[Journal of the Faculty of Science, Hokkaidô University, Japan, Ser. VII, Vol. II, No. 2, 1964]

A Compact Expression for Displacement of Dispersive RAYLEIGH Waves in a Layer Overlying a Half Space

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(Received Sept. 30, 1964)

Abstract

An observation point as well as a line source of P or S wave have been considered respectively in each layer, a superficial one and a half space. Any displacement consists of three factors: the first is related to the condition of the source, the second does to that of the observation and the third is independent of the source and observation conditions but shows purely the character of surface waves. The reciprocal relation between an observation point and a source point has been pointed out on the result obtained.

1. A general expression for displacement potentials

In the present paper, such two dimentional two layers will be considered as that shown in Figs. 1 and 2 where velocities of P and S waves are taken as

$$v_{pj}=\{(\lambda_j+2\mu_j)/\rho_j\}^{1/2} \quad \text{and} \quad v_{sj}=(\mu_j/\rho_j)^{1/2}. \tag{1.1}$$

Taking angular frequency and angular wave number in x-direction as ω and ξ, we will use following notations:

$$h_j=\omega/v_{pj},\ k_j=\omega/v_{sj},\ \alpha_j=(h_j^2-\xi^2)^{1/2},\ \beta_j=(k_j^2-\xi^2)^{1/2} \tag{1.2}$$

where the subscript j is 1 or 2 and means the quantity relating to the j th layer.

Taking up a line source

$$\phi_{0\nu}=B_{0\nu}\exp\{\pm i\,\alpha_\nu(z-E)\} \quad \text{or} \quad \psi_{0\nu}=D_{0\nu}\exp\{\pm i\,\beta_\nu\,(z-E)\}$$
$$\text{for} \quad z\gtrless E \tag{1.3}$$

at $z=E$, we can write secondary diplacement potentials generated from $z=0$ and H as follows:

$$\left.\begin{aligned}\phi_1&=A_1 e^{i\alpha_1 z}+B_1 e^{-i\alpha_1 z}, & \psi_1&=C_1 e^{i\beta_1 z}+D_1 e^{-i\beta_1 z},\\ \phi_2&=B_2 e^{-i\alpha_2 z}, & \psi_2&=D_2 e^{-i\beta_2 z}\end{aligned}\right\} \tag{1.4}$$

in which the common coefficient $\exp\{i(\omega t-\xi x)\}$ is omitted. The subscript ν means that any source is lain in the ν th layer. In response to that notation,

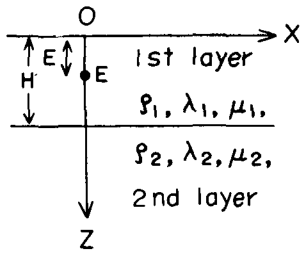

Fig. 1. A source lies in the first layer.

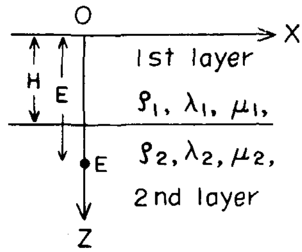

Fig. 2. A source lies in the second layer.

$\phi_j, A_j, \ldots, D_j$ in (1.4) must be written in detail as ϕ_j and so on. But subscript ν will sometimes be omitted for simplicity.

Further, we use more notations:

$$l_j = \alpha_j/\xi\,, \quad m_j = \beta_j/\xi\,, \quad n_j = m_j^2 - 1\,, \quad \chi = \mu_2/\mu_1\,, \tag{1.5}$$

$$\Delta = \begin{pmatrix} n_1 & -n_1 & 2m_1 & 2m_1 & 0 & 0 \\ 2l_1 & 2l_1 & -n_1 & n_1 & 0 & 0 \\ -e^{i\alpha_1 H} & e^{-i\alpha_1 H} & m_1 e^{i\beta_1 H} & m_1 e^{-i\beta_1 H} & 1 & m_2 \\ l_1 e^{i\alpha_1 H} & l_1 e^{-i\alpha_1 H} & e^{i\beta_1 H} & -e^{-i\beta_1 H} & l_2 & -1 \\ n_1 e^{i\alpha_1 H} & -n_1 e^{-i\alpha_1 H} & 2m_1 e^{i\beta_1 H} & 2m_1 e^{-i\beta_1 H} & -\chi n_2 & 2\chi m_2 \\ 2l_1 e^{i\alpha_1 H} & 2l_1 e^{-i\alpha_1 H} & -n_1 e^{i\beta_1 H} & n_1 e^{-i\beta_1 H} & 2\chi l_2 & \chi n_2 \end{pmatrix},$$

$$[\mathrm{A}_j] = \begin{pmatrix} A_1 \\ -B_1 \\ C_1 \\ -D_1 \\ B_2 e^{-i\alpha_2 H} \\ D_2 e^{-i\beta_2 H} \end{pmatrix}, \quad \frac{[\mathrm{P1}]}{B_{01}} = \begin{pmatrix} -n_1 e^{-i\alpha_1 E} \\ -2l_1 e^{-i\alpha_1 E} \\ e^{i\alpha_1 (E-H)} \\ l_1 e^{i\alpha_1 (E-H)} \\ -n_1 e^{i\alpha_1 (E-H)} \\ 2l_1 e^{i\alpha_1 (E-H)} \end{pmatrix}, \quad \frac{[\mathrm{S1}]}{D_{01}} = \begin{pmatrix} -2m_1 e^{-i\beta_1 E} \\ n_1 e^{-i\beta_1 E} \\ m_1 e^{i\beta_1 (E-H)} \\ -e^{i\beta_1 (E-H)} \\ 2m_1 e^{i\beta_1 (E-H)} \\ n_1 e^{i\beta_1 (E-H)} \end{pmatrix},$$

$$\frac{[\mathrm{P2}]}{B_{02}} = \begin{pmatrix} 0 \\ 0 \\ -e^{i\alpha_2 (H-E)} \\ l_2 e^{i\alpha_2 (H-E)} \\ \chi n_2 e^{i\alpha_2 (H-E)} \\ 2\chi l_2 e^{i\alpha_2 (H-E)} \end{pmatrix}, \quad \frac{[\mathrm{S2}]}{D_{02}} = \begin{pmatrix} 0 \\ 0 \\ m_2 e^{i\beta_2 (H-E)} \\ e^{i\beta_2 (H-E)} \\ 2\chi m_2 e^{i\beta_2 (H-E)} \\ -\chi n_2 e^{i\beta_2 (H-E)} \end{pmatrix}. \tag{1.6}$$

The boundary conditions tell that stresses must be zero on $z=0$ and stresses as well as displacements must be continuous on $z=H$. Then the following simultaneous equation written by the above matrices should be satisfied,

$$\Delta\,[A_j] = [P\,\nu] \tag{1.7}$$

in which $[P\nu]$ represents $[P\nu]$ or $[S\nu]$ in (1.6).

Every coefficient on the right hand side of (1.4) will be easily expressed by the above matrices as follows:

$$\left.\begin{array}{l} A_1 = \det \varDelta_{A1}/\det \varDelta\,,\ -B_1 = \det \varDelta_{B1}/\det \varDelta\,,\ C_1 = \det \varDelta_{C1}/\det \varDelta\,, \\ -D_1 = \det \varDelta_{D1}/\det \varDelta\,,\ B_2 \exp(-i\,\alpha_2 H) = \det \varDelta_{B2}/\det \varDelta\,, \\ D_2 \exp(-i\,\beta_2 H) = \det \varDelta_{D2}/\det \varDelta \end{array}\right\} \tag{1.8}$$

in which $\det \varDelta_{A1}$, for instance, is the determinant of the matrix where the first column of $\varDelta$ is replaced by matrix [P1].

Putting

$$(10) = \begin{pmatrix} n_1 & -2\,m_1 \\ 2\,l_1 & n_1 \end{pmatrix} \quad \text{and} \quad (12) = \begin{vmatrix} -1 & m_1 & 1 & m_2 \\ l_1 & 1 & l_2 & -1 \\ n_1 & 2\,m_1 & -\chi\,n_2 & 2\,\chi\,m_2 \\ 2\,l_1 & -n_1 & 2\,\chi\,l_2 & \chi\,n_2 \end{vmatrix}, \tag{1.9}$$

the present author[1)] has shown

$$\det \varDelta = \det(10)\cdot\det(12)\cdot M \tag{1.10}$$

and all determinants $\det \varDelta_{A1}$, ... are also proportional to det (10)·det (12). Therefore this common coefficient will be omitted hereafter.

Taking PP, PS, SP and SS reflection coefficients[2)] on $z=0$ and H in the first layer as

$$A,\ B,\ C,\ D \quad \text{and} \quad A',\ B',\ C',\ D',$$

the present author has expressed M in (1.10) as

$$M = e^{ip} + J\,e^{-ip} - A\,(A'\,e^{iq} + D'\,e^{-iq}) - 2\,B\,C' \tag{1.11}$$

in which

$$p = (\alpha_1 + \beta_1)\,H,\quad q = -\,(\alpha_1 - \beta_1)\,H \quad \text{and} \quad J = A'\,D' - B'\,C'\,. \tag{1.12}$$

In the previous papers[1),3),4)], M was multiplied by $\exp\{i(\alpha_1+\beta_1)H\}$. No confusion will occur by omitting the common coefficient $\exp\{i(\alpha_1+\beta_1)H\}$ from the previous M, $\det \varDelta_{A1}$ and so on.

Following the above process, $A_1, B_1, \ldots, D_2$ in (1.4) can be determined by (1.7). Therefore general expressions for displacement potentials may be obtained by the following operations:

$$\left.\begin{aligned} \Phi_{jp\nu} &= \int_{-\infty}^{\infty} (\phi_{o\nu} + \phi_j) \frac{d\xi}{\alpha_\nu}, \quad \Psi_{jp\nu} = \int_{-\infty}^{\infty} \psi_j \frac{d\xi}{\alpha_\nu}, \\ \Phi_{js\nu} &= \int_{-\infty}^{\infty} \phi_j \frac{d\xi}{\beta_\nu}, \quad \Psi_{js\nu} = \int_{-\infty}^{\infty} (\psi_{o\nu} + \psi_j) \frac{d\xi}{\beta_\nu}. \end{aligned}\right\} \quad (1.13)$$

where $\phi_{o\nu} = \psi_{o\nu} = 0 \quad \text{for} \quad j \neq \nu.$

Among various waves contained in integral expression (1.13), the waves which satisfy the characteristic equation

$$M(\omega, \xi) = 0 \qquad (1.14)$$

will be called dispersive RAYLEIGH waves. Equation (1.14) makes the right hand side of (1.11) zero.

Displacement potentials of dispersive RAYLEIGH waves are to be derived from poles in the integral of (1.13) and may be calculated as follows:

$$\left.\begin{aligned} [\Phi_1]^{P\nu}_{M=0} &= \pi i (\alpha_\nu M_\xi)^{-1} \left[(\det \Delta_{A1}) e^{i\alpha_1 z} + (-\det \Delta_{B1}) e^{-i\alpha_1 z}\right]^{P\nu}_{M=0}, \\ [\Psi_1]^{P\nu}_{M=0} &= \pi i (\alpha_\nu M_\xi)^{-1} \left[(\det \Delta_{C1}) e^{i\beta_1 z} + (-\det \Delta_{D1})^{-i\beta_1 z}\right]^{P\nu}_{M=0}, \\ [\Phi_1]^{S\nu}_{M=0} &= \pi i (\beta_\nu M_\xi)^{-1} \left[(\det \Delta_{A1}) e^{i\alpha_1 z} + (-\det \Delta_{B1}) e^{-i\alpha_1 z}\right]^{S\nu}_{M=0}, \\ [\Psi_1]^{S\nu}_{M=0} &= \pi i (\beta_\nu M_\xi)^{-1} \left[(\det \Delta_{C1}) e^{i\beta_1 z} + (-\det \Delta_{D1}) e^{-i\beta_1 z}\right]^{S\nu}_{M=0} \end{aligned}\right\} \quad (1.15)$$

and

$$\left.\begin{aligned} [\Phi_2]^{P\nu}_{M=0} &= \pi i (\alpha_\nu M_\xi)^{-1} \left[(\det \Delta_{B2}) e^{-i\alpha_2 z}\right]^{P\nu}_{M=0}, \\ [\Psi_2]^{P\nu}_{M=0} &= \pi i (\alpha_\nu M_\xi)^{-1} \left[(\det \Delta_{D2}) e^{-i\beta_2 z}\right]^{P\nu}_{M=0}, \\ [\Phi_2]^{S\nu}_{M=0} &= \pi i (\beta_\nu M_\xi)^{-1} \left[(\det \Delta_{B2}) e^{-i\alpha_2 z}\right]^{S\nu}_{M=0}, \\ [\Psi_2]^{S\nu}_{M=0} &= \pi i (\beta_\nu M_\xi)^{-1} \left[(\det \Delta_{D2}) e^{-i\beta_2 z}\right]^{S\nu}_{M=0} \end{aligned}\right\} \quad (1.16)$$

where ν is 1 or 2 and M_ξ means $\partial M(\omega, \xi)/\partial \xi$.

2. Displacement potentials in the first layer generated from a line source of P wave in the first layer

Putting

$$\left.\begin{aligned} &ⓛ = A' e^{iq} - A\,J\,e^{-ip}\,, \quad ⓜ = e^{ip} - J\,e^{-ip} + A\,A'\,e^{iq} - A\,D'\,e^{-iq}\,, \\ &ⓝ = A\,e^{ip} - D'\,e^{-iq} \end{aligned}\right\} \tag{2.1}$$

we have, from (1.5), (1.6), (1.7) and (1.14),

$$\left.\begin{aligned} &B_{01}^{-1}\,(\det \Delta_{A1})_{M=0}^{P1} = ⓛ\, e^{i\alpha_1 E} + \frac{1}{2}\,ⓜ\, e^{-i\alpha_1 E}\,, \\ &B_{01}^{-1}\,(-\det \Delta_{B1})_{M=0}^{P1} = \frac{1}{2}\,ⓜ\, e^{i\alpha_1 E} + ⓝ\, e^{-i\alpha_1 E}\,, \\ &B_{01}^{-1}\,(\det \Delta_{C1})_{M=0}^{P1} = C^{-1}\Big\{\Big(\frac{1}{2}ⓜ - ⓛ\,A\Big)e^{i\alpha_1 E} + \Big(ⓝ - \frac{1}{2}ⓜ\,A\Big)e^{-i\alpha_1 E}\Big\}\,, \\ &B_{01}^{-1}\,(-\det \Delta_{D1})_{M=0}^{P1} = C^{-1}\Big\{\Big(\frac{1}{2}ⓜ\,A - ⓛ\Big)e^{i\alpha_1 E} + \Big(ⓝ\,A - \frac{1}{2}ⓜ\Big)e^{-i\alpha_1 E}\Big\}\,. \end{aligned}\right\} \tag{2.2}$$

However, if (1.14) is satisfied, the next relation will be found,

$$\exp(-2\,i\,\theta) = ⓛ\Big/\Big(\frac{1}{2}\,ⓜ\Big) = \frac{1}{2}\,ⓜ/ⓝ\,. \tag{2.3}$$

Therefore (2.2) may be rewritten as follows:

$$\left.\begin{aligned} &B_{01}^{-1}\,(\det \Delta_{A1})_{M=0}^{P1} = ⓜ\cos(\alpha_1 E - \theta)\cdot e^{-i\theta}\,, \\ &B_{01}^{-1}\,(-\det \Delta_{B1})_{M=0}^{P1} = ⓜ\cos(\alpha_1 E - \theta)\cdot e^{i\theta}\,, \\ &B_{01}^{-1}\,(\det \Delta_{C1})_{M=0}^{P1} = ⓜ\cos(\alpha_1 E - \theta)\cdot C^{-1}\,(e^{i\theta} - A\,e^{-i\theta})\,, \\ &B_{01}^{-1}\,(-\det \Delta_{D1})_{M=0}^{P1} = ⓜ\cos(\alpha_1 E - \theta)\cdot C^{-1}\,(A\,e^{i\theta} - e^{-i\theta})\,, \end{aligned}\right\} \tag{2.4}$$

resulting in

$$B_{01}^{-1}\left[(\det \Delta_{A1})\,e^{i\alpha_1 z} + (-\det \Delta_{B1})\,e^{-i\alpha_1 z}\right]_{M=0}^{P1} = 2\,ⓜ\cos(\alpha_1 E - \theta)\cdot\cos(\alpha_1 z - \theta)\,,$$

$$B_{01}^{-1}\left[(\det \Delta_{C1})\, e^{i\beta_1 z} + (-\det \Delta_{D1})\, e^{-i\beta_1 z}\right]_{M=0}^{P1} = i\,2\,ⓜ \cos(\alpha_1 E - \theta)\cdot C^{-1}$$
$$\times \left\{\sin(\beta_1 z + \theta) - A \sin(\beta_1 z - \theta)\right\}.$$

At last, the first and the second equations in (1.15) become

$$\left.\begin{aligned} B_{01}^{-1}\,[\Phi_1]_{M=0}^{P1} &= i\,(2\,\pi/M_\xi)\,(ⓜ/\alpha_1)\cos(\alpha_1 E-\theta)\cos(\alpha_1 z-\theta)\,,\\ B_{01}^{-1}\,[\Psi_1]_{M=0}^{P1} &= -\,(2\,\pi/M_\xi)\,(ⓜ/\alpha_1)\cos(\alpha_1 E-\theta)\cdot C^{-1}\\ &\quad\times\left\{\sin(\beta_1 z+\theta)-A\sin(\beta_1 z-\theta)\right\}.\end{aligned}\right\} \tag{2.5}$$

On the other hand, the present author[3] has obtained the relation,

$$\{M_\xi(\omega,\xi)\}^{-1} = -\,(\xi/H)\,(c\,U^{-1}-1)\,(\alpha_1\,ⓜ + \beta_1\,ⓜ) \tag{2.6}$$

in which c and U mean respectively phase velocity and $d\omega/d\xi$ and

$$ⓜ = e^{ip} - J\,e^{-ip} - (A\,A'\,e^{iq} - A\,D'\,e^{-iq})\,. \tag{2.7}$$

If we compare (2.1) with (2.7), refering to (1.11), we find ⓜ will become ⓜ by replacing α_1 by β_1 and A' by D'. When the same replacements as those for ⓜ are made for ⓛ and ⓝ, we have

$$ⓛ = D'\,e^{-iq} - A\,J\,e^{-ip} \quad\text{and}\quad ⓝ = A\,e^{ip} - A'\,e^{iq}\,. \tag{2.8}$$

Since (1.14) is again satisfied, we have the similar relation to (2.3),

$$\exp(-2\,i\,\theta') = ⓛ\Big/\left(\frac{1}{2}ⓜ\right) = \frac{1}{2}\,ⓜ/ⓝ\,. \tag{2.9}$$

3. Displacement potentials in the first layer generated from a line source of S wave in the first layer

Using notations given by (2.7) and (2.8), we have, through the same process as that obtaining (2.2),

$$D_{01}^{-1}\,(\det \Delta_{A1})_{M=0}^{S1} = B^{-1}\left\{\left(\frac{1}{2}ⓜ - ⓛ\,A\right)e^{i\beta_1 E} + \left(ⓝ - \frac{1}{2}ⓜ\,A\right)e^{-i\beta_1 E}\right\},$$

$$D_{01}^{-1}\,(-\det \Delta_{B1})_{M=0}^{S1} = B^{-1}\left\{\left(\frac{1}{2}ⓜ\,A - ⓛ\right)e^{i\beta_1 E} + \left(ⓝ A - \frac{1}{2}ⓜ\right)e^{-i\beta_1 E}\right\},$$

$$D_{01}^{-1}\,(\det \Delta_{C1})_{M=0}^{S1} = ⓛ\,e^{i\beta_1 E} + \frac{1}{2}\,ⓜ\,e^{-i\beta_1 E}\,,$$

$$D_{01}^{-1}\,(-\det \varDelta_{\mathrm{D1}})_{M=0}^{\mathrm{S1}} = \frac{1}{2}\,\textcircled{m'}\, e^{i\beta_1 E} + \textcircled{n'}\, e^{-i\beta_1 E}\,.$$

When (1.14) is satisfied, these equations may be rewritten by the use of (2.9):

$$\left.\begin{aligned}
D_{01}^{-1}\,(\det \varDelta_{\mathrm{A1}})_{M=0}^{\mathrm{S1}} &= \textcircled{m'} \cos(\beta_1 E - \theta') \cdot B^{-1}\,(e^{i\theta'} - A\, e^{-i\theta'})\,,\\
D_{01}^{-1}\,(-\det \varDelta_{\mathrm{B1}})_{M=0}^{\mathrm{S1}} &= \textcircled{m'} \cos(\beta_1 E - \theta') \cdot B^{-1}\,(A\, e^{i\theta'} - e^{-i\theta'})\,,\\
D_{01}^{-1}\,(\det \varDelta_{\mathrm{C1}})_{M=0}^{\mathrm{S1}} &= \textcircled{m'} \cos(\beta_1 E - \theta') \cdot e^{-i\theta'}\,,\\
D_{01}^{-1}\,(-\det \varDelta_{\mathrm{D1}})_{M=0}^{\mathrm{S1}} &= \textcircled{m'} \cos(\beta_1 E - \theta')\, e^{i\theta'}\,.
\end{aligned}\right\} \tag{3.1}$$

Therefore the third and the fourth equations in (1.15) become

$$\begin{aligned}
D_{01}^{-1}\,[\varPhi_1]_{M=0}^{\mathrm{S1}} &= -\,(2\,\pi/M_\xi)\,(\textcircled{m'}/\beta_1) \cos(\beta_1 E - \theta') \cdot B^{-1}\\
&\qquad \times \left\{\sin(\alpha_1 z + \theta') - A \sin(\alpha_1 z - \theta')\right\}, \qquad (3.2)\\
D_{01}^{-1}\,[\varPsi_1]_{M=0}^{\mathrm{S1}} &= i\,(2\,\pi/M_\xi)\,(\textcircled{m'}/\beta_1) \cos(\beta_1 E - \theta') \cos(\beta_1 z - \theta')\,.
\end{aligned}$$

However, we can see

$$\left.\begin{aligned}
&\frac{1}{2}\,\textcircled{m} - \textcircled{l}\, A = \frac{1}{2}\,\textcircled{m'} - \textcircled{l'}\, A\,, \quad \frac{1}{2}\,\textcircled{m}\, A - \textcircled{l} = \textcircled{n'} - \frac{1}{2}\,\textcircled{m'}\, A\,,\\
&\textcircled{n} - \frac{1}{2}\,\textcircled{m}\, A = \frac{1}{2}\,\textcircled{m'}\, A - \textcircled{l'}\,, \quad \textcircled{n}\, A - \frac{1}{2}\,\textcircled{m} = \textcircled{n'}\, A - \frac{1}{2}\,\textcircled{m'}\,,
\end{aligned}\right\} \tag{3.3}$$

resulting in

$$\textcircled{n} - \textcircled{l} = \textcircled{n'} - \textcircled{l'}\,. \tag{3.4}$$

As we have from (2.3) and (2.9)

$$1 - 4\, e^{-4i\theta} = (\textcircled{n} - \textcircled{l})/\textcircled{n} \quad \text{and} \quad 1 - 4\, e^{-4i\theta'} = (\textcircled{n'} - \textcircled{l'})/\textcircled{n'}\,,$$

we can know with the aid of (3.4),

$$\sin 2\,\theta/\sin 2\,\theta' = \textcircled{n'}\, e^{-2i\theta'}/(\textcircled{n}\, e^{-2i\theta}) = \textcircled{m'}/\textcircled{m}\,. \tag{3.5}$$

Moreover, we have from (2.3), (2.9) and (3.3)

$$i \tan\theta = (1 - e^{-2i\theta})/(1 + e^{-2i\theta})$$

$$=\left\{\left(n A-\frac{1}{2} m\right)-\left(\frac{1}{2} m A-l\right)\right\}\left\{\left(n A-\frac{1}{2} m\right)+\left(\frac{1}{2} m A-l\right)\right\}^{-1}$$

$$=\left\{\left(n A-\frac{1}{2} m\right)-\left(n-\frac{1}{2} m A\right)\right\}\left\{\left(n A-\frac{1}{2} m\right)+\left(n-\frac{1}{2} m A\right)\right\}^{-1}$$

$$=(A-1)(A+1)^{-1}\left(n+\frac{1}{2} m\right)\left(n-\frac{1}{2} m\right)^{-1}=(i \tan \theta')^{-1}(A-1)(A+1)^{-1} .$$

$$\therefore \quad \tan \theta \tan \theta'=(1-A)(1+A)^{-1} . \tag{3.6}$$

Using (3.5) and (3.6), we arrive at the following relations:

$$\left.\begin{aligned}
&\sin(\beta_1 z+\theta)-A \sin(\beta_1 z-\theta)=(1+A)(\sin\theta/\cos\theta')\cos(\beta_1 z-\theta'),\\
&\sin(\alpha_1 z+\theta')-A\sin(\alpha_1 z-\theta')=(m/m)(1+A)(\sin\theta/\cos\theta')\cos(\alpha_1 z-\theta),\\
&m/\beta_1=(m/\alpha_1)\{(1+A)/C\}^2(\sin\theta/\cos\theta')^2 .
\end{aligned}\right\} \tag{3.7}$$

Putting

$$X=2\pi m/(\alpha_1 M_\xi) \quad \text{and} \quad Y=\{(1+A)/C\}(\sin\theta/\cos\theta'), \tag{3.8}$$

we see $Y^2=(\alpha_1/\beta_1)(m/m)$ and we can rewrite (2.5) and (3.2) as follows:

$$\left.\begin{aligned}
B_{01}^{-1}[\Phi_1]_{M=0}^{P1}&=i X\cos(\alpha_1 E-\theta)\cos(\alpha_1 z-\theta),\\
B_{01}^{-1}[\Psi_1]_{M=0}^{P1}&=-X Y\cos(\alpha_1 E-\theta)\cos(\beta_1 z-\theta'),
\end{aligned}\right\} \tag{3.9}$$

$$\left.\begin{aligned}
D_{01}^{-1}[\Phi_1]_{M=0}^{S1}&=X Y\cos(\beta_1 E-\theta')\cos(\alpha_1 z-\theta),\\
D_{01}^{-1}[\Psi_1]_{M=0}^{S1}&=i X Y^2\cos(\beta_1 E-\theta')\cos(\beta_1 z-\theta') .
\end{aligned}\right\} \tag{3.10}$$

Even if we exchange E for z, the right hand sides of the first equation in (3.9) and the second equation in (3.10) remain as they were. On the other hand, the second equation in (3.9) changes the sign of the first equation in (3.10) by exchanging E for z. In the latter case, the source of P wave is changed for that of S wave and the wave observed does from S to P. It may be due to the relation between reflection coefficients, $B=-(\alpha_1/\beta_1)C$ and $B'=-(\alpha_1/\beta_1)C'$ that the second equation in (3.9) has a different sign from the first equation in (3.10). Following the above considerations, we see that E and z are reciprocal in (3.9) and (3.10).

Equations (3.9) and (3.10) can be arranged in a somewhat compact form

$$\left.\begin{aligned}[\Phi_1]^{\nu=1}_{M=0} &= [\Phi_1]^{\mathrm{P1}}_{M=0} + [\Phi_1]^{\mathrm{S1}}_{M=0} = i\, Z_1\, X \cos(\alpha_1 z - \theta)\,, \\ [\Psi_1]^{\nu=1}_{M=0} &= [\Psi_1]^{\mathrm{P1}}_{M=0} + [\Psi_1]^{\mathrm{S1}}_{M=0} = -\, Z_1\, X\, Y \cos(\beta_1 z - \theta')\end{aligned}\right\} \tag{3.11}$$

in which

$$Z_1 = B_{01} \cos(\alpha_1 E - \theta) - i\, Y\, D_{01} \cos(\beta_1 E - \theta')\,. \tag{3.12}$$

4. Displacements in the first layer generated from a line source in the first layer

Taking components of displacement in x- and z-directions as u_j and w_j, we can express them as

$$u_j = \partial \phi_j / \partial x + \partial \psi_j / \partial z \quad \text{and} \quad w_j = \partial \phi_j / \partial z - \partial \psi_j / \partial x\,. \tag{4.1}$$

When P and S waves are generated at the same time from a line source in the first layer, every component of displacement consists of two parts:

$$u_1^{\nu=1} = u_1^{\mathrm{P1}} + u_1^{\mathrm{S1}} \quad \text{and} \quad w_1^{\nu=1} = w_1^{\mathrm{P1}} + w_1^{\mathrm{S1}}\,. \tag{4.2}$$

Substituting (3.11) into (4.1), we have

$$u_1^{\nu=1} = \xi\, Z_1\, X\, U_1 \quad \text{and} \quad w_1^{\nu=1} = -\, i\, \xi\, Z_1\, X\, W_1 \tag{4.3}$$

in which

$$\left.\begin{aligned}U_1 &= \cos(\alpha_1 z - \theta) + (\beta_1/\xi)\, Y \sin(\beta_1 z - \theta')\,, \\ W_1 &= (\alpha_1/\xi) \sin(\alpha_1 z - \theta) + Y \cos(\beta_1 z - \theta')\,.\end{aligned}\right\} \tag{4.4}$$

In (4.4) Z_1 is related to conditions of the source and U_1 or W_1 does to those of observation. On the other hand, X does not depend on conditions of the source as well as on those of observation, but shows purely the character of surface waves.

As we had the next boundary conditions:

$$u_1^{\nu=1} = u_2^{\nu=1} \quad \text{and} \quad w_1^{\nu=1} = w_2^{\nu=1} \quad \text{on} \quad z = H\,, \tag{4.5}$$

we have, from (4.3) and (4.5),

$$\left.\begin{aligned}[B_2\, e^{-i\alpha_2 H}]^{\nu=1}_{M=0} &= \{1 + (\alpha_2/\xi)(\beta_2/\xi)\}^{-1} Z_1\, X\, \{i\, U_1 + (\beta_2/\xi)\, W_1\}_{z=H}\,, \\ [D_2\, e^{-i\beta_2 H}]^{\nu=1}_{M=0} &= \{1 + (\alpha_2/\xi)(\beta_2/\xi)\}^{-1} Z_1\, X\, \{i(\alpha_2/\xi)\, U_1 - W_1\}_{z=H}\,.\end{aligned}\right\} \tag{4.6}$$

Displacement potentials in the second layer will be easily obtained, from (4.6), as follows:

$$\left.\begin{aligned}[\Phi_2]^{\nu=1}_{M=0} &= e^{-i\alpha_2(z-H)}\{1+(\alpha_2/\xi)(\beta_2/\xi)\}^{-1} Z_1 X \{i U_1 + (\beta_2/\xi) W_1\}_{z=H}\,, \\ [\Psi_2]^{\nu=1}_{M=0} &= e^{-i\beta_2(z-H)}\{1+(\alpha_2/\xi)(\beta_2/\xi)\}^{-1} Z_1 X \{i(\alpha_2/\xi) U_1 - W_1\}_{z=H}\,.\end{aligned}\right\} \tag{4.7}$$

5. Displacement in the second layer generated from a line source in the first layer

By a different process from that described at the end of the previous section, we can get B_2 and D_2 directly from (1.6) and (1.7):

$$\begin{aligned}B_{01}^{-1}[B_2 e^{-i\alpha_2 H}]^{=1}_{M=0} &= [\{E' e^{-i\alpha_1 H}(e^{ip} - A D' e^{-iq} - B C') + G' e^{-i\beta_1 H}(A' B e^{iq} \\ &\quad + A B')\} e^{i\alpha_1 E} + \{E' e^{-i\alpha_1 H}(A e^{ip} - D' e^{-iq}) + G' e^{-i\beta_1 H} \\ &\quad \times (B e^{ip} + B')\} e^{-i\alpha_1 E}]/M\,, \\ B_{01}^{-1}[D_2 e^{-i\beta_2 H}]^{=1}_{M=0} &= [\{F' e^{-i\alpha_1 H}(e^{ip} - A D' e^{-iq} - B C') + H' e^{-i\beta_1 H}(A' B e^{iq} \\ &\quad + A B')\} e^{i\alpha_1 E} + \{F' e^{-i\alpha_1 H}(A e^{ip} - D' e^{-iq}) + H' e^{-i\beta_1 H} \\ &\quad \times (B e^{ip} + B')\} e^{-i\alpha_1 E}]/M\end{aligned}$$

where E', F', G' and H' are PP, PS, SP and SS refraction coefficients on $z=H$ from the first to the second layer[1)]

If condition (1.14) is satisfied, the above equations can be rewritten by the use of (2.1) and (2.3) as follows:

$$\left.\begin{aligned}B_{01}^{-1}[B_2 e^{-i\alpha_2 H}]^{\mathrm{P1}}_{M=0} &= i(\pi/M_\xi)(m/\alpha_1)\cos(\alpha_1 E - \theta) \\ &\quad \times \{E' e^{-i(\alpha_1 H - \theta)} + (G' e^{-i\beta_1 H}) C^{-1}(A e^{i\theta} - e^{-i\theta})\}\,, \\ B_{01}^{-1}[D_2 e^{-i\beta_2 H}]^{\mathrm{P1}}_{M=0} &= i(\pi/M_\xi)(m/\alpha_1)\cos(\alpha_1 E - \theta) \\ &\quad \times \{F' e^{-i(\alpha_1 H - \theta)} + (H' e^{-i\beta_1 H}) C^{-1}(A e^{i\theta} - e^{-i\theta})\}\,.\end{aligned}\right\} \tag{5.1}$$

The equivalent expression to (5.1) can be also obtained by ray theoretical considerations:

$$\left.\begin{aligned} B_{01}^{-1}\,[B_2\,e^{-i\alpha_2 H}]^{\mathrm{P1}} &= E'\,\left\{e^{i\alpha_1(E-H)} + B_1\,e^{-i\alpha_1 H}\right\} + (G'\,e^{-i\beta_1 H})\,D_1\,, \\ B_{01}^{-1}\,[D_2\,e^{-i\beta_2 H}]^{\mathrm{P1}} &= F'\,\left\{e^{i\alpha_1(E-H)} + B_1\,e^{-i\alpha_1 H}\right\} + (H'\,e^{-i\beta_1 H})\,D_1\,. \end{aligned}\right\} \tag{5.2}$$

Substituting (1.8) and (2.2) into (5.2), we see this will coincide with (5.1).

When S wave is generated from the origin, we have, by the similar process to that arriving at (5.1),

$$\left.\begin{aligned} D_{01}^{-1}\,[B_2\,e^{-i\alpha_2 H}]_{M=0}^{\mathrm{S1}} &= i\,(\pi/M_\xi)\,(\textcircled{m'}/\beta_1)\cos(\beta_1 E - \theta') \\ &\quad\times\left\{(E'\,e^{-i\alpha_1 H})\,B^{-1}\,(A\,e^{i\theta'} - e^{-i\theta}) + G'\,e^{-i(\beta_1 H - \theta')}\right\}, \\ D_{01}^{-1}\,[D_2\,e^{-i\beta_2 H}]_{M=0}^{\mathrm{S1}} &= i\,(\pi/M_\xi)\,(\textcircled{m'}/\beta_1)\cos(\beta_1 E - \theta') \\ &\quad\times\left\{(F'\,e^{-i\alpha_1 H})\,B^{-1}\,(A\,e^{i\theta'} - e^{-i\theta}) + H'\,e^{-i(\beta_1 H - \theta')}\right\}. \end{aligned}\right\} \tag{5.3}$$

At any rate, displacement potentials in the second layer may be expressed as follows:

$$\left.\begin{aligned} B_{01}^{-1}\,[\Phi_2]_{M=0}^{\mathrm{P1}} &= i\,(\pi/M_\xi)\,(\textcircled{m}/\alpha_1)\,\left\{E'\,e^{-i(\alpha_1 H - \theta)} + (G'\,e^{-i\beta_1 H})\,C^{-1}\right. \\ &\quad\times\left.(A\,e^{i\theta} - e^{-i\theta})\right\}\cos(\alpha_1 E - \theta)\,e^{-i\alpha_2(z-H)}\,, \\ B_{01}^{-1}\,[\Psi_2]_{M=0}^{\mathrm{P1}} &= i\,(\pi/M_\xi)\,(\textcircled{m}/\alpha_1)\,\left\{F'\,e^{-i(\alpha_1 H - \theta)} + (H'\,e^{-i\beta_1 H})\,C^{-1}\right. \\ &\quad\times\left.(A\,e^{i\theta} - e^{-i\theta})\right\}\cos(\alpha_1 E - \theta)\,e^{-i\beta_2(z-H)}\,, \end{aligned}\right\} \tag{5.4}$$

$$\left.\begin{aligned} D_{01}^{-1}\,[\Phi_2]_{M=0}^{\mathrm{S1}} &= i\,(\pi/M_\xi)\,(\textcircled{m'}/\beta_1)\,\left\{(E'\,e^{-i\alpha_1 H})\,B^{-1}\,(A\,e^{i\theta'} - e^{-i\theta})\right. \\ &\quad\left.+\,G'\,e^{-i(\beta_1 H - \theta')}\right\}\cos(\beta_1 E - \theta')\,e^{-i\alpha_2(z-H)}\,, \\ D_{01}^{-1}\,[\Psi_2]_{M=0}^{\mathrm{S1}} &= i\,(\pi/M_\xi)\,(\textcircled{m'}/\beta_1)\,\left\{(F'\,e^{-i\alpha_1 H}))\,B^{-1}\,(A\,e^{i\theta'} - e^{-i\theta})\right. \\ &\quad\left.+\,H'\,e^{-i(\beta_1 H - \theta')}\right\}\cos(\beta_1 E - \theta')\,e^{-i\beta_2(z-H)}\,. \end{aligned}\right\} \tag{5.5}$$

However, we know the next relation from (3.5), (3.6) and (3.8)

$$\left.\begin{aligned} C^{-1}\,(A\,e^{i\theta} - e^{-i\theta}) &= i\,C^{-1}\,(1 + A)\,(\sin\theta/\cos\theta')\,e^{i\theta'} = i\,Y\,e^{i\theta'}\,, \\ B^{-1}\,(A\,e^{i\theta'} - e^{-i\theta'}) &= i\,(\textcircled{m}/\textcircled{m'})\,B^{-1}\,(1 + A)\,(\sin\theta/\cos\theta')\,e^{i\theta} \\ &= -\,i\,(\beta_1/\alpha_1)\,(\textcircled{m}/\textcircled{m'})\,Y\,e^{i\theta}\,. \end{aligned}\right\} \tag{5.6}$$

Therefore we have

$$\left.\begin{aligned}
[\Phi_2]^{P1}_{M=0}+[\Phi_2]^{S1}_{M=0}&=i\,Z_1\,X\cdot\frac{1}{2}\left\{E'\,e^{-i(\alpha_1 H-\theta)}+i\,Y\,G'\,e^{-i(\beta_1 H-\theta')}\right\}e^{-i\alpha_2(z-H)}\,,\\
[\Psi_2]^{P1}_{M=0}+[\Psi_2]^{S1}_{M=0}&=i\,Z_1\,X\cdot\frac{1}{2}\left\{F'\,e^{-i(\alpha_1 H-\theta)}+i\,Y\,H'\,e^{-i(\beta_1 H-\theta')}\right\}e^{-i\beta_2(z-H)}\,.
\end{aligned}\right\}\tag{5.7}$$

Physically, (4.7) should coincide with (5.7) by the next equations:

$$[\Phi_2]^{\nu=1}_{M=0}=[\Phi_2]^{P1}_{M=0}+[\Phi_2]^{S1}_{M=0}\quad\text{and}\quad[\Psi_2]^{\nu=0}_{M=0}=[\Psi_2]^{P1}_{M=0}+[\Psi_2]^{S1}_{M=0}\,,$$

that is

$$\left.\begin{aligned}
\frac{1}{2}\left\{E'\,e^{-i(\alpha_1 H-\theta)}+i\,Y\,G'\,e^{-i(\beta_1 H-\theta')}\right\}&=\left\{1+(\alpha_2/\xi)\,(\beta_2/\xi)\right\}^{-1}\\
&\quad\times\left\{U_1-i\,(\beta_2/\xi)\,W_1\right\}_{z=H}\,,\\
\frac{1}{2}\left\{F'\,e^{-i(\alpha_1 H-\theta)}+i\,Y\,H'\,e^{-i(\beta_1 H-\theta')}\right\}&=\left\{1+(\alpha_2/\xi)\,(\beta_2/\xi)\right\}^{-1}\\
&\quad\times\left\{(\alpha_2/\xi)\,U_1+i\,W_1\right\}_{z=H}\,.
\end{aligned}\right\}\tag{5.8}$$

6. Displacement potentials generated from a source in the second layer

Denoting PP and PS refraction coefficients on $z=H$ from the second to the first layer by E'' and F''[1], we may arrive at, from (1.6), (1.7) and (1.8),

$$\begin{aligned}
B_{02}^{-1}\,[A_1]^{P2}_{M=0}&=e^{-i\alpha_2(E-H)}\left\{E''\,e^{-i\alpha_1 H}\,(e^{ip}-A\,D'\,e^{-iq}-B\,C')\right.\\
&\qquad\left.+F''\,e^{-i\beta_1 H}\,(A\,C'+C\,A'\,e^{iq})\right\}/M\,,\\
B_{02}^{-1}\,[B_1]^{P2}_{M=0}&=e^{-i\alpha_2(E-H)}\left\{E''\,e^{-i\alpha_1 H}\,(A\,e^{ip}-D'\,e^{-iq})+F''\,e^{-i\beta_1 H}\,(C\,e^{ip}-C')\right\}/M\,,\\
B_{02}^{-1}\,[C_1]^{P2}_{M=0}&=e^{-i\alpha_2(E-H)}\left\{E''\,e^{-i\alpha_1 H}\,(B\,D'\,e^{-iq}+A\,B')\right.\\
&\qquad\left.+F''\,e^{-i\beta_1 H}\,(e^{ip}-A\,A'\,e^{iq}-B\,C')\right\}/M\,,\\
B_{02}^{-1}\,[D_1]^{P2}_{M=0}&=e^{-i\alpha_2(E-H)}\left\{E''\,e^{-i\alpha_1 H}\,(B\,e^{ip}+B')+F''\,e^{-i\beta_1 H}\,(A\,e^{ip}-A'\,e^{iq})\right\}/M\,.
\end{aligned}$$

When condition (1.14) is satisfied, the above equations can be rewritten, by the use of (2.3) and (2.9), as follows:

$$B_{02}^{-1}\,[A_1]_{M=0}^{P2} = i\,(\pi/M_\xi)\,(\text{ⓜ}/\alpha_2)\cdot\frac{1}{2}\left\{E''\,e^{-i(\alpha_1 H-\theta)} + F''\,e^{-i\beta_1 H}\right.$$
$$\left.\times B^{-1}\,(A\,e^{i\theta} - e^{-i\theta})\right\}\,e^{-i\alpha_2(E-H)}\,e^{-i\theta}\,,$$

$$B_{02}^{-1}\,[B_1]_{M=0}^{P2} = i\,(\pi/M_\xi)\,(\text{ⓜ}/\alpha_2)\cdot\frac{1}{2}\left\{E''\,e^{-i(\alpha_1 H-\theta)} + F''\,e^{-i\beta_1 H}\right.$$
$$\left.\times B^{-1}\,(A\,e^{i\theta} - e^{-i\theta})\right\}\,e^{-i\alpha_2(E-H)}\,e^{i\theta}\,,$$

$$B_{02}^{-1}\,[C_1]_{M=0}^{P2} = i\,(\pi/M_\xi)\,(\text{ⓜ}/\alpha_2)\cdot\frac{1}{2}\left\{E''\,e^{-i\alpha_1 H}\cdot C^{-1}\,(A\,e^{i\theta'} - e^{-i\theta})\right.$$
$$\left.+ F''\,e^{-i(\beta_1 H-\theta')}\right\}\,e^{-i\alpha_2(E-H)}\,e^{-i\theta'}\,,$$

$$B_{02}^{-1}\,[D_1]_{M=0}^{P2} = i\,(\pi/M_\xi)\,(\text{ⓜ}/\alpha_2)\cdot\frac{1}{2}\left\{E''\,e^{-i\alpha_1 H}\cdot C^{-1}\,(A\,e^{i\theta'} - e^{-i\theta})\right.$$
$$\left.+ F''\,e^{-i(\beta_1 H-\theta')}\right\}\,e^{-i\alpha_2(E-H)}\,e^{i\theta'}\,.$$

Refering to the previous paper[2], we see

$$\left.\begin{aligned} E'' &= (\rho_2/\rho_1)\,(\alpha_2/\alpha_1)\,E'\,, & F'' &= -\,(\rho_2/\rho_1)\,(\alpha_2/\beta_1)\,G'\,,\\ G'' &= -\,(\rho_2/\rho_1)\,(\beta_2/\alpha_1)\,F'\,, & H'' &= (\rho_2/\rho_1)\,(\beta_2/\beta_1)\,H' \end{aligned}\right\} \quad (6.1)$$

where G' and H' are SP and SS refraction coefficients on $z=H$ from the second to the first layer.

Therefore we reach

$$\left.\begin{aligned} B_{02}^{-1}\,(\rho_1/\rho_2)\,[\Phi_1]_{M=0}^{P2} &= i\,(\pi/M_\xi)\,(\text{ⓜ}/\alpha_1)\,\{E'\,e^{-i(\alpha_1 H-\theta)}\\ &\quad + G'\,e^{-i\beta_1 H}\,(A\,e^{i\theta} - e^{-i\theta})\,C^{-1}\}\cdot e^{-i\alpha_2(E-H)}\cos(\alpha_1 z-\theta)\,,\\ B_{02}^{-1}\,(\rho_1/\rho_2)\,[\Psi_1]_{M=0}^{P2} &= -\,i\,(\pi/M_\xi)\,(\text{ⓜ}/\beta_1)\,\{E'\,e^{-i\alpha_1 H}\,(A\,e^{i\theta'}\\ &\quad - e^{-i\theta'})\,B^{-1} + G'\,c^{-i(\beta_1 H-\theta')}\}\cdot e^{-\alpha_2(E-H)}\cos(\beta_1 z-\theta')\,. \end{aligned}\right| \quad (6.2)$$

If E is exchanged for z, the right hand side of the first equation in (6.2) coincides with that in (5.4), but the right hand side of the second equation in (6.2) does with negative expression of the first equation in (5.5). These reciprocities have been expected at (6.1).

Utilizing these reciprocal relations, we can easily obtain, from the second equations in (5.4) and (5.5), displacement potentials in the first layer generated from a source of S wave in the second layer:

$$
\left.\begin{aligned}
D_{02}^{-1}(\rho_1/\rho_2)[\Phi_1]_{M=0}^{S2} &= -i(\pi/M_\xi)(ⓜ/\alpha_1)\{F' e^{-i(\alpha_1 H-\theta)} \\
&\quad + H' e^{-i\beta_1 H}(A e^{i\theta} - e^{-i\theta}) C^{-1}\} \cdot e^{-i\beta_2(E-H)} \cos(\alpha_1 z - \theta), \\
D_{02}^{-1}(\rho_1/\rho_2)[\Psi_1]_{M=0}^{S2} &= i(\pi/M_\xi)(ⓜ/\beta_1)\{F' e^{-i\alpha_1 H}(A e^{i\theta'} - e^{-i\theta'}) B^{-1} \\
&\quad + H' e^{-i(\beta_1 H-\theta')}\} \cdot e^{-i\beta_2(E-H)} \cos(\beta_1 z - \theta').
\end{aligned}\right\} \tag{6.3}
$$

Substituting (5.6) into (6.2) and (6.3), we can arrange them in

$$
\left.\begin{aligned}
[\Phi_1]_{M=0}^{\nu=2} &= [\Phi_1]_{M=0}^{P2} + [\Phi_1]_{M=0}^{S2} = i Z_2 X \cos(\alpha_1 z - \theta), \\
[\Psi_1]_{M=0}^{\nu=2} &= [\Psi_1]_{M=0}^{P2} + [\Psi_1]_{M=0}^{S2} = -Z_2 X Y \cos(\beta_1 z - \theta')
\end{aligned}\right\} \tag{6.4}
$$

in which

$$
\begin{aligned}
Z_2 = \frac{1}{2}(\rho_2/\rho_1)\big[&B_{02} e^{-i\alpha_2(E-H)}\{E' e^{-i(\alpha_1 H-\theta)} + i Y G' e^{-i(\beta_1 H-\theta')}\} \\
&- D_{02} e^{-i\beta_2(E-H)}\{F' e^{-i(\alpha_1 H-\theta)} + i Y H' e^{-i(\beta_1 H-\theta')}\}\big].
\end{aligned} \tag{6.5}
$$

When we obtain B_2 and D_2 directly from (1.6), (1.7) and (1.8), it is difficult to express them with reflection and refraction coefficients already known. According to ray theoretical considerations, however, we seen on $z=H$

$$
\left.\begin{aligned}
B_2 e^{-i\alpha_2 H} &= A'' B_{02} e^{-i\alpha_2(E-H)} + E' e^{-i\alpha_1 H} B_1 + G' e^{-i\beta_1 H} D_1, \\
D_2 e^{-i\beta_2 H} &= B'' B_{02} e^{-i\alpha_2(E-H)} + F' e^{-i\alpha_1 H} B_1 + H' e^{-i\beta_1 H} D_1
\end{aligned}\right\} \tag{6.6}
$$

where A'' and B'' are PP and PS reflection coefficients on $z=H$ from the second to the first layer.

Picking up coefficients of $\exp(-i\alpha_1 z)$ and $\exp(-i\beta_1 z)$ in (6.4) which correspond to B_1 and D_1, and substituting them into (6.6), we have

$$
\left.\begin{aligned}
[\Phi_2]_{M=0}^{\nu=2} &= i Z_1 X \cdot \frac{1}{2}\{E' e^{-i(\alpha_1 H-\theta)} + i Y G' e^{-i(\beta_1 H-\theta')}\} e^{-i\alpha_2(z-H)}, \\
[\Psi_2]_{M=0}^{\nu=2} &= i Z_2 X \cdot \frac{1}{2}\{F' e^{-i(\alpha_1 H-\theta)} + i Y H' e^{-i(\beta_1 H-\theta')}\} e^{-i\beta_2(z-H)}.
\end{aligned}\right\} \tag{6.7}
$$

If we exchange Z_2 for Z_1, (6.7) becomes (5.7). Because (5.7) can be expressed by the form of (4.7), as was already described, (6.4) and (6.7) can also be expressed by the same notations as those were used in (4.7). Substituting (5.8) into Z_2 in (6.5), we have

$$
\begin{aligned}
Z_2 = (\rho_2/\rho_1) \left\{1 + (\alpha_2/\xi)(\beta_2/\xi)\right\}^{-1} \big[& B_{02}\, e^{-i\alpha_2(E-H)} \left\{U_1 - i(\beta_2/\xi) W_1\right\}_{z=H} \\
& - D_{02}\, e^{-i\beta_2(E-H)} \left\{(\alpha_2/\xi) U_1 + i W_1\right\}_{z=H} \big] .
\end{aligned} \tag{6.8}
$$

7. A compact expression for displacement

Arranging various components of displacement in a compact form, we have

$$
\left.
\begin{aligned}
u_1^{\nu=1} &= \xi Z_1 X U_1 \quad \text{and} \quad w_1^{\nu=1} = -i \xi Z_1 X W_1 , \\
u_1^{\nu=2} &= \xi Z_2 X U_1 \quad \text{and} \quad w_1^{\nu=2} = -i \xi Z_2 X W_1
\end{aligned}
\right\} \tag{7.1}
$$

when observations are made in the first layer;

$$
\left.
\begin{aligned}
u_2^{\nu=1} &= \xi Z_1 X U_2 \quad \text{and} \quad w_2^{\nu=1} = \xi Z_1 X W_2 , \\
u_2^{\nu=2} &= \xi Z_2 X U_2 \quad \text{and} \quad w_2^{\nu=2} = \xi Z_2 X W_2
\end{aligned}
\right\} \tag{7.2}
$$

when observations are made in the second layer. In (7.1) and (7.2), Z_1, U_1 and W_1 are given in (3.12) and (4.4) whereas Z_2 is given in (6.8) and

$$
\left.
\begin{aligned}
U_2 = \left\{1 + (\alpha_2/\xi)(\beta_2/\xi)\right\}^{-1} \big[& e^{-i\alpha_2(z-H)} \left\{U_1 - i(\beta_2/\xi) W_1\right\}_{z=H} \\
& + e^{-i\beta_2(z-H)} (\beta_2/\xi) \left\{(\alpha_2/\xi) U_1 + i W_1\right\}_{z=H} \big] , \\
W_2 = \left\{1 + (\alpha_2/\xi)(\beta_2/\xi)\right\}^{-1} \big[& e^{-i\alpha_2(z-H)} (\alpha_2/\xi) \left\{U_1 - i(\beta_2/\xi) W_1\right\}_{z=H} \\
& - e^{-i\beta_2(z-H)} \left\{(\alpha_2/\xi) U_1 + i W_1\right\}_{z=H} \big] .
\end{aligned}
\right\} \tag{7.3}
$$

In the above expressions amplitudes of Z_1 and Z_2 are respectively measured in a unit of any displacement potential. If a unit is taken as any component of displacement at an origin, $B_{0\nu}$ *etc* must be replaced by $i\xi B_{0\nu}$ and so on. Thus, (7.1) and (7.2) become

$$
\left.
\begin{aligned}
u_1^{\nu=1} &= i Z_1' X U_1 \quad \text{and} \quad w_1^{\nu=1} = Z_1'' X W_1 , \\
u_1^{\nu=2} &= i Z_2' X U_1 \quad \text{and} \quad w_1^{\nu=2} = Z_2'' X W_1
\end{aligned}
\right\} \tag{7.4}
$$

$$
\left.
\begin{aligned}
u_2^{\nu=1} &= i Z_1' X U_2 \quad \text{and} \quad w_2^{\nu=1} = i Z_1'' X W_2 , \\
u_2^{\nu=2} &= i Z_2' X U_2 \quad \text{and} \quad w_2^{\nu=2} = i Z_2'' X W_2
\end{aligned}
\right\} \tag{7.5}
$$

where

$$\left.\begin{aligned} Z_1' &= B_{01} \cos(\alpha_1 E - \theta) - D_{01} (\beta_1/\xi) Y \sin(\beta_1 E - \theta'), \\ Z_1'' &= B_{01} (\alpha_1/\xi) \sin(\alpha_1 E - \theta) + D_{01} Y \cos(\beta_1 E - \theta'), \end{aligned}\right\} \quad (7.6)$$

$$\left.\begin{aligned} Z_2' &= \{1 + (\alpha_2/\xi)(\beta_2/\xi)\}^{-1} [B_{02} e^{-i\alpha_2(E-H)} \{U_1 - i(\beta_2/\xi) W_1\}_{z=H} \\ &\quad + D_{02} e^{-i\beta_2(E-H)} (\beta_2/\xi) \{(\alpha_2/\xi) U_1 + i W_1\}_{z=H}], \\ Z_2'' &= \{1 + (\alpha_2/\xi)(\beta_2/\xi)\}^{-1} [B_{02} e^{-i\alpha_2(E-H)} (\alpha_2/\xi) \{U_1 \\ &\quad - i(\beta_2/\xi) W_1\}_{z=H} - D_{02} e^{-i\beta_2(E-H)} \{(\alpha_2/\xi) U_1 + i W_1\}_{z=H}]. \end{aligned}\right\} \quad (7.7)$$

If E is replaced by z in company with B_{ov} and D_{ov}, all components of displacement are reciprocal by that exchange.

Factor X is common to all components of displacement and is expressed, from (2.6) and (3.8), as

$$X = -2\pi (\xi H)^{-1} (c U^{-1} - 1) \{(\alpha_1/\xi)^2 + (\beta_1/\xi)^2 Y^2\}^{-1}. \quad (7.8)$$

This is the most fundamental quantity for representing the relation between amplitude and period of dispersive RAYLEIGH waves.

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