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Ray-theoretical Construction of Dispersive RAYLEIGH Waves (Continued)

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Abstract

Following the previous paper (TAZIME, 1958), reflecting coefficients on solid-solid interface have been expressed explicitly by matrices. Several relations between them are obtained. Using reflecting coefficients, displacement potentials of dispersive RAYLEIGH waves are expressed, besides the characteristic equation.

6. General expressions of reflecting coefficients on the lower boundary in the superficial layer.

(A', B', C', D') in (1.4) can be expressed in the general form as follows:

$$\left.\begin{aligned} A' &= A_1/B_1 = \Delta_A'/\Delta',\ B' = \Delta_B'/\Delta' ,\\ C' &= A_1/D_1 = \Delta_C'/\Delta',\ D' = \Delta_D'/\Delta' . \end{aligned}\right\} \tag{6.1}$$

Denoting

$$\left.\begin{aligned} P_2 &= \left(\frac{\beta_2^2}{\xi^2} - 1\right)^2 + 4\,\frac{\alpha_2}{\xi}\,\frac{\beta_2}{\xi}, \quad Q_2 = \left(\frac{\beta_2^2}{\xi^2} - 1\right) - 2\,\frac{\alpha_2}{\xi}\,\frac{\beta_2}{\xi},\\ R_2 &= 1 + \frac{\alpha_2}{\xi}\,\frac{\beta_2}{\xi}, \quad S = \left(\frac{c}{v_{s1}} \cdot \frac{c}{v_{s2}}\right)^2 \end{aligned}\right\} \tag{6.2}$$

matrices in (6.1) will be found as

$$\left.\begin{aligned} \Delta' &= \left(\frac{\mu_2}{\mu_1}\right)^2 P_2\left(1 + \frac{\alpha_1}{\xi}\,\frac{\beta_1}{\xi}\right) - 2\left(\frac{\mu_2}{\mu_1}\right) Q_2\left\{\left(\frac{\beta_1^2}{\xi^2} - 1\right) - 2\,\frac{\alpha_1}{\xi}\,\frac{\beta_1}{\xi}\right\}\\ &\quad + R_2\left\{\left(\frac{\beta_1^2}{\xi^2} - 1\right)^2 + 4\,\frac{\alpha_1}{\xi}\,\frac{\beta_1}{\xi}\right\} + \left(\frac{\mu_2}{\mu_1} S\right)\left(\frac{\alpha_1}{\xi}\,\frac{\beta_2}{\xi} + \frac{\beta_1}{\xi}\,\frac{\alpha_2}{\xi}\right),\\ -\Delta_A' &= \left(\frac{\mu_2}{\mu_1}\right)^2 P_2\left(1 - \frac{\alpha_1}{\xi}\,\frac{\beta_1}{\xi}\right) - 2\left(\frac{\mu_2}{\mu_1}\right) Q_2\left\{\left(\frac{\beta_1^2}{\xi^2} - 1\right) + 2\,\frac{\alpha_1}{\xi}\,\frac{\beta_1}{\xi}\right\}\\ &\quad + R_2\left\{\left(\frac{\beta_1^2}{\xi^2} - 1\right)^2 - 4\,\frac{\alpha_1}{\xi}\,\frac{\beta_1}{\xi}\right\} - \left(\frac{\mu_2}{\mu_1} S\right)\left(\frac{\alpha_1}{\xi}\,\frac{\beta_2}{\xi} - \frac{\beta_1}{\xi}\,\frac{\alpha_2}{\xi}\right),\\ -\Delta_D' &= \left(\frac{\mu_2}{\mu_1}\right)^2 P_2\left(1 - \frac{\alpha_1}{\xi}\,\frac{\beta_1}{\xi}\right) - 2\left(\frac{\mu_2}{\mu_1}\right) Q_2\left\{\left(\frac{\beta_1^2}{\xi^2} - 1\right) + 2\,\frac{\alpha_1}{\xi}\,\frac{\beta_1}{\xi}\right\}\\ &\quad + R_2\left\{\left(\frac{\beta_1^2}{\xi^2} - 1\right)^2 - 4\,\frac{\alpha_1}{\xi}\,\frac{\beta_1}{\xi}\right\} + \left(\frac{\mu_2}{\mu_1}\right) S\left(\frac{\alpha_1}{\xi}\,\frac{\beta_2}{\xi} - \frac{\beta_1}{\xi}\,\frac{\alpha_2}{\xi}\right), \end{aligned}\right| \tag{6.3}$$

$$\Delta_B' = 2(\alpha_1/\xi)\Delta_{BC}', \quad -\Delta_C'=2(\beta_1/\xi)\Delta_{BC}'$$

where

$$\Delta_{BC}'=\left(\frac{\mu_2}{\mu_1}\right)^2 P_2 - \left(\frac{\mu_2}{\mu_1}\right) Q_2 \left\{\left(\frac{\beta_1^2}{\xi^2}-1\right)-2\right\}-2R_2\left(\frac{\beta_1^2}{\xi^2}-1\right).$$

When $\mu_2/\mu_1=0$ in (6.3), one sees from (1.21) that

$$A' = A, \quad B' = -B, \quad C' = -C, \quad D' = A, \tag{6.4}$$

coinciding with (3.7) which was the relation in a plate.

Recalling (1.21) again, one will easily find, as in (3.6), that

$$BC' = B'C \tag{6.5}$$

also in such a general case as that of (6.3).

Using the next notations,

$$K = \left(\frac{\mu_2}{\mu_1}\right)^2 P_2 - 2\left(\frac{\mu_2}{\mu_1}\right) Q_2\left(\frac{\beta_1^2}{\xi^2}-1\right) + R_2\left(\frac{\beta_1^2}{\xi^2}-1\right)^2$$

and

$$L = \left(\frac{\mu_2}{\mu_1}\right)^2 P_2 + 4\left(\frac{\mu_2}{\mu_1}\right) Q_2 + 4R_2, \tag{6.6}$$

one may express (6.3) more simply,

$$\Delta' = K + \frac{\alpha_1}{\xi}\frac{\beta_1}{\xi} L + \left(\frac{\mu_2}{\mu_1} S\right)\left(\frac{\alpha_1}{\xi}\frac{\beta_2}{\xi} + \frac{\beta_1}{\xi}\frac{\alpha_2}{\xi}\right),$$

$$-\Delta_A' = K - \frac{\alpha_1}{\xi}\frac{\beta_1}{\xi} L - \left(\frac{\mu_2}{\mu_1} S\right)\left(\frac{\alpha_1}{\xi}\frac{\beta_2}{\xi} - \frac{\beta_1}{\xi}\frac{\alpha_2}{\xi}\right), \tag{6.7}$$

$$-\Delta_D' = K - \frac{\alpha_1}{\xi}\frac{\beta_1}{\xi} L + \left(\frac{\mu_2}{\mu_1} S\right)\left(\frac{\alpha_1}{\xi}\frac{\beta_2}{\xi} - \frac{\beta_1}{\xi}\frac{\alpha_2}{\xi}\right).$$

One sees, from (6.2), (6.3) and (6.6), that

$$P_2 R_2 - Q_2^2 = \frac{\alpha_2}{\xi}\frac{\beta_2}{\xi}\left(\frac{c}{v_{s2}}\right)^4$$

and

$$(\Delta_{BC}')^2 = KL - \frac{\alpha_2}{\xi}\frac{\beta_2}{\xi}\left(\frac{\mu_2}{\mu_1} S\right)^2. \tag{6.8}$$

Here one has, from (6.3), (6.7) and (6.8),

$$\Delta_A'\,\Delta_D' - \Delta_B'\,\Delta_C' = \Delta'\,\Delta^* \tag{6.9}$$

in which

$$\Delta^* = K + \frac{\alpha_1}{\xi}\frac{\beta_1}{\xi} L - \left(\frac{\mu_2}{\mu_1} S\right)\left(\frac{\alpha_1}{\xi}\frac{\beta_2}{\xi} + \frac{\beta_1}{\xi}\frac{\alpha_2}{\xi}\right) \tag{6.10}$$

conjugates with Δ'.

Therefore another useful relation in addition to (6.5) is obtained from (6.1) and (6.9),

$$A'D'-B'C' = (\Delta_A'/\Delta')(\Delta_D'/\Delta')-(\Delta_B'/\Delta')(\Delta_C'/\Delta') = \Delta^*/\Delta' . \quad (6.11)$$

If

$$v_{s2}/v_{s1}>1 ,$$

the right hand sides of eqs. (6.3) must have extremely large values, since

$$\mu_2/\mu_1 = (\rho_2/\rho_1)(v_{s2}/v_{s1})^2 . \quad (6.12)$$

It will not be preferable for numerical calculation to treat large numerical values. In the case when $v_{s2}/v_{s1}>1$, the following expressions will be used in place of (6.3), for the purpose of avoiding the above mentioned difficulty.

$$\left.\begin{aligned}
&\left(\frac{\mu_1}{\mu_2}\right)^2\Delta' = K'+\frac{\alpha_1}{\xi}\frac{\beta_1}{\xi}L'+\frac{\rho_1}{\rho_2}\left(\frac{c}{v_{s2}}\right)^4\left(\frac{\alpha_1}{\xi}\frac{\beta_2}{\xi}+\frac{\beta_1}{\xi}\frac{\alpha_2}{\xi}\right),\\
&\left(\frac{\mu_1}{\mu_2}\right)^2\Delta^* = K'+\frac{\alpha_1}{\xi}\frac{\beta_1}{\xi}L'-\frac{\rho_1}{\rho_2}\left(\frac{c}{v_{s2}}\right)^4\left(\frac{\alpha_1}{\xi}\frac{\beta_2}{\xi}+\frac{\beta_1}{\xi}\frac{\alpha_2}{\xi}\right),\\
&-\left(\frac{\mu_1}{\mu_2}\right)^2\Delta_A' = K'-\frac{\alpha_1}{\xi}\frac{\beta_1}{\xi}L'-\frac{\rho_1}{\rho_2}\left(\frac{c}{v_{s2}}\right)^4\left(\frac{\alpha_1}{\xi}\frac{\beta_2}{\xi}-\frac{\beta_1}{\xi}\frac{\alpha_2}{\xi}\right),\\
&-\left(\frac{\mu_1}{\mu_2}\right)^2\Delta_D' = K'-\frac{\alpha_1}{\xi}\frac{\beta_1}{\xi}L'+\frac{\rho_1}{\rho_2}\left(\frac{c}{v_{s2}}\right)^4\left(\frac{\alpha_1}{\xi}\frac{\beta_2}{\xi}-\frac{\beta_1}{\xi}\frac{\alpha_2}{\xi}\right),\\
&\left(\frac{\mu_1}{\mu_2}\right)^2\Delta_B' = 2\frac{\alpha_1}{\xi}\left(\frac{\mu_1}{\mu_2}\right)^2\Delta_{BC}', \quad -\left(\frac{\mu_1}{\mu_2}\right)^2\Delta_C' = 2\frac{\beta_1}{\xi}\left(\frac{\mu_1}{\mu_2}\right)^2\Delta_{BC}'
\end{aligned}\right\} \quad (6.13)$$

where

$$\left.\begin{aligned}
&K'P_2-2\left(\frac{\rho_1}{\rho_2}\right)Q_2\left\{\left(\frac{c}{v_{s2}}\right)^2-2\left(\frac{v_{s1}}{v_{s2}}\right)^2\right\}+R_2\left\{\left(\frac{c}{v_{s2}}\right)^2-2\left(\frac{v_{s1}}{v_{s2}}\right)^2\right\}^2,\\
&L' = P_2+4\left(\frac{\rho_1}{\rho_2}\right)Q_2\left(\frac{v_{s1}}{v_{s2}}\right)^2+4\left(\frac{\rho_1}{\rho_2}\right)^2R_2\left(\frac{v_{s1}}{v_{s2}}\right)^4\\
&\left(\frac{\mu_1}{\mu_2}\right)^2\Delta_{BC}' = L'-\left(\frac{c}{v_{s2}}\right)^2\left\{\left(\frac{\rho_1}{\rho_2}\right)Q_2+2\left(\frac{\rho_1}{\rho_2}\right)^2R_2\left(\frac{v_{s1}}{v_{s2}}\right)^2\right\}.
\end{aligned}\right\} \quad (6.14)$$

Reflecting coefficients given by (6.1) will not be changed if use is made of the right hand sides of (6.13) instead of the right hands sides of (6.3).

Now all wave numbers in z-direction must be either real or purely imaginary if ω and ξ are real. Various combination of them should be investigated in the region occupied by c. In Table 1 all combinations possible are shown. Nine types of combinations, (i), ... (v), ... (iii'), will be enough for the following study, although six cases are considered in Table 1.

Table 1. $\bar{\alpha}_j=\sqrt{h_j^2-\xi^2}\,,\quad \hat{\alpha}_j=\sqrt{\xi^2-h_j^2}$
$\bar{\beta}_j=\sqrt{k_j^2-\xi^2}\,,\quad \hat{\beta}_j=\sqrt{\xi^2-k_j^2}$

case (A) $v_{s1}<v_{p1}<v_{s2}<v_{p2}$

type	region	α_2	β_2	α_1	β_1
(i)	$v_{p2}<c$	$\bar{\alpha}_2$	$\bar{\beta}_2$	$\bar{\alpha}_1$	$\bar{\beta}_1$
(ii)	$v_{s2}<c<v_{p2}$	$-i\hat{\alpha}_2$	$\bar{\beta}_2$	$\bar{\alpha}_1$	$\bar{\beta}_1$
(iii)	$v_{p1}<c<v_{s2}$	$-i\hat{\alpha}_2$	$-i\hat{\beta}_2$	$\bar{\alpha}_1$	$\bar{\beta}_1$
(iv)	$v_{s1}<c<v_{p1}$	$-i\hat{\alpha}_2$	$-i\hat{\beta}_2$	$-i\hat{\alpha}_1$	$\bar{\beta}_1$
(v)	$c<v_{s1}$	$-i\hat{\alpha}_2$	$-i\hat{\beta}_2$	$-i\hat{\alpha}_1$	$-i\hat{\beta}_1$

case (B) $v_{s1}<v_{s2}<v_{p1}<v_{p2}$

type	region	α_2	β_2	α_1	β_1
(i)	$v_{p2}<c$				
(ii)	$v_{p1}<c<v_{p2}$	$-i\hat{\alpha}_2$	$\bar{\beta}_2$	$\bar{\alpha}_1$	$\bar{\beta}_1$
(iii)	$v_{s2}<c<v_{p1}$	$-i\hat{\alpha}_2$	$\bar{\beta}_2$	$-i\hat{\alpha}_1$	$\bar{\beta}_1$
(iv)	$v_{s1}<c<v_{s2}$	$-i\hat{\alpha}_2$	$-i\hat{\beta}_2$	$-i\hat{\alpha}_1$	$\bar{\beta}_1$
(v)	$c<v_{s1}$	$-i\hat{\alpha}_2$	$-i\hat{\beta}_2$	$-i\hat{\alpha}_1$	$-i\hat{\beta}_1$

case (C) $v_{s2}<v_{s1}<v_{p1}<v_{p2}$

type	region	α_2	β_2	α_1	β_1
(ii)	$v_{p1}<c<v_{p2}$	$-i\hat{\alpha}_2$	$\bar{\beta}_2$	$\bar{\alpha}_1$	$\bar{\beta}_1$
(iii′)	$v_{s1}<c<v_{p1}$	$-i\hat{\alpha}_2$	$\bar{\beta}_2$	$-i\hat{\alpha}_1$	$\bar{\beta}_1$
(iv′)	$v_{s2}<c<v_{s1}$	$-i\hat{\alpha}_2$	$\bar{\beta}_2$	$-i\hat{\alpha}_1$	$-i\hat{\beta}_1$

case (D) $v_{s2}<v_{p2}<v_{s1}<v_{p1}$

type	region	α_2	β_2	α_1	β_1
(ii′)	$v_{s1}<c<v_{p1}$	$\bar{\alpha}_2$	$\bar{\beta}_2$	$-i\hat{\alpha}_1$	$\bar{\beta}_1$
(iii″)	$v_{p2}<c<v_{s1}$	$\bar{\alpha}_2$	$\bar{\beta}_2$	$-i\hat{\alpha}_1$	$-i\hat{\beta}_1$
(iv′)	$v_{s2}<c<v_{p2}$	$-i\hat{\alpha}_2$	$\bar{\beta}_2$	$-i\hat{\alpha}_1$	$-i\hat{\beta}_1$

case (E) $v_{s2}<v_{s1}<v_{p2}<v_{p1}$

type	region	α_2	β_2	α_1	β_1
(ii′)	$v_{p2}<c<v_{p1}$	$\bar{\alpha}_2$	$\bar{\beta}_2$	$-i\bar{\alpha}_1$	$\bar{\beta}_1$
(iii′)	$v_{s1}<c<v_{p2}$	$-i\hat{\alpha}_2$	$\bar{\beta}_2$	$-i\hat{\alpha}_1$	$\bar{\beta}_1$
(iv′)	$v_{s2}<c<v_{s1}$	$-i\hat{\alpha}_2$	$\bar{\beta}_2$	$-i\hat{\alpha}_1$	$-i\hat{\beta}_1$

case (F) $v_{s1}<v_{s2}<v_{p2}<v_{p1}$

type	region	α_2	β_2	α_1	β_1
(ii′)	$v_{p2}<c<v_{p1}$	$\hat{\alpha}_2$	$\bar{\beta}_2$	$-i\hat{\alpha}_1$	$\bar{\beta}_1$
(iii′)	$v_{s2}<c<v_{p2}$	$-i\hat{\alpha}_2$	$\bar{\beta}_2$	$-i\hat{\alpha}_1$	$\bar{\beta}_1$
(iv)	$v_{s1}<c<v_{s2}$	$-i\hat{\alpha}_2$	$-i\hat{\beta}_2$	$-i\hat{\alpha}_1$	$\bar{\beta}_1$

7. Practical expressions of reflecting coefficients.

(i) (6.2) will become

$$\left.\begin{aligned} P_2 &= \left(\frac{\bar\beta_2^2}{\xi^2}-1\right)^2+4\frac{\bar\alpha_2}{\xi}\frac{\bar\beta_2}{\xi}, \\ Q_2 &= \left(\frac{\bar\beta_2^2}{\xi^2}-1\right)-2\frac{\bar\alpha_2}{\xi}\frac{\bar\beta_2}{\xi}, \quad R_2=1+\frac{\bar\alpha_2}{\xi}\frac{\bar\beta_2}{\xi} \end{aligned}\right\} \tag{7.1}$$

and (6.7) will become

$$\left.\begin{aligned} \Delta' &= K+\frac{\bar\alpha_1}{\xi}\frac{\bar\beta_1}{\xi}L+\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\left(\frac{\bar\alpha_1}{\xi}\frac{\bar\beta_2}{\xi}+\frac{\bar\beta_1}{\xi}\frac{\bar\alpha_2}{\xi}\right), \\ \Delta^* &= K+\frac{\bar\alpha_1}{\xi}\frac{\bar\beta_1}{\xi}L-\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\left(\frac{\bar\alpha_1}{\xi}\frac{\bar\beta_2}{\xi}+\frac{\bar\beta_1}{\xi}\frac{\bar\alpha_2}{\xi}\right), \\ -\Delta_A' &= K-\frac{\bar\alpha_1}{\xi}\frac{\bar\beta_1}{\xi}L-\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\left(\frac{\bar\alpha_1}{\xi}\frac{\bar\beta_2}{\xi}-\frac{\bar\beta_1}{\xi}\frac{\bar\alpha_2}{\xi}\right), \\ -\Delta_D' &= K-\frac{\bar\alpha_1}{\xi}\frac{\bar\beta_1}{\xi}L+\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\left(\frac{\bar\alpha_1}{\xi}\frac{\bar\beta_2}{\xi}-\frac{\bar\beta_1}{\xi}\frac{\bar\alpha_2}{\xi}\right). \end{aligned}\right\} \tag{7.2}$$

All reflecting coefficients A', B', C' and D' have real values.

(ii)

$$\left.\begin{aligned} P_2 &= \left(\frac{\bar\beta_2^2}{\xi^2}-1\right)^2-4i\frac{\hat\alpha_2}{\xi}\frac{\bar\beta_2}{\xi}, \\ Q_2 &= \left(\frac{\bar\beta_2^2}{\xi^2}-1\right)+2i\frac{\hat\alpha_2}{\xi}\frac{\bar\beta_2}{\xi}, \quad R_2=1-i\frac{\hat\alpha_2}{\xi}\frac{\bar\beta_2}{\xi}, \end{aligned}\right\} \tag{7.3}$$

$$\left.\begin{aligned} \Delta' &= \bar K+\frac{\bar\alpha_1}{\xi}\frac{\bar\beta_1}{\xi}\bar L+\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\bar\alpha_1}{\xi}\frac{\bar\beta_1}{\xi}-i\left\{\hat K+\frac{\bar\alpha_1}{\xi}\frac{\bar\beta_1}{\xi}\hat L+\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\bar\beta_1}{\xi}\frac{\hat\alpha_2}{\xi}\right\}, \\ \Delta^* &= \bar K+\frac{\bar\alpha_1}{\xi}\frac{\bar\beta_1}{\xi}\bar L-\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\bar\alpha_1}{\xi}\frac{\bar\beta_2}{\xi}-i\left\{\hat K+\frac{\bar\alpha_1}{\xi}\frac{\bar\beta_1}{\xi}\hat L-\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\bar\beta_1}{\xi}\frac{\hat\alpha_2}{\xi}\right\}, \\ -\Delta_A' &= \bar K-\frac{\bar\alpha_1}{\xi}\frac{\bar\beta_1}{\xi}\bar L-\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\bar\alpha_1}{\xi}\frac{\bar\beta_2}{\xi}-i\left\{\hat K-\frac{\bar\alpha_1}{\xi}\frac{\bar\beta_1}{\xi}\hat L+\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\bar\beta_1}{\xi}\frac{\hat\alpha_2}{\xi}\right\}, \\ -\Delta_D' &= \bar K-\frac{\bar\alpha_1}{\xi}\frac{\bar\beta_1}{\xi}\bar L+\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\bar\alpha_1}{\xi}\frac{\bar\beta_2}{\xi}-i\left\{\hat K-\frac{\bar\alpha_1}{\xi}\frac{\bar\beta_1}{\xi}\hat L-\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\bar\beta_1}{\xi}\frac{\hat\alpha_2}{\xi}\right\} \end{aligned}\right\} \tag{7.4}$$

where

$$\left.\begin{aligned} \bar K &= \left\{\frac{\mu_2}{\mu_1}\left(\frac{\bar\beta_2^2}{\xi^2}-1\right)+\left(2-\frac{c^2}{v_{s1}^2}\right)\right\}^2, \quad \hat K=\frac{\hat\alpha_2}{\xi}\frac{\bar\beta_2}{\xi}\left\{2\left(\frac{\mu_2}{\mu_1}-1\right)+\frac{c^2}{v_{s1}^2}\right\}^2, \\ \bar L &= \left\{\frac{\mu_2}{\mu_1}\left(\frac{\bar\beta_2^2}{\xi^2}-1\right)+2\right\}^2, \quad \hat L=4\frac{\hat\alpha_2}{\xi}\frac{\bar\beta_2}{\xi}\left(\frac{\mu_2}{\mu_1}-1\right)^2. \end{aligned}\right\} \tag{7.5}$$

All reflecting coefficients have complex values.

(iii)

$$\left.\begin{aligned} P_2 &= \left(\frac{\hat{\beta}_2^2}{\xi^2}+1\right)^2 - 4\,\frac{\hat{\alpha}_2}{\xi}\,\frac{\hat{\beta}_2}{\xi}\,, \\ Q_2 &= -\left\{\left(\frac{\hat{\beta}_2^2}{\xi^2}+1\right) - 2\,\frac{\hat{\alpha}_2}{\xi}\,\frac{\hat{\beta}_2}{\xi}\right\}, \qquad R_2 = 1 - \frac{\hat{\alpha}_2}{\xi}\,\frac{\hat{\beta}_2}{\xi} \end{aligned}\right\} \tag{7.6}$$

$$\left.\begin{aligned} \Delta' &= K + \frac{\bar{\alpha}_1}{\xi}\,\frac{\bar{\beta}_1}{\xi}\,L - i\,\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\left(\frac{\bar{\alpha}_1}{\xi}\,\frac{\hat{\beta}_2}{\xi} + \frac{\bar{\beta}_1}{\xi}\,\frac{\hat{\alpha}_2}{\xi}\right) = |\Delta'|e^{i\varepsilon}\,, \\ \Delta^* &= K + \frac{\bar{\alpha}_1}{\xi}\,\frac{\bar{\beta}_1}{\xi}\,L + i\,\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\left(\frac{\bar{\alpha}_1}{\xi}\,\frac{\hat{\beta}_2}{\xi} + \frac{\bar{\beta}_1}{\xi}\,\frac{\hat{\alpha}_2}{\xi}\right) = |\Delta'|e^{-i\varepsilon}\,, \\ -\Delta_A' &= K - \frac{\bar{\alpha}_1}{\xi}\,\frac{\bar{\beta}_1}{\xi}\,L + i\,\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\left(\frac{\bar{\alpha}_1}{\xi}\,\frac{\hat{\beta}_2}{\xi} - \frac{\bar{\beta}_1}{\xi}\,\frac{\hat{\alpha}_2}{\xi}\right) = |-\Delta_A'|e^{i\varepsilon'}, \\ -\Delta_D' &= K - \frac{\bar{\alpha}_1}{\xi}\,\frac{\bar{\beta}_1}{\xi}\,L - i\,\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\left(\frac{\bar{\alpha}_1}{\xi}\,\frac{\hat{\beta}_2}{\xi} - \frac{\bar{\beta}_1}{\xi}\,\frac{\hat{\alpha}_2}{\xi}\right) = |-\Delta_A'|e^{-i\varepsilon'} \end{aligned}\right\} \tag{7.7}$$

where

$$\left.\begin{aligned} \tan\varepsilon &= -\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\left(\frac{\bar{\alpha}_1}{\xi}\,\frac{\hat{\beta}_2}{\xi} + \frac{\bar{\beta}_1}{\xi}\,\frac{\hat{\alpha}_2}{\xi}\right)\Big/\left(K + \frac{\bar{\alpha}_1}{\xi}\,\frac{\bar{\beta}_1}{\xi}\,L\right), \\ \tan\varepsilon' &= \frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\left(\frac{\bar{\alpha}_1}{\xi}\,\frac{\hat{\beta}_2}{\xi} + \frac{\bar{\beta}_1}{\xi}\,\frac{\hat{\alpha}_2}{\xi}\right)\Big/\left(K - \frac{\bar{\alpha}_1}{\xi}\,\frac{\bar{\beta}_1}{\xi}\,L\right). \end{aligned}\right\} \tag{7.8}$$

Quadrants occupied by ε and ε' must be decided respectively by signs of Re Δ' and Im Δ' and by signs of Re $(-\Delta_A')$ and Im $(-\Delta_A')$.

Using these notations, ε and ε', one has

$$A' = \Delta_A'/\Delta' = -\Gamma\exp\{i(\varepsilon'-\varepsilon)\} \text{ and } D' = \Delta_D'/\Delta' = -\Gamma\exp\{-i(\varepsilon'+\varepsilon)\} \tag{7.9}$$

in which $\Gamma = |-\Delta_A'|/|\Delta'|$.

Reconsidering (1.21), (1.22) and (6.11), one will obtain several relations between reflecting coefficients,

$$\left.\begin{aligned} &A'D' - B'C' = \Delta^*/\Delta' = \exp(-2i\varepsilon)\,, \quad A'D' = \Gamma^2\exp(-2i\varepsilon)\,, \\ &B'C\exp(i\varepsilon) = \pm(1-A^2)^{1/2}(1-\Gamma^2)^{1/2} \text{ for } \Delta_{BC}' \gtrless 0\,. \end{aligned}\right\} \tag{7.10}$$

(iv)

$$\begin{aligned} \Delta' &= K - \frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\hat{\alpha}_1}{\xi}\,\frac{\hat{\beta}_2}{\xi} - i\left\{\frac{\hat{\alpha}_1}{\xi}\,\frac{\bar{\beta}_1}{\xi}\,L + \frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\bar{\beta}_1}{\xi}\,\frac{\hat{\alpha}_2}{\xi}\right\} = |\Delta'|e^{i\varepsilon}\,, \\ \Delta^* &= K + \frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\hat{\alpha}_1}{\xi}\,\frac{\hat{\beta}_2}{\xi} - i\left\{\frac{\hat{\alpha}_1}{\xi}\,\frac{\bar{\beta}_1}{\xi}\,L - \frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\bar{\beta}_1}{\xi}\,\frac{\hat{\alpha}_2}{\xi}\right\} = |\Delta^*|e^{-i\varepsilon'}\,, \\ -\Delta_A' &= K + \frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\hat{\alpha}_1}{\xi}\,\frac{\bar{\beta}_2}{\xi} + i\left\{\frac{\hat{\alpha}_1}{\xi}\,\frac{\bar{\beta}_1}{\xi}\,L - \frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\bar{\beta}_1}{\xi}\,\frac{\hat{\alpha}_2}{\xi}\right\} = |\Delta^*|e^{i\varepsilon'}\,, \end{aligned}$$

$$-\Delta_D' = K - \frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4 \frac{\hat{\alpha}_1}{\xi}\frac{\hat{\beta}_2}{\xi} + i\left\{\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_1}{\xi}L + \frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4 \frac{\bar{\beta}_1}{\xi}\frac{\hat{\alpha}_2}{\xi}\right\} = |\Delta'|e^{-i\varepsilon} \tag{7.11}$$

where

$$\left.\begin{aligned} \tan\varepsilon &= -\left\{\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_1}{\xi}L + \frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4 \frac{\bar{\beta}_1}{\xi}\frac{\hat{\alpha}_2}{\xi}\right\}\Big/\left\{K - \frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4 \frac{\hat{\alpha}_1}{\xi}\frac{\hat{\beta}_2}{\xi}\right\}, \\ \tan\varepsilon' &= \left\{\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_1}{\xi}L - \frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4 \frac{\bar{\beta}_1}{\xi}\frac{\hat{\alpha}_2}{\xi}\right\}\Big/\left\{K + \frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4 \frac{\hat{\alpha}_1}{\xi}\frac{\hat{\beta}_2}{\xi}\right\}. \end{aligned}\right\} \tag{7.12}$$

Here one has

$$\left.\begin{aligned} A' = -\Gamma \exp i(\varepsilon' - \varepsilon),\ D' = -\exp(-2i\varepsilon) \\ \text{and } \Delta^*/\Delta' = \Gamma \exp\{-i(\varepsilon' + \varepsilon)\} \end{aligned}\right\} \tag{7.13}$$

in which $\Gamma = |\Delta^*|/|\Delta'|$.

Moreover one will get

$$\left.\begin{aligned} &A'D' - B'C' = \Gamma \exp\{-i(\varepsilon' + \varepsilon)\}, \\ &B'C \exp\{-i(\delta - \varepsilon)\} = \pm 2i\{\Gamma \sin 2\delta \sin(\varepsilon' - \varepsilon)\}^{1/2} \text{ for } (c/v_{s1} - \sqrt{2})\,\Delta_{BC} \gtrless 0 \end{aligned}\right\} \tag{7.14}$$

in which $A = -\exp(2i\delta)$ as that in the case of (5.7).

(v)

$$\left.\begin{aligned} \Delta' &= K - \frac{\hat{\alpha}_1}{\xi}\frac{\hat{\beta}_1}{\xi}L - \frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\left(\frac{\hat{\alpha}_1}{\xi}\frac{\hat{\beta}_2}{\xi} + \frac{\hat{\beta}_1}{\xi}\frac{\hat{\alpha}_2}{\xi}\right), \\ \Delta^* &= K - \frac{\hat{\alpha}_1}{\xi}\frac{\hat{\beta}_1}{\xi}L + \frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\left(\frac{\hat{\alpha}_1}{\xi}\frac{\hat{\beta}_2}{\xi} + \frac{\hat{\beta}_1}{\xi}\frac{\hat{\alpha}_2}{\xi}\right), \\ -\Delta_A' &= K + \frac{\hat{\alpha}_1}{\xi}\frac{\hat{\beta}_1}{\xi}L + \frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\left(\frac{\hat{\alpha}_1}{\xi}\frac{\hat{\beta}_2}{\xi} - \frac{\hat{\beta}_1}{\xi}\frac{\hat{\alpha}_2}{\xi}\right), \\ -\Delta_D' &= K + \frac{\hat{\alpha}_1}{\xi}\frac{\hat{\beta}_1}{\xi}L - \frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\left(\frac{\hat{\alpha}_1}{\xi}\frac{\hat{\beta}_2}{\xi} - \frac{\hat{\beta}_1}{\xi}\frac{\hat{\alpha}_2}{\xi}\right). \end{aligned}\right\} \tag{7.15}$$

All reflecting coefficients have again real values.

(ii′)

$$\left.\begin{aligned} \Delta' &= K + \frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\bar{\beta}_1}{\xi}\frac{\bar{\alpha}_2}{\xi} - i\left\{\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_1}{\xi}L + \frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_2}{\xi}\right\} = |\Delta'|e^{-i\varepsilon}, \\ \Delta^* &= K - \frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\bar{\beta}_1}{\xi}\frac{\bar{\alpha}_2}{\xi} - i\left\{\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_1}{\xi}L - \frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_2}{\xi}\right\} = |\Delta^*|e^{i\varepsilon'}, \\ -\Delta_A' &= K + \frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\bar{\beta}_1}{\xi}\frac{\bar{\alpha}_2}{\xi} + i\left\{\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_1}{\xi}L + \frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_2}{\xi}\right\} = |\Delta'|e^{i\varepsilon}, \\ -\Delta_D' &= K - \frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\bar{\beta}_1}{\xi}\frac{\bar{\alpha}_2}{\xi} + i\left\{\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_1}{\xi}L - \frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_2}{\xi}\right\} = |\Delta^*|e^{-i\varepsilon'}. \end{aligned}\right|$$

(7.16)

where

$$\left.\begin{aligned}\tan\varepsilon&=\left\{\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_1}{\xi}L+\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_2}{\xi}\right\}\Big/\left\{K+\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\bar{\beta}_1}{\xi}\frac{\bar{\alpha}_2}{\xi}\right\},\\ \tan\varepsilon'&=-\left\{\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_1}{\xi}L-\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_2}{\xi}\right\}\Big/\left\{K-\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\bar{\beta}_1}{\xi}\frac{\bar{\alpha}_2}{\xi}\right\}.\end{aligned}\right\}\quad(7.17)$$

Here one has

$$A'=-\exp(2i\varepsilon)\text{ and }-\Delta_D'/\Delta^*=\exp(-2i\varepsilon').$$

Therefore

$$\left.\begin{aligned}&D'=(\Delta_D'/\Delta^*)(\Delta^*/\Delta)=-\Gamma\exp\{i(\varepsilon-\varepsilon')\},\\&A'D'-B'C'=\Gamma\exp\{i(\varepsilon+\varepsilon')\},\\&B'C\exp(-i\varepsilon)=\pm\{2i\Gamma\sin(\varepsilon+\varepsilon')\}^{1/2}.\end{aligned}\right\}\quad(7.18)$$

(iii′)

$$\left.\begin{aligned}\Delta'&=\bar{K}-\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_1}{\xi}\bar{L}-i\left\{\hat{K}+\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_1}{\xi}\hat{L}+\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\left(\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_2}{\xi}+\frac{\bar{\beta}_1}{\xi}\frac{\hat{\alpha}_2}{\xi}\right)\right\},\\ \Delta^*&=\bar{K}-\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_1}{\xi}\bar{L}-i\left\{\hat{K}+\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_1}{\xi}\hat{L}-\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\left(\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_2}{\xi}+\frac{\bar{\beta}_1}{\xi}\frac{\hat{\alpha}_2}{\xi}\right)\right\},\\ -\Delta_A'&=\bar{K}+\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_1}{\xi}\bar{L}-i\left\{\hat{K}-\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_1}{\xi}\hat{L}-\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\left(\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_2}{\xi}-\frac{\bar{\beta}_1}{\xi}\frac{\hat{\alpha}_2}{\xi}\right)\right\},\\ -\Delta_D'&=\bar{K}+\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_1}{\xi}\bar{L}-i\left\{\hat{K}-\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_1}{\xi}\hat{L}+\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\left(\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_2}{\xi}-\frac{\bar{\beta}_1}{\xi}\frac{\hat{\alpha}_2}{\xi}\right)\right\}.\end{aligned}\right\}$$

(iv′)

$$\left.\begin{aligned}\Delta&=\bar{K}-\frac{\hat{\alpha}_1}{\xi}\frac{\hat{\beta}_1}{\xi}\bar{L}-\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\hat{\beta}_1}{\xi}\frac{\hat{\alpha}_2}{\xi}-i\left\{\hat{K}-\frac{\hat{\alpha}_1}{\xi}\frac{\hat{\beta}_1}{\xi}\hat{L}+\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_2}{\xi}\right\},\\ \Delta^*&=\bar{K}-\frac{\hat{\alpha}_1}{\xi}\frac{\hat{\beta}_1}{\xi}\bar{L}+\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\hat{\beta}_1}{\xi}\frac{\hat{\alpha}_2}{\xi}-i\left\{\hat{K}-\frac{\hat{\alpha}_1}{\xi}\frac{\hat{\beta}_1}{\xi}\hat{L}-\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_2}{\xi}\right\},\\ -\Delta_A'&=\bar{K}+\frac{\hat{\alpha}_1}{\xi}\frac{\hat{\beta}_1}{\xi}\bar{L}-\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\hat{\beta}_1}{\xi}\frac{\hat{\alpha}_2}{\xi}-i\left\{\hat{K}+\frac{\hat{\alpha}_1}{\xi}\frac{\hat{\beta}_1}{\xi}\hat{L}-\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_2}{\xi}\right\},\\ -\Delta_D'&=\bar{K}+\frac{\hat{\alpha}_1}{\xi}\frac{\hat{\beta}_1}{\xi}\bar{L}+\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\hat{\beta}_1}{\xi}\frac{\hat{\alpha}_2}{\xi}-i\left\{\hat{K}+\frac{\hat{\alpha}_1}{\xi}\frac{\hat{\beta}_1}{\xi}\hat{L}+\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_2}{\xi}\right\}.\end{aligned}\right\}$$

(iii″)

$$\Delta'=K-\frac{\hat{\alpha}_1}{\xi}\frac{\hat{\beta}_1}{\xi}L-i\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\left(\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_2}{\xi}+\frac{\hat{\beta}_1}{\xi}\frac{\bar{\alpha}_2}{\xi}\right),$$

$$\left.\begin{aligned}\Delta^* &= K-\frac{\hat{\alpha}_1}{\xi}\frac{\hat{\beta}_1}{\xi}L+i\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\left(\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_2}{\xi}+\frac{\hat{\beta}_1}{\xi}\frac{\bar{\alpha}_2}{\xi}\right),\\ -\Delta_A' &= K+\frac{\hat{\alpha}_1}{\xi}\frac{\hat{\beta}_1}{\xi}L+i\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\left(\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_2}{\xi}-\frac{\hat{\beta}_1}{\xi}\frac{\bar{\alpha}_2}{\xi}\right),\\ -\Delta_D' &= K+\frac{\hat{\alpha}_1}{\xi}\frac{\hat{\beta}_1}{\xi}L-i\frac{\rho_2}{\rho_1}\left(\frac{c}{v_{s1}}\right)^4\left(\frac{\hat{\alpha}_1}{\xi}\frac{\bar{\beta}_2}{\xi}-\frac{\hat{\beta}_1}{\xi}\frac{\bar{\alpha}_2}{\xi}\right).\end{aligned}\right\}$$

8. Numerical values of reflecting coefficients.

Numerical values of reflecting coefficients have been calculated respectively for nine geological conditions which are classified in Table 2.

Table. 2. $\rho_1/\rho_2=1$, $v_{p1}/v_{p2}=1/4$.

σ_2 \ σ_1	0.25	0.48	0.50
0.25	0.25 – 0.25	0.48 – 0.25	0.50 – 0.25
0.48	0.25 – 0.48	0.48 – 0.48	0.50 – 0.48
0.50	0.25 – 0.50	0.48 – 0.50	0.50 – 0.50

Absolute values and arguments of A' and D' are exhibited in Fig. 2 when $\sigma_2=0.25$ and in Fig, 3 when $\sigma_2=0.48$. Full and dotted lines in these figures correspond respectively to cases when $\sigma_1=0.25$ and 0.48, but chain lines do to case when $\sigma_1=0.50$.

In types (iii) and (iv), numerical values of $\mathcal{E}$, $\mathcal{E}'$, Γ and Δ'_{BC} will be more useful than those of reflecting coefficients themselves. Therefore they are also exhibited in Figs. 4 and 5.

9. General expressions of the characteristic equation and of displacement potentials in the superficial layer.

Putting $M=0$ in (1.24), one has a general form of the characteristic equation,

$$2B'C=e^{i(\alpha_1+\beta_1)H}+(\Delta^*/\Delta')e^{-i(\alpha_1+\beta_1)H}-A\left\{A'e^{-i(\alpha_1-\beta_1)H}+D'e^{i(\alpha_1-\beta_1)H}\right\}, \quad (9.1)$$

according to (6.11).

On the other hand, one sees that

$$B'C=\pm\left\{(1-A^2)(\Delta^*/\Delta'-A'D')\right\}^{1/2}, \quad (9.2)$$

because he has, from (1.21) and (1.22)

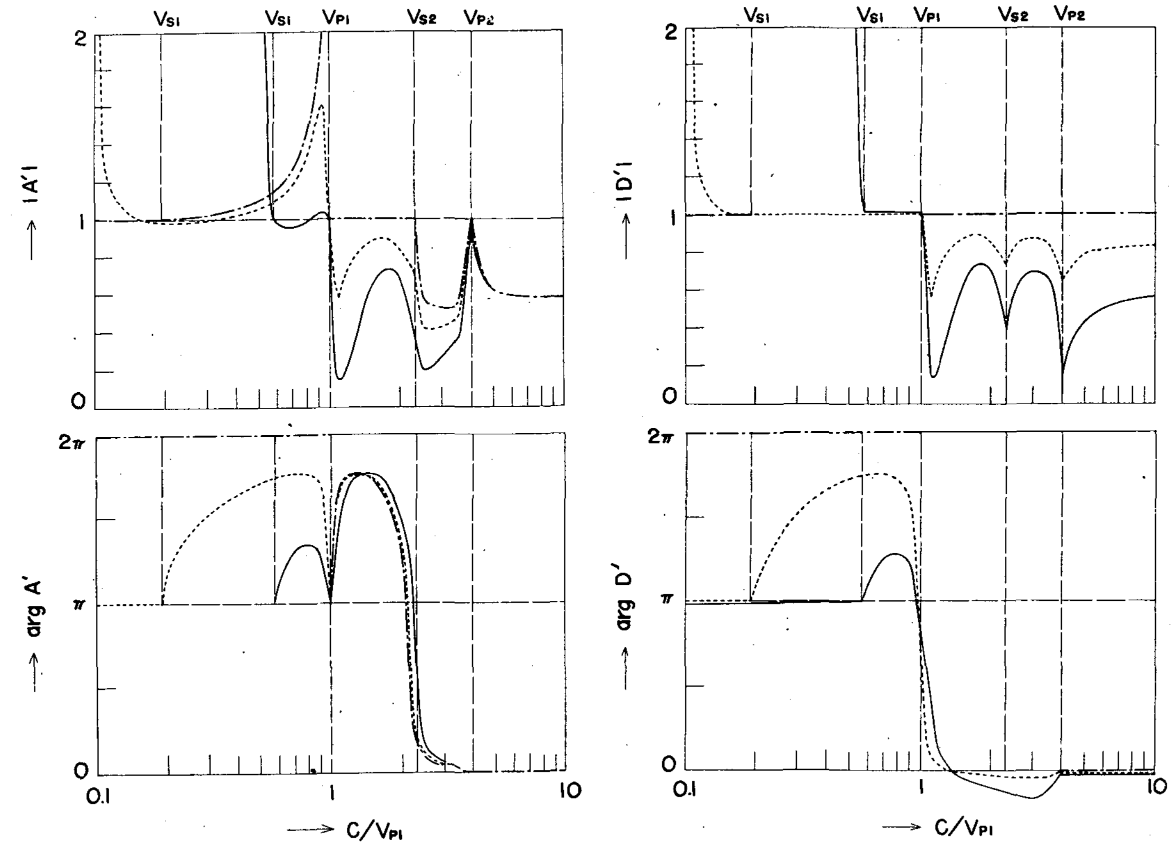

Fig. 2. Absolute values and arguments of A' and D' for conditions 0.25–0.25, 0.48–0.25 and 0.50–0.25 in Table 2.

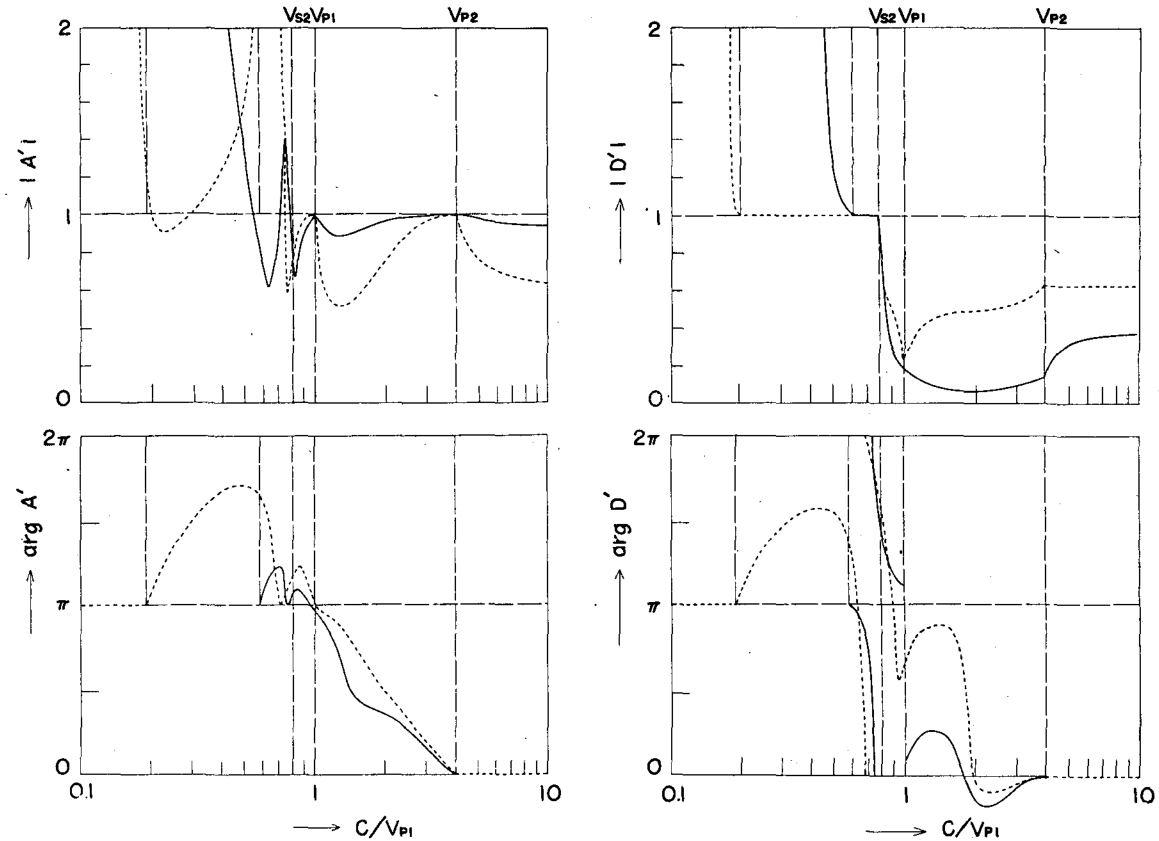

Fig. 3. Absolute values and arguments of *A'* and *D'* for conditions 0.25–0.48, 0.48–0.48 and 0.50–0.48 in Table 2.

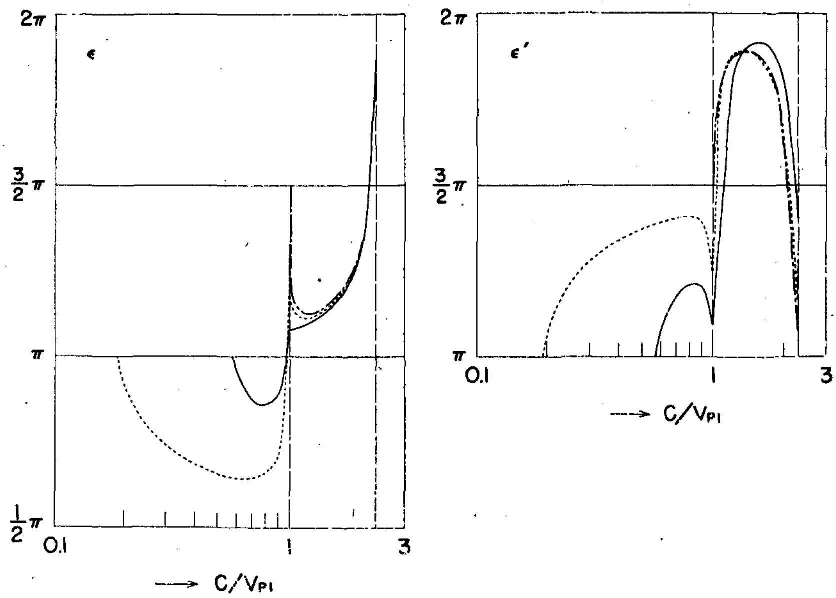

Fig. 4. (a). $\sigma_2=0.25$

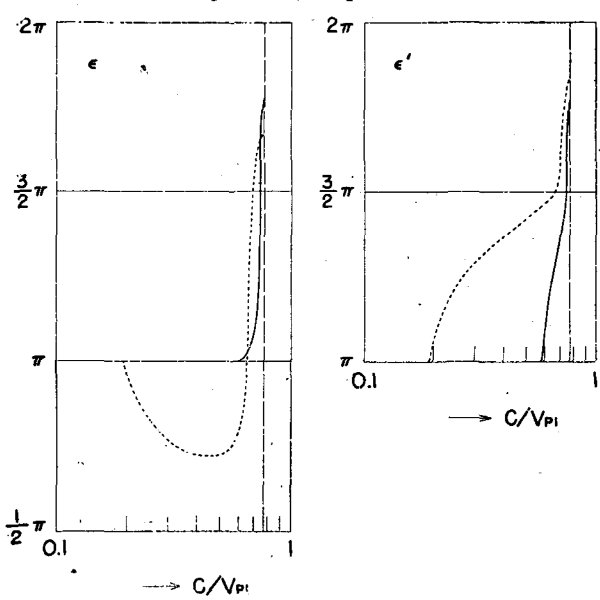

Fig. 4. (b). $\sigma_2=0.48$

Fig. 4. Numerical values of ε and ε' when $\rho_1/\rho_2=1$, $v_{p1}/v_{p2}=1/4$, $\sigma_2=0.25$ and 0.48, σ_1 being 0.25, 0.48 and 0.50.

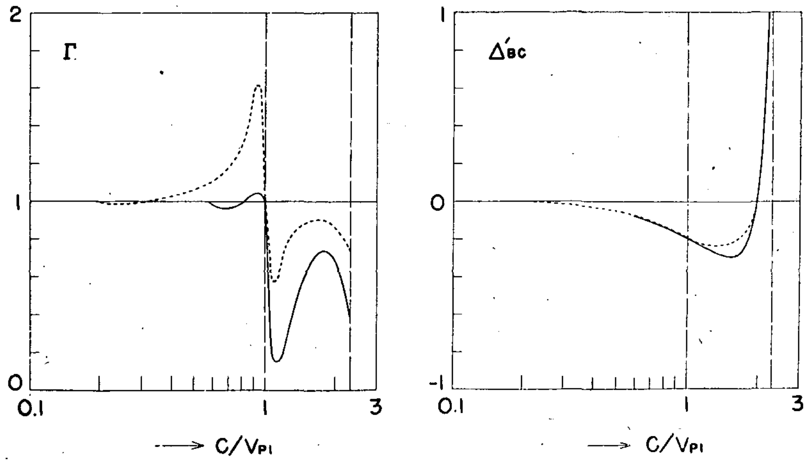

Fig. 5. (a). $\sigma_2=0.25$

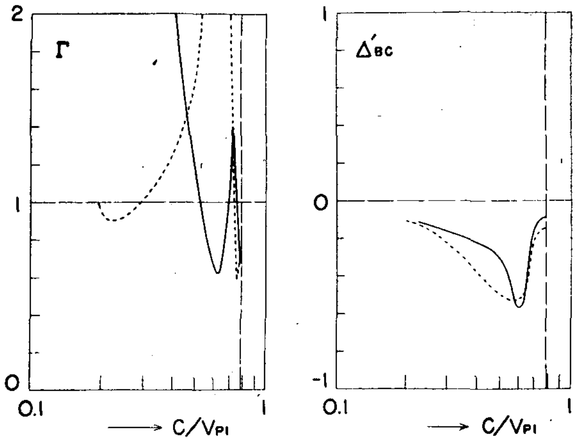

Fig. 5. (b) $\sigma_2=0.48$

Fig. 5. Numerical values of Γ and Δ_{BC}' when $\rho_1/\rho_2=1$, $v_{p1}/v_{p2}=1/4$, $\sigma_2=0.25$ and 0.48, σ_1 being 0.25, 0.48 and 0.50.

$$B=-\frac{\alpha_1}{\beta_1}C=4\frac{\alpha_1}{\xi}\left(\frac{c^2}{v_{s1}^2}-2\right)\left\{\left(\frac{c^2}{v_{s1}^2}-2\right)^2+4\frac{\alpha_1}{\xi}\frac{\beta_1}{\xi}\right\}^{-1}=\pm\left(\frac{\alpha_1}{\beta_1}\right)^{1/2}(1-A^2)^{1/2} \tag{9.3}$$

and from (6.3) and (6.11),

$$B'=2\frac{\alpha_1}{\xi}\frac{\Delta_{BC}'}{\Delta'}=\pm\left(\frac{\alpha_1}{\beta_1}\right)^{1/2}\left(\frac{\Delta^*}{\Delta'}-A'D'\right)^{1/2}. \tag{9.4}$$

Anyone of the $\pm$ signs of the right hand sides in (9.2) to (9.4) must be decided by $\pm$ signs of the left hand sides in (9.3) and (9.4).

When (1.23), (6.11) and (9.1) are used, numerators of ϕ_1 and ψ_1 in (1.1) and the factor ϑM in (2.3) will be calculated as follows:

$$\text{numerator of } \phi_1 e^{i(\alpha_1+\beta_1)H} = \cos\alpha_1 (E-z)\left[\{e^{i(\alpha_1+\beta_1)H}-(\Delta^*/\Delta)\, e^{-i(\alpha_1+\beta_1)H}\}\right.$$
$$\left.+A\,\{A'\,e^{-i(\alpha_1-\beta_1)H}-D'\,e^{i(\alpha_1-\beta_1)H}\}\right]+\{A'\,e^{i\alpha_1(E+z-H)+i\beta_1H}-D'\,e^{-i\alpha_1(E+z-H)-i\beta_1H}\}$$
$$-A\,\{(\Delta^*/\Delta)\,e^{i\alpha_1(E+z-H)-i\beta_1H}-e^{-i\alpha_1(E+z-H)+i\beta_1H}\}\,, \tag{9.5}$$

$$\text{numerator of } \psi_1 e^{i(\alpha_1+\beta_1)H} = B\left[\{(\Delta^*/\Delta)\,e^{-i\alpha_1(H-E)-i\beta_1(H-z)}+e^{i\alpha_1(H-E)+i\beta_1(H-z)}\}\right.$$
$$\left.+\{A'\,e^{-i\alpha_1(H-E)+i\beta_1(H-z)}+D'\,e^{i\alpha_1(H-E)-i\beta_1(H-z)}\}\right]+B'\left[\{e^{i(\alpha_1E+\beta_1z)}+e^{-i(\alpha_1E+\beta_1z)}\}\right.$$
$$\left.+A\{e^{i(\alpha_1E-\beta_1z)}+e^{-i(\alpha_1E-\beta_1z)}\}\right], \tag{9.6}$$

$$(\vartheta M)\,e^{i(\alpha_1+\beta_1)H}=i\,H\left[(\alpha_1+\beta_1)\{e^{i(\alpha_1+\beta_1)H}-(\Delta^*/\Delta)\,e^{-i(\alpha_1+\beta_1)H}\}\right.$$
$$\left.+A\,(\alpha_1-\beta_1)\,\{A'e^{-i(\alpha_1-\beta_1)H}-D'\,e^{i(\alpha_1-\beta_1)H}\}\right]. \tag{9.7}$$

After these preliminary calculations, one will reach

$$\begin{Bmatrix}\phi_1\\ \psi_1\end{Bmatrix}_{M=0}=\pi i\begin{Bmatrix}(9.5)\\(9.6)\end{Bmatrix}_{M=0}\left\{\alpha_1\frac{\partial M\,e^{i(\alpha_1+\beta_1)H}}{\partial\xi}\right\}^{-1}_{M=0}$$
$$=-\pi i\,\omega\left(\frac{1}{U}-\frac{1}{c}\right)\begin{Bmatrix}(9.5)\\(9.6)\end{Bmatrix}_{M=0}\left\{\alpha_1\,(9.7)\right\}^{-1}_{M=0}, \tag{9.8}$$

following the procedure described in (4.4) to (4.5).

It must be noticed that (9.1) for every type, except for (v) in Table 1, contains generally complex quantities. Therefore one has, in practice, the next simultaneous equations from (9.1),

$$\mathrm{Re}\ M(\omega,\xi)=0 \text{ and } \mathrm{Im}\ M(\omega,\xi)=0\,. \tag{9.9}$$

However (9.9) can have no real root, if the two equations are independent each other.

Referring to section 7, one obtains two independent equations from (9.1) in types (i), (ii), (ii′), (iii′), (iv′) and (iii″) but only one independent equation in types (iii), (iv) and (v). In the latter three types alone, therefore, can (9.1) have real roots. Looking at Table 1, one finds that c must be smaller than v_{s2} in these types. This means that no energy is propagated into the lower layer when the characteristic equation has real roots.

From now on, types (iii), (iv) and (v) alone will be considered in the present paper, treating only real roots of the characteristic equation.

10. Practical expressions of the characteristic equation, displacement potentials and displacements.

(iii) Putting (7.9) and (7.10) into (9.1), one has

$$\cos Ⓧ + A\,\Gamma \cos Ⓨ = \pm (1-A^2)^{1/2} (1-\Gamma)^{1/2} \text{ for } \Delta_{BC}' \lessgtr 0 \qquad (10.1)$$

in which

$$Ⓧ = (\bar{\alpha}_1 + \bar{\beta}_1) H + \varepsilon \text{ and } Ⓨ = (\bar{\alpha}_1 - \bar{\beta}_1) H - \varepsilon' .$$

Equation (10.1) is an expression identical with that obtained by TOLSTOY and USDIN (1953).

Putting (7.9) and (7.10) into (9.7), one sees that (9.8) will become

$$\left.\begin{aligned} [\phi_1]_{M=0} &= -\pi (\bar{\alpha}_1 H M_{\xi H})^{-1} \big[\cos \bar{\alpha}_1 (E-z) \{\sin Ⓧ + A\,\Gamma \sin Ⓨ\} \\ &\quad - A \sin \{\bar{\alpha}_1 (E+z) - Ⓧ\} - \Gamma \sin \{\bar{\alpha}_1 (E+z) - Ⓨ\}\big], \\ [\psi_1]_{M=0} &= \pi i (\bar{\alpha}_1 H M_{\xi H})^{-1} \big[B\{\cos (\bar{\alpha}_1 E + \bar{\beta}_1 z - Ⓧ) \\ &\quad - \Gamma \cos (\bar{\alpha}_1 E - \bar{\beta}_1 z - Ⓨ)\} + B' e^{i\varepsilon} \{\cos (\bar{\alpha}_1 E + \bar{\beta}_1 z) \\ &\quad + A \cos (\bar{\alpha}_1 E - \bar{\beta}_1 z)\}\big] \end{aligned}\right\} \qquad (10.2)$$

where

$$\left.\begin{aligned} (M_{\xi H})^{-1} &= (c/U-1) \big\{(\bar{\alpha}_1/\xi) (\sin Ⓧ + A\,\Gamma \sin Ⓨ) \\ &\quad + (\bar{\beta}_1/\xi) (\sin Ⓧ - A\,\Gamma \sin Ⓨ)\big\}^{-1}, \end{aligned}\right\} \qquad (10.3)$$

$$B = (\bar{\alpha}_1/\bar{\beta}_1)^{1/2} (1-A^2)^{1/2}$$

$$\text{and } B' \exp (i\,\varepsilon) = \pm (\bar{\alpha}_1/\bar{\beta}_1)^{1/2} (1-\Gamma^2)^{1/2} \text{ for } \Delta_{BC}' \gtrless 0 . \qquad (10.4)$$

Four components of the displacement will be easily calculated from (10.2) as follows :

$$H\left[\frac{\partial \phi_1}{\partial x}\right]_{M=0} = i \frac{\xi}{\bar{\alpha}_1} \frac{\pi}{M_{\xi H}} \big[\cos \bar{\alpha}_1 (E-z) (\sin Ⓧ + A\,\Gamma \sin Ⓨ)$$

$$- A \sin \{\bar{\alpha}_1 (E+z) - Ⓧ\} - \Gamma \sin \{\bar{\alpha}_1 (E+z) - Ⓨ\}\big],$$

$$H\left[\frac{\partial \phi_1}{\partial z}\right]_{M=0} = -\frac{\pi}{M_{\xi H}} \big[\sin \bar{\alpha}_1 (E-z) (\sin Ⓧ + A\,\Gamma \sin Ⓨ)$$

$$- A \cos \{\bar{\alpha}_1 (E+z) - Ⓧ\} - \Gamma \cos \{\bar{\alpha}_1 (E+z) - Ⓨ\}\big],$$

$$H\left[\frac{\partial \psi_1}{\partial x}\right]_{M=0} = \frac{\xi}{\bar{\alpha}_1} \frac{\pi}{M_{\xi H}} \big[B \{\cos (\bar{\alpha}_1 E + \bar{\beta}_1 z - Ⓧ) - \Gamma \cos (\bar{\alpha}_1 E - \bar{\beta}_1 z - Ⓨ)\}$$

$$+ B' e^{i\varepsilon} \{\cos (\bar{\alpha}_1 E + \bar{\beta}_1 z) + A \cos (\bar{\alpha}_1 E - \bar{\beta}_1 z)\}\big],$$

$$H\left[\frac{\partial \psi_1}{\partial z}\right]_{M=0} = i \frac{\bar{\beta}_1}{\bar{\alpha}_1} \frac{\pi}{M_{\xi H}} \big[-B\{\sin (\bar{\alpha}_1 E + \bar{\beta}_1 z - Ⓧ) + \Gamma \sin (\bar{\alpha}_1 E - \bar{\beta}_1 z - Ⓨ)\}$$

$$+B' e^{i\varepsilon} \{-\sin(\hat{\alpha}_1 E+\bar{\beta}_1 z)+A \sin(\hat{\alpha}_1 E-\bar{\beta}_1 z)\}] .$$

(iv)

Putting (7.13) and (7.14) into (9.1), one has

$$\sin(\bar{\beta}_1 H-\delta+\varepsilon)-\Gamma e^{-2\hat{\alpha}_1 H} \sin(\bar{\beta}_1 H+\delta+\varepsilon')$$
$$=\pm 2\,\Gamma^{1/2} e^{-\hat{\alpha}_1 H}\{\sin 2\delta \sin(\varepsilon'-\varepsilon)\}^{1/2} \text{ for } (c/v_{s1}-\sqrt{2})\,\Delta_{BC}' \gtrless 0 . \quad (10.5)$$

Putting (7.13) and (7.14) into (9.5) to (9.7), one sees that (9.8) will become

$$\left.\begin{aligned}
[\phi_1]_{M=0} &= -\pi(\hat{\alpha}_1 H M_{\xi H})^{-1}[\cosh\hat{\alpha}_1(E-z)\{\sin(\bar{\beta}_1 H-\delta+\varepsilon) \\
&\quad +\Gamma e^{-2\hat{\alpha}_1 H}\sin(\bar{\beta}_1 H+\delta+\varepsilon')\}-\{e^{-\hat{\alpha}_1(E+z)}\sin(\bar{\beta}_1 H+\delta+\varepsilon) \\
&\quad +\Gamma e^{-\hat{\alpha}_1(2H-E-z)}\sin(\bar{\beta}_1 H-\delta+\varepsilon')\}] , \\
[\psi_1]_{M=0} &= -\pi(\hat{\alpha}_1 H M_{\xi H})^{-1}[B e^{-i\delta}\{e^{-\hat{\alpha}_1 E}\sin(\bar{\beta}_1 H-\bar{\beta}_1 z+\varepsilon') \\
&\quad -\Gamma e^{-\hat{\alpha}_1(2H-E)}\sin(\bar{\beta}_1 H-\bar{\beta}_1 z+\varepsilon')\} \\
&\quad +B' e^{i\varepsilon}\{e^{-\hat{\alpha}_1(H-E)}\sin(\bar{\beta}_1 z-\delta)-e^{-\hat{\alpha}_1(H+E)}\sin(\bar{\beta}_1 z+\delta)\}]
\end{aligned}\right\} \quad (10.6)$$

where

$$(M_{\xi H})^{-1} = -(c/U-1)[(\hat{\alpha}_1/\xi)\{\sin(\bar{\beta}_1 H-\delta+\varepsilon)+\Gamma e^{-2\hat{\alpha}_1 H}\sin(\bar{\beta}_1 H+\delta+\varepsilon')\}$$
$$+(\bar{\beta}_1/\xi)\{\cos(\bar{\beta}_1 H-\delta+\varepsilon)-\Gamma e^{-2\hat{\alpha}_1 H}\cos(\bar{\beta}_1 H+\delta+\varepsilon')\}]^{-1}, \quad (10.8)$$

and

$$\left.\begin{aligned}
B e^{-i\delta} &= -4i(\hat{\alpha}_1/\xi)\cos\delta\,(c^2/v_{s1}^2-2)^{-1} \\
&= \pm i(2\hat{\alpha}_1/\bar{\beta}_1)^{1/2}(\sin 2\delta)^{1/2} \text{ for } c/v_{s1} \lesseqgtr \sqrt{2} \\
B' e^{i\varepsilon} &= -2i(\hat{\alpha}_1/\xi)(|\Delta'|)^{-1}\Delta_{BC}' \\
&= \pm i(2\hat{\alpha}_1/\bar{\beta}_1)^{1/2}\{\Gamma\sin(\varepsilon'-\varepsilon)\}^{1/2} \text{ for } \Delta_{BC}' \lesseqgtr 0 .
\end{aligned}\right\} \quad (10.9)$$

Four components of the displacement will be easily calculated from (10.6) as follows:

$$H\left[\frac{\partial\phi_1}{\partial x}\right]_{M=0} = i\frac{\xi}{\hat{\alpha}_1}\cdot\frac{\pi}{M_{\xi H}}[\cosh\hat{\alpha}_1(E-z)\{\sin(\bar{\beta}_1 H-\delta+\varepsilon)$$
$$+\Gamma e^{-2\hat{\alpha}_1 H}\sin(\bar{\beta}_1 H+\delta+\varepsilon')\}-\{e^{-\hat{\alpha}_1(E+z)}\sin(\bar{\beta}_1 H+\delta+\varepsilon)$$
$$+\Gamma e^{-\hat{\alpha}_1(2H-E-z)}\sin(\bar{\beta}_1 H-\delta+\varepsilon')\}] ,$$

$$H\left[\frac{\partial\phi_1}{\partial z}\right]_{M=0} = \frac{\pi}{M_{\xi H}}[\sinh\hat{\alpha}_1(E-z)\{\sin(\bar{\beta}_1 H-\delta+\varepsilon)$$
$$+\Gamma e^{-2\hat{\alpha}_1 H}\sin(\bar{\beta}_1 H+\delta+\varepsilon')\}-\{e^{-\hat{\alpha}_1(E+z)}\sin(\bar{\beta}_1 H+\delta+\varepsilon)$$
$$-\Gamma e^{-\hat{\alpha}_1(2H-E-z)}\sin(\bar{\beta}_1 H-\delta+\varepsilon')\}] ,$$

$$H\left[\frac{\partial\psi_1}{\partial x}\right]_{M=0} = i\frac{\xi}{\hat\alpha_1}\cdot\frac{\pi}{M_{\xi H}}\Big[B e^{-i\delta}\{e^{-\hat\alpha_1 E}\sin(\bar\beta_1 H-\bar\beta_1 z+\varepsilon)$$
$$-e^{-\hat\alpha_1(2H-E)}\sin(\bar\beta_1 H-\bar\beta_1 z+\varepsilon')\}$$
$$+B' e^{i\varepsilon}\{e^{-\hat\alpha_1(H-E)}\sin(\bar\beta_1 z-\delta)-e^{-\hat\alpha_1(H+E)}\sin(\bar\beta_1 z+\delta)\}\Big],$$

$$H\left[\frac{\partial\psi_1}{\partial z}\right]_{M=0} = -\frac{\bar\beta_1}{\hat\alpha_1}\cdot\frac{\pi}{M_{\xi H}}\Big[B e^{-i\delta}\{-e^{-\hat\alpha_1 E}\cos(\bar\beta_1 H-\bar\beta_1 z+E)$$
$$+e^{-\hat\alpha_1(2H-E)}\cos(\bar\beta_1 H-\bar\beta_1 z+\varepsilon')\}$$
$$+B' e^{i\varepsilon}\{e^{-\hat\alpha_1(H-E)}\cos(\bar\beta_1 z-\delta)-e^{-\hat\alpha_1(H+E)}\cos(\bar\beta_1 z+\delta)\}\Big].$$

(v)

Since $(\rho_2/\rho_1)\ (c/v_{s1})^4 \ll 1$ in this type,

$$\Delta^*/\Delta' \approx 1 \text{ and } A' \approx D'.$$

Therefore (9.1) will be reduced to

$$B'C = \cosh(\hat\alpha_1+\hat\beta_1)H - AA'\cosh(\hat\alpha_1-\hat\beta_1)H. \tag{10.10}$$

Numerical calculations show that

$$B'C \approx AA',$$

because

$$|A| \gg 1 \text{ and } |A'| \gg 1.$$

Thus (10.10) will be approximated by

$$\cosh(\hat\alpha_1+\hat\beta_1)H = AA'\{\cosh(\hat\alpha_1-\hat\beta_1)H+1\}. \tag{10.11}$$

This equation can have no real root, if $AA' \leqq 0$ as in region $0<c<v_{R1}$ where v_{R1} means velocity of RAYLEIGH waves when H is infinitely thick.

Putting (7.15) into (9.5) to (9.7), one sees that (9.8) will become

$$\left.\begin{aligned}
[\phi_1]_{M=0} = &-\pi(\hat\alpha_1 H M_{\xi H})^{-1}\big[\cosh\hat\alpha_1(E-z)\{\sinh(\hat\alpha_1+\hat\beta_1)H\\
&-AA'\sinh(\hat\alpha_1-\hat\beta_1)H+A'\sinh(\hat\alpha_1 E+\hat\alpha_1 z-\hat\alpha_1 H+\hat\beta_1 H)\\
&-A\sinh(\hat\alpha_1 E+\hat\alpha_1 z-\hat\alpha_1 H-\hat\beta_1 H)\big],\\
[\psi_1]_{M=0} = &-\pi(\hat\alpha_1 H M_{\xi H})^{-1}\big[B\{\cosh(\hat\alpha_1 H-\hat\alpha_1 E+\hat\beta_1 H-\hat\beta_1 z)\\
&+A'\cosh(\hat\alpha_1 H-\hat\alpha_1 E-\hat\beta_1 H+\hat\beta_1 z)\}+B'\{\cosh(\hat\alpha_1 E+\hat\beta_1 z)\\
&+A\cosh(\hat\alpha_1 E-\hat\beta_1 z)\}\big]
\end{aligned}\right\}\tag{10.12}$$

where

$$(M_{\xi H})^{-1} = -(c/U-1)\left[(\hat{\alpha}_1/\xi)\{\sinh(\hat{\alpha}_1+\hat{\beta}_1)H - AA'\sinh(\hat{\alpha}_1-\hat{\beta}_1)H\}\right.$$
$$\left. -(\hat{\beta}_1/\xi)\{\sinh(\hat{\alpha}_1+\hat{\beta}_1)H + AA'\sinh(\hat{\alpha}_1-\hat{\beta}_1)H\}\right]^{-1}, \qquad (10.13)$$
$$B = -i(\hat{\alpha}_1/\hat{\beta}_1)^{1/2}A \text{ and } B' = i(\hat{\alpha}_1/\beta_1)^{1/2}A'. \qquad (10.14)$$

Four components of the displacement will be easily calculated from (10.12) as follows:

$$H\left[\frac{\partial\phi_1}{\partial x}\right]_{M=0} = i\frac{\xi}{\hat{\alpha}_1}\cdot\frac{\pi}{M_{\xi H}}\left[\cosh\hat{\alpha}_1(E-z)\cdot\{\sinh(\hat{\alpha}_1+\hat{\beta}_1)H\right.$$
$$-AA'\sinh(\hat{\alpha}_1-\hat{\beta}_1)H\} + A'\sinh(\hat{\alpha}_1E+\hat{\alpha}_1z-\hat{\alpha}_1H+\hat{\beta}_1H)$$
$$\left.-A\sinh(\hat{\alpha}_1E+\hat{\alpha}_1z-\hat{\alpha}_1H-\hat{\beta}_1H)\right],$$

$$H\left[\frac{\partial\phi_1}{\phi z}\right]_{M=0} = -\frac{\pi}{M_{\xi H}}\left[\sinh\hat{\alpha}_1(E-z)\{\sinh(\hat{\alpha}_1+\hat{\beta}_1)H\right.$$
$$-AA'\sinh(\hat{\alpha}_1-\hat{\beta}_1)H\} + A'\cosh(\hat{\alpha}_1E+\hat{\alpha}_1z-\hat{\alpha}_1H+\hat{\beta}_1H)$$
$$\left.-A\cosh(\hat{\alpha}_1E+\hat{\alpha}_1z-\hat{\alpha}_1H-\hat{\beta}_1H)\right],$$

$$H\left[\frac{\partial\psi_1}{\partial x}\right]_{M=0} = i\frac{\xi}{\hat{\alpha}_1}\cdot\frac{\pi}{M_{\xi H}}\left[B\{\cosh(\hat{\alpha}_1H-\hat{\alpha}_1E+\hat{\beta}_1H-\hat{\beta}_1z)\right.$$
$$+A'\cosh(\hat{\alpha}_1H-\hat{\alpha}_1E-\hat{\beta}_1H+\hat{\beta}_1z)\} + B'\{\cosh(\hat{\alpha}_1E+\hat{\beta}_1z)$$
$$\left.+A\cosh(\hat{\alpha}_1E-\hat{\beta}_1z)\}\right],$$

$$H\left[\frac{\partial\psi_1}{\partial z}\right]_{M=0} = -\frac{\hat{\beta}_1}{\hat{\alpha}_1}\cdot\frac{\pi}{M_{\xi H}}\left[B\{-\sinh(\hat{\alpha}_1H-\hat{\alpha}_1E+\hat{\beta}_1H-\hat{\beta}_1z)\right.$$
$$+A'\sinh(\hat{\alpha}_1H-\hat{\alpha}_1E-\hat{\beta}_1H+\hat{\beta}_1z)\} + B'\{\sinh(\hat{\alpha}_1E+\hat{\beta}_1z)$$
$$\left.-A\sinh(\hat{\alpha}_1E-\hat{\beta}_1z)\}\right].$$

Numerical calculations were often confined in the relative amplitude, because it has been considered most troublesome to find numerical values of the common coefficient M_ξ in each displacement.

However, the present paper shows that calculation of M_ξ is rather more easy than that of quantities given by crotchets in the expression of the displacement.

It is to be noted that the amplitude characteristics can be obtained from the phase characteristics of these waves and *vice versa*, as those relations stated by the theory of the amplifier.

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