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Dispersive RAYLEIGH Waves in a Layer Overlying a Half Space

Kyozi TAZIME

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Abstract

The characteristic equation, displacement potentials and displacements appropriate to the topic have been expressed with reflection coefficients in references 1) and 2). Numerical values of reflection coefficients for some geological conditions given in Table 2 of reference 2) are illustrated in Figs. 2 to 5 of that paper.

Now several properties of dispersive RAYLEIGH waves may be investigated in reference to that table.

1. Dispersion curves as to phase velocity

According to the consideration given in 2), real roots in the characteristic equation must exist in regions (iii), (iv) and (v).

(iii) $v_{s2} > c > v_{p1}$

In agreement with reference 3), the characteristic equation can be expressed in the next form,

$$\left[\sin^2 \frac{1}{2}\left\{(\bar{\beta}_1+\bar{\alpha}_1) H+\varepsilon\right\} - l^2\right]^{1/2} = \mp (-A\Gamma)^{1/2} \sin \frac{1}{2}\left\{(\bar{\beta}_1-\bar{\alpha}_1) H+\varepsilon'\right\} \quad (1.1)$$

or

$$\left[l^2 - \sin^2 \frac{1}{2}\left\{(\bar{\beta}_1+\bar{\alpha}_1) H+\varepsilon\right\}\right]^{1/2} = \mp (A\,\Gamma)^{1/2} \sin \frac{1}{2}\left\{(\bar{\beta}_1-\bar{\alpha}_1) H+\varepsilon'\right\} \quad (1.2)$$

where

$$l^2 = \frac{1}{2}\left\{1 + A\,\Gamma \mp (1-A^2)^{1/2}(1-\Gamma^2)^{1/2}\right\} \text{ for } \Delta'_{\mathrm{BC}} \gtrless 0\,, \quad (1.3)$$

in which numerical value of l may be seen in Fig. 1.

An exmaple of graphical solution of (1.1) is shown in Fig. 2 where the left hand side of (1.1) is so drawn with thick lines that it will coincide with $\cos \frac{1}{2}\left\{(\bar{\beta}_1+\bar{\alpha}_1)H+\varepsilon\right\}$ if $l=1$. On the other hand, the right hand sides having negative or positive sign are drawn respectively as a thin full or a thin broken line in this figure. Thus roots of (1.1) are classified into two groups, respectively indicated by black and white circles.

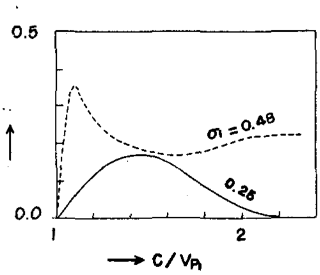

Fig. 1. The graph of l given by (1.3). $\rho_2/\rho_1=1$, $v_{p2}/v_{p1}=4$, $\sigma_2=0.25$.

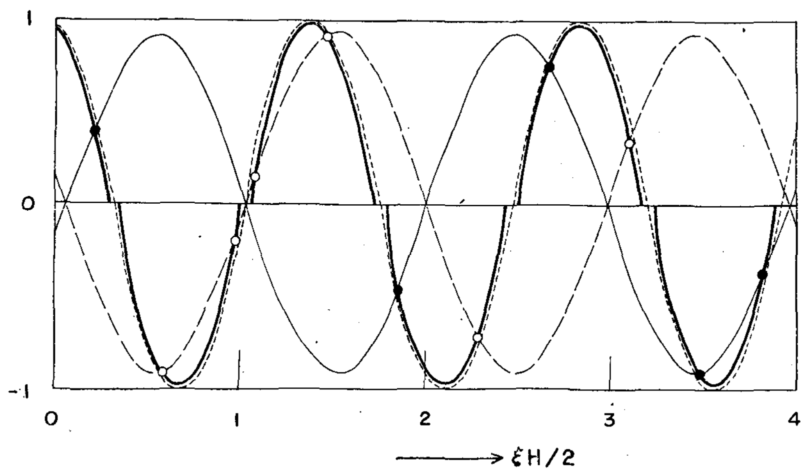

Fig. 2. An example of graphical solution of the characteristic equation. $\rho_2/\rho_1=1$, $v_{p2}/v_{p1}=4$, $\sigma_1=0.48$, $\sigma_2=0.25$, $c/v_{p1}=1.5$.

(iv) and (v) $v_{p1}>c$

In these regions shapes of dispersion curves are rather more simple than that of (iii). Special considerations will be omitted here.

Dispersion curves are illustrated by thick full and dotted lines in Figs. 3 to 6 for several cases tabulated in Table 1 in which primes mean figures

Table 1, $\rho_2/\rho_1=1$, $v_{p2}/v_{p1}=4$.

σ_2 \ σ_1	0.25	0.48	0.50
0.25	Fig. 3	Fig. 4	chain lines in Figs. 3 and 4
0.48	Fig. 5	Fig. 6	no real root
0.50	Fig. 5′	Fig. 4′	chain lines in Figs. 4′ and 5′

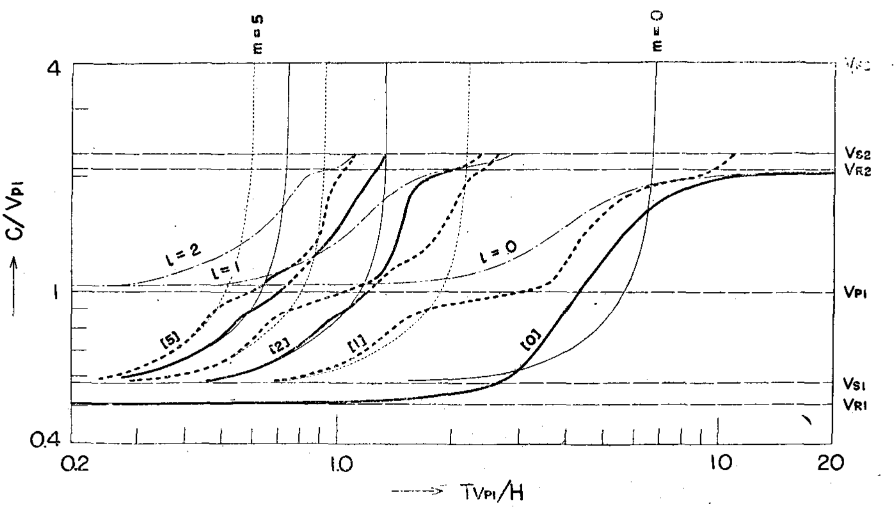

Fig. 3. $\sigma_1=0.25$, $\sigma_2=0.25$.

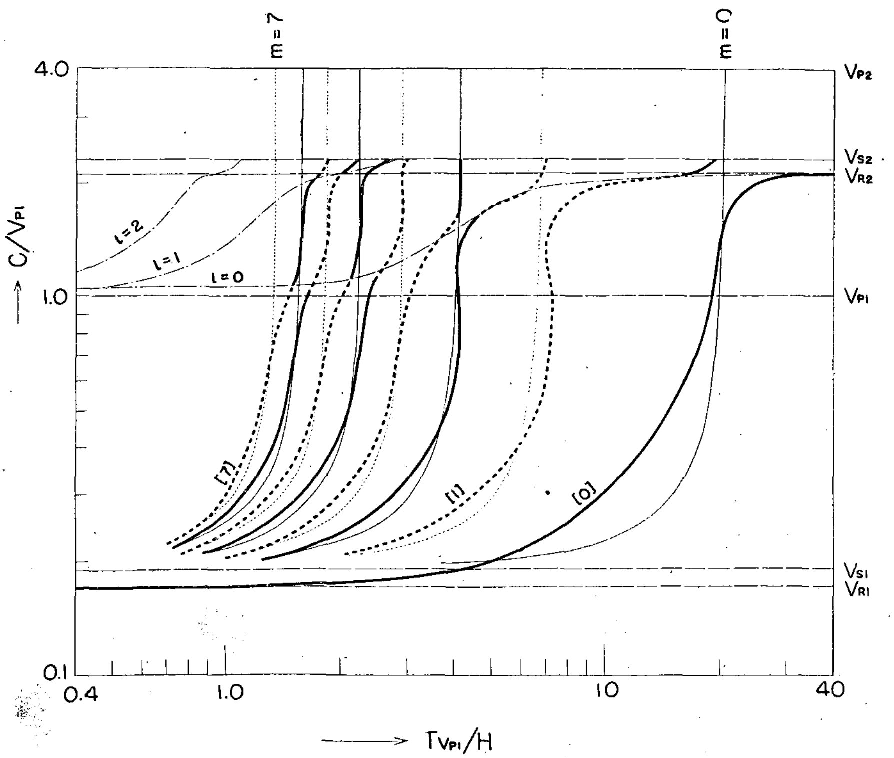

Fig. 4. $\sigma_1=0.48$, $\sigma_2=0.25$.

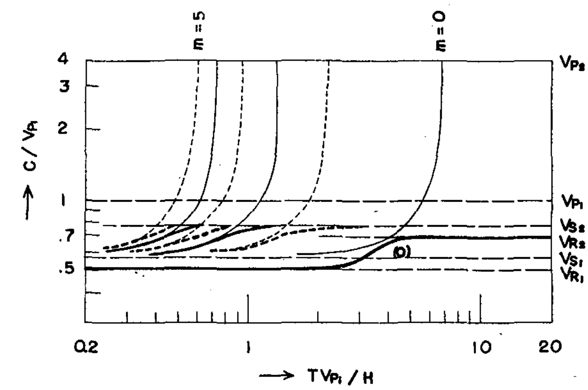

Fig. 5. $\sigma_1=0.25$, $\sigma_2=0.48$.

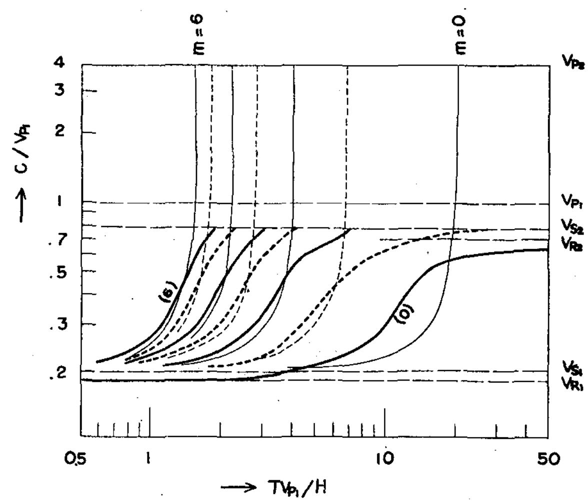

Fig. 6. $\sigma_1=0.48$, $\sigma_2=0.48$.

in reference 4). The thick full line shows the trace of the black circles while the thick dotted line does that of the white circles indicated in Fig. 2. Chain lines, on the other hand, are dispersion curves for normal mode waves in a liquid layer overlying a solid bottom. It must be noted in this stage that thick full and dotted lines should exchange their classifications with each other whenever they cross chain lines. This fact will be called "the first rule for construction of dispersion curves".

2. Transition from solid-solid to liquid-solid surface waves

When POISSON's ratio σ_1 becomes 0.50,

$BC=B'C'=0$, $A=D=-1$ and $D'=1$.

In this case, the characteristic equation obtained in reference 1),

$$M = 1-\{AA'\,e^{-2i\alpha_1 H}+DD'\,e^{-2i\beta_1 H} + (B'C'-A'D')e^{-2i(\alpha_1+\beta_1)H}+2\,B'C\,e^{-i(\alpha_1+\beta_1)H}\} = 0\,, \tag{2.1}$$

will be reduced to

$$1 - A'\exp(-2\,i\,\bar{\alpha}_1 H) = 0 \tag{2.2}$$

or

$$\cos\bar{\beta}_1 H = 0 \text{ that is } \bar{\beta}_1 H = \frac{\pi}{2}(2m+1)\,. \tag{2.3}$$

Eq. (2.2) is nothing but the characteristic equation for normal mode waves in a liquid layer overlying a solid bottom and is shown by chain lines in Figs. 3 and 4. Eq. (2.3), on the other hand, coincides with the characteristic equation of LOVE waves when μ_2/μ_1 becomes infinitely large. Even orders of (2.3) are shown by thin full lines and odd ones by thin dotted lines in Figs. 3 to 6.

One may see in reference 3) that (2.2) and (2.3) have the next combinations with each other,

$$\left.\begin{array}{l}\text{odd orders of (2.2) with even orders of (2.3)} \\ \text{but} \\ \text{even orders of (2.2) with odd orders of (2.3).}\end{array}\right\} \tag{2.4}$$

These combinations of (2.4) will be called "the second rule for construction of dispersion curves", because (2.2) and (2.3) might be taken as base lines for general dispersion curves.

One sees again in Figs. 3 and 4 that the nearer σ_1 approaches to 0.50, the larger part of thick lines coincides with thin lines which are taken as base lines. Moreover, it may be expected that amplitudes in (2.3) will be zero when σ_1 becomes 0.50. These circumstances are similar to that of the case when $\sigma_2=0.50$ described in reference 4).

3. Transition from solid-solid to liquid-liquid surface waves

Two different courses can be taken into consideration for this transition as shown by Fig. 7. Transitions from solid-solid to liquid-solid and from solid-liquid to liquid-liquid can be recognized with real roots of the characteristic equation, as one has seen in the previous section. These transitions are indicated by full lines in Fig. 7. On the other hand, transitions from solid-solid to solid-liquid and from liquid-solid to liquid-liquid must be recognized

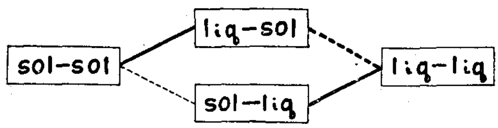

Fig. 7. Two courses of the transition from solid-solid to liquid-liquid surface waves.

with complex roots of the characteristic equation, as one may see in reference 5). Those transitions are connected by dotted lines in Fig. 7 where thick lines mean transitions already investigated but the thin one does a transition which has not yet been studied numerically.

4. Dispersion curves as to group velocity

From now on, in the present paper, POISSON's ratio σ_2 will be fixed to 0.25. Group velocity has been obtained graphically from dispersion curves as to phase velocity and is illustrated by thick lines in Figs. 8 and 9, where thin chain lines indicate group velocities for liquid-solid waves. The part which has steep inclination corresponds to that part for the phase velocity curve coinciding approximately with (2.3).

Since amplitudes of that part will become zero when σ_1 reaches 0.50,

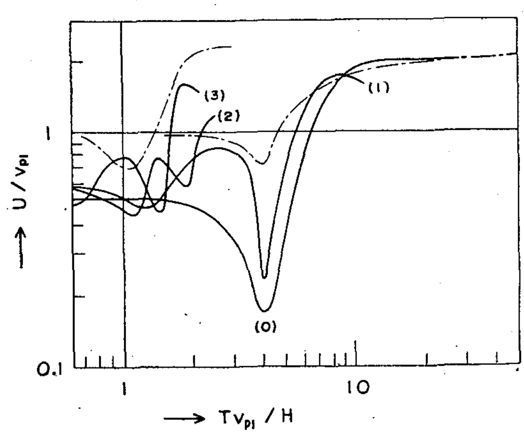

Fig. 8. $\sigma_1=0.25$, $\sigma_2=0.25$.

a group velocity curve for liquid-solid waves will be constructed from that for solid-solid waves of various orders.

In Fig. 9 the group velocity curve for the fourth order, for example, is very complicated, because one part of it is affected by liquid-liquid waves of the first order but the other part is affected by those of the second order.

Sometimes extremely small group velocity appears in dispersion curves when σ_1 approaches to 0.50. However this phase may be neglected, because it can not be propagated even if it has a large amplitude.

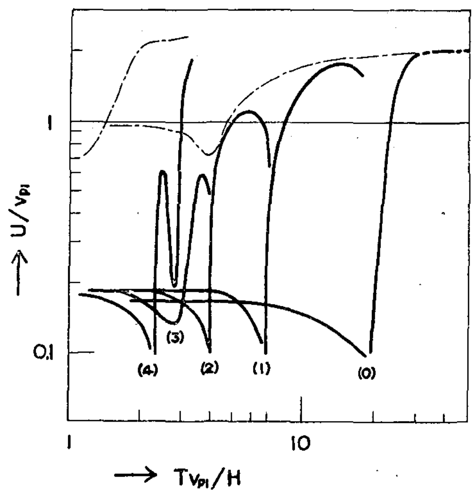

Fig. 9. $\sigma_1=0.48$, $\sigma_2=0.25$.

5. Discussion on amplitudes

Expressions for displacement potentials have had a common coefficient $1/M_\xi(\omega, \xi)$.

Now

$$M(\omega, \xi) = M(c, \xi) = 0 \tag{5.1}$$

in which

$$\omega = \xi c \ . \tag{5.2}$$

From (5.1) and (5.2), one sees

$$U = \frac{d\omega}{d\xi} = -\frac{M_\xi(\omega, \xi)}{M_\omega(\omega, \xi)} \quad \text{and} \quad \frac{U-c}{\xi} = \frac{dc}{d\xi} = -\frac{M_\xi(c, \xi)}{M_c(c, \xi)} \ . \tag{5.3}$$

Therefore

$$\frac{M_\xi(c,\xi)}{M_\xi(\omega,\xi)}\cdot\frac{M_\omega(\omega,\xi)}{M_c(c,\xi)}=\frac{1}{U}\cdot\frac{dc}{d\xi}=\frac{1}{\xi}\left(1-\frac{c}{U}\right). \tag{5.4}$$

Taking ξ as a constant, however, one has

$$\xi M_\omega(\omega,\xi)=M_c(c,\xi), \tag{5.5}$$

because $\partial\omega/\partial c=\xi$ from (5.2).

Inserting (5.5) into (5.4), one obtains

$$\frac{1}{M_\xi(\omega,\xi)}=-\left(\frac{c}{U}-1\right)\frac{1}{M_\xi(c,\xi)}=-\left(\frac{c}{U}-1\right)\frac{1}{HM_{\xi H}(c,\xi)} \tag{5.6}$$

where $M_{\xi H}$ means $\partial M/\partial(\xi H)$.

Here one must recall to mind that $\partial M(c,\xi)/\partial(\xi H)$ has been already given in Fig. 2 in the process of getting dispersion curves. Thus the common coefficient is to be easily obtained, at least logically. Unfortunately the relation between the common coefficient and period is not so simple in cases of solid-solid waves that one must take special care in numerical calculations.

Another difficulty exists in the expression of displacement potential in which coefficients containing only E and z are not separated from each other. In region (iii), displacement potentials may be rewritten as

$$\begin{aligned}[\phi_1]_{M=0}=&-\pi(\bar\alpha_1 M_\xi)^{-1}\{ⓐ\sin\bar\alpha_1 E\sin\bar\alpha_1 z+ⓑ\cos\bar\alpha_1 E\cos\bar\alpha_1 z\\&+ⓒ\sin\bar\alpha_1(E+z)\},\end{aligned} \tag{5.7}$$

$$\begin{aligned}[\psi_1]_{M=0}=&\,i\pi(\bar\alpha_1 M_\xi)^{-1}\{-ⓓ\sin\bar\alpha_1 E\sin\bar\beta_1 z\\&+ⓔ\cos\bar\alpha_1 E\cos\bar\beta_1 z+ⓕ\sin\bar\alpha_1 E\cos\bar\beta_1 z+ⓖ\cos\bar\alpha_1 E\sin\bar\beta_1 z\}\end{aligned} \tag{5.8}$$

where

$ⓐ=(1-A)(\sin ⓧ-\Gamma\sin ⓨ)$, $\quad ⓑ=(1+A)(\sin ⓧ+\Gamma\sin ⓨ)$,

$ⓒ=-(A\cos ⓧ+\Gamma\cos ⓨ)$, $\quad ⓓ=B(\cos ⓧ+\Gamma\cos ⓨ)+(1-A)B'e^{i\varepsilon}$,

$ⓔ=B(\cos ⓧ-\Gamma\cos ⓨ)+(1+A)B'e^{i\varepsilon}$, $\quad ⓕ=B(\sin ⓧ-\Gamma\sin ⓨ)$,

$ⓖ=B(\sin ⓧ+\Gamma\sin ⓨ)$.

But, ⓐ, ⓑ, ..., ⓖ are also altered considerably with change of period. Therefore the numerical value of $(M_\xi)^{-1}$ ⓐ, for instance, needs precise calculations of M_ξ as well as of ⓐ.

If σ_1 becomes 0.50,

$$\varepsilon+\varepsilon' \to 3\pi,\ \cos(\bar{\alpha}_1 H+\varepsilon) \to 0 \text{ or } \cos\bar{\beta}_1 H \to 0$$

and

$$\sin ⓧ - \Gamma \sin ⓨ \to 2\cos\bar{\beta}_1 H \sin(\bar{\alpha}_1 H+\varepsilon)\ ,$$

$$\sin ⓧ + \Gamma \sin ⓨ \to 2\sin\bar{\beta}_1 H \cos(\bar{\alpha}_1 H+\varepsilon)\ ,$$

$$A\cos ⓧ + \Gamma\cos ⓨ \to -2\cos\bar{\beta}_1 H \cos(\bar{\alpha}_1 H+\varepsilon)\ .$$

On the other hand,

$$M_{\xi H}(c,\xi) \to -2\{(\bar{\alpha}_1/\xi)\cos\bar{\beta}_1 H \sin(\bar{\alpha}_1 H+\varepsilon)+(\bar{\beta}_1/\xi)\sin\bar{\beta}_1 H\cos(\bar{\alpha}_1 H+\varepsilon)\}\ .$$

Therefore the first term is most important on the left hand side of (5.7) whilst $[\psi_1]_{M=0}$ may be trivial when σ_1 approaches to 0.50.

The coefficient of $\sin\bar{\alpha}_1 E\ \sin\bar{\alpha}_1 z$ alone has been calculated here when $\sigma_1=0.48$, because the amplitude of $[\phi_1]_{M=0}$ in region (iv) has been found also trivial.

If the original displacment potential is not sinusoidal but a pulse, $[\phi_1]_{M=0}$ must be proportional to

$$x^{-1/2}\,U\,(|\mathrm{d}\,U/\mathrm{d}\omega|)^{-1/2}\,T \tag{5.9}$$

when $\mathrm{d}U/\mathrm{d}\omega$ is different from zero. In Fig. 10,

$$(c-U)\,v_{P1}^{-1}\,(\alpha_1/\xi)^{-1}\,\{M_{\xi H}(c,\xi)\}^{-1}(\sin ⓧ - \Gamma\sin ⓨ) \tag{5.10}$$

is illutstrated by appropriate width of group velocity curve. The upper

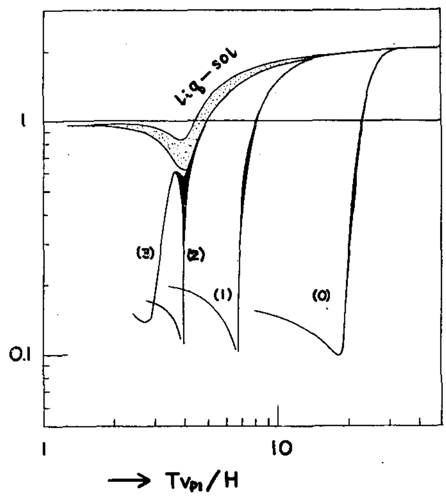

Fig. 10. Approximated amplitude of ϕ_1, when $\sigma_1=0.48$ and $\sigma_2=0.25$.

shadow zone exhibits $(c\text{-}U)/v_{p1}$ for liquid-solid waves of the fundamental mode on its group velocity curve, using the same scale as (5.10).

It must be noted in Fig. 10 that the property of band-pass is more apparent for $\sigma_1=0.48$ than for 0.50. Some period having predominant amplitude coincides nearly with $Tv_{p1}/H=4/(2l+1)$ and another with $Tv_{s1}/H=4/(2m+1)$.

It has been a matter of surprise that plots of observed group velocities are often partial to either side of the minimum group velocity. This question may be recognized in Fig. 10 which shows amplitudes will predominate partially on either side of it.

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