_{L^\infty_H L^p_{x_3}(\mathbb{T}^3)\to L^\infty_H L^q_{x_3}(\mathbb{T}^3)} \le Ct^{-\frac{k+1}{2}-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})},
\end{aligned}$$

where C is independent of ε. We can prove these estimates by getting the integral kernel formulas of the operators and applying the Young inequality to them.

The top term $\partial_t w - \Delta w$ causes regularity loss because, in the theory of analytic semigroups, it is impossible to estimate I_2 of (2.2) without additional regularity assumptions. However, it is enough to assume additional regularity for the horizontal direction to v, see Theorem 1.1. To find this, we derive the equation that w satisfies via the div-free condition in the primitive equations

$$\begin{aligned}
\partial_t w - \Delta w = &\int_{-\pi}^{x_3} \mathrm{div}_H\ (\tilde{v}\cdot\nabla_H\tilde{v})\,dz + \mathrm{div}_H\left(\int_{-\pi}^{x_3}\tilde{v}dz\cdot\nabla_H\overline{v}\right)\\
&+ \mathrm{div}_H\left(\overline{v}\cdot\nabla_H\int_{-\pi}^{x_3}\tilde{v}dz\right) + \mathrm{div}_H(w\tilde{v})\\
&+ \mathrm{div}_H\left(\int_{-\pi}^{x_3}(\mathrm{div}_H\tilde{v})\tilde{v}dz\right)\\
&- \frac{1}{2}(x_3-\pi)\mathrm{div}_H\int_{-\pi}^{x_3}\tilde{v}\cdot\nabla_H\tilde{v} + (\mathrm{div}_H\,\tilde{v})\tilde{v}dz,
\end{aligned} \tag{2.3}$$

where $\tilde{v} = v - \overline{v}$ and $\overline{v}$ is the horizontal average. The terms on the right-hand side can be estimated by v if v is regular enough with respect to x'.

Using the estimates of the composite operators, we have quadratic bounds for I_1. The term I_2 is bounded by quadratic terms of v from estimates for the right-hand side of (2.3). Therefore we obtain $I_2 \le O(\varepsilon)$. Using the Fujita-Kato principle, we obtain (1.4).

The following corollary implies the global-well posedness of (1.3) for small ε.

Corollary 2.1 ([4]). *Under the same assumptions for $T, q, u_0, \varepsilon, \varepsilon_0$, if $\varepsilon < \varepsilon_0$, then there exists a unique solution $u_\varepsilon = (v_\varepsilon, w_\varepsilon) \in C_t C_H L^q_{x_3}(\mathbb{T}^3 \times (0,T))$ to (1.3) such that*

$$\begin{aligned}&\sup_{0<t<T} \|v_\varepsilon\|_{L^\infty_H L^1_{x_3}(\mathbb{T}^3)} + \sup_{0<t<T} t^{1/2} \|\nabla v_\varepsilon\|_{L^\infty_H L^1_{x_3}(\mathbb{T}^3)} \\ &+ \sup_{0<t<T} \|\varepsilon w_\varepsilon\|_{L^\infty_H L^1_{x_3}(\mathbb{T}^3)} + \sup_{0<t<T} t^{1/2} \|\varepsilon \nabla w_\varepsilon\|_{L^\infty_H L^1_{x_3}(\mathbb{T}^3)} \\ &< \infty. \end{aligned} \tag{2.4}$$

Acknowledgement

The talk and abstract are based on the joint work with Prof. Kashiwabara of the University of Tokyo.

References

[1] P. Azérad and F. Guillén, Mathematical justification of the hydrostatic approximation in the primitive equations of geophysical fluid dynamics, SIAM J. Math. Anal., 33, 4, (2001), 847–859.

[2] C. Cao and E. S. Titi, Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics, Ann. of Math. (2), 166, 1, (2007), 245–267.

[3] K. Furukawa, Y. Giga, M. Hieber, A. Hussein, T. Kashiwabara, and M. Wrona, Rigorous justification of the hydrostatic approximation for the primitive equations by scaled Navier-Stokes equations, Nonlinearity 33„ no. 12, 2020, 6502–-6516.

[4] K. Furukawa, T. Kashiwabara, JUSTIFICATION OF THE HYDROSTATIC APPROXIMATION OF THE PRIMITIVE EQUATIONS IN ANISOTROPIC SPACE $L^\infty_H L^q(\mathbb{T}^3)$, Advances in Mathematical Sciences and Applications, to appear.

[5] Y. Giga, M. Gries, M. Hieber, A. Hussein, and T. Kashiwabara, Analyticity of solutions to the primitive equations, Math. Nachr. 293(2), (2020), 284-304.

[6] Y. Giga, M. Gries, M. Hieber, A. Hussein, and T. Kashiwabara, Bounded H^∞-calculus for the hydrostatic Stokes operator on L^p-spaces and applications, Proc. Amer. Math. Soc., 145, 9, (2017), 3865–3876.

[7] Y. Giga, M. Gries, M. Hieber, A. Hussein, and T. Kashiwabara, The hydrostatic Stokes semigroup and well-posedness of the primitive equations on spaces of bounded functions, J. Funct. Anal., 279, 3, (2020), 108561, 46.

[8] Y. Giga, M. Gries, M. Hieber, A. Hussein, and T. Kashiwabara, The Primitive Equations in the scaling invariant space $L^\infty(L^1)$, J. Evol. Equ., (2021), https://doi.org/10.1007/s00028-021-00716-z.

[9] F. Guillén-González, N. Masmoudi, and M. A. Rodríguez-Bellido, Anisotropic estimates and strong solutions of the primitive equations, Differential Integral Equations, 14, 1, (2001), 1381–1408.

[10] M. Hieber and T. Kashiwabara, Global strong well-posedness of the three dimensional primitive equations in l^p-spaces, Archive Rational Mech. Anal., (2016), 1077–1115.

[11] J. Li and E. S. Titi, The primitive equations as the small aspect ratio limit of the Navier-Stokes equations: rigorous justification of the hydrostatic approximation, J. Math. Pures Appl. (9) 124, (2019), 30–58.

Forward and inverse analyses for anisotropic elastic and viscoelastic systems

Gen Nakamura*

1 Introduction

In this talk, I will concern about forward and inverse analyses for anisotropic elastic systems and anisotropic viscoelastic systems, especially for the dynamical case. In most of inverse problems, basic tools for them are the unique continuation property (UCP) of solutions and the exact boundary controllability (EBC). The first one is well known in PDE. The second one is to find a boundary control which connect the given initial state to the given final state. Moving from a scalar equation to a system and the system becomes anisotropic, it becomes hard to have these tools. Nevertheless, in order to have these tools, it is necessary to consider a good setting that does not lose generality as much as possible.

The settings which I propose here are as follows. For having the Holmgren-John UCP, the setting would be piecewise homogeneous coefficients or piecewise analytic coefficients. This UCP should be the optimal one which can give the optimal time for continuing the zero set of solutions in terms of the travel time of the slowest wave. I do believe that this setting is general enough because it includes the first order finite element model, and it can also describe the medium discontinuity. Combining the Holmgren-John UCP with some geometric tools and algebraic tools, it is possible to obtain some global uniqueness result for the coefficient identification problem (referred as inverse coefficient problem which is abbreviated by ICP) by using the so-called localized Neumann to Dirichlet map (ND-map) as measured data, and also some local recovery of coefficients if we know the medium discontinuity. Here the geometric tools are the theory of analytic sets ([3]) and continuity of symmetric axis of tensors ([12]), and the algebraic tools are the Stroh formalism ([20]) and the theory of matrix polynomials ([10]). This ICP result gives a mathematical foundation for the vibroseis reflection exploration in geophysics.

The EBC can imply the observability inequality which estimates the initial data by some lateral boundary integral of the solution to the initial boundary value problem

*Departments of Mathematics, Hokkaido University and Research Center of Mathematics for Social Creativity, Research Institute for Electronic Science, Hokkaido University (Email:gnaka@math.sci.hokudai.ac.jp)

for a system. Further, this inequality is useful to analyze the stability estimate for the inverse source problem for the system, which is the first step toward analyzing the stability estimate for the ICP ([1]). The biggest advantage of the EBC is that it is not necessary to assume the piecewise homogeneous or analytic condition for the coefficients of the system.

For an elastic system coupled with a conductivity equation, i.e. piezoelectric system, it is possible to obtain the EBC by the Russell principle ([11]). This principle says whenever there is a uniform decay estimate of solutions for the initial boundary value problem as the time goes to infinity for a hyperbolic equation or system, it is possible to have the EBC. Then, the result given in ([11]) is very interesting, because the piezoelectric system is not hyperbolic and it has to be anisotropic due to the anisotropy of materials which have piezoelectric property. Further, the conductivity equation is giving physically very natural dissipation which yields the uniform decay of solutions. As for the viscoelastic system, the viscosity gives the dissipation but it is an obstruction for having the time reversibility in the complete sense. Hence, it is interesting to analyze the possibility of having the EBC for the viscoelastic system.

The rest of this paper is organized as follows. In Section 2, anisotropic elastic/ viscoelastic systems are introduces. Then in Section 3, some results and possible results will be presented for the ICP for elastic/viscoelastic systems with some explanation of the tools for their proofs. In Section 4, I give a very short description of current situation of the ongoing study on the EBC for viscoelastic systems.

2 Anisotropic elastic/viscoelastic systems

Elasticity, more precisely the linear elasticity is a property of small mechanical deformation of solids expressed by Hooke's law. For the one space dimensional case, it is given as the linear relation between stress σ and strain ϵ of a spring (see Figure 1) give as

$$\sigma = C\epsilon \ \text{(constitutive equation)},$$

where C is called the spring constant.

Figure 1

Its three space dimensional generalization is given as

$$\sigma_{ij} = \sum_{i,j=1}^{3} C_{ijkl}\epsilon_{kl},\ 1 \le i, j \le 3 \quad \text{i.e. } \sigma = C :: \epsilon, \tag{2.1}$$

where $\sigma := (\sigma_{ij})$ and $\epsilon := (\epsilon_{kl})$ are the stress and strain, respectively. Also, $C := (C_{ijkl})$ is called the elasticity tensor. In terms of the displacement vector $u = (u_1, u_2, u_2)^t$ with respect to the Cartesian coordinates $(x_1, x_2, x_3)^t$, ϵ is given as $\epsilon := \frac{1}{2}\{\nabla u + (\nabla u)^t\}$ with the gradient $\nabla u := (\partial_j u_i) = (\partial u_i / \partial x_j)$ of u, where the superscript t denotes the transpose.

It is physically natural to assume the following symmetry and strong convexity conditions for $C = C(x) \in L^\infty(\Omega)$ with a bounded domain $\Omega \subset \mathbb{R}^3$ taken as a reference domain with Lipschitz smooth boundary Γ for a meanwhile.

symmetry:

$$C_{ijkl}(x) = C_{ijlk}(x) \ (\textit{minor symmetry}), \ C_{ijkl}(x) = C_{klij}(x) \ (\textit{major symmetry})$$

for any $i, j, k, l \in \{1, 2, 3\}$ and a.e. $x \in \overline{\Omega}$.

strong convexity:

$$^\exists\delta > 0 : {}^\forall\text{symmetric}\,\epsilon := (\epsilon_{ij}),\ ^\forall\,\text{a.e.}\,x \in \Omega,\ (C(x) :: \epsilon) : \epsilon \geq \delta(\epsilon : \epsilon) := \delta(\sum_{i,j=1}^{3} \epsilon_{ij}\epsilon_{ij}).$$

If C does not depend on position, it is called homogeneous. Also, if C does not depend on direction, it is called isotropic and given as $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj})$ with Lamé moduli $\lambda = \lambda(x)$, $\mu = \mu(x)$ satisiying the strong convexity condition $^\exists\delta > 0 :\ 3\lambda + 2\mu \geq \delta$ a.e. $x \in \Omega$, where δ_{ij} is Kronecker's delta.

There are many viscoelastic materials around us, for examples biological tissues, polymer(plastic, rubber, $\cdots$) and many earth materials. There are three typical viscoelastic models called Maxwell model, Kelvin Voigt model and Zener model. For the one space dimensional case, they can be described by using spring and dashpot (see Figure 2). The constitutive equations of these models are given as

$$\begin{cases} \sigma(t) = C\epsilon(t) - \int_0^t e^{-\eta^{-1}C(t-\tau)}\eta^{-1}C^2\epsilon(\tau)\,d\tau & \text{(Maxwell)}, \\ \sigma(t) = C\epsilon(t) + \eta\,\partial_t\epsilon(t) & \text{(Kelvin-Voigt)}, \\ \sigma(t) = \sum_{j=1}^{2}\{C_j\epsilon(t) - \int_0^t e^{-\eta_j^{-1}C_j(t-\tau)}\eta_j^{-1}C_j^2\,\epsilon(\tau)\,d\tau\} & \text{(Zener)}, \end{cases}$$

where C, C_1, C_2 and η, η_1, η_2 are spring constants of springs and viscosities of dashpots, respectively. In deriving these constitutive equations, it is assumed that $\epsilon(0) = 0$ and the stress applied to the dashpot is proportional to the strain. Note that the Kelvin-Voigt model is the only model which does not contain any integral term, referred as memory term, in its constitutive equation. In the rest of this talk, I will focus on the case the constitutive systems which have a memory term.

The three space dimensional generalization of such constitutive equations are given as

$$\sigma(t) = C\epsilon(t) - \int_0^t G(t-\tau)\epsilon(\tau)\,d\tau, \tag{2.2}$$

where $C = C(x)$ and -1 multiplied to $G(t) = G(x, t)$ are the so-called instantaneous elastic tensor and is the $t-$ derivative of the relaxation tensor, respectively.

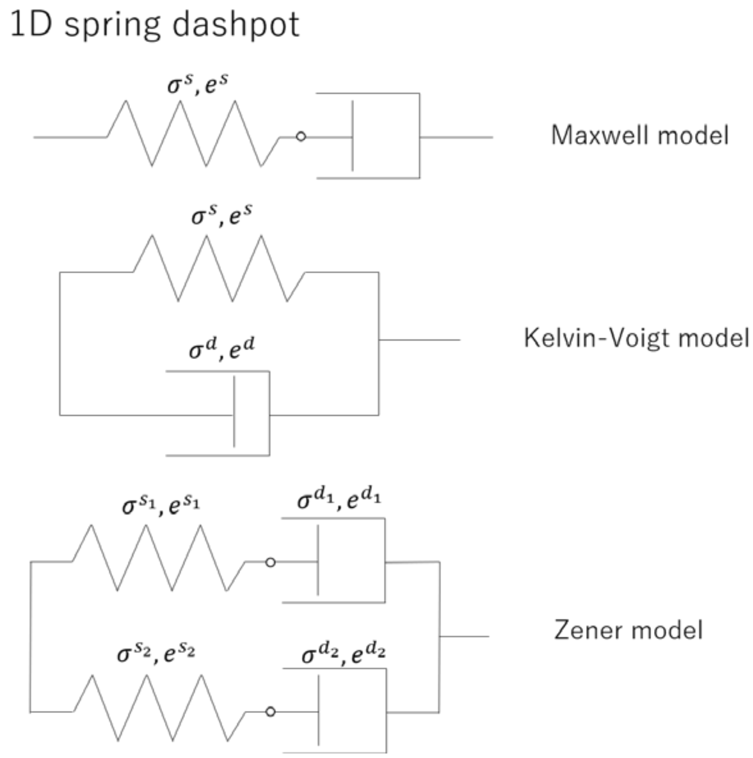


Figure 2

Again, it is physically natural to assume the symmetry for C. As for G, it is natural to assume the symmetry likewise for C, $\dot{G} := \partial_t G \le 0$ and the following inequality which corresponds to the strong convexity in relation with C :

$$^\exists\nu_0 > 0 : {}^\forall\text{symmetric}\,\epsilon,\ {}^\forall\text{a.e.}\,x \in \Omega,\ \{(C(x) - \int_0^\infty G(x,\tau)\,d\tau) :: \epsilon\} : \epsilon \ge \nu_0|\epsilon|^2, \quad (2.3)$$

where $|\epsilon|^2 := \epsilon : \epsilon$.

Besides these we assume the following technical conditions:

$$\begin{cases} 0 \le G \in C^2([0,\infty); L^\infty(\Omega)), \\ ^\exists p > 1, {}^\exists\kappa_j > 0, j = 1,2,3 \ : \\ \quad -\kappa_1\, G \le \dot{G} \le -\kappa_2\, G^{1+1/p},\ \ \ddot{G} := \partial_t^2 G \le \kappa_3\, G^{1+1/p}, \\ \quad \sup_{t\in(0,\infty)}(1+t)^p\|G(\cdot,t) + |\dot{G}(\cdot,t)|\|_{L^\infty(\Omega)} < \infty. \end{cases} \quad (2.4)$$

Now, for a density $\rho \in L^\infty(\Omega)$ with $\operatorname{esssup}_{x\in\Omega}\rho(x) > 0$, the elastodynamical system and viscoelastic system with source term F are both given in the form:

$$\rho\partial_t^2 u - \operatorname{div}\sigma = F \text{ in } \Omega_T := \Omega \times (0,T) \quad (2.5)$$

It is quite standard to show the well-posedness of the initial boundary value problem under some conditions on the source term, boundary data and initial data at initial time $t = 0$ even in the case that the boundary condition is a mixed type boundary condition (see [7], [21]).

3 ICP for elastic/viscoelastic systems

In order to state our results, we first define several terminologies.

Definition 3.1

(i) *Consider finitely many subdomains* $D_\alpha \subset \Omega$, $\alpha \in A$ *such that*

$$\bar{\Omega} = \cup_{\alpha \in A} \overline{D_\alpha}, \quad D_\alpha \cap D_\beta = \emptyset \;\; if \;\; \alpha \neq \beta.$$

$\{D_\alpha\}_{\alpha \in A}/\{\overline{D_\alpha}\}_{\alpha \in A}$ *and each* ∂D_α *are called cover of* Ω *and interface, respectively. Assume that each interface is piecewise analytic. In this situation, if* ρ, C *(resp.* ρ, C, G*) are homogeneous (resp. analytic) on each* $\overline{D_\alpha}$*, we say* (ρ, C) *(resp.* (ρ, C, G)*) is piecewise homogeneous (resp. analytic).*

(ii) *Yello colored subdomains in Figure 3 are called a chain connecting* D_α *to a subdomain, say* D_{α_1} *such that* $\Sigma \Subset \partial D_{\alpha_1} \cap \Gamma$*, where* Σ *is an open connected set where we conduct our measurements.*

(iii) *If* $\overline{\Omega}$ *consists of simplices, we say* Ω *is grid decomposed or has a grid decomposition.*

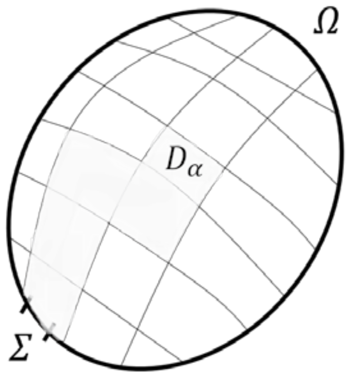


Figure 3

Next, let us see how much the UCP holds when the elastic body and the viscoelastic body are anisotropic. Unfortunately, few results are known about having the UCP under physically acceptable assumptions. For a transversally isotropic elastic system with $C^\infty(\overline{\Omega})$. the UCP is available under the following condition:
Let the x_3-axis is the symmetric axis without loss of generality. Then, the condition is given as

$$^\exists\, \delta > 0 : A,\, C,\, L,\, (A \pm N)/2,\, (A+N)C - 2K^2 \geq \delta \text{ on } \overline{\Omega},$$

where

$$\begin{aligned} &A = C_{1111} = C_{2222},\; N = C_{1122},\; K = C_{1133} = C_{2233}, \\ &C = C_{3333},\; L = C_{1313} = C_{2323},\; (A-N)/2 = C_{1212} \end{aligned}$$

and all other C_{ijkl} are zero([13]). Here a tensor is transversally isotropic if it has only one axis of symmetry at each point and is invariant with respect to rotation about that axis of symmetry. So, the symmetric axis depends on position, and it is not desirable to assume an existence of a globally fixed axis. This difficulty of proving the UCP for anisotropic elastic systems as well as for anisotropic viscoelastic systems led to consider the case that the coefficients are piecewise analytic. In this case we can have the Holmgren-John UCP not only for elastic systems ([9]) but also for viscoelastic systems ([8]) in a more geometrically and physically precise form as follows. By using the smallest positive characteristic root of the characteristic equation

$$\det\bigl(\rho(x)\tau^2 - \sum_{j,l=1}^{3} C_{ijkl}(x)\xi_j\xi_l\bigr) = 0 \text{ with respect to } \tau$$

for $\xi \in T_x^*D_\alpha \setminus 0$, we can define a family of norms $N_x \subset T_xD_\alpha \equiv \mathbb{R}^3$ continuous with respect to x. Hence, we can define a distance $d(x,y) := \inf_\gamma \int_0^1 N_{\gamma(t)}(\frac{d\gamma(t)}{dt})\, dt$ between x and y in D_α, where $\gamma \in C^2([0,1]; D_\alpha)$ is a non-selfintersecting regular curve such that $\gamma(0) = x$, $\gamma(1) = y$. The physical meaning of $d(x,y)$ is the travel time of the slowest wave between x and y. Then, an improved version of the Holmgren-John UCP is as follows.

Theorem 3.1 *Let $u \in H^2(D_\alpha \times (0,T))$ be a solution of $\rho\partial_t^2 u - div\,\sigma = 0$ in $D_\alpha \times (0,T)$ with zero Cauchy data on $\Sigma_\alpha \times (0,T)$, where Σ_α is a C^2-regular open connected subset of the boundary ∂D_α of D_α. Then, we have $u(x, T/2) = 0$ for any $x \in D_\alpha$ such that $d(x,y) < T/2$ for some $y \in \Sigma_\alpha$.*

This UCP combined with the so-called the determination at the boundary can be used to propagate the ND-map along a tubular set of a curve (see the region between the two red curves in Figure 4). Here, for example, the determination at the boundary for D_{α_1} is to identify ρ and the tensors in D_{α_1} from the ND-map on $\Sigma \subset \partial D_{\alpha_1}$ over $(0,T)$. Namely, the set of Cauchy data considered on $\Sigma \times (0,T)$ of solutions of the initial boundary value problem for (2.5) with homogeneous initial condition at $t = 0$ whose Neumann data (=tractions) are supported in $\Sigma \times (0,T)$.

Then, we have the following result which has been already proved for elastic systems ([4]) and should be proved in a similar way for viscoelastic systems.

Theorem 3.2 *Assume (ρ, C) (resp. (ρ, C, G)) is piecewise homogeneous. Also, assume that each regular part of $\partial D_\alpha \setminus \Gamma$ and Σ are curved (i.e. strong curvature condition). Then, (ρ, C) (resp. (ρ, C, G)) is uniquely identified by the ND-map defined on $\Sigma \times (0,T)$. If C (resp. C, G) is transversally isotropic (resp. and also having the same symmetric axis for C and G), we can drop the strong curvature condition. Further, Ω can be replaced by a region of interest R and we can estimate the time for identifying (ρ, C) (resp. (ρ, C, G)) in R.*

Remark 3.3 *We give several remarks to Theorem 3.2 and its proof.*

(i) *Note that we did not assume interfaces $\partial D_\alpha \setminus \Gamma\,(\alpha \in A)$ are known. So, for instance considering anisotropic elastic systems, let two covers $\{D_\alpha\}_{\alpha\in A}$, $\{E_\beta\}_{\beta\in B}$ of Ω and the associated piecewise homogeneous (ρ, C) and (ζ, E) give the same ND-map. Suppose $\{D_\alpha\}_{\alpha\in A}$ and $\{E_\beta\}_{\beta\in B}$ are generated by black thin curves and blue thin curves, respectively (see Figure 4). Then, we need to show that we can have a region such as a region between the two red curves in Figure 4 such that interfaces of this region decomposed by the above thin curves are analytic submanifolds. This can be shown by using the theory of subanalytic sets ([3]).*

(ii) *The determination at the boundary used in the proof of Theorem 3.2 can be done using the theory of pseudo-differential operators. For instance, the determination at Σ for elastic systems is to basically compute the principal symbol $p_h(x, \xi)$, $x \in \Sigma$, $\xi \in T_x^*\Sigma$ of the finite time Laplace transform of the ND-map, where $h > 0$ is the reciprocal of the Laplace variable. The strong curvature condition is necessary to recover ρ, C in D_{α_1} from p_h.*

(iii) *Removing the strong curvature condition is based on the continuous dependency of the symmetric axis of any transversally isotropic tensor ([12]).*

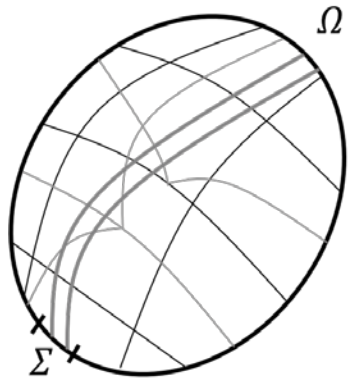


Figure 4

Before closing this section, if Ω is a grid decomposed domain, there is a result which is able to obtain by the method of the determination at the boundary and the Holmgren-John UCP. The result whicn we want to have is a local recovery of piecewise homogeneous (ρ, C) in Ω from the ND-map taken as our measured data by assuming that C is transversally isotropic. The outline of the proof is as follows. Even it can be obtained for viscoelasticity systems, for simplicity, consider a transversally isotropic elastic system with (ρ, C). By the Holmgren-John UCP, there exists $T^* > 0$ such that we can have the approximate controllability, that is the set $\{u^f(\cdot, T^*), -\partial_t u^f(\cdot, T^*) : f \in C_0^\infty(\Sigma\times(0, T^*))\}$ is dense in $H^1(\Omega)\times L^2(\Omega)$, where u^f is the solution for the initial boundary value problem for (2.5) with $F = 0$ with homogeneous initial condition, mixed type boundary condition consisting of zero displacement and inhomogeneous traction f. Then, for any $T > T^*$ the ND-map defined over $(0, T)$ can be extended to $(0, \infty)$. Taking the Laplace transformation with respect to time, our ICP can be transformed to the corresponding ICP for the elliptic system

$$\rho v - h^2 \mathrm{div}\, \sigma(v) = 0 \text{ in } \Omega$$

(see [5]). Then, we can have the Lipschitz stability estimate of identifying (ρ, C) from the ND-map (see [6] and [14]). This immediately implies the local recovery. That is the convergence of the Levenberg-Marquardt method and the Landweber method. Further, by using the result in [2], it is possible to replace the ND-map to finitely many measurements.

4 EBC for viscoelastic systems

This is an ongoing joint work with de Hoop, Lin without having any result confident enough to present here at this stage. First of all under the conditions given in Section 2 for the viscoelastic system (2.5) with $F = 0$ and homogeneous mixed type boundary condition consisting of homogneous displacement boundary condition and a modified version of homogeneous traction boundary condition, solutions of the initial boundary value problem for the system decay uniformly in polynomial order. Note that the modified part of this homogeneous traction boundary condition will be used as controls in Russell's principle. This decaying property of solutions can be proved by adapting the proof given in ([18]), which was not easy at all because ([18]) is very confusing. However, solutions of the dual problem grow which is a big problem for Russell's principle. We made various trials and errors to improve Russell's principle for viscoelastic systems, but we haven't completely succeeded yet.

After that, I came across a wonderful paper by Narukawa ([17]). His idea is to regard isotropic viscoelastic systems as a perturbation of isotropic elastic systems which have EBC using Russell's principle. Unfortunately, this paper was a preliminary report without any detail proof. Since I could not find a full version of this paper, I contacted him and heard that he did not publish any full version of the paper. However, he informed me that the detail of the proof is similar to that in his another paper on the EBC for thermoelastic systems ([16]). His idea is quite natural and really amazing.

In order to apply Narukawa's idea to show the exact controllability for anisotropic viscoelastic systems, the top priority is to show the EBC for anisotropic elastic systems by using Russell's principle under the most general setting. This itself is very interesting. We hope to have a success in our joint work.

References

[1] K. Ammari, M. Choulli and F. Triki, A unified approach to solving some inverse problems for evolution equations by using observability inequalities, CSIAM

Trans. Appl. Math., 1 (2020) pp. 207-239.

[2] G. Alberti and M. Santacesaria, Infinite-dimensional inverse problems with finite measurements, Arch. Rational Mech. Anal. 243 (2021) pp. 1-31.

[3] E. Bierstone and P. Milman, Semialgebraic and subanalytic sets, IHES Pub. Math. 67 (1988) pp. 5-42.

[4] C. Carstea, G. Nakamura and L. Oksanen, Uniqueness for the inverse boundary value problem of piecewise homogeneous anisotropic elasticity in the time domain, Trans. Amer. Math. Soc., 373 (2020) pp. 3423-3443.

[5] M. de Hoop, G. Nakamura and J. Zhai, Unique recovery of piecewise analytic density and stiffness tensor from the elastic-wave Dirichlet-to-Neumann map, SIAM J. Appl. Math., 79(2019) pp. 2359-2384.

[6] M. de Hoop, T. Furuya, C-L. Lin, G. Nakamura and M. Vashisth, Local recovery of a piecewise constant anisotropic conductivity in EIT on domains with exposed corners (preprint).

[7] G.Duvaut and J. Lions, Inequalities in mechanics and physics, Springer, Berlin (1976).

[8] M. Eller, N. Honda and G. Nakamura, Global unique continuation from the boundary for a system of viscoelasticity with analytic coefficients and a memory term (submitted).

[9] M.Eller and D. Toundykov, A global Holmgren theorems for multidimensional hyperbolic equations, Appl. Anal.,91 (2012) pp. 69-90.

[10] I.Gohberg, P.Lancaster and L. Rodman, Matrix polynomial, Academic Press, Inc., New York (1982).

[11] B. Kapitonov, B. Miara and G. Menzala, Stabilization of a layered piezoelectric 3-body by boundary dissipation, ESIAM. COCV, 12 (2006) pp. 198-215.

[12] L. Klimes, Determination of the reference symmetry axis of a generally anisotropic medium which is approximately transversally isotropic, Stud. Geophy. Geodl, 60 (2016) pp. 391-402.

[13] C-L. Lin and G. Nakamura, Conditional stability for the hexagonal anisotropic elastic dynamical system, Comm. PDE, 34 (2009) pp.1251-1264.

[14] A. Lechleiter and A. Rieder, Newton regularizations for impedance tomography: convergence by local injectivity, Inverse Problems, 24 (2008) 18 pp.

[15] G. Nakamura and M. Oliva, Exponential decay of solutions to initial boundary value problem for anisotropic visco-elastice systems, Rend. Istit. Mat. Univ. Trieste, 48 (2016) pp. 5-19.

[16] K. Narukawa, Boundary value control of thermoelastic systems, Hiroshima Math. J.,13 (1983) pp. 227-272.

[17] K. Narukawa, Exact and admissible controllability of viscoelastic systems with boundary controls, IFAC 9th Tiennial World Congress, Budapest, Hungary, 1984.

[18] J. Rivera and E. Lapa, Decay rates of solutions of an anisotropic inhomogeneous n-dimensional equation with polynomially decay Kernels, Comm. Math. Phys., 177 (1996) pp, 583-602.

[19] D. Russell, Controllability and stability theory for linear partial differential equations: Recent progress and open questions, SIAM Review, 20 (1978) pp.639-739.

[20] K. Tanuma, Stroh formalism and Rayleigh waves, J. Elasticity, 89 (2007) 159 pp.

[21] J. Wloka, Partial differential equations, Cambridge Univ.'Press, Cambridge (1987).

Type II blow-up and asymptotic profile of Keller-Segel model

Maolin Zhou*
Chern Institute of Mathematics, Nankai University,
No. 94, Weijin Road, Tianjin 300071, P. R. China.

Abstract

In this paper, we studied the blow-up rate and blow-up porfile of radially decreasing solutions of the following parabolic-elliptic Keller-Segel-Patlak system in space dimensions $N \geq 3$:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \mathbb{R}^N, \ t \in (0,T), \\ 0 = \Delta v + u, & x \in \mathbb{R}^N, \ t \in (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{0.1}$$

In [25], P. Souplet and M. Winkler show thatthe final profile satisfies $c|x|^{-2} \leq U(x) \leq C|x|^{-2}$ where $U(x) := \lim_{t \to T^-} u(x,t)$ with convergence in L^1 under suitalbe assumptions on the initial data. They leave an open question that whether the limit $\lim_{x \to 0} |x|^2 U(x)$ exists under these conditions. The first goal of this paper is to solve this open problem in the whole space case under suitable conditions. The main idea of our proof is to construct a family of backward self-simlar solutions and then apply zero number theory to the problem.

When $n = 2$, based on the method of the zero number theory, we give a new proof of the fact which has been established in [16]: each blowup is type II in radial case.

Key words: Keller-Segel-Patlak system, Chemotaxis, Blowup profile, Zero number theory
MSC 2010: 92C17, 35B40, 35B44, 35K40

*Corresponding author's E-mail: 019153@nankai.edu.cn

1 Introduction

In this paper, we consider the blowup profile of the following Keller-Segel-Patlak system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \mathbb{R}^N,\ t \in (0,T), \\ 0 = \Delta v + u, & x \in \mathbb{R}^N,\ t \in (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $T > 0$ is the maximal existence time.
For the initial data, similar to [25] we assume the following condition:

$$0 \le u_0 \in L^\infty(\Omega),\ u_0 \text{ is radially symmetric and nonincreasing with respect to} |x|. \tag{1.2}$$

This system can be considered as a simplified model of the classical Keller-Segel model that established by Keller and Segel in 1970s ([15]). The Keller-Segel type chemotaxis model is very famous and has been well studied during the last four decades (see the surveys [1, 12, 13]). It is well-known that the all the solution of Keller-Segel model are global bounded in dimension one and there exist blowup solutions for high dimensions. Since there are a lot of research relate to all kinds of Keller-Segel type chemotaxis models, it is hard to list all of them in details. We will only recall these results related to the simplified model (1.1).
When $N = 2$, it is well known that there exists a critical total mass 8π: if $\|u_0\|_{L^1} \le 8\pi$ all the solutions are global in time (see, e.g.,[2, 3, 4, 5, 24]), and if $\|u_0\|_{L^1} > 8\pi$, then all classical solutions with finite second moment blow up in finite time [14, 19]. For the blowup rate, it was shown in [11, 17] that there exist radial type II blowup solutions (recall that the blowup is called of type I if $\limsup_{t\to T}(T-t)\|u(t)\|_{L^\infty} < \infty$ and of type II otherwise). The blowup profile was considered in [24] (see also [11, 22]), it is proved that the blowup solution will concentrate at least 8π mass into a single point. Some further studies of the structure of blowup solutions could be found in [21, 7].
For the case $N \ge 3$, there exists blowup solutions (see e.g. [10, 23, 25]). The blowup rate maybe depends on N: When $3 \le N \le 9$, there exists both type I blowup solutions (see e.g. N. Mizoguchi, T. Senba [18]) and type II blowup solutions (this case was formally established in Brenner et. al. [6] and M.A. Herrero, E. Medina, J.J.L. Velázquez [9]); If $N \ge 11$, there exist radial type II blowup solutions [11, 17]. In order to understand the blowup profile of blowup solutions, a family of backward self-similar blowup solutions with profile $u(x,T) \sim \frac{C}{|x|^2},\ |x| \to 0$ and some other properties were constructed by M.A. Herrero, E. Medina, J.J.L. Velázquez [9, 23] (also see T. Senba [23] for a further vestigation). For more general solutions (not only restrict to the self-similar case), Y. Giga, N. Mizoguchi and T. Senba[8] obtained a blowup profile for type I blowup solutions of (1.1) in self-similar sense

$$\lim_{t\to T}(T-t)u(y\sqrt{T-t}) = V(y),\ \text{uniformly for } y \text{ bounded.}$$

where V is the profile of some backward self-similar solutions. This result implies that the blowup solutions have similar properties of the backward self-similar solutions. But, since this result is relate to the "microscopic" scale y, the shape of the solution when t is close to the blowup time is still unclear. To understand the shape of blowup solution, P. Souplet and M. Winkler [25] shown that the blowup profile is also similar to the self-similar solution, that is, like $C|x|^{-2}$. More specifically, if $T < \infty$ the blowup set $B(u_0)$ of (u,v) is defined by

$$B(u_0) := \{x_0 \in \bar{\Omega}; u(x_j,t_j) + |\nabla v(x_j,t_j)| \to \infty \text{for some subsequence } (x_j,t_j) \to (x_0,T)\},$$

they obtain the following result (in their paper, this result is established not only for the whole space but also for the bounded domain).

Proposition 1.1 *Let $N \geq 3$. Assume that u_0 satisfies (1.2), $T < \infty$ and $B(u_0) \neq \mathbb{R}^N$.*

(i) There exists $C > 0$ such that

$$u(x,t) \leq \frac{C}{|x|^2}, \ 0 < |x| \leq 1, \ 0 < t < T. \tag{1.3}$$

Moreover, we have $B(u_0) = \{0\}$, the final blowup profile $U(x) := \lim_{t \to T} u(x,t)$ exists for all $x \in \mathbb{R}^N - \{0\}$, where convergence also take place in $L^1(B_1)$, and U satisfies

$$U(x) \leq \frac{C}{|x|^2}, \ 0 < |x| \leq 1. \tag{1.4}$$

(ii) If we assume in addition that $u_0 \in C^1(\mathbb{R})$, $r^{n-1}u_0'(r) + u_0(r)\int_0^r u_0(s)s^{n-1}ds \geq 0$. Then there exist $c, \eta > 0$, such that

$$U(x) \geq c|x|^{-2}, \ x \in B_\eta - \{0\}$$

Remark 1.2 *It can be see from their proof process, the assumption "u_0 is nonincreasing in $r = |x|$" can be weaken into "$w_0 = r^{-N}\int_0^r s^{N-1}u_0(s)ds$ is nonincreasing in r" in (i).*

When $U(x)$ is the profile of those backward self-similar solutions, the estimate $c \leq |x|^2 U(x) \leq C$ can be improved to a uniformly form, that is, the limit $\lim_{x \to 0} |x|^2 U(x)$ exists. Inspired by this, P. Souplet and M. Winkler guess this uniform blow-up profile is valid for more general solutions. The main target of this paper is to give a positive answer to this question.
Before state our main results, we introduce some notations, some of them are the same as those in [25].
Let $w(r,t) = r^{-n}\int_0^r s^{n-1}u(s,t)\,\mathrm{ds}$, it is routine to check that w satisfies

$$w_t = w_{rr} + \frac{N+1}{r}w_r + (nw + rw_r)w. \tag{1.5}$$

and

$$w(r,0) = r^{-N}\int_0^r s^{N-1}u_0(s)\,\mathrm{ds} := w_0(r). \tag{1.6}$$

Let $W(r) = \lim_{t \to T} w(r,t)$ whenever it exists and assume that w_0 satisfies

$$\text{The following limit exists} : \lim_{r \to \infty} w_0(r)r^2 \in [0, \infty]. \tag{1.7}$$

Our first result can be stated as follows:

Theorem 1.3 *Let $n \geq 3$. Suppose that u_0 satisfies (1.2) and (1.7), w is single point blow-up solution of (1.5) with $T < \infty$. Then the following limit exists*

$$\lim_{r \to 0} W(r)r^2 := \alpha \in [0, \infty].$$

If we assume in addition that $w_0(r)$ is nonincreasing in r, we can draw further conclusions that

$$\lim_{r\to 0} W(r)r^2 := \alpha \in [0,\infty)$$

and

$$\lim_{r\to 0} U(r)r^2 = (N-2)\alpha \in [0,\infty).$$

The next result has been established in [16], we will prove it in a different way.

Theorem 1.4 *Let $n=2$. Assume (u,v) be a radial solution of 1.1 blowing up in finite time. Then the blowup of (u,v) is of type II.*

References

[1] N. Bellomo, A. Bellouquid, Y. Tao, M. Winkler: *Toward a Mathematical Theory of Keller-Segel Models of Pattern Formation in Biological Tissues.* Math. Mod. Meth. Appl. Sci. **25**, 1663-1763 (2015)

[2] A. Blanchet, V. Calvez, J. A. Carrillo: *Convergence of the mass-transport steepest descent scheme for the subcritical Patlak-Keller-Segel model,* SIAM J. Numer. Anal. **2**, 691-721 (2008)

[3] A. Blanchet, V. Calvez, J. A. Carrillo: *Functional inequalities, thick tails and asymptotics for the critical mass Patlak-Keller-Segel model,* J. Func. Anal. **103**, 146-178 (1993)

[4] A. Blanchet, J. A. Carrillo, N. Masmoudi: *Infinite time aggregation for the critical Patlak-Keller-Segel model in $\mathbb{R}^2$,* Comm. Pure Appl. Math. **61**, 1449-1481 (2008)

[5] A. Blanchet, J. Dolbeault, B. Perthame: *Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solution,* Electron. J. Differential Equations **44**, 32pp (2006)

[6] M.P. Brenner, P. Constantin, L.P. Kadanoff, A. Schenkel, S.C. Venkataramani: *Diffusion, attraction and collapse,* Nonlinearity **12**, 1071–1098 (1999)

[7] T. Ghoul, N. Masmoudi: *Infinite time aggregation for the critical Patlak-Keller-Segel model in $\mathbb{R}^2$,* Comm. Pure Appl. Math. **61**, 1449-1481 (2018)

[8] Y. Giga, N. Mizoguchi, T. Senba: *Asymptotic behavior of type I blowup solutions to a parabolic-elliptic system of drift-diffusion type,* Arch. Ration. Mech. Anal. **201**, 549-573 (2011)

[9] M.A. Herrero, E. Medina, J.J.L. Velázquez: *Finite-time aggregation into a single point in a reaction–diffusion system,* Nonlinearity **10**, 1739–1754 (1997)

[10] M.A. Herrero, E. Medina, J.J.L. Velázquez: *Self-Similar blowup for a reaction diffusion system,* Journal of Computational Applied Mathematics **97**, 99–119 (1998)

[11] M.A. Herrero, J.J.L. Velázquez: *Singularity patterns in a chemotaxis model,* Math. Ann. **306**, 583-623 (1996)

[12] T. Hillen, K. J. Painter: *A users guide to PDE models for chemotaxis.* J. Math. Biol. **58**, 183-217 (2009)

[13] D. Horstmann: *From 1970 until present: The Keller–Segel model in chemotaxis and its consequences: Jahresber I.* Jahresberichte Deutsch. Math.-Verein **105**, 103-165 (2003)

[14] W. Jäger, S. Luckhaus: *On explosions of solutions to a system of partial differential equations modelling chemotaxis,* Trans. Amer. Math. Soc. **329**, 819-824 (1992)

[15] E.F. Keller, L.A. Segel: *Initiation of slime mold aggregation viewed as an instability.* J. Theoret. Biol. **26**, 399-415 (1970)

[16] N. Mizoguchi: *Refined Asymptotic Behavior of Blowup Solutions to a Simplified Chemotaxis System,* Comm. Pure Appl. Math. **75**, 1870-1886 (2022)

[17] N. Mizoguchi, T. Senba: *Type II blowup solutions to a parabolic–elliptic system,* Adv. Math. Sci. Appl. **17**, 505-545 (2007)

[18] N. Mizoguchi, T. Senba: *A sufficient condition for type I blowup in a parabolic–elliptic system,* J. Differential Equations **250**, 182-203 (2011)

[19] T. Nagai: *Blow-up of radially symmetric solutions to a chemotaxis system,* Adv. Math. Sci. Appl. **191**, 17-51 (2002)

[20] P. Quittner, P. Souplet: *Superlinear Parabolic Problems–Blow-up, Global Existence and Steady States,* Birkhäuser Advanced Texts: Basel Boston Berlin, 2007.

[21] P. Raphaël, R. Schweyer: *On the stability of critical chemotactic aggregation,* Math. Ann. **359**, 267–377 (2014)

[22] T. Senba: *Type II blowup of solutions to a simplified Keller-Segel system in two dimensional domains,* Nonlinear Anal. **66**, 1817-1839 (2007)

[23] T. Senba: *Blowup behavior of radial solutions to Jäger-Luckhaus system in high dimensional domain,* Funkcial. Ekvac. **48**, 247-271 (2005)

[24] T. Senba, T. Suzuki: *Weak solutions to a parabolic-elliptic system of chemotaxis,* J. Funct. Anal. **191**, 17-51 (2002)

[25] P. Souplet, M. Winkler:*Blow-up profiles for the parabolic-elliptic Keller-Segel system in dimensions $n \geq 3$,* Comm. Math. Phys. **367**, 665-681 (2019)

A geometric singular perturbation analysis of regularised reaction-nonlinear diffusion models including shocks

Bronwyn Bradshaw-Hajek[1], Ian Lizarraga[2], Robert Marangell[2], and Martin Wechselberger*[2]

[1]UniSA STEM, University of South Australia, Mawson Lakes, South Australia, Australia
[2]School of Mathematics and Statistics, University of Sydney, NSW, Australia

1 Motivation

Wave fronts are ubiquitous in nature. In the context of population dynamics, such waves may be viewed as representing patterns or structure in migrating populations. Reaction-diffusion equations, such as the extensively studied Fisher equation [5], are used to model population growth dynamics combined with a simple Fickian diffusion process, and are typically capable of exhibiting travelling wave solutions.
In cell migration, advection (or transport) is another important model mechanism. It may represent, e.g., tactically-driven movement, where cells migrate in a directed manner in response to a concentration gradient. Such a concentration gradient develops, for example, in a soluble fluid (chemotaxis) or as a gradient of cellular adhesion sites or of substrate-bound chemoattractants (haptotaxis). Well studied examples of individual cells exhibiting directed motion in response to a chemical gradient include bacteria chemotactically migrating towards a food source. Wound healing, angiogenesis or malignant tumor invasion are just a few examples of chemotactic and/or haptotactic cell movement where the migrating cells form part of a dense population of cells as may be found in tissues. Such migrating cell populations not only form travelling waves but may also develop sharp interfaces in the wave form.
From a classical PDE point of view, these advection-reaction models may represent hyperbolic balance laws (hyperbolic conservation laws with source terms), and the formation of shock fronts is well known. In general, shocks are problematic because as the wave front steepens (and a shock forms) the solution becomes multivalued and physically nonsensical. The model breaks down and it becomes impossible to compute the temporal evolution of the solution [9].
To account for shocks, modellers have employed the technique of *regularisation* – adding small higher order terms to these models to smooth out the shocks. In the context of hyperbolic conservation/balance laws, these are usually small viscous (diffusive) regularisations, e.g., the viscous Burgers equation. Due to dissipative mechanisms, these physical shocks are observed as narrow transition regions with steep gradients of field variables. Mathematically, questions of existence

*corresponding author & presenter (martin.wechselberger@sydney.edu.au)

and uniqueness of such viscous shock profiles are fundamental.[1]

Another source for the formation of sharp interfaces can be found in density-dependent nonlinear diffusion processes. Through sensing the local cell density, cells make informed decisions, i.e., they perform a 'biased walk'. This could lead to, e.g., the tendency to cluster or aggregate with other nearby cells; think of flocking or swarming which might be perceived as an advantageous situation for the cell population. Such aggregation mechanism can be achieved through, e.g., negative (or backward) diffusion. Such reaction-nonlinear diffusion (RND) models may form shocks. Again, modellers have employed the technique of *regularisation* – adding small higher order terms to these models to 'smooth' the shocks, but these are not so well-known, at least in the bioscientific community. Possible shock formation in such regularised RND models is the main focus of this presentation, and we will use the tools from geometric singular perturbation theory and dynamical systems theory to tackle this problem.

2 The setup for RND Models

We start by considering a dimensionless reaction–nonlinear diffusion model of the form

$$u_t = (D(u)u_x)_x + f(u) = \Phi(u)_{xx} + f(u) \tag{1}$$

where $x \in \mathbb{R}$ denotes the spatial domain, $t \in \mathbb{R}_+$ denotes the time domain, $u \in \mathbb{R}_+$ denotes a (population/agent) density, $D(u)$ models a (population/agent) density dependent diffusivity. $\Phi(u)$ is an anti-derivative of $D(u)$, i.e. $\Phi'(u) = D(u)$, referred to as the *potential.* The (dimensionless) population/agent density u is scaled such that $u \in [0,1]$ forms the domain of interest where $u = 1$ is the carrying capacity of the population/agent density. This domain of interest is also reflected in the reaction term $f(u)$ which is modelled either as *logistic growth*, $f(u) = f_l(u) = u(1-u)$, or as *bistable growth*, $f(u) = f_b(u) = u(u-\alpha)(1-u)\,, 0 < \alpha < 1$.
We focus on RND models where not only diffusion is present but also *aggregation* (or *backward diffusion*) [2, 6, 4, 10, 11]. The heuristic motivation for this modelling assumption is based on the observation that population tend to cluster for, e.g., safety (to avoid easy predation). By imposing different motility rates for agents that are isolated, compared to other agents, one obtains density dependent nonlinear diffusion [7]. Aggregation will manifest itself in these models in sign changes of the density dependent diffusion coefficient $D(u)$. The simplest density-dependent nonlinear diffusion coefficient model that we consider is of the polynomial form

$$D(u) = \beta(u-\gamma_1)(u-\gamma_2) \tag{2}$$

with $0 < \gamma_1 < \gamma_2 < 1$, i.e., diffusion-aggregation-diffusion (DAD) in the domain of interest. For sparse population density diffusive behaviour is assumed, while for intermediate population density aggregation will happen which again turns into diffusive behaviour for large population densities (close to carrying capacity).

[1] Another option is dispersive regularisation, e.g., the KdV equation. Note that both regularisations (viscous and dispersive) deal with the same equation (inviscid Burgers) and create very different outcomes.

2.1 Shock fronts in the RND model (1)

It is well-known that in the case of a nonlinear advection-reaction model which defines a hyperbolic balance law (under certain assumptions) shock formation is a well-known phenomenon. Similarly, negative diffusitivity $D(u)$ can also cause shock formation in an RND model [4, 12, 13].

Let us look for one of the simplest coherent structures in such RND models (1), travelling waves with wave speed $c \in \mathbb{R}$ that connect the asymptotic end states $u_- = 1 \to u_+ = 0$, i.e., population/agents invade the unoccupied domain with constant speed. A travelling wave analysis introduces a co-moving frame $z = x - ct$ in (1), $c \in \mathbb{R}$. Stationary solutions, i.e., $u_t = 0$, in this co-moving frame include travelling waves/fronts, and they are found as special (heteroclinic) solutions of the corresponding ODE problem $-cu_z - (D(u)u_z)_z = f(u)$. Define $v := -cu - D(u)u_z$ to obtain the corresponding 2D dynamical system

$$\begin{aligned} u_z &= -\frac{(v + cu)}{D(u)} \\ v_z &= f(u)\,. \end{aligned}$$

We are interested in a travelling front/wave in this system corresponding to a heteroclinic connection from one steady state ($u = 1$) to the other ($u = 0$) or vice versa. In our setup of the RND model, such a solution (if it exists) cannot be smooth, because the zeros of the diffusion coefficient $D(u)$ which exist in the relevant domain of interest $u \in [0, 1]$ define singularities in this problem. To avoid these singularities, discontinuous jumps (shocks) would be necessary to define 'weak' solutions of the original PDE problem (1).

2.2 Regularisations of RND models

To account for these shocks, we employ the technique of regularisation to this RND problem. Regularisation of RND models is typically considered in one of two ways [11, 12]. The first method of regularisation accounts for *viscous relaxation* by adding a small temporal change in the diffusivity:

$$u_t = (\Phi(u) + \varepsilon u_t)_{xx} + f(u), \qquad 0 \leq \varepsilon \ll 1. \tag{3}$$

The second of these involves adding a small change in the potential to account for *nonlocal effects*, leading to:

$$u_t = (\Phi(u) - \varepsilon^2 u_{xx})_{xx} + f(u), \qquad 0 \leq \varepsilon \ll 1. \tag{4}$$

These regularisation techniques have been widely employed in models of chemical phase-separation, though they have gone relatively unnoticed in biological models until very recently.

In this presentation, we study the possible effects of *both* regularisations in a single RND model, i.e.,

$$u_t = (\Phi(u) + \varepsilon a u_t - \varepsilon^2 u_{xx})_{xx} + f(u), \qquad 0 \leq \varepsilon \ll 1, a \geq 0. \tag{5}$$

Since we only consider small perturbative regularisations $0 < \varepsilon \ll 1$, these models are so-called *singularly perturbed systems* and, as a consequence, the powerful machinery of geometric singular perturbation theory (GSPT) is applicable [3, 8, 16], as we shall explain in this presentation.

Remark 2.1 *This regularised RND model* (5) *can be derived fom the history dependent energy functional*

$$E(u) = \int_\Omega \left(F(u) + \varepsilon a \int_0^t u_s^2 ds + \frac{\varepsilon^2}{2}|u_x|^2 \right) dx\,,$$

where $F(u) = \int \Phi(u)du$ is the free energy density function of the homogeneous state. The interfacial energy, $\frac{\varepsilon^2}{2}|u_x|^2$, introduces smoothing effects in regions with large gradients, and so does the memory term, $\varepsilon a \int_0^t u_s^2 ds$, which can be interpreted as visco-elastic potential energy; see, e.g., [18].

Remark 2.2 *Continuum macroscale models can also be derived from lattice-based microscale models; see [7] for leading order RND models and [1] for regularised RND models (albeit more complicated).*

3 Travelling wave analysis of the regularised RND model (5)

We derive conditions based on the specific functions $D(u)$ and $f(u)$ that lead to travelling waves with sharp interfaces (shocks) in one spatial dimension. We introduce a travelling wave coordinate $z = x - ct$ for waves with speed $c \in \mathbb{R}$. This transforms the regularised RND model (5) into a fourth order ordinary differential equation

$$-cu_z = \Phi(u)_{zz} - \varepsilon a c u_{zzz} - \varepsilon^2 u_{zzzz} + f(u)\,, \tag{6}$$

which we can recast as a *singularly perturbed dynamical system* in standard form

$$\begin{aligned} \varepsilon u_z &= \hat{u} \\ \varepsilon \hat{u}_z &= w + \Phi(u) - \delta \hat{u} \\ v_z &= f(u) \\ w_z &= v + cu\,. \end{aligned} \tag{7}$$

where $(u, \hat{u}) \in \mathbb{R}^2$ are 'fast' variables, $(v, w) \in \mathbb{R}^2$ are 'slow' variables, and $\varepsilon \ll 1$ is the singular perturbation parameter.
Rescaling the 'slow' independent travelling wave variable $dz = \varepsilon dy$ in (7) gives the equivalent fast system

$$\begin{aligned} u_y &= \hat{u} \\ \hat{u}_y &= w + \Phi(u) - \delta \hat{u} \\ v_y &= \varepsilon f(u) \\ w_y &= \varepsilon(v + cu)\,. \end{aligned} \tag{8}$$

with the 'fast' independent travelling wave variable y. These equivalent dynamical systems (7) respectively (8) have a symmetry

$$(\hat{u}, v, c, y) \leftrightarrow (-\hat{u}, -v, -c, -y), \quad \text{resp.} \quad (\hat{u}, v, c, z) \leftrightarrow (-\hat{u}, -v, -c, -z)\,.$$

The aim is to use methods from GSPT to analyse the travelling wave problem in its 'slow' and 'fast' singular limit, and to obtain results on the existence (and stability) of travelling waves in the full regularised RND problem.

3.1 The limit on the 'fast' scale - the layer problem

We begin with the 'fast' system (8). Here the limit $\varepsilon \to 0$ gives the *layer problem*

$$
\begin{aligned}
u_y &= \hat{u} \\
\hat{u}_y &= w + \Phi(u) - \delta\hat{u} \\
v_y = w_y &= 0\,,
\end{aligned}
\tag{9}
$$

i.e., (v, w) are considered parameters. Hence, the flow is along two-dimensional fast fibers $\mathcal{L} := \{(u, \hat{u}, v, w) \in \mathbb{R}^4 : (v, w) = const\}$. The set of equilibria of the layer problem,

$$
S := \{(u, \hat{u}, v, w) \in \mathbb{R}^4 : \hat{u} = \hat{u}(u, v) = 0,\; w = w(u, v) = -\Phi(u)\}\,, \tag{10}
$$

forms the two-dimensional *critical manifold* of the problem which is a graph over (u, v)-space. The stability property of this set of equilibria S is determined by the two non-trivial eigenvalues of the layer problem, i.e., the eigenvalues of the Jacobian evaluated along S,

$$
J = \begin{pmatrix} 0 & 1 \\ D(u) & -\delta \end{pmatrix}\,. \tag{11}
$$

This matrix has $\mathrm{tr}J = -\delta$ and $\det J = -D(u)$. Hence, for $D(u) > 0$ equilibria are of saddle-type while for $D(u) > 0$ equilibria are of focus/node/centre-type. Loss of normal hyperbolicity happens along the one-dimensional set(s)

$$
F := \{(u, \hat{u}, v, w) \in S : D(u) = 0\}\,. \tag{12}
$$

In the assumed diffusion-aggregation-diffusion (DAD) setup of (2), we have $F = F_l \cup F_r = \{(u, \hat{u}, v, w) \in S : u = \gamma_1\} \cup \{(u, \hat{u}, v, w) \in S : u = \gamma_2\}$, i.e., we have a splitting of the critical manifold $S = S_s^l \cup F_l \cup S_m \cup F_r \cup S_s^r$ where

$$
\begin{aligned}
S_s^l &:= \{(u, \hat{u}, v, w) \in \mathbb{R}^4 : \hat{u} = \hat{u}(u, v) = 0,\; w = w(u, v) = -\Phi(u),\; u < \gamma_1\} \\
S_s^r &:= \{(u, \hat{u}, v, w) \in \mathbb{R}^4 : \hat{u} = \hat{u}(u, v) = 0,\; w = w(u, v) = -\Phi(u),\; u > \gamma_2\}
\end{aligned}
$$

denote saddle-type outer branches and, for $\delta \neq 0$,

$$
S_m := \{(u, \hat{u}, v, w) \in \mathbb{R}^4 : \hat{u} = \hat{u}(u, v) = 0,\; w = w(u, v) = -\Phi(u),\; \gamma_1 < u < \gamma_2\}
$$

denotes the (node/focus-type) middle branch. For $\delta = 0$ this middle branch S_m is of centre-type (where normal hyperbolicity is lost as well).

3.1.1 The $\delta = 0$ case

In this case, the layer problem (9),

$$
\begin{aligned}
u_y &= \hat{u} \\
\hat{u}_y &= w + \Phi(u)
\end{aligned}
\tag{13}
$$

is Hamiltonian with

$$
H(u, \hat{u}) = \frac{\hat{u}^2}{2} - \int (w + \Phi(u))du\,. \tag{14}
$$

Trajectories of this layer problem are confined to level sets of the Hamiltonian (14), i.e., $H(u, \hat{u}) = k$. Possible trajectories that are able to connect equilibrium points on different branches of the critical manifold S are confined to the saddle branches $S_s^{l/r}$ including the boundaries $F_{l/r}$. The corresponding equilibrium points $p_{l/r} = (u_{l/r}, 0, v_{l/r}, -\Phi(u_{l/r})) \in S_s^{l/r} \cup F_{l/r}$ of such connections must fulfill $v_l = v_r$ and $\Phi(u_l) = \Phi(u_r)$ since v and w are constant.

Remark 3.1 *This creates a bound on possible w-values, $w \in [-\Phi(u_{f-}), -\Phi(u_{f+})]$ where $D(u_{f\mp}) = 0$, i.e., confined to region between the local extrema of Φ.*

Without loss of generality, set $H(u_l, \hat{u} = 0) = 0$, i.e., $H(u, \hat{u}) = \frac{\hat{u}^2}{2} - \int_{u_l}^{u}(w + \Phi(u))du$. Then $H(u_r, \hat{u} = 0)$ must be equal zero as well for the existence of a layer connection between these two points. This constraint leads to the well-known *'equal area rule'* (see, e.g. [12]),

$$\int_{u_l}^{u_r} (w_h + \Phi(u))du = 0 \,. \tag{15}$$

This rule allows for $S_s^{l/r}$ to $S_s^{r/l}$ connections, but not to the boundaries $F_{l/r}$ or the centre-type middle branch S_m. Due to the symmetry ($\hat{u} \leftrightarrow -\hat{u}$), there exists automatically a pair of such heteroclinic connections for fixed $w = w_h$, i.e., $\Gamma_+(w_h, 0) : p_l \to p_r$ and $\Gamma_-(w_h, 0) : p_r \to p_l$.

Remark 3.2 *The equal area rule* (15) *determines the value $w = w_h$ for which this integral vanishes. For $a = 0$, it is independent of the possible wave speed $c \in \mathbb{R}$.*

3.1.2 The small $|\delta|$ case

For sufficiently small $|\delta| > 0$, we show that nearby heteroclinic connections to the same asymptotic end states still exist. This is done via a *Melnikov-type* argument; see, e.g., [15, 17]:

Define $x = (u, \hat{u})^\top$ and $f(x; w, \delta) = (\hat{u}, w + \Phi(u) - \delta\hat{u})^\top$ such that the layer problem is given in vector form by $x' = f(x; w, \delta)\,, x \in \mathbb{R}^2$. This system possesses heteroclinic orbits $\Gamma_\pm(y)$ for $w = w_h$ and $\delta = 0$, i.e., $\Gamma' = f(\Gamma_\pm; w_h, 0)$. Let $x = \Gamma_\pm + X$, $X \in \mathbb{R}^2$ which transforms the layer problem to the non-autonomous problem $X' = A(y)X + g(X, y; w, \delta)$ with the non-autonomous matrix $A(y) := D_x f(\Gamma_\pm; w_h, 0)$ and the nonlinear remainder $g(X, y; w, \delta) = f(\Gamma_\pm + X; w, \delta) - f(\Gamma_\pm; w_h, 0) - A(y)X\,.$ The linear equation $X' = A(y)X$ is the *variational equation* along $\Gamma_\pm$. The corresponding *adjoint equation* is given by $\Psi' + A^\top(y)\Psi = 0\,.$ Solutions of the variational and its adjoint equation preserve a constant angle along $\Gamma_\pm$, i.e., $D_y(\Psi^\top(y)X(y)) = 0, \forall y \in \mathbb{R}\,.$ We can use this fact to define a splitting of the vector space $\mathbb{R}^2$ along $\Gamma_\pm$. Without loss of generality, we define it at $y = 0$ in the following way: $\mathbb{R}^2 = \operatorname{span}\{f(\Gamma_\pm(0); w_h, 0)\} \oplus W$ where W is spanned by the solutions of the adjoint equation that decay exponentially for $y \to \pm\infty$; here, this space is one-dimensional and we denote the corresponding solution by $\psi(y) = (\psi_1(y), \psi_2(y))^\top$.

We measure the distance $\Delta \in \mathbb{R}$ between the one-dimensional stable and unstable manifolds emanating from the saddle-equilibria in a suitable cross section Σ. We denote these manifold segments by $X_\pm$. Based on our setup, we choose $\Sigma = W$. This distance function depends on the system parameters, i.e., $\Delta = \Delta(w, \delta)$. In the previous section, we established $\Delta(w_h, 0) = 0$. In general, one cannot solve $\Delta(w, \delta) = 0$ explicitly. Thus one aims to solve $\Delta(w, \delta) = 0$ near $(w, \delta) = (w_h, 0)$ approximately by means of the implicit function theorem: e.g., if $D_w\Delta(w_h, 0) \neq 0$ then $w = w_h(\delta) = w_h + b\delta + O(\delta^2)$ solves $\Delta(w_h(\delta), \delta) = 0$ for $\delta \in (-\delta_0, +\delta_0)$. The leading order expansion parameter b is then given by

$$b = -\frac{D_\delta\Delta(w_h, 0)}{D_w\Delta(w_h, 0)}\,.$$

These first-order expansion terms of the distance function Δ are known as first-order *Melnikov integrals.* They can be calculated as follows:

$$(D_w\Delta(w_h,0), D_\delta\Delta(w_h,0)) = (\int_{-\infty}^{\infty}(\psi(s)^\top D_w f(\Gamma_\pm(0);w_h,0))ds, \int_{-\infty}^{\infty}(\psi(s)^\top D_\delta f(\Gamma_\pm(0);w_h,0))ds)$$

We have $D_w f(\Gamma_\pm(0);w_h,0) = (0,1)^\top$ and, hence,

$$D_w\Delta(w_h,0) = \int_{-\infty}^{\infty}(\psi(s)^\top D_w f(\Gamma_\pm(0);w_h,0))ds = \int_{-\infty}^{\infty}\psi_2(s)ds \neq 0,$$

based on the geometric observation that the ψ_2-component does not change sign along $\Gamma_\pm$. The measure is well-defined since $\psi_2(y)$ is decaying exponentially for $y \to \pm\infty$. Hence, $w = w_h(\delta) = w_h + b\delta + O(\delta^2)$ solves $\Delta(w(\delta),\delta) = 0$ for $\delta \in (-\delta_0, +\delta_0)$. We also have $D_\delta f(\Gamma_\pm(0);w_h,0) = (0, -\hat{u}(y))^\top$ and, hence,

$$D_\delta\Delta(w_h,0) = \int_{-\infty}^{\infty}(\psi(s)^\top D_\delta f(\Gamma_\pm(0);w_h,0))ds = -\int_{-\infty}^{\infty}\hat{u}(s)\psi_2(s)ds \neq 0\,,$$

based on a similar geometric observation as above, i.e., both terms do not change sign under the variation along $\Gamma_\pm$. Hence,

$$b = \frac{D_\delta\Delta(w_h,0)}{D_w\Delta(w_h,0)} = -\frac{\int_{-\infty}^{\infty}\hat{u}(s)\psi_2(s)ds}{\int_{-\infty}^{\infty}\psi_2(s)ds} \neq 0$$

and we have a leading order affine solution $w(\delta)$ to $\Delta(w,\delta) = 0$ near $(w_h, 0)$.

Remark 3.3 *Only for $(w,\delta) = (w_h, 0)$ there exist two heteroclinics $\Gamma_\pm$ simultaneously. For fixed small $\delta \neq 0$, the two heteroclinics exist for distinct w-values. There is also the symmetry $\delta \leftrightarrow -\delta$. Thus one only needs to continue one heteroclinic in (w,δ)-space. The other is given through the symmetry.*

Remark 3.4 *The leading order linear growth found in the Melnikov analysis cannot continue indefinitely since the saddle equilibria $p_{l/r}$ are confined to w-values between the local extrema of Φ; see Remark 3.1. These extrema indicate saddle-node bifurcations of equilibria.*

3.1.3 The $\delta = O(1)$ case

What is the fate of the heteroclinic branches established in the previous section? Do they exist for large $|\delta|$ as well? Note, the heteroclinic orbits are confined to the upper (Γ_+) or lower (Γ_+) half-plane in $(u,\hat{u})$-space. In these half-planes, the u-motion is monotone. Hence, all heteroclinics $\Gamma_\pm$ are graphs over the u-coordinate chart in $(u,\hat{u})$-space, i.e., $\Gamma_\pm : \hat{u}(u) : u \in (u_l, u_r)$. We consider Γ_+. Such a heteroclinic orbit $\hat{u}(u)$ must fulfill

$$\begin{aligned}
\frac{d\hat{u}}{du} &= \frac{w + \Phi(u) - \delta\hat{u}}{\hat{u}}\,, \quad \forall u \in (u_l, u_r) \\
\implies \frac{d}{du}(\frac{\hat{u}^2}{2}) &= \frac{d}{du}\int(w + \Phi(u) - \delta\hat{u})du\,, \quad \forall u \in (u_l, u_r) \\
\implies \frac{\hat{u}^2}{2} &= \int_{u_l}^{u}(w + \Phi(u) - \delta\hat{u})du\,, \quad \forall u \in (u_l, u_r)\,.
\end{aligned}$$

For $u \to u_l$, the last line is fulfilled since $\hat{u}(u_l) = 0$. For $u \to u_r$, where $\hat{u}(u_r) = 0$, we obtain a condition for the existence of a heteroclinic orbit,

$$\boxed{\int_{u_l}^{u_r} (w + \Phi(u))du = \delta \int_{u_l}^{u_r} \hat{u}(u)\, du\,,} \tag{16}$$

which, for $\delta = 0$, gives the equal area rule as established previously. For $\delta \neq 0$ this formula provides a generalised 'equal area rule', i.e., the left hand side must move away from its 'equal area' position given for $w = w_h(0)$ to counteract the right hand side contribution. This gives $w = w_h(\delta)$.
For sufficiently large $|\delta| = \delta_m$, w will reach its limit w_{sn} where one of the saddle equilibria $p_{l/r}$ goes through a saddle-node bifurcation. Until then, the heteroclinic connection is along the hyperbolic direction, but afterwards it will be along the centre direction which is non-unique and, hence, replaces the codimension-one role of the w variation. Hence, for fixed $w = w_{sn}$ and for sufficiently large $|\delta| > \delta_m$, there exists always a heteroclinic orbit.

Remark 3.5 *For $w = w_{sn}$, the rhs of (16) is fixed. One concludes that for sufficiently large $|\delta| > \delta_m$, there is a $\hat{u}(u)$ that fulfills the generalised equal are rule, i.e., that fixes the right hand side $\delta \int \hat{u}\, du$ to the correct value.*

Figure 1 summarizes our results on the existence of shocks in the regularised RND model, i.e., the solution branches of $\Delta(w, \delta) = 0$. The important insight here is that viscous relaxation regularisation is dominant for $|\delta| > |\delta_m|$

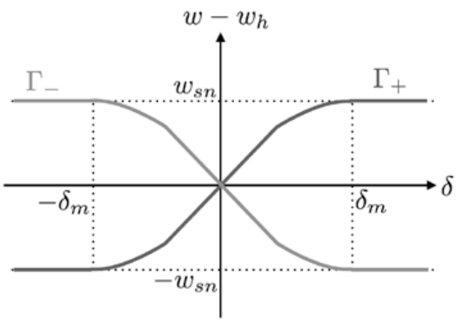


Figure 1: sketch of complete bifurcation diagram for heteroclinic connections $\Gamma_\pm$ in (δ, w)-space centered at $(w_h, 0)$.

3.2 The limit on the slow scale - the reduced problem

For the slow system (7), the limit $\varepsilon \to 0$ gives the *reduced problem*

$$\begin{aligned} 0 &= \hat{u} \\ 0 &= w + \Phi(u) - \delta\hat{u} \\ v_z &= f(u) \\ w_z &= v + cu\,. \end{aligned} \tag{17}$$

It describes the 'evolution' of the slow variables (v, w) constrained to the 2D critical manifold S (10). Here, the critical manifold S is given as a graph over the (u, v)-space, i.e., $S : \psi(u, v) \in \mathbb{R}^4$.

Therefore, we aim to study the corresponding reduced flow on S in this (u,v)-coordinate chart. By definition, the main requirement on the reduced vector field $R(u,v) \in \mathbb{R}^2$ is that, when mapped onto S via $D\psi$ it has to correspond to the (leading order) slow component of the full four-dimensional vector field constraint to S, i.e.,

$$D\psi(u,v)R(u,v) = \Pi^S G(\psi(u,v)) = (\frac{v+cu}{-D(u)}, 0, f(u), v+cu)^\top$$

where $\Pi^S G(\psi(u,v))$ is the (oblique) projection of the vector field $G = (0,0,f(u),v+cu)^\top$ onto the tangent bundle TS of the critical manifold S along fast fibres spanned by $\{(1,0,0,0)^\top, (0,1,0,0)^\top\}$. Thus the reduced vector field $R(u,v)$ in the (u,v)-coordinate chart is given by

$$\begin{aligned} -D(u)u_z &= v + cu \\ v_z &= f(u)\,. \end{aligned} \tag{18}$$

Note that this dynamical system is singular along the folds $F^\pm$ where $D(u) = 0$. To be able to study the reduced problem (18) in a neighbourhood of $F^\pm$, we make an auxiliary state-dependent time transformation $dz = D(u)d\zeta$ which gives the so-called *desingularised problem*

$$\begin{aligned} u_\zeta &= -(v + cu) \\ v_\zeta &= D(u)f(u)\,. \end{aligned} \tag{19}$$

This problem is topologically equivalent to (18) on the saddle-type outer branches of S while one has to reverse the orientation on the middle node-focus-centre-type branch of S to obtain the equivalent flow. We classify all singularities of the reduced problem (18) by analysing the auxiliary system, the desingularised problem (19).

Remark 3.6 *We emphasize that the desingularised system is only a proxy system to study the problem near the folds. To completely understand the orginal flow near the folds, one has to use additional techniques such as the blow-up method [14].*

The asymptotic end states of the travelling waves form equilibrium states of the desingularised (and the reduced) problem defined by $f(u_\pm) = 0$, and $v_\pm = -cu_\pm$. Our focus is on asymptotic end states given by the equilibria $(u_\mp, v_\mp) = (1, -c)$ and $(u_\pm, v_\pm) = (0,0)$. In the case of the bistable reaction term, there exists an additional equilibrium in the domain of interest defined by $f(u_b = \alpha) = 0$ which gives $(u_b, v_b) = (\alpha, -c\alpha)$. We assume $\gamma_1 < \alpha < \gamma_2$, i.e. this additional equilibrium is located on the middle branch S_m. The Jacobian evaluated at any of these equilibria $(u_{\pm,b}, v_{\pm,b})$ is given by

$$J = \begin{pmatrix} -c & -1 \\ D(u_{\pm,b})f'(u_{\pm,b}) & 0 \end{pmatrix}$$

which has tr $J = -c$ and det $J = D(u_{\pm,b})f'(u_{\pm,b})$. The types of equilibria of the reduced problem are summarized in the following table.

Remark 3.7 *The auxiliary system* (19) *defines another type of singularities for the reduced problem through $D(u) = 0$ which exist on the folds $F_{l/r}$ and are known as* folded singularities*; see, e.g., [16]. If time permits, I will briefly discuss this in my presentation.*

$D(u)$	$f(u)$	(u_-, v_-)	(u_+, v_+)	(u_b, v_b), $\gamma_1 < \alpha < \gamma_2$
DAD	logistic	Saddle	stable NF	-
DAD	bistable	Saddle	Saddle	Saddle

Table 1: Type of equilibria on critical manifold S

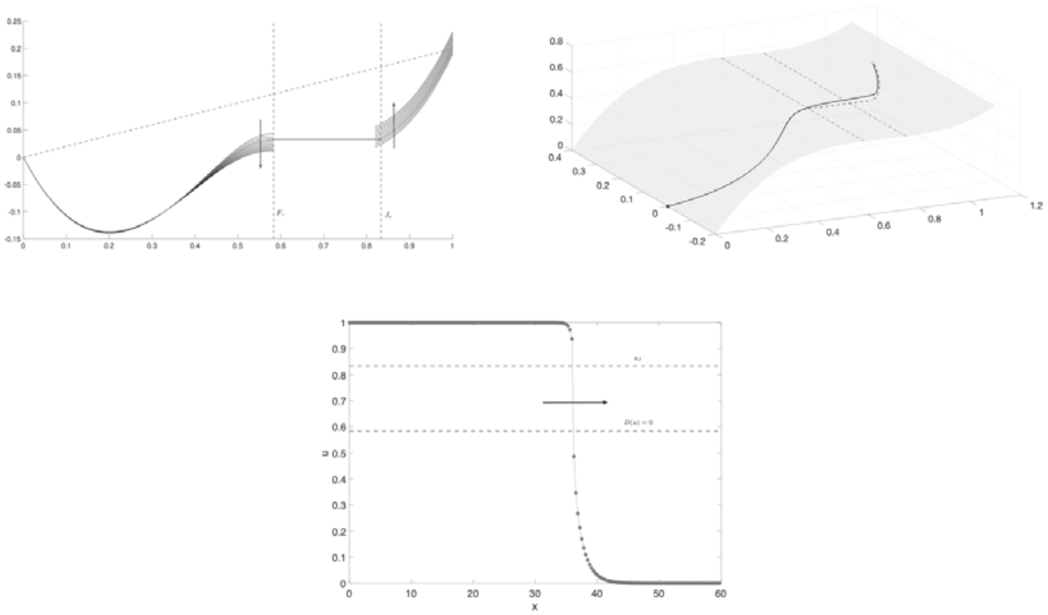


Figure 2: (Top Left) construction of singular heteroclinic orbit in (u, v) sub-space for $\delta > \delta_m$. The shock connects from $J_a \in S_s^r$ to F^- with asymptotic end states $u_- = 1$ and $u_+ = 0$. (Top Right) shows the singular heteroclinic orbit (dashed red, $\varepsilon = 0$) as well as the perturbed shock-fronted travelling wave (black solid, $0 < \varepsilon \ll 1$) in (u, v, w)-space. (Bottom) The corresponding travelling wave profile with sharp interface for $0 < \varepsilon \ll 1$

3.3 Construction of heteroclinic orbits that give shock waves

The final task is to concatenate solutions form the two limiting problems to construct singular heteroclinic orbits. Figure 2 shows an example for $|\delta| > |\delta_m|$; see Figure 1. In this case, the shock location is at (one of) the folds of the critical manifold. From a regularisation pint-of-view, this indicates that viscous relaxation (u_{xxt}-term) dominates the nonlocal effects (u_{xxxx}-term).
The power of geometric singular perturbation theory is to show the persistence of travelling waves with smooth and sharp interfaces (shocks) constructed above under sufficiently small perturbations $0 < \varepsilon \ll 1$ by means of geometric properties of invariant manifolds. Figure 2 (Bottom) indicates that this is indeed the case.

Remark 3.8 *If time permits, I will also discuss some of the intricacies of numerical schemes to resolve the predicted analytical shock location. Similarly, if time permits, I briefly discuss (spectral) stability properties of these solutions.*

Acknowledgement

This work has been funded by the Australian Research Council DP200102130 grant.

References

[1] K. Anguige and C. Schmeiser. A one-dimensional model of cell diffusion and aggregation, incorporating volume filling and cell-to-cell adhesion. *Journal of Mathematical Biology*, 58(3):395–427, jun 2008.

[2] J. W. Cahn. On spinoidal decomposition. *Acta. Metall.*, 9(795-801), 1961.

[3] N. Fenichel. Geometric singular perturbation theroy. *J Differential Equations*, 31:53–98, 1979.

[4] A. E. Fernando, K. A. Landman, and M. J. Simpson. Nonlinear diffusion and exclusion processes with contact interactions. *Physical Review E*, 81, January 2010.

[5] R. A. Fisher. The wave of advance of advantageous genes. *Annals of Eugenics*, 7:355–369, 1937.

[6] J. Hilliard. Spinodal decomposition. *in* phase transformations. *Am. Soc. Metals*, pages 497–560, 1970.

[7] S. T. Johnston, R. E. Baker, D. L. S. McElwain, and M. J. Simpson. Co-operation, competition and crowding: a discrete framework linking allee kinetics, nonlinear diffusion, shocks and sharp-frontec travelling waves. *Scientific Reports*, 7(42134), 2017.

[8] C. K. R. T. Jones. Geometric singular perturbation theory. *in Dynamical Systems, Springer Lecture Notes Math.*, 1609:44–120, 1995.

[9] P. K. Maini, L. Malaguti, C. Marcelli, and S. Matucci. Aggregative movement and front propagation for bi-stable population models. *Mathematical Models and Methods in Applied Sciences*, 17(9):1351–1368, 2007.

[10] J.D. Murray. *Mathematical Biology I: An introduction*, volume 1 of *Interdisciplinary Applied Mathematics*. Springer, Hong Kong, 3rd edition, 2002.

[11] V. Padron. Effect of aggregation on population recovery modelled by a forward-backward pseudoparabolic equation. *Transactions of the American Mathematical Society*, 356(7):2739–2756, 2003.

[12] R. L. Pego. Front migration in the nonlinear Cahn–Hilliard equation. *Proc. R. Soc. Lond. A*, 422:261–278, 1989.

[13] C. J. Penington, B. D. Hughes, and K. A. Landman. Building macroscale models from microscale probablistic models: A general probablistic approach for nonlinear diffusion and multispecies phenomena. *Physical Review E*, 84, October 2011.

[14] P. Szmolyan and M. Wechselberger. Canards in $\mathbb{R}^3$. *J Differential Equations*, 177:419–453, 2001.

[15] André Vanderbauwhede. Bifurcation of degenerate homoclinics. *Results in Mathematics*, 21(1-2):211–223, mar 1992.

[16] M. Wechselberger and G. J. Pettet. Folds, canards and shocks in advection-reaction-diffusion models. *Nonlinearity*, 23:1949–1969, 2010.

[17] Martin Wechselberger. Extending Melnikov theory to invariant manifolds on non-compact domains. *Dynamical Systems*, 17(3):215–233, aug 2002.

[18] T.P. Witelski. The structure of internal layers for unstable nonlinear diffusion equations. *Studies in Applied Mathematics*, 96:277–300, 1996.

Stability analysis for 2-mode solutions for mass-conserved reaction diffusion system

Tsubasa Sukekawa[1]

Department of mathematics, Faculty of science, Hokkaido university
e-mail: sukekawa-tsubasa@eis.hokudai.ac.jp

1 Introduction

In this study, we consider mass-conserved reaction diffusion system. This system is introduced for various mathematical models, for example a conceptial model for cell polarity [4], [6], [1]. We consider reaction diffusion system defined on interval $\Omega := (-K/2, K/2)$ as following:

$$\begin{cases} \partial_t u = d\partial_x^2 u + f(u,v), & t > 0, \quad x \in \Omega, \\ \tau \partial_t v = \partial_x^2 v - f(u,v), & t > 0, \quad x \in \Omega, \end{cases} \tag{1}$$

where, $u = u(t,x), v = v(t,x)$ are real valued unknown functions, $f \in C^3(\mathbb{R}^2)$, d, τ, K are positive constants.

In this study, we consider homogenious neumann boundary conditions

$$\text{(N.B.C)} \quad \partial_x u = \partial_x v = 0 \quad (t > 0, x = \pm K/2), \tag{2}$$

or periodic boundary conditions

$$\text{(P.B.C)} \quad \begin{array}{ll} u(t,-K/2) = u(t,K,2),\ \partial_x u(t,-K/2) = \partial_x u(t,K/2) & (t > 0), \\ v(t,-K/2) = v(t,K,2),\ \partial_x v(t,-K/2) = \partial_x v(t,K/2) & (t > 0). \end{array} \tag{3}$$

Significant feature of system (1) is mass conservation law; under (N.B.C), or (P.B.C), any classical solution (u,v) for (1) has following conserved quantity

$$\frac{1}{K}\int_\Omega (u(t,x) + \tau v(t,x))dx = \langle u(t,\cdot)\rangle + \tau\langle v(t,\cdot)\rangle,$$

where $\langle \phi \rangle := \frac{1}{K}\int_\Omega \phi(x)dx$. We can check that $\frac{\mathrm{d}}{\mathrm{d}t}(\langle u \rangle + \tau \langle v \rangle) = 0$, using boundary conditions.

We consider the stability of stationary solution for (1). Since stationary problem of (1) with (N.B.C) or (P.B.C) can be transformed into scalar equation, the stationary solutions are characterized by n-mode solution.

[1]This is joint work with Shin-Ichiro Ei (Hokkaido university)

Definition 1.1. *Let $P(x) = (u^*(x), v^*(x))$ be stationary solution to (1) with (N.B.C), or (P.B.C) and $n \geq 1$ be positive integer. In the case of (N.B.C), P is n-mode solution, if each u^*, v^* has just $(n-1)$ critical points in Ω. For the case of (P.B.C), P is n-mode solution, if each u^*, v^* has just $(2n-1)$ critical points in Ω. If both of u^* and v^* are constant functions, P is called 0-mode solution.*

In other words, 1-mode solution means that each of u^* and v^* is monotone increasing, or decreasing function on Ω for (N.B.C), and each u^* and v^* is unimodal on Ω for (P.B.C) (Fig. 1).

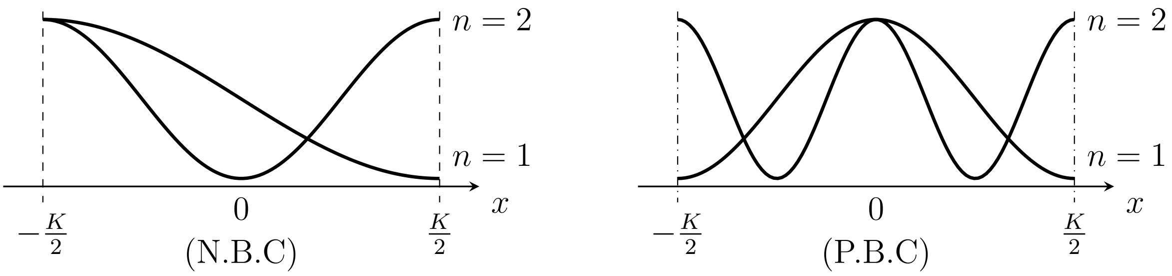


Figure 1: Sketch of n-mode solution

We note that when we get a 1-mode solution on the half interval $(-K/2, 0)$, saying P^1, 2-mode solution P^2 is constructed by extending P^1 symmetrically about $x = 0$:

$$P^2(x) = \begin{cases} P^1(x) & (x \in (-K/2, 0)), \\ P^1(-x) & (x \in [0, K/2)). \end{cases} \tag{4}$$

Remark 1.2. *For general case $n \geq 3$, we can get n-mode solution by similar mannars. We treat this case in future work.*

For specific case of reaction term $f(u, v)$, parameters and boundary conditions, reaction diffusion system (1) addmits Lyapunov function. Then we can apply variational techniques for investigating the stability of stationary solutions. Moreover, it is proved that stable stationary solution under (N.B.C), or (P.B.C) must be 0-mode, or 1-mode solution[5], [2], [3].

From previous studies, we have following question.

Q. Let n be positive integer and $n \geq 2$. Does stable n-mode solution exist for (1) under (N.B.C), or (P.B.C)?

However, reacion diffusion system (1) addmits time-periodic solution for certain reaction term, for example $f(u, v) = a_1(u + \tau v)\{(du + v)(u + \tau v)\}$, $(a_1, a_2 > 0)$[6], [7]. Lyapunov function does not exist in this case, because if there exist Lyapunov function, time-periodic solution can't exist. Hence, it seems difficult to apply variational techniques to this situation.

Our study aims to investigate the stability of 2-mode solution of system (1). For this purpose, we introduce another reaction diffusion systems, which is formulated as following:

$$\begin{cases} T\partial_t \boldsymbol{u}_i = \boldsymbol{\mathcal{L}}(\boldsymbol{u}_i), & t > 0, x \in \Omega_i, \\ D\partial_x \boldsymbol{u}_1 = \varepsilon A(\boldsymbol{u}_2 - \boldsymbol{u}_1) = D\partial_x \boldsymbol{u}_2, & t > 0, x = 0, \\ \partial_x \boldsymbol{u}_1 = 0, & t > 0, x = -K/2, \\ \partial_x \boldsymbol{u}_2 = 0, & t > 0, x = K/2. \end{cases} \tag{5}$$

$$\begin{aligned} &\boldsymbol{u}_i(t,x) := \begin{pmatrix} u_i(t,x) \\ v_i(t,x) \end{pmatrix}, T := \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix}, D := \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}, A := \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, \\ &\boldsymbol{\mathcal{L}}(\boldsymbol{u}_i) := D\partial_x^2 \boldsymbol{u}_i + \boldsymbol{F}(\boldsymbol{u}_i), \boldsymbol{F}(\boldsymbol{u}_i) := \begin{pmatrix} f(u_i, v_i) \\ -f(u_i, v_i) \end{pmatrix}. \end{aligned} \tag{6}$$

where $i \in \{1, 2\}, \Omega_1 := (-K/2, 0), \Omega_2 := (0, K/2), \alpha > 0, \varepsilon > 0$, and $(u_i(t,x), v_i(t,x))$ is defined on each interval Ω_i. We call the systems(5) reaction diffusion compartment model.

This model consists of reaction diffusion systems defined on independent intervals Ω_i, and these systems are coupled by the boundary conditions at $x = 0$, which represent diffusive coupling derived from Fick's law. New parameter α represents the ratio of constants of coupling, and ε represents the strength of coupling.

Reaction diffusion compartment model has the propeties which are similar to original system (1). First, the systems (5) hold mass conservation law. Let s_1 and s_2 be as follows:

$$s_i(t) := \frac{2}{K} \int_{\Omega_i} (u_i(t,x) + \tau v_i(t,x)) dx = \langle u_i \rangle_i + \tau \langle v_i \rangle_i,$$

where $\langle \phi \rangle_i := \frac{2}{K} \int_{\Omega_i} \phi \, dx$. From the boundary conditions of (5), we can check that $\frac{\mathrm{d}}{\mathrm{d}t}(s_1 + s_2) = 0$. Second, we can regard any 2-mode solution to (1) with (N.B.C) as stationary solution to (5), since derivative of 2-mode solution become 0 at $x = 0$.

Let us explain the relation between system (1) and compartment model (5). When we consider the case $\varepsilon \to +\infty$, we can formally derive that $u_2(t,0) - u_1(t,0) \to 0$ and $v_2(t,0) - v_1(t,0) \to 0$ from the boundary conditions at $x = 0$. Then we expect that the dynamics of solutions of (5) is related to these of original system, especially the stability of 2-mode stationary solutions. Hence we consider linearized eigenvalue problem for (5).

We note that if $\varepsilon = 0$, the reaction diffusion compartment model is same to reaction diffusion systems defined on separeted intervals Ω_i with (N.B.C), which namely there are no diffusive coupling at $x = 0$. When $0 < \varepsilon \ll 1$, we regard the boundary condition as the pertubation to the system with (N.B.C), and ε as the parameter of the pertubation. From above consideration, we fomally derive an eigenvalue in neiborhood of origin by regular pertubation with respect to ε.

Finally, we note that this abstract is reporting an on-going work of this study and organized as follows: in section 2, we introduce basic properties and assumptions for 1-mode stationary solution define on Ω_1. In section 3, we formulate linearized eigenvalue problem for (5) around 2-mode solution. In section 4, we fomally derive an eigenvalue in neiborhood of origin in the case $0 < \varepsilon \ll 1$.

Remark 1.3. *Previous studies treat not only (N.B.C), but also (P.B.C). In this study, however, we focus on (N.B.C) only. This is because 2-mode solution with (P.B.C) is obtained by extending a 2-mode solution with (N.B.C) define on half interval symetrically, then if the later is unstable under (N.B.C) in linearized sense, the former become unstable too. Hence, we focus on (N.B.C) in this paper.*

2 Preliminaries

In this section, we introduce the basic properties and assumptions for stationary solutions. Firstly, we consider stationary problem defined on Ω_1:

$$\begin{cases} du'' + f(u,v) = 0, & x \in \Omega_1, \\ v'' - f(u,v) = 0, & x \in \Omega_1, \\ u' = v' = 0, & x = -K/2, 0, \\ \langle u \rangle_1 + \tau \langle v \rangle_1 = s. \end{cases} \tag{7}$$

Note that $' := \frac{\mathrm{d}}{\mathrm{d}x}$ and the last equation of (7) come from the mass conservation law. By regarding s as the parameter of stationary solution, we let $(u^*(x;s), v^*(x;s))$ stands for the solution of (7).

Adding first and second equations, we get the relation $(du+v)'' = 0$ on Ω_1. Thus, for any solution (u,v) of (7), there is certain constant c, such that the solution satisfies the relation

$$du + v \equiv c \quad (x \in \Omega_1), \tag{8}$$

because of neumann boundary conditions. Thus, function u satisfies

$$\begin{cases} du'' + f(u, c - du) = 0, & x \in \Omega_1, \\ u' = 0, & x = -K/2, 0. \end{cases} \tag{9}$$

Standard shooting method implies that u is n-mode solution on Ω_1, namely u has just $(n-1)$ critical points for certain positeve integer n on Ω_1, or u is constant solution. Then v is n-mode solution with the same sense, because of the relation (8).

Taking spatial average of (8), the constant c is calculated by

$$c = d\langle u \rangle_1 + \langle v \rangle_1 = \frac{s}{\tau} - \frac{1 - d\tau}{\tau} \langle u \rangle_1.$$

Therefore, the stationary problem (7) corresponds to the following scalar non-local equation one-to-one:

$$\begin{cases} du'' + f\left(u, -du + \dfrac{s}{\tau} - \dfrac{1-d\tau}{\tau}\langle u\rangle_1\right) = 0, & x \in \Omega_1, \\ \hfill u' = 0, & x = -K/2, 0. \end{cases} \tag{10}$$

We suppose for (10) that:

(H.1) There is the solution $u^*(x;s)$ for (10) at $s=\bar{s}$. u^* is monotone decreasing function on Ω_1 and continuously differentiable with respect to s in a neighborhood of $\bar{s}$.

Let $v^*(x;s) = -du^*(x;s) + \frac{s}{\tau} - \frac{1-d\tau}{\tau}\langle u^*\rangle_1$, then $P^1(x;s) := (u^*(x;s), v^*(x;s))$ is 1-mode solution for (7).

When we assume **(H.1)**, P^1 satisfies the relation $\langle u^*_s\rangle_1 + \tau\langle v^*_s\rangle_1 = 1$ by differentiating the last equation of (7) and constant c, which is define by (8) become continuouly differentiable function of s in neiborhood of $\bar{s}$, then we represent $c = c(s)$.

Next we consider linerized eigenvalue problm for 1-mode solution P^1 to (7):

$$\lambda\Phi = \lambda\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \begin{pmatrix} d\phi'' + f^*_u(x)\phi + f^*_v(x)\psi \\ \tau^{-1}\{\psi'' - f^*_u(x)\phi - f^*_v(x)\psi\}\end{pmatrix} =: L_{\mathrm{N}}\Phi, \tag{11}$$

where $f^*_u(x) := f_u(u^*(x;\bar{s}), v^*(x;\bar{s})), f^*_v(x) := f_v(u^*(x;\bar{s}), v^*(x;\bar{s}))$. L_N is operator in $L^2(\Omega_1)\times L^2(\Omega_1)$ with the domain

$$\mathcal{D}(L_{\mathrm{N}}) = \left\{\Phi = (\phi,\psi) \in H^2(\Omega_1)\times H^2(\Omega_1); \partial_x\Phi = 0\ (x=-K/2, 0)\right\}. \tag{12}$$

Let $\sigma(L_{\mathrm{N}})$ be spectrum of L_N.

If $d\tau = 1$, linearized equations (11) turns to:

$$\lambda\begin{pmatrix}\phi\\ \eta\end{pmatrix} = \begin{pmatrix} d\phi'' + \{f^*_u(x) - df^*_v(x)\}\phi + f^*_v(x)\eta \\ d\eta''\end{pmatrix}, \tag{13}$$

where $\eta := \phi + \tau\psi$. When $\eta = 0$, the first equation of (13) become linerized eigenvalue problem corresponding to (9). From the theory of sturm-liouville eigenvalue problem[8], there is positive real eigenvalue for L_N. Hence P^1 is always unstable in the case $d\tau = 1$.

We suppose that $d\tau \neq 1$ through this paper and:

(H.2) There exists $\gamma > 0$, such that $\sigma(L_{\mathrm{N}}) = \{0\}\cup\Sigma_0, \Sigma_0 \subset \{\lambda\in\mathbb{C}; \mathrm{Re}\,\lambda < -\gamma\}$. Moreover, 0 is simple eigenvalue and the corresponding eigenfunction is P^1_s.

This assumption is natural and essential to our study. Because if there is an eigenvalue of L_N with positive real part, the 2-mode solution P^2 defined by (4) become unstable in linearized sense, this is trivial situation.

3 Linerized eigenvalue problem for 2-mode solution

From hypothesis **(H.1)**, we obtain non-constant stationary solution for (5)

$$P^2(x;\bar{s}) := \begin{cases} P^1(x;\bar{s}) & (x \in \Omega_1), \\ P^1(-x;\bar{s}) & (x \in \Omega_2). \end{cases}$$

Although $P^2(x;s)$ is not defined at $x = 0$, P^2 can be expanded uniquely to continuous functions defined on Ω.

Let $\Phi(x) := \begin{cases} \Phi_1(x) & (x \in \Omega_1), \\ \Phi_2(x) & (x \in \Omega_2), \end{cases}$ $\Phi_i(x) := (\phi_i(x), \psi_i(x)), (x \in \Omega_i)$. The linearized eigenvalue problem for P^2 is formulated as below:

$$\begin{cases} \lambda T\Phi_i = L\Phi_i, & x \in \Omega_i, \\ D\partial_x\Phi_1 = \varepsilon A(\Phi_2 - \Phi_1) = D\partial_x\Phi_2, & x = 0, \\ \partial_x\Phi_1 = 0, & x = -K/2, \\ \partial_x\Phi_2 = 0, & x = K/2, \end{cases} \tag{14}$$

where $L\Phi_i := D\partial_x^2\Phi_i + \boldsymbol{F}'(P^2)\Phi_i$, $\boldsymbol{F}'(P^2)$ is Jacobian matrix of $\boldsymbol{F}$ at P^2.

Φ can be decomposed into even function and odd function on Ω_i, namely we represent $\Phi = \Phi_\mathrm{e} + \Phi_\mathrm{o}$ where

$$\Phi_\mathrm{e} := \begin{cases} \frac{1}{2}\{\Phi_1(x) + \Phi_2(-x)\} & (x \in \Omega_1), \\ \frac{1}{2}\{\Phi_1(-x) + \Phi_2(x)\} & (x \in \Omega_2), \end{cases} \qquad \Phi_\mathrm{o} := \begin{cases} \frac{1}{2}\{\Phi_1(x) - \Phi_2(-x)\} & (x \in \Omega_1), \\ \frac{1}{2}\{-\Phi_1(-x) + \Phi_2(x)\} & (x \in \Omega_2). \end{cases}$$

It is easy to see that $\Phi_\mathrm{e}(x) = \Phi_\mathrm{e}(-x), \Phi_\mathrm{o}(x) = -\Phi_\mathrm{o}(-x)$ $(x \in \Omega_1 \cup \Omega_2)$.

Conversely, if certain functions Φ_e and Φ_o satisfy

$$\begin{cases} \lambda T\Phi_\mathrm{e} &= L\Phi_\mathrm{e}, & x \in \Omega_1, \\ \partial_x\Phi_\mathrm{e} &= 0, & x = -K/2, 0, \end{cases} \tag{15}$$

$$\begin{cases} \lambda T\Phi_\mathrm{o} &= L\Phi_\mathrm{o}, & x \in \Omega_1, \\ \partial_x\Phi_\mathrm{o} &= 0, & x = -K/2, \\ D\partial_x\Phi_\mathrm{o} &= -2\varepsilon A\Phi_\mathrm{o}, & x = 0, \end{cases} \tag{16}$$

with cetain number λ, then $\Phi := \Phi_\mathrm{e} + \Phi_\mathrm{o}$ and λ satisfy (14).

The problem (15) is same to (11). **(H.2)**, implies that if $\Phi_\mathrm{e} \not\equiv 0$, λ is an eigenvalue of L_N. Therefore, we may focus on the problem (16) for searching an eigenvalue in a neiborhood of 0.

4 Formal derivation of eigenvalue

In this section, we formally derive an eigenvalue in neiborhood of 0 for the case $0 < \varepsilon \ll 1$. Let $\Phi(x) = (\phi(x), \psi(x)), x \in \Omega_1$ and consider the problem:

$$\begin{cases} \lambda T\Phi = L\Phi, & x \in \Omega_1, \\ \partial_x \Phi = 0, & x = -K/2, \\ D\partial_x \Phi = -2\varepsilon A\Phi, & x = 0, \\ \langle \phi \rangle_1 + \tau \langle \psi \rangle_1 = 1. \end{cases} \tag{17}$$

When $\varepsilon = 0$, we obtain

$$\lambda(\phi + \tau\psi) = (d\phi + \psi)'',$$

then $\langle \phi \rangle_1 + \tau \langle \psi \rangle_1 = 0$, when $\lambda \neq 0$. Hence there unique solution $(0, P_s^1)$ to (17) from the view of **(H.2)**.

Let $0 < \varepsilon \ll 1$, we suppose that eigenvalue and eigenfunction of (17) can be expanded as follows:

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots$$
$$\Phi = \Phi_0 + \varepsilon\Phi_1 + \varepsilon^2\Phi_2 + \cdots$$

Substituting λ and Φ into (17), the equations of the lowest order is represented by:

$$O(1) \begin{cases} \lambda_0 T\Phi_0 = L\Phi_0, & (x \in \Omega_1), \\ \partial_x \Phi_0 = 0, & (x = -K/2, 0), \\ \langle \phi_0 \rangle_1 + \tau \langle \psi_0 \rangle_1 = 1. \end{cases} \tag{18}$$

Then, we obtain $(\lambda_0, \Phi_0) = (0, P_s^1)$.

The equations of the next order is:

$$O(\varepsilon) \begin{cases} \lambda_0 T\Phi_1 + \lambda_1 T\Phi_0 = L\Phi_1, & (x \in \Omega_1), \\ \partial_x \Phi_1 = 0, & (x = -K/2), \\ D\partial_x \Phi_1 = -2A\Phi_0, & (x = 0), \\ \langle \phi_1 \rangle_1 + \tau \langle \psi_1 \rangle_1 = 0. \end{cases} \tag{19}$$

Since $(\lambda_0, \Phi_0) = (0, P_s^1)$, the first equation of (19) turns to

$$\lambda_1 \begin{pmatrix} u_s^* \\ \tau v_s^* \end{pmatrix} = \begin{pmatrix} d\phi_1'' + f_u^*(x)\phi_1 + f_v^*(x)\psi_1 \\ \psi_1'' - f_u^*(x)\phi_1 - f_v^*(x)\psi_1 \end{pmatrix}. \tag{20}$$

Adding the above equations,

$$\lambda_1(u_s^* + \tau v_s^*) = (d\phi_1'' + \psi_1'').$$

Taking spatial average, we obtain

$$\lambda_1 = -\frac{4}{K}(\alpha u_s^*(0; \bar{s}) + v_s^*(0; \bar{s})) \tag{21}$$

Let $w^*(x;\bar{s}) := u^*(x;\bar{s})+\tau v^*(x;\bar{s}),\ c(\bar{s}) := du^*(x;\bar{s})+v^*(x;\bar{s})$, then P^1 is represented by:

$$\begin{pmatrix} u^*(x;s) \\ v^*(x;s) \end{pmatrix} = \frac{1}{1-d\tau}\left\{ \begin{pmatrix} 1 \\ -d \end{pmatrix} w^*(x;s) + \begin{pmatrix} -\tau \\ 1 \end{pmatrix} c(s) \right\}$$

By using above representation, (21) is transformed into:

$$\lambda_1 = -\frac{4}{(1-d\tau)K}\left\{(\alpha-d)w_s^*(0;\bar{s}) + (1-\alpha\tau)c_s(\bar{s})\right\} \tag{22}$$

When we choose $\alpha = d$, $\lambda_1 = -\frac{4}{K}c_s(\bar{s})$. Hence the sign of λ_1 is opposite to $c_s(\bar{s})$ in this case.

Remark 4.1. *From the fomal calculation, we expect that there is small, but non-zero eigenvalue in neiborhood of origin for sufficiently small ε. We are attempting to prove above expectation by implicit function theorem, which is still on-going.*

Remark 4.2. *The relation* (22) *suggests that if* $c_s(\bar{s}) < 0$, P^2 *become unstable. This relation is firstly indicated in the study of cell polarity model[1], and the physical meaning of it is discussed [6]. When $\alpha = d$, our study may include their results. However, when $\alpha \neq d$, the stability is possibly changed. In fact, we observed that an* 2*-mode solution, which is unstable for the case $\alpha = d$, become stable for certain $\alpha > d$ in numerical simulation.*

Acknowledgement

This work was supported by JST SPRING, Grant Number JPMJSP2119.

References

[1] Shuji Ishihara, Mikiya Otsuji, and Atsushi Mochizuki. Transient and steady state of mass-conserved reaction-diffusion systems. *Physical Review E*, 75(1):015203, 2007.

[2] Shuichi Jimbo and Yoshihisa Morita. Lyapunov function and spectrum comparison for a reaction–diffusion system with mass conservation. *Journal of Differential Equations*, 255(7):1657–1683, 2013.

[3] Evangelos Latos, Yoshihisa Morita, and Takashi Suzuki. Stability and spectral comparison of a reaction–diffusion system with mass conservation. *Journal of Dynamics and Differential Equations*, 30(2):823–844, 2018.

[4] Yoichiro Mori, Alexandra Jilkine, and Leah Edelstein-Keshet. Wave-pinning and cell polarity from a bistable reaction-diffusion system. *Biophysical journal*, 94(9):3684–3697, 2008.

[5] Yoshihisa Morita. Spectrum comparison for a conserved reaction-diffusion system with a variational property. *J. Appl. Anal. Comput*, 2:57–71, 2012.

[6] Mikiya Otsuji, Shuji Ishihara, Carl Co, Kozo Kaibuchi, Atsushi Mochizuki, and Shinya Kuroda. A mass conserved reaction–diffusion system captures properties of cell polarity. *PLoS computational biology*, 3(6):e108, 2007.

[7] Takashi Okuda Sakamoto. Hopf bifurcation in a reaction–diffusion system with conservation of mass. *Nonlinearity*, 26(7):2027, 2013.

[8] Anton Zettl. *Sturm-liouville theory*. Number 121. American Mathematical Soc., 2012.

PDEs, Geometric measure theory and Ricci curvature on nonsmooth spaces

Shouhei Honda *

Abstract

The purpose of this abstract is to provide precise backgrounds on my talk at The 47th Sapporo Symposium on Partial Differential Equations. The talk includes a survey on recent developments on metric measure spaces with Ricci curvature bounded below. As an application to the smooth setting, I will explain how to obtain failure of L^p-Calderón-Zygmund inequality on a smooth Riemannian manifold with nonnegative Ricci (more strongly sectional) curvature for $p > 2$.

Contents

1 Smooth Ricci curvature

1.1 Unweighted case

Let (M^n, g) be a Riemannian manifold of dimension n, namely M^n is a smooth manifold of dimension n, and g determines an inner product g_x on the tangent space T_xM^n at each point $x \in M^n$, where the map $x \mapsto g_x$ behaves smoothly with respect to $x \in M^n$. A typical example is a nonempty open subset Ω^n in $\mathbb{R}^n$ with the canonical Riemannian metric g_{Ω^n} as follows:

$$g_{\Omega^n} = \sum_{i=1}^{n} \mathrm{d}x_i \otimes \mathrm{d}x_i, \tag{1.1}$$

*Tohoku University, shouhei.honda.e4@tohoku.ac.jp

namely

$$g_{\Omega^n}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \delta_{ij}, \quad T_x\Omega^n = \mathrm{span}\left\{\left(\frac{\partial}{\partial x_i}\right)_x\right\}. \tag{1.2}$$

Then the Ricci curvature Ric^g of (M^n, g) is originally defined by taking a trace of the curvature operator R^g. Since it is characterized by a tensor T satisfying the following two conditions, we adopt the following as the definition, which is a starting observation of this abstract:

1. (Symmetry) T is a symmetric tensor of type $(0, 2)$ over M^n, namely T determines a symmetric bilinear form of the tangent space T_xM^n at each $x \in M^n$:

$$T_x : T_xM^n \times T_xM^n \to \mathbb{R}. \tag{1.3}$$

   Moreover the map $x \mapsto T_x$ behaves smoothly with respect to $x \in M^n$.

2. (Bochner identity) For any $f \in C^\infty(M^n)$, we have

$$\frac{1}{2}\Delta^g|\nabla^g f|^2 = |\mathrm{Hess}_f^g|^2 + g(\nabla^g\Delta^g f, \nabla f) + T(\nabla^g f, \nabla^g f), \tag{1.4}$$

   where $\nabla^g f, \Delta^g f$ denote the gradient vector field, the Laplacian of f, respectively, associated with g.

It is worth pointing out that the Laplacian $\Delta^g f$ is characterized by a smooth function φ on M^n satisfying the integration-by-parts formula:

$$\int_{M^n} g(\nabla^g h, \nabla^g f)\,\mathrm{dvol}^g = -\int_{M^n} h\varphi\,\mathrm{dvol}^g, \quad \forall h \in C_c^\infty(M^n), \tag{1.5}$$

where vol^g denotes the Riemannian volume measure on M^n, namely it coincides with the n-dimensional Hausdorff measure $\mathscr{H}^n$ with respect to the Riemannian distance d^g induced by g.

It follows from a direct calculation that for (Ω^n, g_{Ω^n}) as above, the Riemannian volume measure coincides with the restriction of the Lebesgue measure, the Ricci curvture vanishes and we have

$$\nabla^{g_{\Omega^n}} f = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_i}, \quad \Delta^{g_{\Omega^n}} f = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}. \tag{1.6}$$

We say that the Ricci curvature Ric^g of (M^n, g) is bounded below by $K \in \mathbb{R}$ if

$$\mathrm{Ric}^g \geq Kg \tag{1.7}$$

holds as bilinear forms. Thanks to observations above, it is trivial to check that (1.7) is equivalent to satisfying

$$\frac{1}{2}\int_{M^n} \Delta^g h \cdot |\nabla^g f|^2\,\mathrm{dvol}^g \geq \int_{M^n} h\left(|\mathrm{Hess}_f^g|^2 + g(\nabla^g\Delta^g f, \nabla f) + K|\nabla^g f|^2\right)\mathrm{dvol}^g \tag{1.8}$$

for all $f, h \in C_c^\infty(M^n)$ with $h \geq 0$.

On the other hand, the Cauchy-Schwarz inequality implies

$$|\Delta^g f| = \left|\langle \mathrm{Hess}_f^g, g\rangle\right| \leq |\mathrm{Hess}_f^g| \cdot |g| = |\mathrm{Hess}_f^g| \cdot \sqrt{n}. \tag{1.9}$$

In particular for any $N \geq n$ we have

$$|\mathrm{Hess}_f^g|^2 \geq \frac{(\Delta^g f)^2}{N}. \tag{1.10}$$

Thus combining this with (1.8) shows

$$\frac{1}{2}\int_{M^n} \Delta^g h \cdot |\nabla^g f|^2 \,\mathrm{dvol}^g \geq \int_{M^n} h \left(\frac{(\Delta^g f)^2}{N} + g(\nabla^g \Delta^g f, \nabla^g f) + K|\nabla^g f|^2 \right) \mathrm{dvol}^g \tag{1.11}$$

for all $N \geq n$ and $f, h \in C_c^\infty(M^n)$ with $h \geq 0$. Let us emphasize that for all $N > 0$ and $K \in \mathbb{R}$, (1.7) is equivalent to satisfying (1.7) with $n \leq N$.

There is a rich analytic/geometric comparison theory for complete (M^n, d^g) with (1.7). For example if (1.7) is satisfied in the case when $K = 0$, then for a fixed $x \in M^n$, the ratio:

$$\frac{\mathrm{vol}^g(B_r^g(x))}{\omega_n r^n} \tag{1.12}$$

is not increasing as a function of $r > 0$, which is called the Bishop-Gromov inequality, where $B_r^g(x)$ denote the open ball centered at x of radius r with respect to d^g and ω_n denotes the volume of a ball in $\mathbb{R}^n$ of radius 1. Moreover it is known that the following hold:

- (Bishop inequality) the ratio (1.12) is bounded above by 1;
- (Rigidity) if the ratio (1.12) is equal to 1 for some $r > 0$, then the ball is isometric to the corresponding Euclidean ball.

Note that it is essential to assume the completeness of (M^n, d^g) in order to get such comparison results. In fact, for instance, it is easily checked that for a suitable noncovex domain Ω^n in $\mathbb{R}^n$ (recall that the Ricci curvature vanishes), the desired monotonicity for (1.12) is not satisfied.

1.2 Weighted case

Let us focus on the weighted versions of the discussions in the previous subsection. Fix a Riemannian manifold (M^n, g) of dimension n and a smooth function $\varphi \in C^\infty(M^n)$. Our starting point is to consider the weighted Riemannian volume measure vol_φ^g associated with φ:

$$\mathrm{vol}_\varphi^g(A) := \int_A e^{-\varphi} \,\mathrm{dvol}^g. \tag{1.13}$$

Then letting

$$\Delta_\varphi^g f := \Delta^g f - g(\nabla^g f, \nabla^g \varphi), \tag{1.14}$$

we have

$$\int_{M^n} g(\nabla^g h, \nabla^g f) \,\mathrm{dvol}_\varphi^g = -\int_{M^n} h \Delta_\varphi^g f \,\mathrm{dvol}_\varphi^g, \quad \forall h \in C_c^\infty(M^n). \tag{1.15}$$

Moreover, as in the unweighted case, this integration-by-parts formula characterize $\Delta_\varphi^g f$. Thus this observation allows us to call $\nabla^g f, \Delta_\varphi^g f$ the gradient vector field, the Laplacian of f, respectively, associated with g, φ.

Then it is natural to ask;

(Q) What is the Ricci curvature associated with g, φ?

In order to give an answer, let us observe the following identity:

$$\begin{aligned}&\frac{1}{2}\Delta^g_\varphi|\nabla^g f|^2\\ &= |\mathrm{Hess}^g_f|^2 + g(\nabla^g\Delta^g_\varphi f, \nabla^g f) + \frac{(\Delta^g_\varphi f - \Delta^g f)^2}{a} + \left(\mathrm{Ric}^g + \mathrm{Hess}^g_\varphi - \frac{1}{a}\,\mathrm{d}\varphi\otimes\mathrm{d}\varphi\right)(\nabla^g f, \nabla^g f)\end{aligned} \tag{1.16}$$

for any $a \neq 0$. Recalling $|\mathrm{Hess}^g_f|^2 \geq (\Delta^g f)^2/n$ with an elemental inequality

$$\frac{x^2}{b} + \frac{(y-x)^2}{a} \geq \frac{y^2}{a+b}, \quad \forall a > 0,\ \forall b > 0,\ \forall x \in \mathbb{R},\ \forall y \in \mathbb{R}, \tag{1.17}$$

the RHS of (1.16) is bounded below by

$$\frac{(\Delta^g_\varphi f)^2}{n+a} + g(\nabla^g\Delta^g_\varphi f, \nabla^g f) + K|\nabla^g f|^2 \tag{1.18}$$

if $a > 0$ and

$$\mathrm{Ric}^g + \mathrm{Hess}^g_\varphi - \frac{1}{a}\,\mathrm{d}\varphi\otimes\mathrm{d}\varphi \geq Kg. \tag{1.19}$$

Thus under assuming $a > 0$ and (1.19), applying also (1.15), we have

$$\frac{1}{2}\int_{M^n}\Delta^g_\varphi h\cdot|\nabla^g f|^2\,\mathrm{dvol}^g_\varphi \geq \int_{M^n} h\left(\frac{(\Delta^g_\varphi f)^2}{n+a} + g(\nabla^g\Delta^g_\varphi f, \nabla^g f) + K|\nabla^g f|^2\right)\mathrm{dvol}^g_\varphi \tag{1.20}$$

for all $f, h \in C^\infty_c(M^n)$ with $h \geq 0$. Moreover, as in the unweighted case, (1.20) characterizes (1.19).

Under assuming (1.19) with $a > 0$ and the completeness of (M^n, d^g), there is also a righ analytic/geometric comparison theory for $(M^n, \mathsf{d}^g, \mathrm{vol}^g_\varphi)$. Namely, the answer to the question (Q) is the LHS of (1.19) with a parameter $a > 0$.

For example in the case when $K = 0$, for a fixed $x \in M^n$, the ratio

$$\frac{\mathrm{vol}^g_\varphi(B_r(x))}{r^{n+a}} \tag{1.21}$$

is not increasing as a function of $r > 0$ which is a generalization of the Bishop-Gromov inequality as explained in (1.12) to the weighted case. However let us emphasize that a Bishop type inequality in the weighted case is not satisfed in general because (1.21) diverge as $r \to 0^+$ whenever $a > 0$.

2 Nonsmooth Ricci curvature

The main purpose of this section is to give a generalization of the observation above to the nonsmooth setting. Fix a complete separable metric space (X, d) and a locally finite Borel measure $\mathfrak{m}$ on X with full support. Such a triple $(X, \mathsf{d}, \mathfrak{m})$ is called a metric measure space. Define the Cheeger energy $\mathsf{Ch} : L^2(X, \mathfrak{m}) \to [0, \infty]$ by

$$\mathsf{Ch}(f) := \inf_{\|f_i - f\|_{L^2(X,\mathfrak{m})}\to 0}\left\{\liminf_{i\to\infty}\int_X \mathrm{lip}^2 f_i\,\mathrm{d}\mathfrak{m} :\ f_i \in \mathrm{Lip}_b(X, \mathsf{d}) \cap L^2(X, \mathfrak{m})\right\}, \tag{2.1}$$

where

$$\text{lip} f(x) := \begin{cases} \limsup_{y \to x} \frac{|f(y)-f(x)|}{\mathsf{d}(y,x)} & \text{if } x \in X \text{ is not isolated,} \\ 0 & \text{otherwise.} \end{cases}$$

Then, the Sobolev space $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is defined as the finiteness domain of Ch. By looking at the optimal sequence in (2.1) one can identify a canonical object $|\nabla f|$, called the minimal relaxed slope, which is local on Borel sets (namely $|\nabla f_1| = |\nabla f_2|$ for $\mathfrak{m}$-a.e. on $\{f_1 = f_2\}$) and provides an integral representation to Ch:

$$\mathsf{Ch}(f) = \int_X |\nabla f|^2 \, \mathrm{d}\mathfrak{m} \qquad \forall f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m}).$$

We are now in a position to introduce the definition of satisfying

"the Ricci curvature is bounded below by K and the dimension is bounded above by N"

for the metric measure space $(X, \mathsf{d}, \mathfrak{m})$ in a synthetic sense based on the discussions in the previous section. Since we usually call a metric measure space satisfying this condition an RCD(K, N) space, let us use this terminology from now on.

Definition 2.1 (RCD(K, N) space)**.** For any $K \in \mathbb{R}$ and any $N \in (0, \infty]$, a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is said to be an RCD(K, N) space, or an RCD space for short if the following four conditions are satisfied.

1. (Volume growth) There exist $x \in X$ and $C > 1$ such that $\mathfrak{m}(B_r(x)) \leq Ce^{Cr^2}$ holds for any $r > 0$.
2. (Inifinitesimally Hilbertian property) Ch is a quadratic form. In particular, the function
$$\langle \nabla f_1, \nabla f_2 \rangle := \lim_{\epsilon \to 0} \frac{|\nabla (f_1 + \epsilon f_2)|^2 - |\nabla f_1|^2}{2\epsilon}$$
provides a symmetric bilinear form on $H^{1,2}(X, \mathsf{d}, \mathfrak{m}) \times H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ with values in $L^1(X, \mathfrak{m})$, and
$$\mathcal{E}(f_1, f_2) := \int_X \langle \nabla f_1, \nabla f_2 \rangle \, \mathrm{d}\mathfrak{m}, \qquad \forall f_1, f_2 \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$$
defines a strongly local Dirichlet form.
3. (Sobolev-to-Lipschitz property) Any $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ with $|\nabla f| \leq 1$ for $\mathfrak{m}$-a.e. has an 1-Lipschitz representative.
4. (Bochner inequality) For any $f \in D(\Delta)$ with $\Delta f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ we have
$$\frac{1}{2} \int_X \Delta \varphi \cdot |\nabla f|^2 \, \mathrm{d}\mathfrak{m} \geq \int_X \varphi \left(\frac{(\Delta f)^2}{N} + \langle \nabla \Delta f, \nabla f \rangle + K |\nabla f|^2 \right) \mathrm{d}\mathfrak{m} \tag{2.2}$$
for any $\varphi \in D(\Delta) \cap L^\infty(X, \mathfrak{m})$ with $\varphi \geq 0$, $\Delta \varphi \in L^\infty(X, \mathfrak{m})$, where
$$D(\Delta) := \{ f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m}) \, : \text{there exists } h \in L^2(X, \mathfrak{m}) \text{ such that } \mathcal{E}(f, g) = -\int_X hg \, \mathrm{d}\mathfrak{m} \text{ for all } g \in H^{1,2}(X, \mathsf{d}, \mathfrak{m}) \}$$
and $\Delta f := h$ for any $f \in D(\Delta)$.

See for instance [A19, AGS14, AMS19, CM21, G13, G15, EKS15, LV09, St06a, St06b] for fundamental results on $\mathrm{RCD}(K,N)$ spaces. Let us emphasize that as seen in the previous section, for a complete Riemannian manifold (M^n, g) and $\varphi \in C^\infty(M^n)$, a metric measure space $(M^n, \mathsf{d}^g, \mathrm{vol}^g_\varphi)$ is an $\mathrm{RCD}(K,N)$ space if and only if $n \le N$ and

$$\mathrm{Ric}^g + \mathrm{Hess}^g_\varphi - \frac{\mathrm{d}\varphi \otimes \mathrm{d}\varphi}{N-n} \ge Kg, \tag{2.3}$$

where if $n = N$, then (2.3) is understood as that φ is constant with $\mathrm{Ric}^g \ge Kg$.

As a synthetic counterpart of the unweighted case, let us introduce the following notion defined in [DePhG18], where the reason why we use the terminology "noncollapsed" comes from a convergence theory of Riemannian manifolds (c.f. [CC97, CC00a, CC00b]). There is no serious misleading if we use "unweighted" instead of using "noncollapsed". However we follow the standard terminology in this field.

Definition 2.2 (Noncollapsed space)**.** Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space for some $K \in \mathbb{R}$ and some $N \in (0,\infty)$. We say that it is noncollapsed if $\mathfrak{m}$ coincides with the N-dimensional Hausdorff measure $\mathscr{H}^N$.

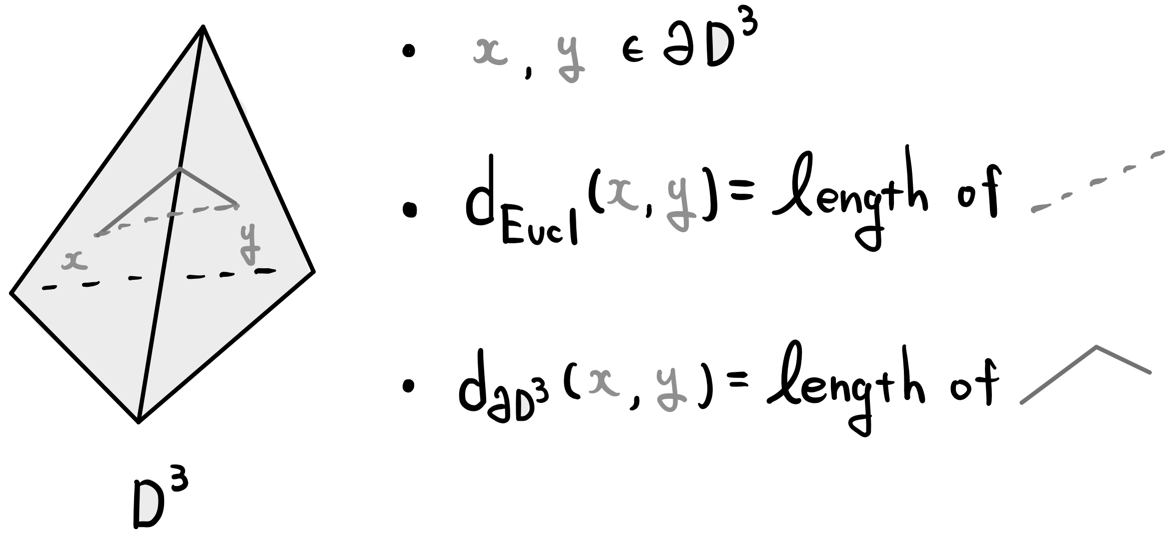
In order to obtain the structure results on $\mathrm{RCD}(K,N)$ spaces, Geometric measure theory provides very useful techniques. For example, the intrinsic version of Reifenberg's method established in [CC97] (originally in [R62]) allows us to prove that any noncollapsed $\mathrm{RCD}(K,N)$ space is a topological manifold of dimension N except for a closed $\mathfrak{m}$-null set.

Next let us introduce an important example.

Example 2.3 (Alexandrov space)**.** It is well-known from [BGP92] that there exists a synthetic notion of satisfying;

"sectional curvature bounded below by K"

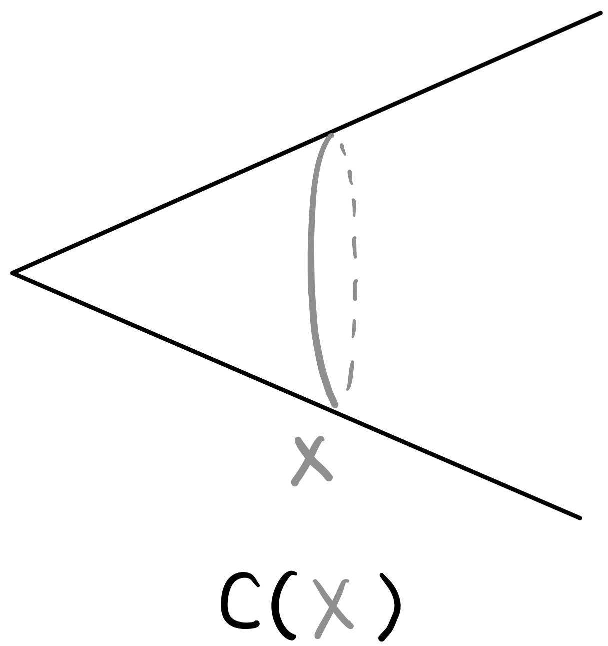
for metric space (X, d). Such a metric space satisfying this condition is called an Alexandrov space of curvature bounded below by K. Typical examples of Alexandrov spaces of nonnegative curvature include convex body D^n in $\mathbb{R}^n$ with the Euclidean distance $\mathsf{d}_{\mathrm{Eucl}}$ and its boundary ∂D^n with the intrinsic distance $\mathsf{d}_{\partial D^n}$.



It is proved in [P11, ZZ10] that any n-dimensional Alexandrov space of curvature bounded below by K equipped with $\mathscr{H}^n$ is an $\mathrm{RCD}(K(n-1), n)$ space. In particular $(D^n, \mathsf{d}_{\mathrm{Eucl}}, \mathscr{H}^n)$ is an $\mathrm{RCD}(0, n)$ space and $(\partial D^n, \mathsf{d}_{\partial D^n}, \mathscr{H}^{n-1})$ is an $\mathrm{RCD}(0, n-1)$ space.

It is also known from [KL16, LS22] that the metric structure (X, d) of an $\mathrm{RCD}(K, N)$ space $(X, \mathsf{d}, \mathfrak{m})$ with $N \leq 2$ is an Alexandrov space of curvature bounded below by K. In particular if the space is 1-dimensional, then the metric structure is isometric to a circle or an (possibly infinite) interval.

Let us mention that $\mathrm{RCD}(K, N)$ conditions are stable under suitable operations including, quotients by isometric actions, warped products, and limits with respect to measured Gromov-Hausdorff convergence. See [GKMS18, GMS13, K15]. In connection with this, let us discuss on harmonic functions on the (metric) cone $C(X)$ over an RCD space $(X, \mathsf{d}, \mathfrak{m})$, which is a typical example of warped products.



Firstly, let us mention that $\mathbb{R}^n$ is the cone over $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n; |x|_{\mathbb{R}^n} = 1\}$ and that any harmonic function on $\mathbb{R}^n$ with polynomial growth comes from eigenfunctions of $-\Delta^{g_{\mathbb{S}^{n-1}}}$ on $\mathbb{S}^{n-1}$, where $g_{\mathbb{S}^{n-1}}$ denotes the canonical Riemanniam metric on $\mathbb{S}^{n-1}$.

This observation can be generalized to $C(X)$ under assuming that $(X, \mathsf{d}, \mathfrak{m})$ has a positive Ricci curvature. In particular, if (X, d) is "small" in some sense, then there is no nontrivial linear growth harmonic function on $C(X)$, which plays a key role in the topic we will explain in the next section.

Finally let us end this section by menthioning that one of the main themes in this topic is to find nontrivial continuous functions on the space of all $\mathrm{RCD}(K, N)$ spaces as many as possible. We can find such functions via solutions of (nonlinear) PDEs (in particular, p-Laplace equations). See [AH17, GMS13, H17, H18].

See for instance [ABS19, BGHZ21, BMS22, BNS22, BPS21, G22, H20a, HP22, MS21, MS22] for recent developments on geometric analysis on $\mathrm{RCD}(K, N)$ spaces.

3 Failure of L^p-Calderón-Zygmund inequality

Let us go back to the smooth unweighted setting.

Let (M^n, g) be a complete (namely (M^n, d^g) is complete) Riemannian manifold of dimension n. For $1 \le p \le \infty$, we say that (M^n, g) satisfies an L^p-Calderón-Zygmund inequality if there exists $C > 0$ such that

$$\|\mathrm{Hess}_f^g\|_{L^p} \le C\left(\|f\|_{L^p} + \|\Delta^g f\|_{L^p}\right), \quad \forall f \in C_c^\infty(M^n). \tag{3.1}$$

Of course this inequality plays an important role to get regularity estimates on elliptic PDEs. Note that for the Euclidean space $(\mathbb{R}^n, g_{\mathbb{R}^n})$, it satisfies an L^p-Calderón-Zygmund inequality if and only if $1 < p < \infty$. See for instance [GT01, O62].

In connection with Ricci curvature lower bound, applying Bochner's inequality (1.8), (M^n, g) satisfies an L^2-Calderón-Zygmund inequality if the Ricci curvature is bounded below by a constant. Let us introduce a conjecture raised in [G16];

(C) For any $1 < p < \infty$, an L^p-Calderón-Zygmund inequality is satisfied if the space has nonnegative Ricci curvature.

Note that if we do not assume any curvature condition. then there is a couterexample for the validity of L^p-Calderón-Zygmund inequality for all $1 < p < \infty$. See [L21].

However some recently obtained examples ([HMRV21, MV21, DePhZ22]) show that the conjecture (C) is not true. In order to show this, the functional analysis with respect to measured Gromov-Hausdorff convergence of $\mathrm{RCD}(K, N)$ spaces ([AH17, AH18, GMS13, H15]) play a key role.

In this talk I will explain precisely how to apply the study of RCD spaces to this topic. Let us emphasize here that the validity of (3.1) is closely related to the density of $C_c^\infty(M^n)$ in Sobolev spaces $W^{k,p}(M^n)$ of higher order derivatives and that harmonic functions with polynomial growth on cones, as explained in the previous section, play a role in the proof. We refer [P20] for a survey on this topic.

References

[A19] L. Ambrosio: *Calculus, heat flow and curvature-dimension bounds in metric measure spaces*, Proceedings of the ICM 2018, Vol. 1, World Scientific, Singapore, (2019), 301–340.

[ABS19] L. Ambrosio, E. Bruè, D. Semola: *Rigidity of the 1-Bakry-Émery inequality and sets of finite perimeter in* RCD *spaces.*. Geom. Funct. Anal. **29** (2019), no. 4, 949—1001.

[AGS14] L. Ambrosio, N. Gigli, G. Savaré: *Metric measure spaces with Riemannian Ricci curvature bounded from below*, Duke Math. J. **163** (2014), 1405–1490.

[AH17] L. Ambrosio, S. Honda: *New stability results for sequences of metric measure spaces with uniform Ricci bounds from below*, in Measure Theory in Non-Smooth Spaces, 1–51, De Gruyter Open, Warsaw, 2017.

[AH18] L. Ambrosio, S. Honda: *Local spectral convergence in* $\mathrm{RCD}^*(K,N)$ *spaces*, Nonlinear Anal. **177** Part A (2018), 1–23.

[AMS19] L. Ambrosio, A. Mondino, G. Savaré: *Nonlinear diffusion equations and curvature conditions in metric measure spaces*, Mem. Amer. Math. Soc. **262** (2019), no. 1270.

[BGHZ21] C. Brena, N. Gigli, S. Honda, X. Zhu: *Weakly noncollapsed* RCD *spaces are strongly noncollpased.* arXiv:2110.02420.

[BMS22] E. Bruè, A. Mondino, D. Semola: *The metric measure boundary of spaces with Ricci curvature bounded below.* arXiv:2205.10609 (2022).

[BNS22] E. Bruè, A. Naber, D. Semola: *Boundary regularity and stability for spaces with Ricci bounds below.* Invent. Math. **228** (2022), 777–891.

[BPS21] E. Bruè, E. Pasqualetto, D. Semola: *Rectifiability of the reduced boundary for sets of finite perimeter over* $\mathrm{RCD}(K,N)$ *spaces.* to appear in J. Euor. Math. Soc.

[BS20] E. Brué, D. Semola: *Constancy of dimension for* $\mathrm{RCD}(K,N)$ *spaces via regularity of Lagrangian flows*, Comm. Pure and Appl. Math. **73** (2020), 1141–1204.

[BGP92] Y. Burago, M. Gromov and G. Perelman *A. D. Alexandrov spaces with curvature bounded below*, Uspekhi Mat. Nauk. **47**: 2 (1992) 3-51.

[CM21] F. Cavalletti, E. Milman: *The Globalization Theorem for the Curvature Dimension Condition.* Invent. Math. **226** (2021), no. 1, 1–137

[CC97] J. Cheeger, T. H. Colding: *On the structure of spaces with Ricci curvature bounded below. I*, J. Differential Geom. **46** (1997), 406–480.

[CC00a] J. Cheeger, T. H. Colding: *On the structure of spaces with Ricci curvature bounded below, II.* J. Differential Geom. **54** (2000), 13–35.

[CC00b] J. Cheeger, T. H. Colding: *On the structure of spaces with Ricci curvature bounded below, III.* J. Differential Geom. **54** (2000), 37–74.

[DePhG18] G. De Philippis, N. Gigli: *Non-collapsed spaces with Ricci curvature bounded below*, Journal de l'École polytechnique, **5** (2018), 613–650.

[DePhZ22] G. De Philippis, J.N.-Zimbrón: *The behavior of harmonic functions at singular points of* RCD *spaces*, manuscripta mathematica, **5** (2022), 14pp. https://doi.org/10.1007/s00229-021-01365-9

[EKS15] M. Erbar, K. Kuwada, K.-T. Sturm: *On the equivalence of the entropic curvature-dimension condition and Bochner's inequality on metric measure spaces*, Invent. Math. **201** (2015), 993–1071.

[GKMS18] F. Galaz-Garcia, M. Kell, A. Mondino, G. Sosa: *On quotients of spaces with Ricci curvature bounded below.* J. Func. Anal. Volume **275**, Issue 6, 15 (2018), 1368-1446.

[G13] N. Gigli: *The splitting theorem in non-smooth context*, arXiv:1302.5555.

[G15] N. Gigli: *On the differential structure of metric measure spaces and applications*, Mem. Amer. Math. Soc. **236** (2015), no. 1113.

[G18] N. Gigli: *Nonsmooth differential geometry – An approach tailored for spaces with Ricci curvature bounded from below*, Mem. Amer. Math. Soc. **251** (2018), no. 1196.

[G22] N. Gigli: *On the regularity of harmonic maps from* $\mathrm{RCD}(K,N)$ *to* $\mathrm{CAT}(0)$ *spaces and related results.* arXiv: 2204.04317.

[GMS13] N. Gigli, A. Mondino, G. Savaré: *Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows*, Proceedings of the London Mathematical Society **111** (2015), 1071–1129.

[GT01] D.Gilbarg, N.S.Trudinger: *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag GmbH Germany, part of Springer Nature 2001.

[G16] B. Güneysu: *Sequences of Laplacian cut-off functions*, J. Geom. Anal. **26** (2016), no. 1, 171–184.

[H15] S. Honda: *Ricci curvature and* L^p*-convergence*, J. Reine Angew Math. **705** (2015), 85–154.

[H17] S. Honda: *Spectral convergence under bounded Ricci curvature* J. Funct. Anal. **273** (2017), 1577–1662.

[H18] S. Honda: *Elliptic PDEs on compact Ricci limit spaces and applications* Mem. Amer. Math. Soc. **253** (2018), 1211.

[H20a] S. Honda: *New differential operator and non-collapsed* RCD *spaces*, Geom. Topol. **24** (2020), no. 4, 2127–2148.

[HMRV21] S. Honda, L. Mari, M. Rimoldi, G. Veronelli: *Density and non-density of* $C_c^\infty \hookrightarrow W^{k,p}$ *on complete manifolds with curvature bounds.* Nonlinear Anal. **211** (2021), Paper No. 112429, 26 pp.

[HP22] S. Honda, Y. Peng: *A note on the topological stability theorem from* RCD *spaces to Riemannian manifolds.* arXiv:2202.06500

[K15] C. Ketterer: *Cones over metric measure spaces and the maximal diameter theorem*, J. Math. Pures Appl. **103** (2015), 1228–1275.

[KL16] Y. Kitabeppu, S. Lakzian: *Characterization of low dimensional* $\mathrm{RCD}^*(K,N)$ *spaces*, Anal. Geom. Metr. Spaces, **4** (2016) 187–215.

[L21] S. Li: *Counterexamples to the* L^p*-Calderón-Zygmund estimate on open manifolds.* Ann. Global Anal. Geom. **57** (2021), no. 1, 61–70.

[LV09] J. Lott, C. Villani: *Ricci curvature for metric-measure spaces via optimal transport*, Ann. of Math. **169** (2009), 903–991.

[LS22] A. Lytchak, S. Stadler: *Ricci curvature in dimension 2.* to appear in J. Eur. Math. Soc.

[MV21] L. Marini, G. Veronelli: *The* L^p*-Calderón-Zygmund estimate on non-compact manifolds of positive curvature.* Ann. Global Anal. Geom. **60** (2021), no. 2, 253–267.

[MS21] A. Mondino, D. Semola, *Weak Laplacian bounds and minimal boundaries in non-smooth spaces with Ricci curvature lower bounds*, arXiv:2107.12344.

[MS22] A. Mondino, D. Semola, *Lipschitz continuity and Bochner-Eells-Sampson inequality for harmonic maps from* $\mathrm{RCD}(K,N)$ *spaces to* $\mathrm{CAT}(0)$ *spaces*, arXiv:2202.01590

[O62] D. Ornstein: *A non-equality for differential operators in the* L^1 *norm*, Arch. Rational Mech. Anal. **11** (1962), 40–49.

[P11] A. Petrunin: *Alexandrov meets Lott-Villani-Sturm*, Munster J. Math. **4** (2011), 53-64.

[P20] S. Pigola: *Global Calderón-Zygmund inequalities on complete Riemannian manifolds,*, arXiv:2011.03220v2.

[R62] E. R. Reifenberg: *Solution of the Plateau problem for m-dimensional surfaces of varying topological type*, Acta Math. **104** (1962) 1-92.

[St06a] K.-T. Sturm: *On the geometry of metric measure spaces, I*, Acta Math. **196** (2006), 65–131.

[St06b] K.-T. Sturm: *On the geometry of metric measure spaces, II*, Acta Math. **196** (2006), 133–177.

[ZZ10] H.-C. Zhang, X.-P. Zhu: *Ricci curvature on Alexandrov spaces and rigidity theorems*, Comm. Anal. Geom. **18** (2010), no. 3, 503–553.

HORIZONTALLY QUASICONVEX ENVELOPE IN THE HEISENBERG GROUP

XIAODAN ZHOU
OKINAWA INSTITUTE OF SCIENCE AND TECHNOLOGY GRADUATE UNIVERSITY

This talk is concerned with a PDE-based approach to the horizontally quasiconvex (h-quasiconvex for short) envelope of a given continuous function in the Heisenberg group. We provide a characterization for upper semicontinuous, h-quasiconvex functions in terms of the viscosity subsolution to a first-order nonlocal Hamilton-Jacobi equation. We also construct the corresponding envelope of a continuous function by iterating the nonlocal operator. One important step in our arguments is to prove the uniqueness and existence of viscosity solutions to the Dirichlet boundary problem for the nonlocal Hamilton-Jacobi equation. Applications of our approach to the h-convex hull of a given set in the Heisenberg group are discussed as well. This talk is based on a joint work with Antoni Kijowski (OIST) and Qing Liu (OIST).

1. Background and motivation

Convex analysis is a classical and fundamental topic with numerous applications in various fields of mathematics and beyond. In contrast to the extensive literature on convex analysis in the Euclidean space, less is known about the case in a general geometric setting such as sub-Riemannian manifolds. This work is mainly concerned with a PDE method to deal with a certain weak type of convexity for both sets and functions in the first Heisenberg group $\mathbb{H}$.

The Heisenberg group $\mathbb{H}$ is $\mathbb{R}^3$ endowed with the non-commutative group multiplication

$$(x_p, y_p, z_p) \cdot (x_q, y_q, z_q) = \left(x_p + x_q, y_p + y_q, z_p + z_q + \frac{1}{2}(x_p y_q - x_q y_p)\right),$$

for all $p = (x_p, y_p, z_p)$ and $q = (x_q, y_q, z_q)$ in $\mathbb{H}$. The differential structure of $\mathbb{H}$ is determined by the left-invariant vector fields

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \quad X_3 = \frac{\partial}{\partial z}.$$

One may easily verify the commuting relation $X_3 = [X_1, X_2] = X_1X_2 - X_2X_1$. Let

$$\mathbb{H}_0 = \{h \in \mathbb{H} : h = (x, y, 0) \ \text{ for } x, y \in \mathbb{R}\}.$$

For any $p \in \mathbb{H}$, the set

$$\mathbb{H}_p = \{p \cdot h : \ h \in \mathbb{H}_0\}$$

is called the horizontal plane through p. It is clear that $\mathbb{H}_p = \operatorname{span}\{X_1(p), X_2(p)\}$ for every $p \in \mathbb{H}$. See [9] for a detailed introduction of the Heisenberg group.

Our primary interest is to understand how to convexify a given bounded set in the Heisenberg group $\mathbb{H}$, that is, we aim to find the smallest convex set that contains the

[14] and later studied in [27, 6, 1] etc. A set $E \subset \mathbb{H}$ is said to be weakly h-convex if the horizontal segment connecting any two points in E lies in E; see also Definition 2.1. Hereafter we call such a set an h-convex set for simplicity of terminology.

There are several other types of set convexity in $\mathbb{H}$ defined with different kinds of convex combination of two points. One natural notion is based on geodesics in $\mathbb{H}$. A set $E \subset \mathbb{H}$ is said to be geodetically convex if, for every pair of points $p, q \in E$, the set E contains all geodesics joining p and q. This notion is known to be a very strong one; the geodetically convex hull of any three points that are not on the same geodesic in $\mathbb{H}$ has to be the whole group [26]. A different notion, called strong h-convexity [14] or twisted convexity [6], uses the dilation of group quotient to combine two points. It is still a quite strong notion, much stronger than the Euclidean convexity. In general a strongly h-convex hull of a bounded set consisting of more than two points could be unbounded [6]. We refer the reader to [27, 6] for related discussions on these convexity notions.

The notion of (weak) h-convexity is obviously weaker than the Euclidean convexity as well as the other aforementioned notions because of the restriction on horizontal segments. Such relaxation gives rise to unexpected properties. An h-convex set can even be disconnected, as pointed out in [7]. One simple example of h-convex sets is the union of two points $(0, 0, 1)$ and $(0, 0, -1)$ in $\mathbb{H}$. It is h-convex, because no horizontal segments exist to connect the points. A less trivial example of disconnected h-convex sets is presented in Example 2.2. Such an unusual character makes it challenging to find h-convex hull of a given set in $\mathbb{H}$. In contrast to the Euclidean situation, where one can generate the convex hull of a set by simply connecting every pair of points in the set with a segment, it is not straightforward at all how to describe and construct the h-convex hull of a set. This motivates us to search possible analytic methods to solve this problem.

A closely related problem we also intend to discuss is constructing the horizontally quasiconvex (or simply h-quasiconvex) envelope of a given function in an h-convex domain $\Omega \subset \mathbb{H}$. An h-quasiconvex function in Ω is defined to be a function whose sublevel sets are all h-convex. It is equivalent to saying that

$$u(w) \leq \max\{u(p), u(q)\} \tag{1.1}$$

holds for any $p \in \Omega$, $q \in \mathbb{H}_p \cap \Omega$ and $w \in [p, q]$. This notion is introduced in [30]. It is also studied in [7], where such functions are called weakly h-quasiconvex functions. We again suppress the term "weakly" in this paper to avoid redundancy. Similar to the case of sets, for any function in Ω, it is not trivial in general how one can find its h-quasiconvex envelope in a constructive way.

We remark that a stronger notion of function convexity in $\mathbb{H}$, called horizontal convexity, is introduced by [14] and [23]. Various properties and generalizations of such convex functions are discussed in [2, 18, 31, 24, 8, 3, 25] etc. The corresponding convex envelope and its applications to convexity properties of sub-elliptic equations are studied in [22].

2. Notations and preliminaries

We list several notations that are often used in the work. Throughout this work, $|\cdot|_G$ stands for the Korányi gauge, i.e., for $p = (x, y, z) \in \mathbb{H}$

$$|p|_G = \left((x^2 + y^2)^2 + 16z^2\right)^{\frac{1}{4}}.$$

The Korányi gauge induces a left invariant metric d_H on $\mathbb{H}$ with

$$d_H(p,q) = |p^{-1}\cdot q|_G \quad p,q\in\mathbb{H}.$$

We also use the right invariant metric $\tilde{d}_H$, defined by

$$\tilde{d}_H(p,q) = |p\cdot q^{-1}|_G, \quad p,q\in\mathbb{H}. \tag{2.1}$$

The associated distances between a point $p\in\mathbb{H}$ and a set $E\subset\mathbb{H}$ are respectively denoted by $d_H(p,E)$ and $\tilde{d}_H(p,E)$. For two sets $D,E\subset\mathbb{H}$, we write $d_H(D,E)$ and $\tilde{d}_H(D,E)$ to denote respectively the Hausdorff distances between D and E with respect to the metrics d_H and $\tilde{d}_H$, i.e, for $d=d_H$ or $d=\tilde{d}_H$,

$$d(D,E) = \max\left\{\sup_{p\in D} d(p,E),\ \sup_{p\in E} d(p,D)\right\}.$$

We denote by $B_r(p)$ the open gauge ball in $\mathbb{H}$ centered at $p\in\mathbb{H}$ with radius $r>0$, that is,

$$B_r(p) = \{q\in\mathbb{H} : |p^{-1}\cdot q|_G < r\},$$

while $\tilde{B}_r(p)$ represents the corresponding right-invariant metric ball.

Let δ_λ denote the non-isotropic dilation in $\mathbb{H}$ with $\lambda\geq 0$, that is, $\delta_\lambda(p) = (\lambda x,\lambda y,\lambda^2 z)$ for $p=(x,y,z)\in\mathbb{H}$. We write $\delta_\lambda(E)$ to denote the dilation of a given set $E\subset\mathbb{H}$, that is, $\delta_\lambda(E) = \{\delta_\lambda(p) : p\in E\}$.

Let us also go over the definition of h-convex sets. We restrict the original definition proposed in [14] for general Carnot groups to the case of $\mathbb{H}$.

Definition 2.1 (Definition 7.1 in [14])**.** We say, that a set $E\subset\mathbb{H}$ is h-convex if for every $p\in E$ and $q\in\mathbb{H}_p\cap E$, the horizontal segment $[p,q]$ joining p and q stays in E.

As pointed out in [14, Proposition 7.4], any gauge ball $B_R(p)$ with $p\in\mathbb{H}$ and $R>0$ is h-convex. The notion of h-convex sets is in fact very weak. There are numerous h-convex sets in $\mathbb{H}$ that are obviously not convex in the Euclidean sense.

Example 2.2 (Disconnected h-convex sets). Denote by $\pi(0,\rho)$ the planar open disk centered at the origin with radius $\rho>0$, i.e.,

$$\pi(0,\rho) := \{(x,y)\in\mathbb{R}^2 : x^2+y^2<\rho^2\}.$$

Let us consider a disconnected set $E = (\pi(0,r)\times\{0\})\cup(\pi(0,R)\times\{t\})$, where $r,R,t>0$ are given. Such a set E is h-convex under appropriate conditions on r and R. To see this, we take the horizontal plane

$$Z_t = \{(x,y,z)\in\mathbb{H} : z=t\}$$

and compute the distance between $q_t=(0,0,t)$ and $\mathbb{H}_p\cap Z_t$ for each point $p=(x_p,y_p,0)\in\pi(0,r)\times\{0\}$. It turns out that

$$d_H(q_t,\mathbb{H}_p\cap Z_t) = \frac{2t}{\sqrt{x_p^2+y_p^2}} \geq \frac{2t}{r}.$$

If $d_H(q_t,\mathbb{H}_p\cap Z_t)\geq R$, then none of the horizontal planes of points $p\in\pi(0,r)\times\{0\}$ in the lower disk will intersect the upper disk $\pi(0,R)\times\{t\}$. This means that E is h-convex if $2t\geq rR$. It is obvious that in general E is not connected and thus cannot be convex as a subset of $\mathbb{R}^3$. It is also clear that E is no longer h-convex if $2t<rR$.

Let us also recall from [7] the definition of h-quasiconvex functions in $\mathbb{H}$.

Definition 2.3 (Definition 4.3 in [7])**.** Suppose that $\Omega \subset \mathbb{H}$ is h-convex. We say, that a function $u : \Omega \to \mathbb{R}$ is h-quasiconvex if (1.1) holds for every $p \in \Omega$, $q \in \mathbb{H}_p \cap \Omega$ and $w \in [p, q]$. In other words, u is h-quasiconvex if for every $\lambda \in \mathbb{R}$ the sublevel set $\{w \in \Omega : u(w) \leq \lambda\}$ is h-convex.

2.1. **Viscosity characterization of h-quasiconvexity.** The following characterization of h-quasiconvexity is known for smooth functions [7, Theorem 4.5]. We provide a generalized result in the nonsmooth case by extending [5, Proposition 2.2] to the Heisenberg group.

Theorem 2.4 (Viscosity characterization of h-quasiconvexity)**.** *Let $\Omega \subset \mathbb{H}$ be open and h-convex and $u : \Omega \to \mathbb{R}$ be upper semicontinuous. Then, u is h-quasiconvex if and only if whenever there exist $p \in \Omega$ and $\varphi \in C^1(\Omega)$ such that $u - \varphi$ attains a maximum at p,*

$$\langle \nabla_H \varphi(p), (p^{-1} \cdot \xi)_h \rangle \leq 0 \quad \textit{holds for any } \xi \in \mathbb{H}_p \cap \Omega \textit{ satisfying } u(\xi) < u(p). \tag{2.2}$$

Remark 2.5. It is worth mentioning that, as in [7], we can also express the inner product term $\left\langle \nabla_H \varphi(p), (p^{-1} \cdot \xi)_h \right\rangle$ in (2.2) by $\langle \nabla \varphi(p), \xi - p \rangle$. This is possible because of the condition that $\xi \in \mathbb{H}_p$. We shall maintain the expression on the left hand side to emphasize possible generalization of our results in general Carnot groups, which is not elaborated in this paper.

Proof of Theorem 2.4. Let us prove the necessity of (2.2) by contradiction. Suppose, that u is upper semicontinuous h-quasiconvex function, $\varphi \in C^1(\Omega)$ is such that $u - \varphi$ attains a maximum at $p \in \Omega$, and there exists $\xi \in \mathbb{H}_p \cap \Omega$ with $u(\xi) < u(p)$ such that

$$\langle \nabla_H \varphi(p), (p^{-1} \cdot \xi)_h \rangle > 0.$$

Then, for $\lambda > 0$ small enough and $w = p \cdot \delta_\lambda(\xi^{-1} \cdot p)$ there holds $u(w) < u(p)$. Indeed, the directional derivative of φ at p in the direction $p^{-1} \cdot \xi$ is positive, and hence $\varphi(w) < \varphi(p)$ for $\lambda > 0$ small enough. Since $u - \varphi$ attains a maximum at p we obtain $u(p) > u(w)$. We conclude with $\xi \in \mathbb{H}_w$, $p \in [w, \xi]$ and $u(p) > \max\{u(w), u(\xi)\}$, which contradicts the h-quasiconvexity of u.

Now we are left with proving sufficiency of (2.2). Suppose that u is not h-quasiconvex. Then, without loss of generality there exists a point $\xi = (x_\xi, y_\xi, 0) \in \mathbb{H}_0$ such that

$$\max_{[0,\xi]} u > \max\{u(0), u(\xi)\}.$$

Denote by Z the set of maximizers of u on the segment $[0, \xi]$. By the upper semicontinuity of u there exists $R \in (0, |\xi|/4)$ small enough such that

$$\min\{d_H(0, Z), d_H(\xi, Z)\} > R$$

(i.e. Z is in the relative interior of $[0, \xi]$) and $u(q) < \max_{[0,\xi]} u$ for any $q \in B_R(0) \cup B_R(\xi)$. Let

$$\mathcal{C} := \{q \in \Omega : d_H(q, [0, \xi]) < R, 0 < \langle q, \xi \rangle < |\xi|^2\}$$

be a cylindrical neighborhood of the segment $[0, \xi]$. We define φ_n by

$$\varphi_n(p) = \frac{1}{n} \langle p, \xi \rangle + n \left((x_p y_\xi - y_p x_\xi)^2 + z_p^2 \right), \quad p \in \Omega.$$

As $n \to \infty$, $u - \varphi_n \to u$ pointwise in $[0, \xi]$ and $u - \varphi_n \to -\infty$ elsewhere in $\overline{\mathcal{C}}$. Then there exists a sequence $p_n = (x_{p_n}, y_{p_n}, z_{p_n}) \in \overline{\mathcal{C}}$ such that

$$\max_{\overline{\mathcal{C}}} (u - \varphi_n) = u(p_n) - \varphi_n(p_n),$$

which converges to a point in Z via a subsequence. Let us index the subsequence still by n for notational simplicity. Suppose that the subsequence $p_n \to (tx_\xi, ty_\xi, 0)$ as $n \to \infty$ for some $t \in (R/|\xi|, 1 - R/|\xi|)$. Let us consider $w_n = (x_{p_n}/t, y_{p_n}/t, z_{p_n}) \in \mathbb{H}_{p_n}$. Observing that

$$d_H(\xi, w_n) = |\xi^{-1} \cdot w_n|_G \to 0,$$

we get $u(w_n) < u(p_n)$ for n large enough.

Let us compute $\nabla_H \varphi_n(p_n)$ as follows:

$$\nabla_H \varphi_n(p_n) = \left(\frac{x_\xi}{n} + 2ny_\xi(x_{p_n} y_\xi - y_{p_n} x_\xi) - ny_{p_n} z_{p_n}, \frac{y_\xi}{n} - 2nx_\xi(x_{p_n} y_\xi - y_{p_n} x_\xi) + nx_{p_n} z_{p_n} \right).$$

Since

$$(p_n^{-1} \cdot w_n)_h = \left(\frac{1}{t} - 1 \right) (x_{p_n}, y_{p_n}),$$

we have

$$\langle \nabla_H \varphi_n(p_n), (p_n^{-1} \cdot w_n)_h \rangle = \left(\frac{1}{t} - 1 \right) \left(\frac{\langle p_n, \xi \rangle}{n} + 2n(x_{p_n} y_\xi - y_{p_n} x_\xi)^2 \right) > 0,$$

which contradicts (2.2). □

Theorem 2.4 amounts to saying that $u \in USC(\Omega)$ is h-quasiconvex if (3.1) holds in the viscosity sense, that is,

$$\sup\{\langle \nabla_H \varphi(p), (p^{-1} \cdot \xi)_h \rangle : \xi \in \mathbb{H}_p \cap \Omega,\ u(\xi) < u(p)\} \le 0$$

whenever there exist $p \in \Omega$ and $\varphi \in C^1(\Omega)$ such that $u - \varphi$ attains a maximum at p. As a standard remark in the viscosity solution theory, the maximum here can be replaced by a local maximum or a strict maximum.

2.2. **H-quasiconvex envelope.** In what follows, we introduce the h-quasiconvex envelope of a given function.

Definition 2.6 (Definition of h-quasiconvex envelope)**.** Let Ω be an h-convex domain in $\mathbb{H}$ and $f : \Omega \to \mathbb{R}$ be a given function. We say, that $Q(f)$ is the h-quasiconvex envelope of f if it is the greatest h-quasiconvex function majorized by f, that is

$$Q(f)(p) := \sup\{g(p) : g \le f \text{ and } g \text{ is h-quasiconvex}\}.$$

By definition, we can easily see that $Q(f)$ is monotone in f; namely, if $Q(f)$ and $Q(g)$, then $Q(f) \le Q(g)$ in Ω provided that $f \le g$ in Ω. Moreover, $Q(f)$ is also stable with respect to f.

Proposition 2.7 (Stability of h-quasiconvex envelope)**.** *Suppose that Ω is an h-convex domain in $\mathbb{H}$ and $f, g : \Omega \to \mathbb{R}$ are given functions. Assume that both $Q(f)$ and $Q(g)$ exist in Ω. Then there holds*

$$\sup_\Omega |Q(f) - Q(g)| \le \sup_\Omega |f - g|. \tag{2.3}$$

Proof. Let $M := \sup_{\overline{\Omega}} |f - g|$. Since $f - M \le g$ in $\overline{\Omega}$, by the monotonicity of Q, we get

$$Q(f - M) \le Q(g) \quad \text{in } \Omega. \tag{2.4}$$

Noticing that $Q(f) - M$ is h-quasiconvex and $Q(f) \le f$ in Ω, by Definition (2.6), we deduce that

$$Q(f) - M \le Q(f - M) \quad \text{in } \Omega,$$

which, by (2.4), yields

$$Q(f) - M \leq Q(g) \quad \text{in } \Omega.$$

Exchanging the roles of f and g, we conclude the proof of (2.3). □

Let us now discuss how to find the h-convex envelope of a given function. A straightforward method is to employ a convexification operator. For an h-convex domain $\Omega \subset \mathbb{H}$ and $f : \Omega \to \mathbb{R}$, let $T[f]$ be given by

$$T[f](w) = \inf\left\{\max\{f(p), f(q)\} : w \in [p,q],\ p \in \Omega,\ q \in \Omega \cap \mathbb{H}_p\right\}, \text{ for } w \in \Omega. \tag{2.5}$$

It is clear that $\inf_\Omega f \leq T[f] \leq f$ in Ω. Also, it is easily seen that $T[f] = f$ in Ω if and only if f is h-quasiconvex.

This operator is inspired by its Euclidean analogue, which is given by

$$T_{eucl}(w) = \inf\left\{\max\{f(p), f(q)\} : w \in [p,q], p \in \Omega, q \in \Omega\right\}, \quad w \in \mathbb{R}^3.$$

In the Euclidean case, where the quasiconvex envelope, written as $Q_{eucl}(f)$, satisfies

$$Q_{eucl}(f) = T_{eucl}[f]$$

in a bounded convex domain Ω. In contrast, the following example shows that in the Heisenberg group, in general $T[f]$ is not necessarily an h-quasiconvex function.

Example 2.8. Let $f : \mathbb{H} \to \mathbb{R}$ be defined as $f(p) = |1 - z^2|$ for $p = (x, y, z) \in \mathbb{H}$. One can compute directly to get, for $p = (x, y, z)$,

$$T[f](p) = \begin{cases} z^2 - 1 & |z| \geq 1, \\ 0 & |z| < 1 \text{ and } (x,y) \neq (0,0), \\ 1 - z^2 & |z| < 1 \text{ and } (x,y) = (0,0), \end{cases}$$

which fails to be quasiconvex. In fact, letting $p = (x, y, t)$, $q = (-x, -y, t)$ with $|t| < 1$, we see that $q \in \mathbb{H}_p$ and at $w = (0, 0, t) \in [p, q]$, we have

$$T[f](w) = 1 - t^2 > 0 = \max\{T[f](p), T[f](q)\}.$$

However, if we apply the operator one more time, then we have, for $p = (x, y, z) \in \mathbb{H}$,

$$T^2[f](p) = \begin{cases} z^2 - 1 & |z| \geq 1, \\ 0 & |z| < 1. \end{cases}$$

It is not difficult to see that $Q(f) = T^2[f]$ in $\mathbb{H}$. Indeed, Noticing that $T^2[f]$ is h-quasiconvex, by definition we have $T^2[f] \leq Q(f)$ in $\mathbb{H}$. On the other hand, the reverse inequality $Q(f) \leq T^2[f]$ can be obtained by applying the operator T twice to the inequality $Q(f) \leq f$.

It turns out that in general one can obtain the quasiconvex envelope by iterating the operator T. Such type of iteration is also used in [22] to construct the h-convex envelope of a given continuous function in the Heisenberg group.

Theorem 2.9 (Iterative scheme with direct convexification)**.** *Let Ω be an h-convex domain in $\mathbb{H}$. Suppose that f is bounded from below. Let T be the operator given by* (2.5)*. Then $T^n[f] \to Q(f)$ pointwise in Ω as $n \to \infty$.*

Proof. Notice that by the monotonicity of $T^n[f]$ in n and boundedness of f from below, the pointwise limit of $T^n[f]$ exists. Let us denote it by F, i.e.,

$$F := \lim_{n\to\infty} T^n[f].$$

Let us fix $\varepsilon > 0$, $p \in \Omega$, $q \in \Omega \cap \mathbb{H}_p$ and $w \in [p, q]$. For n sufficiently large, there holds

$$F(p) \geq T^n[f](p) - \varepsilon, \qquad F(q) \geq T^n[f](q) - \varepsilon.$$

Moreover, we have

$$\max\{T^n[f](p), T^n[f](q)\} \geq T^{n+1}[f](w) \geq F(w),$$

and therefore

$$\max\{F(p), F(q)\} \geq F(w) - \varepsilon.$$

Letting $\varepsilon \to 0$, we deduce that F is h-quasiconvex and thus $F \leq Q(f)$ in Ω.

On the other hand, for any $w \in \Omega$ and $p \in \Omega$, $q \in \Omega \cap \mathbb{H}_p$ such that $w \in [p, q]$, there holds

$$\max\{f(p), f(q)\} \geq \max\{Q(f)(p), Q(f)(q)\} \geq Q(f)(w).$$

It follows that $T[f] \geq Q(f)$ in Ω. We can iterate this argument obtain $T^n[f] \geq Q(f)$ in Ω for every n. Hence, sending $n \to \infty$, we are led to $F \geq Q(f)$ holds in Ω, which completes the proof. $\square$

As shown in Example 2.8, $T^n[f] = Q(f)$ may hold for a finite n. We do not know in general how many iterations one needs to run to obtain $Q(f)$. It would be interesting to find a condition to guarantee the finiteness of n.

3. Main results

Inspired by the Euclidean results in [5], in this work we provide a PDE-based approach to investigate h-convex hulls and h-quasiconvex envelopes. Our study starts from an improved characterization of h-quasiconvex functions, Theorem 2.4 that an upper semicontinuous (USC) function u is h-quasiconvex if and only if

$$\sup\{\left\langle \nabla_H \varphi(p), (p^{-1} \cdot \xi)_h \right\rangle : \xi \in \mathbb{H}_p \cap \Omega,\ u(\xi) < u(p)\} \leq 0 \tag{3.1}$$

whenever there exist $p \in \Omega$ and $\varphi \in C^1(\Omega)$ such that $u - \varphi$ attains a maximum at p.

Further developing the generalized characterization, we adopt an iterative scheme to find the h-quasiconvex envelope of a continuous function f, denoted by $Q(f)$, in a bounded h-convex domain Ω under a particular Dirichlet boundary condition. The iteration is implemented by solving a sequence of nonlocal Hamilton-Jacobi equations, where the Hamiltonian is given by the left hand side of (3.1), that is,

$$H(p, u(p), \nabla u(p)) = \sup\{\left\langle \nabla_H u(p), (p^{-1} \cdot \xi)_h \right\rangle : \xi \in \mathbb{H}_p \cap \Omega,\ u(\xi) < u(p)\}.$$

We briefly describe our scheme in what follows.

Let $f \in C(\overline{\Omega})$ be a given function satisfying

$$\begin{cases} f = K & \text{on } \partial\Omega, \quad (3.2) \\ f \leq K & \text{in } \overline{\Omega} \quad (3.3) \end{cases}$$

with $K \in \mathbb{R}$. This set of conditions resembles the coercivity assumption on f when $K > 0$ is taken large. It guarantees the existence of an h-quasiconvex function $\underline{f} \in C(\overline{\Omega})$ such that $\underline{f} \leq f$ in $\overline{\Omega}$ and $\underline{f} = K$ on $\partial\Omega$. This in turn implies the existence of $Q(f)$ taking the same boundary value.

We set $u_0 = f$ in Ω and take u_n $(n = 1, 2, \ldots)$ to be the unique viscosity solution of

$$u_n + H(p, u_n, \nabla_H u_n) = u_{n-1} \quad \text{in } \Omega \tag{3.4}$$

satisfying the same set of conditions as in (3.2)–(3.3), that is,

$$\begin{cases} u_n = K & \text{on } \partial\Omega, & (3.5)\\ u_n \leq K & \text{in } \overline{\Omega}. & (3.6)\end{cases}$$

It turns out that u_n is a nonincreasing sequence and converges uniformly to $Q(f)$ as $n \to \infty$. This is in fact our main theorem.

Theorem 3.1 (Iterative scheme for envelope)**.** *Suppose that Ω is a bounded h-convex domain in $\mathbb{H}$ and $f \in C(\overline{\Omega})$ satisfying* (3.2)–(3.3) *for some $K \in \mathbb{R}$. Let $Q(f) \in USC(\Omega)$ be the h-quasiconvex envelope of f in Ω. Let $u_0 = f$ in Ω and u_n be the unique solution of* (3.4) *satisfying* (3.5)(3.6) *for $n \geq 1$. Then $u_n \to Q(f)$ uniformly in $\overline{\Omega}$ as $n \to \infty$.*

Such type of nonlocal schemes is proposed in [5] in the Euclidean case for general Dirichlet data. We remark that in the Euclidean space a similar class of nonlocal equations depending on the level sets of the unknown is also studied for applications in geometric evolutions and front propagation [10, 11, 28, 19]. Although our PDE looks analogous to theirs, the well-posedness in the sub-Riemannian case is not straightforward at all. The main difference from the Euclidean case lies in an additional constraint that requires $\xi \in \mathbb{H}_p \cap \Omega$, which depends on the space variable p. This extra constraint brings us much difficulty in proving the comparison principle for (3.4). It is the coercivity-like setting (3.5)(3.6) that enables us to overcome the difficulty and obtain the uniqueness of solutions.

The existence of viscosity solutions, on the other hand, can be handled in a standard way by adapting Perron's method [13]. Once the sequence u_n is determined iteratively for all $n = 1, 2, \ldots$, Theorem 3.1 can be proved by applying a stability argument for viscosity solutions.

It is worth mentioning two variants of our main result above. Instead of adopting the restrictive setting (3.5)(3.6), one can alternatively impose a slightly stronger h-convexity assumption on Ω to solve (3.4) with general boundary data and obtain the scheme convergence as in Theorem 3.1. The strengthened convexity assumption demands Ω be h-convex up to its boundary; namely, the horizontal segment $[p, q) \in \Omega$ for any $p \in \Omega$ and $q \in \partial\Omega$. For bounded sets, it is easy to see that h-convex up to its boundary condition implies the set is h-convex; in general, it is not clear if there two notions of convexity of sets are equivalent.

Another possible modification of the scheme is to consider the maximal subsolution of (3.4) rather than its solutions at each step. Although we only get pointwise convergence of the scheme in this case, it allows us to avoid the uniqueness issue and to construct the h-quasiconvex envelopes for a general class of upper semicontinuous functions, even in an unbounded domain Ω.

As an application of our constructive methods for h-quasiconvex envelopes, we study the h-convex hull of a given bounded set in $\mathbb{H}$. A key ingredient is the so-called level set formulation, which plays an important role in the study of geometric evolutions [12, 16, 17]. We can apply the same idea to our problem, since the nonlocal Hamiltonian is actually a geometric operator, homogeneous in u. Suppose that E is a bounded open set in $\mathbb{H}$. Take a bounded h-convex domain Ω such that $E \subset \Omega$. We next choose a defining function $f \in C(\overline{\Omega})$ such that

$$E = \{p \in \Omega : f(p) < 0\} \tag{3.7}$$

and (3.2)(3.3) hold for some $K > 0$. It turns out that the h-convex hull of E coincides with the zero sublevel set of $Q(f)$, i.e.,

$$\mathrm{co}(E) = \{p \in \Omega : Q(f) < 0\}. \tag{3.8}$$

We remark that $\mathrm{co}(E)$ is independent of the choices of f and Ω. As long as (3.7) together with (3.2)(3.3) holds, $\mathrm{co}(E)$ obtained in (3.8) will not change. We state the second main result below.

Theorem 3.2 (Level set method for h-convex hull)**.** *Let $E \subset \mathbb{H}$ be a bounded open (resp., closed) set. Let $\Omega = B_R(0)$ with $R > 0$ large such that $\mathrm{co}(E) \subset\subset \Omega$. Assume that $f \in C(\mathbb{H})$ satisfies*

$$E = \{p \in \mathbb{H} : f(p) < 0\} \quad (\textit{resp.},\ \ E = \{p \in \mathbb{H} : f(p) \leq 0\}). \tag{3.9}$$

Assume also that there exists $K > 0$ such that

$$f \equiv K \quad \textit{in } \mathbb{H} \setminus B_R(0). \tag{3.10}$$

Let $Q(f)$ be the h-quasiconvex envelope of f. Then, $Q(f) \in C(\mathbb{H})$ and $Q(f)$ satisfies

$$Q(f) \equiv K \quad \textit{in } \mathbb{H} \setminus B_R(0) \tag{3.11}$$

and

$$\mathrm{co}(E) = \{p \in \Omega : Q(f)(p) < 0\} \quad (\textit{resp.},\ \ \mathrm{co}(E) = \{p \in \Omega : Q(f)(p) \leq 0\}). \tag{3.12}$$

This PDE approach leads us to a better understanding about h-convex hulls. One application is about the inclusion principle. By definition, it is easily seen that $\mathrm{co}(D) \subset \mathrm{co}(E)$ holds for any sets $D, E \subset \mathbb{H}$ satisfying $D \subset E$. We also establish a quantitative version of the inclusion principle in $\mathbb{H}$. For any bounded open (or closed) sets $D, E \subset \mathbb{H}$, we obtain

$$\inf\left\{\tilde{d}_H(p,q) : p \in \mathrm{co}(D), q \in \mathbb{H} \setminus \mathrm{co}(E)\right\} \geq \inf\left\{\tilde{d}_H(p,q) : p \in D, q \in \mathbb{H} \setminus E\right\}. \tag{3.13}$$

where $\tilde{d}_H$ denotes the right invariant gauge metric in $\mathbb{H}$; see (2.1). This property amounts to saying that taking h-convex hulls of two sets in $\mathbb{H}$, one contained in the other, does not reduce the shortest $\tilde{d}_H$ distance between their boundaries. If E contains the right invariant δ-neighborhood of D for some $\delta > 0$, then $\mathrm{co}(E)$ also contains the right invariant δ-neighborhood of $\mathrm{co}(D)$.

While such a result can be obtained comparatively easily in the Euclidean case, the proof is more involved in the Heisenberg group. Our proof is based on comparing the h-quasiconvex envelopes of defining functions of both sets combined with arguments involving sup-convolutions. It is not clear to us whether one can replace $\tilde{d}_H$ by the left invariant gauge metric d_H. This problem is related to the h-convexity preserving property for solutions of evolution equations in the Heisenberg group; see some partial results in [21, 22].

The details of the proofs and more results are available in [20]. The well-posedness of the nonlocal Hamilton-Jacobi equation, including the uniqueness and existence of viscosity solutions is presented in Section 3. Our PDE-based iterative scheme and the proof of Theorem 3.1 are given in Section 4. Applications of our results to the h-convex hull of a given open or closed set including the proof of Theorem 3.2 are given in Section 5.

Acknowledgments

The work of the speaker was supported by JSPS Grant-in-Aid for Research Activity Start-up (No. 20K22315) and JSPS Grant-in-Aid for Early-Career Scientists (No. 22K13947).

References

[1] G. Arena, A. O. Caruso, and R. Monti. Regularity properties of H-convex sets. *J. Geom. Anal.*, 22(2):583–602, 2012.

[2] Z. M. Balogh and M. Rickly. Regularity of convex functions on Heisenberg groups. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 2(4):847–868, 2003.

[3] M. Bardi and F. Dragoni. Subdifferential and properties of convex functions with respect to vector fields. *J. Convex Anal.*, 21(3):785–810, 2014.

[4] G. Barles, H. M. Soner, and P. E. Souganidis. Front propagation and phase field theory. *SIAM J. Control Optim.*, 31(2):439–469, 1993.

[5] E. N. Barron, R. Goebel, and R. R. Jensen. The quasiconvex envelope through first-order partial differential equations which characterize quasiconvexity of nonsmooth functions. *Discrete Contin. Dyn. Syst. Ser. B*, 17(6):1693–1706, 2012.

[6] A. Calogero, G. Carcano, and R. Pini. Twisted convex hulls in the Heisenberg group. *J. Convex Anal.*, 14(3):607–619, 2007.

[7] A. Calogero, G. Carcano, and R. Pini. On weakly H-quasiconvex functions on the Heisenberg group. *J. Convex Anal.*, 15(4):753–766, 2008.

[8] A. Calogero and R. Pini. Horizontal normal map on the Heisenberg group. *J. Nonlinear Convex Anal.*, 12(2):287–307, 2011.

[9] L. Capogna, D. Danielli, S. D. Pauls, and J. T. Tyson. *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*, volume 259 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2007.

[10] P. Cardaliaguet. On front propagation problems with nonlocal terms. *Adv. Differential Equations*, 5(1-3):213–268, 2000.

[11] P. Cardaliaguet. Front propagation problems with nonlocal terms. II. *J. Math. Anal. Appl.*, 260(2):572–601, 2001.

[12] Y. G. Chen, Y. Giga, and S. Goto. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *J. Differential Geom.*, 33(3):749–786, 1991.

[13] M. G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):1–67, 1992.

[14] D. Danielli, N. Garofalo, and D.-M. Nhieu. Notions of convexity in Carnot groups. *Comm. Anal. Geom.*, 11(2):263–341, 2003.

[15] F. Dragoni, N. Garofalo, and P. Salani. Starshapedness for fully non-linear equations in Carnot groups. *J. Lond. Math. Soc. (2)*, 99(3):901–918, 2019.

[16] L. C. Evans and J. Spruck. Motion of level sets by mean curvature. I. *J. Differential Geom.*, 33(3):635–681, 1991.

[17] Y. Giga. *Surface evolution equations*, volume 99 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 2006. A level set approach.

[18] C. E. Gutiérrez and A. Montanari. On the second order derivatives of convex functions on the Heisenberg group. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 3(2):349–366, 2004.

[19] T. Kagaya, Q. Liu, and H. Mitake. Quasiconvexity preserving property for fully nonlinear nonlocal parabolic equations. in preparation.

[20] A. Kijowski, Q. Liu and X. Zhou. Horizontally quasiconvex envelope in the Heisenberg group. arXiv:2205.01993

[21] Q. Liu, J. J. Manfredi, and X. Zhou. Lipschitz continuity and convexity preserving for solutions of semilinear evolution equations in the Heisenberg group. *Calc. Var. Partial Differential Equations*, 55(4):55:80, 2016.

[22] Q. Liu and X. Zhou. Horizontal convex envelope in the Heisenberg group and applications to subelliptic equations. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 22(4):2039–2076, 2021.

[23] G. Lu, J. J. Manfredi, and B. Stroffolini. Convex functions on the Heisenberg group. *Calc. Var. Partial Differential Equations*, 19(1):1–22, 2004.

[24] V. Magnani. Lipschitz continuity, Aleksandrov theorem and characterizations for H-convex functions. *Math. Ann.*, 334(1):199–233, 2006.

[25] V. Magnani and M. Scienza. Characterizations of differentiability for h-convex functions in stratified groups. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 13(3):675–697, 2014.
[26] R. Monti and M. Rickly. Geodetically convex sets in the Heisenberg group. *J. Convex Anal.*, 12(1):187–196, 2005.
[27] M. Rickly. *On questions of existence and regularity related to notions of convexity in Carnot groups. Ph.D. Thesis*. PhD thesis, Mathematisches Institut, Bern, 2005.
[28] D. Slepčev. Approximation schemes for propagation of fronts with nonlocal velocities and Neumann boundary conditions. *Nonlinear Anal.*, 52(1):79–115, 2003.
[29] H. M. Soner. Motion of a set by the curvature of its boundary. *J. Differential Equations*, 101(2):313–372, 1993.
[30] M.-B. Sun and X.-P. Yang. Some properties of quasiconvex functions on the Heisenberg group. *Acta Math. Appl. Sin. Engl. Ser.*, 21(4):571–580, 2005.
[31] C. Wang. Viscosity convex functions on Carnot groups. *Proc. Amer. Math. Soc.*, 133(4):1247–1253 (electronic), 2005.

(Xiaodan Zhou) Analysis on Metric Spaces Unit, Okinawa Institute of Science and Technology Graduate University, Okinawa 904-0495, Japan, `xiaodan.zhou@oist.jp`

STABLE BIG BANG SINGULARITY FORMATION IN GENERAL RELATIVITY

TODD A. OLIYNYK

Extended Abstract

Background: Since the 1920's, it has been known that the spatially homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes generically develop curvature singularities in the contracting time direction along spacelike hypersurfaces, known as *big bang singularities*, both in vacuum and for a wide range of matter models. For many years, it remained unclear if the big bang singularities were physically relevant. It was thought by some that big bang singularities were due to the unphysical assumption of spatial homogeneity and that they would disappear in non-homogenous spacetimes, or in other words, big bang singularities were *unstable* under nonlinear perturbations as solutions to the Einstein field equations. A partial resolution to this situation came in 1967 when Hawking established his singularity theorem that guarantees a cosmological spacetime will be geodesically incomplete for a large class of matter models and initial data sets, including highly anisotropic ones. While Hawking's singularity theorem guarantees that cosmological spacetimes are geodesically incomplete (i.e. at least one observer will experience something pathological at a finite time in the past) for a large class of initial data sets, it is silent on the cause of the geodesic incompleteness. It has been widely anticipated that the geodesics incompleteness is due to the formation of curvature singularities, and it is an outstanding problem in mathematical cosmology to rigorously establish the conditions under which this expectation is true and to understand the dynamical behaviour of cosmological solutions near singularities.

The well known BKL-conjecture [4, 18] suggests that singularities should generically be big bang singularities that are spacelike and oscillatory. However, currently there is little in the way of rigorous arguments beyond the work of [23] in the spatially homogeneous setting to support the BKL-picture. While it is worth noting that there is some numerical support [1, 8, 13, 14, 15, 16, 25], recent work on *spikes* [5, 7, 19, 20, 22] and *weak null singularities* [9, 21] indicate that the BKL-picture is incomplete. At the moment, understanding cosmological solutions near generic singularities seems out of reach. However, the situation improves considerably for cosmological spacetimes that exhibit *asymptotically velocity term dominated* (AVTD) behaviour [11, 17] near the singularity. By definition, AVTD singularities are special types of big bang type singularities that are *spacelike and non-oscillatory.*

Near FLRW stable big bang singularity formation. The Einstein-scalar field equations are given by[1]

$$\bar{R}_{ij} = 2\bar{\nabla}_i\phi\bar{\nabla}_j\phi, \tag{1}$$

$$\Box_{\bar{g}}\phi = 0, \tag{2}$$

where $\bar{R}_{ij}$ and $\bar{\nabla}_i$ are the Ricci curvature tensor and Levi-Civita connection of the spacetime metric $\bar{g} = \bar{g}_{ij}\theta^i \otimes \theta^j$, ϕ is the scalar field, and $\Box_{\bar{g}} = \bar{g}^{ij}\bar{\nabla}_i\bar{\nabla}_j$ is the wave operator. The first instance where the breakdown in the Hawking singularity theorem was shown for an open set of initial data to be due to a big bang singularity along which the curvature blows was in the article [24] by Rodnianski and Speck, which first appeared as a preprint in 2014. In particular Rodnianski and Speck established that (i) FLRW solutions to the Einstein-scalar field equations are stable in the contracting direction under small, nonlinear perturbations, (ii) the perturbed solutions all terminate in a big bang singularity characterized by curvature blow up, and (iii) that the big bang singularities exhibit AVTD (i.e. non-oscillatory) behaviour.

Conformal Einstein-scalar field equations. Rather than solving the Einstein-scalar field equations (1)-(2) directly, it turns out to be useful to replace the scalar field ϕ with a scalar field τ defined by

$$\phi = \sqrt{\frac{n-1}{2(n-2)}}\ln(\tau)$$

[1]We use lower case Latin indices, e.g. i, j, k, which run from 0 to $n-1$ to index frame indices. For example, if e_i is a local basis, then a vector field X can be expressed locally as $X = X^i e_i$. Letting θ^i denote the dual basis, i.e. $\theta^i(e_j) = \delta^i_j$, then a one form ω can be expressed locally as $\omega = \omega_i\theta^i$

and the physical spacetime metric $\bar{g}_{ij}$ with a (unphysical) conformal spacetime metric g_{ij} defined via

$$\bar{g}_{ij} = \tau^{\frac{2}{n-2}} g_{ij}.$$

In terms of these variables, the Einstein-scalar field equations can be expressed as

$$R_{ij} = \frac{1}{\tau}\nabla_i\nabla_j\tau, \tag{3}$$

$$\Box_g \tau = 0, \tag{4}$$

where R_{ij} and ∇_j are the Einstein tensor and Levi-Civita connection of the conformal metric g_{ij}, respectively. In the following, I will refer to these equations as the *conformal Einstein-scalar field equations*. It is important to note the conformal Einstein-scalar field equations (3)-(4) are completely equivalent to the Einstein-scalar field equations (1)-(2) and their solutions are in a one-to-one correspondence via the transformation

$$\left\{\bar{g}_{ij} = \tau^{\frac{2}{n-2}} g_{ij},\, \phi = \sqrt{\frac{n-1}{2(n-2)}}\ln(\tau)\right\}.$$

Initial value problem for the conformal Einstein-scalar field equations. Before further discussing the Rodnianski-Speck big bang stability result, I recall the initial value problem (IVP) for the Einstein field equations. Initial data is specified on a $(n-1)$-dimensional manifold Σ, which I will take to be diffeomorphic to a $(n-1)$-dimensional torus $\mathbb{T}^{n-1}$. Here, n is the spacetime dimension of which $n=4$ is the most important as it corresponds to reality. Now, Σ should be viewed as a spacelike hypersurface of an ambient spacetime manifold M, i.e. a spatial slice of M at a fixed moment of time. The picture one should have in mind is that there exists a time function[2]

$$t : M \longrightarrow \mathbb{R}$$

such that the spacetime manifold M can be decomposed as

$$M = \bigcup_{\tilde{t}\in(0,t_0]} t^{-1}(\{\tilde{t}\})$$

where

$$t^{-1}(\{\tilde{t}\}) \cong \mathbb{T}^{n-1} \quad \text{and} \quad \Sigma = t^{-1}(\{t_0\}).$$

The big bang singularity is then taken to occur along the hypersurface $t^{-1}(\{0\})$ that lies on the boundary of M.

Initial data for the conformal Einstein-scalar field equations consists of specifying the following fields on Σ:

$$\{g|_\Sigma = \mathring{g}, \mathrm{L}_n\, g|_\Sigma = \grave{g}, \tau|_\Sigma = \mathring{\tau}, \mathrm{L}_n\,\tau|_\Sigma = \grave{\tau}\}, \tag{5}$$

where $n = n^i e_i$ is a vector field satisfying $n(dt) = 1$ and L_n is the Lie derivative along n. However, not all of this initial data can be chosen freely. As is well known the initial data must satisfy the constraint equations[3]

$$n_i\Big(R_{ij} - \frac{1}{\tau}\nabla_i\nabla_j\tau\Big)\Big|_\Sigma = 0. \tag{6}$$

Remark 1. The *geometric initial data* on Σ consists of the fields $\{\mathtt{g}, \mathtt{K}, \mathring{\tau}, \grave{\tau}\}$ where[4] $\mathtt{g} = \mathtt{g}_{\Lambda\Omega} d\hat{x}^\Lambda \otimes d\hat{x}^\Omega$ is the *spatial metric* and $\mathtt{K} = \mathtt{K}_{\Lambda\Omega} d\hat{x}^\Lambda \otimes d\hat{x}^\Omega$ is the *second fundamental form*, which are determined from the initial data $\{\mathring{g}_{\mu\nu}, \grave{g}_{\mu\nu}, \mathring{\tau}, \grave{\tau}\}$ via

$$\mathtt{g}_{\Lambda\Omega} = \mathring{g}_{\Lambda\Omega} \quad \text{and} \quad \mathtt{K}_{\Lambda\Omega} = \frac{1}{2\mathtt{N}}(\grave{g}_{\Lambda\Omega} - 2\mathtt{D}_{(\Lambda}\mathtt{b}_{\Omega)}),$$

respectively. Here,

$$\mathtt{b}_\Lambda = \mathring{g}_{0\Lambda} \quad \text{and} \quad \mathtt{N}^2 = -\mathring{g}_{00} + \mathtt{b}^\Lambda \mathtt{b}_\Lambda$$

define the *shift* $\mathtt{b} = \mathtt{b}_\Lambda d\hat{x}^\Lambda$ and *lapse* $\mathtt{N}$, respectively, $\mathtt{D}_\Lambda$ denotes the Levi-Civita connection of the spatial metric $\mathtt{g}_{\Lambda\Omega}$, and we have used the inverse metric $\mathtt{g}^{\Lambda\Omega}$ of $\mathtt{g}_{\Lambda\Omega}$ to raise indices, e.g. $\mathtt{b}^\Lambda = \mathtt{g}^{\Lambda\Omega}\mathtt{b}_\Omega$. The importance of the geometric initial data is that it represents the physical part (i.e. non-gauge) of the initial data. Moreover, the gravitational constraint equations (6) can be formulated entirely in terms of the geometric initial data.

[2]The existence of such a time function needs to be established as part of any big bang stability proof.

[3]Here and in the following, the conformal metric $g = g_{ij}\theta^i \otimes \theta^j$ will be used to raise and lower the frame indices. For example, if $X = X^i e_i$ is a vector filed then $X_i = g_{ij}X^j$ denotes the components of the one-form $X^\flat = X_i\theta^i$ obtained from X using g. Similarity, given a one form $\omega = \omega_i\theta^i$, the inverse conformal metric $g^{-1} = g^{ij} e_i \otimes e_j$ $((g^{ij}) = (g_{ij})^{-1})$ can be used to construct the vector field $\omega^\sharp = \omega^i e_i$ where $\omega^i = g^{ij}\omega_j$.

[4]Lower case Greek indices, e.g. α, β, γ, will run from 0 to $n-1$ and will be used to label coordinate indices associated to a coordinate system (x^μ). Upper case Greek indices, e.g. Λ, Σ, Γ, will run from 1 to $n-1$ and will be used to label coordinate indices associated to the "spatial coordinates" (x^Λ), while the "time" coordinate x^0 will often be denoted by t.

Once suitable initial data (5) that satisfies the constraint equations (6) has been specified, then the conformal Einstein-scalar field equations can be solved, generally only "locally-in-time" for large initial data, to yield a solution $\{g, \tau\}$ on

$$M_{t_1,t_0} = \bigcup_{\tau\in(t_1,t_0]} t^{-1}(\{\tau\})$$

for some $t_1 \in (0, t_0]$. It is important to note that if $\psi : M_{t_1,t_0} \longrightarrow M_{t_1,t_0}$ is a diffeomorphism that leaves the initial data set fixed, then $\{\psi_* g, \psi_* \tau\}$ will also satisfy the conformal Einstein-scalar field equations, and as a consequence, solutions to the conformal Einstein-scalar field equations are not unique in the usual sense. However, they are unique in the geometric sense that the only source of non-uniqueness is due to the action by initial data preserving diffeomorphism on solutions. In this sense, the solutions are unique up to diffeomorphism, which is a mathematical manifestation of Einstein's physical notion of covariance.

Remark 2. While establishing the local-in-time existence of solutions to the IVP for the conformal Einstein-scalar field equations is now well understood, establishing the existence of solutions to the IVP that terminate in a big bang singularity is much more difficult. There are a number of serious technical challenges that need to be overcome in order to establish a result of this type. I will discuss some of these over the course of my talk.

Gauge reductions of the conformal Einstein-scalar field equations. From a PDE point of view, it is not possible to solve conformal Einstein-scalar field equations directly, since when they are expressed as a system of PDEs, they are not of a well-defined type due to the diffeomorphism invariance. To break the invariance, a *gauge* needs to be imposed and there is an infinite number of ways to do this. Choosing a gauge is an art and the choice of gauge is often crucial to establishing the existence of solutions to the Einstein equations, especially in the neighbourhood of singularities or globally to future or past. In this talk, I will only consider wave gauges in any detail. These gauges are defined by first introducing a background metric

$$\mathring{g} = \mathring{g}_{ij}\theta^i \otimes \theta^j$$

on M. Letting $\mathcal{D}_i$ denote the Levi-Civita connection of $\mathring{g}$ and introducing a vector field $X = X^k e_k$ via

$$X^k = \frac{1}{2} g^{ij} g^{kl} \big(2\mathcal{D}_i g_{jl} - \mathcal{D}_l g_{ij}\big),$$

the (wave gauge) reduced version of the conformal Einstein-scalar field equations (3)-(4) are defined by

$$-2R_{ij} + 2\nabla_{(i}X_{j)} = -\frac{2}{\tau}\nabla_i\nabla_j\tau, \tag{7}$$

$$\Box_g \tau = 0. \tag{8}$$

The virture of these equations over the conformal Einstein-scalar field equations is that they now form a coupled system of quasi-linear wave equations. In particular, the "local-in-time" existence and uniqueness of solutions to (7)-(8) in M_{t_1,t_0} generated from the initial data (5) can be obtained from standard hyperbolic PDE theory.

Now, in general, solutions of the reduced conformal Einstein-scalar field equations will not solve the conformal Einstein-scalar field equations. However, if the initial data (5) is chosen so that it satisfies

$$X^k|_\Sigma = 0 \tag{9}$$

in addition to the constraint equations (6), then it can be shown that solutions of reduced conformal Einstein-scalar field equations (7)-(8) will necessarily satisfy

$$X^k = 0 \quad \text{in } M_{t_1,t_0},$$

and by virtue of this, will also satisfy the conformal Einstein-scalar field equations (3)-(4).

Remark 3. It is always possible for a given choice of geometric initial data $\{\mathtt{g}, \mathtt{K}, \mathring{\tau}, \dot{\tau}\}$ to choose the remaining initial data, which is non-physical, so that the wave gauge constraints (9) are satisfied.

Remark 4. It is worth noting that Rodnianski and Speck do not use the conformal Einstein-scalar field equations in the proof of their stability result [24]; they work directly with the Einstein-scalar field equations. Also, they do not use a wave gauge. Instead, they employ a gauge that consists of a constant mean curvature time foliation along with a zero shift condition. One drawback of this gauge is that it is non-local and as a consequence the big bang stability results of Rodnianski-Speck cannot be localized; I will say more about this fact below.

FLRW and Kasner spacetimes. In the conformal picture, the *Kasner-scalar field spacetimes* are exact solutions $\{g = \breve{g}, \tau = \breve{\tau}\}$ of the conformal Einstein-scalar field equations (3)-(4) on the spacetime manifold $\mathbb{R}_{>0} \times \mathbb{T}^{n-1}$ that are defined by

$$\breve{g} = -t^{\breve{r}_0} dt \otimes dt + \sum_{\Lambda=1}^{n-1} t^{\breve{r}_\Lambda} dx^\Lambda \otimes dx^\Lambda \quad \text{and} \quad \breve{\tau} = t, \tag{10}$$

where the (x^Λ) are period coordinates on $\mathbb{T}^{n-1}$ and $t = x^0$ is a time coordinate. The constants $\breve{r}_\mu$ in the above expressions are known as the *Kasner exponents* and are defined by

$$\breve{r}_0 = \frac{1}{\breve{P}}\sqrt{\frac{2(n-1)}{n-2}} - \frac{2(n-1)}{n-2} \quad \text{and} \quad \breve{r}_\Lambda = \frac{1}{\breve{P}}\sqrt{\frac{2(n-1)}{n-2}}\breve{q}_\Lambda - \frac{2}{n-2}, \tag{11}$$

where $0 < \breve{P} \leq \sqrt{(n-2)/(2(n-1))}$ and the $\breve{q}_\Lambda$ satisfy the *Kasner relations*

$$\sum_{\Lambda=1}^{n-1} \breve{q}_\Lambda = 1 \quad \text{and} \quad \sum_{\Lambda=1}^{n-1} \breve{q}_\Lambda^2 = 1 - 2\breve{P}^2.$$

The physical Kasner metric is defined by $\bar{g}_{\mu\nu} = t^{\frac{2}{n-2}} \breve{g}_{\mu\nu}$ and its curvature invariants $\bar{R}_{\mu\nu}\bar{R}^{\mu\nu}$ and $\bar{R} = \bar{g}^{\mu\nu}\bar{R}_{\mu\nu}$ are given explicitly by

$$\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} = \left(\frac{n-1}{n-2}\right)^2 t^{-4\frac{n-1}{n-2} - 2\breve{r}_0} \quad \text{and} \quad \bar{R} = -\frac{n-1}{n-2} t^{-2\frac{n-1}{n-2} - \breve{r}_0},$$

respectively. It is clear from these formulas that $\bar{R}_{\mu\nu}\bar{R}^{\mu\nu}$ and $\bar{R} = \bar{g}^{\mu\nu}\bar{R}_{\mu\nu}$ blow-up along the spacelike hypersurface $\{0\} \times \mathbb{T}^{n-1}$, which defines the *Kasner big bang singularity.*

Kasner spacetimes where the constants q_Λ are all the same equal to

$$\breve{q}_\Lambda = \frac{1}{n-1}$$

coincide with *FLRW spacetimes.* In this situation, the conformal Kasner-scalar field solution (10) simplifies to

$$\breve{g} = -dt \otimes dt + \delta_{\Lambda\Omega} dx^\Lambda \otimes dx^\Omega \quad \text{and} \quad \breve{\tau} = t. \tag{12}$$

Kasner big bang stability. For a given $t_0 > 0$, each Kasner-scalar field solution (10) determines a corresponding geometric initial data set

$$\{\breve{\mathtt{g}}, \breve{\mathtt{K}}, \mathring{\breve{\tau}} = t_0, \grave{\breve{\tau}} = 1\}$$

on the initial hypersurface

$$\Sigma = \{t_0\} \times \mathbb{T}^{n-1}.$$

Definition 5. The big bang singularity of a Kasner-scalar field solution (10) is said to be *stable* if there exist constants $t_0 > 0$ and $\delta > 0$ such that for all geometric initial $\{\mathtt{g}, \mathtt{K}, \mathring{\tau}, \grave{\tau}\}$ data satisfying[5]

$$\|(\mathtt{g}, \mathtt{K}, \mathring{\tau}, \grave{\tau}) - (\breve{\mathtt{g}}, \breve{\mathtt{K}}, t_0, 1)\| < \delta$$

there exists a (geometrically) unique C^2 solution $\{g, \tau\}$ to the conformal Einstein-scalar field equations (3)-(4) on

$$M = \bigcup_{\tilde{t} \in (0, t_0]} \Sigma_{\tilde{t}}$$

such that the following hold:

(a) $\Sigma_{\tilde{t}} \cong \mathbb{T}^{n-1}$ for all $\tilde{t} \in (0, t_0]$ and $\Sigma_{t_0} = \Sigma$.
(b) The *time function* $t : M \longrightarrow \mathbb{R}$, defined by $t|_{\Sigma_{\tilde{t}}} = \tilde{t}$, is C^1 and extends continuously to the closure of M. Furthermore, dt is uniformly timelike on M, i.e. there exists a $c > 0$ such that $|dt|_g^2 < c$ on M.
(c) There exists at least one curvature invariant $\bar{\mathcal{R}}$ (e.g. $\bar{g}^{\mu\nu}\bar{R}_{\mu\nu}$, $\bar{R}^{\mu\nu}\bar{R}_{\mu\nu}$, $\bar{R}^{\mu\nu\alpha\beta}\bar{R}_{\mu\nu\alpha\beta}$, etc.) of the physical metric $\bar{g} = t^{\frac{2}{n-2}} g$ that is unbounded everywhere along the hypersurface $\Sigma_0 = t^{-1}(\{0\})$, i.e.

$$\inf_M (\bar{\mathcal{R}}) = \infty.$$

The stability result [24] of Rodnianski and Speck can now be stated informally as follows.

Theorem 6. *For spacetime dimensions $n \geq 4$, the big bang singularity of the FLRW-scalar field solution* (12) *is stable.*

[5]Here, the norm $\|\cdot\|$ should be interpreted as a H^s Sobolev norm on Σ with $s > 0$ sufficiently large.

Heuristic arguments [2, 3, 10] suggest that the big bang singularity of a Kasner-scalar field solution will be stable and exhibit AVTD behaviour (i.e. non-oscillatory behaviour near the singularity) if and only if the constants $\breve{q}_\Lambda$ from the Kasner exponents (11) satisfy the inequality

$$\max\{\, \breve{q}_\Lambda + \breve{q}_\Sigma - \breve{q}_\Gamma \,|\, \Lambda, \Sigma, \Gamma \in \{1, 2, \dots, n-1\},\ \Lambda < \Sigma \,\} < 1. \tag{13}$$

It is worth noting that if this condition is not satisfied, then it is expected that the behaviour of the metric and matter field near the singularity will be exhibit a chaotic, oscillatory behaviour, known as *mixmaster behaviour*, in agreement with the BKL conjecture. The analysis of these spacetimes remains out of reach with current techniques.

Recently in [12], Rodnianski, Speck and Fournodavlos have generalized the FLRW-stablity result from Theorem 6 to establish the stability of the big bang singularity of Kasner-scalar field solutions whose Kasner exponents satisfy (13).

Theorem 7. *For spacetime dimensions $n \geq 4$, the big bang singularity of a Kasner-scalar field solution* (10) *is stable provided its exponents satisfy* (13).

Remark 8. More detailed behaviour of the perturbed solutions near their big bang singularities has been derived in [12, 24]. In particular, the AVTD behaviour of the singularities has been established and blow-up rates for the curvature near the singularities are computed.

Localized big bang stability. One important question that is not addressed by Theorems 6 and 7 is whether or not big bang stability is a local property. By a local property, we mean that if the geometric initial data is changed in a localized region of the initial hypersurface Σ, will this change lead to a corresponding local change on the singular hypersurface $\Sigma_0 = t^{-1}(\{0\})$ or will all of Σ_0 be affected? One reason to believe that the effects might be local is that there exist gauges, e.g. wave gauges, that lead to reduced Einstein-scalar field equations that are a hyperbolic system of PDEs. The hyperbolicity of the reduced equations implies a finite propagation speed, and consequently, that local changes in the initial data should lead to local changes in the solutions at later times.

A precise statement of localized big bang stability is as follows.

Definition 9. The big bang singularity of a Kasner-scalar field solution (10) is said to be *locally stable*, if for any open, connected set $\tilde{\Sigma} \subset\subset \Sigma$, there exist constants $t_0 > 0$ and $\delta > 0$ such that for all geometric initial data $\{\mathtt{g}, \mathtt{K}, \mathring{\tau}, \dot{\tau}\}$ on $\tilde{\Sigma}$ satisfying[6]

$$\|(\mathtt{g}, \mathtt{K}, \mathring{\tau}, \dot{\tau}) - (\breve{\mathtt{g}}, \breve{\mathtt{K}}, t_0, 1)\| < \delta$$

there exists a (geometrically) unique C^2 solution $\{g, \tau\}$ to the conformal Einstein-scalar field equations (3)-(4) on

$$\tilde{M} = \bigcup_{\tilde{t} \in (0, t_0]} \tilde{\Sigma}_{\tilde{t}}$$

such that the following hold:

(a) $\tilde{\Sigma}_{\tilde{t}} \cong \tilde{\Sigma}$ for all $\tilde{t} \in (0, t_0]$ and $\tilde{\Sigma}_{t_0} = \tilde{\Sigma}$.

(b) The *time function* $t : \tilde{M} \longrightarrow \mathbb{R}$, defined by $t|_{\tilde{\Sigma}_{\tilde{t}}} = \tilde{t}$, is C^1 and extends continuously to the closure of $\tilde{M}$. Furthermore, dt is uniformly timelike on $\tilde{M}$, i.e. there exists a $c > 0$ such that $|dt|^2_g < c$ on $\tilde{M}$.

(c) There exists at least one curvature invariant $\bar{\mathcal{R}}$ (e.g. $\bar{g}^{\mu\nu}\bar{R}_{\mu\nu}$, $\bar{R}^{\mu\nu}\bar{R}_{\mu\nu}$, $\bar{R}^{\mu\nu\alpha\beta}\bar{R}_{\mu\nu\alpha\beta}$, etc.) of the physical metric $\bar{g} = t^{\frac{2}{n-2}} g$ that is unbounded everywhere along the hypersurface $\tilde{\Sigma}_0 = t^{-1}(\{0\})$, i.e.

$$\inf_{\tilde{M}}(\bar{\mathcal{R}}) = \infty.$$

The main reason that the stability Theorems 6 and 7 do not address the question of local stability is that they were established using a gauge that lead to a reduced system with an infinite propagation speed. The infinite propagation speed was introduced into the reduced system through the use of a time function t, known as a *CMC (Constant Mean Curvature) foliation*, that is chosen according to

$$t = -\frac{1}{\mathrm{tr}\mathtt{K}}$$

where $\mathrm{tr}\mathtt{K} = \mathtt{g}^{\Lambda\Gamma}\mathtt{K}_{\Lambda\Gamma}$ is the mean curvature of the hypersurfaces $\Sigma_{\tilde{t}} = t^{-1}\{\tilde{t}\}$. This choice of gauge played a critical role in the proofs of Theorems 6 and 7, and remained an open question if this choice of gauge was essential to establishing big bang stability. If it turned out that it was, then big bang singularities would not be locally stable.

[6]Here, the norm $\|\cdot\|$ should be interpreted as a H^s Sobolev norm on $\tilde{\Sigma}$ with $s > 0$ sufficiently large.

In the article [6], my collaborator Florian Beyer and I revisited the big bang stability question using a wave gauge and a different time function that led to a reduced system with a finite propagation speed. Using this new reduction, we were able to prove a localized version of the Rodnianski-Speck FLRW big bang stability theorem, Theorem 6.

Theorem 10. *For the spacetime dimensions $n \geq 3$, the big bang singularity of the FLRW-scalar field solution* (12) *is locally stable.*

Florian and I are currently investigating the possibility of establishing a localized version of the Fournodavlos-Rodnianski-Speck Kasner big bang stability theorem, Theorem 7.

Plan for the talk. I will begin the talk by introducing the FLRW and Kasner solutions of the Einstein-scalar field equations, which are exact, spatially homogeneous solutions that play a distinguished role in the analysis of big bang singularities. After briefly providing context for the FLRW and Kasner solutions in the historical development of the field of cosmology, I will define what it means for a FLRW/Kasner big bang singularity to be *stable*. With this notion in hand, I will then discuss in some detail the recent influential FLRW and Kasner big bang stability proofs of Rodnianski-Speck and Fournodavlos-Rodnianski-Speck. Once aspect of these stability results that I will pay particular attention to is their global nature. To conclude the talk, I will discuss some recent work done in collaboration with Florian Beyer where we improve the Rodnianski-Speck FLRW big bang stability result by establishing that the FRLW big bang is locally stable, which is a significantly stronger notion of stability with important physical consequences that I will briefly discuss. I will make an effort to identify the essential role that PDE techniques play in our localized FLRW big bang stability result. Time permitting, I will also briefly discuss open questions and future directions for research.

References

1. L. Andersson, H. van Elst, W. C. Lim, and C. Uggla, *Asymptotic Silence of Generic Cosmological Singularities*, Phys. Rev. Lett. **94** (2005), no. 5, 051101.
2. J. Barrow, *Quiescent cosmology*, Nature **272** (1978), 211–215.
3. V. A. Belinskii and I. M. Khalatnikov, *Effect of scalar and vector fields on the nature of the cosmological singularity*, Ž. Èksper. Teoret. Fiz. **63** (1972), 1121–1134.
4. V. A. Belinskii, I. M. Khalatnikov, and E. M. Lifshitz, *Oscillatory approach to a singular point in the relativistic cosmology*, Adv. Phys. **19** (1970), no. 80, 525–573.
5. Beverly K Berger and Vincent Moncrief, *Numerical investigation of cosmological singularities*, Physical Review D **48** (1993), no. 10, 4676–4687.
6. F. Beyer and T.A. Oliynyk, *Localized big bang stability for the Einstein-scalar field equations*, preprint [arXiv:2112.07730], 2021.
7. A A Coley and W C Lim, *Spikes and matter inhomogeneities in massless scalar field models*, Classical and Quantum Gravity **33** (2016), no. 1, 015009.
8. Joshua Curtis and David Garfinkle, *Numerical simulations of stiff fluid gravitational singularities*, Physical Review D **72** (2005), no. 6, 064003.
9. Mihalis Dafermos and Jonathan Luk, *The interior of dynamical vacuum black holes I: The C^0-stability of the Kerr Cauchy horizon*, (2017), Preprint. arXiv:1710.01722.
10. T. Damour, M. Henneaux, A. D. Rendall, and M. Weaver, *Kasner-Like Behaviour for Subcritical Einstein-Matter Systems*, Ann. Henri Poincaré **3** (2002), 1049–1111.
11. D. M. Eardley, E. Liang, and R. K. Sachs, *Velocity-Dominated Singularities in Irrotational Dust Cosmologies*, J. Math. Phys. **13** (1972), no. 1, 99.
12. G. Fournodavlos, I. Rodnianski, and J. Speck, *Stable Big Bang formation for Einstein's equations: The complete sub-critical regime*, (2020), Preprint. arXiv:2012.05888.
13. David Garfinkle, *Harmonic coordinate method for simulating generic singularities*, Physical Review D **65** (2002), no. 4, 044029.
14. ———, *Simulations of Generic Singularities in Harmonic Coordinates*, The Conformal Structure of Space-Time: Geometry, Analysis, Numerics, Springer, 2002, pp. 349–358.
15. ———, *Numerical Simulations of Generic Singularities*, Physical Review Letters **93** (2004), no. 16, 6.
16. ———, *Numerical simulations of general gravitational singularities*, Classical and Quantum Gravity **24** (2007), no. 12, S295–S306.
17. J. Isenberg and V. Moncrief, *Asymptotic behavior of the gravitational field and the nature of singularities in Gowdy spacetimes*, Ann. Phys. **199** (1990), no. 1, 84–122.
18. E. M. Lifshitz and I. M. Khalatnikov, *Investigations in relativistic cosmology*, Adv. Phys. **12** (1963), no. 46, 185–249.
19. Woei Chet Lim, *New explicit spike solutions—non-local component of the generalized Mixmaster attractor*, Classical and Quantum Gravity **25** (2008), no. 4, 045014.
20. Woei Chet Lim, Lars Andersson, David Garfinkle, and Frans Pretorius, *Spikes in the mixmaster regime of G_2 cosmologies*, Physical Review D **79** (2009), no. 12, 123526.
21. Jonathan Luk, *Weak null singularities in general relativity*, Journal of the American Mathematical Society **31** (2018), 1–63.
22. Alan D Rendall and Marsha Weaver, *Manufacture of Gowdy spacetimes with spikes*, Classical and Quantum Gravity **18** (2001), no. 15, 2959–2975.
23. H. Ringstöm, *The Bianchi IX attractor*, Ann. Henri Poincaré **2** (2001), 405–500.

24. I. Rodnianski and J. Speck, *Stable Big Bang formation in near-FLRW solutions to the Einstein-scalar field and Einstein-stiff fluid systems*, Sel. Math. New Ser. **24** (2018), no. 5, 4293–4459.
25. M. Weaver, B. K. Berger, and J. Isenberg, *Oscillatory Approach to the Singularity in Vacuum T^2 Symmetric Spacetimes*, The Ninth Marcel Grossmann Meeting (Vahe G. Gurzadyan, Robert T. Jantzen, and Remo Ruffini, eds.), December 2002, pp. 1011–1012.

School of Mathematical Sciences, 9 Rainforest Walk, Monash University, VIC 3800, Australia
Email address: todd.oliynyk@monash.edu

Small energy stabilization of 1D Klein-Gordon equation with potential

Masaya Maeda

1 Introduction and main theorem

In this talk, we will discuss a partial extension of the result of Bambusi-Cuccagna [1] (which is a generalization of the result by Soffer-Weinstein [11]) to a one dimensional setting. The equation which we will be studying is the nonlinear Klein-Gordon equation (NLKG) with potential in one dimension:

$$\partial_t^2 u + L_1 u + f(u), \ u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \tag{1.1}$$

where

$$L_1 = -\partial_x^2 + V + m,$$

the potential V is a Schwartz function and f is a smooth function with $f(0) = f'(0) = 0$.

We will assume that the Schrödinger operator L_1 has N eigenvalues

$$(0 <)\lambda_1^2 < \cdots < \lambda_N^2 (< m^2).$$

Notice that by the eigenvalues, corresponding linear Klein-Gordon equation will have periodic or quasi-periodic solutions with the form

$$\sum_{j=1}^{N} a_j \cos(\lambda_j t + \theta_j)\phi_j,$$

where ϕ_j are the eigenfunction of L_1 associated to λ_j^2. However, for the nonlinear problem with small initial data we can show that there are no such solutions.

Theorem 1.1 (Cuccagna-M.-Scrobogna, in preparation)**.** *Under several assumptions (exponential decay of V, nonlinear Fermi Golden Rule condition, repulsivity of the potential after Darboux transform etc.), there exists $\delta_0 > 0$ s.t. if $\|(u_0, u_1)\|_{H^1 \times L^2} < \delta_0$, then for any $R > 0$, we have*

$$\lim_{t \to \infty} \|(u(t), \partial_t u(t))\|_{H^1 \times L^2(-R,R)} = 0,$$

where u is the solution of (1.1) *with $(u(0), \partial_t u(0)) = (u_0, u_1)$.*

Remark 1.2. The Fermi Golden Rule assumption is given in Assumption 2.2 below.

By Theorem 1.1, even thought small solutions behave like linear solution for long time, one see that the asymptotic behavior of the nonlinear solutions is completely different. Such phenomena was first studied by Sigal [10], who showed that there are no periodic (in time) solution exists in some sense. The dynamical result, which is to show that all solutions locally decay and therefore no periodic solution can exists, was first given by Soffer-Weinstein [11]. The result of Sigal and Soffer-Weinstein were rather restricted in the sense they were considering only the case $N = 1$ and $m < 2\lambda_1$. For general spectral configuration, Bambusi-Cuccagna [1] proved Theorem 1.1 for the 3 dimensional case. In their work, the remainder is proved to scatter. For related works and the development in this field, we refer [2] and reference therein.

The difficulty in the one dimensional problem lie in the fact that the remainder is not expected to scatter. That is, for example if $f(u) = u^p$ with $p = 2$ or 3, it is a long range nonlinearity and one do not expect that the remainder behaves as a free wave. Thus, the strategy used in the 3 dimensional case, which is to use Strichartz estimate, seems to be hopeless.

Instead of relying on Strichartz estimate, for the proof of Theorem 1.1, we use virial estimate following the works by Kowalzyck-Martel-Munoz(-Van den Bosch) [5, 6, 7, 8, 9] combined with the notion of *refined profile* developed for the study of solitons of nonlinear Schrödigner equations having many internal modes [3, 4].

2 Strategy of the proof of Theorem 1.1

In the following, we write

$$\mathbf{u} = (u, \partial_t u).$$

As usual for asymptotic stability problems, we introduce a coordinate in the $H^1 \times L^2$ neighborhood of the origin such as

$$\mathbf{u} = \boldsymbol{\phi}[\mathbf{z}] + \boldsymbol{\eta},$$

where $\mathbf{z} \in \mathbb{C}^N$ and $\boldsymbol{\eta} \in H^1 \times L^2$ satisfies some orthogonality condition. The function $\boldsymbol{\phi}[\mathbf{z}]$ is the refined profile which is like a linear combination of periodic solutions

$$\boldsymbol{\phi}[\mathbf{z}] \sim \sum_{j=1}^{N} \left(z_j \boldsymbol{\phi}_j + \overline{z_j \boldsymbol{\phi}_j} \right),$$

where $\boldsymbol{\phi}_j = (\phi_j, \mathrm{i}\lambda_j \phi_j)$ but with higher order corrections to encode the nonlinear interaction.

By such coordinate, the problem is to estimate $\mathbf{z}$ and $\boldsymbol{\eta}$. The main estimate will be as follows:

$$\sum_{\substack{(m_1,\cdots,m_N)\in\mathbb{N}^N_{\le \lambda_1^{-1}(m+1)+1}, \\ \sum_{j=1}^N \lambda_j m_j > m}} \|z_1^{m_1}\cdots z_N^{m_N}\|_{L^2(\mathbb{R})} + \|e^{-\kappa|x|}\boldsymbol{\eta}\|_{L^2(\mathbb{R},H^1\times L^2)} \le C\|(u_0,u_1)\|_{H^1\times L^2}, \tag{2.1}$$

where κ and C are some constants.

To obtain (2.1), we need three estimates, which are

1. The 1st virial estimate,
2. The 2nd virial estimate,
3. The Fermi Golden Rule estimate.

Here, we will only explain the 1st viral estimate and the Fermi Golden Rule estimate. In the following, we will express the 1st term of (2.1) as $\sum_{\mathbf{m}\in\mathbf{R}_m} \|\mathbf{z}^{\mathbf{m}}\|_{L^2(\mathbb{R})}$.

2.1 1st virial estimate

We will consider the norms

$$\|\boldsymbol{\eta}\|_{\boldsymbol{\Sigma}_A} := \left\|\exp\left(\frac{2}{A}|x|\right)\eta_1'\right\|_{L^2} + A^{-1}\left\|\exp\left(\frac{2}{A}|x|\right)\boldsymbol{\eta}\right\|_{L^2} \text{ and}$$
$$\|\boldsymbol{\eta}\|_{\boldsymbol{L}^2_{-\kappa}} := \|\exp(\kappa|x|)\boldsymbol{\eta}\|_{L^2},$$

where $A \gg 1$ will be chosen after all the estimates are done.

We fix an even function $\chi \in C_0^\infty(\mathbb{R}, [0,1])$ satisfying

$$1_{[-1,1]} \le \chi \le 1_{[-2,2]} \text{ and } x\chi'(x) \le 0$$

and we set

$$\chi_A := \chi(\cdot/A). \tag{2.2}$$

For the above χ, we set

$$\zeta_A(x) := \exp\left(-\frac{|x|}{A}(1-\chi(x))\right),$$
$$\varphi_A(x) := \int_0^x \zeta_A^2(y)\,dy \text{ and}$$
$$S_A := \frac{1}{2}\varphi_A' + \varphi_A\partial_x.$$

We will consider the virial functional

$$\mathcal{I}_{1\mathrm{st}} := \frac{1}{2}\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\boldsymbol{\eta}, S_A\boldsymbol{\eta}\right),$$

Differentiating $\mathcal{I}_{1\mathrm{st}}$ by time, we will obtain the following estimate:

Lemma 2.1. *We have*

$$\|\exp\left(\frac{2}{A}|x|\right)\eta_1'\|_{L^2}^2 + A^{-2}\|\exp\left(\frac{2}{A}|x|\right)\eta_1\|_{L^2}^2$$
$$\lesssim -\dot{\mathcal{I}}_{1\mathrm{st},1} + A^2\delta\|\boldsymbol{\eta}\|_{\boldsymbol{\Sigma}_A}^2 + \|\boldsymbol{\eta}\|_{L^2_{-\kappa}}^2 + \sum_{\mathbf{m}\in\mathbf{R}_m}|\mathbf{z}^{\mathbf{m}}|^2.$$

Lemma 2.1 provides an estimate for the Σ_A norm. Σ_A norm cannot control the local norm $L^2_{-\kappa}$ so for the estimate for this norm, we will need another virial estimate (2nd virial estimate).

From 2.1 and similar estimate coming from another virial functional, we have

$$\|\boldsymbol{\eta}\|_{L^2(I,\boldsymbol{\Sigma}_A)} \lesssim \delta + \|\boldsymbol{\eta}\|_{L^2(I,\boldsymbol{L}^2_{-\kappa})} + \sum_{\mathbf{m}\in\mathbf{R}_m}\|\mathbf{z}^{\mathbf{m}}\|_{L^2}.$$

2.2 Fermi Golden Rule estimate

To explain the Fermi Golden Rule estimate, we restrict our attention to the case $N=1$. We set $M \ge 2$ so that $M\lambda_1 > m > (M-1)\lambda_1$ (we will assume $n\lambda_1 \ne m$ for $m \in \mathbb{N}$). In this situation, we can construct an approximate periodic solution $\boldsymbol{\phi}[z]$ of (1.1) (which is the refined profile) satisfying

$$D\boldsymbol{\phi}[z]\widetilde{z} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\left(\begin{pmatrix}L_1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}f(\phi_1[z])\\ 0\end{pmatrix} + (z^N + \overline{z}^N)\mathbf{G} + \mathbf{R}[z]\right), \tag{2.3}$$

where

$$D\boldsymbol{\phi}[z]\widetilde{z} = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \boldsymbol{\phi}[z+\epsilon\widetilde{z}],$$

and $\widetilde{z}$ and $\mathbf{R}[z]$ satisfies

$$|\widetilde{z} - M\lambda_1 z| \lesssim |z|^2$$

and

$$\|e^{a\langle x\rangle}\mathbf{R}[z]\|_{H^1} \lesssim |z|^{M+1}$$

for some $a > 0$.

Using (2.3), we can explicitly state the Fermi Golden Rule assumption as follows.

Assumption 2.2 (Fermi Golden Rule). There exists a bounded solution $\mathbf{g}$ of

$$\begin{pmatrix} 0 & 1 \\ -L_1 & 0 \end{pmatrix} \mathbf{g_m} = \mathrm{i}M\lambda_1 \mathbf{g_m}$$

s.t.

$$\langle \mathbf{G}, \mathbf{g}\rangle = \gamma \neq 0.$$

Remark 2.3. By replacing $\mathbf{g}$ to $-\mathbf{g}$ if necessary, we can assume $\gamma > 0$.

As the virial estimate, we consider the following functional.

$$\mathcal{J}_{\mathrm{FGR}} := \Omega(\boldsymbol{\eta}, \chi_A(z^M + \overline{z}^M)\mathbf{g}),$$

where χ_A is given in (2.2).

Computing the time derivative of $\mathcal{J}_{\mathrm{FGR}}$, we have the following estimate.

Lemma 2.4. *We have*

$$\left|\dot{\mathcal{J}}_{\mathrm{FGR}} - \left\langle \left(z^M + \overline{z}^M\right)\mathbf{G}, \left(z^M + \overline{z}^M\right)\mathbf{g}\right\rangle\right| \lesssim A^{-1/2}\left(|z^M|^2 + \|\boldsymbol{\eta}\|^2_{\boldsymbol{\Sigma}_A}\right).$$

Since the nonresonant terms like z^{2M} can be ignored, from Lemma 2.4, we obtain the estimate:

$$\gamma\|z^M\|_{L^2} \lesssim \|(u_0, u_1)\|_{H^1\times L^2} + \|\boldsymbol{\eta}\|_{L^2\boldsymbol{\Sigma}_A}.$$

2.3 Bootstrapping

In the end of the day, bootstrapping all 3 estimates, we will obtain

$$\|\boldsymbol{\eta}\|_{L^2\Sigma_A} + \|z^M\|_{L^2} \lesssim \|(u_0, u_1)\|_{H^1\times L^2},$$

which shows a decay in time averaged sense. To update the time averaged decay to a decay in the form

$$\lim_{t\to\infty}\left(\|\boldsymbol{\eta}\|_{\Sigma_A} + |z^M|\right) = 0,$$

we use energy estimates. From this, going back to the original coordinate, we have the conclusion of Theorem 1.1.

References

[1] Dario Bambusi and Scipio Cuccagna, On dispersion of small energy solutions to the nonlinear Klein Gordon equation with a potential, Amer. J. Math. **133** (2011), no. 5, 1421–1468.

[2] S. Cuccagna and M. Maeda, A survey on asymptotic stability of ground states of nonlinear Schrödinger equations II, Discrete Contin. Dyn. Syst. Ser. S 14 (2021), no. 5, 1693–1716.

[3] S. Cuccagna and M. Maeda, Coordinates at small energy and refined profiles for the Nonlinear Schrödinger Equation, Ann. PDE 7 (2021), no. 2, Paper No. 16, 34 pp.

[4] S. Cuccagna and M. Maeda, Revisiting asymptotic stability of solitons of nonlinear Schrödinger equations via refined profile method, Journal of Evolution Equations volume 22, Article number: 51 (2022).

[5] M. Kowalczyk and Y. Martel, Kink dynamics under odd perturbations for $(1+1)$-scalar field models with one internal mode, preprint arXiv:2203.04143v1.

[6] M. Kowalczyk, Y. Martel and C. Muñoz, Kink dynamics in the ϕ^4 model: asymptotic stability for odd perturbations in the energy space, J. Amer. Math. Soc. 30 (2017), 769–798.

[7] M. Kowalczyk, Y. Martel and C. Muñoz, Nonexistence of small, odd breathers for a class of nonlinear wave equations, Lett. Math. Phys. 107 (2017), 921–931.

[8] M. Kowalczyk, Y. Martel and C. Muñoz, Soliton dynamics for the 1D NLKG equation with symmetry and in the absence of internal modes, arXiv:1903.12460, to appear in Jour. Eur. Math Soc.

[9] M. Kowalczyk, Y. Martel, C. Muñoz, and H. Van Den Bosch, A sufficient condition for asymptotic stability of kinks in general $(1+1)$-scalar field models, Ann. PDE **7** (2021), no. 1, Paper No. 10, 98.

[10] I. M. Sigal, Nonlinear wave and Schrödinger equations. I. Instability of periodic and quasiperiodic solutions, Comm. Math. Phys. **153** (1993), no. 2, 297–320.

[11] A.Soffer and M.I.Weinstein, Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations, Invent. Math. 136 (1999), 9–74.