Existence of minimal solutions to quasilinear elliptic equations with several sub-natural growth terms ☆

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Abstract

We study the existence of positive solutions to quasilinear elliptic equations of the type

$$-\Delta_p u = \sigma u^q + \mu \quad \text{in } \mathbb{R}^n,$$

in the sub-natural growth case $0 < q < p - 1$, where $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$ is the p-Laplacian with $1 < p < n$, and σ and μ are nonnegative Radon measures on $\mathbb{R}^n$. We construct minimal generalized solutions under certain generalized energy conditions on σ and μ. To prove this, we give new estimates for interaction between measures. We also construct solutions to equations with several sub-natural growth terms using the same methods.

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1. Introduction and main results

In this paper, we consider the model quasilinear elliptic problem

$$\begin{cases} -\Delta_p u = \sigma u^q + \mu, \quad u > 0 \quad \text{in } \mathbb{R}^n, \\ \liminf_{|x| \to \infty} u(x) = 0, \end{cases} \tag{1.1}$$

in the sub-natural growth case $0 < q < p - 1$, where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p-Laplacian with $1 < p < n$, and σ and μ are nonnegative Radon measures on $\mathbb{R}^n$. We construct minimal generalized solutions to (1.1) under certain generalized energy conditions on σ and μ.

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When $\mu = 0$, Eq. (1.1) becomes

$$\begin{cases} -\Delta_p u = \sigma u^q, \quad u > 0 \quad \text{in } \mathbb{R}^n, \\ \liminf_{|x|\to\infty} u(x) = 0. \end{cases} \tag{1.2}$$

This equation is related to the trace inequality

$$\|u\|_{L^{1+q}(\mathbb{R}^n,d\sigma)} \le C\|\nabla u\|_{L^p(\mathbb{R}^n)} \quad \forall u \in C_c^\infty(\mathbb{R}^n), \tag{1.3}$$

where $\|\cdot\|_{L^{1+q}(\mathbb{R}^n,d\sigma)}$ is the L^{1+q} norm with respect to the measure σ. Cascante, Ortega and Verbitsky [12] and Verbitsky [27] proved that (1.3) holds if and only if

$$\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\sigma)^{\frac{(1+q)(p-1)}{p-1-q}} \, d\sigma < \infty, \tag{1.4}$$

where $\mathbf{W}_{1,p}\sigma$ is the Wolff potential of σ which is defined by

$$\mathbf{W}_{1,p}\sigma(x) := \int_0^\infty \left(\frac{\sigma(B(x,r))}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}, \quad x \in \mathbb{R}^n.$$

Cao and Verbitsky [10] showed that there exists a unique finite energy solution $u \in \dot{W}_0^{1,p}(\mathbb{R}^n)$ to (1.2) under (1.4), where $\dot{W}_0^{1,p}(\mathbb{R}^n)$ is the homogeneous Sobolev space. They also proved the necessity of (1.4). Seesanea and Verbitsky [25] extend such results to Eq. (1.1); there exists a unique finite energy solution $u \in \dot{W}_0^{1,p}(\mathbb{R}^n)$ to (1.1) if and only if (1.4) and

$$\int_{\mathbb{R}^n} \mathbf{W}_{1,p}\mu \, d\mu < \infty \tag{1.5}$$

are fulfilled. Treating general measure data $\mu \ge 0$ causes problems about interaction between σ and μ. The key to proof was to control them in the dual of $\dot{W}_0^{1,p}(\mathbb{R}^n)$.

However, Eq. (1.2) has various infinite energy solutions. In fact, in the classic paper by Brezis and Kamin [8], the existence and uniqueness of bounded solutions to (1.2) was proved under $p = 2$ and $\|\mathbf{I}_2\sigma\|_{L^\infty(\mathbb{R}^n)} < \infty$. Here, $\mathbf{I}_2\sigma$ is the Newtonian potential of σ. Their solutions do not belong to $\dot{W}_0^{1,2}(\mathbb{R}^n)$ in general. Boccardo and Orsina [6] treated elliptic equations with singular coefficients and applied concept of *renormalized solutions*. Their solutions are also called *p-superharmonic functions* in now. For details of such generalized solutions, see [18, 19, 3, 5, 13, 4, 16, 17].

Recently, the study of generalized solutions to (1.2) has made significant progress. Cao and Verbitsky [9] defined the *intrinsic* Wolff potential $\mathbf{K}_{1,p,q}\sigma$ of σ and proved that there exists a minimal p-superharmonic solution to (1.2) if and only if the potentials $\mathbf{W}_{1,p}\sigma$ and $\mathbf{K}_{1,p,q}\sigma$ are not identically infinite. Unfortunately, behavior of $\mathbf{K}_{1,p,q}\sigma$ can not be easily calculated from its definition. Cao and Verbitsky [11] constructed weak solutions in $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ under a certain capacity condition and gave two-sided pointwise estimates of such solutions. Seesanea and Verbitsky [23] gave a sufficient condition for the existence of L^r-integrable p-superharmonic solutions. From existence of such solutions, behavior of the potentials is derived conversely. For very recent progress in the study of $\mathbf{K}_{1,p,q}\sigma$, see [28, 29].

In this paper, we extend results in [23] to Eq. (1.1). We consider the following conditions:

$$\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\sigma)^{\frac{(\gamma+q)(p-1)}{p-1-q}} \, d\sigma < \infty, \tag{1.6}$$

$$\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\mu)^{\gamma} \, d\mu < \infty, \tag{1.7}$$

where $0 \leq \gamma < \infty$. We denote by $\nu[u]$ the Riesz measure of a p-superharmonic function u and interpret (1.1) as $\nu[u] = \sigma u^q + \mu$ (see Definition 4.1). Our main result is as follows.

Theorem 1.1. *Let $1 < p < n$, $0 < q < p - 1$. Assume that (1.6) and (1.7) hold for some $0 < \gamma < \infty$ and that $(\sigma, \mu) \not\equiv (0, 0)$. Then there exists a minimal p-superharmonic solution u to (1.1). Moreover, u satisfies*

$$\int_{\mathbb{R}^n} u^{\gamma} \, d\nu[u] < \infty \tag{1.8}$$

and belongs to $L^{r,\rho}(\mathbb{R}^n)$, where $r = n(p-1+\gamma)/(n-p)$ and $\rho = p - 1 + \gamma$.

Remark 1.2. (i) Conversely, it follows from [10, Theorem 2.3] that if there exists a p-superharmonic supersolution to (1.1) satisfying (1.8), then (1.6) and (1.7) must be fulfilled. (ii) As in [9, Theorem 1.1], when $p \geq n$, there is no nontrivial supersolution to (1.1).

Here, $L^{r,\rho}(\mathbb{R}^n)$ denotes the Lorentz space with respect to the Lebesgue measure. The authors do not know the same statement even if $\sigma = 0$. Theorem 1.1 includes the existence theorems in [23] and [25] as the special cases $\mu = 0$ and $\gamma = 1$. In general, our generalized solutions do not belong to $\dot{W}_0^{1,p}(\mathbb{R}^n)$, so we can not use the dual of $\dot{W}_0^{1,p}(\mathbb{R}^n)$ to control interaction between σ and μ. Hence, we derive an estimate of interaction directly using Wolff potentials (see Theorem 3.1). One of the authors used similar arguments for Green potentials in [24]. However, such arguments do not work for nonlinear potentials. To overcome this difficulty, we use tools of nonlinear potential theory. Theorem 3.1 can also be regarded as a generalization of (1.3). The Lorentz estimate for solutions is a direct consequence of it.

We also give variants of Theorem 1.1. Theorem 5.1 and Proposition 5.2 are analogs of Theorem 1.1 for $\gamma = \infty$ and $\gamma = 0$, respectively. In such cases, similar interaction between σ and μ do not appear from difference of energy structures. Theorem 6.1 is a generalization of Theorem 1.1 to equations of the form

$$\begin{cases} -\Delta_p u = \displaystyle\sum_{m=1}^{M} \sigma^{(m)} u^{q_m} + \mu, \quad u > 0 \quad \text{in } \mathbb{R}^n, \\ \displaystyle\liminf_{|x|\to\infty} u(x) = 0. \end{cases} \tag{1.9}$$

This result is new even if $p = 2$ and $\gamma = 1$. However, the spirit of proof is the same as Theorem 1.1. We also show the uniqueness of finite energy solutions.

Organization of the paper

In Section 2, we collect some facts of nonlinear potential theory to be used later. In Section 3, we prove an estimate for mutual energy and collect its consequences. In Section 4, we prove Theorem 1.1 using the results in the previous section. In Sections 5 and 6, we give some variants of Theorem 1.1.

Notation

We use the following notation in this paper. Let Ω be a domain (connected open subset) in $\mathbb{R}^n$.

- $B(x, R) := \{y \in \mathbb{R}^n \colon |x - y| < R\}$.
- For $B = B(x, R)$ and $\lambda > 0$, we write $\lambda B := B(x, \lambda R)$.
- $|A|$:= the Lebesgue measure of a measurable set A.
- $\mathbf{1}_A(x)$:= the indicator function of A.
- $C_c^\infty(\Omega)$:= the set of all infinitely-differentiable functions with compact support in Ω.
- $\mathcal{M}^+(\Omega)$:= the set of all nonnegative Radon measure on Ω.
- $A \approx B$ means $c_1 A \le B \le c_2 A$ for some constants $0 < c_1 \le c_2 < \infty$ independent of A and B.

For $\mu \in \mathcal{M}^+(\Omega)$, we denote by $L^p(\Omega, d\mu)$ the L^p space with respect to μ. When μ is the Lebesgue measure, we write $L^p(\Omega, dx)$ as $L^p(\Omega)$ simply. For a Banach space X, we denote by X^* the dual of X. We denote by c and C various constants with and without indices.

2. Preliminaries

2.1. Function spaces

Let Ω be a domain in $\mathbb{R}^n$, and let $1 < p < \infty$. The Sobolev space $W^{1,p}(\Omega)$ ($W^{1,p}_{\mathrm{loc}}(\Omega)$) is the space of all weakly differentiable functions u such that $u \in L^p(\Omega)$ and $|\nabla u| \in L^p(\Omega)$ ($u \in L^p_{\mathrm{loc}}(\Omega)$ and $|\nabla u| \in L^p_{\mathrm{loc}}(\Omega)$). The space $W^{1,p}_0(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$.

We denote by $\dot{W}^{1,p}_0(\Omega)$ the set of all functions $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ such that $|\nabla u| \in L^p(\Omega)$, and $\|\nabla(\varphi_j - u)\|_{L^p(\Omega)} \to 0$ as $j \to \infty$ for a sequence $\{\varphi_j\}_{j=1}^\infty \subset C_c^\infty(\Omega)$. The space $\dot{W}^{1,p}_0(\Omega)$ is called the homogeneous Sobolev space (or Dirichlet space). When $1 < p < n$ or when Ω is bounded, we define the norm of $\dot{W}^{1,p}_0(\Omega)$ by $\|\nabla \cdot\|_{L^p(\Omega)}$. If Ω is bounded, then $\dot{W}^{1,p}_0(\Omega) = W^{1,p}_0(\Omega)$ by the Poincaré inequality. The following basic properties of functions in Sobolev spaces are parenthetically used in our arguments. Their proofs are similar to the ones of [16, Theorems 1.18, 1.20 and 1.24].

Lemma 2.1. *Suppose that $u \in \dot{W}_0^{1,p}(\Omega)$.*

(i) *Let $f \in C^1(\mathbb{R})$ and $f(0) = 0$. Assume that f' is bounded on the range of u. Then $f(u) \in \dot{W}_0^{1,p}(\Omega)$ and $\nabla f(u) = f'(u)\nabla u$ a.e. in Ω.*

(ii) *Let $w = \min\{\max\{u, m\}, M\}$, where m and M are constants satisfying $m \le 0 \le M$. Then $w \in \dot{W}_0^{1,p}(\Omega)$ and $\nabla w = \mathbf{1}_{\{m<u<M\}}\nabla u$ a.e. in Ω.*

(iii) *Assume also that u is bounded. Suppose that $v \in W_{\mathrm{loc}}^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfies $|\nabla v| \in L^p(\Omega)$. Then $uv \in \dot{W}_0^{1,p}(\Omega)$ and $\nabla(uv) = v\nabla u + u\nabla v$ a.e. in Ω.*

We also recall notion of Lorentz spaces [14].

Definition 2.2. Let f be a measurable function on Ω, and let $0 < r, \rho \le \infty$. We define the *Lorentz norm* of f by

$$\|f\|_{L^{r,\rho}(\Omega)} = \begin{cases} \left(\int_0^\infty \left(t^{\frac{1}{r}} f^*(t) \right)^\rho \frac{dt}{t} \right)^{\frac{1}{\rho}} & \text{if } \rho < \infty, \\ \sup\limits_{t>0} t^{\frac{1}{r}} f^*(t) & \text{if } \rho = \infty, \end{cases}$$

where f^* is the *decreasing rearrangement* of f which is defined by

$$f^*(t) = \inf\{\alpha > 0 \colon |\{x \in \Omega \colon |f(x)| > \alpha\}| \le t\}.$$

The space of all f with $\|f\|_{L^{r,\rho}(\Omega)} < \infty$ is denoted by $L^{r,\rho}(\Omega)$ and is called the *Lorentz space* with indices r and ρ.

2.2. p-Laplacian and p-superharmonic functions

For $u \in W_{\mathrm{loc}}^{1,p}(\Omega)$, we define the p-Laplacian Δ_p in the weak (distributional) sense, i.e., for every $\varphi \in C_c^\infty(\Omega)$,

$$\langle -\Delta_p u, \varphi \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx.$$

A function $u \in W_{\mathrm{loc}}^{1,p}(\Omega)$ is called as *p-harmonic* if u is a continuous weak solution to

$$-\Delta_p u = 0 \quad \text{in } \Omega. \tag{2.1}$$

For basic properties of the p-Laplacian including comparison principles for weak solutions and solvability of Dirichlet problems, we refer to [16, Chapter 3] and [21].

To treat measure data problems, we introduce p-superharmonic functions. A function $u \colon \Omega \to (-\infty, \infty]$ is called *p-superharmonic* if u is lower semicontinuous in Ω, is not identically infinite in any component of Ω, and u satisfies the comparison principle on each subdomain $D \Subset \Omega$; if $h \in C(\overline{D})$ is p-harmonic in D and if $u \ge h$ on ∂D, then $u \ge h$ in D.

By [16, Theorem 7.22], if u and v are p-superharmonic in Ω and if $u \le v$ a.e. in Ω, then $u(x) \le v(x)$ for all $x \in \Omega$. If $u \in W_{\mathrm{loc}}^{1,p}(\Omega)$ is a supersolution to

(2.1), then it has a lower semicontinuous representative and can be regarded as a p-superharmonic function up to taking such a representative (see [16, Theorems 3.63 and 7.25]). If u is a p-superharmonic function in Ω, then its truncation $\min\{u,k\}$ is a supersolution to (2.1) for each $k > 0$. Hence, there exists a unique Radon measure $\nu[u]$ such that

$$\int_\Omega |\,\mathrm{D}u|^{p-2}\,\mathrm{D}u \cdot \nabla\varphi \, dx = \int_\Omega \varphi \, d\nu[u] \quad \forall \varphi \in C_c^\infty(\Omega),$$

where $\mathrm{D}u$ is the *very weak gradient* of u which is defined by

$$\mathrm{D}u := \lim_{k\to\infty} \nabla \min\{u,k\}.$$

The measure $\nu[u]$ is called the *Riesz measure* of u. By definition, if $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, then $\mathrm{D}u = \nabla u$ and $\nu[u] = -\Delta_p u$ in the sense of weak solutions.

We say that a property holds *quasieverywhere* (q.e.) if it holds except on a set of p-capacity zero. Here, for $E \subset \mathbb{R}^n$, the (Sobolev) p-capacity is defined by

$$C_p(E) = \inf \int_{\mathbb{R}^n} (|u|^p + |\nabla u|^p) \, dx,$$

where the infimum is taken over all $u \in W^{1,p}(\mathbb{R}^n)$ such that $u = 1$ in a neighborhood of E. We note that every $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ has a *quasicontinuous* representative, which coincides with u quasieverywhere and that every p-superharmonic function u is quasicontinuous (see, e.g., [16, Theorems 4.4 and 10.9]). Henceforth, we assume that u is always chosen to be quasicontinuous.

2.3. Wolff potentials

Definition 2.3. Let $1 < p < \infty$. The Wolff potential $\mathbf{W}_{1,p}\sigma$ of $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$ is defined by

$$\mathbf{W}_{1,p}\sigma(x) := \int_0^\infty \left(\frac{\sigma(B(x,r))}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}.$$

This nonlinear potential was first introduced by Havin and Maz'ya [22]. For any $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$, $\mathbf{W}_{1,p}\sigma(x)$ is a lower semicontinuous function of x (see [15]). By a simple calculation, for any $\sigma, \mu \in \mathcal{M}^+(\mathbb{R}^n)$ and $\gamma, \beta \geq 0$,

$$\mathbf{W}_{1,p}(\gamma\sigma + \beta\mu)(x) \leq c(p) \left(\gamma^{\frac{1}{p-1}} \mathbf{W}_{1,p}\sigma(x) + \beta^{\frac{1}{p-1}} \mathbf{W}_{1,p}\mu(x) \right) \quad \forall x \in \mathbb{R}^n. \tag{2.2}$$

Also, the following weak maximum principle for Wolff potentials holds:

$$\mathbf{W}_{1,p}\sigma(x) \leq c \sup_{\operatorname{supp}\sigma} \mathbf{W}_{1,p}\sigma \quad \forall x \in \mathbb{R}^n, \tag{2.3}$$

where $c = c(n,p) > 0$ (see, e.g., [29]). Moreover, by the lower semicontinuity of $\mathbf{W}_{1,p}\sigma$, we observe that $\sup_{\operatorname{supp}\sigma} \mathbf{W}_{1,p}\sigma = \|\mathbf{W}_{1,p}\sigma\|_{L^\infty(\mathbb{R}^n, d\sigma)}$.

It was shown in [27, Theorem 1.11] that for any $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$,

$$\int_K \frac{d\sigma}{(\mathbf{W}_{1,p}\sigma)^{p-1}} \leq C \mathrm{cap}_p(K, \mathbb{R}^n)$$

for any compact set $K \subset \mathbb{R}^n$, where $C = C(n,p)$ is a constant and $\mathrm{cap}_p(K, \mathbb{R}^n)$ is the variational p-capacity of $(K, \mathbb{R}^n)$. From this inequality, one can easily deduce (by using a similar argument in [9, Lemma 3.6]) that (1.6) implies σ must be absolutely continuous with respect to the p-capacity, that is, $\sigma(E) = 0$ whenever $C_p(E) = 0$ for every Borel set $E \subset \mathbb{R}^n$.

The following two-sided Wolff potential bounds were established by Kilpeläinen and Malý [18, 19].

Theorem 2.4 ([19, Theorem 1.6])**.** *Let $1 < p < \infty$. Suppose that u is a nonnegative p-superharmonic function in $2B = B(x, 2R)$ and that μ is the Riesz measure of u. Then*

$$\frac{1}{c_K} \mathbf{W}_{1,p}^R \mu(x) \leq u(x) \leq c_K \left(\inf_B u + \mathbf{W}_{1,p}^{2R} \mu(x) \right),$$

where $c_K = c_K(n,p) \geq 1$ and $\mathbf{W}_{1,p}^R \mu$ is the truncated Wolff potential of μ which is defined by

$$\mathbf{W}_{1,p}^R \mu(x) := \int_0^R \left(\frac{\mu(B(x,r))}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}.$$

3. Estimate for mutual energy and its consequences

The following Wolff energy estimate is our key ingredient.

Theorem 3.1. *Let $1 < p < n$, $0 < \gamma < \infty$ and $-\gamma < q < p-1$. Then for any $\mu, \sigma \in \mathcal{M}^+(\mathbb{R}^n)$,*

$$\begin{aligned} &\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\mu)^{\gamma+q} \, d\sigma \\ &\leq C \left(\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\mu)^{\gamma} \, d\mu \right)^{\frac{\gamma+q}{p-1+\gamma}} \left(\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\sigma)^{\frac{(\gamma+q)(p-1)}{p-1-q}} \, d\sigma \right)^{\frac{p-1-q}{p-1+\gamma}}, \end{aligned}$$

where C is a positive constant depending only on n, p, γ and q.

To derive this estimate, we prove the following simple lemma.

Lemma 3.2. *Let $1 < p < \infty$. Let $u \in \dot{W}_0^{1,p}(\Omega)$, and let $v \in W_{\mathrm{loc}}^{1,p}(\Omega)$ be a nonnegative p-superharmonic function in Ω. Assume also that $\|\nabla v\|_{L^p(\Omega)}$ and $\nu[v](\Omega)$ are finite. Then*

$$\int_\Omega |u|^p v^{1-p} \, d\nu[v] \leq \int_\Omega |\nabla u|^p \, dx.$$

Proof. We set $u_+^M = \min\{u_+, M\}$ and $v_M = v + M^{-1}$, where $u_+ = \max\{u, 0\}$ and M is a positive constant. Then $(u_+^M)^p (v_M)^{1-p} \in \dot{W}_0^{1,p}(\Omega)$ by Lemma 2.1. Since $\mathrm{D}v = \nabla v = \nabla v_M \in L^p(\Omega)$, by density arguments as in [16, Lemma 21.14],

$$\int_\Omega \varphi \, d\nu[v] = \int_\Omega |\nabla v_M|^{p-2} \nabla v_M \cdot \nabla \varphi \, dx \quad \forall \varphi \in \dot{W}_0^{1,p}(\Omega).$$

Substituting $(u_+^M)^p (v_M)^{1-p}$ into φ and using a Picone type inequality (see [2, Theorem 1.1]), we get

$$\int_\Omega (u_+^M)^p (v_M)^{1-p} \, d\nu[v] \le \int_\Omega |\nabla (u_+^M)|^p \, dx.$$

Taking the limit $M \to \infty$, we arrive at

$$\int_\Omega u_+^p v^{1-p} \, d\nu[v] \le \int_\Omega |\nabla u_+|^p \, dx.$$

Applying the same argument to $u_- = (-u)_+$, we get the desired estimate. $\square$

Lemma 3.3. *Let $1 < p < \infty$, $0 < \gamma < \infty$ and $-\gamma < q < p - 1$. Let $u \in \dot{W}_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, and let $v \in W_{\mathrm{loc}}^{1,p}(\Omega)$ be a nonnegative p-superharmonic function in Ω. Assume also that $\|\nabla v\|_{L^p(\Omega)}$ and $\nu[v](\Omega)$ are finite. Then*

$$\begin{aligned} &\int_\Omega |u|^{\gamma+q} \, d\nu[v] \\ &\le C \left(\int_\Omega |\nabla u|^p |u|^{\gamma-1} \, dx \right)^{\frac{\gamma+q}{p-1+\gamma}} \left(\int_\Omega v^{\frac{(\gamma+q)(p-1)}{p-1-q}} \, d\nu[v] \right)^{\frac{p-1-q}{p-1+\gamma}}, \end{aligned}$$

where C is a positive constant depending only on p, γ and q.

Proof. Without loss of generality, we may assume that both integrals on the right-hand side are finite and that $u \ge 0$. Applying Hölder's inequality to $d\omega = v^{1-p} d\nu[v]$, we get

$$\begin{aligned} &\int_\Omega u^{\gamma+q} \, d\nu[v] = \int_\Omega u^{\gamma+q} v^{p-1} \, d\omega \\ &\le \left(\int_\Omega u^{p-1+\gamma} \, d\omega \right)^{\frac{\gamma+q}{p-1+\gamma}} \left(\int_\Omega v^{\frac{(p-1+\gamma)(p-1)}{p-1-q}} \, d\omega \right)^{\frac{p-1-q}{p-1+\gamma}}. \end{aligned} \tag{3.1}$$

For each $\epsilon > 0$, we set $w_\epsilon = (u^{\frac{p-1+\gamma}{p}} - \epsilon)_+$. Then by Lemmas 2.1 and 3.2,

$$\int_\Omega w_\epsilon^p v^{1-p} \, d\nu[v] \le \int_\Omega |\nabla w_\epsilon|^p \, dx.$$

Since

$$\nabla w_\epsilon = \frac{p-1+\gamma}{p} \nabla u u^{\frac{\gamma-1}{p}} \mathbf{1}_{\{u^{\frac{p-1+\gamma}{p}} > \epsilon\}} \quad \text{a.e. in } \Omega,$$

by the monotone convergence theorem,

$$\int_\Omega u^{p-1+\gamma} v^{1-p}\, d\nu[v] \le \left(\frac{p-1+\gamma}{p}\right)^p \int_\Omega |\nabla u|^p u^{\gamma-1}\, dx. \tag{3.2}$$

From (3.1) and (3.2), we obtain the desired estimate. □

Let $1 < p < n$. For $k \in \mathbb{N}$, we set $\mu_k = \mathbf{1}_{\Omega(\mu,k)}\mu$, where

$$\Omega(\mu, k) = \{x \in \mathbb{R}^n : \mathbf{W}_{1,p}\mu(x) \le k\} \cap \overline{B(0, 2^k)}. \tag{3.3}$$

Then

$$\int_{\mathbb{R}^n} \mathbf{W}_{1,p}\mu_k\, d\mu_k = \int_{\Omega(\mu,k)} \mathbf{W}_{1,p}\mu_k\, d\mu \le k\mu(\Omega(\mu, k)) < \infty$$

which is equivalent to $\mu_k \in (\dot{W}_0^{1,p}(\mathbb{R}^n))^*$ by the Hedberg-Wolff theorem (see [15] or [1, Theorem 4.5.4]). Thus, there exists a unique nonnegative p-superharmonic function $u_k \in \dot{W}_0^{1,p}(\mathbb{R}^n)$ satisfying $-\Delta_p u_k = \mu_k$ in $\mathbb{R}^n$ and $\liminf_{|x|\to\infty} u_k(x) = 0$. Moreover, by Theorem 2.4 and (2.3),

$$\|u\|_{L^\infty(\mathbb{R}^n)} \le c_K \|\mathbf{W}_{1,p}\mu_k\|_{L^\infty(\mathbb{R}^n)} \le Ck.$$

Remark 3.4. For $\mu \in \mathcal{M}^+(\mathbb{R}^n)$, the p-capacity of $\{x \in \mathbb{R}^n : \mathbf{W}_{1,p}\mu(x) = \infty\}$ is zero by Theorem 2.4 (see [20, Remark 3.7]). Thus, if μ is absolutely continuous with respect to the p-capacity, then $\mathbf{1}_{\Omega(\mu,k)} \uparrow \mathbf{1}_{\mathbb{R}^n}$ $d\mu$-a.e., and $\mathbf{W}_{1,p}\mu_k(x) \uparrow \mathbf{W}_{1,p}\mu(x)$ for all $x \in \mathbb{R}^n$.

Proof of Theorem 3.1. We may assume that both integrals on the right-hand side are finite without loss of generality. Thus, μ and σ are absolutely continuous with respect to the p-capacity. For each $k \in \mathbb{N}$, put $\mu_k = \mathbf{1}_{\Omega(\mu,k)}\mu$ and $\sigma_k = \mathbf{1}_{\Omega(\sigma,k)}\sigma$, where $\Omega(\mu, k)$ and $\Omega(\sigma, k)$ are defined by (3.3). Let $u_k, v_k \in \dot{W}_0^{1,p}(\mathbb{R}^n)$ be the bounded finite energy p-superharmonic functions satisfying $-\Delta_p u_k = \mu_k$ and $-\Delta_p v_k = \sigma_k$ in $\mathbb{R}^n$, respectively. By Theorem 2.4, $u_k \approx \mathbf{W}_{1,p}\mu_k$ and $v_k \approx \mathbf{W}_{1,p}\sigma_k$ in $\mathbb{R}^n$. By Lemma 2.1, $(u_k^\gamma - \epsilon)_+ \in \dot{W}_0^{1,p}(\mathbb{R}^n)$ for any $\epsilon > 0$. Since μ_k is the Riesz measure of u_k,

$$\begin{aligned}\int_{\mathbb{R}^n} (u_k^\gamma - \epsilon)_+\, d\mu_k &= \int_{\mathbb{R}^n} |\nabla u_k|^{p-2}\nabla u_k \cdot \nabla(u_k^\gamma - \epsilon)_+\, dx \\ &= \gamma \int_{\{u_k^\gamma > \epsilon\}} |\nabla u_k|^p u_k^{\gamma-1}\, dx.\end{aligned}$$

By the monotone convergence theorem,

$$\begin{aligned}\int_{\mathbb{R}^n} u_k^\gamma\, d\mu_k &= \lim_{\epsilon\to 0} \int_{\mathbb{R}^n} (u_k^\gamma - \epsilon)_+\, d\mu_k \\ &= \lim_{\epsilon\to 0} \gamma \int_{\{u_k^\gamma > \epsilon\}} |\nabla u_k|^p u_k^{\gamma-1}\, dx = \gamma \int_{\mathbb{R}^n} |\nabla u_k|^p u_k^{\gamma-1}\, dx.\end{aligned}$$

Consequently, we have the following estimates:

$$\begin{aligned}\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\mu_k)^{\gamma+q}\, d\sigma_k &\approx \int_{\mathbb{R}^n} u_k^{\gamma+q}\, d\sigma_k,\\ \int_{\mathbb{R}^n} v_k^{\frac{(\gamma+q)(p-1)}{p-1-q}}\, d\sigma_k &\approx \int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\sigma_k)^{\frac{(\gamma+q)(p-1)}{p-1-q}}\, d\sigma_k,\\ \int_{\mathbb{R}^n} |\nabla u_k|^p u_k^{\gamma-1}\, dx &\approx \int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\mu_k)^{\gamma}\, d\mu_k.\end{aligned}$$

Here, the constants in equivalence depend only on n, p, γ and q. Combining these estimates and Lemma 3.3, we get

$$\begin{aligned}&\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\mu_k)^{\gamma+q}\, d\sigma_k\\ &\leq C\left(\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\mu_k)^{\gamma}\, d\mu_k\right)^{\frac{\gamma+q}{p-1+\gamma}}\left(\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\sigma_k)^{\frac{(\gamma+q)(p-1)}{p-1-q}}\, d\sigma_k\right)^{\frac{p-1-q}{p-1+\gamma}}.\end{aligned}$$

By Remark 3.4, $\mathbf{W}_{1,p}\mu_k(x) \uparrow \mathbf{W}_{1,p}\mu(x)$ for all $x \in \mathbb{R}^n$ and $\mathbf{1}_{\Omega(\sigma,k)} \uparrow \mathbf{1}_{\mathbb{R}^n}$ $d\sigma$-a.e. Therefore,

$$(\mathbf{W}_{1,p}\mu_k)^{\gamma+q}\mathbf{1}_{\Omega(\sigma,k)} \uparrow (\mathbf{W}_{1,p}\mu)^{\gamma+q} \quad d\sigma\text{-a.e.}$$

Using the monotone convergence theorem, we arrive at the desired estimate. □

The following quasi-triangle inequality is a direct consequence of Theorem 3.1. When $\gamma = 1$, it readily follows from the Hedberg-Wolff theorem.

Corollary 3.5. *Let $1 < p < n$ and $0 < \gamma < \infty$. Then for any $\mu, \nu \in \mathcal{M}^+(\mathbb{R}^n)$,*

$$\begin{aligned}&\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}(\mu+\nu))^{\gamma}\, d(\mu+\nu)\\ &\approx \int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\mu)^{\gamma}\, d\mu + \int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\nu)^{\gamma}\, d\nu,\end{aligned}$$

where the constants in equivalence depend only on n, p and γ.

Proof. Each of the right-hand side is controlled by the left-hand side. Let us estimate the left-hand side. By (2.2),

$$\begin{aligned}&\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}(\mu+\nu))^{\gamma}\, d(\mu+\nu)\\ &\leq C\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\mu)^{\gamma}\, d\mu + C\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\nu)^{\gamma}\, d\nu\\ &+ C\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\mu)^{\gamma}\, d\nu + C\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\nu)^{\gamma}\, d\mu.\end{aligned}$$

Applying Theorem 3.1 with $q = 0$, we can estimate the latter two terms by other two. Then the assertion follows from Young's inequality. □

Corollary 3.6. *Let $1 < p < n$ and $0 < \gamma < \infty$. Then*

$$\|\mathbf{W}_{1,p}\mu\|_{L^{r,\rho}(\mathbb{R}^n)} \le C \left(\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\mu)^\gamma \, d\mu \right)^{\frac{1}{p-1+\gamma}} \tag{3.4}$$

for any $\mu \in \mathcal{M}^+(\mathbb{R}^n)$, where $r = n(p-1+\gamma)/(n-p)$, $\rho = p-1+\gamma$ and C is a positive constant depending only on n, p and γ.

Proof. It is known that for any $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$,

$$\mathbf{W}_{1,p}\sigma(x) \le c(n,p)\mathbf{V}_{1,p}\sigma(x)$$

for all $x \in \mathbb{R}^n$, where $\mathbf{V}_{1,p}\sigma := \mathbf{I}_1[(\mathbf{I}_1\sigma)^{\frac{1}{p-1}} dx]$ is the Havin-Maz'ya potential of σ (see [15]). Therefore, by boundedness of Riesz potentials (see, e.g., [14, Theorem 1.4.19]),

$$\|\mathbf{W}_{1,p}(f dx)\|_{L^{r,\rho}(\mathbb{R}^n)} \le C \|f\|^{\frac{1}{p-1}}_{L^{\frac{r}{r-\gamma},\frac{\rho}{\rho-\gamma}}(\mathbb{R}^n)}$$

for any nonnegative $f \in L^{\frac{r}{r-\gamma},\frac{\rho}{\rho-\gamma}}(\mathbb{R}^n)$. Hence, by Hölder's inequality,

$$\begin{aligned}
\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}(f dx))^\gamma f \, dx &\le \|(\mathbf{W}_{1,p}(f dx))^\gamma\|_{L^{\frac{r}{\gamma},\frac{\rho}{\gamma}}(\mathbb{R}^n)} \|f\|_{L^{\frac{r}{r-\gamma},\frac{\rho}{\rho-\gamma}}(\mathbb{R}^n)} \\
&= \|\mathbf{W}_{1,p}(f dx)\|^\gamma_{L^{r,\rho}(\mathbb{R}^n)} \|f\|_{L^{\frac{r}{r-\gamma},\frac{\rho}{\rho-\gamma}}(\mathbb{R}^n)} \\
&\le C \|f\|^{1+\frac{\gamma}{p-1}}_{L^{\frac{r}{r-\gamma},\frac{\rho}{\rho-\gamma}}(\mathbb{R}^n)}.
\end{aligned} \tag{3.5}$$

Combining (3.5) and Theorem 3.1, we get

$$\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\mu)^\gamma f \, dx \le C \left(\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\mu)^\gamma \, d\mu \right)^{\frac{\gamma}{p-1+\gamma}} \|f\|_{L^{\frac{r}{r-\gamma},\frac{\rho}{\rho-\gamma}}(\mathbb{R}^n)}.$$

Thus, by a dual characterization of $L^{\frac{r}{\gamma},\frac{\rho}{\gamma}}(\mathbb{R}^n)$,

$$\begin{aligned}
&\|\mathbf{W}_{1,p}\mu\|^\gamma_{L^{r,\rho}(\mathbb{R}^n)} = \|(\mathbf{W}_{1,p}\mu)^\gamma\|_{L^{\frac{r}{\gamma},\frac{\rho}{\gamma}}(\mathbb{R}^n)} \\
&\le C \sup \left\{ \int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\mu)^\gamma f \, dx \colon \|f\|_{L^{\frac{r}{r-\gamma},\frac{\rho}{\rho-\gamma}}(\mathbb{R}^n)} \le 1, f \ge 0 \right\} \\
&\le C \left(\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\mu)^\gamma \, d\mu \right)^{\frac{\gamma}{p-1+\gamma}}.
\end{aligned}$$

This completes the proof. □

Remark 3.7. The inequality (3.5) gives a refinement of [23, Corollary 1.2]. Suppose that $1 < p < n$, $0 < q < p-1$ and $0 < \gamma < \infty$. Let f and g be nonnegative functions on $\mathbb{R}^n$.

1. If $d\sigma = f dx$ and if

$$f \in L^{s,t}(\mathbb{R}^n) \text{ with } s = \frac{n(p-1+\gamma)}{n(p-1-q)+p(\gamma+q)},\ t = \frac{p-1+\gamma}{p-1-q},$$

then (1.6) is fulfilled.

2. If $d\mu = g dx$ and if

$$g \in L^{s,t}(\mathbb{R}^n) \text{ with } s = \frac{n(p-1+\gamma)}{n(p-1)+p\gamma},\ t = \frac{p-1+\gamma}{p-1},$$

then (1.7) is fulfilled.

Hence, in view of Theorem 1.1, there exists a minimal p-superharmonic solution $u \in L^{r,\rho}(\mathbb{R}^n)$ to (1.1), where $r = n(p-1+\gamma)/(n-p)$ and $\rho = p-1+\gamma$.

4. Construction of minimal solutions to (1.1)

Throughout, we assume that $1 < p < n$ and $0 < q < p-1$. Our definition of generalized solutions is as follows:

Definition 4.1. Let Ω be a domain in $\mathbb{R}^n$, and let $\sigma, \mu \in \mathcal{M}^+(\Omega)$. A nonnegative function u is said to be a *p-superharmonic solution* (*supersolution*) to the equation

$$-\Delta_p u = \sigma u^q + \mu \quad \text{in } \Omega$$

if u is p-superharmonic in Ω, $u \in L^q_{\text{loc}}(\Omega, d\sigma)$, and $\nu[u] = u^q d\sigma + \mu$ ($\nu[u] \geq u^q d\sigma + \mu$). We say that a nontrivial p-superharmonic solution u to (1.1) is *minimal* if $w \geq u$ in $\mathbb{R}^n$ whenever w is a nontrivial p-superharmonic supersolution to (1.1).

The following theorem was established by Cao and Verbitsky [10, 9]. It gives pointwise lower estimates of supersolutions to (1.2).

Theorem 4.2 ([10, Theorem 2.3])**.** *Let $w \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$ be a positive p-superharmonic supersolution to* (1.2)*. Then*

$$w \geq c_0 (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}},$$

where c_0 is a constant depending only on n, p and q.

To construct minimal solutions to (1.1), we consider a family of solutions to localized problems and solve the localized problems using a sub- and supersolution method. We shall need the following weighted norm inequality.

Lemma 4.3 ([23, Lemma 2.1])**.** *Assume that* (1.6) *holds with $0 < \gamma < \infty$. Then for any $f \in L^{\frac{\gamma+q}{q}}(\mathbb{R}^n, d\sigma)$,*

$$\|\mathbf{W}_{1,p}(f d\sigma)\|_{L^{\gamma+q}(\mathbb{R}^n, d\sigma)} \leq C \|f\|^{\frac{1}{p-1}}_{L^{\frac{\gamma+q}{q}}(\mathbb{R}^n, d\sigma)},$$

where C is a constant depending only on n, p, q, γ and the upper bound of (1.6).

Lemma 4.4. *Let $B = B(x_0, R)$ be a ball in $\mathbb{R}^n$. Assume that* (1.6) *and* (1.7) *hold with $0 < \gamma < \infty$. Assume also that*

$$\sup_{\operatorname{supp}\sigma} \mathbf{W}_{1,p}\sigma < \infty, \quad \operatorname{supp}\sigma \subset \overline{B}$$

and

$$\sup_{\operatorname{supp}\mu} \mathbf{W}_{1,p}\mu < \infty, \quad \operatorname{supp}\mu \subset \overline{B}.$$

Then there exists a nonnegative p-superharmonic function u satisfying

$$-\Delta_p u = \sigma u^q + \mu \quad \text{in } 2B \tag{4.1}$$

and

$$\min\{u, l\} \in \dot{W}_0^{1,p}(2B) \quad \forall l > 0.$$

Moreover, u satisfies the following properties:

(i) *The solution u belongs to $L^{\gamma+q}(2B, d\sigma) \cap L^{r,\rho}(2B)$. Moreover,*

$$\left(\int_{2B} (\mathbf{W}_{1,p}(u^q d\sigma))^\gamma u^q \, d\sigma\right)^{\frac{1}{p-1+\gamma}} + \|u\|_{L^{r,\rho}(2B)} \leq C,$$

where C is a constant depending only on n, p, q, γ and the bounds of (1.6) *and* (1.7)*.*

(ii) *For every $x \in B$,*

$$u(x) \geq \frac{1}{C}\left\{(\mathbf{W}^{\frac{R}{2}}_{1,p}\sigma(x))^{\frac{p-1}{p-1-q}} + \mathbf{W}^{\frac{R}{2}}_{1,p}\mu(x)\right\},$$

where C is a positive constant depending only on n, p and q.

(iii) *If w is a p-superharmonic supersolution to* (1.1)*, then $w \geq u$ in $2B$.*

(iv) *Let u_1 be a p-superharmonic function which defined by* (4.4)*. Assume that w is a nonnegative p-superharmonic supersolution to* (4.1) *and that $w \geq u_1$ in $2B$. Then $w \geq u \geq u_1$ in $2B$.*

Proof. **Step 1.** We construct approximate solutions $\{u_j\}_{j=1}^\infty$. By assumptions on σ and μ, along with the Hedberg-Wolff theorem ([1, Theorem 4.5.4]), σ and μ belong to $(\dot{W}_0^{1,p}(2B))^*$. Therefore, there exists a bounded finite energy p-superharmonic function $v \in \dot{W}_0^{1,p}(2B) \cap L^\infty(2B)$ satisfying

$$-\Delta_p v = \tilde{\sigma} := \left(\frac{p-1-q}{p-1}\right)^{p-1} \sigma \quad \text{in } 2B.$$

Then for any $\beta \geq 1$, $v^\beta \in \dot{W}_0^{1,p}(2B) \cap L^\infty(2B)$. Moreover,

$$\int_{2B} |\nabla v^\beta|^{p-2} \nabla v^\beta \cdot \nabla\varphi \, dx \leq \beta^{p-1} \int_{2B} |\nabla v|^{p-2} \nabla v \cdot \nabla(v^{(\beta-1)(p-1)}\varphi) \, dx$$

for any nonnegative $\varphi \in C_c^\infty(2B)$. In other words,

$$-\Delta_p v^\beta \leq \beta^{p-1} v^{(\beta-1)(p-1)} \tilde{\sigma} \quad \text{in } 2B$$

in the sense of distribution. Let

$$u_0 = c_1 v^{\frac{p-1}{p-1-q}}, \tag{4.2}$$

where $c_1 := \min\{c_0 c_K^{\frac{(1-p)}{p-1-q}}, 1\}$. Here, c_K and c_0 are the constants in Theorems 2.4 and 4.2, respectively. Then $u_0 \in \dot{W}_0^{1,p}(2B) \cap L^\infty(2B)$ and

$$-\Delta_p u_0 \leq (c_1)^{p-1-q} \sigma u_0^q \leq \sigma u_0^q \quad \text{in } 2B.$$

Moreover, by Theorem 2.4,

$$u_0 \leq c_0 (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}} \quad \text{in } 2B. \tag{4.3}$$

We define a sequence of p-superharmonic functions $\{u_j\}_{j=1}^\infty$ by

$$-\Delta_p u_{j+1} = \sigma u_j^q + \mu, \quad \text{in } 2B,\ j = 0, 1, 2, \ldots. \tag{4.4}$$

Assume that u_j is bounded for some $j \geq 0$. Then the measure $\sigma u_j^q + \mu$ belongs to the dual of $\dot{W}_0^{1,p}(2B)$, and $\mathbf{W}_{1,p}(u_j^q d\sigma + d\mu)$ is bounded. By the comparison principle for weak solutions, $u_{j+1} \geq u_j$ for all $j \geq 0$. Therefore, $\{u_j\}_{j=1}^\infty$ is defined as an increasing sequence of bounded finite energy p-superharmonic functions. By Theorem 2.4, for every $x \in B$,

$$\begin{aligned} u_1(x) &\geq \max\left\{ u_0(x), \frac{1}{c_K} \mathbf{W}_{1,p}^{\frac{R}{2}} \mu(x) \right\} \\ &\geq \frac{1}{C} \left\{ (\mathbf{W}_{1,p}^{\frac{R}{2}}\sigma)^{\frac{p-1}{p-1-q}}(x) + \mathbf{W}_{1,p}^{\frac{R}{2}}\mu(x) \right\}. \end{aligned} \tag{4.5}$$

Assume that w is a nonnegative p-superharmonic supersolution to (1.1). Then by Theorem 4.2 and (4.3),

$$u_0 \leq c_0 (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}} \leq w \quad d\sigma\text{-a.e. in } 2B.$$

Since σ is absolutely continuous with respect to the p-capacity, it follows from the comparison principle for renormalized solutions (see [9, Lemma 5.2]) that $w \geq u_1$ in $2B$. Thus, by induction,

$$w \geq u_j \quad \text{in } 2B \tag{4.6}$$

for all $j \geq 1$. The same argument is valid if w satisfies assumptions in (iv).
Step 2. We give bounds of $\{u_j\}_{j=1}^\infty$. For simplicity, we denote by u_j the zero extension of u_j again. By (4.3) and (1.6), $u_0 \in L^{\gamma+q}(2B, d\sigma)$. Assume that

$u_j \in L^{\gamma+q}(2B, d\sigma)$ for some $j \geq 0$. Then by Theorem 2.4, the comparison principle and (2.2),

$$\begin{aligned}\|u_{j+1}\|_{L^{\gamma+q}(2B,d\sigma)} &\leq c_K \|\mathbf{W}_{1,p}(u_j^q d\sigma + d\mu)\|_{L^{\gamma+q}(2B,d\sigma)} \\ &\leq C\|\mathbf{W}_{1,p}(u_j^q d\sigma)\|_{L^{\gamma+q}(2B,d\sigma)} + C\|\mathbf{W}_{1,p}\mu\|_{L^{\gamma+q}(2B,d\sigma)} \\ &= C\|\mathbf{W}_{1,p}(u_j^q d\sigma)\|_{L^{\gamma+q}(\mathbb{R}^n,d\sigma)} + C\|\mathbf{W}_{1,p}\mu\|_{L^{\gamma+q}(\mathbb{R}^n,d\sigma)}.\end{aligned}$$

By Lemma 4.3,

$$\|\mathbf{W}_{1,p}(u_j^q d\sigma)\|_{L^{\gamma+q}(\mathbb{R}^n,d\sigma)} \leq C\|u_j^q\|^{\frac{1}{p-1}}_{L^{\frac{\gamma+q}{q}}(\mathbb{R}^n,d\sigma)} = C\|u_j\|^{\frac{q}{p-1}}_{L^{\gamma+q}(2B,d\sigma)}. \tag{4.7}$$

Moreover, by Theorem 3.1 and (1.7),

$$\|\mathbf{W}_{1,p}\mu\|_{L^{\gamma+q}(\mathbb{R}^n,d\sigma)} \leq C.$$

Thus,

$$\|u_{j+1}\|_{L^{\gamma+q}(2B,d\sigma)} \leq C(\|u_j\|^{\frac{q}{p-1}}_{L^{\gamma+q}(2B,d\sigma)} + 1).$$

Since $q < p-1$, by Young's inequality and monotonicity of u_j,

$$\|u_{j+1}\|_{L^{\gamma+q}(2B,d\sigma)} \leq C. \tag{4.8}$$

Then by Hölder's inequality and Lemma 4.3,

$$\left(\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}(u_j^q d\sigma))^\gamma u_j^q \, d\sigma\right)^{\frac{1}{p-1+\gamma}} \leq C\|u_j^q\|^{\frac{1}{p-1}}_{L^{\frac{\gamma+q}{q}}(\mathbb{R}^n,d\sigma)} = C\|u_j\|^{\frac{q}{p-1}}_{L^{\gamma+q}(2B,d\sigma)}.$$

Hence, by (4.8),

$$\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}(u_j^q d\sigma))^\gamma u_j^q \, d\sigma \leq C. \tag{4.9}$$

Fix $l > 0$. Testing (4.4) with $\min\{u_{j+1}, l\}$, we get

$$\begin{aligned}\int_{2B} |\nabla \min\{u_{j+1}, l\}|^p \, dx &\leq l\left(\int_{2B} u_j^q \, d\sigma + \mu(2B)\right) \\ &\leq l\left(\|u_j\|^q_{L^{\gamma+q}(2B,d\sigma)} \sigma(2B)^{\frac{\gamma}{\gamma+q}} + \mu(2B)\right).\end{aligned}$$

By using (4.8) again,

$$\int_{2B} |\nabla \min\{u_{j+1}, l\}|^p \, dx \leq Cl. \tag{4.10}$$

Here, the constant C depends also on $\sigma(2B)$ and $\mu(2B)$, but not on $j \in \mathbb{N}$.
Step 3. Let

$$u = \lim_{j\to\infty} u_j.$$

By (4.10) and the Poincaré inequality, u is not identically infinite. Hence, u is p-superharmonic in $2B$ by [16, Lemma 7.3]. Moreover, by the weakly continuity result in [26, Theorem 3.1], we have $\nu[u] = \sigma u^q + \mu$. Applying the monotone convergence theorem, we have

$$\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}(u^q d\sigma))^\gamma u^q \, d\sigma \leq C. \tag{4.11}$$

By Corollary 3.5,

$$\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\nu[u])^\gamma d\nu[u] \approx \int_{\mathbb{R}^n} (\mathbf{W}_{1,p}(u^q d\sigma))^\gamma u^q \, d\sigma + \int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\mu)^\gamma \, d\mu.$$

Thus, by Theorem 2.4, Corollary 3.6, (4.11) and (1.7),

$$\|u\|_{L^{r,\rho}(2B)} \leq c_K \|\mathbf{W}_{1,p}\nu[u]\|_{L^{r,\rho}(\mathbb{R}^n)} \leq C.$$

Also, by (4.10), $\min\{u, l\} \in \dot{W}_0^{1,p}(2B)$ for all $l > 0$. The properties (ii), (iii) and (iv) follow from (4.5) and (4.6). □

Proof of Theorem 1.1. For each $k \in \mathbb{N}$, we consider measures $\sigma_k = \mathbf{1}_{\Omega(\sigma,k)}\sigma$ and $\mu_k = \mathbf{1}_{\Omega(\mu,k)}\mu$, where $\Omega(\sigma, k)$ and $\Omega(\mu, k)$ are defined by (3.3). Using Lemma 4.4, we construct a sequence of p-superharmonic functions $\{u_k\}_{k=1}^\infty$ satisfying

$$-\Delta_p u_k = \sigma_k u_k^q + \mu_k \quad \text{in } B(0, 2^{k+1}) \tag{4.12}$$

and

$$\min\{u, l\} \in W_0^{1,p}(B(0, 2^{k+1})) \quad \forall l > 0.$$

By the comparison principle for weak solutions and (iv) in Lemma 4.4,

$$u_{k+1} \geq u_k \quad \text{in } B(0, 2^{k+1}).$$

We denote by u_k the zero extension of u_k again.

Let $u = \lim_{k\to\infty} u_k$. Then by the monotone convergence theorem,

$$\|u\|_{L^{r,\rho}(\mathbb{R}^n)} = \lim_{k\to\infty} \|u_k\|_{L^{r,\rho}(\mathbb{R}^n)} \leq C. \tag{4.13}$$

By Remark 3.4, $u_k^q \mathbf{1}_{\Omega(\sigma,k)} \uparrow u^q$ $d\sigma$-a.e. Hence, using the monotone convergence theorem twice, we get

$$\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}(u^q d\sigma))^\gamma u^q \, d\sigma \leq C. \tag{4.14}$$

By (4.13), u is not identically infinite, and $\liminf_{|x|\to\infty} u(x) = 0$. Therefore, u is p-superharmonic in $\mathbb{R}^n$ and $\nu[u] = \sigma u^q + \mu$. By Theorem 2.4 and Corollary 3.5,

$$\begin{aligned} \int_{\mathbb{R}^n} u^\gamma \, d\nu[u] &\approx \int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\nu[u])^\gamma \, d\nu[u] \\ &\approx \int_{\mathbb{R}^n} (\mathbf{W}_{1,p}(u^q d\sigma))^\gamma u^q \, d\sigma + \int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\mu)^\gamma \, d\mu. \end{aligned}$$

Thus, (1.8) is finite by (4.14) and (1.7).

Fix any $x \in \mathbb{R}^n$. Then by (*ii*) in Lemma 4.4,

$$u(x) \geq u_k(x) \geq \frac{1}{C}\left\{(\mathbf{W}_{1,p}^{2^{k-1}}\sigma_k)^{\frac{p-1}{p-1-q}}(x) + (\mathbf{W}_{1,p}^{2^{k-1}}\mu_k)(x)\right\}$$

whenever $x \in B(0, 2^k)$. Passing to the limit $k \to \infty$ and applying the monotone convergence theorem to the right-hand side, we get

$$u(x) \geq \frac{1}{C}\left\{(\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}}(x) + (\mathbf{W}_{1,p}\mu)(x)\right\}.$$

Hence, u is positive in $\mathbb{R}^n$. If w is a p-superharmonic supersolution to (1.1), then $w \geq u_k$ for all $k \in \mathbb{N}$ by (*iii*) in Lemma 4.4. Therefore, $w \geq u$. This implies that u is minimal. □

Corollary 4.5. *Assume that there exists a positive p-superharmonic supersolution $w \in L^q_{\mathrm{loc}}(\mathbb{R}^n, d\sigma)$ to* (1.1) *satisfying* (1.8) *for some $\gamma > 0$. Then there exists a minimal p-superharmonic solution $u \in L^{r,\rho}(\mathbb{R}^n)$ to* (1.1) *satisfying* (1.8) *and $w \geq u$ in $\mathbb{R}^n$.*

Proof. In this case, (1.6) and (1.7) are fulfilled by Theorems 4.2 and 2.4, respectively. Hence there exists a minimal p-superharmonic solution u to (1.1) by Theorem 1.1. By minimality of u, $w \geq u$ in $\mathbb{R}^n$. □

Corollary 4.6. *Assume that the assumptions of Theorem 1.1 hold with $\gamma \geq 1$. Let u be the minimal solution in Theorem 1.1. Then u belongs to $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ and satisfies* (1.1) *in the sense of weak solutions.*

Proof. We give additional estimates for solutions $\{u_j\}_{j=1}^{\infty}$ in Lemma 4.4. By Corollary 3.5,

$$\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\nu[u_j])^{\gamma} d\nu[u_j] \approx \int_{\mathbb{R}^n} (\mathbf{W}_{1,p}(u_j^q d\sigma))^{\gamma} u_j^q \, d\sigma + \int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\mu)^{\gamma} \, d\mu.$$

The right-hand side is estimated by (*i*) in Lemma 4.4 and (1.7). Thus, by Theorem 2.4,

$$\begin{aligned}\|u_{j+1}\|_{L^{\gamma}(2B,d\mu)} &\leq c_K \|\mathbf{W}_{1,p}\nu[u_j]\|_{L^{\gamma}(2B,d\mu)} \\ &\leq c_K \|\mathbf{W}_{1,p}\nu[u_j]\|_{L^{\gamma}(\mathbb{R}^n,d\nu[u_j])} \leq C.\end{aligned}$$

Fix a ball $B(x_0, R_0) \subset B$, and take a nonnegative function $\eta \in C_c^{\infty}(B(x_0, 2R_0))$ satisfying $\eta \equiv 1$ on $B(x_0, R_0)$ and $|\nabla \eta| \leq C/R_0$. Testing (4.4) with $u_{j+1}\eta^p$ and using Young's inequality, we get

$$\begin{aligned}&\int_{2B} |\nabla u_{j+1}|^p \eta^p \, dx \\ &\leq C\left(\int_{2B} u_{j+1}^p |\nabla \eta|^p \, dx + \int_{2B} u_{j+1}\eta^p u_j^q \, d\sigma + \int_{2B} u_{j+1}\eta^p \, d\mu\right) \\ &\leq C\left(\frac{C}{R_0^p}\int_{B(x_0,2R_0)} u_{j+1}^p \, dx + \int_{B(x_0,2R_0)} u_{j+1}^{1+q} \, d\sigma + \int_{B(x_0,2R_0)} u_{j+1} \, d\mu\right).\end{aligned}$$

Since $\gamma \geq 1$, by (i) in Lemma 4.4 and Hölder's inequality,

$$\|u_{j+1}\|_{L^p(B(x_0,2R_0))} \leq CR_0^{\frac{n}{p}-\frac{n-p}{p-1+\gamma}}\|u\|_{L^{r,\rho}(2B)} \leq C$$

and

$$\|u_{j+1}\|_{W^{1,p}(B(x_0,R_0))} \leq C'.$$

Here, the constant C' depends also on R_0, $\sigma(\overline{B(x_0,2R_0)})$ and $\mu(\overline{B(x_0,2R_0)})$, but not on $j \in \mathbb{N}$ and B. Therefore, $\lim_{j\to\infty} u_j = u \in W^{1,p}(B(x_0,R_0))$ and $\|u\|_{W^{1,p}(B(x_0,R_0))} \leq C'$. Applying the same limit argument to $\{u_k\}_{k=1}^{\infty}$ in Theorem 1.1, we see that $\lim_{k\to\infty} u_k = u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$. □

5. Remarks for the endpoint cases

When $\gamma = \infty$, we replace (1.6) and (1.7) with the following conditions:

$$\|\mathbf{W}_{1,p}\sigma\|_{L^\infty(\mathbb{R}^n,d\sigma)} < \infty, \tag{1.6'}$$

$$\|\mathbf{W}_{1,p}\mu\|_{L^\infty(\mathbb{R}^n,d\mu)} < \infty. \tag{1.7'}$$

Theorem 5.1. *Let $1 < p < n$ and $0 < q < p-1$. Let $\sigma, \mu \in \mathcal{M}^+(\mathbb{R}^n)$ with $(\sigma,\mu) \not\equiv (0,0)$. Assume that (1.6′) and (1.7′) hold. Then there exists a minimal bounded weak solution $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ to (1.1). Conversely, if there exists a bounded p-superharmonic supersolution $u \in L^q_{\mathrm{loc}}(\mathbb{R}^n, d\sigma)$ to (1.1), then (1.6′) and (1.7′) are fulfilled.*

Proof. We consider solutions to (4.1). As in the proof of Lemma 4.4, let $\{u_j\}_{j=1}^{\infty}$ be the sequence of p-superharmonic functions defined in (4.2) and (4.4). Then by (1.6′) and (2.3),

$$\|\mathbf{W}_{1,p}(u_j^q d\sigma)\|_{L^\infty(2B)} \leq C\|u_j\|_{L^\infty(2B,d\sigma)}^{\frac{q}{p-1}} \tag{5.1}$$

for all $j \geq 0$. Replacing (4.7) by (5.1) and using (1.7′), we get

$$\|u_{j+1}\|_{L^\infty(2B)} \leq C\left(\|u_j\|_{L^\infty(2B)}^{\frac{q}{p-1}} + 1\right).$$

Therefore,

$$\|u_j\|_{L^\infty(2B)} \leq C$$

for all $j \geq 1$ and $u = \lim_{j\to\infty} u_j$ is a bounded solution to (4.1). Using solutions to (4.12), we construct a p-superharmonic function u in $\mathbb{R}^n$ satisfying $\nu[u] = \sigma u^q + \mu$ in $\mathbb{R}^n$. Then by the bound of solutions to (4.12), u is bounded on $\mathbb{R}^n$. Moreover, by Theorem 2.4,

$$u_k(x) \leq c_K \mathbf{W}_{1,p}\nu[u_k](x) \leq C\|u\|_{L^\infty(\mathbb{R}^n)}^{\frac{q}{p-1}}\mathbf{W}_{1,p}\sigma(x) + C\mathbf{W}_{1,p}\mu(x)$$

for all $k \geq 1$ and for all $x \in \mathbb{R}^n$. By [9, Corollary 3.2], this implies that $\liminf_{|x|\to\infty} u(x) = 0$. From the arguments in the proof of Theorem 1.1, it follows that u is a minimal p-superharmonic solution to (1.1). Since u is a bounded p-superharmonic function, $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ by [16, Theorem 7.25].

The converse part follows from Theorems 2.4 and 4.2. □

The case of $\gamma = 0$ is more delicate. We give a sufficient condition for the existence of minimal solutions to (1.1).

Proposition 5.2. *Let $1 < p < n$ and $0 < q < p-1$. Let $\sigma, \mu \in \mathcal{M}^+(\mathbb{R}^n)$ with $(\sigma, \mu) \not\equiv (0,0)$. Assume that there exists a positive function $w \in L^q(\mathbb{R}^n, d\sigma)$ satisfying*

$$w = \mathbf{W}_{1,p}(w^q d\sigma) \quad \text{in } \mathbb{R}^n.$$

Assume also that μ is finite and absolutely continuous with respect to the p-capacity. Then there exists a minimal p-superharmonic solution $u \in L^q(\mathbb{R}^n, d\sigma) \cap L^{\frac{n(p-1)}{n-p}, \infty}(\mathbb{R}^n)$ to (1.1). Moreover, the Riesz measure of u is finite.

Proof. As above, we consider the localized problem (4.1) and its approximate solutions $\{u_j\}_{j=1}^{\infty}$. Then by [9, Theorem 4.4], we have

$$\|\mathbf{W}_{1,p}(u_j^q d\sigma)\|_{L^q(2B, d\sigma)} \leq C \|u_j\|_{L^q(2B, d\sigma)}^{\frac{q}{p-1}}$$

and

$$\|\mathbf{W}_{1,p}\mu\|_{L^q(2B, d\sigma)} \leq C.$$

From Theorem 2.4 and (2.2), it follows that

$$\|u_{j+1}\|_{L^q(2B, d\sigma)} \leq C(\|u_j\|_{L^q(2B, d\sigma)}^{\frac{q}{p-1}} + 1).$$

Thus, $u = \lim_{j\to\infty} u_j$ belongs to $L^q(2B, d\sigma)$, and its Riesz measure $\nu[u]$ satisfies

$$\left(\int_{2B} d\nu[u]\right)^{\frac{1}{p-1}} = \left(\int_{2B} u^q \, d\sigma + \int_{2B} d\mu\right)^{\frac{1}{p-1}} \leq C.$$

Since $\min\{u, l\} \in W_0^{1,p}(2B)$ for all $l > 0$, it follows from [3, Lemma 4.1] that

$$\|u\|_{L^{\frac{n(p-1)}{n-p}, \infty}(2B)} \leq C. \tag{5.2}$$

Using solutions to (4.12), we construct a p-superharmonic function u in $\mathbb{R}^n$ satisfying $\nu[u] = \sigma u^q + \mu$. Then by (5.2) and the Fatou property of the $L^{r,\infty}$ norm (see, e.g., [14, p.14]),

$$\|u\|_{L^{\frac{n(p-1)}{n-p}, \infty}(\mathbb{R}^n)} \leq C.$$

Hence $\liminf_{|x|\to\infty} u(x) = 0$. Using the same argument as in the proof of Theorem 1.1, we see that u is a minimal p-superharmonic solution to (1.1). By the monotone convergence theorem, $u \in L^q(\mathbb{R}^n, d\sigma)$. Hence, the Riesz measure of u is finite. □

6. Elliptic equations with several sub-natural growth terms

Theorem 6.1. *Let $1 < p < n$. Let $\{q_m\}_{m=1}^M$ be positive constants satisfying*

$$0 < q_m < p-1, \quad m = 1, \dots, M. \tag{6.1}$$

Let $\{\sigma^{(m)}\}_{m=1}^M$ and μ be Radon measures on $\mathbb{R}^n$ such that $(\sigma^{(1)}, \dots, \sigma^{(m)}, \mu) \neq (0, \dots, 0, 0)$. Assume that there exists a positive constant $0 < \gamma < \infty$ such that

$$\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\sigma^{(m)})^{\frac{(\gamma+q_m)(p-1)}{p-1-q_m}} \, d\sigma^{(m)} < \infty, \quad m = 1, \dots, M, \tag{6.2}$$

$$\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\mu)^{\gamma} \, d\mu < \infty. \tag{6.3}$$

Then there exists a minimal p-superharmonic solution u to (1.9). Moreover, u satisfies (1.8) and belongs to $L^{r,\rho}(\mathbb{R}^n)$, where $r = n(p-1+\gamma)/(n-p)$ and $\rho = p-1+\gamma$.

Proof. First, we consider localized problems. For $k \in \mathbb{N}$, we consider measures $\sigma_k^{(m)} = \mathbf{1}_{\Omega(\sigma^{(m)},k)}\sigma$ and $\mu_j = \mathbf{1}_{\Omega(\mu,j)}\mu$, where $\Omega(\sigma^{(m)}, k)$ and $\Omega(\mu, k)$ are defined by (3.3). For simplicity of notation, we temporarily write $B(0, 2^k)$, $\sigma_k^{(m)}$ and μ_k as B, $\sigma^{(m)}$ and μ respectively.

Next, we construct a p-superharmonic function u satisfying

$$-\Delta_p u = \sum_{m=1}^{M} \sigma^{(m)} u^{q_m} + \mu \quad \text{in } 2B \tag{6.4}$$

and

$$\min\{u, l\} \in \dot{W}_0^{1,p}(2B) \quad \forall l > 0.$$

Let $\{v^{(m)}\} \subset \dot{W}_0^{1,p}(2B) \cap L^\infty(2B)$ be the weak solutions to

$$-\Delta_p v^{(m)} = \left(\frac{p-1-q_m}{p-1} \right)^{p-1} \sigma^{(m)} \quad \text{in } 2B.$$

For $m = 1, \dots M$, put $u_0^{(m)} = c_1^{(m)} (v^{(m)})^{\frac{p-1}{p-1-q_m}}$, where $\{c_1^{(m)}\}_{m=1}^M \subset (0,1]$ are constants depending only on n, p and q. Then $u_0^{(m)}$ satisfy

$$-\Delta_p u_0^{(m)} \leq \sigma_k^{(m)} (u_0^{(m)})^{q_m} \quad \text{in } 2B.$$

As in the proof of Lemma 4.4, we can choose small $\{c_1^{(m)}\}_{m=1}^M$ such that

$$w \geq u_0^{(m)} \quad \text{in } 2B \quad (m = 1, \dots M) \tag{6.5}$$

whenever w is a nonnegative p-superharmonic supersolution to (1.9). We define an increasing sequence of p-superharmonic functions $\{u_j\}_{j=1}^\infty \subset \dot{W}_0^{1,p}(2B)$ by

$$-\Delta_p u_1 = \sum_{m=1}^{M} \sigma^{(m)} (u_0^{(m)})^{q_m} + \mu$$

and

$$-\Delta_p u_{j+1} = \sum_{m=1}^{M} \sigma^{(m)} u_j^{q_m} + \mu, \quad j = 1, 2, \ldots. \tag{6.6}$$

By Theorem 2.4 and (2.2), for each m,

$$\begin{aligned}&\|u_{j+1}\|_{L^{\gamma+q_m}(2B, d\sigma^{(m)})}\\&\leq C \sum_{l=1}^{M} \|\mathbf{W}_{1,p}(u_j^{q_l} d\sigma^{(l)})\|_{L^{\gamma+q_m}(\mathbb{R}^n, d\sigma^{(m)})} + C\|\mathbf{W}_{1,p}\mu\|_{L^{\gamma+q_m}(\mathbb{R}^n, d\sigma^{(m)})}.\end{aligned}$$

Since $\sigma^{(m)}$ satisfies (6.2), Theorem 3.1 yields

$$\begin{aligned}\|\mathbf{W}_{1,p}(u_j^{q_l} d\sigma^{(l)})\|_{L^{\gamma+q_m}(\mathbb{R}^n, d\sigma^{(m)})} &= \left(\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}(u_j^{q_l} d\sigma^{(l)}))^{\gamma+q_m} \, d\sigma^{(m)} \right)^{\frac{1}{\gamma+q_m}}\\&\leq C \left(\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}(u_j^{q_l} d\sigma^{(l)}))^{\gamma} u_j^{q_l} \, d\sigma^{(l)} \right)^{\frac{1}{p-1+\gamma}}.\end{aligned}$$

Moreover, since $\sigma^{(l)}$ satisfies (6.2), by Hölder's inequality and Lemma 4.3,

$$\begin{aligned}\left(\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}(u_j^{q_l} d\sigma^{(l)}))^{\gamma} u_j^{q_l} \, d\sigma^{(l)} \right)^{\frac{1}{p-1+\gamma}} &\leq C \|u_j^{q_l}\|_{L^{\frac{\gamma+q_l}{q_l}}(\mathbb{R}^n, d\sigma^{(l)})}^{\frac{1}{p-1}}\\&= C \|u_j\|_{L^{\gamma+q_l}(2B, d\sigma^{(l)})}^{\frac{q_l}{p-1}}.\end{aligned} \tag{6.7}$$

By (6.2), (6.3) and Theorem 3.1,

$$\|\mathbf{W}_{1,p}\mu\|_{L^{\gamma+q_m}(\mathbb{R}^n, d\sigma^{(m)})} \leq C.$$

Therefore,

$$\sum_{m=1}^{M} \|u_{j+1}\|_{L^{\gamma+q_m}(2B, d\sigma^{(m)})} \leq C \sum_{m=1}^{M} \|u_j\|_{L^{\gamma+q_m}(2B, d\sigma^{(m)})}^{\frac{q_m}{p-1}} + C.$$

By (6.1) and Young's inequality, this implies that

$$\sum_{m=1}^{M} \|u_{j+1}\|_{L^{\gamma+q_m}(2B, d\sigma^{(m)})} \leq C.$$

Using (6.7) again, we get a uniform bound corresponding to (4.9). Let $u = \lim_{j\to\infty} u$. Then u satisfies (6.4). Moreover, by Corollary 3.6,

$$\sum_{m=1}^{M} \left(\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}(u^{q_m} d\sigma^{(m)}))^{\gamma} u^{q_m} \, d\sigma^{(m)} \right)^{\frac{1}{p-1+\gamma}} + \|u\|_{L^{r,\rho}(2B)} \leq C.$$

By Theorem 2.4, for any $x \in B$,

$$
\begin{aligned}
u(x) &\geq \max\left\{u_0^{(1)}(x), \ldots, u_0^{(M)}(x), \frac{1}{c_K}\mathbf{W}_{1,p}^{\frac{R}{2}}\mu(x)\right\} \\
&\geq \frac{1}{C}\left\{\sum_{m=1}^{M}(\mathbf{W}_{1,p}^{\frac{R}{2}}\sigma^{(m)})^{\frac{p-1}{p-1-q}}(x) + \mathbf{W}_{1,p}^{\frac{R}{2}}\mu(x)\right\}.
\end{aligned} \tag{6.8}
$$

If w is a nonnegative p-superharmonic supersolution to (1.9), then by (6.5) and induction,

$$
w \geq u \quad \text{in } 2B. \tag{6.9}
$$

Finally, we construct a minimal p-superharmonic solution to (1.9). For each $k \geq 1$, let u_k be a p-superharmonic function satisfying (6.4). Passing to the limit $k \to \infty$, we get a p-superharmonic function $u \in L^{r,\rho}(\mathbb{R}^n)$ satisfying $\nu[u] = \sum_{m=1}^{M}\sigma^{(m)}u^{q_m} + \mu$. By Theorem 2.4 and Corollary 3.5,

$$
\begin{aligned}
\int_{\mathbb{R}^n} u^\gamma \, d\nu[u] &\approx \int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\nu[u])^\gamma \, d\nu[u] \\
&\approx \sum_{m=1}^{M}\int_{\mathbb{R}^n}(\mathbf{W}_{1,p}(u^{q_m}d\sigma^{(m)}))^\gamma u^{q_m}\, d\sigma^{(m)} + \int_{\mathbb{R}^n}(\mathbf{W}_{1,p}\mu)^\gamma \, d\mu.
\end{aligned}
$$

Hence, u satisfies (1.8). Positivity and minimality of u follow from (6.8) and (6.9), respectively. Thus, u is a minimal p-superharmonic solution to (1.9). ☐

We also prove the uniqueness of finite energy solutions using a convexity argument as in [10] and [25].

Corollary 6.2. *Assume that* (6.1)-(6.3) *hold with* $\gamma = 1$. *Then there exists a unique finite energy weak solution* $u \in \dot{W}_0^{1,p}(\mathbb{R}^n)$ *satisfying* (1.9).

Proof. Existence of a minimal solution follow from Theorem 6.1. Testing (6.6) with u_{j+1} and using monotonicity of $\{u_j\}_{j=1}^\infty$, we have

$$
\begin{aligned}
\int_{2B} |\nabla u_{j+1}|^p \, dx &= \sum_{m=1}^{M}\int_{2B} u_{j+1}u_j^{q_m}\, d\sigma^{(m)} + \int_{2B} u_{j+1}\, d\mu \\
&\leq \sum_{m=1}^{M} \|u_{j+1}\|_{L^{1+q_m}(2B, d\sigma^{(m)})}^{1+q_m} + \|u_{j+1}\|_{L^1(2B,d\mu)} \leq C,
\end{aligned}
$$

where C is a constant depending only on n, p and (6.1)-(6.3). Thus, the limit function u belongs to $\dot{W}_0^{1,p}(\mathbb{R}^n)$.

Let us prove uniqueness. For simplicity, we put $\sigma^{(0)} = \mu$ and $q_0 = 0$. Let $u, v \in \dot{W}_0^{1,p}(\mathbb{R}^n)$ be weak solutions to (1.9). Without loss of generality, we may assume that u and v are quasicontinuous on $\mathbb{R}^n$ and that v is minimal. Hence, $u \geq v$ q.e. in $\mathbb{R}^n$. Since each $\sigma^{(m)}$ is absolutely continuous with respect to the p-capacity, $u \geq v$ $d\sigma^{(m)}$-a.e. for all $m = 0, 1, \ldots, M$.

Testing the equations of u with u, we have

$$\int_{\mathbb{R}^n} |\nabla u|^p \, dx = \sum_{m=0}^{M} \int_{\mathbb{R}^n} u^{1+q_m} \, d\sigma^{(m)}. \tag{6.10}$$

By the same way,

$$\int_{\mathbb{R}^n} |\nabla v|^p \, dx = \sum_{m=0}^{M} \int_{\mathbb{R}^n} v^{1+q_m} \, d\sigma^{(m)}. \tag{6.11}$$

For $t \in [0,1]$, we set

$$\lambda_t(x) := \left((1-t)v^p(x) + tu^p(x)\right)^{\frac{1}{p}}.$$

Then

$$0 \le v \le \lambda_t \le u \quad \text{q.e. in } \mathbb{R}^n. \tag{6.12}$$

Moreover, by the hidden convexity (see [7, Proposition 2.6]),

$$\int_{\mathbb{R}^n} |\nabla \lambda_t|^p \, dx \le (1-t) \int_{\mathbb{R}^n} |\nabla v|^p \, dx + t \int_{\mathbb{R}^n} |\nabla u|^p \, dx. \tag{6.13}$$

Combining (6.10), (6.11) and (6.13), we get

$$\int_{\mathbb{R}^n} \frac{|\nabla \lambda_t|^p - |\nabla \lambda_0|^p}{t} \, dx \le \sum_{m=0}^{M} \left(\int_{\mathbb{R}^n} u^{1+q_m} \, d\sigma^{(m)} - \int_{\mathbb{R}^n} v^{1+q_m} \, d\sigma^{(m)} \right).$$

By the inequality

$$|a|^p - |b|^p \ge p|b|^{p-2} b \cdot (a-b) \quad (a, b \in \mathbb{R}^n),$$

we have

$$|\nabla \lambda_t|^p - |\nabla \lambda_0|^p \ge p|\nabla \lambda_0|^{p-2} \nabla \lambda_0 \cdot (\nabla \lambda_t - \nabla \lambda_0).$$

Therefore,

$$\begin{aligned} &p \int_{\mathbb{R}^n} |\nabla v|^{p-2} \nabla v \cdot \frac{\nabla(\lambda_t - \lambda_0)}{t} \, dx \\ &\le \sum_{m=0}^{M} \left(\int_{\mathbb{R}^n} u^{1+q_m} \, d\sigma^{(m)} - \int_{\mathbb{R}^n} v^{1+q_m} \, d\sigma^{(m)} \right). \end{aligned} \tag{6.14}$$

On the other hand, using methods in [16, Lemma 1.25], we can see that $\lambda_t \in \dot{W}_0^{1,p}(\mathbb{R}^n)$ from (6.12) and (6.13). Testing the equation of v with $\lambda_t - \lambda_0 \in \dot{W}_0^{1,p}(\mathbb{R}^n)$, we get

$$\int_{\mathbb{R}^n} |\nabla v|^{p-2} \nabla v \cdot \nabla(\lambda_t - \lambda_0) \, dx = \sum_{m=0}^{M} \int_{\mathbb{R}^n} (\lambda_t - \lambda_0) v^{q_m} \, d\sigma^{(m)}. \tag{6.15}$$

Combining (6.14) and (6.15), we obtain

$$\sum_{m=0}^{M} p \int_{\mathbb{R}^n} \frac{(\lambda_t - \lambda_0)}{t} v^{q_m}\, d\sigma^{(m)} \le \sum_{m=0}^{M} \left(\int_{\mathbb{R}^n} u^{1+q_m}\, d\sigma^{(m)} - \int_{\mathbb{R}^n} v^{1+q_m}\, d\sigma^{(m)} \right).$$

Note that $\lambda_t - \lambda_0 \ge 0$ $d\sigma^{(m)}$-a.e. for all m. Therefore, by Fatou's lemma,

$$\int_{\mathbb{R}^n} \left(\frac{u^p}{v^{p-1}} - v \right) v^{q_m}\, d\sigma^{(m)} \le \liminf_{t \to 0} p \int_{\mathbb{R}^n} \frac{(\lambda_t - \lambda_0)}{t} v^{q_m}\, d\sigma^{(m)}$$

for each m. Hence, passing to the limit $t \to 0$, we arrive at

$$\sum_{m=0}^{M} \int_{\mathbb{R}^n} \left(\frac{u^p}{v^{p-1}} - v \right) v^{q_m}\, d\sigma^{(m)} \le \sum_{m=0}^{M} \left(\int_{\mathbb{R}^n} u^{1+q_m}\, d\sigma^{(m)} - \int_{\mathbb{R}^n} v^{1+q_m}\, d\sigma^{(m)} \right),$$

or equivalently

$$\sum_{m=0}^{M} \int_{\mathbb{R}^n} \left(\frac{u^p v^{q_m}}{v^{p-1}} - u^{1+q_m} \right) d\sigma^{(m)} \le 0.$$

By minimality of v, $v^{q_m - p+1} \ge u^{q_m - p+1}$ $d\sigma^{(m)}$-a.e. for all m, and hence, each integral on the left-hand side is nonnegative. Thus, $u = v$ $d\sigma^{(m)}$-a.e. for all m.

Testing the equations of u and v with $u - v \in \dot{W}_0^{1,p}(\mathbb{R}^n)$, we get

$$\int_{\mathbb{R}^n} \left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \cdot \nabla (u - v)\, dx = 0.$$

This implies that $\nabla u = \nabla v$ a.e. in $\mathbb{R}^n$ (see, e.g., [16, Lemma 5.6]), and therefore, $u = v$ in $\dot{W}_0^{1,p}(\mathbb{R}^n)$. □

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[1] D. R. Adams and L. I. Hedberg. *Function spaces and potential theory*, volume 314 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1996.

[2] W. Allegretto and Y. X. Huang. A Picone's identity for the p-Laplacian and applications. *Nonlinear Anal.*, 32(7):819–830, 1998.

[3] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and J. L. Vázquez. An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 22(2):241–273, 1995.

[4] M. F. Bidaut-Véron. Removable singularities and existence for a quasilinear equation with absorption or source term and measure data. *Adv. Nonlinear Stud.*, 3(1):25–63, 2003.

[5] L. Boccardo, T. Gallouët, and L. Orsina. Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 13(5):539–551, 1996.

[6] L. Boccardo and L. Orsina. Sublinear equations in L^s. *Houston J. Math.*, 20(1):99–114, 1994.

[7] L. Brasco and G. Franzina. Convexity properties of Dirichlet integrals and Picone-type inequalities. *Kodai Math. J.*, 37(3):769–799, 2014.

[8] H. Brezis and S. Kamin. Sublinear elliptic equations in $\mathbf{R}^n$. *Manuscripta Math.*, 74(1):87–106, 1992.

[9] D. Cao and I. Verbitsky. Nonlinear elliptic equations and intrinsic potentials of Wolff type. *J. Funct. Anal.*, 272(1):112–165, 2017.

[10] D. T. Cao and I. E. Verbitsky. Finite energy solutions of quasilinear elliptic equations with sub-natural growth terms. *Calc. Var. Partial Differential Equations*, 52(3-4):529–546, 2015.

[11] D. T. Cao and I. E. Verbitsky. Pointwise estimates of Brezis-Kamin type for solutions of sublinear elliptic equations. *Nonlinear Anal.*, 146:1–19, 2016.

[12] C. Cascante, J. M. Ortega, and I. E. Verbitsky. Trace inequalities of Sobolev type in the upper triangle case. *Proc. London Math. Soc. (3)*, 80(2):391–414, 2000.

[13] G. Dal Maso, F. Murat, L. Orsina, and A. Prignet. Renormalized solutions of elliptic equations with general measure data. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 28(4):741–808, 1999.

[14] L. Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2008.

[15] L. I. Hedberg and T. H. Wolff. Thin sets in nonlinear potential theory. *Ann. Inst. Fourier (Grenoble)*, 33(4):161–187, 1983.

[16] J. Heinonen, T. Kilpeläinen, and O. Martio. *Nonlinear potential theory of degenerate elliptic equations*. Dover Publications, Inc., Mineola, NY, 2006. Unabridged republication of the 1993 original.

[17] T. Kilpeläinen, T. Kuusi, and A. Tuhola-Kujanpää. Superharmonic functions are locally renormalized solutions. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 28(6):775–795, 2011.

[18] T. Kilpeläinen and J. Malý. Degenerate elliptic equations with measure data and nonlinear potentials. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 19(4):591–613, 1992.

[19] T. Kilpeläinen and J. Malý. The Wiener test and potential estimates for quasilinear elliptic equations. *Acta Math.*, 172(1):137–161, 1994.

[20] T. Kilpeläinen and X. Xu. On the uniqueness problem for quasilinear elliptic equations involving measures. *Rev. Mat. Iberoamericana*, 12(2):461–475, 1996.

[21] D. Kinderlehrer and G. Stampacchia. *An introduction to variational inequalities and their applications*, volume 88 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980.

[22] V. G. Maz′ja and V. P. Havin. A nonlinear potential theory. *Uspehi Mat. Nauk*, 27(6):67–138, 1972.

[23] A. Seesanea and I. E. Verbitsky. Solutions in Lebesgue spaces to nonlinear elliptic equations with subnatural growth terms. *Algebra i Analiz*, 31(3):216–238, 2019.

[24] A. Seesanea and I. E. Verbitsky. Solutions to sublinear elliptic equations with finite generalized energy. *Calc. Var. Partial Differential Equations*, 58(1):Art. 6, 21, 2019.

[25] A. Seesanea and I. E. Verbitsky. Finite energy solutions to inhomogeneous nonlinear elliptic equations with sub-natural growth terms. *Adv. Calc. Var.*, 13(1):53–74, 2020.

[26] N. S. Trudinger and X.-J. Wang. On the weak continuity of elliptic operators and applications to potential theory. *Amer. J. Math.*, 124(2):369–410, 2002.

[27] I. E. Verbitsky. Nonlinear potentials and trace inequalities. In *The Maz′ya anniversary collection, Vol. 2 (Rostock, 1998)*, volume 110 of *Oper. Theory Adv. Appl.*, pages 323–343. Birkhäuser, Basel, 1999.

[28] I. E. Verbitsky. Quasilinear elliptic equations with sub-natural growth terms and nonlinear potential theory. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, 30(4):733–758, 2019.

[29] I. E. Verbitsky. Wolff's inequality for intrinsic nonlinear potentials and quasilinear elliptic equations. *Nonlinear Analysis*, 2020. `https://doi.org/10.1016/j.na.2019.04.015`.