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Model Seismology on Characteristics of Surface Waves Generated from a Sinusoidal Source of a Finite Duration

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Abstract

We cannot always define group velocity by $d\omega/d\xi$, because this value sometimes becomes zero or negative. Indeed various wave groups may be supposed, but anyone of them will be important which can have considerable amplitudes.

Sometimes a wave group will predominate in practice when $d\omega/d\xi$ coincides with x/t, but sometimes it will not. In the latter case, none or another wave group may be observed.

Surface waves were generated from a pulse and from sinusoidal waves of a finite duration in the same two dimentional model. Characteristics, namely group velocity and relative amplitude, of them have been coincided with each other. The group velocity defined by x/t in practice has been found to coincide also with a part of the calculated curve giving $d\omega/d\xi$. On the part, condition $d\omega/d\xi=x/t$ must be satisfied. However, the other part of dispersion curves calculated leaves some questions in future.

1. Conception of group velocity

1.1 We feel often some difficulties in treating group velocity when we compare observed data of surface waves with theoretical dispersion curves.

The solution of elastic wave equation yields to anyone of the next two manners: one is an approximate solution attributed to singular points on the complex plane of an integral and the other is the exact solution. The latter is to be similar to the result of a model experiment, so it must be difficult to follow a physical process to reaching the result. On the other hand, the former oughts to have various physical meanings.

Conceptions of group velocity have been obtained in an analytical process to finding an approximate solution. A group velocity may be sometimes coincided with velocity of energy propagated[1]. Indeed group velocity is a very benifit conception for getting general understanding of wave phenomena, but this conception is never the almighty. Any definition of group velocity should have own limit to be applied. How far the general understanding may be extended? To this question, very few considerations have been given by now.

Solving characteristic equation for dispersive RAYLEIGH waves, we see some parts where

$$d\omega/d\xi \leqq 0 \tag{1.1}$$

on dispersion curves, as shown in Fig. 1, being ω angular frequency and ξ angular wave number in parallel direction to the surface.

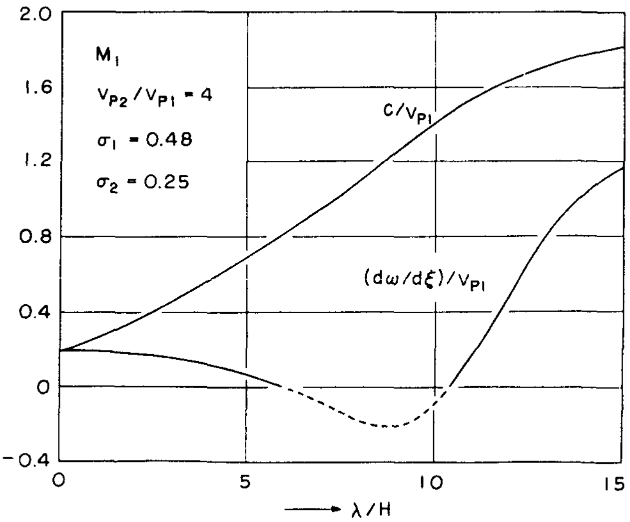

Fig. 1. Dispersion curves for the first higher mode of RAYLEIGH waves.

If we define group velocity without notice by

$$U = d\omega/d\xi, \tag{1.2}$$

equation (1.1) means zero or negative group velocity. Yet, negative group velocity can be hardly understood physically.

Definition (1.2) is delivered in many books written by LAMB[2], RAYLEIGH[3], BULLEN[4] and EWING *et al.*[5] But no inquiry into this definition is given in these books.

BRILLOUIN said in his book[6] that group velocity could be defined by (1.2), if $\Delta\omega/\Delta\xi$ had a definite value $d\omega/d\xi$ at the limit when $\Delta\xi$ became zero. In that case, spectrum of the wave group should be confined within a finite range. BRILLOUIN dealt with only absolute values of phase and group velocities, while the reason in general was not clarified.

The above books emphasized importance and application of definition (1.2) but didn't describe detailed logics of obtaining the definition.

1.2 STRATTON[7] started from the equation,

$$\psi(x, t) = \int_{-\infty}^{\infty} A(\xi) \exp\{i(\omega t - \xi x)\} d\xi . \tag{1.3}$$

This expression is one of the most general form of a solution for a wave equation. If we take ω as a function of ξ and use A (z, ξ) in place of A (ξ), the solution for surface waves in two dimentional space is also to be formulated as (1.3).

If spectrum $A(\xi)$ has a sharp peak, we see

$$A(\xi) \approx 0 \quad \text{for} \quad \xi < \xi^* - \delta\xi,\ \xi > \xi^* + \delta\xi \quad \text{and} \quad \delta\xi \approx 0 . \tag{1.4}$$

In this case, ω can be expanded near ξ^* by TAYLOR series as follows:

$$\omega(\xi) \approx \omega(\xi^*) + (\xi - \xi^*)(d\omega/d\xi)_{\xi=\xi^*} + \cdots, \tag{1.5}$$

neglecting higher orders than the second term.

Considering (1.4) when we substitute (1.5) into (1.3), we have

$$\begin{aligned} \psi(x, t) &= \bar{\psi} \exp\{i(\omega^* t - \xi^* x)\} , \\ \bar{\psi} &= \int_{\xi^*-\delta\xi}^{\xi^*+\delta\xi} A(\xi) \exp[i(\xi - \xi^*)\{(d\omega/d\xi)^* t - x\}] d\xi \end{aligned} \tag{1.6}$$

where additional notation * means the value corresponding to $\xi = \xi^*$.

Noticing the phenomena on (x, t)-plane, we see the amplitude of $\bar{\psi}$ is constant on the straight line

$$(d\omega/d\xi)^* t - x = C(\xi^*) \tag{1.7}$$

in which C is a constant independent of x and t.

This constant amplitude must propagate with velocity $(d\omega/d\xi)^*$. If we call the wave train having the constant amplitude a wave group, $(d\omega/d\xi)^*$ means the velocity of the wave group thus defined.

JEFFREYS[8] started also from (1.3). Putting

$$x f(\xi) = i(\omega t - \xi x) , \tag{1.8}$$

he had from (1.3)

$$\psi(x, t) = \int_{-\infty}^{\infty} A(\xi) \exp\{x f(\xi)\} d\xi . \tag{1.9}$$

If "x is large", (1.10)

the right hand side of (1.9) can be calculated approximately by means of the method of steepest descent. The saddle point in this case is given by the next condition,

$$\mathrm{d\,f}(\xi)/\mathrm{d}\,\xi=0 \quad \text{that is} \quad \mathrm{d}\,\omega/\mathrm{d}\,\xi=x/t. \tag{1.11}$$

Owing to condition (1.10), we have a large amplitude in (1.9) near $\xi=\xi_0$ which satisfies (1.11). The large amplitude thus obtained should propagate with velocity $(\mathrm{d}\omega/\mathrm{d}\xi)_{\xi=\xi_0}$. If we call this velocity a group velocity, the group velocity never become negative, because we assumed in (1.3) x and t to be positive.

It must be remembered that the wave group defined by (1.7) could not always have a large amplitude. In order to determine the value of C, another condition should be added to (1.7) so that $\bar{\psi}$ may have a large amplitude. In order to calculate the right hand side of (1.9), we should assume that $\mathrm{A}(\xi)$ might be constant near $\xi=\xi_0$. This assumption seems to be inconsistent with (1.4). Emphasizing condition (1.10), however, we can make $\delta\xi$ very small, keeping the assumption active.

STRATTON made a serious view of spectrum $\mathrm{A}(\xi)$. On the other hand, JEFFREYS thought much of $\exp\{(x\,\mathrm{f}(\xi)\}$. To tell the truth, definition of group velocity (1.11) should not be available without comparison of two spectra $\mathrm{A}(\xi)$ and $\exp\{x\ \mathrm{f}\ (\xi)\}$.

Taking account of the two spectra at the same time, we must consider $\mathrm{C}(\xi^*)$ in (1.7) to be a function of x. If x will become very large, the wave group satisfying (1.11) must be predominant.

1.3 Thin full lines shown in Fig. 2(a) exhibit time-distance curves of every peak for surface waves which were investigated by HAMADA[9]. Among these wave groups, the ranges are marked by thick full lines where peaks and troughs having respectively periods of 16μs and 20μs seem to reach. The chain line is drawn in such a way that a straight line may pass centers of each full line through the origin. The two chain lines can be taken as time-distance relations for wave groups whose periods are respectively 16μs and 20μs.

Where epicentral distance x is large, the center of each wave group coincides well with the chain line. Therefore the reciprocal of inclination of the chain line might be corresponded with the group velocity (1.11) defined by JEFFREYS.

On the other hand, where x is small, the center of each wave group runs off the chain line and x/t of the former has a different value from that of the latter. In this region, no wave satisfying (1.11) can be predominated but the wave group defined by (1.7) may be observed in practice, being C not zero.

By the same procedure as that of Fig. 2(a), thick full lines of Fig. 2(b) have been obtained where only breadth H of the model differs to each others.

Noticing in these figures the epicentral distance x_0 where the center of wave groups coincides with the chain line for the first time, we find that x_0 is smaller for wave group having short period than for a wave group whose period is long. This fact seems to be natural, because the smaller the period, the shorter the wave-length. A wave group having a long period becomes predominant and x_0 becomes large with increase of thickness of the layer.

Group velocities observed by HAMADA were mean values of x/t for each wave train surrounded by dotted lines in Fig. 2. According to his figure, the mean value observed coincides almost with calculated value $d\omega/d\xi$, at least within the range of periods observed. This fact is apt to be considered as a

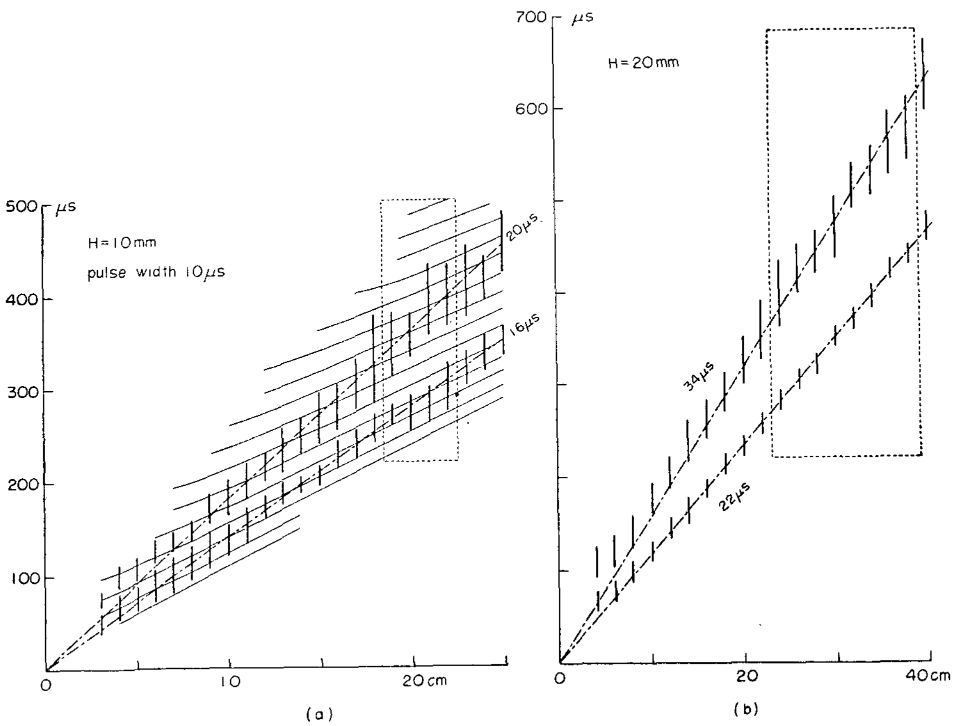

Fig. 2. Time-distance relations of wave groups observed by HAMADA's model experiment.

matter of course. To tell the truth, however, there is no definite reason why x/t must coincide with $d\omega/d\xi$, as described in section **1.2**, if there is no condition other than that given already. It is to be anticipated for the first time when the wave group satisfying conditions (1.10) and (1.11) may be expected by any other reason to have large amplitude that x/t will coincide with $(d\omega/d\xi)$. But this anticipation is, so to speak, an ideal phenomenon. It has been not yet known how far the ideal may be available in practice.

Observed value x/t by HAMADA seems in Fig. 2 to be obtained at so large epicentral distances that x satisfies condition (1.10). Conversely speaking, we can say that it has been ascertained by the coincidence of observed x/t with calculated $d\omega/d\xi$ that the wave group observed by HAMADA would be that satisfying (1.10) as well as (1.11). From this view point, we meet with a question whether any essential wave group satisfying (1.11) may exist or not on the part where x/t could not be observed although $d\omega/d\xi$ was calculated.

Making general considerations, apart from the experimental result by HAMADA, we cannot permit that any part of dispersion curves may satisfy (1.11) where $d\omega/d\xi$ becomes zero or negative. This part, therefore never correspond to the wave group defined by JEFFREYS. Indeed every part of dispersion curves may have a possibility of satisfying (1.11) where $d\omega/d\xi$ is positive, but the possibility must be a question to be examined. Even if condition (1.11) might be satisfied, moreover, we cannot know from this simple reason whether the wave group may have a predominant amplitude or not.

2. Experiments with a source generating sinusoidal waves of a finite duration

2.1 Characteristics of surface waves are used to be explained on basis of FOURIER components of a wave train observed. When a source generates a pulse, it may be apprehended that a spectrum of the pulse should have influences on characteristics of a wave group. In order to avoid this anxiety, it will be desirable that the experiment is to be carried out by sinusoidal waves in place of a pulse. If sinusoidal waves of an infinite duration are used, however, each wave group cannot be observed separately. Therefore sinusoidal waves continuing τ in time must be taken up:

$$f(t) = \begin{cases} 0 & \text{for} \quad |t| > \tau/2\,, \\ \cos\omega^* t & \text{for} \quad |t| < \tau/2\,. \end{cases} \tag{2.1}$$

The spectrum of (2.1) is expressed by

$$A(\omega) = (1/\pi)\,(\omega^* - \omega)^{-1} \sin\{(\omega^* - \omega)\,(\tau/2)\} \tag{2.2}$$

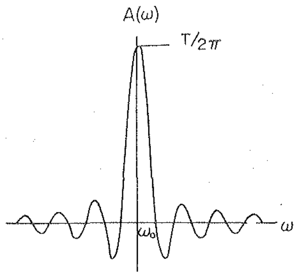

Fig. 3. The spectrum for sinusoidal waves of a finite duration.

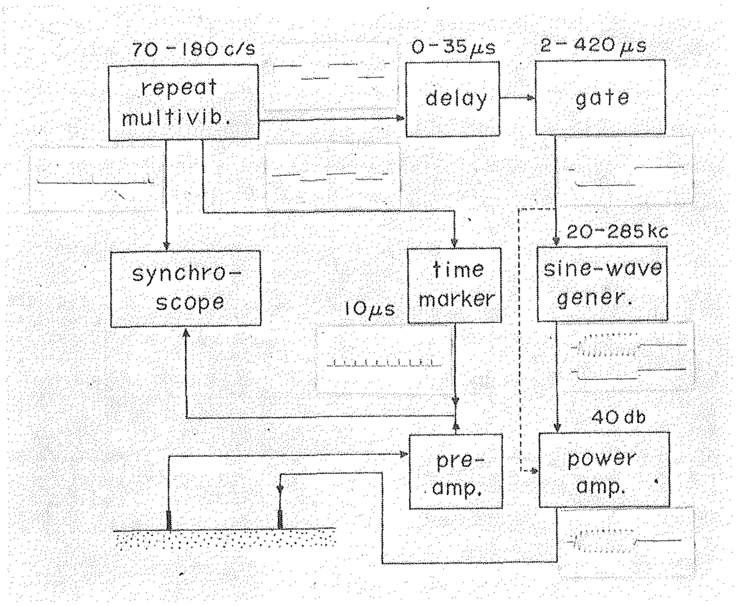

Fig. 4. The block-diagram of the apparatus for generating sinusoidal waves of a finite duration.

and is exhibited in Fig. 3. The larger τ of (2.1), the sharper the peak of $A(\omega)$ near $\omega=\omega^*$, resulting in the same condition as (1.4). Thus, according to STRATTON's opinion described in **1.2**, a wave group will be expected on the straight line given by (1.7).

2.2 The block-diagram for the apparatus which were used in the present experiment is exhibited in Fig. 4. The synchro-scope was that on the market (IWASAKI, SS-5302). The frequency band of the sine-wave generator lies between 20 kc and 285 kc. A duration of oscillations is contralled by the gate circuit. Durations of oscillations are repeated with frequencies from 70 c/s to 180 c/s by the repeat multivibrator. For the purpose of synchronizing the time mark with the break out time, oscillations of sinusoidal waves are put back 0 to 35 μs by the delay circuit. The electrical wave form of out-put from each block is also shown in the figure. The circuit is so designed that the out-put from the gate may be also used as a source of a rectangular wave, as shown by a dotted line.

The arrangement for observing the wave form of direct P-wave is exhibited in Fig. 5(a). The receiver is attached under a plastic block of 1 cm^3 which is welded, at depth 80 mm from the surface, to a large plastic plate whose size is $900\times600\times2$ mm^3. Fig. 5(b) illustrates the arrangement for observing surface waves, where the source as well as the receiver are set on the surface of a superficial layer. The superficial layer in the present model is made of a plastic plate whose thickness is equal to that of the lower layer. The latter is made of an alminum plate whose size is $2000\times1000\times2$ mm^3.

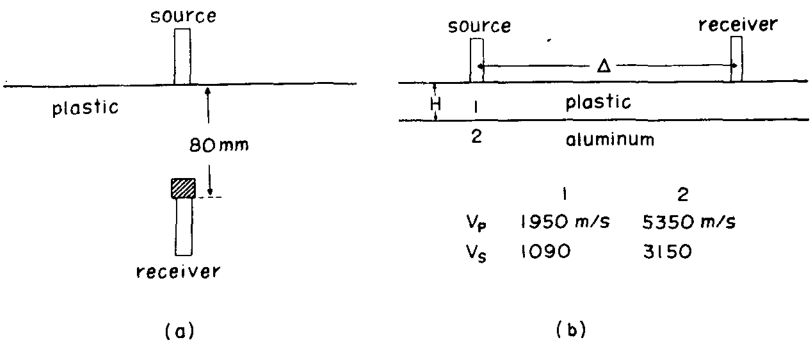

Fig. 5. The arrangement of the source and the receiver on the two dimentional model.

2.3 Direct P-waves as well as surface waves observed are illustrated in Fig. 6, which were generated from the same sinusoidal source having various periods and durations. The surface waves were observed at a place 20 cm distant from the source on two models: breadth of the superficial layer was 15 mm for one and 10 mm for the other. When those three types of waves were observed, it was tried to keep condition of oscillations constant for each period. Gain of the amplifier was so adjusted that amplitudes of P-waves might not be

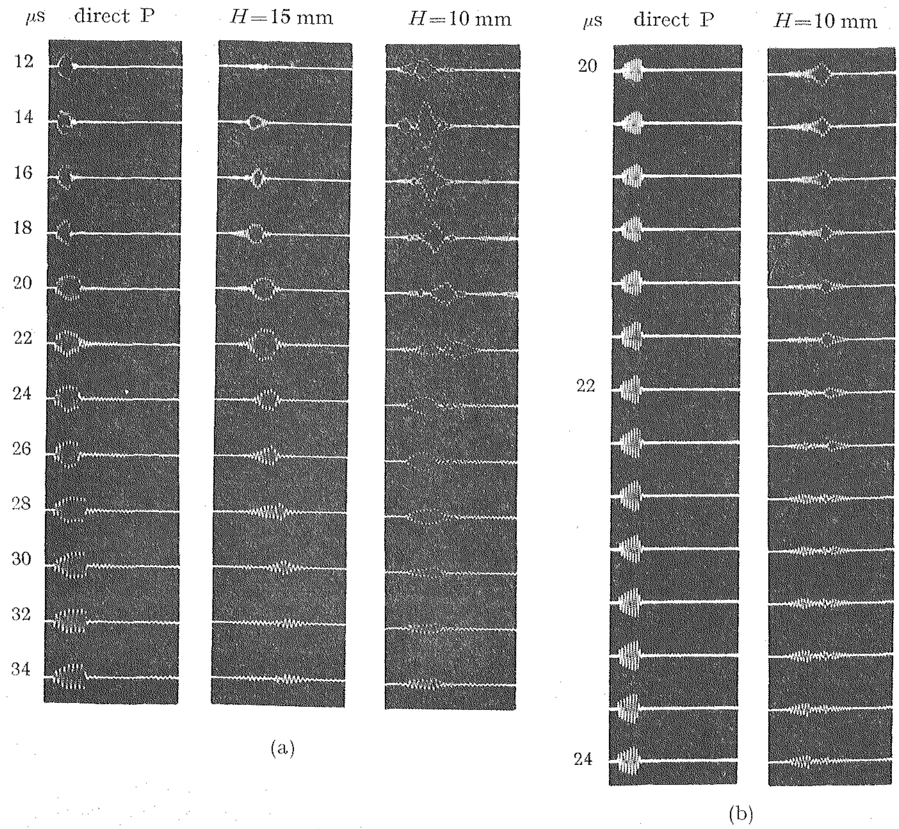

Fig. 6. Records of elastic waves for various periods.

varied by shifting period. The break out time was made common to each record.

Looking at Fig. 6(a), we see surface waves predominant in records denoted by 15 mm and 10 mm. Arrival times of these surface waves are changed by period. Amplitudes of the surface waves are also changed while they have the maximum for any period. These facts will suggest characteristics of surface waves.

Looking at the record, for instance that of period 20μs for the model H= 15 mm, in which surface waves have specially large amplitude, we see the wave form resembles closely that of direct P-waves. Not only the period but also wave numbers of both waves are coincided to each other.

Appearances of wave groups are abruptly varied between 22μs and 24μs in the record for $H=10$ mm of Fig. 6(a). At a glance, it may be considered that the wave group has a minimum group velocity near 22μs. In order to confirm this, observations were made precisely between 20μs and 24μs, resulting in Fig. 6(b). This figure shows that the precedent wave group disappears near 24μs and a new wave group seems to grow up from there.

2.4 Spectra of P-waves generated by a sinusoidal source of a finite duration are illustrated in Fig. 7 which shows a sharp peak for each period given by the oscillator. These spectra are similar to an ideal one exhibited in Fig. 3.

Next, variation of the wave form was observed, keeping period 20μs and epicentral distance 20 cm, by changing only breadth of the gate. The

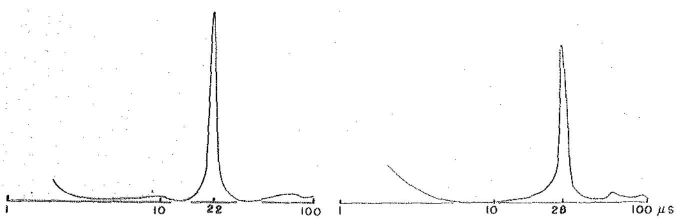

Fig. 7. Spectra of direct P-waves.

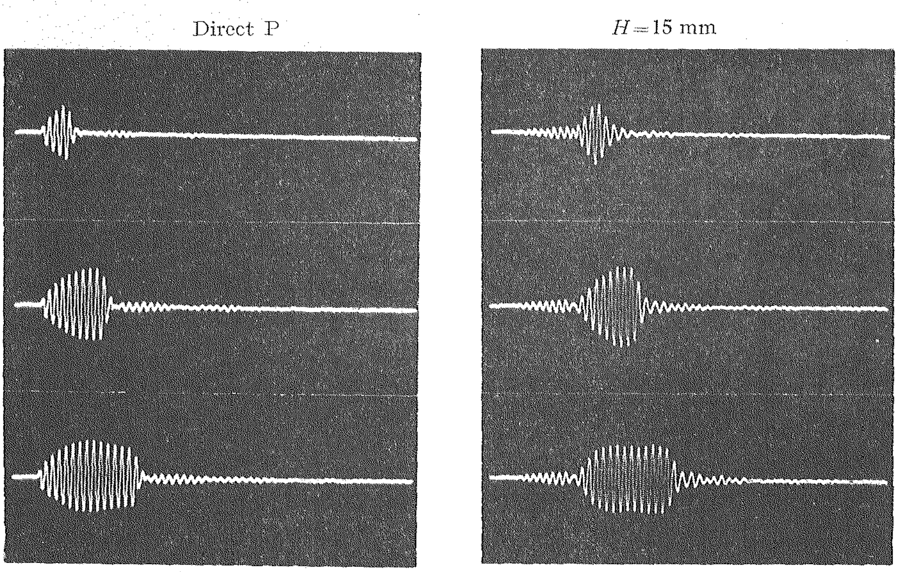

Fig. 8. Relations between the wave form and width of the gate.

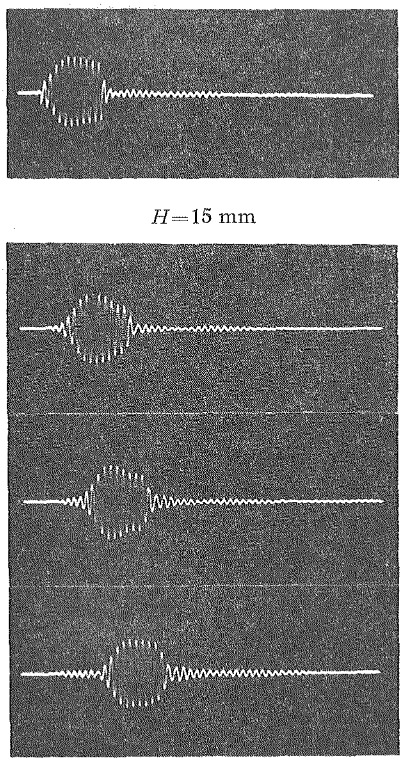

Fig. 9. Relations between the wave form and epicentral distance.

results are illustrated in Fig. 8. Comparing P-waves with surface waves, we see that the period and wave numbers of both waves are equal to each other and amplitude ratio of surface waves to direct P-waves is unchanged by different breadths of the gate.

The last test was carried out, keeping period as well as breadth of the gate constant, by changing only epicentral distance. The records obtained are illustrated in Fig. 9. Arrival time is delayed but wave form is scarcely changed with increase of epicentral distance.

2.5 Infering the spectrum of direct P-waves illustrated in Fig. 6, we can consider that wave groups observed in the present experiment may be built up by STRATTON's theory. However, we had another anticipation at the end of **1.2**: a wave group having a large amplitude might be expected when C in (1.7) became zero, if epicentral distance was large.

Treating the present observed data, we have tentatively taken x/t as a group velocity in which x means epicentral distance and t is the difference between the arrival time of the center of a surface wave group and that of P-waves. In the present case, x is fixed upon 20 cm. Group velocities thus obtained are illustrated by black and white circles in Fig. 10. White circles are corresponded to the wave group in Fig. 6 which was developped by larger periods than $24\mu s$. The curve drawn by a full line is $d\omega/d\xi$ which was calculated for the fundamental mode of dispersive RAYLEIGH waves in each model.

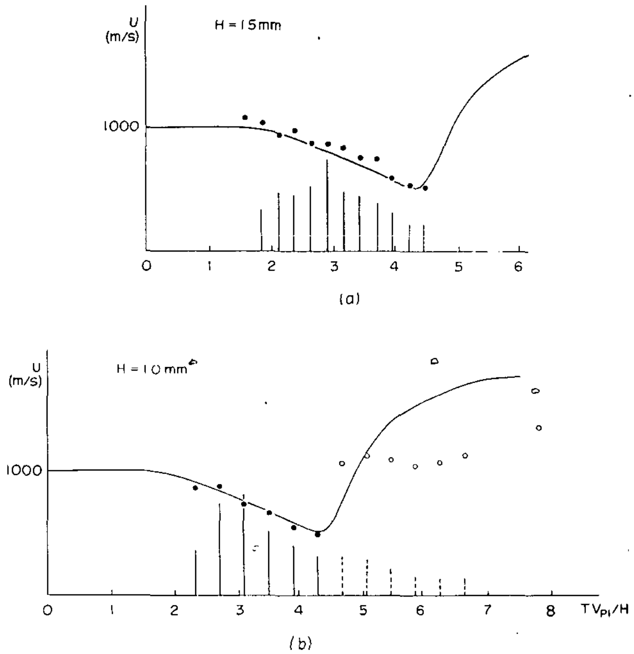

Fig. 10. Relation between group velocities, amplitudes and periods.

The graph by sticks in Fig. 10 illustrates relative amplitudes where sticks drawn by full lines correspond to black circles and those by dotted lines to white circles.

It must be noticed that black circles as well as sticks of full lines have coincided with the experimental result by HAMADA[10]. Therefore it seems that the records of surface wave groups illustrated in Fig. 6 correspond to the thick full lines at an epicentral distance in Fig. 2. Because Figs. 2 and 6 have been obtained from the experiments for the same model, numerical value

20 cm of the present epicentral distance seems to satisfy the assumption given in (1.10), in spite of so sharp spectrum used in the experiment as that shown in Fig. 7.

In conclusion, we have had the next relation:

$$\text{observed } x/t \text{ is equal to calculated } d\omega/d\xi \tag{2.3}$$

on the part drawn by black circles. On the other hand, the part drawn by white circles runs off the above relation.

2.6 Relation (2.3) is not satisfied beyond the period corresponding to the minimum of $d\omega/d\xi$ in the experiment by KNOPOFF *et al*[11] who investigated also sinusoidal waves of a finite duration. They attributed the disagreement to errors in their observation.

Indeed white circles having large periods had large errors in the present measurement, but it seemed in Fig. 6 (b) that black and white circles in Fig. 10 belonged to different wave groups each other.

The larger the period, the larger the phase velocity, resulting in large wave-length. It will become difficult for waves having large wave-length to satisfy assumption (1.10).

Wave groups on the part drawn by black circles should satisfy (1.11) but those by white circles might not. If we have such a point of view as that, two wave groups shown by black and white circles are to be different from each other.

It lies moreover at our hearts that velocities indicated by white circles are not varied by change of period but coincide almost with the velocity of S-wave in the superficial layer. As explained at the end of 1.3, condition (1.11) cannot always be satisfied throughout the range of dispersion curves illustrating calculated $d\omega/d\xi$.

The property of the wave group shown by white circles could not be pursued farther in the present experiment. The authors think that any problem left in the consideration of **1.2** will be clarified by full investigations of the wave group in question.

Appendix

The electrical circuit of the sine-wave generator is exhibited in Fig. 11 which was contained in the block diagram of the present apparatus.

Acknowledgements The apparatus used in the present experiment was designed by Dr. Nozomu DEN, Department of Geophysics in Hokkaido

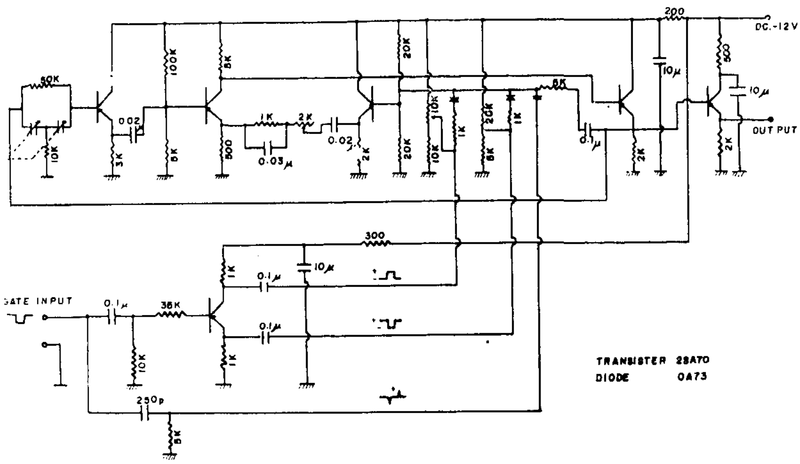

Fig. 11. The electrical circuit of the sine-wave generator.

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