[Journal of the Faculty of Science, Hokkaidô University, Japan, Ser. VII, Vol. II, No. 2, 1964]

Energies of Elastic Surface-waves

Kyozi TAZIME

(Received Sept. 30, 1964)

Abstract

Flow of energy density is conserved on reflection and refraction of SH-waves. Energy flow of LOVE waves is equal to the product of energy and so-called "group velocity". This relation is satisfied also by dispersive RAYLEIGH waves. It has been found, however, that definition $U = d\omega/d\xi$ does not directly mean group velocity but does energy velocity.

1. Conservation of energy flow on reflection and refraction.

Considering reflection and refraction of SH-waves shown in Fig. 1, we see that displacements within the first and second layers are expressed respectively as follows:

$$\left.\begin{aligned}\psi_1 &= A_1 \cos(\omega t - \xi x - \eta_1 z) + B_1 \cos(\omega t - \xi x + \eta_1 z)\,,\\ \psi_2 &= A_2 \cos(\omega t - \xi x - \eta_2 z)\end{aligned}\right\} \qquad (1.1)$$

where

$$\left.\begin{aligned}&k_1^2 = \xi^2 + \eta_1^2\,,\quad k_2^2 = \xi^2 + \eta_2^2\,,\quad k_1 = \omega/v_1\,,\quad k_2 = \omega/v_2\,,\\ &v_1 = (\mu_1/\rho_1)^{1/2} \quad \text{and} \quad v_2 = (\mu_2/\rho_2)^{1/2}\,.\end{aligned}\right\} \qquad (1.2)$$

Since the boundary conditions on $z=0$ are

$$\psi_1 = \psi_2 \quad \text{and} \quad \mu_1 \partial\psi_1/\partial z = \mu_2 \partial\psi_2/\partial z\,, \qquad (1.3)$$

the reflection and refraction coefficients are given by

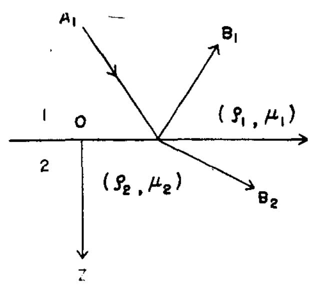

Fig. 1. Reflection and refraction of SH-waves.

$$\left.\begin{aligned} K_{12} &\equiv B_1/A_1 = \{1-(\mu_2\eta_2)/(\mu_1\eta_1)\}\{1+(\mu_2\eta_2)/(\mu_1\eta_1)\}^{-1}, \\ 1+K_{12} &\equiv A_2/A_1 = 2\{1+(\mu_2\eta_2)/(\mu_1\eta_1)\}^{-1}. \end{aligned}\right\} \quad (1.4)$$

Substituting (1.4) into (1.1), we see that the incidence, reflection and refraction waves are expressed as follows:

$$\left.\begin{aligned} \psi_i &= \cos(\omega t - \xi x - \eta_1 z), \\ \psi_l &= K_{12}\cos(\omega t - \xi x + \eta_1 z), \\ \psi_r &= (1+K_{12})\cos(\omega t - \xi x - \eta_2 z) \end{aligned}\right\} \quad (1.5)$$

where A_1 is taken as unity.

The flow of energy density passing a plane of a unit area perpendicular to z-axis in a unit time is given by

$$f = \mu\,(\partial\psi/\partial z)\,(\partial\psi/\partial t)\,. \quad (1.6)$$

Therefore, the flows of energy densities derived from (1.5) are respectively

$$\left.\begin{aligned} f_i &= -\omega\mu_1\eta_1\sin^2(\omega t - \xi x - \eta_1 z), \\ f_l &= \omega\mu_1\eta_1 K_{12}^2\sin^2(\omega t - \xi x + \eta_1 z), \\ f_r &= -\omega\mu_2\eta_2(1+K_{12})^2\sin^2(\omega t - \xi x - \eta_2 z) \end{aligned}\right\} \quad (1.7)$$

where flow f_l is opposite in z-direction to the others.

Taking up mean values during one period, we have

$$\bar{f}_i = \frac{1}{2}\omega\mu_1\eta_1, \quad \bar{f}_l = \frac{1}{2}\omega\mu_1\eta_1 K_{12}^2 \quad \text{and} \quad \bar{f}_r = \frac{1}{2}\omega\mu_1\eta_1(1+K_{12})^2. \quad (1.8)$$

However, as we have seen, in (1.4),

$$(\mu_2\eta_2)/(\mu_1\eta_1) = (1-K_{12})/(1+K_{12}),$$

we can know

$$\bar{f}_i = \bar{f}_l + \bar{f}_r \quad (1.9)$$

which shows the flow of energy density must be conserved on reflection and refraction of SH-waves.

Even if we consider the flow of energy density passing through a plane, as that shown by dotted lines in Fig. 2, perpendicular to x-axis in place of z-one, we are able to obtain the same result as (1.9). It must be remembered, however, that areas of the planes are to be adjusted by η_1/ξ and η_2/ξ respectively.

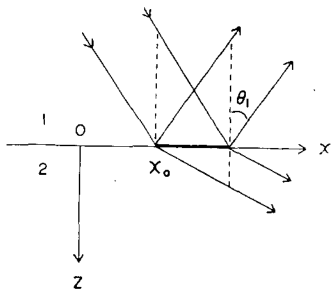

Fig. 2. The flow of energy density.

Energy density of SH-waves is given by

$$w = \frac{1}{2}\rho\,(\partial\psi/\partial t) + \frac{1}{2}\mu\left\{(\partial\psi/\partial x)^2 + (\partial\psi/\partial t)^2\right\}. \tag{1.10}$$

Substituting the first equation of (1.5) into (1.10) and taking (1.2) into account, we have

$$w_i = \rho_1\,\omega^2 \sin^2(\omega t - \xi x - \eta_1 z)\,,$$

resulting in the average value during a period

$$\bar{w}_i = \frac{1}{2}\rho_1\,\omega^2\,. \tag{1.11}$$

Therefore we have, from (1.8) and (1.11),

$$\bar{J}_i \sec\theta_1/\bar{w}_i = v_1\,. \tag{1.12}$$

The numerator on the left hand side of (1.12) means the flow of energy density in the progressive direction of incident waves. As to the other waves, we have also

$$\bar{J}_l \sec\theta_1/\bar{w}_l = v_1 \quad \text{and} \quad \bar{J}_r \sec\theta_2/\bar{w}_r = v_2\,. \tag{1.13}$$

Substituting (1.12) and (1.13) into (1.9), we have

$$\bar{w}_i = \bar{w}_l + \bar{w}_r\,(\sin 2\theta_2/\sin 2\theta_1) \tag{1.14}$$

which shows that the energy density should not be conserved.

We have easily arrived at principle of conservation as to energy flow from the two conditions stated in (1.3). Speaking about physical conceptions, however, we may expect that principle of conservation concerning energy must be superior to any other condition.

Substituting

$$\psi_i = \cos(\omega t - \xi x - \eta_1 z)\,,$$

$$\psi_l = B_1 \cos(\omega t - \xi x + \eta_1 z)\,, \tag{1.15}$$

and

$$\psi_r = A_2 \cos(\omega t - \xi x - \eta_2 z)$$

into the relation

$$\mu_1 (\partial \psi_i/\partial z)(\partial \psi_i/\partial t) = \mu_1 (\partial \psi_l/\partial z)(\partial \psi_l/\partial t) + \mu_2 (\partial \psi_r/\partial z)(\partial \psi_r/\partial t) \tag{1.16}$$

which is equivalent to (1.9), we have

$$1 = B_1^2 + (\mu_2 \eta_2)(\mu_1 \eta_1)^{-1} A_2^2\,. \tag{1.17}$$

We cannot obtain (1.3) from (1.17) alone. Anyone of (1.3), however, may be obtained by combining (1.17) with the other condition of it.

Adopting the first condition in (1.3), for instance, we have from (1.16)

$$1 + B_1 = A_2$$

which makes (1.17) as follows:

$$1 - B_1 = (\mu_2 \eta_2)(\mu_1 \eta_1)^{-1} A_2 \tag{1.18}$$

in which

$$B_1 = \left\{1 - (\mu_2 \eta_2)(\mu_1 \eta_1)^{-1}\right\}\left\{1 + (\mu_2 \eta_2)(\mu_1 \eta_1)^{-1}\right\}\,.$$

Equation (1.18) is nothing but the second condition of (1.3). If we adopt the other condition on $z=0$ than the second of (1.3), flow of energy density cannot be conserved.

2. Energy of Love waves

Physical meaning of (1.12) tells

$$\text{(energy flow)} = \text{(energy) (energy velocity)} \tag{2.1}$$

which may be compared with the similar relation for water flow. Equation (2.1) is also explained by RAYLEIGH[1] as well as by BRILLOUIN[2] as to waves in dispersive media. BRILLOUIN describes that energy velocity might coincide with group velocity, if the former is real and positive.

Let us consider LOVE waves in a layer overlying a half space, we have the characteristic equation

$$M(\omega, \xi) = 0 \tag{2.2}$$

in which

$$M(\omega, \xi) = \tan \eta_1 H - (\mu_2/\mu_1)(\hat{\eta}_2/\eta_1), \quad \hat{\eta}_2 = i\,\eta_2 \tag{2.3}$$

and H means the thickness of a superficial layer.

Displacements in each layer can be expressed as follows[3]:

$$\left.\begin{aligned}\psi_1 &= -2\pi A(\xi)\cos\eta_1 E\cos\eta_1 z\exp\{i(\omega t-\xi x)\},\\ \psi_2 &= -2\pi A(\xi)\cos\eta_1 E\cos\eta_1 H\exp\{-\hat{\eta}_2(z-H)\}\exp\{i(\omega t-\xi x)\}\end{aligned}\right\} \tag{2.4}$$

where

$$2\pi A(\xi) = 2\pi(\xi H)^{-1}(\xi/\eta_1)^2(cU^{-1}-1)\,. \tag{2.5}$$

Dealing with real roots in equation (2.2), we see

$$\Gamma = -2\pi A(\xi)\cos\eta_1 E \tag{2.6}$$

is also real. Therefore real parts on the right hand side of (2.4) may be expressed by

$$\left.\begin{aligned}\mathcal{R}\,\psi_1 &= \Gamma\cos\eta_1 z\cos(\omega t-\xi x),\\ \mathcal{R}\,\psi_2 &= \Gamma\cos\eta_1 H\exp\{-\hat{\eta}_2(z-H)\}\cos(\omega t-\xi x)\,.\end{aligned}\right\} \tag{2.7}$$

From now on, notation $\mathcal{R}$ will be omitted, no confusion being expected. Of course, equation of motion

$$\rho(\partial^2\psi/\partial t^2) = \mu\nabla^2\psi \tag{2.8}$$

is satisfied by (2.7). Moreover, we see the relation concerning principle of energy conservation[4]

$$\partial w/\partial t + \partial(wU_x)/\partial x + \partial(wU_z)/\partial z = 0 \tag{2.9}$$

in which U_x and U_z mean respectively components of energy velocity in x and z-directions. But waves given by (2.7) do not propagate in z-direction, so the third term on the left hand side of (2.9) will be neglected.

We have from (1.10) and (2.8)

$$\partial w/\partial t = \partial\{\mu(\partial\psi/\partial x)(\partial\psi/\partial t)\}/\partial x\,. \tag{2.10}$$

Comparing this with (2.9), we see

$$\partial\{wU_x + \mu(\partial\psi/\partial x)(\partial\psi/\partial t)\}/\partial x = 0\,,$$

resulting in

$$wU_x + \mu(\partial\psi/\partial x)(\partial\psi/\partial t) = C(t, z)\,.$$

Because this must be satisfied by any x, $C(t, z)$ should be zero. Therefore we reach

$$wU_x + \mu(\partial\psi/\partial x)(\partial\psi/\partial t) = 0$$

or

$$U_x = -\mu(\partial\psi/\partial x)(\partial\psi/\partial t)/w\,. \tag{2.11}$$

This is the similar relation to that shown by BULLEN[4] who treated one dimensional problems.

Substituting the first expression of (2.7) into (1.6) and (1.10), we have

$$f_1 = -\mu_1 (\partial \psi_1/\partial x)(\partial \psi_1/\partial t) = \Gamma^2 \mu_1 \xi \omega \cos^2 \eta_1 z \sin^2 (\omega t - \xi x),$$

$$w_1 = (\Gamma^2/2) \rho_1 \xi^2 \{(c^2 + v_1^2) \cos^2 \eta_1 z \sin^2 (\omega t - \xi x) + (c^2 - v_1^2) \sin^2 \eta_1 z \cos^2 (\omega t - \xi x)\}$$

where $c = \omega/\xi$.

Adopting mean values during a period, we have

$$\bar{f}_1 = (\Gamma^2/2) \mu_1 \xi \omega \cos^2 \eta_1 z,$$

$$\bar{w}_1 = (\Gamma^2/4) \rho_1 \xi^2 (c^2 + v_1^2 \cos 2 \eta_1 z).$$

These must be integrated through the superficial layer,

$$\int_0^H \bar{f}_1 \, dz = (\Gamma^2/4) \mu_1 \xi \omega \{H + (\sin \eta_1 H \cos \eta_1 H)/\eta_1\},$$

$$\int_0^H \bar{w}_1 \, dz = (\Gamma^2/4) \mu_1 \xi^2 \{H (c/v_1)^2 + (\sin \eta_1 H \cos \eta_1 H)/\eta_1\}.$$

Likewise we have

$$\int_H^\infty \bar{f}_2 \, dz = (\Gamma^2/4) \mu_2 \xi \omega \cos^2 \eta_1 H/\hat{\eta}_2,$$

$$\int_H^\infty \bar{w}_2 \, dz = (\Gamma^2/4) \mu_2 \xi^2 \cos^2 \eta_1 H/\hat{\eta}_2.$$

Therefore the total quantities are given by

$$F \equiv \int_0^H \bar{f}_1 \, dz + \int_H^\infty \bar{f}_2 \, dz = (\Gamma^2/4) \mu_1 \xi \omega \{H + (\sin \eta_1 H \cos \eta_1 H)/\eta_1 + (\mu_2/\mu_1) \cos^2 \eta_1 H/\hat{\eta}_2\},$$

$$W \equiv \int_0^H \bar{w}_1 \, dz + \int_H^\infty \bar{w}_2 \, dz = (\Gamma^2/4) \mu_1 \xi^2 \{H (c/v_1)^2 + (\sin \eta_1 H \cos \eta_1 H)/\eta_1 + (\mu_2/\mu_1) \cos^2 \eta_1 H/\hat{\eta}_2\}.$$

However, we see from (2.2) and (2.3)

$$(\sin \eta_1 H \cos \eta_1 H)/\eta_1 + (\mu_2/\mu_1) \cos^2 \eta_1 H/\hat{\eta}_2 = (c/v_1)^2 \{(\sin \eta_1 H \cos \eta_1 H)/\eta_1 + (\rho_2/\rho_1) \cos^2 \eta_1 H/\hat{\eta}_2\},$$

so we have at last

$$\left.\begin{aligned} F &= (\mu_1/4)\,(\Gamma\cos\eta_1 H)^2\,\xi\,\omega\,\{H\sec^2\eta_1 H + (\tan\eta_1 H)/\eta_1 + (\mu_2/\mu_1)/\hat{\eta}_2\}\,, \\ W &= (\mu_1/4)\,(\Gamma\cos\eta_1 H)^2\,(\omega/v_1)^2\,\{H\sec^2\eta_1 H + (\tan\eta_1 H)/\eta_1 + (\rho_2/\rho_1)/\hat{\eta}_2\}\,. \end{aligned}\right\} \tag{2.12}$$

Substituting these into (2.11), we have

$$U_x = (v_1^2/c)\,\{H\sec^2\eta_1 H + (\tan\eta_1 H)/\eta_1 + (\mu_2/\mu_1)/\hat{\eta}_2\} \times \{H\sec^2\eta_1 H + (\tan\eta_1 H)/\eta_1 + (\rho_2/\rho_1)/\hat{\eta}_2\}^{-1} \tag{2.13}$$

which shows that energy velocity is always real and positive when wave number ξ is real and positive.

If group velocity U may be defined by

$$U = \mathrm{d}\,\omega/\mathrm{d}\,\xi\,, \tag{2.14}$$

this can be expressed by

$$\mathrm{d}\,\omega/\mathrm{d}\,\xi = -M_\xi(\omega,\xi)/M_\omega(\omega,\xi)$$

in which M_ξ and M_ω are able to be calculated from (2.2) and (2.3);

$$\left.\begin{aligned} M_\xi &\equiv \partial M/\partial\xi = -(\xi/\eta_1)\,\{H\sec^2\eta_1 H + (\tan\eta_1 H)/\eta_1 + (\mu_2/\mu_1)/\hat{\eta}_2\}\,, \\ M_\omega &\equiv \partial M/\partial\omega = (1/v_1^2)\,(\omega/\eta_1)\,\{H\sec^2\eta_1 H + (\tan\eta_1 H)/\eta_1 + (\rho_2/\rho_1)/\hat{\eta}_2\}\,. \end{aligned}\right\} \tag{2.15}$$

Substituting these into the right hand side of (2.14) and comparing the result with the right hand side of (2.13), we find

$$\text{energy velocity } U_x = \text{group velocity } U. \tag{2.16}$$

Using (2.15), we can rewrite (2.12) as follows:

$$\left.\begin{aligned} F &= -(\mu_1/4)\,(\Gamma\cos\eta_1 H)^2\,\eta_1\,\omega\,M_\xi\,, \\ W &= (\mu_1/4)\,(\Gamma\cos\eta_1 H)^2\,\eta_1\,\omega\,M_\omega\,. \end{aligned}\right\} \tag{2.17}$$

On the other hand, we have from (2.13) and (2.16)

$$H\sec^2\eta_1 H + (\tan\eta_1 H)/\eta_1 + (\mu_2/\mu_1)/\hat{\eta}_2 = (\eta_1/\xi)^2\,H\,(\sec^2\eta_1 H)\,(c\,U^{-1}-1)^{-1}\,.$$

Substituting this into the first equation of (2.12) and using (2.5) and (2.6), we arrive at

$$\left.\begin{aligned} F &= (\pi^2/H)\,\mu_1\cos^2\eta_1\,E\,(\xi/\eta_1)^2\,c\,(c\,U^{-1}-1)\,, \\ W &= (\pi^2/H)\,\mu_1\cos^2\eta_1\,E\,(\xi/\eta_1)^2\,c\,U^{-1}\,(c\,U^{-1}-1) \end{aligned}\right\} \tag{2.18}$$

which shows that energy flow F and energy W have respectively a peak when group velocity has the minimum value. The sharpness of the peak of energy flow is equivalent to that of amplitude function[3] $2\pi A(\xi)$, but the peak of energy must be sharper than that of amplitude function.

3. Energy of dispersive RAYLEIGH waves in a layer overlying a half space absolutely rigid

3.1 Expression of group velocity

Characteristic equation may be expressed by[5]

$$M(\omega, \xi) = \cos 2p - A A' \cos 2q - B C' = 0 . \tag{3.1.1}$$

Now we see the following relations:

$$\left.\begin{aligned} &\partial(\alpha, \beta)/\partial\xi = -\xi/\alpha, \ -\xi/\beta , \\ &\partial(\alpha, \beta)/\partial\omega = -(1/c)(c/v_p)^2(\partial\alpha/\partial\xi), \ -(1/c)(c/v_s)^2(\partial\beta/\partial\xi) , \end{aligned}\right\} \tag{3.1.2}$$

$$\left.\begin{aligned} &\partial(\alpha/\xi, \beta/\xi)/\partial\xi = -(1/\alpha)(c/v_p)^2, \ -(1/\beta)(c/v_s)^2 , \\ &\partial(\alpha/\xi, \beta/\xi)/\partial\omega = -(1/c)\,\partial(\alpha/\xi)/\partial\xi, \ -(1/c)\,\partial(\beta/\xi)/\partial\xi . \end{aligned}\right\} \tag{3.1.3}$$

Because reflection and refraction coefficients, for instance A, are functions of α/ξ and β/ξ alone,

$$\partial A/\partial\xi = \partial A/\partial(\alpha/\xi)\cdot\partial(\alpha/\xi)/\partial\xi + \partial A/\partial(\beta/\xi)\cdot\partial(\beta/\xi)/\partial\xi$$

and we have, owing to the second relation of (3.1.3),

$$\begin{aligned} \partial A/\partial\omega &= \partial A/\partial(\alpha/\xi)\cdot\partial(\alpha/\xi)/\partial\omega + \partial A/\partial(\beta/\xi)\cdot\partial(\beta/\xi)/\partial\omega \\ &= -(1/c)(\partial A/\partial\xi) . \end{aligned} \tag{3.1.4}$$

Therefore we reach, from (3.1.1), (3.1.2) and (3.1.4),

$$\begin{aligned} M_\xi = \xi H &\{(\sin 2p + A A' \sin 2q)/\alpha + (\sin 2p - A A' \sin 2q)/\beta\} \\ &- \{(A A')_\xi \cos 2q + (B C')_\xi\} , \end{aligned} \tag{3.1.5}$$

$$\begin{aligned} M_\omega = -(\xi H/c) &\{(c/v_p)^2(\sin 2p + A A' \sin 2q)/\alpha + (c/v_s)^2(\sin 2p \\ &- A A' \sin 2q)/\beta\} + (1/c)\{(A A')_\xi \cos 2q + (B C')_\xi\} . \end{aligned} \tag{3.1.6}$$

On the other hand, employing the following notations,

$$Ⓐ = \frac{1}{2}\{(h/\alpha)^2 + (k/\beta)^2\} \quad \text{and} \quad Ⓐ = Ⓐ - 2k^2/(\beta^2 - \xi^2) , \tag{3.1.7}$$

we see that

$$\partial A/\partial\xi = (B C/\xi)\,Ⓐ, \ \partial B/\partial\xi = (B/\xi)\left[Ⓐ A - \frac{1}{2}\{(\xi/\alpha)^2 - (\xi/\beta)^2\}\right] ,$$

$$\partial C/\partial \xi = (C/\xi)\left[ⓐ A + \frac{1}{2}\{(\xi/\alpha)^2 - (\xi/\beta)^2\}\right]$$

and in case of $\mu_2 = \infty$

$$\partial A'/\partial \xi = (B' C'/\xi)\, ⓐ, \quad \partial B'/\partial \xi = (B'/\xi)\left[ⓐ A - \frac{1}{2}\{(\xi/\alpha)^2 - (\xi/\beta)^2\}\right],$$

$$\partial C'/\partial \xi = (C'/\xi)\left[ⓐ A' + \frac{1}{2}\{(\xi/\alpha)^2 - (\xi/\beta)^2\}\right].$$

Using these expressions with that of (3.1.1), we obtain

$$(A\,A')_\xi \cos 2q + (B\,C')_\xi = \xi^{-1}\{ⓐ(A\cos 2p - A'\cos 2q) + ⓐ(A'\cos 2p - A\cos 2q)\}. \tag{3.1.8}$$

Substituting (3.1.8) into (3.1.5) and (3.1.6), we have the explicit expression of group velocity

$$U = -M_\xi(\omega, \xi)/M_\omega(\omega, \xi). \tag{3.1.9}$$

If anyone or either of α and β might be purely imaginary, the above calculations have no requirement of repetition from the beginning. It will be enough for getting new results to put imaginary values, in place of real ones, into the result already obtained.

3.2 Energy in the superficial layer

Displacement potentials of elastic surface-waves may be expressed in general as follows[5]:

$$\left.\begin{aligned} \Phi &= (\mathrm{a}\cos\alpha z + \mathrm{b}\sin\alpha z)\cos(\omega t - \xi x), \\ \Psi &= (\mathrm{c}\cos\beta z + \mathrm{d}\sin\beta z)\sin(\omega t - \xi x). \end{aligned}\right\} \tag{3.2.1}$$

Putting x and z-components of displacement

$$u = \partial\Phi/\partial x + \partial\Psi/\partial z \quad \text{and} \quad w = \partial\Phi/\partial z - \partial\Psi/\partial x, \tag{3.2.2}$$

we have the next expression for energy density[6]

$$w = \frac{1}{2}\rho\{(\partial u/\partial t)^2 + (\partial w/\partial t)^2\} + \frac{1}{2}\left[(\lambda + 2\mu)(\partial u/\partial x + \partial w/\partial z)^2 + \mu\{(\partial w/\partial x + \partial u/\partial z)^2 - 4(\partial u/\partial x)(\partial w/\partial z)\}\right]. \tag{3.2.3}$$

As we know

$$\nabla^2\Phi = \partial u/\partial x + \partial w/\partial z \quad \text{and} \quad \nabla^2\Psi = \partial u/\partial z - \partial w/\partial x,$$

we can rewrite (3.2.3) to

$$w/(\rho\,\omega^2) = \frac{1}{2}\left[(1/\omega^2)\left\{(\partial u/\partial t)^2 + (\partial w/\partial t)^2\right\} + h^2\,\Phi^2 + k^2\,\Psi^2\right.$$
$$\left. + (4/k^2)\left\{(\partial w/\partial x)(\partial u/\partial z) - (\partial u/\partial x)(\partial w/\partial z)\right\}\right], \qquad (3.2.4)$$

using

$$(\nabla^2 + h^2)\,\Phi = 0 \quad \text{and} \quad (\nabla^2 + k^2)\,\Psi = 0 \qquad (3.2.5)$$

in case of stationary motion as that given by (3.2.1).

The same notation w is employed for z-component of displacement as well as for energy density, but no confusion will occur because the dimention differs each other.

Putting

$$\phi = \mathrm{a}\cos\alpha z + \mathrm{b}\sin\alpha z \quad \text{and} \quad \psi = \mathrm{c}\cos\beta z + \mathrm{d}\sin\beta z \qquad (3.2.6)$$

and substituting (3.2.1) and (3.2.3) into (3.2.4), we have

$$w/(\rho\,\omega^2) = \frac{1}{2}\left[(h^2 + \xi^2 - 4\,\xi^2\,\alpha^2\,k^{-2})\,\phi^2 + 2\,\xi\left\{1 + 2\,(\xi^2 - \alpha^2)\,k^{-2}\right\}\phi\,(\partial\psi/\partial z)\right.$$
$$\left. + (1 + 4\,\xi^2\,k^{-2})\,(\partial\psi/\partial z)^2\right]\cos^2(\omega t - \xi x) + \frac{1}{2}\left[(k^2 + \xi^2 - 4\,\xi^2\,\beta^2\,k^{-2})\right.$$
$$\times\,\psi^2 + 2\,\xi\left\{1 + 2\,(\xi^2 - \beta^2)\,k^{-2}\right\}\psi\,(\partial\phi/\partial z) + (1 + 4\,\xi^2\,k^{-2})$$
$$\left.\times\,(\partial\phi/\partial z)^2\right]\sin^2(\omega t - \xi x)$$
$$= \frac{1}{2}\left\{[1] + [2]\cos 2\,(\omega t - \xi x)\right\}. \qquad (3.2.7)$$

When we take account of the mean value $\bar{w}$ of energy density during a period, the second term on the right hand side of (3.2.7) is to disappear. In this case, we have

$$2\,\bar{w}/(\rho\,\omega^2) = [1] \equiv [11] + [12] + [13] \qquad (3.2.8)$$

in which

$$[11] \equiv \frac{1}{2}\left\{(h^2 + \xi^2 - 4\,\xi^2\,\alpha^2\,k^{-2})\,\phi^2 + (1 + 4\,\xi^2\,k^{-2})\,(\partial\phi/\partial z)^2\right\}$$

$$= \frac{1}{2}\left[h^2\,(\mathrm{a}^2 + \mathrm{b}^2) + \xi^2\,(1 - 4\,\alpha^2\,k^{-2})\left\{(\mathrm{a}^2 - \mathrm{b}^2)\cos 2\,\alpha z + 2\,\mathrm{a}\,\mathrm{b}\sin 2\,\alpha z\right\}\right],$$

$$[12] \equiv \frac{1}{2}\left\{(k^2 + \xi^2 - 4\,\xi^2\,\beta^2\,k^{-2})\,\psi^2 + (1 + 4\,\xi^2\,k^{-2})\,(\partial\psi/\partial z)^2\right\}$$

$$= \frac{1}{2}\left[k^2\,(\mathrm{c}^2 + \mathrm{d}^2) + \xi^2\,(1 - 4\,\beta^2\,k^{-2})\left\{(\mathrm{c}^2 - \mathrm{d}^2)\cos 2\,\beta z + 2\,\mathrm{c}\,\mathrm{d}\sin 2\,\beta z\right\}\right],$$

$$[13] \equiv \xi[\{1 + 2(\xi^2 - \alpha^2)k^{-2}\}\phi(\partial\psi/\partial z) + \{1 + 2(\xi^2 - \beta^2)k^{-2}\}\psi(\partial\phi/\partial z)]$$
$$= (\xi/2)\,[(\mathrm{a\,d+b\,c})\,\{1 + 2(\xi^2 - \alpha\beta)k^{-2}\}(\alpha+\beta)\cos(\alpha+\beta)z + (\mathrm{a\,d-b\,c})\}$$
$$\times\{1+2(\xi^2+\alpha\beta)k^{-2}\}(\beta-\alpha)\cos(\beta-\alpha)z - (\mathrm{a\,c-b\,d})\{1+2(\xi^2-\alpha\beta)k^{-2}\}$$
$$\times(\alpha+\beta)\sin(\alpha+\beta)z - (\mathrm{a\,c+b\,d})\{1+2(\xi^2+\alpha\beta)k^{-2}\}(\beta-\alpha)\sin(\beta-\alpha)z].$$

Denoting thickness of the superficial layer by H and putting

$$2p = (\alpha + \beta)H \quad \text{and} \quad 2q = (\beta - \alpha)H, \tag{3.2.9}$$

we have

$$\int_0^H [11]\,dz = \frac{1}{2}(\mathrm{a}^2 + \mathrm{b}^2)h^2 H + \int_0^H [112]\,dz \tag{3.2.10}$$

in which [112] is the second term on the right hand side of [11] and

$$\int_0^H [112]\,dz = \frac{1}{4}(\xi^2/\alpha)(1-4\alpha^2k^{-2})\,[(\mathrm{a}^2-\mathrm{b}^2)\sin 2(p-q) + 2\mathrm{ab}\{1-\cos 2(p-q)\}].$$

Likewise we have

$$\int_0^H [12]\,dz = \frac{1}{2}(\mathrm{c}^2 + \mathrm{d}^2)k^2 H + \int_0^H [122]\,dz \tag{3.2.11}$$

in which

$$\int_0^H [122]\,dz = \frac{1}{4}(\xi^2/\beta)(1-4\beta^2k^{-2})\,[(\mathrm{c}^2-\mathrm{d}^2)\sin 2(p+q) + 2\mathrm{cd}\{1-\cos 2(p+q)\}]$$

and

$$\int_0^H [13]\,dz = (\xi/2)\{1 + 2(\xi^2 - \alpha\beta)k^{-2}\}\{(\mathrm{ad}+\mathrm{bc})\sin 2p + (\mathrm{ac}-\mathrm{bd})\cos 2p\}$$
$$+ (\xi/2)\{1 + 2(\xi^2 + \alpha\beta)k^{-2}\}\{(\mathrm{ad}-\mathrm{bc})\sin 2q + (\mathrm{ac}+\mathrm{bd})\cos 2q\}$$
$$- \xi\{(1 + 2\xi^2k^{-2})\mathrm{ac} + 2\alpha\beta k^{-2}\mathrm{bd}\}. \tag{3.2.12}$$

Therefore we arrive at

$$2W/(\rho\omega^2) \equiv \int_0^H \{2\bar{w}/(\rho\omega^2)\}\,dz = (H/2)\{(\mathrm{a}^2 + \mathrm{b}^2)h^2 + (\mathrm{c}^2 + \mathrm{d}^2)k^2\}$$
$$+ \int_0^H \{[112] + [122] + [13]\}\,dz. \tag{3.2.13}$$

3.3 Energy flow through the superficial layer

Putting stress components as

$$X_x \equiv \lambda(\partial u/\partial x + \partial w/\partial z) + 2\mu(\partial u/\partial x),\ X_z = Z_x \equiv \mu(\partial w/\partial x + \partial u/\partial z)$$

and

$$Z_z \equiv \lambda\,(\partial u/\partial x + \partial w/\partial z) + 2\mu(\partial w/\partial z)\,,$$

we have a compact expression for time variation of energy density

$$\partial w/\partial t = \partial\left\{X_x\,(\partial u/\partial t) + X_z\,(\partial w/\partial t)\right\}/\partial x + \partial\left\{Z_x\,(\partial u/\partial t) + Z_z\,(\partial w/\partial t)\right\}/\partial z\,, \tag{3.3.1}$$

differentiating (3.2.3) with respect to time.

Comparing this with (2.10) for LOVE waves, we see that flows of energy densities are to be defined by the negative values in the braces of the first and the second terms on the right hand side of (3.3.1).

Normal mode waves, however, calls the flow in x-direction

$$f = -\left\{X_x\,(\partial u/\partial t) + X_z\,(\partial w/\partial t)\right\} \tag{3.3.2}$$

alone in question.

Substituting (3.2.1) and (3.2.3) into (3.3.2), we have

$$\begin{aligned} f/(\rho\,\omega^3) &= (\xi\,\phi + \partial\psi/\partial z)\left\{(1 - 2\,\alpha^2 k^{-2})\,\phi + 2\,\xi\,k^{-2}\,(\partial\psi/\partial z)\right\}\cos^2(\omega t - \xi x) \\ &\quad + (\xi\,\psi + \partial\phi/\partial z)\left\{(1 - 2\,\beta^2 k^{-2})\,\psi + 2\,\xi\,k^{-2}\,(\partial\phi/\partial z)\right\}\sin^2(\omega t - \xi x) \\ &= \frac{1}{2}\left\{[1'] + [2']\cos 2\,(\omega t - \xi x)\right\}. \end{aligned} \tag{3.3.3}$$

When we take account of the mean value during a period, the second term on the right hand side of (3.3.3) is to disappear. In this case, we have

$$2\bar{f}/(\rho\,\omega^3) = [1'] \equiv [11'] + [12'] + [13'] \tag{3.3.4}$$

in which

$$[11']\,\xi^{-1} \equiv (1 - 2\,\alpha^2 k^{-2})\,\phi^2 + 2\,k^{-2}\,(\partial\phi/\partial z)^2 = \frac{1}{2}\,(\mathrm{a}^2 + \mathrm{b}^2) + \xi^{-2}\,[112]\,,$$

$$[12']\,\xi^{-1} \equiv (1 - 2\,\beta^2 k^{-2})\,\psi^2 + 2\,k^{-2}\,(\partial\psi/\partial z)^2 = \frac{1}{2}\,(\mathrm{c}^2 + \mathrm{d}^2) + \xi^{-2}\,[122]$$

and $[13']\,\xi^{-1} = \xi^{-2}\,[13]\,.$

Therefore we have

$$[1'] = (\xi/2)\,(\mathrm{a}^2 + \mathrm{b}^2 + \mathrm{c}^2 + \mathrm{d}^2) + \xi^{-1}\left\{[112] + [122] + [13]\right\}.$$

3.4 A layer overlying a half space absolutely rigid

Because, in this case,

$$A^2 - B\,C = 1 \quad \text{and} \quad A'^2 - B'\,C' = 1\,,$$

the following relations are obtained from (3.1.1):

$$B'(1 \pm A) + B(\cos 2p \pm A' \cos 2q) = (1/C)(A \pm 1)(A \cos 2p - A' \cos 2q), \quad (3.4.1)$$

$$B(1 \pm A') + B'(\cos 2p \pm A \cos 2q) = (1/C')(A' \pm 1)(A' \cos 2p - A \cos 2q), \quad (3.4.2)$$

$$(1 - A^2)(\sin^2 2p - A'^2 \sin^2 2q) = (A \cos 2p - A' \cos 2q)^2. \quad (3.4.3)$$

In the expressions for displacement potentials already obtained[5], putting the common coefficient

$$\Gamma \equiv -\pi/(a M_\xi) = -(\pi/H)(\omega/\alpha^2)(U^{-1} - c^{-1})\{\sin 2p + A A' \sin 2q + (\beta/\alpha)(\sin 2p - A A' \sin 2q)\}^{-1}, \quad (3.4.4)$$

we can express, a, b, c and d in (3.2.1) with use of (3.4.1) as follows:

$$a\Gamma^{-1} = (1 + A)(\sin 2p + A' \sin 2q)\cos \alpha E - (A \cos 2p - A' \cos 2q)\sin \alpha E,$$

$$b\Gamma^{-1} = -(A \cos 2p - A' \cos 2q)\cos \alpha E + (1 - A)(\sin 2p - A' \sin 2q)\sin \alpha E,$$

$$c = -b(1 + A)/C \quad \text{and} \quad d = -a(1 - A)/C.$$

Using (3.4.3), moreover, we easily arrive at the following results:

$$a^2 + b^2 = 2(\sin 2p + A A' \sin 2q)\Gamma^2 [\alpha E],$$

$$a^2 - b^2 = 2(A \sin 2p + A' \sin 2q)\Gamma^2 [\alpha E],$$

$$ab = -(A \cos 2p - A' \cos 2q)\Gamma^2 [\alpha E],$$

$$c^2 + d^2 = 2(\alpha/\beta)(\sin 2p - A A' \sin 2q)\Gamma^2 [\alpha E],$$

$$c^2 - d^2 = 2(\alpha/\beta)(A \sin 2p - A' \sin 2q)\Gamma^2 [\alpha E],$$

$$cd = -(\alpha/\beta)(A \cos 2p - A' \cos 2q)\Gamma^2 [\alpha E],$$

$$ad = -(1/C)(1 - A^2)(\sin 2p + A' \sin 2q)\Gamma^2 [\alpha E],$$

$$bc = -(1/C)(1 - A^2)(\sin 2p - A' \sin 2q)\Gamma^2 [\alpha E],$$

$$ac = (1/C)(1 + A)(A \cos 2p - A' \cos 2q)\Gamma^2 [\alpha E],$$

$$bd = (1/C)(1 - A)(A \cos 2p - A' \cos 2q)\Gamma^2 [\alpha E]$$

where

$$[\alpha E] \equiv (a\Gamma^{-1})\cos \alpha E + (b\Gamma^{-1})\sin \alpha E$$
$$= \{(1 + A)^{1/2}(\sin 2p + A' \sin 2q)^{1/2}\cos \alpha E \pm (1 - A)^{1/2}(\sin 2p - A' \sin 2q)^{1/2}\sin \alpha E\}^2, \quad \text{for} \quad A \cos 2p - A' \cos 2q \lessgtr 0.$$

Substituting the above values into (3.2.13), we have

$$(H/2)\left\{(\mathrm{a}^2+\mathrm{b}^2)\,h^2+(\mathrm{c}^2+\mathrm{d}^2)\,k^2\right\}=\alpha\,\xi^2\,H\left\{(c/v_p)^2\,(\sin 2p+A\,A'\sin 2q)/\alpha\right.$$
$$\left.+\,(c/v_s)^2\,(\sin 2p-A\,A'\sin 2q)/\beta\right\} \qquad (3.4.5)$$

in which the common coefficient $\Gamma^2[\alpha E]$ is omitted.

Likewise we have

$$\int_0^H\left\{[112]+[122]\right\}\mathrm{d}\,z=\alpha\,\xi^2\left\{\frac{1}{2}\,(\alpha^{-2}+\beta^{-2})-4\,k^{-2}\right\}(A+A')\,(\cos 2q-\cos 2p)$$

and by help of (3.4.2)

$$\int_0^H[13]\,\mathrm{d}\,z=\xi\left[(B-B')\left\{1+2\,(\xi^2-\alpha\,\beta)\,k^{-2}\right\}(1-\cos 2p)\right.$$
$$\left.+\,(A'\,B-A\,B')\left\{1+2\,(\xi^2+\alpha\,\beta)\,k^{-2}\right\}(1-\cos 2q)\right]$$
$$=\alpha\left\{2\,\xi^2\,(1+2\,\xi^2\,k^{-2})\,(\beta^2-\xi^2)^{-1}+(\beta^2-\xi^2)\,k^{-2}\right\}(A\cos 2p-A'\cos 2q)$$
$$-\,\alpha\,(1+4\,\xi^2\,k^{-2})\,(A'\cos 2p-A\cos 2q)\,.$$

Therefore, recalling (3.1.7) and (3.1.8) in minds, we have

$$\int_0^H\left\{[112]+[122]+[13]\right\}\mathrm{d}\,z=-\,\alpha\left\{Ⓐ\,(A'\cos 2p-A\cos 2q)\right.$$
$$\left.+\,Ⓐ\,(A\cos 2p-A'\cos 2q)\right.=-\,\alpha\,\xi\left\{(A\,A')_\xi\cos 2q+(B\,C')_\xi\right\}. \qquad (3.4.6)$$

Substituting (3.4.5) and (3.4.6) into (3.2.13) and comparing the result with (3.1.6), we find

$$2\,W/(\rho\,\omega^3)=-\,\alpha\,\Gamma^2\,[\alpha\,E]\,M_\omega\,. \qquad (3.4.7)$$

Next, we have from (3.3.3)

$$(\xi\,H/2)\left\{(\mathrm{a}^2+\mathrm{b}^2)+(\mathrm{c}^2+\mathrm{d}^2)\right\}=\alpha\,\xi\,H\left\{(\sin 2p+A\,A'\sin 2q)/\alpha\right.$$
$$\left.+\,(\sin 2p-A\,A'\sin 2q)/\beta\right\} \qquad (3.4.8)$$

and

$$\int_0^H\left\{[112']+[122']+[13']\right\}\mathrm{d}\,z=\xi^{-1}\int_0^H\left\{[112]+[122]+[13]\right\}\mathrm{d}\,z$$
$$=-\alpha\left\{(A\,A')_\xi\cos 2q+(B\,C')_\xi\right\}. \qquad (3.4.9)$$

Substituting (3.4.8) and (3.4.9) into (3.3.3) and comparing the result with (3.1.5), we find

$$2F/(\rho\,\omega^3) = a\,\Gamma^2\,[a\,E]\,M_\xi\,. \tag{3.4.10}$$

in which $F = \int_0^H \bar{f}\,\mathrm{d}z$.

4. Remarks

4.1 Negative velocity of energy

Using (3.1.9) and (3.4.4), we can rewrite (3.4.7) and (3.4.10) as follows:

$$2\,W/(\pi^2\,\rho\,\omega^3) = (a\,M_\xi)^{-1}\,[a\,E]\,U^{-1} \quad \text{and} \quad 2\,F/(\pi^2\,\rho\,\omega^3) = (a\,M_\xi)^{-1}\,[a\,E]\,, \tag{4.1.1}$$

resulting in

$$F = W\,U \tag{4.1.2}$$

which tells the same meaning as (2.1). We see that U defined by (3.1.9) means energy velocity.

Energy velocity of LOVE waves is always real and positive, as we noticed in (2.12). On the other hand, energy velocity of dispersive RAYLEIGH waves may sometimes be negative, as that shown by dotted lines in Fig. 3. This dues to the reason why energy flow might sometimes be negative, although energy is certainly positive. To tell the truth, (3.1.9) does not define group velocity by itself but does energy velocity. It must be remembered that JEFFREYS's definition[7] tells if $d\omega/d\xi$ is equal to x/t, then this means group velocity.

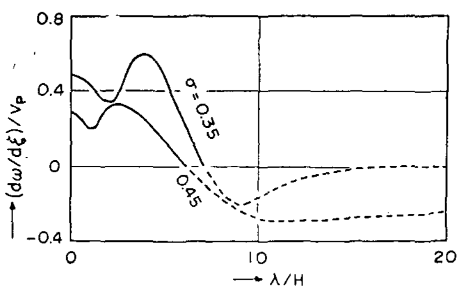

Fig. 3. Dispersion curves for the first higher mode of RAYLEIGH waves when $\sigma_2=\infty$, being σ POISSON's ratio and λ wave-length.

4.2 Leaking modes

The characteristic equation, for example (2.2) or (3.1.1), has not only real roots but also complex ones which mean wave number in x-direction is expressed by

$$\xi = \bar{\xi} + i\,\hat{\xi} \tag{4.2.1}$$

where $\bar{\xi}$ and $\hat{\xi}$ are real.

If ω is real, therefore,

$$c = \omega/\xi = \bar{c} + i\,\hat{c} \tag{4.2.2}$$

becomes also complex. In the above expression, c has been taken as phase velocity. But complex value of c will bring some confusion.

Let us consider

$$\psi\,(t, x) = \exp\left\{i\,(\omega\,t - \xi\,x)\right\}, \tag{4.2.3}$$

we see

$$\begin{aligned}\psi\,(t + \delta\,t, x + \delta\,x) &= \exp\left\{i\,(\omega\,t - \xi\,x) + i\,(\omega\,\delta\,t - \xi\,\delta\,x)\right\} \\ &= \psi\,(t, x)\exp\left\{i\,(\omega\ \ \delta\,t - \xi\,\delta\,x)\right\}.\end{aligned} \tag{4.2.4}$$

Therefore (4.2.4) must have the same wave form as that of (4.2.3), if

$$\delta\,x/\delta\,t = \omega/\xi\,, \tag{4.2.5}$$

being ξ real.

When ξ is complex, we have

$$\psi\,(t + \delta\,t, x + \delta\,x) = \psi\,(t, x)\exp\,(\hat{\xi}\,x)\exp\left\{i\,(\omega\,\delta\,t - \bar{\xi}\,\delta\,x)\right\} \tag{4.2.6}$$

which cannot have the same wave form exactly as that of (4.2.3) but maintains the form $\psi(t, x)\ \exp\ (\hat{\xi}\ x)$, if

$$\delta\,x/\delta\,t = \omega/\bar{\xi}\,. \tag{4.2.7}$$

It will be rather preferable to define phase velocity by

$$c = \delta\,x/\delta\,t \tag{4.2.8}$$

than to do by (4.2.2). As x and t are always real, phase velocity c becomes always real.

In order to distinguish this from one defined by (4.2.2), marking the latter with *, we have

$$c^* = \omega/\xi = \bar{c} + i\,\hat{c} \tag{4.2.9}$$

in which

$$\bar{c} = (\omega/\bar{\xi})\left\{1 + (\hat{\xi}/\bar{\xi})^2\right\}^{-1} \quad\text{and}\quad \hat{c} = (\omega/\hat{\xi})\left\{1 + (\bar{\xi}/\hat{\xi})^2\right\}^{-1} \tag{4.2.10}$$

are real quantities.

On the other hand, we find

$$c = \bar{c}\left\{1 + (\hat{\xi}/\bar{\xi})^2\right\}. \tag{4.2.11}$$

The above mentioned c^* has no physical meaning but it is a benefit quantity for calculation.

Following (4.2.9), if we define group velocity by

$$U^* = \mathrm{d}\,\omega/\mathrm{d}\,\xi\,, \tag{4.2.12}$$

U^* will become complex when ξ is complex. In this case we cannot define group velocity by U^*, because we meet with a difficulty of interpreting U^*. We ought to define group velocity by

$$U = x/t\,, \tag{4.2.13}$$

following (4.2.8). In this case, the next expression will be suggested from (4.2.7):

$$U = x/t = \mathrm{d}\,\omega/\mathrm{d}\,\bar{\xi}\,. \tag{4.2.14}$$

Appendix

Terms neglected in (3.2.7) and (3.3.3) will be written down:

$[2] = [21] + [22] + [23]$

in which

$$[21] \equiv \frac{1}{2}\left\{(h^2 + \xi^2 - 4\,\xi^2\,\alpha^2\,k^{-2})\,\phi^2 - (1 + 4\,\xi^2\,k^{-2})\,(\partial\,\phi/\partial\,z)^2\right\}$$

$$= \frac{1}{2}\left[\xi^2\,(\mathrm{a}^2 + \mathrm{b}^2)\,(1 - 4\,\alpha^2\,k^{-2}) + h^2\left\{(\mathrm{a}^2 - \mathrm{b}^2)\cos 2\,\alpha\,z + 2\,\mathrm{a\,b}\sin 2\,\alpha\,z\right\}\right],$$

$$[22] \equiv \frac{1}{2}\left\{-\,(k^2 + \xi^2 - 4\,\xi^2\,\beta^2\,k^{-2})\,\psi^2 + (1 + 4\,\xi^2\,k^{-2})\,(\partial\,\psi/\partial\,z)^2\right\}$$

$$-\frac{1}{2}\left[\xi^2\,(\mathrm{c}^2 + \mathrm{d}^2)\,(1 - 4\,\beta^2\,k^{-2}) + k^2\left\{(\mathrm{c}^2 - \mathrm{d}^2)\cos 2\,\beta\,z + 2\,\mathrm{c\,d}\sin 2\,\beta\,z\right\}\right],$$

$$[23] \equiv \xi\left[\left\{1 + 2\,(\xi^2 - \alpha^2)\,k^{-2}\right\}\phi\,(\partial\,\psi/\partial\,z) - \left\{1 + 2\,(\xi^2 - \beta^2)\,k^{-2}\right\}\psi\,(\partial\,\phi/\partial\,z)\right]$$

$$= (\xi/2)\left[(\mathrm{ad} - \mathrm{bc})\,(\alpha + \beta)\left\{1 + 2\,(\xi^2 - \alpha\,\beta)\,k^{-2}\right\}\cos(\alpha + \beta)\,z + (\mathrm{ad} + \mathrm{bc})\right.$$

$$\times\left\{1 + 2\,(\xi^2 + \alpha\,\beta)\,k^{-2}\right\}(\beta - \alpha)\cos(\beta - \alpha)\,z - (\mathrm{ac} + \mathrm{bd})\left\{1 + 2\,(\xi^2 - \alpha\,\beta)\,k^{-2}\right\}$$

$$\left.\times\,(\alpha + \beta)\sin(\alpha + \beta)\,z - (\mathrm{ac} - \mathrm{bd})\left\{1 + 2\,(\xi^2 + \alpha\,\beta)\,k^{-2}\right\}(\beta - \alpha)\sin(\beta - \alpha)\,z\right].$$

$[2'] = [21'] + [22'] + [23']$

in which

$$[21']\,\xi^{-1} = \frac{1}{2}\left\{(\mathrm{a}^2 + \mathrm{b}^2)\,(1 - 4\,\alpha^2\,k^{-2}) + (\mathrm{a}^2 - \mathrm{b}^2)\cos 2\,\alpha\,z + 2\,\mathrm{a\,b}\sin 2\,\alpha\,z\right\}$$

$$[22'] \, \xi^{-1} = -\frac{1}{2}\left\{(c^2 + d^2)(1 - 4\,\beta^2 k^{-2}) + (c^2 - d^2)\cos 2\,\beta\,z + 2\,c\,d\sin 2\,\beta\,z\right\},$$

$$[23'] = [23]\,\xi^{-1}.$$

References

1) RAYLEIGH, J.W.S.: On Progressive Waves, Theory of Sound, **1** (1926), 475.
2) BRILLOUIN, Léon: Wave Propagation in Periodic Structures, DOVER (1953), 69–80.
3) TAZIME, Kyozi: Minimum Group Velocity, Maximum Amplitude and Quarter Wave- length Law, J. Phys. Earth, **5** (1957), 44.
4) BULLEN, K.E.: Theory of Seismology, C.U.P. (1947), 64–65.
5) TAZIME, Kyozi: Ray-theoretical Construction of Dispersive RAYLEIGH Waves, J. Phys. Earth, **6** (1958), 88.
6) LOVE, A.E.H.: Mathematical Theory of Elasticitiy, C.U.P. (1934), 102.
7) JEFFREYS, H.: Operational Methods, C.U.P. (1931), 88.