



Title	Freeness for multiarrangements of hyperplanes over arbitrary fields
Author(s)	Torielli, Michele
Citation	Communications in algebra, 50(1), 292-299 https://doi.org/10.1080/00927872.2021.1956514
Issue Date	2021-07-30
Doc URL	http://hdl.handle.net/2115/86436
Rights	This is an Accepted Manuscript of an article published by Taylor & Francis in Communications in algebra Published online: 30 Jul 2021, available online: http://www.tandfonline.com/10.1080/00927872.2021.1956514
Type	article (author version)
File Information	Commun. Algebr. 50-1_292-299.pdf



[Instructions for use](#)

FREENESS FOR MULTIARRANGEMENTS OF HYPERPLANES OVER ARBITRARY FIELDS

MICHELE TORIELLI

ABSTRACT. In this paper, we study the class of free multiarrangements of hyperplanes. Specifically, we investigate the relations between freeness over a field of finite characteristic and freeness over \mathbb{Q} .

Keywords: Hyperplane arrangements, Multiarrangements of hyperplanes, Freeness, Modular approach, Logarithmic derivations.

2010 MSC: 32S22, 52C35.

1. INTRODUCTION

Let V be a vector space of dimension l over a field \mathbb{K} . Fix a system of coordinates (x_1, \dots, x_l) of V^* . We denote by $S = S(V^*) = \mathbb{K}[x_1, \dots, x_l]$ the symmetric algebra. A hyperplane arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$ is a finite collection of hyperplanes in V .

The theory of freeness of hyperplane arrangements is a key notion which connects arrangement theory with algebraic geometry and combinatorics. By definition, an arrangement is free if and only if its module of logarithmic derivations is a free module. A lot is known about free arrangements, however there is still some mystery around the notion of freeness. The notion of freeness was introduced by Saito in [11] for the case of hypersurfaces in the analytic category. The special case of hyperplane arrangements was first studied by Terao in [13], where he showed that we can pass from analytic to algebraic considerations. In [17], Ziegler extended this theory to the class of multiarrangements of hyperplanes, i.e. arrangements in which each hyperplane has a non-negative integer multiplicity. In addition, Ziegler related the notions of free arrangements and free multiarrangements. Since their introduction, free (multi)arrangements have been intensively studied in connection with the famous Terao's conjecture. See for example [15], [1], [4], [12], [16], [2], [9] and [10].

Date: July 13, 2021.

The purpose of this paper is to extend the work of [8] in order to study the connections between freeness of multiarrangements over a field of characteristic zero and over a finite field, and to describe in which cases the two situations are related and how.

2. PRELIMINARES ON HYPERPLANE ARRANGEMENTS

In this section, we recall the terminology, the basic notations and some fundamental results related to multiarrangements of hyperplanes.

Let \mathbb{K} be a field. A finite set of affine hyperplanes $\mathcal{A} = \{H_1, \dots, H_n\}$ in \mathbb{K}^l is called a **hyperplane arrangement**. For each hyperplane H_i we fix a defining polynomial $\alpha_i \in S = \mathbb{K}[x_1, \dots, x_l]$ such that $H_i = \alpha_i^{-1}(0)$, and let $Q(\mathcal{A}) = \prod_{i=1}^n \alpha_i$. An arrangement \mathcal{A} is called **central** if each H_i contains the origin of \mathbb{K}^l . In this case, each polynomial $\alpha_i \in S$ is linear homogeneous, and hence $Q(\mathcal{A})$ is homogeneous of degree n . In this paper, we will only consider central arrangements.

Definition 2.1. A *multiarrangement of hyperplanes* is a pair $(\mathcal{A}, \mathbf{m})$ of a central arrangement \mathcal{A} with a map $\mathbf{m}: \mathcal{A} \rightarrow \mathbb{Z}_{\geq 0}$, called the **multiplicity**.

We also put $Q(\mathcal{A}, \mathbf{m}) = \prod_{i=1}^n \alpha_i^{\mathbf{m}(H_i)}$ and $|\mathbf{m}| = \sum_{i=1}^n \mathbf{m}(H_i)$.

The theory of multiarrangements is a generalization of the one of arrangements. In fact, an arrangement \mathcal{A} can be identified with $(\mathcal{A}, \mathbf{1})$ a multiarrangement with constant multiplicity $\mathbf{m} \equiv 1$, which is sometimes called a **simple arrangement**.

We denote by $\text{Der}_{\mathbb{K}^l} = \{\sum_{i=1}^l f_i \partial_{x_i} \mid f_i \in S\}$ the S -module of **polynomial vector fields** on \mathbb{K}^l (or S -derivations). Let $\delta = \sum_{i=1}^l f_i \partial_{x_i} \in \text{Der}_{\mathbb{K}^l}$. Then δ is said to be **homogeneous of polynomial degree d** if f_1, \dots, f_l are homogeneous polynomials of degree d in S . In this case, we write $\text{pdeg}(\delta) = d$.

Definition 2.2. Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement in \mathbb{K}^l . Define the **module of vector fields logarithmic tangent** to \mathcal{A} with multiplicity \mathbf{m} (or *logarithmic vector fields*) by

$$D(\mathcal{A}, \mathbf{m}) = \{\delta \in \text{Der}_{\mathbb{K}^l} \mid \delta(\alpha_i) \in \langle \alpha_i^{\mathbf{m}(H_i)} \rangle, \forall i\}.$$

The module $D(\mathcal{A}, \mathbf{m})$ is obviously a graded S -module and plays a central role in the theory of free multiarrangements of hyperplanes. In general, in contrast to the case of simple arrangements, $D(\mathcal{A}, \mathbf{m})$ does not coincide with $\{\delta \in \text{Der}_{\mathbb{K}^l} \mid \delta(Q(\mathcal{A})) \in \langle Q(\mathcal{A}, \mathbf{m}) \rangle\}$.

Definition 2.3. A multiarrangement $(\mathcal{A}, \mathbf{m})$ in \mathbb{K}^l is said to be **free with exponents** (e_1, \dots, e_l) if and only if $D(\mathcal{A}, \mathbf{m})$ is a free S -module and there exists a basis $\delta_1, \dots, \delta_l \in D(\mathcal{A}, \mathbf{m})$ such that $\text{pdeg}(\delta_i) = e_i$, or equivalently $D(\mathcal{A}, \mathbf{m}) \cong \bigoplus_{i=1}^l S(-e_i)$.

As described in [17], since $D(\mathcal{A}, \mathbf{m})$ is a reflexive S -module, then any multiarrangement in \mathbb{K}^2 is free.

Example 2.4. Consider the multiarrangement $(\mathcal{A}, \mathbf{m})$ in \mathbb{R}^2 with $Q(\mathcal{A}, \mathbf{m}) = x^2y^2(x - y)$. Then $(\mathcal{A}, \mathbf{m})$ is free with exponents $(2, 3)$. In fact, $D(\mathcal{A}, \mathbf{m})$ is free and it is generated by

$$\delta_1 = x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y},$$

$$\delta_2 = (x - y)y^2 \frac{\partial}{\partial y}.$$

Remark 2.5. Consider $\delta_1, \dots, \delta_l \in D(\mathcal{A}, \mathbf{m})$, then $\det(\delta_i(x_j))_{i,j}$ is divisible by $Q(\mathcal{A}, \mathbf{m})$.

In [11], after introducing the notion of freeness, Saito described one of the most famous characterizations of freeness, see also [17]. Saito's criterion checks if $(\mathcal{A}, \mathbf{m})$ is free or not by looking at the determinant of the coefficient matrix of $\delta_1, \dots, \delta_l \in D(\mathcal{A}, \mathbf{m})$. Notice that the original statement is for characteristic zero. However, as noted in [14], this statement holds true for any characteristic.

Theorem 2.6 (Saito's criterion). *Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement in \mathbb{K}^l and $\delta_1, \dots, \delta_l$ homogeneous elements of $D(\mathcal{A}, \mathbf{m})$. Then the following facts are equivalent*

- (1) $D(\mathcal{A}, \mathbf{m})$ is free with basis $\delta_1, \dots, \delta_l$, i.e. $D(\mathcal{A}, \mathbf{m}) = S \cdot \delta_1 \oplus \dots \oplus S \cdot \delta_l$.
- (2) $\det(\delta_i(x_j))_{i,j} = cQ(\mathcal{A}, \mathbf{m})$, where $c \in \mathbb{K} \setminus \{0\}$.
- (3) $\delta_1, \dots, \delta_l$ are linearly independent over S and $\sum_{i=1}^l \text{pdeg}(\delta_i) = |\mathbf{m}|$.

Directly from Saito's criterion, we obtain that if $(\mathcal{A}, \mathbf{m})$ is a free multiarrangement in \mathbb{K}^l with exponents (e_1, \dots, e_l) , then $\deg(Q(\mathcal{A}, \mathbf{m})) = |\mathbf{m}| = \sum_{i=1}^l e_i$.

Notice that in contrast to the theory of simple arrangements, see [5], the exponents of a free multiarrangement are not combinatorial in general, as shown in the following example.

Example 2.7 ([17]). Consider the multiarrangement $(\mathcal{A}, \mathbf{m})$ in \mathbb{R}^2 with defining polynomial $Q(\mathcal{A}, \mathbf{m}) = x^3y^3(x - y)(x + y)$. Then $(\mathcal{A}, \mathbf{m})$ is free with exponents $(3, 5)$. However, if we consider the multiarrangement $(\mathcal{A}', \mathbf{m})$ in \mathbb{R}^2 with $Q(\mathcal{A}', \mathbf{m}) = x^3y^3(x - y)(x + 4y)$, then also $(\mathcal{A}', \mathbf{m})$ is free but with exponents $(4, 4)$.

Given a multiarrangement $(\mathcal{A}, \mathbf{m})$ in \mathbb{K}^l with $\mathcal{A} = \{H_1, \dots, H_n\}$, let $M(\mathcal{A}, \mathbf{m})$ be the S -submodule of the free S -module S^n defined by

$$M(\mathcal{A}, \mathbf{m}) = \langle (\alpha_1^{\mathbf{m}(H_1)}, 0, \dots, 0), \dots, (0, \dots, 0, \alpha_n^{\mathbf{m}(H_n)}) \rangle.$$

Consider the $n \times l$ matrix $A(\mathcal{A}) = (\frac{\partial \alpha_i}{\partial x_j})_{i,j}$ with coefficients in S . We can then define the multiplication map of S -modules

$$\varphi_{(\mathcal{A}, \mathbf{m})}: S^l \longrightarrow S^n / M(\mathcal{A}, \mathbf{m}) \quad (1)$$

defined by $(g_1, \dots, g_l)^t \mapsto A(\mathcal{A})(g_1, \dots, g_l)^t$.

With this construction, it is trivial to prove the following

Proposition 2.8. *Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement in \mathbb{K}^l . Then*

$$D(\mathcal{A}, \mathbf{m}) \cong \text{Ker}(\varphi_{(\mathcal{A}, \mathbf{m})}).$$

3. FROM CHARACTERISTIC 0 TO CHARACTERISTIC p

Similarly to [8] and [6], from now on we will assume that $(\mathcal{A}, \mathbf{m})$ is a multiarrangement in \mathbb{Q}^l . After getting rid of the denominators, we can suppose that $\alpha_i \in \mathbb{Z}[x_1, \dots, x_l]$ for all $i = 1, \dots, n$, and hence that $Q(\mathcal{A}, \mathbf{m}) = \prod_{i=1}^n \alpha_i^{\mathbf{m}(H_i)} \in \mathbb{Z}[x_1, \dots, x_l]$. Moreover, we can also assume that there exists no prime number p that divides any α_i .

Let p be a prime number. Consider the image of each α_i under the canonical homomorphism $\pi_p: \mathbb{Z}[x_1, \dots, x_l] \longrightarrow \mathbb{F}_p[x_1, \dots, x_l]$. By assumption, $\pi_p(\alpha_i) \neq 0$ for all $i = 1, \dots, n$.

Definition 3.1. *Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement in \mathbb{Q}^l . We will call a prime number p **good** for $(\mathcal{A}, \mathbf{m})$ if $\pi_p(\alpha_i) \neq \pi_p(\alpha_j)$ for all $i \neq j$.*

Similarly to the case of simple arrangements, described in [8], we have the following.

Lemma 3.2. *There is a finite number of primes p that are not good for $(\mathcal{A}, \mathbf{m})$.*

Proof. By definition, p is not good for $(\mathcal{A}, \mathbf{m})$ if and only if $\pi_p(\alpha_i) = \pi_p(\alpha_j)$ for some $i \neq j$ and this can happen only for a finite number of primes. \square

Let now p be a good prime for $(\mathcal{A}, \mathbf{m})$, and consider $(\mathcal{A}_p, \mathbf{m})$ the multiarrangement in \mathbb{F}_p^l defined by $\pi_p(Q(\mathcal{A}, \mathbf{m}))$. By construction, $Q(\mathcal{A}_p, \mathbf{m}) = \pi_p(Q(\mathcal{A}, \mathbf{m})) = \prod_{i=1}^n \pi_p(\alpha_i)^{\mathbf{m}(H_i)} \neq 0$ and it has degree $|\mathbf{m}|$.

Theorem 3.3. *If $(\mathcal{A}, \mathbf{m})$ is a free multiarrangement in \mathbb{Q}^l with exponents (e_1, \dots, e_l) , then $(\mathcal{A}_p, \mathbf{m})$ is a free multiarrangement in \mathbb{F}_p^l with exponents (e_1, \dots, e_l) , for all good primes except possibly a finite number of them.*

Proof. Let $\Delta = \{\delta_1, \dots, \delta_l\}$ be a basis of $D(\mathcal{A}, \mathbf{m})$ with $\text{pdeg}(\delta_i) = e_i$ for all $i = 1, \dots, l$. By getting rid of the denominators, we can assume that every polynomial that appears in each $\delta \in \Delta$ is in $\mathbb{Z}[x_1, \dots, x_l]$. Hence we can consider $\bar{\delta}_1, \dots, \bar{\delta}_l \in \text{Der}_{\mathbb{F}_p^l}$ the image of $\delta_1, \dots, \delta_l$. We can assume that $p \nmid \delta$ for all $\delta \in \Delta$, and hence $\bar{\delta} \neq 0$ for all $\delta \in \Delta$. This implies that $\text{pdeg}(\bar{\delta}_i) = \text{pdeg}(\delta_i) = e_i$ for all $i = 1, \dots, l$.

Fix one $j = 1, \dots, n$. Then by the definition of $D(\mathcal{A}, \mathbf{m})$, for each $\delta \in \Delta$ there exists $h_j \in \mathbb{Z}[x_1, \dots, x_l]$ such that $\delta(\alpha_j) = h_j \alpha_j^{\mathbf{m}(H_j)}$. If we apply π_p to this expression we obtain that $\bar{\delta}(\pi_p(\alpha_j)) = \pi_p(\delta(\alpha_j)) = \pi_p(h_j \alpha_j^{\mathbf{m}(H_j)}) = \pi_p(h_j) \pi_p(\alpha_j)^{\mathbf{m}(H_j)}$. Since this holds for all α_j , then $\bar{\delta} \in D(\mathcal{A}_p, \mathbf{m})$.

By Theorem 2.6, since in each $\delta \in \Delta$ every polynomial that appears has only integer coefficients, there exists $c \in \mathbb{Z} \setminus \{0\}$ such that $\det(\delta_i(x_j))_{i,j} = cQ(\mathcal{A}, \mathbf{m})$. If we apply π_p to the previous equality we get that $\det(\bar{\delta}_i(x_j))_{i,j} = \pi_p(\det(\delta_i(x_j))_{i,j}) = \pi_p(cQ(\mathcal{A}, \mathbf{m})) = \pi_p(c)Q(\mathcal{A}_p, \mathbf{m})$. Hence if p does not divide c , we have that $\pi_p(c) \in \mathbb{F}_p \setminus \{0\}$ and hence again by Theorem 2.6, we have that $\bar{\delta}_1, \dots, \bar{\delta}_l$ are a basis of $D(\mathcal{A}_p, \mathbf{m})$. This proves that $(\mathcal{A}_p, \mathbf{m})$ is free with exponents (e_1, \dots, e_l) . \square

By Lemma 3.2, the number of non-good primes is finite. Hence we have the following.

Corollary 3.4. *Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement in \mathbb{Q}^l and p a large prime number. If $(\mathcal{A}, \mathbf{m})$ is free in \mathbb{Q}^l with exponents (e_1, \dots, e_l) , then $(\mathcal{A}_p, \mathbf{m})$ is free in \mathbb{F}_p^l with exponents (e_1, \dots, e_l) .*

If we consider the second multiarrangement of Example 2.7 over a different field, we have the following.

Example 3.5. *Consider the multiarrangement $(\mathcal{A}', \mathbf{m})$ of Example 2.7 with $\mathbb{K} = \mathbb{Q}$. All prime numbers $p \neq 2$ are good for $(\mathcal{A}', \mathbf{m})$. A direct computation shows that $(\mathcal{A}', \mathbf{m})$ is free with exponents $(4, 4)$. However, if we consider $((\mathcal{A}')_3, \mathbf{m})$ as a multiarrangement in \mathbb{F}_3^2 , then it is still free but its exponents are $(3, 5)$. This is because if we can consider the matrix*

$$\begin{pmatrix} x^4 & 13xy^3 - 12y^4 \\ x^3y & -3xy^3 + 4y^4 \end{pmatrix}$$

as the coefficient matrix of a basis of $D(\mathcal{A}', \mathbf{m})$, its determinant is equal to $-3Q(\mathcal{A}', \mathbf{m})$. A direct computation shows that if we take another basis of $D(\mathcal{A}', \mathbf{m})$ with only integer coefficients, then the determinant of the coefficient matrix is equal to $cQ(\mathcal{A}', \mathbf{m})$ with $c \in 3\mathbb{Z} \setminus \{0\}$. This is why over \mathbb{F}_3 the exponents of $(\mathcal{A}', \mathbf{m})$ change.

The following is an example of a free multiarrangements in \mathbb{Q}^l that is not free in \mathbb{F}_p^l , for some good prime p .

Example 3.6. Consider the multiarrangement $(\mathcal{A}, \mathbf{m})$ in \mathbb{Q}^3 with defining polynomial $Q(\mathcal{A}, \mathbf{m}) = x^2y^2z^2(x-y)^2(x-z)^2(y-z)^2$. In this situation, all prime numbers are good for $(\mathcal{A}, \mathbf{m})$, and $(\mathcal{A}, \mathbf{m})$ is free with exponents $(4, 4, 4)$. However, if we consider $(\mathcal{A}_2, \mathbf{m})$ as a multiarrangement in \mathbb{F}_2^3 , then $(\mathcal{A}_2, \mathbf{m})$ is free with exponents $(2, 4, 6)$. On the other hand, if we consider $(\mathcal{A}_3, \mathbf{m})$ as a multiarrangement in \mathbb{F}_3^3 , then $(\mathcal{A}_3, \mathbf{m})$ is not free. This is because the determinant of a coefficient matrix of a basis of $D(\mathcal{A}, \mathbf{m})$ is equal to $cQ(\mathcal{A}, \mathbf{m})$ with $c \in 18\mathbb{Z} \setminus \{0\}$. In particular, if we consider $p \neq 2, 3$ a prime number and $(\mathcal{A}_p, \mathbf{m})$ as a multiarrangement in \mathbb{F}_p^3 , then $(\mathcal{A}_p, \mathbf{m})$ is free with exponents $(4, 4, 4)$.

4. FROM CHARACTERISTIC p TO CHARACTERISTIC 0

As in Section 3, we will assume that $(\mathcal{A}, \mathbf{m})$ is a multiarrangement in \mathbb{Q}^l , that $\alpha_i \in \mathbb{Z}[x_1, \dots, x_l]$ for all $i = 1, \dots, n$, and that there exists no prime number p that divides any α_i . Moreover, let $S = \mathbb{Q}[x_1, \dots, x_l]$, $S_{\mathbb{Z}} = \mathbb{Z}[x_1, \dots, x_l]$ and $S_p = \mathbb{F}_p[x_1, \dots, x_l]$.

Similarly to the construction of the map (1) at the end of Section 2, by our assumptions on the α_i , we can consider $M(\mathcal{A}, \mathbf{m})$ as $S_{\mathbb{Z}}$ -submodule of $S_{\mathbb{Z}}^n$, and $A(\mathcal{A})$ as matrix with coefficients in $S_{\mathbb{Z}}$. Hence we can construct the map of $S_{\mathbb{Z}}$ -modules

$$\varphi_{\mathbb{Z}}: S_{\mathbb{Z}}^l \longrightarrow S_{\mathbb{Z}}^n / M(\mathcal{A}, \mathbf{m}) \quad (2)$$

defined by $(g_1, \dots, g_l)^t \mapsto A(\mathcal{A})(g_1, \dots, g_l)^t$. It is trivial to see that

Lemma 4.1. *Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement in \mathbb{Q}^l . Then*

$$D(\mathcal{A}, \mathbf{m}) \cong \text{Ker}(\varphi_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

For any integer $k \geq 1$, the canonical homomorphism $\pi_p: S_{\mathbb{Z}} \longrightarrow S_p$ extends naturally to the homomorphism $\pi_p^k: S_{\mathbb{Z}}^k \longrightarrow S_p^k$. If we assume that p is a good prime for $(\mathcal{A}, \mathbf{m})$, and that \mathcal{A} and \mathcal{A}_p have the same orders under the canonical morphism, then $\pi_p^n(M(\mathcal{A}, \mathbf{m})) = M(\mathcal{A}_p, \mathbf{m})$. This implies that we can construct the following commutative diagram

$$\begin{array}{ccccc} \text{Ker}(\varphi_{\mathbb{Z}}) & \xhookrightarrow{i} & S_{\mathbb{Z}}^l & \xrightarrow{\varphi_{\mathbb{Z}}} & S_{\mathbb{Z}}^n / M(\mathcal{A}, \mathbf{m}) \\ \downarrow \pi_p^l & & \downarrow \pi_p^l & & \downarrow \pi_p^n \\ D(\mathcal{A}_p, \mathbf{m}) & \xhookrightarrow{i_p} & S_p^l & \xrightarrow{\varphi(\mathcal{A}_p, \mathbf{m})} & S_p^n / M(\mathcal{A}_p, \mathbf{m}) \end{array} \quad (3)$$

The introduction of the commutative diagram (3) allows us to describe in which situations the freeness of a multiarrangement over \mathbb{F}_p implies the freeness over \mathbb{Q} .

Theorem 4.2. *Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement in \mathbb{Q}^l . Let p be a good prime number for $(\mathcal{A}, \mathbf{m})$ and assume that the map*

$$\pi_p^l: \text{Ker}(\varphi_{\mathbb{Z}}) \longrightarrow D(\mathcal{A}_p, \mathbf{m})$$

is surjective. If $(\mathcal{A}_p, \mathbf{m})$ is free in \mathbb{F}_p^l with exponents (e_1, \dots, e_l) , then $(\mathcal{A}, \mathbf{m})$ is free in \mathbb{Q}^l with exponents (e_1, \dots, e_l) .

Proof. Let $\delta_1, \dots, \delta_l$ be a basis of $D(\mathcal{A}_p, \mathbf{m})$ such that $\text{pdeg}(\delta_i) = e_i$. Since the map $\pi_p^l: \text{Ker}(\varphi_{\mathbb{Z}}) \longrightarrow D(\mathcal{A}_p, \mathbf{m})$ is surjective, there exist $\tilde{\delta}_1, \dots, \tilde{\delta}_l$ in $\text{Ker}(\varphi_{\mathbb{Z}}) \setminus \{0\}$ such that $\pi_p^l(\tilde{\delta}_i) = \delta_i$. We can assume that each $\tilde{\delta}_i$ is homogeneous. Clearly $\text{pdeg}(\tilde{\delta}_i) = e_i$.

By Lemma 4.1, we can consider $\tilde{\delta}_1, \dots, \tilde{\delta}_l$ as elements of $D(\mathcal{A}, \mathbf{m})$. By Remark 2.5, we have that $\det(\tilde{\delta}_i(x_j)_{i,j}) = hQ(\mathcal{A}, \mathbf{m})$ for some $h \in S_{\mathbb{Z}} \subseteq S$. On the other hand, $\deg(\det(\tilde{\delta}_i(x_j)_{i,j})) = \sum_{k=1}^l \text{pdeg}(\tilde{\delta}_k) = \sum_{k=1}^l e_k = |\mathbf{m}| = \deg(Q(\mathcal{A}, \mathbf{m}))$. This implies that $h \in \mathbb{Z} \subseteq \mathbb{Q}$. Suppose that $h = 0$, then $\det(\tilde{\delta}_i(x_j)_{i,j}) = 0$. If we apply the map $\pi_p: S_{\mathbb{Z}} \longrightarrow S_p$ to this equality we obtain $0 = \pi_p(\det(\tilde{\delta}_i(x_j)_{i,j})) = \det(\delta_i(x_j)_{i,j})$. However this is impossible, since, by Theorem 2.6, $\det(\delta_i(x_j)_{i,j}) = cQ(\mathcal{A}_p, \mathbf{m})$ for some $c \in \mathbb{F}_p \setminus \{0\}$. This implies that $\det(\tilde{\delta}_i(x_j)_{i,j}) = hQ(\mathcal{A}, \mathbf{m})$ for some $h \in \mathbb{Z} \setminus \{0\}$. If we apply Theorem 2.6 to $\tilde{\delta}_1, \dots, \tilde{\delta}_l$, we obtain that they form a basis of $D(\mathcal{A}, \mathbf{m})$, and hence that \mathcal{A} is free with exponents (e_1, \dots, e_l) . \square

In general, it might not be easy to check directly the surjectivity of the map $\pi_p^l: \text{Ker}(\varphi_{\mathbb{Z}}) \longrightarrow D(\mathcal{A}_p, \mathbf{m})$. However, we can obtain information by looking at $\text{coker}(\varphi_{\mathbb{Z}})$.

Proposition 4.3. *Let $(\mathcal{A}, \mathbf{m})$ be a multiarrangement in \mathbb{Q}^l and p a good prime for $(\mathcal{A}, \mathbf{m})$. If p is not a zero divisor of $\text{coker}(\varphi_{\mathbb{Z}})$, then the map $\pi_p^l: \text{Ker}(\varphi_{\mathbb{Z}}) \longrightarrow D(\mathcal{A}_p, \mathbf{m})$ is surjective.*

Proof. Assume to the contrary that there exists $\delta \in D(\mathcal{A}_p, \mathbf{m}) \setminus \pi_p^l(\text{Ker}(\varphi_{\mathbb{Z}}))$. Since the map $\pi_p^l: S_{\mathbb{Z}}^l \longrightarrow S_p^l$ is surjective, there exists $\tilde{\delta} \in S_{\mathbb{Z}}^l \setminus \text{Ker}(\varphi_{\mathbb{Z}})$ such that $\pi_p^l(\tilde{\delta}) = i_p(\delta)$. Since $i_p(\delta) \neq 0$, then p does not divide $\tilde{\delta}$. By construction, $\varphi_{\mathbb{Z}}(\tilde{\delta}) \neq 0$ and $\pi_p^n(\varphi_{\mathbb{Z}}(\tilde{\delta})) = \varphi_{(\mathcal{A}_p, \mathbf{m})}(i_p(\delta)) = 0$. This implies that $\varphi_{\mathbb{Z}}(\tilde{\delta}) = pv$, for some non-zero $v \in S_{\mathbb{Z}}^n/M(\mathcal{A}, \mathbf{m})$. To conclude we just need to show that v does not belong to the image of the map $\varphi_{\mathbb{Z}}$, and hence that p is a zero divisor of the cokernel of $\varphi_{\mathbb{Z}}$, leading to a contradiction.

Suppose now that there exists $\sigma \in S_{\mathbb{Z}}^l$ such that $\varphi_{\mathbb{Z}}(\sigma) = v$. Since p does not divide $\tilde{\delta}$, then $\tilde{\delta} - p\sigma \neq 0$. Moreover, $\varphi_{\mathbb{Z}}(\tilde{\delta} - p\sigma) = 0$, and hence $\tilde{\delta} - p\sigma \in \text{Ker}(\varphi_{\mathbb{Z}})$. By construction, $\pi_p^l(\tilde{\delta} - p\sigma) = \delta$, and hence $\delta \in \pi_p^l(\text{Ker}(\varphi_{\mathbb{Z}}))$, but this is impossible. Hence, v does not belong to the image of the map $\varphi_{\mathbb{Z}}$, as claimed. \square

Putting together Theorem 4.2 and Proposition 4.3, we obtain the following result.

Theorem 4.4. *Let $(\mathcal{A}, \mathfrak{m})$ be a multiarrangement in \mathbb{Q}^l . Let p be a good prime number for $(\mathcal{A}, \mathfrak{m})$ that is not a zero divisor of $\text{coker}(\varphi_{\mathbb{Z}})$. If $(\mathcal{A}_p, \mathfrak{m})$ is free in \mathbb{F}_p^l with exponents (e_1, \dots, e_l) , then $(\mathcal{A}, \mathfrak{m})$ is free in \mathbb{Q}^l with exponents (e_1, \dots, e_l) .*

In Theorem 4.4, the assumption that the prime p is not a zero divisor in the cokernel of the map $\varphi_{\mathbb{Z}}$ is fundamental. In fact we have the following.

Example 4.5. *Consider the multiarrangement $(\mathcal{A}, \mathfrak{m})$ in \mathbb{Q}^3 with defining polynomial $Q(\mathcal{A}, \mathfrak{m}) = x^2y^2(x-y)^2(x-z)^2(y-z)^2$. $(\mathcal{A}, \mathfrak{m})$ is not free and 2 is a zero divisor of in the cokernel of the map $\varphi_{\mathbb{Z}}$. In fact, $\varphi_{\mathbb{Z}}((x^2, y^2, z^2)^t) = (2x^2, 2y^2, 2(x^2 - xy), 2(x^2 - xz), 2(y^2 - yz))^t$, however a direct computation shows that $(x^2, y^2, x^2 - xy, x^2 - xz, y^2 - yz)^t$ is not in the image of $\varphi_{\mathbb{Z}}$. On the other hand, the multiarrangement $(\mathcal{A}_2, \mathfrak{m})$ in \mathbb{F}_2^3 is free with exponents $(2, 4, 4)$.*

In general, given M a finitely generated $S_{\mathbb{Z}}$ -module, the number of zero divisor is infinite. However, if we restrict our attention to zero divisors that are prime numbers, we have the following.

Proposition 4.6. *Let M a finitely generated $S_{\mathbb{Z}}$ -module. Then the number of distinct prime numbers that are zero divisors in M is finite.*

Proof. By Theorem 14.4 of [3], there exists $a \in \mathbb{Z} \setminus \{0\}$ such that $M[a^{-1}]$ is a free $\mathbb{Z}[a^{-1}]$ -module. This implies that the set of distinct prime numbers that are zero divisors in M is included in the set of distinct prime numbers that divide a , that is finite by the unique factorization theorem. \square

By applying Proposition 4.6 to the cokernel of the map $\varphi_{\mathbb{Z}}$, the number of prime numbers that are zero divisors in $\text{coker}(\varphi_{\mathbb{Z}})$ is finite. Hence, putting together Corollary 3.4 and Theorem 4.2, we have the following.

Corollary 4.7. *Let $(\mathcal{A}, \mathfrak{m})$ be a multiarrangement in \mathbb{Q}^l and p a large prime number. $(\mathcal{A}_p, \mathfrak{m})$ is free in \mathbb{F}_p^l with exponents (e_1, \dots, e_l) if and only if $(\mathcal{A}, \mathfrak{m})$ is free in \mathbb{Q}^l with exponents (e_1, \dots, e_l) .*

In [8], the authors studied the freeness of simple arrangements and in Theorem 6.1, they proved the following.

Theorem 4.8 ([8]). *Let $\mathcal{A} = (\mathcal{A}, \mathbf{1})$ be a simple central arrangement in \mathbb{Q}^l and $J(\mathcal{A})_{\mathbb{Z}}$ the ideal of $S_{\mathbb{Z}}$ generated by $Q(\mathcal{A}) = Q(\mathcal{A}, \mathbf{1})$ and its partial derivatives. Let p be a good prime number for \mathcal{A} that is not a zero divisor in $S_{\mathbb{Z}}/J(\mathcal{A})_{\mathbb{Z}}$. If \mathcal{A}_p is free in \mathbb{F}_p^l with exponents (e_1, \dots, e_l) , then \mathcal{A} is free in \mathbb{Q}^l with exponents (e_1, \dots, e_l) .*

In the case of simple arrangements, Theorem 4.4 is exactly Theorem 4.8. In fact we have the following.

Proposition 4.9. *Let $\mathcal{A} = (\mathcal{A}, \mathbf{1})$ be a simple central arrangement in \mathbb{Q}^l . Assume that p is a good prime for \mathcal{A} . Then p is a zero divisor of $S_{\mathbb{Z}}/J(\mathcal{A})_{\mathbb{Z}}$ if and only if p is a zero divisor of $\text{coker}(\varphi_{\mathbb{Z}})$.*

Proof. Consider the map of $S_{\mathbb{Z}}$ -modules $\psi: S_{\mathbb{Z}}^l \rightarrow S_{\mathbb{Z}}/\langle Q(\mathcal{A}) \rangle S_{\mathbb{Z}}$ defined by $(g_1, \dots, g_l)^t \mapsto \sum_{i=1}^l g_i \partial Q(\mathcal{A}) / \partial x_i$. By construction, the image of ψ is $J(\mathcal{A})_{\mathbb{Z}} S_{\mathbb{Z}} / \langle Q(\mathcal{A}) \rangle S_{\mathbb{Z}}$, and hence $\text{coker}(\psi) \cong S_{\mathbb{Z}}/J(\mathcal{A})_{\mathbb{Z}}$.

Since $D(\mathcal{A}) = D(\mathcal{A}, \mathbf{1}) = \{\delta \in \text{Der}_{\mathbb{Q}^l} \mid \delta(Q(\mathcal{A})) \in \langle Q(\mathcal{A}) \rangle\}$, we have that $\text{Ker}(\psi) \cong \text{Ker}(\varphi_{\mathbb{Z}})$. By the first isomorphism theorem for modules, the image of ψ and $\varphi_{\mathbb{Z}}$ are isomorphic and hence we have that p is a zero divisor of $S_{\mathbb{Z}}/J(\mathcal{A})_{\mathbb{Z}}$ if and only if p is a zero divisor of $\text{coker}(\varphi_{\mathbb{Z}})$. \square

Acknowledgements. The author would like to thank E. Palezzato and G. Roehrlé for many helpful discussions and the anonymous referee for the useful suggestions. During the preparation of this article the author was supported by JSPS Grant-in-Aid for Early-Career Scientists (19K14493). For details on how to perform the computations with (multi)arrangements, see [7].

REFERENCES

- [1] T. Abe, H. Terao, and M. Wakefield. The Euler multiplicity and addition–deletion theorems for multiarrangements. *Journal of the London Mathematical Society*, 77(2):335–348, 2008.
- [2] A. M. Bigatti, E. Palezzato, and M. Torielli. New characterizations of freeness for hyperplane arrangements. *Journal of Algebraic Combinatorics*, 51(2):297–315, 2020.
- [3] D. Eisenbud. *Commutative algebra with a view toward algebraic geometry*, volume 150. Springer, 1995.
- [4] T. Hoge and G. Röhrle. Ziegler’s multireflection arrangements are free. *Experimental Mathematics*, 23(4):448–451, 2014.
- [5] P. Orlik and H. Terao. *Arrangements of hyperplanes*, volume 300 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.
- [6] E. Palezzato and M. Torielli. Combinatorially equivalent hyperplane arrangements. *Advances in Applied Mathematics*, 128, 2021.
- [7] E. Palezzato and M. Torielli. Hyperplane arrangements in CoCoA. *Journal of Software for Algebra and Geometry*, 9(1):43–54, 2019.
- [8] E. Palezzato and M. Torielli. Free hyperplane arrangements over arbitrary fields. *Journal of Algebraic Combinatorics*, 52(2):237–249, 2020.
- [9] E. Palezzato and M. Torielli. Lefschetz properties and hyperplane arrangements. *Journal of Algebra*, 555:289–304, 2020.
- [10] E. Palezzato and M. Torielli. Localization of plus-one generated arrangements. *Communications in Algebra*, 49(1), 301–309, 2021.

- [11] K. Saito. Theory of logarithmic differential forms and logarithmic vector fields. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 27(2):265–291, 1980.
- [12] D. Suyama, M. Torielli and S. Tsujie. Signed graphs and freeness of the Weyl subarrangements of type B_l . *Discrete Mathematics*, 342(1), 233–249, 2019.
- [13] H. Terao. Arrangements of hyperplanes and their freeness I. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 27(2):293–312, 1980.
- [14] H. Terao. Free arrangements of hyperplanes over an arbitrary field. *Proceedings of the Japan Academy, Series A, Mathematical Sciences*, 59(7):301–303, 1983.
- [15] M. Yoshinaga. Characterization of a free arrangement and conjecture of Edelman and Reiner. *Inventiones mathematicae*, 157(2):449–454, 2004.
- [16] M. Yoshinaga. Freeness of hyperplane arrangements and related topics. In *Annales de la Faculté des sciences de Toulouse: Mathématiques*, volume 23, pages 483–512, 2014.
- [17] G. M. Ziegler. Multiarrangements of hyperplanes and their freeness. *Contemporary Math*, 90:345–358, 1986.

MICHELE TORIELLI, DEPARTMENT OF MATHEMATICS, GI-CORE GSB, HOKKAIDO UNIVERSITY, KITA 10, NISHI 8, KITA-KU, SAPPORO 060-0810, JAPAN.

E-mail address: torielli@math.sci.hokudai.ac.jp